FACULTY
OF MATHEMATICS
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Charles University

MASTER THESIS

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Higher gauge theory

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Study programme:	Physics
Study branch:	Nuclear and Subnuclear Physics

Prague 2020

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I would like to thank to Dr. Branislav Jurco and to my family for their infinite patience.

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Abstract: This thesis gives a short introduction into the higher gauge algebras. We first introduce the BRST formalism in the context of ordinary gauge theories and show the properties that allow us to use it in the context of higher gauge theories. We define the 2-groups and show the correspondence between 2-groups and crossed modules. We then give a brief introduction into the theory of L_∞-algebras - we give account of the graded manifolds and Q-manifolds. We give a short account of Homotopy Maurer-Cartan theory and show that it reduces to the BF theory in case of 4-dimensional manifold and 2-term L_∞-algebra.

Keywords: higher gauge theory L_∞-algebra BRST

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Introduction

One of the central notions in physics is the idea of symmetry. This idea that the laws of physics stay the same even when we change our viewpoint has been a driving concept ever since the days of Galileo and has become the basis of modern physics. In particle physics we encounter symmetry especially in the form of a gauge theory, theory which is described by a Lagrangian that is invariant under local transformations represented by a Lie group. These transformations represent the symmetries between the particles and in order to keep the theory invariant to these symmetries, we have to introduce new fields called gauge bosons into the theory. This adds challenges to quantization of the theory, since the introduced fields are generally tied up by constraints and we have to use new formalisms, known as the Becchi-Rouet-Stora-Tyutin (BRST) and Batalin-Vilkovisky formalisms to efficiently quantise such theory.

The ordinary gauge theory however deals only with point-like particles and many theories concerned with the unification of gravity with the Standard Model (SM), such as the string theory or spin foam models predict the existence of higher-dimensional elementary structures. In order to treat these structures in gauge theory, we have to develop a more general formalism encompassing point-like particles as well as higher dimensional ones. This is the purpose of higher gauge theory. In the higher gauge theory we deal with such problems by categorifying the mathematical structures previously used to describe the ordinary gauge theories. A typical example we will cover in this thesis is a group, which we first describe in terms of category theory and then provide an additional structure by promoting the category into a 2-category.

The higher gauge theory is a very young and very active field of study - most of the advances in the field have taken place in the last few decades. It is also an inspiration for development of new mathematics. Notably it has driven the exploration of L_∞-algebras and has been a driving force in the inquiries into the $(\infty, 1)$-categories. However, since the theory is generally described by concepts of mathematics that are usually unknown to the high energy physics community, so it is still relatively undeveloped and there is still a vast space of possible applications in the theories beyond the SM and in condensed matter physics. In this thesis we would thus like to give a gentle introduction to the theory of higher gauges for ordinary high energy physicists.

We will cover the physical background of the theory, namely the BRST formalism and mathematical prerequisites necessary to understand the language of the theory. We will introduce two different mathematical approaches to higher gauge theory and draw parallels between between them. In particular we will cover 2-groups and L_∞-algebras, which serve to describe the higher gauge theories and show that in the special case of the BF theory they give identical results.

1. Becchi-Rouet-Stora-Tyutin formalism

In this chapter we will describe the physical background for higher gauge theory. As we have mentioned in the Introduction, in order to quantise more complicated gauge theories, we first have to develop a framework that will efficiently describe the gauge theories and their symmetries. In this chapter we will introduce two such frameworks. The first one will be the Becchi-Rouet-Stora-Tyutin framework, which will expose a symmetry present even after the gauge has been fixed. The BRST formalism is suited to deal with the higher gauge theories, since it provides a natural L_∞-algebra structure, which we will describe in the following chapters, however it fails for symmetries which are open off-shell. In order to treat such symmetries we would have to introduce a so called Batalin-Vilkovisky formalism, which deals with these problems by adding new fields - so called 'antifields' - for each field in the theory. Since the BRST formalism will fully suffice for our purposes, we will not cover the BV formalism.

Our exposition will follow a treatment by Weinberg (Weinberg, 1996). A more detailed approach can be found for example in (Gomis et al., 1995).

1.1 Non-Abelian Gauge Theories

We will first very briefly revise the theory of non-abelian gauge theories, setting up notation for a following section.

We say that a theory is invariant under a transformation, when the Lagrangian $\mathscr{L}(\psi)$ of the theory is invariant under the infinitesimal transformation of the fields $\psi_l(x)$

$$\delta\psi_l(x) = i\epsilon^\alpha(x)(t_\alpha)_l{}^m\psi_m(x) \tag{1.1}$$

where $(t_\alpha)_l{}^m$ is a constant matrix representation of the transformation group and $\epsilon^\alpha(x)$ is a real infinitesimal parameter. The transformation group will generally be a Lie group G, and the representation will thus satisfy

- Commutation relations:

$$[t_\alpha, t_\beta] = iC^\gamma{}_{\alpha\beta}t_\gamma \tag{1.2}$$

- Jacobi identity:

$$[[t_\alpha, t_\beta]\, t_\gamma] + [[t_\gamma, t_\alpha]\, t_\beta] + [[t_\beta, t_\gamma]\, t_\alpha] = 0 \tag{1.3}$$

Additionally, in order for the Lagrangian to be invariant with respect to the transformations, we will have to introduce gauge fields that transform as

$$\delta A^\beta_\mu = \partial_\mu\epsilon^\beta + i\epsilon^\alpha C^\beta_{\gamma\alpha}A^\gamma{}_\mu \tag{1.4}$$

and change the derivatives ∂ into covariant derivatives

$$D_\mu = \partial\mu - iA^\beta{}_\mu(x)t \tag{1.5}$$

In order to satisfy the gauge invariance, the Lagrangian of our gauge invariant theory cannot contain mass terms of the gauge field, so the gauge field will be represented only throuugh the covariant derivatives and the gauge field-strength tensors $F^{\alpha}_{\mu\nu}$ defined as follows

$$F^{\gamma}{}_{\mu\nu} := \partial_\mu A^{\gamma}{}_{\nu} - \partial_\nu A^{\gamma}{}_{\mu} + C^{\gamma}{}_{\alpha\beta} A^{\alpha}{}_{\mu} A^{\beta}{}_{\nu} \tag{1.6}$$

The invariant Lagrangian density will then be a function of the fields, their covariant derivatives and the field-strength tensors

$$\mathscr{L} = \mathscr{L}(F^{\alpha}_{\mu\nu}, \psi, D\psi) \tag{1.7}$$

and the action $S[\psi]$ is

$$S[\psi] = \int d^4x \mathscr{L} \tag{1.8}$$

where we integrate over four dimensional spacetime.

We can quantise the theory in the path integral formalism using the De Witt-Faddeev-Popov method, in which the vacuum expectation value of a time-ordered product of gauge-invariant operators $\mathcal{O}_A, \mathcal{O}_B, \cdots$ is equal to the ratio

$$\langle T\{\mathcal{O}_A, \mathcal{O}_B, \cdots\}\rangle_V = \frac{\mathcal{I}(T\{\mathcal{O}_A, \mathcal{O}_B, \cdots\})}{\mathcal{I}(1)} \tag{1.9}$$

$$\mathcal{I}(T\{\mathcal{O}_A, \mathcal{O}_B, \cdots\}) := \int \mu(\phi) T\{\mathcal{O}_A, \mathcal{O}_B, \cdots\} e^{iS(\phi)} \delta(F(\psi)) \det(M_{FP}(\psi)) \tag{1.10}$$

where $\phi_n(x)$ are now a set of ordinary and gauge fields; we will denote the space of all ϕ as $\mathfrak{F}$; $\mu(\phi)$ is a measure in the space of ordinary and gauge fields

$$\mu(\phi) = \prod_{n,x} d\phi_n(x) = \left[\prod_{l,x} d\psi_l(x)\right] \left[\prod_{\alpha,\mu,x} dA^{\mu}{}_{\alpha}(x)\right]. \tag{1.11}$$

The δ is a functional analogue of the δ-distribution. F is a gauge fixing function $F : \mathfrak{F} \to Lie(G)$ from the space of fields to the Lie algebra of the gauge group G , such that each point in $F^{-1}(0)$ represents a different orbit of G. The $\delta(F(\phi))$ thus serves as a restriction of the field space $\mathfrak{F}$ to a particular choice of gauge. Finally the M_{FP} is the Fadeev-Popov matrix defined as

$$(M_{FP})_{\alpha x, \beta y}(\phi) = \frac{\delta F_\alpha(\phi_\epsilon; x)}{\delta \epsilon_\beta(y)} \tag{1.12}$$

where ϕ_ϵ is the field ϕ transformed by a gauge transformation with parameters $\epsilon_\beta(x)$.

As has been proven for example in (Weinberg, 1996), the path integral in equation (1.10) is independent of the gauge-fixing functional F_α. However we still have to fix the gauge in order to integrate only over physical degrees of freedom.

