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Conditional quantile models for asset returns

Bachelor's thesis

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Declaration of Authorship

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Abstract

The literature related to Value at Risk estimation is rich in general. However, majority of papers written on this subject concentrates on the unconditional non-parametric or parametric approach to VaR modelling. This thesis focuses on direct conditional VaR estimation using quantile regression. Thereby imposing no restrictions on the return distribution. We use daily volatility measurements for individual stocks in S&P 500 index and quantile regress them on one-day ahead returns of the entire index. Depending on the quantile selected this estimation produces different confidence levels of Value at Risk. In order to minimize complexity of the final model, regularization methods are applied. To the author's knowledge this specific methodology has not yet been applied in any paper. The main objective is to investigate whether this approach is able to produce sound VaR estimates comparable with different methods usually applied. Our result suggests that quantile regression extended with lasso regularization can be used to produce sound one-day-ahead Value at Risk estimates.

JEL Classification	C22, C58, G15
Keywords	volatility, quantile regression, VaR, GARCH

Title	Conditional quantile models for asset returns

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Abstrakt

Literatura na téma Value at Risk (hodnota v riziku) je obecně bohatá. Nicméně většina prací napsaná na toto téma se soustředí nepodmíněné parametricé a neparametrické modelování VaR. Tato práce se zabývá na přímé modelování podmíněčného VaR pomocí kvantilové regrese. Tato metoda nepředjímá žádná omezení pro rozdělení výnosů. V práci používáme výpočty denní volatility pro všechny akcie v indexu S&P 500 a pomocí kvantilové regrese dále modelujeme podmíněčný VaR pro celý burzovní index. Pro určení optimálního počtu nezávyslých proměnných používáme metod regularizace. Autor práce si dále neni vědom žádné podobné práce zpracované na toto konkrétní téma. Hlavní cíl práce zpočívá ve zkoumání zdali je možné touto metodou dosáhnout uspokojivých VaR odhadů, které budou srovnatelné s jinými, běžně používanými, metodami. Závěry této práce ukazují, že kvantilová regrese použitá společně s lasso regularizací může být použitá pro výpočet jednodenního Value at Risku.

Klasifikace JEL	C22, C58, G15
Klíčová slova	volatilita, kvantilová regrese, VaR, GARCH
Název práce	Podmíněné kvantilové modely pro výnosy aktiv
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Contents

List of Tables

List of Figures

Acronyms

ADF Augmented Dickey-Fuller test

AIC Akaike Information Criterion

ARCH Autoregresive Conditional Heteroskedasticity

BIC Bayesian Information Criterion

CC POF + CIF

CIF Christoffersen's Interval Forecast test

GARCH General Autoregresive Conditional Heteroskedasticity

HS Historical Simulation

MSE Mean Square Error

NYSE New York Stock Exchange

OHLC Open, High, Low, Close

OLS Ordinary Least Squares

PE Prediction Error

POF Proportion of Failures test

QR Quantile Regression

SE Standard Error

TBF POF + TBFI

TBFI Time Between Failures

TUFF Time Until First Failure test

VaR Value at Risk

Bachelor's Thesis Proposal

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Proposed topic	Conditional quantile models for asset returns

Motivation and research question The research question I intend to answer is whether behavior of future stock returns at specific quantile can be written as a linear function of remaining stocks from a S&P 500 index. In other words, if there are some specific groups of stocks that tend to move together or vice versa. We anticipate finding some connection because we expect quantiles of the asset to be influenced by other assets. Finding this linear dependence among equities in a portfolio can help us explain conditional behavior at the tails of return distribution. The quantile regression (QR) was first proposed by Koenker and Basset (1978). Where the basic ordinary least square method provides an estimate of the conditional mean of the endogenous variable, the quantile regression estimates the various conditional quantiles of interest directly. Quantile regression provides us an appropriate methodology to estimate value at risk, since it can be viewed as a conditional function of a given return series. Value at risk (VaR) is an estimate of how much a certain portfolio can lose within a given time period and at a given confidence level. Since its introduction in late 1980s it has become a standard measure of risk for financial institutions or regulators. Mainly due to its simple interpretation and easy usage among wide class of assets. Although the concept of VaR is quite straightforward none of the methodologies developed so far yields satisfactory results (Engle and Manganelli 2004). Therefore, endeavor to find proper methodology behind VaR is significant for future risk management.

Contribution Papers written regarding estimation of VaR via quantile regression tried to explain future returns by past variation and endogenous variables (e.g. Christoffersen at al. 2001). In my paper I would like to extrapolate price changes of remaining shares from endogenous variables and use them as regressors. With this approach we can investigate whether volatility of one stock can be attributable to the price changes of the remaining shares in the S&P 500 index as well as to the past volatility. This will help us clarify behavior of shares during rare market conditions with increased volatility. I use dataset of daily closing prices of stock in the S&P 500 index for the past 10 years. After estimatimation of daily volatility we run regression as proposed above. We enhance prediction with regression shrinkage via the lasso. The nature of this constraint tends to produce coefficients that are exactly 0. That helps us increase accuracy of the model and makes makes it more interpretable (Tibshirani 1996).

Outline

1. Introduction
2. Theoretical Framework
3. Empirical research
4. Conclusion

Core bibliography

1. Christoffersen, P., Hahn, J. and Inoue A. 2001 Testing and comparing Value-at-Risk measures. Journal of Empirical Finance 8:325–342
2. Engle, R.F. and S. Manganelli. 2004. Caviar: Conditional Autoregressive Value at Risk by Regression Quantiles. Journal of Business & Economic Statistics 22:367-381
3. Engle, R.F. and S. Manganelli. 2001. Value at Risk Models in Finance. ECB Working Paper No. 75
4. Christoffersen, P. F., Hahn, J., and Inoue, A. 2001. Testing and Comparing Value-at-Risk Measures. Journal of Empirical Finance 8:325-342.
5. Koenker, R. and Bassett, G. 1978. Regression Quantiles. Econometrica 46:33-50.
6. Koenker, R. and Xiao, Z. 2002. Quantile Autoregression. Journal of the American Statistical Association 101:980-990
7. Tibshirani, R. 1996. Regression Shrinkage and Selection via the Lasso. Journal of the Royal Statistical Society 58:267-288
8. Zikes F. and Barunik J. 2016 Semi-parametric Conditional Quantile Models for Financial Returns and Realized Volatility. Journal of Financial Econometrics 14:185-226

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Chapter 1

Introduction

Financial risk management is becoming one of the main domains of financial markets. One of methods for evaluating market risk is Value at Risk (VaR). It has gained its popularity due to its simple idea and implementation. VaR estimation is convenient as it summarizes the potential loss over specified time horizon in just a single number. The probability of losses exceeding this number is expressed with τ, where τ is usually set as 0.01 or 0.05. The origins of risk measurements in form of adequate capital requirements can be traced back to early 20th century. However, Value at Risk was not introduced to public until 1994. Sir Dennis Weatherstone, a former chairman of J.P.Morgan, and his team developed and publicized a system called RiskMetrics. Ultimately, the value of proprietary VaR measures was recognized by the Basel Committee, which authorized their use by banks for performing regulatory capital calculations (Holton 2002). Nowadays, some of VaR estimations are still based on the assumption of normality in returns. These returns series usually exhibit conditional heteroskedasticity and unconditional normality. Under the assumption of normally distributed returns the estimation of VaR corresponds to the estimation of conditional volatility for specified τ. This can be accomplished by models for conditional heteroskedasticity as introduced by Engle (1982) and then extended by Bollerslev (1986). Nonetheless, there is evidence of returns series being not exactly normally distributed (Rydberg 2000). The distribution of returns tends to be negatively skewed with excess kurtosis and fat tails. This fact can result in failure of VaR models based on Gaussian distribution which is especially troublesome during times of increased market turmoil. For this reason there was need to develop models capable of capturing the real nature of returns distribution more accurately.

The objective of this paper is to compare a Value at Risk estimation obtained from conditional volatility models and Value at Risk estimated utilizing the quantile regression approach. Both models will be based on daily returns on S&P 500 index from 1999 to 2020. Quantile regression, firstly introduced by Koenker & Bassett (1978) is a type of regression analysis. Where the method of least squares is able to estimate the conditional mean of the dependent variable, quantile regression estimates conditional quantiles. This approach is robust to extremely large shocks and is possible to conduct without any specifications on the underlying distribution. This makes quantile regression well suited for estimating Value at Risk. However the efficiency of the QR as with any regression is highly dependent on selected regressors. Our estimated VaR models will be further tested using standard backtesting procedures.

The rest of the paper is organized as follows. In Chapter 2 we present the theoretical framework. That includes introducing the concept of VaR together with current estimation approaches. We then follow with presenting the thought behind quantile regression together and lasso regularization methods. Lastly Chapter 2 elaborates on backtesting methods. Chapter 3 concerns with empirical application. We present dataset that we will be using for our analysis with brief summary of its most notable aspects. Then we elaborate on model selection, present the models estimated together with backtesting results. Conclusion of this paper can be found in Chapter 4.

Chapter 2

Theoretical Framework

We start the chapter by presenting the reader with theory used in this paper. We elaborate on the notion of Value at Risk from practical and statistical point of view. In addition we present approaches to its calculation supplemented by their advantages and shortcomings. Then we proceed with the introduction of quantile regression and its connection to computation of Value at Risk. Further we continue with part concerning lasso regularization method. Finally, we provide the reader with basic backtesting methods applied in this work.

2.1 Value at Risk

The origins of Value at Risk can be traced back to NYSE capital requirements in early 20s. After the introduction of portfolio theory by Markowitz in early 1950s theorist started to develop basic mathematics techniques for VaR measurements. By the 1980 markets have become more volatile and the need for development of more sophisticated VaR measurements have arisen. This has been slowly accomplished due to the availability of increased processing power of computers and larger datasets of historical prices. In the early 1990s J.P.Morgan publicized its RiskMetrics service which made VaR available to professionals and institutions. Ultimately, the VaR has been recognized by the Basel Committee as a mean of regulatory capital calculation (Holton 2002).

