FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

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Parameter Estimation in Stochastic Differential Equations

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Study programme: Mathematics
Study branch: Probability, Mathematical Statistics and Econometrics

Prague 2020

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I would like to express my gratitude to my supervisor prof. Bohdan Maslowski who provided all the guidance I needed while working on this Thesis. I would also like to thank my mum for all the support she has been giving me throughout my studies.

Title: Parameter Estimation in Stochastic Differential Equations

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Abstract: In the Thesis the problem of estimating an unknown parameter in a stochastic differential equation is studied. Linear equations with Volterra process as the source of noise are considered. Firstly, the properties of Volterra processes and the properties of stochastic integral with respect to a Volterra process are presented. Secondly, the properties of the solution to the equation under consideration are discussed. This includes the existence of the strictly stationary solution, the properties of such solution and ergodic results. These results are then generalized to equations with a mixed noise. Ergodic results are used to derive strongly consistent estimators of the unknown parameter.

Keywords: parameter estimation, stochastic differential equations, Ornstein-Uhlenbeck process, Volterra type process

Název práce: Odhad parametru ve stochastických diferenciálních rovnicích

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Abstrakt: V diplomové práci je studován problém odhadu parametru ve stochastických diferenciálních rovnicích. Jsou uvažovány lineární rovnice řízené volterrovským procesem. Nejprve jsou uvedeny vlastnosti volterrovského procesu a vlastnosti stochastikého integrálu vzhledem k volterrovskému procesu. Dále se práce zabývá vlastnostmi řešení uvažované rovnice, včetně existence stationárního řešení a ergodicity. Tyto vlastnosti jsou dále zobecněny pro rovnice s řídícím procesem smíšeného typu. Ergodické výsledky jsou použity pro odvození silně konzistentních odhadů neznámého parametru.

Klíčová slova: odhad parametru, stochastické diferenciální rovnice, Ornstein-Uhlenbeckův proces, volterrovský proces

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Introduction

Consider a stochastic differential equation whose coefficients are known up to a unknown parameter. Suppose we observe a trajectory of a solution of the equation. The task to estimate the unknown parameter using the observed trajectory is called the problem of parameter estimation and it is the topic of this Thesis.

In the Thesis we consider a finite dimensional linear stochastic equations driven by a Volterra process. Volterra processes, introduced in Alòs, Mazet and Nualart [1], have risen to popularity in the recent years. Results concerning existence and regularity of solutions of stochastic differential equations driven by a Volterra process were given in Bonaccorsi and Tudor [3], Čoupek and Maslowski [7], Čoupek, Maslowski and Ondreját [8], Čoupek, Maslowski and Šnupárková [9] and Čoupek [6]. Some articles, instead of a general Volterra process, consider the special cases such as fractional Brownian motion (fBm). Linear stochastic equations in a Hilbert space with a cylindrical fractional Brownian motion are considered by Duncan, Maslowski and Pasik-Duncan in [10] and [12] where some results on the continuity and space regularity of sample paths are given and large time behaviour of solutions is investigated. Similar results for a bilinear equation were established by the same authors in [11]. For other works concerning stationarity and large-time behavior of the solutions see e.g. Maslowski and Nualart [18], Maslowski and Šnupárková [21] or Šnupárková [25].

Returning to the problem of parameter estimation, from the pioneer works of Koski, Akademi and Loges [15] and Huebner and Rozovskii [14] who considered Wiener process as the source of noise, most of recent literature deals with the noise in the form of a fractional Brownian motion. For example the work of Cialenco, Lototsky and Pospíšil [10] deals with space asymptotics for a maximum likelihood estimator. The work of Maslowski and Pospíšil [19] proves the strong consistency of the minimum contrast (MC) estimator considering fBm with trace-class covariance operator as a driving noise. The work of Balde, Es-Sebaiy and Tudor [2] deals with the least squares estimators constructed from the one-dimensional projection of the mild solution to the linear SPDEs driven by fBm and the work of Maslowski and Tudor [22] deals with the same problem but consider an cylindrical fBm as a source of noise. For the results concerning an asymptotic normality of the MC estimator see e.g. [16].