Since it is easier to calculate the path integral $\mathcal{I}$ in the form of a Gaussian, we will further take a Fourier transformation of the $\delta(F(\phi))$

$$\delta(F) = exp\left(-\frac{i}{2\xi} \int d^4x F_\alpha(x) F_\alpha(x)\right) \tag{1.13}$$

with a real parameter ξ. For the purposes of the BRST formalism it is better to rewrite the equation (1.13) as a Fourier integral over so-called 'Nakanishi-Lautrup' fields

$$\delta(F) = \int \left[\prod_{\alpha,x} dh_\alpha(x)\right] \exp\left[\frac{i\xi}{2}\int h_\alpha h_\alpha\right] \exp\left[i\int d^4x F_\alpha h_\alpha\right] \tag{1.14}$$

We can also express the Fadeev-Popov determinant as a path integral

$$\det(M_{FP}) \propto \int \left[\prod_{\alpha,x} d\omega_\alpha(x)\right] \left[\prod_{\alpha,x} d\omega^*_\alpha(x)\right] \exp\left(i\int d^4x d^4y \omega^*_\alpha(x)\omega_\beta(y) \left(M_{FP}\right)_{\alpha x,\beta y}\right) \tag{1.15}$$

where we introduce new fields ω_α. These fields have to be fermionic for theF integral to be proportional to $\det(M_{FP})$ and are scalar with respect to Lorentz transformation. However as can be proven in the BRST formalism, these fields cannot appear in initial or final states and appear only as virtual particles, which is why they can have integer spin and still be fermionic. We call these fields the ghost and antighost fields and introduce the ghost number equal +1 for ω_α and −1 for ω^*_α and 0 for all other fields. The ghost number is then conserved at every vertex.

Defining

$$\Delta_\alpha(x) := \int d^4y \left(M_{FP}\right)_{\alpha x,\beta y} \omega_\beta(y) \tag{1.16}$$

we can further rewrite relation (1.15) as

$$\det(M_{FP}) \propto \int \left[\prod_{\alpha,x} d\omega_\alpha(x)\right] \left[\prod_{\alpha,x} d\omega^*_\alpha(x)\right] \exp\left(i\int d^4x \omega^*_\alpha(x)\Delta_\alpha(x)\right) \tag{1.17}$$

Using the equations (1.14 - 1.17) in the relation (1.10) we finally get

$$\mathcal{I}_{\mathcal{MOD}}(T\left\{\mathcal{O}_A, \mathcal{O}_B, \cdots\right\}) := \int \mu(\phi, \omega, \omega^*, h) T\left\{\mathcal{O}_A, \mathcal{O}_B, \cdots\right\} e^{iS_{MOD}(\phi,\omega,\omega^*,h)} \tag{1.18}$$

with the modified action $S_{MOD}[\phi, \omega, \omega^*, h]$ is now

$$S_{MOD}[\phi, \omega, \omega^*, h] = \int d^4x \left(\mathscr{L} + \omega^*_\alpha \Lambda_\alpha + h_\alpha F_\alpha + \frac{\xi}{2} h_\alpha h_\alpha\right) \tag{1.19}$$

1.2 BRST formalism

The De Witt-Faddeev-Popov method has a great disadvantage - since we have to choose the gauge in order to quantise the theory, we lose the explicit gauge invariance of the theory. This poses problems when proving the renormalisability of the theory, since after choosing the gauge we do not know which counterterms to the ultraviolet divergences are restricted by the gauge symmetry. Also the De Witt-Faddeev-Popov method introduces unphysical ghost fields and it is hard to prove that they only appear as virtual particles. A remedy to these problems is the BRST formalism. This formalism introduces a so called BRST symmetry, which is present even after fixing a gauge.

We define the BRST symmetry as a global symmetry, parametrised by an infinitesimal constant θ that anticommutes with fermionic fields and with ω and ω^*. The BRST transformation has a form

$$\delta_\theta \psi = it_\alpha \theta \omega_\alpha \psi \tag{1.20}$$
$$\delta_\theta A_{\alpha\mu} = \theta D_\mu \omega_\alpha = \theta[\partial_\mu \omega_\alpha + C_{\alpha\beta\gamma} A_{\beta\mu} \omega_\gamma] \tag{1.21}$$
$$\delta_\theta \omega^*_\alpha = -\theta h_\alpha \tag{1.22}$$
$$\delta_\theta \omega_\alpha = -\frac{1}{2}\theta C_{\alpha\beta\gamma} \omega_\beta \omega_\gamma \tag{1.23}$$
$$\delta_\theta h_\alpha = 0 \tag{1.24}$$

To see that the action in equation (1.19) is invariant, we will first note that when acting on ordinary fields ψ and on gauge fields $A_{\alpha\mu}$, the BRST symmetry is in fact a gauge symmetry with a parameter

$$\epsilon_\alpha(x) = \theta \omega_\alpha(x) \tag{1.25}$$

Since the original Lagrangian $\mathscr{L}$ is only a function of the original fields ψ, their covariant derivations $D\psi$ and of the gauge field strengths $F^\alpha_{\mu\nu}$: $\mathscr{L}(F^\alpha_{\mu\nu}, \psi, D\psi)$, it will be automatically invariant to the BRST transformation

$$\delta_\theta \int d^4x \mathscr{L} = 0 \tag{1.26}$$

To prove that the rest of the action S_{MOD} is invariant to the BRST transformation, we first define a BRST operator s as

$$\delta_\theta F := \theta s F \tag{1.27}$$

for arbitrary functional $F(\psi, A, \omega, \omega^*, h)$ of fields. As is shown in (Weinberg, 1996), this operator is nilpotent

$$s(sF) = 0 \tag{1.28}$$

for any functional $F(\psi, A, \omega, \omega^*, h)$.

Next we note that using the definition (1.16), the BRST transformation applied to the gauge-fixing function F is

$$\delta_\theta F_\alpha(x) = \int \left.\frac{\delta F_\alpha(x)}{\delta \epsilon^\beta(y)}\right|_{\epsilon=0} \theta \omega_\beta(y) d^4y \tag{1.29}$$
$$= \theta \int (M_{FP})_{\alpha x, \beta y}\, \omega_\beta(y) d^4y \tag{1.30}$$
$$= \theta \Delta_\alpha(x) \tag{1.31}$$

Using the transformation rules (1.22) and (1.24) we can thus rewrite the last three parts of the modified action S_{MOD} as

$$\int d^4x \left(\omega^*_\alpha \Delta_\alpha + h_\alpha F_\alpha + \frac{1}{2}\xi h_\alpha h_\alpha\right) = \int d^4x \left(s\left(\omega^*_\alpha F_\alpha + \frac{1}{2}\xi \omega^*_\alpha h_\alpha\right)\right) = s\Psi \tag{1.32}$$

where we have introduced the gauge-fixing fermion Ψ

$$\Psi := \int d^4x \left(\omega^*_\alpha F_\alpha + \frac{1}{2}\xi \omega^*_\alpha h_\alpha\right) \tag{1.33}$$

We can thus rewrite the modified action S_{MOD} as

$$S_{MOD} = \int d^4x \mathscr{L} + s\Psi \tag{1.34}$$

and since the BRST operator is nilpotent, we have shown that the modified action S_{MOD} is BRST-invariant.

The equation (1.34) also shows, that the gauge-fixing fermion Ψ can be changed by $s\Psi'$ for an arbitrary functional Ψ' of fields leaving the physical contents of the theory unchanged.

We can also define a new fermionic BRST charge Q via the BRST transformation of any field Ψ

$$\delta_\theta \Psi = i[\theta Q, \Psi] = i\theta[Q, \Psi]_\mp \tag{1.35}$$

In terms of the BRST operator s we thus have

$$is\Psi = [Q, \Psi] \tag{1.36}$$

Since the BRST operator s is nilpotent, we have

$$0 = -ss\Psi = [Q, [Q, \Psi]_\mp]_\mp = [Q^2, \Psi]_- \tag{1.37}$$

for all fields Ψ. This means that the BRST charge is either zero or proportional to the unit operator. However since Q^2 has a non-zero ghost number we have

$$Q^2 = 0 \tag{1.38}$$

The BRST charge thus acts as a derivation in the space of BRST fields. We will see later that the existence of this operator, together with the graded vector structure of the space of fields ensures that we can express the BRST field space as a L_∞-algebra.

2. Mathematical prerequisites

Higher gauge theory has many mathematical facets and can be viewed via different mathematical formulations. In this section we will introduce the mathematical apparatus that will allow us to view the theory in a crossed-module formalism and from a point of view of L_∞-algebras. The former one is a special case of a much higher and more abstract formalism of much more abstract and elegant categorified groups and the theory of the nonabelian gerbes - see for example (Aschieri et al., 2004). However to give a proper introduction to these concepts, this thesis would have to be made into a book. In order to keep the thesis within reasonable bounds, we have instead decided to show, that these crossed modules are in fact in an one-to-one correspondence to L_∞-algebra generated strictly in degrees 1 and 2. In the next chapter we will then apply the L_∞-algebra formalism to a special case of homotopy Maurer-Cartan action and show that the results obtained in the formalism of categorified groups can be obtained also in the L_∞-algebra.

2.1 Category theory

Category theory is a very abstract part of mathematics, even to such an extent that the mathematicians pursuing categories have an endearing term 'abstract nonsense' for many of its parts. It abstracts the notion of relationships between objects as arrows and studies the properties such arrows and relationships between arrows and relationships between relationships between arrows, and so on . . . have. Here we will only provide a very brief and very incomplete review of the parts of category theory needed to attempt to understand the higher gauge theories. An interested reader can find much more complete surveys of this wonderful part of mathematics in e.g. (Mac Lane, 2013) for general introduction to the subject and in (Cisinski, 2019) for an introduction of higher categories in the context of homotopical algebras.