Value at Risk is an assessment of potential loss of a portfolio over a specified time horizon for a given confidence interval under normal market conditions Jorion (1996). It is therefore a conditional quantile of an asset returns distribution. Let us have a series of n identically and independently distributed random variables of financial returns $\{r_t\}_{t=1}^{T}$. Let $F(r)$ be the cumulative distribution function of these returns, $F(r) = P(r_t < r|\mathcal{F}_{t-1})$ conditional on the information set $\mathcal{F}_{t-1}$. Let us assume that $\{r_t\}$ is a random process

$$r_t = \mu + \varepsilon_t$$
$$\varepsilon_t = z_t\sigma_t,\ z_t \sim iid(0,1)$$

where $\sigma^2 = E(z_t^2|\mathcal{F}_{t-1})$ and z_t has conditional distribution function $G(z)$, $G(z) = P(z_t < z|\mathcal{F}_{t-1})$. Then VaR with a given confidence level $\tau \in (0,1)$, expressed by $VaR(\tau)$, is defined at the τ quantile of the distribution of financial returns:

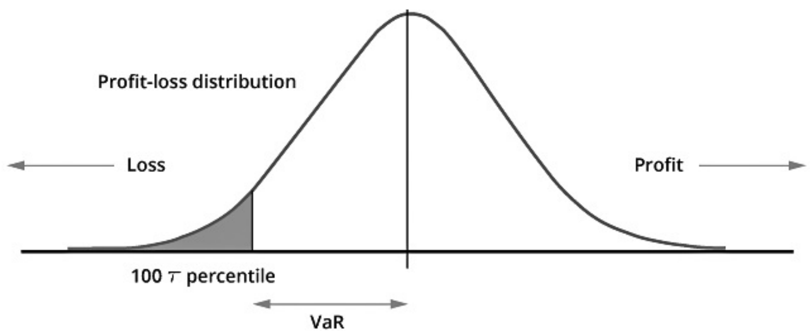
$$F(VaR(\tau)) = P(r_t < -VaR(\tau)|\mathcal{F}_t) = \tau$$

or equivalently:

$$VaR(\tau) = inf\{v|P(r_t < v) = \tau\}$$

In the Figure 2.1 we see VaR depicted on normal distribution of returns. The curve represents profit-loss probability density function. The shaded area on the left represents τ% of the total area under the curve. The VaR is determined by the distance of 100*τ percentile from zero.

Figure 2.1: Value at Risk on Return Distribution



Source: BMEClearing

There are two ways how to estimate this quantile. The first is to invert the distribution function of returns $F(r)$. The second way is to invert the distribution function of innovations, with regard to $G(z)$ and to estimate σ_t^2.

$$VaR(\tau) = F^{-1}(\tau) = \mu + \sigma_t G^{-1}(\tau)$$

Therefore, in estimating VaR we need to first specify $F(r)$ or $G(r)$. There are several methods in estimating these functions. Namely non-parametric,

parametric and semi-parametric. Further, we describe these methods focusing on strengths and shortcoming of each of them.

2.1.1 Non-parametric methods

The non-parametric approach of estimating Value at Risk does not make strong assumptions about the distribution of returns. The essence of this method is looking at the past data and use them to estimate empirical distribution function, which is subsequently used to calculate the VaR. Non-parametric assumptions are based on the assumption that the near future will be similar to the past. Non-parametric methods involve for example Historical Simulation or Monte Carlo simulation. Both of these methods are relatively non-restrictive.

Historical simulation approach is a very common non-parametric method. It looks at the historical returns and uses them to estimate the distribution function $F(r)$. $VaR(\tau)$ is then the τth quantile of this distribution. We can use different time spans and return intervals to estimate the empirical distribution of financial returns. The fact that this method is not based on any particular distribution assumption makes it possible to account for fat tails, skewness and other non-Gaussian characteristics of observed return distribution. This inference of distribution makes this method also very easy to implement. This can however turn out to be treacherous. Since our VaR is calculated merely on past observations we can expect the VaR to be overestimated if our data is taken in times of unusually high volatility and vice versa. Historical simulation also fails to alter the amount of Value at Risk in times of unexpected market turmoil as swiftly as parametric models. That is due to the fact that recent observations are given the same weight as any other observation in our data set. We would thus need a large amount of new and eccentric observations to change our VaR.

Monte Carlo simulation is a general and very flexible approach although more computationally demanding. It is based on simulations of returns according a certain type of distribution. VaR is then calculated as a selected quantile from these simulated processes.

2.1.2 Parametric methods

Variance-covariance & RiskMetrics

Parametric approach, sometimes called analytical, is a simple and straightforward way of estimating Value at Risk. It measures risk by fitting a distribution of returns on historical data and then estimating VaR from the fitted curve. Parametric approach assumes that the innovations and/or the returns of portfolio follow a Gaussian distribution. Under this assumption, the VaR in an $1-\tau\%$ level of confidence is computed as:

$$VaR(\tau) = \mu + \sigma_p G^{-1}(\tau)$$

where $G^{-1}(\tau)$ is the τ-quantile of standard normal distribution. σ_p is a conditional standard deviation of a portfolio. It is calculated using a variance-covariance matrix $\mathbf{\Sigma}$ for all assets and the vector of the weights $\boldsymbol{x}$ of all assets in the portfolio. Using spectral decomposition the variance of the portfolio is thus estimated as $\hat{\sigma}_p^2 = \mathbf{x'\Sigma x}$.

Another parametric method of VaR calculation is the RiskMetrics system by J.P.Morgan (JP Morgan, Reuters 1996). Its computation is the same as above but the way of estimating σ_p differs. The RiskMetrics uses an exponential weighted moving average model to estimate the portfolio variance. The formula is specified as follows with $\lambda = 0.94$ and window size N set as 74.

$$\sigma_p^2 = (1-\lambda)\sum_{n=0}^{N-1}\lambda^j(\varepsilon_{t-n})^2$$

Although both of these approaches are fairly neat, they have some major imperfections. The first being the fact that it assumes the financial returns to be normally distributed. There has been empirical evidence that these returns are not exactly normal. Returns tend to be negatively (left) skewed with fat tails and a peak. This results in underestimation of risk which, in the case of unexpected loss, can have immense consequences (Rydberg 2000).

Second problem concerns the underlying model used for modelling volatility. Although it is capable to capture some volatility features such as volatility clustering, it cannot account for other characteristics. For example the leverage effect or asymmetry volatility (Black & Scholes 1976), (Pagan & Schwert 1990).

Another drawback is that the assumption of independent and identically distributed returns has not been observed in empirical measurements (Hansen 1994).

There has been several attempts to remedy these imperfections. Either by using more sophisticated models that are able to capture the true nature of underlying volatility. Some that is in line with real life observations. This includes families of volatility models such as GARCH, stochastic volatility models or realised volatility models. Or also by investigating different distribution functions which correspond more to the distribution of returns observed in the financial markets.

Volatility models

Volatility models can be divided into three groups. The GARCH family models, the stochastic volatility models and realised volatility models. In this paper we will discuss only GARCH family models as the rest is beyond the scoop of this paper and would provide no use in our empirical research.

Engle (1982) was the first one to introduce a new class of stochastic process called autoregressive conditional heteroskedastic (ARCH) processes. These are zero mean, serially uncorrelated processes with nonconstant variances conditional on the past, but constant unconditional variances. Bollerslev (1986) further extended the ARCH model into Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model. It allows the conditional variance to be dependent not only an the past values of error terms but also on the past values of its own lag. The fundamental GARCH(p, q) model is given as

$$\begin{aligned} r_t &= \mu + \varepsilon_t \\ \varepsilon_t | \mathcal{F}_{t-1} &\sim N(0, \sigma_t^2) \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2. \end{aligned}$$

Where $\mathcal{F}_{t-1}$ denotes the information set up to time $t-1$. The one step ahead GARCH(p, q) conditional variance is given by

$$\sigma_{t+1}^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2. \tag{2.1}$$

Also $p \geq 0$ and $q > 0$. Since variance is positive it must also hold $\alpha_i \geq 0$, $\forall i \in \{0, \ldots, q\}$. Parameter $\alpha = \sum_{i=1}^{q} \alpha_i$ measures the extent to which a volatility shock today feeds through into next period's volatility. It usually ranges between 0.05 (for a relatively stable market) and 0.1 (for a volatile market). Parameter $\beta = \sum_{j=1}^{p} \beta_j$ usually ranges between 0.85 and 0.98 with higher values being associated with lower α (Alexander 2008). Sum $\alpha+\beta$ measures the rate at which persistence of volatility dies over time. As shown by Chan (2010) persistence of volatility which does not die over time occurs when this summation is equal to one. Under this scenario, unconditional variance becomes infinite. For this reason the parameters should also be further constrained such that $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$ to imply weak stationarity.

Although GARCH model is able to capture volatility clusters, it is incapable of differencing between positive and negative shocks. It only takes into account squared residuals. In real world volatility increases at a higher pace when returns are negative rather than positive. There are dozens of non-linear models proposed for solving this drawback such as EGARCH, TGARCH, ect. On the other hand there has been empirical evidence that these models are unable to provide substantially better results in estimating conditional volatility and subsequently Value at Risk than the most basic GARCH(1,1) model. Preferably with student's t-distribution (Orhan & Köksal 2012), (Hansen & Lunde 2005).

2.1.3 Semiparametric methods

Semiparametric approach is a relatively new methodology for calculating VaR. They concentrate on modelling merely the tail of the return distribution in contrast with modelling the whole distribution. One of these methods is a quantile regression approach which will be further detailed in separate chapter.

2.1.4 Criticism

The thought behind Value at Risk is fairly simple. Nevertheless, that has been one of the main sources of its criticism. Since VaR reduces all available information used for the model into just one number we can expect loosing possibly relevant information. We see that VaR does not provide us with the extent of losses which might follow after breaking of the VaR estimate. We are only endowed with information about the potential expected loss which might happen with its associated probability with which this will not be exceeded. This can cause false sense of safety and incorrect interpretation of prevailing risks. However, we can overcome this obstacle with estimating VaR for high confidence levels. We could also possibly aid this issue with a Conditional VaR estimate. This method calculates the expected loss in a case where VaR threshold has been breached (Longin 2001).

As with every model trying to explain real world VaR, estimates are highly dependant on model assumptions. If they are not in line with reality, or are slightly off, the produced VaR can turn out to be imprecise or misleading. VaR estimates could therefore be followed by other risk management techniques to obtain wider overview of possible exposures.

Nonetheless, VaR is still a very popular method applied in risk management. The aim of this thesis is not to asses whether VaR is appropriate method as such, but to determine the suitability of our techniques developed for its calculation. Therefore, we will not take these limitations into consideration as they do not constitute the primary objective of this work.

2.2 Quantile Regression

Nowadays, regression based on minimizing sum of squares is the most widely used in various empirical applications. In contrast with classical ordinary least square regression, which explains expected or conditional expected value, quantile regression focuses on explaining quantile (or conditional quantile) function of independent variable $Q_\tau(y_i)$. Because of that, we are able to capture much more complex information about statistical dependence between the dependent and independent variables. This information would have been lost by concerning only on conditional expected value.

