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Evaluation report on the PhD thesis of Michal Bathory

The Thesis consists of two parts: **(i)** Existence theory in the framework of generalized solutions of a model of viscoelastic fluids in the full thermodynamics framework; **(ii)** existence theory of a simplified model, where the thermal effects are neglected. The underlying model is non–linear and contains the Navier–Stokes system as one of the field equations. Accordingly, the problem of existence of global–in–time solutions for large initial data in the physically relevant 3d case in the framework of strong solutions is completely open. The question is thefore studied for weak (distributional) solutions, where the choice of the function spaces is dictated by the available (known) *a priori* bounds.

The introductory part is devoted to the derivation of the model based on the laws of thermodynamics, and a proper choice of constitutive relations. The basic form of the underlying system of equations is repeated many times in the Thesis. The derivation from the basic principles of thermodynamics is written in detail, which makes the Thesis self–contained. The crucial assumption is, of course, the specific form of the free energy (1.9), taken over from [65], that reduces the internal energy to a linear function of the temperature. On the other hand, this part is basically reproducing the material contained in refs. [27], [53], among others.

Some remarks:

- There is a great piece of information that is "sacrificed" by the choice of the weak formulation of the problem. This fact should not be interpreted as a drawback of the model but rather as inability of the available analytical tools to handle it.

- The text contains various philosophical comments that are rather misleading and should be avoided: "an appropriate existence theory indicates what is the minimal regularity that we can expect" etc.

- The system of equations (2.22) with the form of the internal energy $e = c_v\theta$ is not stated correctly. This consitutive relation results from the special choice of Helmholtz free energy explained later.

The part devoted to the mathematical theory is the heart of the Thesis. The crucial is the weak formulation of the problem based on replacing the energy equation (in some form) by the entropy inequality and the total energy balance. The latter here is also replaced by inequality. This idea is by no means original and has been used many times in the existing literature. In particular, the weak formulation of the Navier–Stokes–Fourier system in ref. [34] is based exactly on the same principle. An application of the same ansatz in the context of incompressible fluids for a system (formally) similar to the present model can be found in:

E.F., M. Frémond, E.Rocca, G. Schimperna: A new approach to non–isothermal models for nematic liquid crystals, Arch. Rational Mech. Anal. 205 (2012), 651–672

Some remarks:

- The remark about the term $\theta B \cdot D\mathbf{v}$ destroying the minimum principle in the heat equation is rather misleading. This term, being linear in θ and therefore vanishing for $\theta = 0$ is still compatible with the minimum principle as long as the other term $B \cdot D\mathbf{v}$ can be controlled. This is, of course, not the case here.

- The energy in an energetically closed system should not be "lost" as the inequality (3.26) indicates. The explanation via [28] is irrelevant as this paper deals with a dissipative system (the heat equation is not included). Anyway, I think the present technique (see the comments below) allows for showing equality in (3.26).

- The symbol $W^{-n,p}$ is usually used to denote the dual to $W_0^{n,p'}$ not to $W^{n,p'}$.

- As the present weak formulation contains only a part of information included in the original system of equations, it would be nice to spent more time on its compatibility or formal interpretation, roughly

$$\text{weak} + \text{smooth} \Rightarrow \text{strong}$$

  The phrase "if we are able to multiply" is meaningless in the weak formulation, "it is easy to show" (page 37) does not help the reader too much either.

- Comments on the fact that the hypotheses concerning growth of the constitutive relations for large values of parameters are not physical but in fact irrelevant as the system will never reach this area are of the same kind as saying that velocity is always bounded by the speed of light.
- Definition 3.1. I am not sure about the meaning of the system in the space dimension $d = 1$ in view of the incompressibility condition.
- Definition 3.1. The energy inequality (3.60) is actually much weaker than a weak formulation of (3.26). Relation (3.60) does not include any information about the time derivative of the total energy.

The main result is Theorem 3.2, where the existence of global in time solutions is claimed. The proof is based on an approximation of Galerkin type for the Navier–Stokes system and suitable regularization of the remaining equations. The system is first solved with the internal energy equation, then positivity of the temperature is used to pass to the total energy balance and the entropy inequality. The limit passage, carried over in two steps, is based on compactness of Sobolev embeddings combined with the standard arguments of Aubin–Lions type.

Remark, suggestion. I think the available bounds are strong enough to show the total energy balance as *equality*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left(\frac{1}{2}|\mathbf{v}|^2 + c_v\theta\right) = \int_{\Omega}\mathbf{f}\cdot\mathbf{v}$$

in $\mathcal{D}'(0,T)$ as the corresponding approximations of $\mathbf{v}$ and θ converge strongly (a.a.).

Theorem 3.3 as stated is not correct (or at least its proof not obvious). The correct statement should read: "there exists a suitable weak solution in the sense of Definition 3.1 ..." It is not obvious if the energy equality holds for *any* weak solution but at least one can be constructed by the method specified in the proof of Theorem 3.2. The same problem appears in the formulation of Theorem 3.4

The last part of the Thesis is independent and forms a separate publication. Here, at least, it would be nice to explain how C^∞_n is interpreted on Lipschitz domain.

Conclusion:

The Thesis contains original results concerning global in time existence of weak solutions to certain models of viscoelastic fluids. The methods used are rather complex, and the proofs very technical. The candidate mastered the apparatus of the modern theory of PDE's and applied it in a creative way. On the basis of this work, I recommend the candidate to be granted the PhD degree at the Faculty of Mathematics of the Charles University in the study branch Mathematical and computer modeling.

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