Definition 1. *A category C consists of:*

- *a collection of objects: $Ob(C)$*
- *for each $X, Y \in Ob(C)$ a set of arrows from X to Y, denoted $C(X, Y)$ or $Hom_C(X, Y)$*
- *for each $X, Y, Z \in Ob(C)$ and each arrow $f \in C(X, Y)$ and $g \in C(Y, Z)$, there is a composition of arrows $gf \in C(X, Z)$*
- *for each $X \in Ob(C)$ there is an arrow $\mathbb{1}_X \in C(X, X)$ called the identity on X*

which satisfy the following two axioms:

- *associativity: for each $f \in C(W, X)$, $g \in C(X, Y)$ and $h \in C(Y, Z)$: $(hg)f = h(gf)$*
- *identity: for each $X, Y \in Ob(C)$ and each $f \in C(X, Y)$: $f\mathbb{1}_X = \mathbb{1}_Y f = f$*

We will frequently use diagrams to visualise the concepts of category theory. In these diagrams the arrows $f \in C(X,Y)$ will be depicted as

$$Y \xleftarrow{f} X$$

so that the composition of arrows takes natural form

$$Z \xleftarrow{f} Y \xleftarrow{g} X \;=\; Z \xleftarrow{fg} X$$

The arrows in the category theory have many names. Ususally they are called either arrows or morphisms and in this thesis we will use both of the terms interchangeably. For an arrow:

$$Y \xleftarrow{f} X$$

we will call X the source of f and Y the target of f.

The most natural example of category is a category of sets Set, where the objects are sets and the morphisms are functions. We distinguish two kinds of categories - small and large ones. The small categories are the ones for which the set of objects and the sets of morphisms are all small, i.e. they are a real set and not a proper class. The large categories are all the others. In this treatise we will deal only with the small categories for simplicity.

We will also denote the set of all arrows in the category C by Hom_C.

Definition 2. *A category where each arrow f has an inverse arrow f^{-1}, such that $ff^{-1} = f^{-1}f = \mathbb{1}$ is called a groupoid.*

The simplest but by no means only example of a groupoid is a group. If we take a groupoid with a single object X, we can take the arrows $f, g \in C(X,X)$ as the elements of the group with a group multiplication being the composition of the arrows $fg \sim f \cdot g$. With the association law and the identity element we will recover all the group axioms.

To make the concept of categories useful we should add some maps between the categories. These maps will respect the identity arrows and composition of arrows in categories and will be called functors.

Definition 3. *A functor F from a category C into category C' consists of*

- *a map between objects $Ob(C)$ and $Ob(C')$ of the categories:*

$$\begin{aligned} F : &Ob(C) \to Ob(C') \\ &X \mapsto F(X) \end{aligned}$$

- *a map which associates an arrow $F(f) \in Hom_{C'}$ to each arrow in $f \in Hom_C$ so that:*
  - *for all $f \in C(A,B)$ and $g \in C(B,C)$ $F(gf) = F(g)F(f)$ for a covariant functor and $F(gf) = F(f)F(g)$ for a contravariant one.*
  - *for all $X \in Ob(C)$ $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$*

A functor will be a very important part of discussion, since we can express many objects as functors from one category to another, with the categories encoding the properties of the object. Unless explicitly specified, we will always presume a functor to be a covariant one.

An example of a functor could be a functor from a group G to Set, which encodes the action of a group on a particular set.

We can view a category as a set endowed with composable maps between the elements of the set. We could take the concept further and add composable maps between the morphisms in the category. If we require that all possible compositions define unambiguous morphisms, we end up with a 2-category.

Definition 4. *A strict 2-category C consists of*

- *a collection of objects $Ob(C)$ also called 0-cells*
- *for all objects X, Y in $Ob(C)$ a set of morphisms $C(X,Y)$ also called 1-cells, the elements of which $f \in C(X,Y)$ we will visualise as*

$$\begin{tikzcd} Y & X \arrow[l, bend right, "f"'] \end{tikzcd}$$

- *for any pair of morphisms f, g in $C(X,Y)$ a set of 2-morphisms - arrows between the morphisms, also called 2-cells which we will label by Greek letters and we will visualise each 2-cell $\alpha : f \Rightarrow g$ as*

$$\begin{tikzcd} Y & X \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

- *for any pair of 0-cells $f \in C(X,Y)$ and $g \in C(Y,Z)$ a composition $fg \in C(X,Z)$*

$$\begin{tikzcd} Z & Y \arrow[l, bend right, "f"'] & X \arrow[l, bend right, "g"'] \end{tikzcd} = \begin{tikzcd} Z & X \arrow[l, bend right, "fg"'] \end{tikzcd}$$

- *for any triplet of 0-cells $f, g, h \in C(X,Y)$ and any pair of 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ a vertical composition $\beta \cdot \alpha$:*

$$\begin{tikzcd} Y & X \arrow[l, bend right=60, "f"'] \arrow[l, "g"'] \arrow[l, bend left=60, "h"] \arrow[l, phantom, shift right=3, "\Downarrow\alpha"] \arrow[l, phantom, shift left=3, "\Downarrow\beta"] \end{tikzcd} = \begin{tikzcd} Y & X \arrow[l, bend right, "f"'] \arrow[l, bend left, "h"] \arrow[l, phantom, "\Downarrow\beta\cdot\alpha"] \end{tikzcd}$$

- *for any quadruplet of 0-cells $f, f' \in C(X,Y)$ and $g, g' \in C(Y,Z)$ and any pair of 2-cells $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ a horizontal composition $\beta \circ \alpha : gf \Rightarrow g'f'$:*

$$\begin{tikzcd} Z & Y \arrow[l, bend right, "g"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\Downarrow\beta"] & X \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\Downarrow\alpha"] \end{tikzcd} = \begin{tikzcd} Z & X \arrow[l, bend right, "gf"'] \arrow[l, bend left, "g'f'"] \arrow[l, phantom, "\Downarrow\beta\circ\alpha"] \end{tikzcd}$$

- *for any 0-cell $X \in Ob(C)$ an identity 1-cell $\mathbb{1}_X \in C(X,X)$ acting as identity for composition of 1-cells*

- *for any identity 1-cell $\mathbb{1}_X \in C(X,X)$ an identity 2-cell $\mathbb{1}_{\mathbb{1}_X}$ acting as identity in horizontal composition of 2-cells*

- *for any identity 1-cell $f \in C(X,Y)$ an identity 2-cell $\mathbb{1}_f$ acting as identity in vertical composition of 2-cells*

with the compositions satisfying the interchange law:

- *for any quadruple of 2-cells we require $(\alpha \cdot \alpha') \circ (\beta \cdot \beta') = (\alpha \circ \beta) \cdot (\alpha' \circ \beta')$*

We will again use the terms 1-cell and morphism and 2-cell and 2-morphism interchangeably. We will adopt the notation from the simple categories and for a 2-cell:

$$\begin{tikzcd} Y & X \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

we will call f the source of α and g the target of α.

This definition may seem a bit long-winded, but all it says is that all the possible map compositions are unique and that for each type of composition there exists a identity element.We can also view the set of morphisms in the definition as a category itself with 2-morphisms as morphisms, with vertical composition being composition of morphisms and endowed with a functor of horizontal composition $\circ : C(B,C) \times C(A,B) \to C(A,C)$.

In a similar fashion we could create the n-categories, which have morphisms, 2-morphisms, ..., up to n-morphisms, with all the compositions unique and with identities with respect to every composition.

As an example of a 2-category we will use the familiar groupoid. First we will define it and then we will show its properties

Definition 5. *A 2-groupoid is a 2-category such that*

- *for each 1-cell $f \in C(X,Y)$ there is an inverse 1-cell f^{-1} such that $ff^{-1} = \mathbb{1}_Y$ and $f^{-1}f = \mathbb{1}_X$*

- *for each 2-cell $\alpha : f \Rightarrow g$ there is a vertical inverse 2-cell α^{-1}_{vert} such that $\alpha \cdot \alpha^{-1}_{vert} = \mathbb{1}_g$ and $\alpha^{-1}_{vert} \cdot \alpha = \mathbb{1}_f$*

From the interchange law and the two conditions in definition 5 we may infer that every 2-cell $\alpha : f \Rightarrow g$ with $f \in C(X,Y)$ also has a horizontal inverse α^{-1}_{hor} such that $\alpha^{-1}_{hor} \circ \alpha = \mathbb{1}_{\mathbb{1}_X}$ and $\alpha \circ \alpha^{-1}_{hor} = \mathbb{1}_{\mathbb{1}_X}$.

An example of 2-groupoid is a 2-group, a 2-groupoid with one object. We will later prove that this 2-group is in fact equivalent to a crossed module. We will however first have to define both of them.

On this example we can see a general method of creating higher structures - we take an ordinary object, we define it in terms of a category and then we promote the category to a n-category and generalise the rules (e.g. existence of an inverse morphism) that defined the object. However the example we have shown is not the most elegant way to express the higher algebraical structures, since in this form we still retain the strict equalities from the definition of a group. A much more elegant way is to define the higher structures via a so called 'vertical categorification', where we allow the aforementioned equalities to hold only up to isomorphism. Using the vertical categorification on our example of groups would lead to so-called 'weak groups'. Unfortunately the mathematical formalism behind the vertical categorification is out of the scope of this thesis, so we will refer an interested reader to a more complete literature, for example this article (Baez and Lauda, 2003).

2.2 Crossed modules

Crossed modules were first introduced by J. H. C. Whitehead (Whitehead, 1946) in the context of homotopy theory. It soon became apparent that they are a very helpful tool even outside the scope of homotopy theory. In 1976 R. Brown and C. B. Spencer (Brown and Spencer, 1976) proved that they are in fact equivalent to 2-groups as we will show here. We will use the crossed modules in the following chapter in a description of BF theory.