2.2.1 Definition

Quantile regression differs from OLS in the way which conditional characteristic of r_t we observe. Let take the financial returns r_t and explanatory variables x_{tp}. In the realm of discrete variables the standard OLS regression model for the average response is

$$\mathbb{E}(r_t) = \beta_0 + \beta_1 x_{t1} + \cdots + \beta_n x_{tn}, \quad t = 1, \ldots, T, \quad n = 1, \ldots, N. \tag{2.2}$$

T represent the length of the returns time series and N is the total amount of regressors. The coefficients β_n's are estimated by solving the least squares minimization problem

$$\min_{\beta_0,\ldots,\beta_n} \sum_{t=1}^{T} \left(r_t - \beta_0 - \sum_{n=1}^{N} \beta_n x_{tn} \right)^2 . \tag{2.3}$$

In contrast, the regression model for quantile level τ of the response is

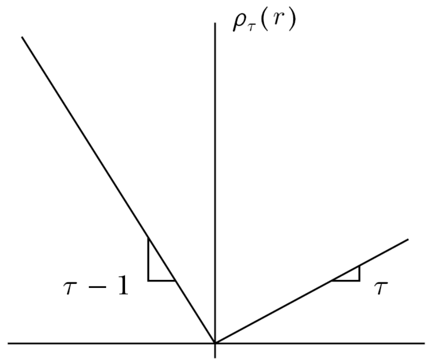
$$Q_\tau(r_t) = \beta_0(\tau) + \beta_1(\tau)x_{t1} + \cdots + \beta_n(\tau)x_{tn} \tag{2.4}$$

and the $\beta_j(\tau)$'s are estimated by solving the minimization problem

$$\min_{\beta_0(\tau),\ldots,\beta_n(\tau)} \sum_{t=1}^{T} \rho_\tau \left(r_t - \beta_0(\tau) - \sum_{n=1}^{N} x_{tn}\beta_j(\tau) \right)^2 \tag{2.5}$$

where $\rho_\tau(r) = \tau \max(r, 0) + (1 - \tau) \max(-r, 0)$. The function $\rho_\tau(r)$ is referred to as the check loss because its shape resembles a check mark as shown in Figure 2.2. For each quantile level τ, the solution to the minimization problem yields a distinct set of regression coefficients. We note that $\tau = 0.5$ corresponds to median regression and $2\rho_{0.5}(r)$ is the absolute value function (Rodriguez & Yau 2007).

Figure 2.2: Quantile Regression ρ Function



Source: Koenker & Hallock (2001)

2.2.2 Volatility estimation

In this section we tackle the problem of estimating capital asset price volatility from easily available public data. Garman & Klass (1980) formulated improved estimators of volatility employing the opening (O), high (H), low (L) and closing

(C) prices. Among several estimators examined they found that most efficient[1] estimator of volatility is

$$\sigma_t^2 \equiv 0.511(u_t - d_t)^2 - 0.019[c_t(u_t + d_t) - 2u_t d_t] - 0.383c_t^2 \qquad (2.6)$$

where $u_t = H_t - O_t, \;\; d_t = L_t - O_t, \;\; c_t = C_t - O_t$.

2.3 Lasso Regularization

Tibshirani (1996) proposed a new method for estimation in linear models - the lasso (least absolute shrinkage and selection operator). It minimizes the residual sum of squares subject to the sum of absolute value of the coefficients being less than a constant. In contrast with other regularization methods, such as ridge regression, it tends to produce some coefficients that are exactly zero and are subsequently completely omitted from regression.

2.3.1 Definition

If we were to perform OLS with minimizing sum of squares we could encounter two drawbacks. First being that the OLS can produce model with low bias but large variance. Sometimes it is suitable to shrink some parameters to zero and for a slight increase of bias reduce variance substantially. With this approach we can avoid overfitting and thus improve the overall prediction accuracy. Second obstacle with classical OLS is in its interpretation. Sometimes we would be better off with smaller amount of regressors but with stronger dependency on the explained variable.

Let us consider dataset (r_t, x_{tn}), $t = 1, 2, \ldots, T$, $n = 1, \ldots N$, where r_t is the regressand for the tth observation. Letting $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n)$, the regression parameter after regularization are defined as

$$(\hat{\beta}_0, \hat{\beta}) = \arg\min \left\{ \sum_{t=1}^{T} \left(r_t - \beta_0 - \sum_n \beta_n x_{tn} \right)^2 \right\}, \quad \text{subject to} \sum_n |\beta_n| \leq k \qquad (2.7)$$

[1]By examining Garman & Klass (1980) we see that they were able to produce even slightly better estimator than the one in 2.6. They accomplished this by incorporating the fraction of the day that trading is closed into the estimator formula

or, equivalently, we minimize

$$\sum_{t=1}^{T}\left(r_t - \beta_0 - \sum_n \beta_n x_{tn}\right)^2 + \lambda \sum_n |\beta_n|. \tag{2.8}$$

Here $k \geq 0$ is a prespecified tuning parameter that determines the amount of regularisation. Further, let $\hat{\beta}^o_n$ be the full OSL estimates and let $k_0 = \sum |\hat{\beta}^o_n|$. Then values $k < k_0$ will shrink the solutions of these coefficients towards zero. Some coefficients may be exactly equal to zero.

Orthonormal design

To further understand nature of the shrinkage can apply orthogonal design case. Let $\mathbf{X}$ bet the $T \times N$ matrix with tnth entry x_{tn}, and suppose that $\mathbf{X}^T\mathbf{X} = \mathbf{I}$.[2] The solution of minimization problem in (2.7) is then

$$\hat{\beta}_n = \text{sign}(\hat{\beta}^o_n)(|\hat{\beta}^o_n| - \gamma)^+ \tag{2.9}$$

where γ is determined by the condition $\sum |\hat{\beta}_n| = k$. In the orthonormal design, the regularization selects p of the largest coefficients in absolute value and sets the rest to zero. For some λ from (2.8) this is equivalent to setting

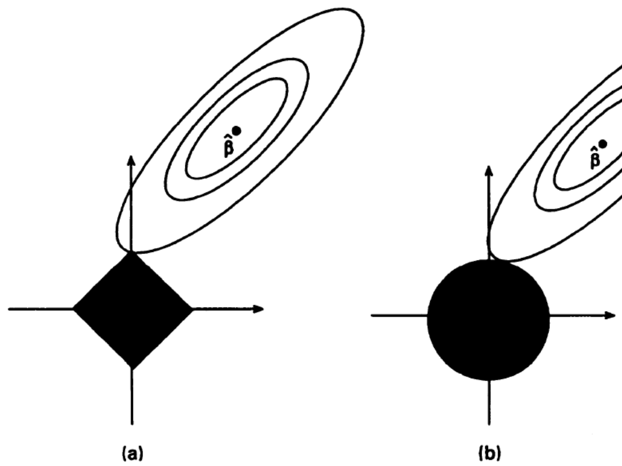
$$\hat{\beta}_n = \begin{cases} \hat{\beta}^o_n & \text{if } |\hat{\beta}^o_n| > \lambda \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Geometry of Lasso

The main difference between lasso and ridge regression is the left hand side of the constraint in 2.7 or 2.8 which is set as $|\beta_n|$ and β_n^2 respectively. The constraint region defined by $|\beta_n|$ is a rotated square for lasso (a). In the case of ridge regression the constraint β_n^2 is a circle (b). This situation is illustrated in Figure 2.3. We see two elliptical contours which are both centred at an OLS estimate $\hat{\beta}$. The solution for lasso is the contour line which touches the rotated square. This may sometimes happen at a corner. This situation results in a coefficient which is equal to zero. On the other hand, if we consider the ridge regression, we see that that there is no counter line that could intercept the circle exactly in corner without intersecting some other part of the circle. Thus, parameters of ridge regression almost never shrink all the way down to zero.

[2] $\mathbf{I}$ - the identity matrix

Figure 2.3: Estimation Picture for (a) Lasso and (b) Ridge Regressions



Source: Tibshirani (1996)

2.3.2 Regularization parameter

Determining the proper value for the regularization parameter λ from formula 2.8 is an important part in ensuring that the model performs well. Tibshirani (1996) further presents three methods for estimation of the lasso regularization parameter. Cross-validation, generalized cross-validation and analytical unbiased estimate of risk. In this paper we will cover only the first method as it is the most convenient and the only one we will be applying in our empirical research.

Suppose we have observations $(R, \mathbf{X}) = (r_t, x_{tn})$ which are drawn from an unknown distribution and

$$R = \eta(\mathbf{X}) + \varepsilon \tag{2.11}$$

where $E(\varepsilon) = 0$ and $\mathrm{Var}(\varepsilon) = \sigma^2$. The mean-squared error of an estimate $\hat{\eta}(\mathbf{X})$ is defined by

$$\mathrm{MSE} = \mathbb{E}[\hat{\eta}(\mathbf{X}) - \eta(\mathbf{X})]^2. \tag{2.12}$$

It is the expected value value taken over the joint distribution of $\mathbf{X}$ and R, with $\hat{\eta}(\mathbf{X})$ fixed. A similar measure is the prediction error of $\hat{\eta}(\mathbf{X})$ given by

$$\mathrm{PE} = \mathbb{E}[R - \hat{\eta}(\mathbf{X})]^2 = \mathrm{ME} + \sigma^2. \tag{2.13}$$

We can estimate the prediction error for the lasso using k-fold cross-validation. The lasso is indexed in terms of the normalized parameter $s = \frac{t}{\sum \hat{\beta}_j^o}$, and the prediction error is estimated for a series of values of s from zero to one. The value of $\hat{s}$ yielding the lowest estimated PE is then selected as a regularization parameter.

2.3.3 Penalized quantile regression

In this part we combine quantile regression together with regularization. The first part of Equation 2.14 is a quantile regression formula from 2.2. The lasso constraint is given similarly as in 2.7.

$$\min_{\beta_0(\tau),\dots,\beta_n(\tau)} \sum_{t=1}^{T} \rho_\tau \left(r_t - \beta_0(\tau) - \sum_{n=1}^{N} x_{tn}\beta_j(\tau) \right)^2, \quad \text{subject to} \sum_n |\beta_n| \leq k \quad (2.14)$$

For further details we refer to Koenker & Mizera (2014).

2.4 Backtesting

At the start of this chapter we introduced different VaR estimation approaches. Since there are drawbacks in each of these methods sound ways of evaluating the risk estimates produced were needed. Naturally, if a risk model does not predict risk accurately, its applicability is questionable. It is thus important to test the quality of models developed with proper techniques. Backtesting is a statistical procedure for validating a set of VaR estimates. Jorion (2001) refers to these checks as "reality checks". If we have confidence level for VaR of 99%, we expect that an exemption should on average occur once in every 100 observations.

The idea behind backtesting is to look at the hit sequence $I_t(\tau)$. It is defined as

$$I_t(\tau) = \begin{cases} 1, & \text{if} \quad r_t < -VaR(\tau) \\ 0, & \text{if} \quad r_t \geq -VaR(\tau). \end{cases} \quad (2.15)$$

$I_t(\tau)$ is basically a column of ones and zeros. If an exemption happens on the tth day, the tth value in the hit sequence is set as one. If no exemption occurs the value is zero. We further see that whether hit happens or not is conditional on the VaR confidence interval that we are modelling.

During backtesting we need to determine whether our model satisfies two properties. Firstly, we need to control the number of exceptions (unconditional coverage). We see that $\sum_{i=1}^{T} I_t$ is the sum of exemptions and T is the number of observations. Their ratio represents our so called failure rate. In an ideal situation the failure rate would correspond to the confidence interval of our VaR. If our model is accurate then the failure rate should converge to the frequency of tail loses, $p = (1 - c)$, as sample size increases (Jorion 2001). This idea can be written as $\mathbb{E}[I_t(\tau)] = P[I_t(\tau) = 1] = \tau$. If $P[I_t(\tau) = 1] > \tau$ the risk is underestimated and if the probability is lesser than τ then the risk is overestimated.