In this Thesis, in order to derive a strongly consistent estimators we employ a method based on ergodicity used in [19], where an infinite-dimensional stochastic differential equation with fractional Brownian motion as a noise is considered. In the Thesis we consider only finite-dimensional case, but we (at least in the beginning) allow the noise to be any α-regular Volterra process. Firstly, we present conditions found in [6] under which a strongly stationary solution exists. Secondly, we find conditions under which the strongly stationary solution is ergodic. As a corollary we obtain a similar ergodic-like result for any solution. We employ these ergodic results to obtain the desired estimators. In order for our main results to not be a special case of results from [19] we slightly generalize our setting and consider a stochastic differential equation with a mixed noise.

Theorem 23 and Theorem 24 are thus a partial generalization of results from [19].

The structure of the Thesis is as follows. In Section 1 we introduce an α-regular Volterra process and a we present a construction of stochastic integral of deterministic functions with respect to an α-regular Volterra process. In the rest of the first chapter we deal with various properties of the integral which will be needed in the following chapters.

In Section 2 we present the results concerning a linear stochastic differential equation of the form

$$\begin{aligned} \mathrm{d}X_t &= AX_t \,\mathrm{d}t + \Phi \,\mathrm{d}B_t, \quad t \geq 0, \\ X_0 &= x_0. \end{aligned}$$

Firstly, we discuss the solution of the equation under consideration. Secondly, we present the results of [6] concerning the existence of a strictly stationary solution. Lastly, following [19], we show the ergodicity of a strictly stationary solution and as a corollary we obtain a similar ergodic result for an arbitrary solution.

In Section 3 we slightly generalize the setting and results of Section 2. We consider a stochastic differential equation of the form

$$\begin{aligned} \mathrm{d}X_t &= AX_t \,\mathrm{d}t + \sum_{i=1}^{p} \Phi^i \,\mathrm{d}B_t^i, \quad t \geq 0, \\ X_0 &= x_0, \end{aligned}$$

and obtain results analogous to those of Section 2. Ergodic results from this chapter are the key ingredient needed in the last chapter.

In the last Section 4 we will consider an equation from Section 4 with added multiplicative parameter in the drift, i.e.

$$\begin{aligned} \mathrm{d}X_t &= \gamma AX_t \,\mathrm{d}t + \sum_{i=1}^{p} \Phi^i \,\mathrm{d}B_t^i, \quad t \geq 0, \\ X_0 &= x_0, \end{aligned}$$

with the unknown parameter γ. Following [19] and using the results from previous chapters we derive a strongly consistent estimators of γ.

The novelty of the Thesis consists in the new proof of ergodicity for stationary solution (Theorem 15). Also the results in Section 3 can probably be considered as new, although they are a rather straightforward generalization of results described in Section 2. As a result of this generalization, the estimators derived in Section 4 partially extend the results found in [19].

The results contained in the Thesis are closely related to GACR grant project no. 19-07140S.

1. Volterra processes

In this preliminary chapter we deal with the definition of a Volterra process, with the definition of a stochastic integral with respect to a Volterra process and we derive properties of such integrals. Most results in this chapter can be found in [5] and [6]. See also [1] and [3].

1.1 Definition and basic properties

We start with the definition of a Volterra process.

Definition. A function $K : \mathbb{R}^2 \to \mathbb{R}, (t,r) \mapsto K(t,r)$ satisfying

1. $K(t,r) = 0$ on $\{t < r\}$,
2. $\forall r \in \mathbb{R} : \lim_{t \to r+} K(t,r) = 0$,
3. $\forall r \in \mathbb{R} : K(\cdot, r)$ is continuously differentiable on (r, ∞),
4. there exists an $\alpha \in (0, \frac{1}{2})$ and $C \in (0,\infty)$ such that
$$\left| \frac{\partial K}{\partial t}(u,r) \right| \leq C(u-r)^{\alpha-1}$$
on $\{r < u\}$

is called an α-regular Volterra kernel.