Definition 6. *A crossed module* (G, H, t, α) *consists of*

- *two groups - G and H*
- *an action of G on H, which is an automorphism*[1]*:*

$$\alpha : G \to Aut(H)$$

- *a group homomorphism $t : H \to G$ satisfying*
  - *(equivariance with respect to conjugation): for each $g \in G$ and each $h \in H$ we have $t(\alpha(g)(h)) = gt(h)g^{-1}$*
  - *(Peiffer identity) for each $h, h' \in H$ we have $\alpha(t(h))h' = hh'h^{-1}$*

The most simple example of a crossed module is a group G with a normal subgroup N, with the maps t being the inclusion

$$t := i : N \to G$$

and α being an action of the group elements of G on the elements of N.

Let us show that that the crossed modules are equivalent to 2-groups, that is - we can reproduce the structure of a 2-group from a structure of a crossed module and vice versa.

Let us first see that each 2-group $\mathcal{G}$ can encode a crossed module.

Since each arrow in a 2-groupoid has an inverse arrow, the set of 1-cells $Hom_{\mathcal{G}}$ is clearly a group, with a group operation being the composition of 1-cells $f, f' \in Hom_{\mathcal{G}}$. Since the 2-group has only one element, we will now depict it as a dot $\bullet$:

$$\bullet \xleftarrow{f} \bullet \xleftarrow{f'} \bullet = \bullet \xleftarrow{ff'} \bullet$$

[1] An automorphism of a group is a bijective homomorphism of a group with itself.

We can thus identify the set $Hom_{\mathcal{G}}$ with the group G from a definition 6.

As the group H from the definition 6 we will take the 2-cells with the source being the identity $\mathbb{1}_\bullet$:

$$\begin{tikzcd}[column sep=large] \bullet & \bullet \arrow[l, bend right=40, "1_\bullet"'] \arrow[l, bend left=40, "t(\eta)"] \arrow[l, phantom, "\Downarrow\eta"] \end{tikzcd}$$

As a group operation we will use the horizontal composition of the 2-cells:

$$\begin{tikzcd}[column sep=large] \bullet & \bullet \arrow[l, bend right=40, "1_\bullet"'] \arrow[l, bend left=40, "t(\eta\circ\eta')"] \arrow[l, phantom, "\Downarrow\eta\eta'"] & = & \bullet & \bullet \arrow[l, bend right=40, "1_\bullet"'] \arrow[l, bend left=40, "t(\eta)"] \arrow[l, phantom, "\Downarrow\eta"] & \bullet \arrow[l, bend right=40, "1_\bullet"'] \arrow[l, bend left=40, "t(\eta')"] \arrow[l, phantom, "\Downarrow\eta'"] \end{tikzcd}$$

This, together with the existence of a horizontal inverse η_{hor}^{-1} makes H into a group.

As the group homomorphism we will take a mapping that assigns to each 2-cell η its target:

$$\begin{tikzcd}[column sep=large] \bullet & \bullet \arrow[l, bend right=40, "1_\bullet"'] \arrow[l, bend left=40, "t(\eta)"] \arrow[l, phantom, "\Downarrow\eta"] \end{tikzcd}$$

From the horizontal composition law, we can see that $t : H \to G$ is in fact a group homomorphism:

$$t(\eta\eta') = t(\eta)t(\eta'). \tag{2.1}$$

As the action α from the definition 6 we take a horizontal conjugation of elements $\eta \in H$ by identity 2-cells $\mathbb{1}_g$:

$$\alpha(g)(\eta) = \mathbb{1}_g \circ \eta \circ \mathbb{1}_{g^{-1}} \tag{2.2}$$

This maps each η with a target $t(\eta)$ to $\alpha(g)(\eta)$ with a target $gt(\eta)g^{-1}$:

$$\begin{tikzcd}[column sep=large] \alpha(g): \bullet & \bullet \arrow[l, bend right=40, "1_\bullet"'] \arrow[l, bend left=40, "t(\eta)"] \arrow[l, phantom, "\Downarrow\eta"] & \longmapsto & \bullet & \bullet \arrow[l, bend right=40, "g"'] \arrow[l, bend left=40, "g"] \arrow[l, phantom, "\Downarrow 1_g"] & \bullet \arrow[l, bend right=40, "1_\bullet"'] \arrow[l, bend left=40, "t(\eta)"] \arrow[l, phantom, "\Downarrow\eta"] & \bullet \arrow[l, bend right=40, "g^{-1}"'] \arrow[l, bend left=40, "g^{-1}"] \arrow[l, phantom, "\Downarrow 1_{g^{-1}}"] \end{tikzcd}$$

It follows trivially that the horizontal conjugation is in fact a group homomorphism

$$\alpha(gg') = \alpha(g)\alpha(g'). \tag{2.3}$$

Since for each pair $\eta, \eta' \in H$ there is an element $\eta'' = \eta \circ \eta'^{-1}_{vert}$ such that

$$\eta = \eta''\eta'$$

we find that this action is onto.

Using the equation (2.3) with $g' = g^{-1}$, we find that

$$\alpha(g)(\eta) = \alpha(g)(\eta') \Rightarrow \eta = \eta'$$

so the action is also one-to-one and thus an automorphism of H.

From the diagram

$$
\begin{tikzcd}
\bullet & \bullet \arrow[l, bend right, "g"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow 1_g"] & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(\eta)"] \arrow[l, phantom, "\Downarrow \eta"] & \bullet \arrow[l, bend right, "g^{-1}"'] \arrow[l, bend left, "g^{-1}"] \arrow[l, phantom, "\Downarrow 1_g^{-1}"] & = & \bullet & \bullet \arrow[l, bend right, "1"'] \arrow[l, bend left, "t(\alpha(g)(\eta))"] \arrow[l, phantom, "\Downarrow \alpha(g)(\eta)"]
\end{tikzcd}
$$

we can infer that the group homomorphism $t : H \to G$ is equivariant with respect to the action α.

The last step is showing that the group homomorphism t and the action α satisfy the Peiffer identity. We prove this by taking $\eta \circ \eta' \circ \eta_{hor}^{-1}$ for any $\eta, \eta' \in H$:

$$
\begin{tikzcd}
\bullet & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(\eta)"] \arrow[l, phantom, "\Downarrow \eta"] & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(\eta')"] \arrow[l, phantom, "\Downarrow \eta'"] & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(\eta^{-1})"] \arrow[l, phantom, "\Downarrow \eta_{hor}^{-1}"]
\end{tikzcd}
$$

we can decompose each η as $\mathbb{1}_{1_\bullet} \cdot \eta$:

$$
\begin{tikzcd}
\bullet & & \bullet \arrow[ll, bend right=60, "1_\bullet"'] \arrow[ll, "t(\eta)"'] \arrow[ll, bend left=60, "t(\eta)"] \arrow[ll, phantom, shift right=3, "\Downarrow \eta"] \arrow[ll, phantom, shift left=3, "\Downarrow 1_{t(\eta)}"]
\end{tikzcd}
$$

thus getting

$$
\begin{tikzcd}
\bullet & & \bullet \arrow[ll, bend right=60, "1_\bullet"'] \arrow[ll, "t(\eta)"'] \arrow[ll, bend left=60, "t(\eta)"] \arrow[ll, phantom, shift right=3, "\Downarrow \eta"] \arrow[ll, phantom, shift left=3, "\Downarrow 1_{t(\eta)}"] & & \bullet \arrow[ll, bend right=60, "1_\bullet"'] \arrow[ll, "1_\bullet"'] \arrow[ll, bend left=60, "t(\eta)"] \arrow[ll, phantom, shift right=3, "\Downarrow 1_{1_\bullet}"] \arrow[ll, phantom, shift left=3, "\Downarrow \eta"] & & \bullet \arrow[ll, bend right=60, "1_\bullet"'] \arrow[ll, "t(\eta^{-1})"'] \arrow[ll, bend left=60, "t(\eta^{-1})"] \arrow[ll, phantom, shift right=3, "\Downarrow \eta^{-1}"] \arrow[ll, phantom, shift left=3, "\Downarrow 1_{t(\eta^{-1})}"]
\end{tikzcd}
$$

Using the interchange law, we can first compose the 2-cells horizontally getting:

$$
\begin{tikzcd}
\bullet & & \bullet \arrow[ll, bend right=60, "1_\bullet"'] \arrow[ll, "1_\bullet"'] \arrow[ll, bend left=60, "t(\eta\eta'\eta^{-1})"] \arrow[ll, phantom, shift right=3, "\Downarrow 1_{1_\bullet}"] \arrow[ll, phantom, shift left=3, "\Downarrow \alpha(t(\eta))(\eta')"]
\end{tikzcd}
$$

Which shows that the Peiffer identity

$$
\alpha(t(\eta))\eta' = \eta\eta'\eta_{hor}^{-1} \tag{2.4}
$$

holds. We have thus shown that each 2-group encodes a crossed module.

With this framework in place it is much easier to show that each crossed module encodes a 2-group $\mathcal{G}$. We take a single-object 2-category and identify the morphisms of the 2-group $\mathcal{G}$ with the elements of the group G. From the group structure of G we see that the composition of the morphisms is associative and unital and that there is an inverse morphism for each morphism present. Next we identify the pairs $(g,h) \in G \times H$ with 2-morphisms $\eta(g,h)$. Each 2-morphism is then defined as

$$\eta(g,h) : g \mapsto t(h)g \tag{2.5}$$

We may also view each 2-morphism as a pair of 2-morphisms:

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(h)"] \arrow[l, phantom, "\Downarrow h"] & \bullet \arrow[l, bend right, "g"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow 1_g"]
\end{tikzcd}$$

Vertical composition of $\eta(g,h)$ and $\eta(g',h')$ is then given by:

$$\eta(g,h) \cdot \eta(g',h') = \eta(g', hh'). \tag{2.6}$$

From the group properties of H it follows that the vertical composition is unital and associative.