Secondly, we need the sequence of hits at two different dates $I_p(\tau)$ and $I_q(\tau)$ to be independently distributed for $p \neq q$ (conditional coverage). That is so because we expect the exemptions to occur randomly and independently on each other. Clusters of hits could signal a functional problem in our model even if was not priorly rejected by conditional coverage tests. Thus, in order to consider our VaR model accurate, the hit sequence obtained from it should satisfy both unconditional and conditional coverage tests.

2.4.1 Unconditional coverage

Unconditional coverage tests look solely at the failure rate. That is the proportion of hits among all VaR estimates. They do not account for the time when the exemptions occur. If the fraction of exemptions is larger than our confidence interval τ, the model overestimates the risk. On the other hand, if the fraction is smaller the model underestimates the risk. Therefore we must conduct statistical analysis to determine whether the amount of errors is in reasonable bounds or not. Since each trading day either produces a VaR violation or not the sequence of hits is more commonly known as Bernoulli trial. The number of exemptions $x = \sum_{i=1}^{T} I_t$ follows a binomial probability distribution:

$$f(x) = \binom{T}{x} p^x (1-p)^{T-x} \tag{2.16}$$

With the number of observation increasing, we can approximate the binomial distribution with a normal distribution

$$z = \frac{x - pT}{\sqrt{p(1-p)T}} \approx N(1, 0) \tag{2.17}$$

where pT is the expected number of exemptions and $\sqrt{p(1-p)T}$ being the variance of exemptions.

Kupiec test

A widely used unconditional coverage test was presented by Kupiec (1995). Kupiec's test (sometimes POF-test)[3] measures whether the amount of exemptions is acceptable. Under the null hypothesis of the model being correct the number of failures follows the binomial distribution with the H_0 for Kupiec-test being

$$H_0 : p = \hat{p} = \frac{x}{T} \tag{2.18}$$

We then use a likelihood-ratio test to examine whether the observed failure rate x/T significantly differs from our expected failure rate p. The test statistic is constructed as

$$LR_{POF} = -2ln \left(\frac{(1-p)^{T-x} p^x}{\left[1 - \left(\frac{x}{T}\right)\right]^{T-x} \left(\frac{x}{T}\right)^x} \right). \tag{2.19}$$

Under the H_0, LR_{POF} is asymptomatically χ^2 distributed with one degree of freedom. So if the value of test statistic exceeds the critial value of χ^2 we reject the null hypothesis and the model is considered to be inaccurate.

TUFF test

Kupiec (1995) suggested another type of unconditional backtesting. The TUFF (time until first failure) test looks at the time v it takes until the first hit occur. Since errors happen independently, the number of periods until first hit occurs should be consistent with the VaR confidence level. The test statistic is similar to the POF test and takes form of likelihood ratio

$$LR_{TUFF} = -2ln \left(\frac{p(1-p)^{v-1}}{\left(\frac{1}{v}\right)\left(1 - \frac{1}{v}\right)^{v-1}} \right). \tag{2.20}$$

The test statistic LR_{TUFF} is again χ^2 distributed with one degree of freedom. If the test statistic is lower than the critical region then the model is accepted. However, some argue that the test actually provides low power in identification of bad VaR models (Hass 2001). For example, if we measured 99% VaR and we conducted the test with 95% confidence interval $\chi^2_{0.95}$, any number v between 7 and 438 would still result in not rejecting the model. This

[3] Proportion of Failures

interval is proportionally the same for every VaR level. Because of its easy application the test can be mostly used just as a preliminary for selection of really bad models. In these cases the usage of further tests would most likely result in rejecting the model after all.

2.4.2 Conditional coverage

Unconditional coverage tests focus merely on the number of exceptions produced in the model. In reality we are also curious about the time when these exceptions occurred. A good VaR model should be able to capture the time periods of increased volatility in a way that the following exemptions happen independently of each other. Clustering of exemptions can cause large losses as these events will tend to be more disastrous than just one exemption once in a while (Christoffersen & Pelletier 2004). Therefore tests for conditional coverage were developed in order to examine not only the number of exemptions but also the conditional dependence of exemptions on each other. In this subsection we present two conditional coverage tests. Interval forecast test developed by Christoffersen (1996) and the mixed Kupiec test introduced by (Hass 2001).

Christoffersen's interval forecast test

Christoffersen (1998) introduced a test to examine whether the probability of observing an exception depends on the time when the exception occurred. Christoffersen's interval forecast (CIF) test similarly to Kupiec test uses of log-likelihood but the test is extended by a separate statistic in order to account for the independence of exemptions in time.

Let us define numbers n_{kl} with their outcome, which is summarized in Table 2.1.

$$n_{kl} = \sum_{i=2}^{n} \mathbf{1}[Y_{i-1} = k, Y_i = l], \quad k, l \in \{0, 1\} \tag{2.21}$$

Table 2.1: CIF Contingency Table

	$Y_{i-1} = 0$	$Y_{i-1} = 1$	
$Y_i = 0$	n_{00}	n_{10}	$n_{00} + n_{10}$
$Y_i = 1$	n_{01}	n_{11}	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	n

Further, we define probabilities

$$\pi_0 = \frac{n_{01}}{n_{00}+n_{01}}, \quad \pi_1 = \frac{n_{11}}{n_{10}+n_{11}}, \quad \pi = \frac{n_{01}+n_{11}}{n_{00}+n_{01}+n_{10}+n_{11}}.$$

π_0 is the probability of having a failure on period t, given that no failure occurred on period $t-1$. π_1 represents the probability of having a failure on period t given that no failure occurred on period $t-1$. π is the probability of having a failure on period t. If the model is accurate, the the probability of exemption should not depend on the previous observation. And hence probabilities π_0 and π_1 should be equal.

The test statistic for independence is given by:

$$\begin{aligned} LR_{CIF} &= -2ln\left(\frac{(1-\pi)^{n_{00}+n_{10}}\ \pi^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}}\ \pi_0^{n_{01}}\ (1-\pi_1)^{n_{10}}\ \pi_1^{n_{11}}}\right) \\ &= -2ln\left(\left(\frac{\pi}{\pi_0}\right)^{n01}\left(\frac{\pi}{\pi_1}\right)^{n11}\left(\frac{1-\pi}{1-\pi_0}\right)^{n00}\left(\frac{1-\pi}{1-\pi_1}\right)^{n10}\right) \end{aligned} \tag{2.22}$$

Now we can combine the Christoffersen's CIF statistic and Kupiec POF test statistic to obtain a conditional coverage mixed test statistic LR_{CC}. With it we can test for independence of errors and failures rate together. This statistic is asymptotically χ^2 distributed with two degrees of freedom.

$$LR_{CC} = LR_{POF} + LR_{CIF} \tag{2.23}$$

This test is only constructed to measure dependency between two consecutive days. In real life there is a chance that dependence between two successive observations would not be enough to capture the conditional interconnection of errors. Therefore Christoffersens's test might not be adequate method for capturing the dependence between all exemptions. To do so we can utilize the next test presented.

Haas's Test

Hass (2001) argues that the dependence of observations between two days is too weak to find significant interconnection. Therefore he proposes an enhanced test for capturing more general forms of time dependence of errors. This test is called Mixed Kupiec's Test or Haas's Time Between Failures (TBFI) test. Lets take time between failures as v_i, $i = 2, \ldots, n$, which is the duration between

i-th and $i-1$th failure. Also let v_1 be the first failure. Then the H_0 is the event of failures being independent of each other and the test statistic is written as

$$LR_{TBFI} = -2ln \sum_{i=2}^{n} \left(\frac{p(1-p)^{v_i - 1}}{\left(\frac{1}{v_i}\right)\left(1 - \frac{1}{v_i}\right)^{v_i - 1}} \right) \tag{2.24}$$

Under H_0 the test statistic is asymptotically χ^2 distributed with n degrees of freedom. We can again combine the statistic with Kupiec's POF test to obtain combined statistic for coverage and independence - the mixed Kupiec's test statistic - which is again χ^2 distributed with $n+1$ degrees of freedom.

$$LR_{TBF} = LR_{POF} + LR_{TBFI} \tag{2.25}$$

Chapter 3

Empirical Research

After introducing the underlying theory in the last chapter, we proceed with the main objective of this thesis, which is the empirical research. In this chapter we apply the previously mentioned methodologies and models to calculate VaR on S&P 500 stock index. The two models used for VaR estimation that we will be comparing are firstly the GARCH model, which estimates the conditional volatility. And secondly, the quantile regression model, which models the particular quantile of returns of S&P 500 that we are interested in. This is done by expressing the return on one day as a linear combination of daily volatilities of all stocks in the index on the day before. The calculation of these daily volatilities was described in Section 2.2.2.

The main research objective is to examine under which conditions can we use quantile regression for estimation of Value at Risk. The reason of selecting GARCH model as a benchmark was its easy implementation and the fact that it is usually able to provide reasonably accurate results. We then use both models to calculate VaRs with 95% and 99% confidence intervals. We can also encounter calculations of VaR for different confidence intervals such as 90%. However, we decided to omit on these levels as the two levels used in this work are the ones most commonly used in practice and literature. We then use the in-sample backtests from part 2.4 to evaluate their individual performance. We have decided to consider only in-sample performance as out-of-sample performance evaluation is beyond the scope of this paper.

The rest of this chapter is organized as follows. In the first section, we introduce the market data we are dealing with. In the second part we provide the reader with regression results for moth models. In the case of GARCH we further use both normal and t-distribution for errors to estimate the volatility and calculate the VaRs parametrically. This method was described in Section 2.1.2. Since the quantile regression models the 1st and 5th quantile directly we do not need to make any assumptions about the underlying distribution. At the end of this chapter we elaborate on the backtesting procedure and test their performance of VaR estimated using in-sample backtests from part 2.4. Lastly, we summarize our empirical findings and summarize the overall performance of our models.

3.1 Data Description

We work with dataset that contains daily open, high, low, and closing (OHLC) prices of S&P 500 stock index obtained from Yahoo Finance. Its is further important to choose appropriate length of data for our analysis. Since we are aiming on examination of tail behaviour which are from definition rare we need to have enough observations of this kind. By taking the dataset of 21 years we are able to incorporate times of increased volatility from both dot-com bubble and the Great Recesion. The time span of our data set is from 5/5/1999 to 5/5/2020 and contains overall 5284 observations.

For our purposes, the daily data are transformed to daily continuously compounded returns. This is because financial time series are usually exposed to exponential growth, and thus log transformation can smooth out (linearise) the time series. The greater is the volatility over multiple holding periods the greater is the difference between arithmetic (non-log) and geometric (log) returns (Hudson & Gregoriou 2010). The daily rate of continuously compounded returns r_t is given using the daily closing prices C_t by the relation

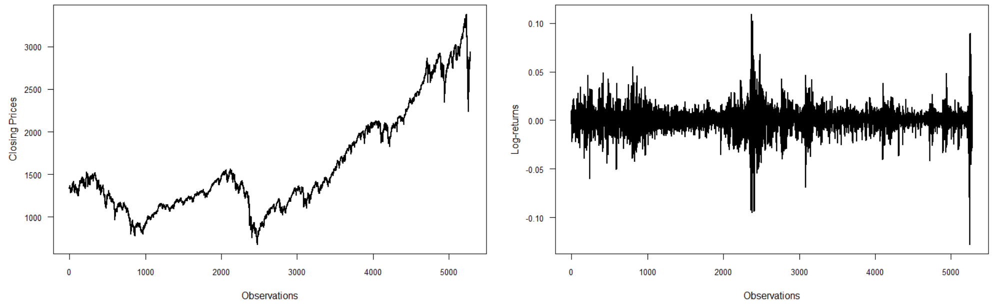
$$r_t = \log\left(\frac{C_t}{C_{t-1}}\right). \tag{3.1}$$

In the next figure we see closing prices 3.1(a) and log-returns 3.1(b) calculated using the formula for returns from Equation 3.1. Both of the time series are plotted for the full time period.

There are several stylized facts exhibited by financial volatility. It is generally found that return distributions of financial assets tend to exhibit phenomena called volatility clusters i.e. periods of higher (lower) volatility tend to be clustered together (Cont 2005). In the Figure 3.1(b) we can easily see volatility clustering during times of increased market turmoil. At the start we

Figure 3.1: S&P500

(a) Closing prices (b) Log-returns



Source: Author's computations.

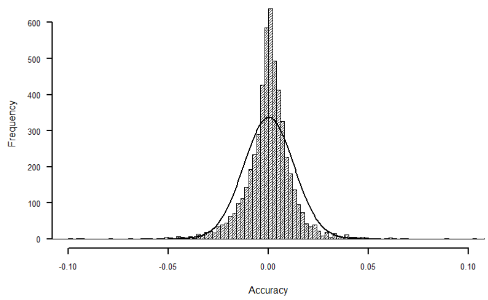
see large cluster as a result of the dot-com bubble together with another events accompanying the start of the century. In the middle we spot a large increase in volatility during the financial crisis started in 2008. The period of increased volatility at the end of our sample is due to 2020 stock market crash during the coronavirus pandemic.

Figure 3.2: S&P500 Compounded Returns

(a) Descriptive statistics

Statistic	S&P 500
Observations	5284
Mean	0.00014
Median	0.00014
Std. dev	0.01251
Minimum	−0.1277
Maximum	0.10957
Skewness	−0.3601
Ex. kurtosis	10.9937

(b) Log-returns density



Source: Author's computations.

Another of these facts is a tendency to observe leptokurtic distributions with fatter tail than in the case of normal distribution. The returns distribution further exhibit positive excess kurtosis. We can confirm these findings by reviewing Figure 3.2(a) which summarizes descriptive statistics of S&P log-returns distribution.

We further applied Augmented Dickey-Fuller test and rejected the null hypothesis of unit root presence with p-value lesser than 0.01. To test normality we we employed Jarque-Bera test. As expected we were able to reject the null with p-value indistinguishable from zero. We also used Ljung-box test to check if autocorrelation exists in squared residuals after using ARMA(p, q) model. If there was no autocorrelation we would not need to use GARCH as ARMA would be sufficient. To determine the order of GARCH model we used Bayesian information criterion which resulted in $(p, q) = (1, 1)$. AIC yielded order of (3,3) and in order to decrease complexity we employed the BIC result. Above all that, since we expect to find serial autocorrelation of high orders the choice is practically identical. The p-value from Ljung-box test on squared residuals from ARMA(1,1) was once again indistinguishable from zero and thus ARCH effects are present. A similar test we could have applied is ARCH LM test, however with such strong results from Ljunx-box we find no need to use it.

3.2 Regression Results

In this section we present the regression results both for GARCH and QR models. We present the estimated parameters together with model selection process.

GARCH

Since GARCH(1,1) is usually considered to be reasonably good model in describing conditional volatility we decided to use it as a bedrock for comparison. The selection of autoregressive p and moving average q parameters was tested using Akaike information criterion with values ranging from one to three. Despite GARCH(1,1) scoring as the second best - only beated by GARCH(1,2) - we still decided to use it. The AIC confirmed that the model is reasonably good and we can further simplify the overall complexity by selecting a model with fewer parameters. Complete results of the model selection are in Table A.1 in appendix. Table 3.1 provides estimates of parameters α_0, α_1 and β_1 of GARCH model. All parameters are statistically significant. The parameter α_1 is estimated as 0.12. It measures the extent to which a volatility shock today feeds through into next period's volatility. Estimation of β_1 yields the result of 0.86 which is large in magnitude but anticipated for this kind of data. It shows that large changes in the volatility will affect future volatilities for a long period of time since the decay is slower. The sum of coefficients $\alpha_1 + \beta_1 = 0.98$ measures the rate at which the volatility effect dies over time. If the sum was equal to one we would observe presence of volatility which does not die over time. Under this scenario, unconditional variance becomes infinite.

Table 3.1: Estimation Results for GARCH(1,1) Model

	Dependent variable:
	log-returns
α_0	0.000588***
	(0.000109)
α_1	0.121636***
	(0.010154)
β_1	0.863421***
	(0.010456)
Observations	5,281
Log Likelihood	−17,050.72
Note:	*p<0.1; **p<0.05; ***p<0.01

Source: Author's computations.

Quantile Regression

The exact model we will be using for our quantile estimation is similar to Equation 2.2. However, since we are trying to predict one-day ahead returns r_t we need to shift our regressors back in time by one day. Since the time span of our data is 21 years we encounter a problem. Some stock from S&P 500 were nor in the index for the full time or some these companies did not even existed for the whole time period. We solve this problem by only considering stock that were in the index for the full time period. Altogether these are 212 stocks represented by x_n. Hence our quantile regression model is described with with Equation 3.2 with $\tau = \{0.05, 0.01\}$; $n = 1, \ldots, 212$; $t = 1, \ldots 5284$.

$$Q_\tau(r_{t+1}) = \beta_0(\tau) + \beta_1(\tau)x_{t1} + \cdots + \beta_n(\tau)x_{tn} \tag{3.2}$$

Since the number of independent variables and coefficients for quantile regression is large we present the results in appendix. Firstly, we start with the regression of the 5th conditional quantile of returns. The estimated coefficients are presented in Table A.2 in Appendix. There are altogether 96 coefficients including the intercept. The absolute difference of all coefficients is −0.0288. Hence, if the volatility of each stock were the same the 95% VaR would be 2.88%. Meaning that there would be only 5% chance that the returns of S&P

on one particular day would be less than -2.88%. The volatilities were estimated using the methodology in Section 2.2.2.

However, the volatility estimation of each stock is hardly the same for all of or them. This is what makes the VaR estimation change in time. The value of any particular coefficient is hard to interpret without knowing the particular volatilities. For example we could look at the estimated coefficient of AT&T (T) of -0.00399 and compare it to Omnicom Group (OMC) which was estimated as 0.00102. If both stocks had the same volatility on one day, the AT&T would outweigh Omnicom Group and increase the Value at Risk. However, if the volatility of Omnicom Group was four times larger the VaR would neutralize. Because of that we take the average volatilities of all stocks left after lasso regularization and multiply them with their estimated coefficients. From Table A.4 we see that these final products are ordered roughly the same as the original coefficients. That is in line with what we would expect and the changes in order are caused by larger or smaller average volatilities of each individual stock.

The estimated coefficients for the first conditional quantile are available in Table A.3 in Appendix. The model estimated 72 coefficients including intercept. The sum of all coefficients is -2.0889 which is surprisingly almost the same as the case of the fifth quantile. This indicates that the main difference between first and fifth quantile must be a difference between stock selected for both regressions.. If we take into consideration the assumption that first quantile should be lower than the fifth quantile, the stock left after regularization should on average have larger volatilities. That difference in volatilities would generate larger VaRs.

The selection of the regularization parameter λ was firstly done using the techniques described in Section 2.3.2. We firstly estimated λ in the framework of OLS regression with lasso regularization and tenfold cross validation. The range of parameters left in the models was from 110 (min MSE) to 60 (estimation within 1SE of MSE). We further utilized these finding in selection of our tuning parameter in the regularized quantile regression. In the Figure A.2 in Appendix we see two graphs for 95% (a) and 99% (b) confidence intervals. Each of these figures plots different λ values on x the axis. For each of these values we plotted the standard deviation of the VaR estimate on the left vertical axis (black circles). On the right vertical axis (red triangles) we have the number of parameters left in the model after regularization. We can clearly see the negative relationship of both standard deviation and number of parameters on λ. Firstly, we see decreasing number of coefficients as λ increases. That is exactly how we expect the model to work according to Section 2.3.1. Further, wee see that the standard deviation of the VaR estimates is larger when we have more parameters in the model. Similar difference in standard deviations can be spotted if we compare the VaRs on Figures 3.5 and 3.6. The VaR estimates for QR tend to oscillate more when compared to the VaR estimates by GARCH model. This may be caused by the intrinsic properties of both methods. Whereas GARCH basically regresses only on two independent variables, the QR has in our cases more that 70 parameters. This increased number or regressors might be responsible for larger volatility of VaR estimates as the regression is more prone to overtiffing. As was pointed out, rapid changes in VaR line suggest considerable changes in volatilities or in the way VaR is estimated. Since the way of estimation is still the same we can account the differences to changes in volatilities. On the contrary, if the VaR line seems to be too smooth our model is probably not adjusting for changes adequately. That might be the case of GARCH. It is complicated to decide if some of these scenarios actually occurred considering only the chart. As λ we selected 0.0012 for the fifth quantile and 0.0006 for the first quantile. It is interesting that the more tailed the modelled quantile is, the less of regularization is needed to leave a similar amount of independent variables in the model. That can be seen if we compare standard deviation of the two figures. The shapes are very close to each other but have the x axis with λ scaled differently.

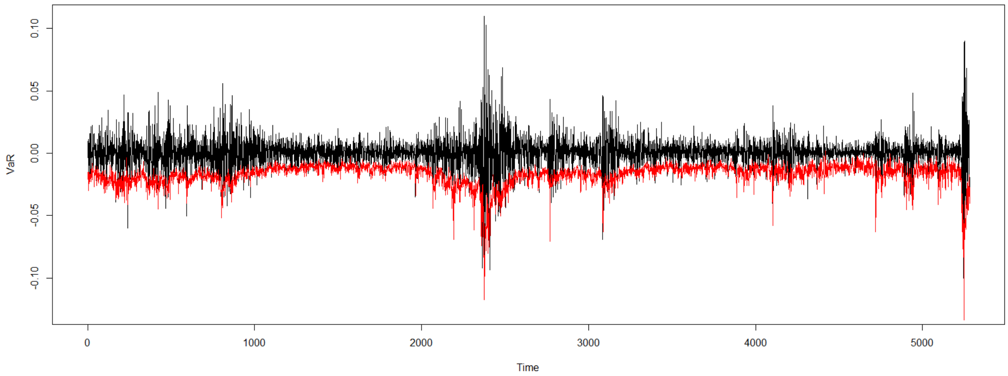
3.3 VaR Estimation

In this section we use calculated conditional volatilities from GARCH model described above and results from QR to estimate one-day Value at Risk. We calculate these predictions for 95% and 99% confidence levels. Calculation method for GARCH is done with approach described in subsection 2.1.2. VaR derived from QR is directly calculated as a result of regression equation from subsection 3.2. We will consider only in-sample performance as out-of-sample performance evaluation is beyond the scope of this paper.