We can infer the horizontal composition from the following diagrams using the Peiffer identity:

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(h)"] \arrow[l, phantom, "\Downarrow h"] & \bullet \arrow[l, bend right, "g"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow 1_g"] & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(h')"] \arrow[l, phantom, "\Downarrow h'"] & \bullet \arrow[l, bend right, "g'"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\Downarrow 1_{g'}"]
\end{tikzcd}$$

$$= \begin{tikzcd}
\bullet & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(h)"] \arrow[l, phantom, "\Downarrow h"] & \bullet \arrow[l, bend right, "g"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow 1_g"] & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(h')"] \arrow[l, phantom, "\Downarrow h'"] & \bullet \arrow[l, bend right, "g^{-1}"'] \arrow[l, bend left, "g^{-1}"] \arrow[l, phantom, "\Downarrow 1_{g^{-1}}"] & \bullet \arrow[l, bend right, "g"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow 1_g"] & \bullet \arrow[l, bend right, "g'"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\Downarrow 1_{g'}"]
\end{tikzcd}$$

$$= \begin{tikzcd}
\bullet & \bullet \arrow[l, bend right, "1_\bullet"'] \arrow[l, bend left, "t(h)"] \arrow[l, phantom, "\Downarrow h"] & & \bullet \arrow[ll, bend right, "1_\bullet"'] \arrow[ll, bend left, "t(\alpha(g)(h'))"] \arrow[ll, phantom, "\Downarrow \alpha(g)(h')"] & \bullet \arrow[l, bend right, "gg'"'] \arrow[l, bend left, "gg'"] \arrow[l, phantom, "\Downarrow 1_{gg'}"]
\end{tikzcd}$$

$$= \begin{tikzcd}
\bullet & & \bullet \arrow[ll, bend right, "1_\bullet"'] \arrow[ll, bend left, "t(h\alpha(g)(h'))"] \arrow[ll, phantom, "\Downarrow h\alpha(g)(h')"] & \bullet \arrow[l, bend right, "gg'"'] \arrow[l, bend left, "gg'"] \arrow[l, phantom, "\Downarrow 1_{gg'}"]
\end{tikzcd}$$

The horizontal composition of $\eta(g,h)$ and $\eta(g',h')$ is then given by:

$$\eta(g,h) \circ \eta(g',h') = \eta(gg', h\alpha(g)(h')) \tag{2.7}$$

From the group properties of G and H we again see that the horizontal composition is unital and associative. We have thus shown that there is a crossed module encoded in every 2-group and vice versa.

Having shown that the cross modules and 2-groups are equivalent, we can now use cross modules to define Lie 2-groups.

Definition 7. *A Lie 2-group is a 2-group $\mathcal{G}$ with the associated cross-module (G, H, t, α), where the groups G and H are Lie groups and the maps $t : H \to G$ and $\alpha : G \to Aut(H)$ are smooth.*

2.3 L_∞-algebras

The history of L_∞-algebras is closely tied up with the study of structure of BRST complex. They were first introduced in 1992 by B Zwiebach in (Zwiebach, 1993). The L_∞-algebras present a structure that generalises the Lie algebras - they are in fact ∞-categorifications of Lie algebra. In this section we will introduce the mathematical tools necessary to express these algebras through differential $\mathbb{Z}$-graded vector spaces. We will also show how the Q-manifold encode such a L_∞-algebra. First we will however have to introduce sheaves and differential graded vector spaces.

2.3.1 Sheaves

We fist define sheaves. These can be defined either via étale bundles or via functors. We will use the definition via functors as it is much more flexible and it shows the previously mentioned concepts of category theory.

Definition 8. *Let X be a topological space. A presheaf F on X is a functor with values in a category C given by the following:*

1. *for each open subset $U \in X$ there is an object $F(U)$ in C*
2. *for each inclusion of open sets $V \subset U$ there is a morphism $res_{V,U} : F(U) \to F(V)$ in C called a restriction morphism such that*
   (a) *for all open subsets U of X $res_{U,U}$ is the identity $\mathbb{1}_U$*
   (b) *for all open subsets W, V, U of X, such that $W \subset V \subset U$ we have*
$$res_{W,V} \circ res_{V,U} = res_{W,U}$$

For an object $s \in F(U)$ define the restriction in C to an open subset $V \subset X$ as $s|_V := res_{V,U}(s)$.

We can also define presheaf much more elegantly by defining fist the category of open sets on X to be the posetal[2] category $\mathcal{O}(X)$ with

- objects are open sets of X
- morphisms are inclusions

With such a definition a C-valued presheaf is the contravariant functor from $\mathcal{O}(X)$ to C. This definition is much more elegant, however it obscures the substance of presheafs, contained in the restriction maps.

Definition 9. *A sheaf S is a presheaf with values in the category of sets satisfying:*

1. *(Locality:) If $(U_i)_{i \in I}$ is an open covering of an open subset $U \subset X$ and if $s, t \in F(U)$ are such that:*
$$s|_{U_i} = t|_{U_i}$$
   for all $i \in I$ then $s = t$.

[2] A posetal category P is a category in which for each pair of objects $X, Y \in Ob(P)$ the set of homomorphisms $P(X, Y)$ contains at most one element.

2. *(Gluing:) If $(U_i)_{i\in I}$ is an open covering of an open subset $U \subset X$ and if for each $i \in I$ we have a section $s_i \in F(U_i)$ such that for each pair i, j*

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$$

then there exists a section $s \in F(U)$ called a gluing such that

$$s|_{U_i} = s_i$$

A presheaf satisfying only the first condition is called a separated presheaf or a monopresheaf.

A typical example of a sheaf is the set of smooth functions $C^\infty(X)$ from a topological space X to real numbers with a restriction map being a literal restriction of functions. If we take a C^k-manifold M then the sheaf of k-times differentiable functions $\mathcal{O}_M$ is called a structure sheaf of the manifold.

2.3.2 Graded vector spaces and graded differential algebras

Another piece of the L_∞-algebra puzzle are the graded vector spaces. These vector spaces introduce an additional structure in a form of grading - the possibility of decomposing the vector space into a direct sum of smaller vector spaces. This additional structure allows us to introduce graded commutative product along with graded Poisson structure and thus provides us with an elegant way of for example including fermionic operators in mathematical formalism of the quantum field theory.

Graded vector fields are of great use in beyond the standard model theories, such as supersymmetry, where a Z_2-graded vector space is used to describe the physical space (Gates Jr et al., 2001).

Definition 10. *Let us have a collection of real vector spaces $(V_i)_{i\in\mathbb{Z}}$.*

- *A $\mathbb{Z}$-graded vector space is a direct sum $V = \bigoplus_{k\in\mathbb{Z}} V_k$.*
- *A non-zero element $v \in V$ which belongs to a single vector space V_i is said to be homogeneous of degree i.*
- *A graded basis $(v_i)_i$ is a sequence of elements of V such that every subsequence $(v_{i_j})_{j\in\mathbb{Z}}$ of all elements of degree k is a basis of V_k*
- *We define a map*

  $$|.| : V \to \mathbb{Z}$$

  assigning degree i to a homogeneous element $v_i \in V_i$.
- *We define a degree shift by $l \in \mathbb{Z}$ as*

  $$V[l] = \bigoplus_{k\in\mathbb{Z}} (V[l])_k \text{ where } (V[l])_k := V_{k+l} \tag{2.8}$$

  Note that this means that if the only non-trivial part of V was V_k for some $k \in \mathbb{Z}$, then the degree-shifted space $V[l]$ will consist only of vectors of degree $k - l$, since only $V[l]_{k-l} = V_k$ is non-trivial.

- *Given two $\mathbb{Z}$-graded vector spaces V and W, their direct sum $V \oplus W$ is defined with*
$$(V \oplus W)_k = (V_i \oplus W_i)$$

- *Given two $\mathbb{Z}$-graded vector spaces V and W, their tensor product $V \otimes W$ is defined with*
$$(V \otimes W)_k = \bigoplus_{l \in \mathbb{Z}} (V_l \otimes W_{k-l})$$

- *Given two $\mathbb{Z}$-graded vector spaces V and W, we say that a map $f : V \to W$ is homogeneous of degree $k \in \mathbb{Z}$ if $f(V_i) \subset W_{i+k}$. We denote the degree of a map f by $|f|$.*

- *A morphism of $\mathbb{Z}$-graded vector spaces is a linear map of degree 0.*

In this thesis we will consider only finite $\mathbb{Z}$-graded vector spaces, for which $dimV_i < \infty$ for all $i \in \mathbb{Z}$.

A typical example of a $\mathbb{Z}$-graded vector space is a space of polynomials, with the homogeneous elements of degree consisting of homogeneous polynomials. We will also see in the next chapter, that the space of fields in the BRST formalism is naturally expressed as a graded vector field, with the ghost number having the meaning of degree of homogeneous element.

Definition 11. *A graded commutative algebra A is a $\mathbb{Z}$-graded vector space that is also an associative unital commutative algebra with the product $A \times A \to A$ being graded commutative, that is for all $a_1, a_2 \in A$ homogeneous we have*

$$a_1 a_2 = (-1)^{|a_1||a_2|} a_2 a_1 \tag{2.9}$$

An example of a graded commutative algebra is a space of differential forms on a manifold X: $\Omega^\bullet(X)$ endowed with an exterior product $\wedge$.