In Figures 3.3 and 3.4 we plotted 95% VaR calculated from GARCH and QR respectively. Blended version of these two charts can be found in appendix Figure A.1. When we consider the VaR-QR 3.3 we see that it tends to be more volatile than the VaR calculated from GARCH 3.4. It might seem like the quantile regression approach overestimates risk. That can be considered as more conservative though also requiring higher amount of available resources.

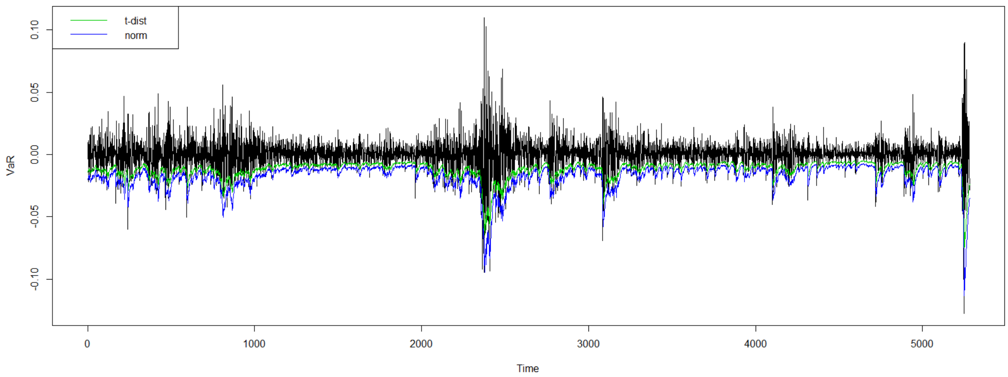
These need to be put aside and making them idle could result in lost profits. On the other hand, if we consider Figure A.1 we see that the VaR-QR is usually within bound of Value at Risks from GARCH model calculated employing normal and student's t-distribution.

Figure 3.3: 95% VaR-QR Calculation



Source: Author's computations.

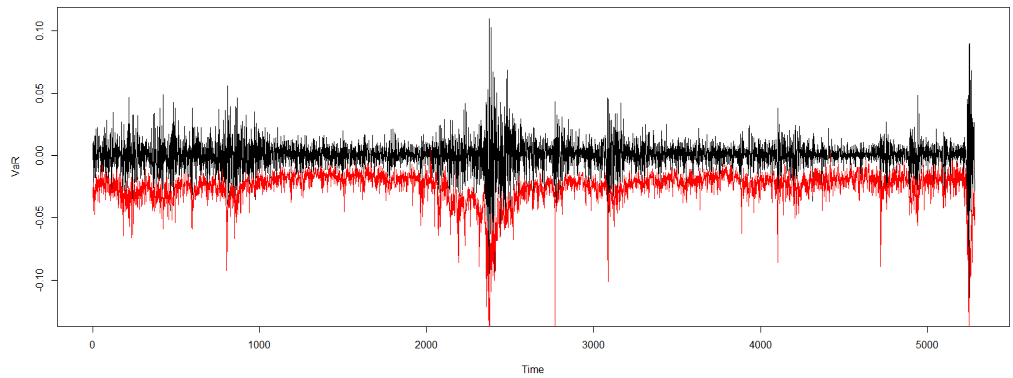
Figure 3.4: 95% VaR-GARCH Calculation



Source: Author's computations.

It is also worth noting that the volatility of Value at risk using quantile regression is largely dependent on the regularization parameter λ. This parameter was described in 2.3, specifically Equation in 2.8. The smaller the regularization we apply the larger the volatility of Value at Risk is. This is probably caused by increased number of regressors left in the model. If we try to fit large number of independent variables on returns, the model results with overfitting even if the number of observations is more than ten times bigger. Let us now consider Figure A.2. The two figures depict negative relationship between λ on x axis. And on the left y axis in red we have the standard deviation of Value at Risk estimates in red. On the right vertical axis we have the number of coefficients that were left after regularization for each λ. The left Figure A.2(a) represents the 95% VaR and the right Figure A.2(b) represents 99% VaR.

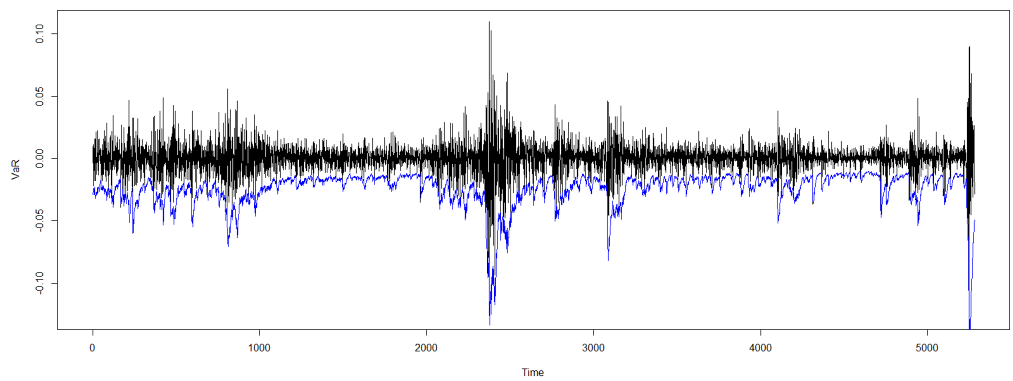
Figure 3.5: 99% VaR-QR Calculation



Source: Author's computations.

To correctly estimate whether the model is overfitted or not we would need to divide our dataset on two parts and perform in-sample and out-of-sample analysis. If the model was truly overfitted the values of VaR in out-of-sample would not pass backtesting procedures. This analysis is however beyond the scope of this paper. Nevertheless, this work can serve as a foundation for further research.

Figure 3.6: 99% VaR-GARCH Calculation (Norm and Std.t Distributed)



Source: Author's computations.

3.4 Backtesting

In this section we present and analyse the results of backtesting tests described in Section 2.4. The table 3.2 summarizes the results of our in-sample backtesting with 95% confidence interval VaR in the left column and 99% confidence interval VaR in the right column. We will firstly start the examination of 95% VaR together with analysing stock that were left in the QR after regularization. Then we provide the reader with similar analysis made for 99% VaR.

In the case of the 95% confidence interval all TUFF tests which look at the time until first failure are unable to reject neither of our three models. The failure rate of the QR is 4.66%. That can be considered a little strict when calculating VaR and means that the model underestimates risk. However this is still in reasonable bounds as shown with POF test with p-value of 0.249. The failure rate of 5.25% is similar in the case of GARCH with normal distribution for residuals. In this case the model tends to overestimate risk. However, the POF p-value of 0.417 is even larger than in the QR case. The GARCH model with t-distributed residual has failure rate of 12.1%. In order for the model to pass in the POF test, the test statistic would need to be lower than 3.841. However, our POF test statistic exceeds this value severalfold. Because of such high inaccuracy in early stage of backtesting, we will discard the model from further discussion as it is far from being optimal for VaR estimation. Next, the CIF test statistic representing the Christoffersen's Interval Forecast test does not exclude any model from further analysis. For QR as well as GARCH are the p-values sufficiently high. The conditional coverage mixed (CC) test, which is a combination of POF statistic and CIF statistic, serves as a test for independence of errors and failure rate together. Again we see that both QR and GARCH models pass this test with p-values of 0.272 and 0.721 respectively. Even though that the p-value of QR is smaller, we cannot say that one model performed particularly better than the other. We can only talk about rejecting a model if a LM statictic exceeds particular $\chi^2_{0.95}$ value. If that does not happen, we treat the models as being equally correct. All of three Hass's tests (TBFI) reject the null hypothesis of failures being independent on each other. This implies that failures are not independent and our model could be inaccurate. It is worth noting that since all of Christoffersen's test did not discover dependence between two subsequent days, there might be a more complex form of dependence in the hit sequence.

Now we focus on the Value at Risk estimation for 99% confidence interval.

Table 3.2: VaR Backtesting Results

95% VaR-GARCH (n)				**99% VaR-GARCH (n)**			
1. violation	after 6 perios			1. violation	after 51 periods		
PoF	5.25%			PoF	1.91%		
	value	$\chi^2_{0.95}$	p-value		value	$\chi^2_{0.95}$	p-value
TUFF[1]	1.0977	3.841	0.294	TUFF	0.3715	3.841	0.543
POF[2]	0.6584	3.841	0.417	POF	35.046	3.841	0.000
CIF[3]	0.0219	3.841	0.882	CIF	1.7737	3.841	0.183
CC[4]	0.6803	5.991	0.712	CC	36.819	5.991	0.000
TBFI[5]	316.82	239.4	0.004	TBFI	185.26	125.5	0.000
TBF[6]	317.89	240.4	0.004	TBF	220.306	126.6	0.000
95% VaR-GARCH (t)				**99% VaR-GARCH (t)**			
1. violation	after 6 perios			1. violation	after 51 periods		
PoF	12.1%			PoF	1.80%		
	value	$\chi^2_{0.95}$	p-value		value	$\chi^2_{0.95}$	p-value
TUFF	1.0977	3.841	0.294	TUFF	0.3715	3.841	0.543
POF	410.23	3.841	0.000	POF	27.525	3.841	0.000
CIF	0.0355	3.841	0.851	CIF	2.3348	3.841	0.126
CC	410.27	5.991	0.000	CC	29.859	5.991	0.000
TBFI	1211.7	700.0	0.000	TBFI	163.85	118.6	0.000
TBF	1621.9	701.0	0.000	TBF	191.38	119.9	0.000
95% VaR-QR				**99% VaR-QR**			
1. violation	after 51 perios			1. violation	after 168 periods		
PoF	4.66%			PoF	1.04%		
	value	$\chi^2_{0.95}$	p-value		value	$\chi^2_{0.95}$	p-value
TUFF	1.2769	3.841	0.258	TUFF	0.3252	3.841	0.568
POF	1.3278	3.841	0.249	POF	0.0905	3.841	0.764
CIF	1.2743	3.841	0.259	CIF	1.1579	3.841	0.4231
CC	2.6021	5.991	0.272	CC	1.2484	5.991	0.050
TBFI	313.49	210.7	0.002	TBFI	325.10	210.7	0.000
TBF	314.82	211.6	0.003	TBF	325.18	211.6	0.001

[1] Time Until First Failure - See Equation 2.20 for Definition
[2] Proportion of Failures - Def: 2.19
[3] Christoffersen's Interval Forecast - Def: 2.22
[4] Proportion of Failures & Christoffersen's Interval Forecast - Def: 2.23
[5] Time Between Failures - Def: 2.24
[6] Proportion of Failures & Time Between Failures - Def: 2.25

Source: Author's computations.