We can also identify the space $\Omega^\bullet(X)$ with the space $C^\infty(T[1]X)$, the space of smooth functions from the tangent bundle of X shifted by 1. We can thus identify the forms dx^α with the coordinate functions ξ^α on the shifted vector bundle. ‘

Definition 12. *A differential graded commutative algebra (A, d) is a graded commutative algebra endowed with a derivation d - a set of differential derivations $d_k : A_k \to A_{k+1}$ of homogeneous degree 1. These derivations satisfy*

$$d_{k+1} \circ d_k = 0 \tag{2.10}$$

and the obey the graded Leibniz rule

$$d(a_1 a_2) = (da_1) a_2 + (-1)^{|a_1|} a_1 (da_2) \tag{2.11}$$

for all $a_1, a_2 \in A$ with a_1 being homogeneous of degree $|a_1|$.

Proceeding with our previous example, the graded commutative algebra of differential forms on a manifold X can be endowed with an exterior derivative d, making $\Omega^\bullet(X)$ into a differential graded commutative algebra. Using the notation from previous example, the space $C^\infty(T[1]X)$ can be endowed with a differential structure as well by introducing the vector field $Q = \xi^\alpha \frac{\partial}{\partial \xi^\alpha}$.

Definition 13. *A morphism $f : (A, d) \to (A', d')$ between two differential graded algebras (A, d) and (A', d') is a collection of maps $f_k : A_k \to A'_k$ with $k \in \mathbb{Z}$ of degree 0 which for all $k \in \mathbb{Z}$ satisfy*

$$f \circ d = d \circ f \tag{2.12}$$

Definition 14. *An isomorphism $f : (A, d) \to (A', d')$ is an invertible morphism between two differential graded algebras (A, d) and (A', d').*

Given a graded vector space V, we define the real tensor algebra of V, denoted as $T(V)$ or $\bigotimes^\bullet V$ as

$$\bigotimes^\bullet V \ := \ \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \ \cdots \tag{2.13}$$

We can define the totally graded symmetric subalgebra of $\bigotimes^\bullet V$ as

$$\bigodot^\bullet V \ := \ \mathbb{R} \oplus V \oplus (V \odot V) \oplus \ \cdots \tag{2.14}$$

and totally graded antisymmetric subalgebra

$$\bigwedge^\bullet V \ := \ \mathbb{R} \oplus V \oplus (V \wedge V) \oplus \ \cdots \tag{2.15}$$

In this setting for a graded-symmetric vector $a_1 \odot a_2 \in V \odot V$ we have

$$a_1 \odot a_2 = (-1)^{|a_1||a_2|} a_2 \odot a_1 \tag{2.16}$$

We will denote the k-fold totally symmetric product of the graded vector space as

$$\bigodot^k V \tag{2.17}$$

and similarly for k-fold totally antisymmetric product and k-fold tensor product.

Definition 15. *We define the symmetric and antisymmetric Koszul signs of a permutation σ $\epsilon(\sigma; v_1, \ldots, v_i)$ and $\chi(\sigma; v_1, \ldots, v_i)$ by equations*

$$v_1 \odot \cdots \odot v_i \ = \ \epsilon(\sigma; v_1, \ldots, v_i) v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)} \tag{2.18}$$

and

$$v_1 \wedge \ldots \wedge v_i \ = \ \chi(\sigma; v_1, \ldots, v_i) v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(i)} \tag{2.19}$$

for all v_j in a graded vector space V.

2.3.3 $\mathbb{Z}$-graded manifolds

We still need to give a few more definitions, before we construct the $\mathbb{Z}$-graded manifold. First we need to give a definition of ideal and the local ringed space.

Definition 16. *For a ring $(R, +, \cdot)$ let $(R, +)$ be its additive group.*

- *A subset I is called a left ideal if:*
  - *(I, +) is a subgroup of (R,+)*

– *for all $r \in R$ and all $x \in I$: $rx \in I$*

- *A right ideal I is defined analogously with all $x \in I$ and $r \in R$ satisfying $xr \in I$. A two-sided ideal is a left ideal that is also a right ideal.*
- *A unit ideal of a ring R is an ideal consisting the whole ring R.*
- *A proper ideal is an ideal I of R such that I is not a unit ideal.*
- *A maximal ideal of a ring $(R, +, \cdot)$ is a proper ideal I for which there is no other J such that $I \subset J$.*

Definition 17. *A ringed space M is a pair $(|M|, \mathcal{S}_M)$ where $|M|$ is a topological space and $\mathcal{S}_M$ is a sheaf of rings on $|M|$ called the structure sheaf of M.*

Definition 18. *A locally ringed space is a ringed space $(|M|, \mathcal{S}_M)$, such that all stalks*[3] *of $\mathcal{S}_M$ have unique maximal ideals.*

Definition 19. *A $\mathbb{Z}$-graded ringed space M is a ringed space for which the structure sheaf $\mathcal{S}_M$ is also a associative unital graded-commutative $\mathbb{Z}$-graded ring.*

Definition 20. *A morphism of $\mathbb{Z}$-graded ringed spaces $f : (|M|, A) \to (|N|, B)$ is a pair of mappings $(|f|, f^*)$, where*

- *$|f| : |M| \to |N|$ is a morphism of topological spaces*
- *$f^* : A \to B$ is a collection of morphisms $(f_V)_{V \subset |N|}$ $f_V : B(V) \to A(f^{-1}(V))$*

A morphism of $\mathbb{Z}$-graded local ringed spaces $f : (|M|, A) \to (|N|, B)$ is a morphism for which the comorphism f^* preserves the maximal ideals

Definition 21. *Let $(|M|, A)$ and $(|N|, B)$ be graded locally ringed spaces. We say that these spaces are locally isomorphic, if for each point $x \in |M|$ there is an open neighbourhood $U \subset |M|$ and an open set $V \subset |N|$ such that there exists an isomorphism of graded locally ringed spaces $f_V : (U, A|_U) \to (V, B|_V)$.*

We can now finally define a $\mathbb{Z}$-graded manifold.

Definition 22. *A real $\mathbb{Z}$-graded manifold is a ringed space $M = (|M|, \mathcal{S}_M)$ where $|M|$ is a real topological manifold, such that for each point $x \in |M|$ there is an open neighbourhood $U \subset |M|$ of x and an open set $U' \subset \mathbb{R}^n$, and $\mathcal{V}_{U'}$ a locally free $\mathbb{Z}$-graded sheaf of $\mathcal{C}^\infty_{U'}$ modules on U'*[4][5]*, such that*

$$(U, \mathcal{S}_M|_U) \;\cong\; (U', \bigodot{}^\bullet \mathcal{V}^*_{U'} \otimes \mathcal{C}^\infty_{U'}). \tag{2.20}$$

[3] We will not define stalks here, since their construction requires direct limits, which are very abstract and would add nothing of value to this thesis. An intuitive notion of stalks is that they are the contents of sheaf when restricted to a single point x of the underlying topological space X. We construct them as limit of restrictions to a sequence of neighbourhoods of the point x with each successive neighbourhood being contained in the previous one.

[4] Once again, we will not define a locally free sheaf, since the details are not very illuminating to the mater at hand. We can intuitively visualise it as a direct sum of structure sheafs mentioned under the definition 9.

[5] $\mathcal{C}^\infty_{U'}$ module is a module of $\mathcal{C}^\infty$ functions from U' to real numbers, in our case with added gradation.

We will denote the set of $\mathbb{Z}$-graded functions from the definition as

$$C^\infty(U) := \bigodot^\bullet \mathcal{V}^*_{U'} \otimes \mathcal{C}^\infty_{U'} \tag{2.21}$$

With this definition we see that the sheaf $\mathcal{S}_M$ looks locally as a set of smooth graded functions from the neighbourhood U to real numbers. If we set $U' = \mathbb{R}^n$ and take a set of generators ξ^α of $\bigodot^\bullet \mathcal{V}^*_{U'}$, we can express the elements f of $C^\infty(\mathbb{R}^n)$ formally as infinite polynomials in the generators ξ^α with the smooth functions $f_0, f_\alpha, f_{\alpha\beta}, \ldots \in \mathcal{C}^\infty(R^n)$ as coefficients:

$$f(x,\xi) \ = \ f_0(x) + \xi^\alpha f_\alpha(x) + \tfrac{1}{2!}\xi^\alpha\xi^\beta f_{\alpha\beta}(x) + \cdots \tag{2.22}$$

We can thus formally decompose the set $C^\infty(U)$ as

$$C^\infty(U) \cong \bigoplus_{k\in\mathbb{Z}} \mathcal{C}^\infty_k(M). \tag{2.23}$$

into a direct sum of homogeneous functions of degree k.

2.3.4 Vector fields on differential manifolds

With the definition of a $\mathbb{Z}$-graded manifold in place, we can construct vector fields on them. The vector fields on a graded manifold are defined analogously to vector fields on ordinary manifolds - we define them through as maps from the set of functions $C^\infty(M)$ on the graded manifold M satisfying a version of Leibniz's law. We however add the graded structure to it and we thus end up with the graded Leibniz law.

Definition 23. *A vector field X^k_V on a $\mathbb{Z}$-graded manifold of degree k is a graded derivation $X^k_V : C^\infty(M) \to C^\infty(M)$, such that for all $f, g \in C^\infty(M)$ homogeneous, the graded Leibniz law holds:*

$$X^k_V(fg) = X^k_V(f)g + (-1)^{k|f|} f X^k_V(g) \tag{2.24}$$

We shall write Vec_M for the set of all vector fields on a $\mathbb{Z}$-graded manifold M.