The failure rate for QR is only 1.04%. That is very close to the expected value of 1%. This was further confirmed by the POF statistic which measures whether the observed failure rate significantly differs from our expected failure rate. The p-value for this test is 0.76. In the case of GARCH the failure rate is almost 2%. Specifically, 1.91% for normally distributed residuals and 1.80% for t-distributed residuals. We observe, that while the residuals for the 5th quantile were better described by normal distribution, in the case of 1st quantile we spot better performance with the t-distributed residuals. The Christoffersen's tests for independence between two continuous days did not reject any model. Now we look at the CC test. It is a sum of POF and CIF statistic with two instead of one degree of freedom. In both cases of GARCH we see that the POF statistic is way too high because of larger amount of hits than expected. As a result, the mixed statistic CC results in rejecting null and the model does not pass the test of joint conditional and unconditional coverage. However in the case of QR the join test did not reject the null. With the p-value of 0.42 we can be confident that the model hit series exhibited both properties of conditional and unconditional coverage. The TBF test examining multiple day dependence result in rejecting the null in all three cases.

Chapter 4

Conclusion

This thesis compares two volatility models that are able to detect volatility clustering and predict time-varying volatility accordingly. We then evaluated them based on their in-sample predicting power of risk measure Value at Risk. In our analysis we focused on VaR estimation modelled directly via quantile regression. In the quantile regression formula we regressed the return of one day on stock volatilities the day before. To test the prediction power of the VaR-QR we decided to use a GARCH model as a benchmark. We used normal and student-t distribution of residuals in the GARCH model.

The first part of this work deals with theoretical framework. It summarizes all methodology that is essential for further analysis. We introduced the concept of Value at Risk and its estimation. Then we followed with examining the quantile regression together with regularization techniques and backtesting methods. In the applied part we calculated VaR of S&P500 stock index using the two methodologies mentioned above. Lastly, we conducted in-sample backtesting to evaluate the performance of both models and compared them to each other.

The goal of this thesis was to show whether we can estimate Value at Risk using this particular quantile regression approach. Since we do not know of any paper providing application implementing QR VaR estimation accompanied with lasso regularization, we were interested in showing whether we can produce VaR estimates at least as good as the ones produced by classical GARCH model. We conducted our research for 95% and 99% confidence interval. Our backtesting results show that in the 95% case the QR and normal GARCH estimates were both equally sound. In the case of 99% confidence interval we noticed that GARCH model with t-distribution performed better than the one with normal distributed residuals. However, it was still inferior to the quantile regression model which performed the best.

Even though our QR models in some case outperformed the benchmark GARCH, there is still some room for improvement in future work. One of the flaws in our QR models is undoubtedly high volatility in its VaR estimates. This problem could be solved through more rigorous methods of model selection. If we also wanted to test the performance of our models more thoroughly it would be suitable to perform also out-of-sample backtesting. Besides these, there are many extensions to both of these models that could be applied. Especially the methodology used during regularization.

Bibliography

Alexander, C, (2008): "Market Risk Analysis, Pricing, Hedging and Trading Financial Instruments." *John Wiley & Sons* : pp. 283

Bollerslev T. (1986): "Generalized Autoregressive Conditional Heteroskedasticity." *Journal of Econometrics* **31**: pp. 307–327

Black, F. & Scholes, M. (1976): "Taxes and the pricing of options." *The Journal of Finance* **32(2)**: pp. 319–332

Cont, R. (2005): "Volatility Clustering in Financial Markets: Empirical Facts and Agent–Based Models." Appeared in *Long memory in economics* by Kirman & Teyssiere

Engle, R.F. (1982): "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation." *Econometrica* **50(4)**: pp. 987–1007

Garman, M.B., Klass, M.J. (1980): "On the Estimation of Security Price Volatilities from Historical Data." *The Journal of Business* **53(1)**: pp. 67–78

Hansen, B. (1994): "Autoregressive Conditional Density Estimation." *International Economic Review* **35(3)**: pp. 705–30

Hansen, P.R. & Lunde, A. (2005): "A forecast comparison of volatility models: does anything beat a GARCH(1,1)?" *Journal of Applied Econometrics* **20(7)**: pp. 873–889

Hass, M. (2001): "New Methods in Backtesting." *Financial Engineering, Research center caesa, Bonn*

Holton, G.A. (2002): "History of Value-at-Risk: 1922-1998." *Method and Hist of Econ Thought*, University Library of Munich, Germany Availible: https://EconPapers.repec.org/RePEc:wpa:wuwpmh:0207001 [Accessed: 17-Apr-2020]

HUDSON, R. & GREGORIOU, A. (2010): "Calculating and Comparing Security Returns is Harder than you Think: A Comparison between Logarithmic and Simple Returns." *International Review of Financial Analysis* **38**

CHAN, N.H. (2010): "Time Series: Applications to Finance with R and S-Plus." *Wiley, 2 edition*: pp. 210

CHRISTOFFERSEN, P. & PELLETIER, D. (2004): "Backtesting Value-at-Risk: A Duration-Based Approach." *Journal of Financial Econometrics* **2**: pp. 84–108

JORION, P. (1996): "Risk:Measuring the Risk in Value at Risk." *Financial Analysts Journal* **52(6)**: pp. 47–56

JORION, P. (2001): "Value at Risk, The :New Benchmark for Managing Financial Risk." *McGraw-Hill, 2nd edition*

JP MORGAN & REUTERS (1996): "RiskMetrics^{TM} - Technical Document." *[Online]* Availible: https://www.msci.com/documents/10199/5915b101-4206-4ba0-aee2-3449d5c7e95a [Accessed: 16-Apr-2020]

KOENKER, R. (2017): "Quantile regression 40 years on." *Annual Review of Economics* **9**: pp. 155–176

KOENKER, R. & BASSETT, G. (1978): "Regression Quantiles." *Econometrica* **46(1)**: pp. 33–50

KOENKER, R. & HALLOCK, K. F. (2001): "Quantile Regression." *Journal of Economic Perspectives* **15(4)**: pp. 143–156

KOENKER, R. & MIZERA, I. (2014): "Convex Optimization in R." *Journal of Statistical Software* **40(5)**

KUPIEC, P. (1995): "Techniques for Verifying the Accuracy of Risk Management Models." *Finance and Economics Discussion Series* **95-24**

LONGIN, F. (2001): "Beyond the VaR." *Journal of Derivatives* **8(4)**: pp. 36–48

ORHAN, M. & KÖKSAL, B. (2012): "A comparison of GARCH models for VaR estimation." *Expert Systems with Applications* **39(3)**: pp. 3582–3592

PAGAN, A. & SCHWERT, G. (1990): "Alternative models for conditional stock volatility." *Journal of Econometrics* **45(1-2)**: pp. 267–290

RODRIGUEZ, R.N. & YAO, Y. (2007): "Five Things You Should Know about Quantile Regression." *Paper SAS525-2017*, Florida State University, Availible: https://support.sas.com/resources/papers/proceedings17/SAS0525-2017.pdf [Accessed: 27-Jul-2020]

RYDBERG, T.H. (2000): "Realistic Statistical Modelling of Financial Data." *International Statistical Review* **68(3)**: pp. 233–258

TIBSHIRANI, R. (1996): "Regression Shrinkage and Selection via the Lasso." *J. R. Statist. Soc. B* **58(1)**: pp. 267–288

Appendix A

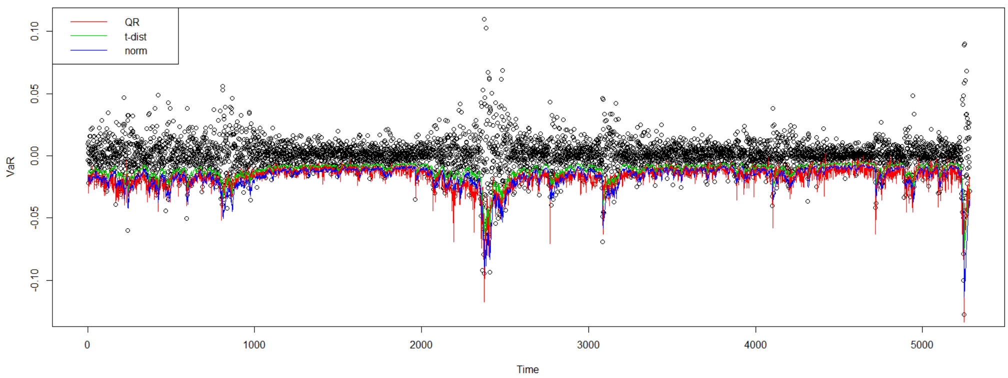
Apendix

Table A.1: Akaike Information Criteria for GARCH(p,q)

p \ q	1	2	3
1	-34060.58	-34062.25	-34034.98
2	-34051.50	-33931.26	-34054.49
3	-34035.56	-34053.94	-34052.28

Source: Author's computations.

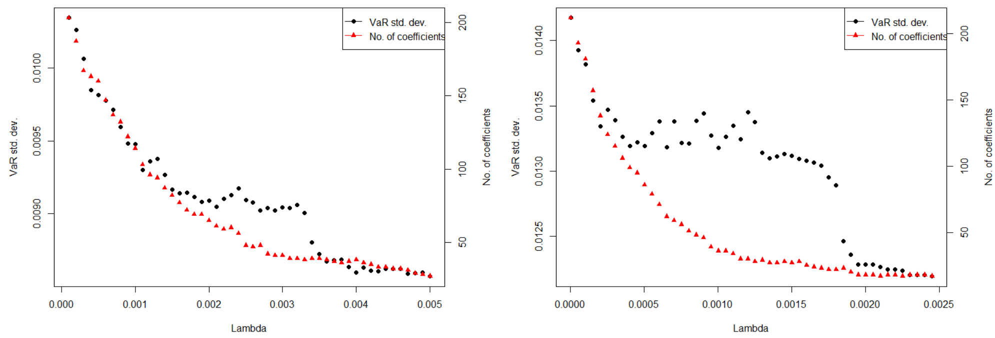
Figure A.1: 95% VaR-QR-GARCH calculation



Source: Author's computations.

Figure A.2: Lambda, Std. Deviation and Number of Coefficients

(a) of 95 % VaR (b) of 99 % VaR



Source: Author's computations.

Table A.2: 95 % QR Parameters

No.	Ticker	Coefficient	GICS.Sector	No.	Ticker	Coefficient	GICS.Sector
1	PGR	0.00420	Financials	49	AON	-0.00019	Financials
2	GPS	0.00321	Consumer Discretionary	50	MDT	-0.00021	Health Care
3	EBAY	0.00301	Consumer Discretionary	51	ETR	-0.00026	Utilities
4	BAC	0.00252	Financials	52	K	-0.00028	Consumer Staples
5	TSN	0.00183	Consumer Staples	53	IBM	-0.00033	Information Technology
6	PHM	0.00178	Consumer Discretionary	54	HAS	-0.00034	Consumer Discretionary
7	SCHW	0.00171	Financials	55	HBAN	-0.00035	Financials
8	IFF	0.00163	Materials	56	ES	-0.00036	Utilities
9	DTE	0.00159	Utilities	57	MTB	-0.00037	Financials
10	HOG	0.00156	Consumer Discretionary	58	CAH	-0.00045	Health Care
11	HIG	0.00146	Financials	59	HON	-0.00046	Industrials
12	CSX	0.00144	Industrials	60	WY	-0.00052	Real Estate
13	DHI	0.00135	Consumer Discretionary	61	CAG	-0.00054	Consumer Staples
14	ALL	0.00121	Financials	62	FITB	-0.00064	Financials
15	OMC	0.00102	Communication Services	63	PNC	-0.00067	Financials
16	TAP	0.00092	Consumer Staples	64	IPG	-0.00067	Communication Services
17	KSS	0.00091	Consumer Discretionary	65	LIN	-0.00068	Materials
18	BEN	0.00084	Financials	66	SWK	-0.00070	Industrials
19	CVS	0.00080	Health Care	67	HRB	-0.00072	Consumer Discretionary
20	ADSK	0.00072	Information Technology	68	DHR	-0.00073	Health Care
21	WBA	0.00071	Consumer Staples	69	GL	-0.00075	Financials
22	SNA	0.00061	Industrials	70	CCL	-0.00078	Consumer Discretionary
23	CMS	0.00054	Utilities	71	LMT	-0.00083	Industrials
24	UNH	0.00052	Health Care	72	BSX	-0.00097	Health Care
25	TROW	0.00052	Financials	73	FE	-0.00116	Utilities
26	SHW	0.00045	Materials	74	NWL	-0.00119	Consumer Discretionary
27	SLB	0.00041	Energy	75	ABT	-0.00120	Health Care
28	SBUX	0.00035	Consumer Discretionary	76	SO	-0.00153	Utilities
29	MSI	0.00035	Information Technology	77	UNM	-0.00156	Financials
30	ROK	0.00033	Industrials	78	EMR	-0.00162	Industrials
31	COP	0.00033	Energy	79	DUK	-0.00168	Utilities
32	PPL	0.00031	Utilities	80	TFC	-0.00171	Financials
33	WAT	0.00027	Health Care	81	PEG	-0.00179	Utilities
34	GWW	0.00021	Industrials	82	NKE	-0.00179	Consumer Discretionary
35	AZO	0.00016	Consumer Discretionary	83	NEM	-0.00183	Materials
36	KLAC	0.00015	Information Technology	84	TGT	-0.00187	Consumer Discretionary
37	AES	0.00014	Utilities	85	ORCL	-0.00195	Information Technology
38	STT	0.00012	Financials	86	WFC	-0.00222	Financials
39	JWN	0.00009	Consumer Discretionary	87	KEY	-0.00223	Financials
40	EIX	0.00008	Utilities	88	intercept	-0.00239	
41	PH	0.00007	Industrials	89	USB	-0.00263	Financials
42	VLO	0.00007	Energy	90	PFE	-0.00280	Health Care
43	BF-B	0.00002	Consumer Staples	91	CVX	-0.00300	Energy
44	QCOM	-0.00004	Information Technology	92	WMT	-0.00311	Consumer Staples
45	WMB	-0.00012	Energy	93	APA	-0.00321	Energy
46	LUV	-0.00017	Industrials	94	T	-0.00399	Communication Services
47	ADM	-0.00018	Consumer Staples	95	MSFT	-0.00443	Information Technology
48	MS	-0.00018	Financials	96	BAX	-0.00501	Health Care

Source: Author's computations.

Table A.3: 99 % QR Parameters

No.	Ticker	Coefficient	GICS.Sector	No.	Ticker	Coefficient	GICS.Sector
1	PPL	0.00730	Utilities	37	WMB	-0.00032	Energy
2	COP	0.00730	Energy	38	MCD	-0.00033	Consumer Discretionary
3	XEL	0.00628	Utilities	39	JPM	-0.00035	Financials
4	GPS	0.00479	Consumer Discretionary	40	IP	-0.00040	Materials
5	TSN	0.00339	Consumer Staples	41	TGT	-0.00040	Consumer Discretionary
6	MAS	0.00324	Industrials	42	HRB	-0.00045	Consumer Discretionary
7	OMC	0.00301	Communication Services	43	AMGN	-0.00050	Health Care
8	PGR	0.00295	Financials	44	CMI	-0.00050	Industrials
9	TXT	0.00270	Industrials	45	MS	-0.00051	Financials
10	DHI	0.00252	Consumer Discretionary	46	IPG	-0.00052	Communication Services
11	GPC	0.00209	Consumer Discretionary	47	FE	-0.00081	Utilities
12	DTE	0.00184	Utilities	48	DE	-0.00093	Industrials
13	CSCO	0.00156	Information Technology	49	IBM	-0.00098	Information Technology
14	ITW	0.00154	Industrials	50	OXY	-0.00125	Energy
15	BSX	0.00153	Health Care	51	DIS	-0.00129	Communication Services
16	BF-B	0.00131	Consumer Staples	52	TJX	-0.00136	Consumer Discretionary
17	TAP	0.00117	Consumer Staples	53	EMR	-0.00159	Industrials
18	PHM	0.00097	Consumer Discretionary	54	LOW	-0.00184	Consumer Discretionary
19	ADSK	0.00087	Information Technology	55	EXC	-0.00186	Utilities
20	AMAT	0.00083	Information Technology	56	MSFT	-0.00219	Information Technology
21	MSI	0.00073	Information Technology	57	NEM	-0.00247	Materials
22	KEY	0.00069	Financials	58	ORCL	-0.00255	Information Technology
23	QCOM	0.00065	Information Technology	59	Intercept	-0.00259	
24	HOG	0.00059	Consumer Discretionary	60	MRO	-0.00308	Energy
25	GWW	0.00054	Industrials	61	HBAN	-0.00363	Financials
26	KSS	0.00049	Consumer Discretionary	62	APA	-0.00364	Energy
27	UNH	0.00040	Health Care	63	DUK	-0.00367	Utilities
28	CNP	0.00026	Utilities	64	GL	-0.00379	Financials
29	AZO	0.00008	Consumer Discretionary	65	CVX	-0.00387	Energy
30	XOM	0.00005	Energy	66	LUV	-0.00418	Industrials
31	MAR	0.00002	Consumer Discretionary	67	CAG	-0.00430	Consumer Staples
32	WAT	0.00001	Health Care	68	BAX	-0.00657	Health Care
33	MTB	-0.00007	Financials	69	WFC	-0.00699	Financials
34	PEG	-0.00022	Utilities	70	WMT	-0.00739	Consumer Staples
35	KMB	-0.00024	Consumer Staples	71	WY	-0.00867	Real Estate
36	TFC	-0.00025	Financials	72	SO	-0.01465	Utilities

Source: Author's computations.

Table A.4: 95 % QR Parametres & Volatilities Product

NO.	Ticker	Value	Security	No.	Ticker	Value	Security
2	GPS	14.33	Gap Inc.	56	ES	-1.35	Eversource Energy
1	PGR	13.03	Progressive Corp.	61	CAG	-1.4	Conagra Brands
8	IFF	12.54	Intl Flavors & Fragrances	49	AON	-1.41	Aon plc
10	HOG	10.9	Harley-Davidson	50	MDT	-1.41	Medtronic plc
11	HIG	10.53	Hartford Financial Svc.Gp.	52	K	-1.42	Kellogg Co.
9	DTE	9.79	DTE Energy Co.	48	MS	-1.47	Morgan Stanley
4	BAC	9.35	Bank of America Corp	64	IPG	-2.03	Interpublic Group
3	EBAY	8.49	eBay Inc.	54	HAS	-2.06	Hasbro Inc.
17	KSS	8.31	Kohl's Corp.	51	ETR	-2.09	Entergy Corp.
26	SHW	7.93	Sherwin-Williams	67	HRB	-2.32	H&R Block
7	SCHW	7.69	Charles Schwab Corporation	72	BSX	-2.87	Boston Scientific
5	TSN	7.59	Tyson Foods	62	FITB	-2.98	Fifth Third Bancorp
14	ALL	7.21	Allstate Corp	69	GL	-3	Globe Life Inc.
6	PHM	6.83	PulteGroup	58	CAH	-3.34	Cardinal Health Inc.
15	OMC	6.53	Omnicom Group	68	DHR	-3.42	Danaher Corp.
35	AZO	6.47	AutoZone Inc	60	WY	-3.47	Weyerhaeuser
20	ADSK	5.87	Autodesk Inc.	59	HON	-3.67	Honeywell Int'l Inc.
13	DHI	5.69	D. R. Horton	70	CCL	-4.33	Carnival Corp.
22	SNA	5.48	Snap-on	74	NWL	-4.56	Newell Brands
16	TAP	5.39	Molson Coors Brewing Company	57	MTB	-4.57	M&T Bank Corp.
24	UNH	5.25	United Health Group Inc.	75	ABT	-4.58	Abbott Laboratories
12	CSX	4.64	CSX Corp.	53	IBM	-4.63	Int. Business Machines
19	CVS	4.57	CVS Health	76	SO	-5.45	Southern Company
21	WBA	4.47	Walgreens Boots Alliance	73	FE	-5.73	FirstEnergy Corp
29	MSI	4.22	Motorola Solutions Inc.	82	NKE	-6.1	Nike
25	TROW	4.06	T. Rowe Price Group	66	SWK	-6.21	Stanley Black & Decker
27	SLB	3.93	Schlumberger Ltd.	77	UNM	-6.38	Unum Group
34	GWW	3.85	Grainger (W.W.) Inc.	65	LIN	-6.53	Linde plc
30	ROK	3.63	Rockwell Automation Inc.	87	KEY	-6.56	KeyCorp
18	BEN	3.6	Franklin Resources	63	PNC	-6.61	PNC Financial Services
33	WAT	3.2	Waters Corporation	81	PEG	-7.17	Public Serv. Enterprise Inc.
31	COP	2.08	ConocoPhillips	80	TFC	-8.11	Truist Financial
36	KLAC	1.68	KLA Corporation	85	ORCL	-8.2	Oracle Corp.
23	CMS	1.67	CMS Energy	78	EMR	-9.52	Emerson Electric Company
28	SBUX	1.14	Starbucks Corp.	86	WFC	-10.17	Wells Fargo
38	STT	1.06	State Street Corp.	90	PFE	-10.62	Pfizer Inc.
32	PPL	0.99	PPL Corp.	89	USB	-11.24	U.S. Bancorp
41	PH	0.76	Parker-Hannifin	79	DUK	-11.34	Duke Energy
39	JWN	0.56	Nordstrom	71	LMT	-11.74	Lockheed Martin Corp.
42	VLO	0.48	Valero Energy	83	NEM	-12.49	Newmont Corporation
37	AES	0.44	AES Corp	84	TGT	-14.56	Target Corp.
40	EIX	0.43	Edison Int'l	94	T	-17.01	AT&T Inc.
43	BF-B	0.05	Brown-Forman Corp.	96	BAX	-20.05	Baxter International Inc.
44	QCOM	-0.33	QUALCOMM Inc.	92	WMT	-22.1	Walmart
45	WMB	-0.54	Williams Cos.	95	MSFT	-25.39	Microsoft Corp.
46	LUV	-0.64	Southwest Airlines	91	CVX	-27.88	Chevron Corp.
47	ADM	-0.73	Archer-Daniels-Midland Co	93	APA	-33.91	Apache Corporation
55	HBAN	-0.78	Huntington Bancshares	88	intercept		

Source: Author's computations.