As a simple example of a graded vector field, we have the Euler vector field Υ which is defined for any homogeneous $f \in C^\infty(M)$ as

$$\Upsilon f = |f| f. \tag{2.25}$$

The Euler vector field is a homogeneous vector field of degree 0.

Definition 24. *A homological vector field $V \in Vec_M$ is a vector field of degree +1, which commutes with itself under the graded commutator*

$$[Q,Q] = 2Q^2 = 0 \tag{2.26}$$

Definition 25. *A Q-manifold is a $\mathbb{Z}$-graded manifold M endowed with a with a homological vector field Q.*

As an example take $\mathbb{Z}$-graded manifold G concentrated in degree 1, that is $\mathbb{Z}$-graded manifold obtained by shifting an ordinary vector space $\mathfrak{g}$ by -1: $G = \mathfrak{g}[1]$. Let ξ^α be the coordinates on the

If we enhance the $\mathbb{Z}$-graded manifolds with a homological vector field Q and add an additional symplectic structure, we also get for a manifold with degree $k \neq -1$ the vector field Q is Hamiltonian.

2.3.5 L_∞-algebras

We have finally enough data to define the L_∞-algebras and to show how they relate to the Q-manifolds.

Definition 26. *A L_∞-algebra consists of*

- *a $\mathbb{Z}$-graded vector space L*
- *for each positive integer n a multilinear totally graded antisymmetric map*

$$\mu_n : L \times \cdots \times L \to L \tag{2.27}$$

of homogeneous degree $2-n$ which satisfies the higher Jacobi identities:

$$\sum_{j+k=i} \sum_{\sigma \in \mathrm{Sh}(j;i)} \chi(\sigma; \ell_1, \ldots, \ell_i)(-1)^k \mu_{k+1}(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(i)}) = 0 \tag{2.28}$$

where $\chi(\sigma; \ell_1, \ldots, \ell_i)$ is the graded Koszul sign defined in 2.19 and Sh *is the set of all $(j;i)$ shuffles σ - that is premutations σ of $\{1, \ldots, i\}$ such that the first j and the last $(i-j)$ images are ordered: $\sigma(1) < \cdots < \sigma(j)$ and $\sigma(j+1) < \cdots < \sigma(i)$.*

The higher Jacobi identities provide us with a generalisation of Jacobi identities. Setting $i = 1$, we get

$$\mu_1(\mu_1(\ell)) = 0 \tag{2.29}$$

and setting $i = 2$ we get

$$\mu_1(\mu_2(\ell_1, \ell_2)) = \mu_2(\mu_1(\ell_1), \ell_2) + (-1)^{|\ell_1|}\mu_2(\ell_1, \mu_1(\ell_2)) \tag{2.30}$$

telling us that μ_1 is a differential respecting the product μ_2.

For $i = 3$ we get a higher Jacobi identity:

$$\begin{aligned} -\mu_1(\mu_3(\ell_1, \ell_2, \ell_3) =& \mu_2(\mu_2(\ell_1, \ell_2), \ell_3) + (-1)^{|\ell_1|(|\ell_2|+|\ell_3|)}\mu_2(\mu_2(\ell_2, \ell_3), \ell_1) \\ &+ (-1)^{|\ell_2|(|\ell_1|+|\ell_3|)}\mu_2(\mu_2(\ell_1, \ell_3), \ell_2) \end{aligned} \tag{2.31}$$

we see that if we set $\mu_i = 0$ for $i \geq 3$ we recover a Lie algebra. Thus the L_∞-algebra is a generalisation of a Lie algebra.

The L_∞-algebras are connected in a way to Q-manifolds. To see this we take the Q-manifold concentrated in degrees $1, \ldots, n$. Such a Q-manifold necessarily has the form $L[1]$ with $L = \bigoplus_{k=-n}^{0} L_k$ being a graded vector space. Let us have ξ^α as the local coordinates of the degree $|\xi^\alpha|$. The most general homological vector field Q can be obtained in the form

$$Q = \sum_{i=1}^{n} \frac{(-1)^{\frac{1}{2}i(i+1)}}{i!} \xi^{\alpha_1} \cdots \xi^{\alpha_i} f_{\alpha_1, \cdots \alpha_i}{}^{\beta} \frac{\partial}{\partial \xi^\beta} \tag{2.32}$$

where $f_{\alpha_1, \cdots \alpha_i}{}^{\beta}$ are constants.

If we now take the basis τ_α of L with the homogeneous degree $|\tau_\alpha| = 1 - |\xi_\alpha|$, we may define the higher product in terms of the constants $f_{\alpha_1, \cdots \alpha_i}{}^{\beta}$:

$$\mu_i(\tau_{\alpha_1}, \ldots, \tau_{\alpha_i}) := f_{\alpha_1, \cdots \alpha_i}{}^{\beta} \tau_\beta \tag{2.33}$$

As has been shown in (Jurčo et al., 2019b), the condition Q^2 then implies that the higher products defined in equation (2.33) satisfy the higher Jacobi identities. We can therefore recover a L_∞-algebra structure from the Q-manifolds.

Definition 27. *A L_∞-algebra endowed with a graded symmetric non-degenerate bilinear pairing*

$$\langle -, - \rangle_L : L \times L \to \mathbb{R} \tag{2.34}$$

which for all $i \in \mathbb{N}$ for al homogeneous $\ell_1, \ldots, \ell_{i+1} \in L$ satisfies

$$\langle \ell_1, \mu_i(\ell_2, \ldots, \ell_{i+1}) \rangle_L \;=\; (-1)^{i+i(|\ell_1|_L+|\ell_{i+1}|_L)+|\ell_{i+1}|_L \sum_{j=1}^{i} |\ell_j|_L} \langle \ell_{i+1}, \mu_i(\ell_1, \ldots, \ell_i) \rangle_L \tag{2.35}$$

is called a cyclic L_∞-algebra.

For a future reference we note that the L_∞-algebra (L, μ) can form a tensor product with a commutative differential graded algebra (A, d), so that the product is a L_∞-algebra. To form this tensor product we define

$$\hat{L} = \bigoplus_{k \in Z} (A \otimes L)_k \quad \text{where} \quad (A \otimes L)_k = \bigoplus_{i+j=k} A_i \otimes L_k. \tag{2.36}$$

For a homogeneous element $a \otimes \ell$ of $\hat{L}$ we thus have

$$|a \otimes \ell|_{\hat{L}} = |a|_A + |\ell|_L \tag{2.37}$$

We define the higher product $\hat{\mu}$ on $\hat{L}$ as

$$\hat{\mu}_1(a_1 \otimes \ell_1) \;:=\; da_1 \otimes \ell_1 + (-1)^{|a_1|_A} a_1 \otimes \mu_1(\ell_1)\ , \tag{2.38}$$

$$\hat{\mu}(a_1 \otimes \ell_1, \ldots, a_i \otimes \ell_i) \;:=\; (-1)^{i \sum_{j=1}^{i} |a_j|_A + \sum_{j=2}^{i} |a_j|_A \sum_{k=1}^{j-1} |\ell_k|_L} \times \tag{2.39}$$

$$\times\, (a_1 \cdots a_i) \otimes \mu_i(\ell_1, \ldots, \ell_i) \tag{2.40}$$

3. Higher gauge theories

In the previous chapters we have shortly revised the physical and mathematical background of the higher gauge theories. In this chapter we will put all the tools we have introduced to work and show, how the higher gauge theories rise from the formalism. In particular we will shortly revise the Maurer Cartan theory and show how it connects with the physical content of quantum field theories.

3.1 Maurer-Cartan Theory

In the first chapter we have revised the BRST formalism. In the process we have introduced a new fermionic BRST charge Q_{BRST}, defined in terms of the BRST symmetry

$$\delta_\theta \Psi = i[\theta Q, \Psi] = i\theta[Q, \Psi]_\mp \tag{3.1}$$

This operator has a ghost number +1 and satisfies

$$Q^2 = 0 \tag{3.2}$$

It thus closely resembles the homological vector field Q from the previous chapter and if we identified the underlying structure of fields with a $\mathbb{Z}$-graded manifold, we could use the formalism developed in the previous chapter to express the gauge symmetries of the theory in L_∞-algebras.

This can easily be done - if we take a look at the (anti-)commutation relations between the fields and the BRST operator and fact, that only the ghost fields have a non-zero ghost number, we can assign a ghost number 0 to the bosonic gauge fields, a ghost number 1 to the fermionic ghost fields, ghost number 2 to the bosonic ghosts of ghosts, that arise from the symmetries of ghosts and so on.

To identify the physical relationships with their mathematical counterparts we use the equations for Q in the contracted form:

$$Qa = -\sum_{l\geq 1} \frac{1}{l!}\mu(a, \dots, a) \tag{3.3}$$

in this form the fields a have to have the homogeneous degree $|a|_{L_\infty} + |a|_{gh} = +1$, where $|a|_{gh}$ is the ghost degree and $|a|_{L_\infty}$ is the L_∞degree of the field. We can thus identify the L_∞ degree as $|a|_{L_\infty} = 1 - |a|_{gh}$. Therefore the gauge fields have the L_∞ degree equal 0, the ghost fields to -1 and so on.

Since we have identified the BRST field space along with the BRST operator with the L_∞-algebra, we can now express the theory of gauge transformation in the framework, which is natural to the L_∞-algebras. This framework is called the Homotopy Maurer-Cartan Theory.

As we have seen before, we can identify the gauge potential a as an element with the L_∞degree +1. We also define the gauge transformations as

$$\delta_{c_0} a = \sum_{l\geq 0} \frac{1}{l!}\mu_{l+1}(a, \dots, a, c_0) \tag{3.4}$$

We can similarly define the higher gauge transformations as

$$\delta_{c_{-k-1}} c_{-k} = \sum_{l \geq 0} \frac{1}{l!} \mu_{l+1}(a, \ldots, a, c_{-k-1}) \tag{3.5}$$

We will further define the curvature $f \in L_2$ as

$$f := \sum_{l \geq 1} \frac{1}{l!} \mu_l(a, \ldots, a) \tag{3.6}$$

Using the higher Jacobi identities we find that the curvature satisfies the Bianchi identity

$$\sum_{l \geq 0} \frac{1}{l!} \mu_{l+1}(a, \ldots, a, f) \tag{3.7}$$

From (3.4) we see that the gauge transformations of the curvature are of form

$$\delta_{c_0} f = \sum_{k \geq 0} \frac{1}{k!} \mu_{k+2}(a, \ldots, a, f, c_0) \tag{3.8}$$

and using the higher Jacobi identities once again we see that

$$[\delta_{c_0}, \delta_{c_0'}]a = a \sum_{l \geq 0} \frac{1}{l!} \mu_{i+2}(a, \ldots, a, c_0, c_0') + \sum_{l \geq 0} \frac{1}{l!} \mu_{l+3}(a, \ldots, a, f, c_0, c_0') \tag{3.9}$$

We thus see that the equations close only if

$$f = 0 \tag{3.10}$$

We will call the gauge potentials satisfying the equation

$$f := \sum_{l \geq 1} \frac{1}{l!} \mu_l(a, \ldots, a) = 0 \tag{3.11}$$

the Maurer-Cartan elements.

The equations of motion $f = 0$ can serve as a constraint for finding the action functional that determines the content of the theory. We will without any further deductions give the Maurer-Cartan action for a cyclic L_∞-algebra and

$$S_{MC} := \sum_{l \geq 1} \frac{1}{(l+1)!} \langle a, \mu_l(a, \ldots, a) \rangle \tag{3.12}$$

3.1.1 Comparison of Maurer-Cartan and BF theories

In their work Baez and Huerta (Baez and Huerta, 2011) have introduced the so-called BF theory. They give the action of the theory in terms of an integral over a 4-dimensional manifold M of a 1-form A and a 2-form B

$$S_{BF}(A, B) = \int_M tr(B \wedge F) \tag{3.13}$$

where $F = dA + A \wedge A$ is a curvature. Baez and Huerta then give an intuitive argument, that the theory is described in terms of a crossed module (G, H, t, α) called the tangent 2-group TG with the properties:

- G is a Lie group
- H is a Lie algebra $\mathfrak{g}$ of the group G
- α is the adjoint representation of G
- t is trivial

The action from equation (3.13) is fairly simple and leads to the equations of motion

$$\underset{\cdot}{B} + [A, B] = 0 \quad F = 0. \tag{3.14}$$

Since the L_∞-algebras contain as a special case of $L = L_0 + L_{-1}$ the algebra of the 2-group (Jurčo et al., 2019a), we can directly compare the BF theory with a special case of Maurer Cartan theory. In this subsection we will thus attempt to compare the two theories - we will give an action for the case of a L_∞-algebra concentrated in degrees 0 and -1 with a higher product μ_3 trivial.

The basic part of the Maurer Cartan theory is a L_∞-algebra. In our thesis we will use as an example the L_∞-algebra to be a tensor product of the de Rham complex $(\Omega^\bullet(X), d\int)$ of forms on a 4-dimensional manifold X without boundary and a finite-dimensional cyclic L_∞-algebra $L = L_0 \oplus L_{-1}$.

First let us take a homogeneous field of degree +1 as our gauge field. Since we take the L_∞-algebra to be a product of the de Rham complex and a L_∞algebra $L = L_0 \oplus L_{-1}$, we can decompose the field into fields $A \in \Omega^1(M, L_0)$ and $B \in \Omega^2(M, L_{-1})$ as shown in equation (2.36). We thus have

$$a = A + B \tag{3.15}$$

In order to write the action we will first express the higher products of a with itself. Using the equation (2.38), we can express $\mu_1(a)$ as

$$\hat{\mu}_1(a) = \hat{\mu}_1(A + B) = -\mu_1(A) + dA + \mu_1(B) + dB \tag{3.16}$$

Since the L degree of μ_1 is +1,the field A has the L-degree 0 and the L_∞-algebra has no homogeneous part L_{+1} it is necessarily $\mu_1(A) = 0$ and thus

$$\hat{\mu}_1(a) = dA + \mu_1(B) + dB \tag{3.17}$$

Using the same rationale we find that for μ_2 with L degree 0 we have $\mu_2(B, B) = 0$ and for μ_3 with degree -1 we have $\mu_3(A, A, B) = \mu_3(A, B, B) = \mu_3(B, B, B) = 0$. By the same reasoning all the higher products μ_i with $i > 3$ are trivial. Using equation (2.39) we thus have:

$$\hat{\mu}_2(a, a) = \mu_2(A, A) + 2\mu_2(A, B) \tag{3.18}$$

$$\hat{\mu}_3(a, a, a) = -\mu_3(A, A, A) \tag{3.19}$$

Inserting these equations into the action (3.12) we get the Maurer-Cartan action

$$
\begin{aligned}
S_{MC} = &\int_X \left\{ \frac{1}{2}\langle A+B, dA+\mu_1(B)+dB\rangle + \frac{1}{3!}\langle A+B, \mu_2(A,A)+2\mu_2(A,B)\rangle \right. \\
&\left. -\frac{1}{4!}\langle A+B, \mu_3(A,A,A)\rangle \right\}
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&= \int_X \langle A, \frac{1}{2}\left(dA+\mu_1(B)+dB\right) + \frac{1}{3!}\mu_2(A,A) + \frac{2}{3!}\mu_2(A,B) - \frac{1}{4!}\mu_3(A,A,A)\rangle \\
&+ \int_X \langle B, \frac{1}{2}\left(dA+\mu_1(B)dB\right) + \frac{1}{3!}\mu_2(A,A) + \frac{2}{3!}\mu_2(A,B) - \frac{1}{4!}\mu_3(A,A,A)\rangle
\end{aligned}
\tag{3.21}
$$

Since we integrate over four dimensional space, all the terms with a total degree other than 4 will vanish

$$
\begin{aligned}
S_{MC} = &\int_X \langle A, \frac{1}{2}dB + \frac{2}{3!}\mu_2(A,B) + \frac{1}{4!}\mu_3(A,A,A)\rangle \\
&+ \int_X \langle B, \frac{1}{2}dA + \frac{1}{2}\mu_1(B) + \frac{1}{3!}\mu_2(A,A)\rangle
\end{aligned}
\tag{3.22}
$$

We can now use the cyclicity of the inner product on the L_∞-algebra together with anticommutativity of the forms

$$
\int_X \langle A, \frac{1}{2}dB + \frac{2}{3!}\mu_2(A,B)\rangle = -\frac{1}{2}\int_X \langle dB, A\rangle + \frac{2}{3!}\int_X \langle B, \mu_2(A,A)\rangle \tag{3.23}
$$

Integrating the first term on the right hand side of equation (3.23) by parts and inserting the result into equation (3.22) be finally obtain the action

$$
S_{MC} = \int_X \langle B, dA + \frac{1}{2}\mu_2(A,A) + \mu_1(B)\rangle + \frac{1}{4!}\int_X \langle A, \mu_3(A,A,A)\rangle \tag{3.24}
$$

We can now compare the action we received with the action of the BF theory. First we note that, since the differential crossed modules correspond to the 2-term L_∞-algebras $L = L_{-1} \oplus L_0$ with a higher product μ_3 trivial. Setting $\mu_3 = 0$ the last term on the right hand side (RHS) of equation (3.24) will vanish. We have also mentioned in the beginning of this chapter, that the BF action corresponds to a crossed module with a trivial homomorphism t. This means that the penultimate term on the RHS of the equation 3.24 will also vanish, since the homomorphism t corresponds to the higher product μ_1. As expected, we thus get the same action as was described by the crossed module TG.

Conclusion

In this thesis we have given a short account of higher gauge algebras. We have described the physical background of the theory and shown that there is a so-called BRST symmetry, which induces a fermionic charge Q_{BRST} of ghost degree $+1$. We have show that this charge is nilpotent, and thus the space of fields endowed with the BRST charge can be interpreted as a Q-manifold.

We have then introduced the mathematical concepts necessary to describe the higher gauge theories. We have shortly revised category theory and defined a strict 2-category and 2-group. We have defined a crossed module and we have shown that the crossed module and 2-group are equivalent.

We have given a short introduction to the Q-manifolds, introducing all the necessary mathematical apparatus that is needed in order to define them.

We have defined L_∞-algebras and we have shown how they correspond to the Q-manifolds.

Then we have shortly described the Maurer-Cartan homotopy theory and how it is introduced in terms of the BRST complex. We have shown that in the special case of two term L_∞-algebra with single non-trivial higher product μ_2 and a 4-dimensional timespace we recover the BF action from the Maurer-Cartan action. This is a clear consequence of the one-to-one correcpondence between the aforementioned two-term L_∞-algebra and the tangent 2-group TG, in terms of which the BF theory is defined.

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