The following lemma will be often needed later.

Lemma 1 ([6], Lemma 2.1)**.** *Let K be an α-regular Volterra kernel. Define*
$$\varphi(u,v) = \int_{-\infty}^{u \wedge v} \frac{\partial K}{\partial t}(u,r) \frac{\partial K}{\partial t}(v,r) \, \mathrm{d}r.$$
Then for $u \neq v$ we have $\varphi(u,v) \leq c_\alpha |u-v|^{2\alpha-1}$ for some $c_\alpha \in (0,\infty)$.

Proof. Claim follows by using 4. from the definition of K and the substitution $z = \frac{v-r}{u-r}$. □

Now for $s_1, t_1, s_2, t_2 \in \mathbb{R}$ define
$$R(s_1,t_1,s_2,t_2) = \int_{\mathbb{R}} (K(t_1,r) - K(s_1,r)) \, (K(t_2,r) - K(s_2,r)) \, \mathrm{d}r.$$
For $s_1 < t_1$ and $s_2 < t_2$ we set
$$R(s_1,t_1,s_2,t_2) := \int_{s_1}^{t_1} \int_{s_2}^{t_2} \varphi(u,v) \, \mathrm{d}u \, \mathrm{d}v$$
and previous lemma implies that $R(s_1,t_1,s_2,t_2)$ is finite.

Definition. A stochastic process $b = (b_t, t \in \mathbb{R})$ is a two-sided α-regular Volterra process, if $b_0 = 0$ $\mathbb{P}$-a.s., b is centred and

$$\mathbb{E}(b_{t_1} - b_{s_1})(b_{t_2} - b_{s_2}) = R(s_1, t_1, s_2, t_2)$$

for every $s_1, t_1, s_2, t_2 \in \mathbb{R}$.

Remark ([6], Remark 2.2)**.** An α-regular Volterra process has a version with locally ε-Hölder continuous trajectories for every $\varepsilon \in (0, \alpha)$. In the sequel we always work with this continuous version.

The following is an example of a Gaussian α-regular Volterra process. For the proofs of statements in the example see e.g. [5], Example 1.26.

Example 1.1.1 (Two-sided fractional Brownian motion)**.** Take $H \in (0,1)$. The two-sided fractional Brownian motion (fBm) with the Hurst index H is defined as a stochastic process $B^H = (B^H_t, t \in \mathbb{R})$ which is continuous centred Gaussian process with $B^H_0 = 0$ $\mathbb{P}$-a.s. and with a covariance function

$$\mathbb{E}\left[B^H_s B^H_t\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right), \quad s, t \in \mathbb{R}.$$

The existence and properties of B^H are well-known. Assume that $H > \frac{1}{2}$. Let

$$c_H = \left(\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$$

where $B(\cdot,\cdot)$ is the Beta function. Let

$$K^H(t,r) = \begin{cases} c_H \int\limits_r^t (u-r)^{H-\frac{3}{2}}\,\mathrm{d}u, & -\infty < r < t, \\ 0, & \text{elsewhere.} \end{cases}$$

Then B^H is a two-sided $(H-\frac{1}{2})$-regular Volterra process with the $(H-\frac{1}{2})$-regular Volterra kernel K^H and the φ_H function from Lemma1 is of the form

$$\varphi_H(u,v) = H(2H-1)|u-v|^{2H-2}.$$

Furthermore, B^H has stationary and reflexive increments.

1.2 Wiener integration

We now proceed with the definition of the integral with respect to an α-regular Volterra process. In what follows we identify function equal almost everywhere. Let $b = (b_t, t \in \mathbb{R})$ be a two-sided α-regular Volterra process. Denote $\mathcal{E}(\mathbb{R})$ the set of $\mathbb{R}$-valued step functions defined on $\mathbb{R}$, i.e. $f \in \mathcal{E}(\mathbb{R})$ if and only if

$$f(t) = \sum_{j=1}^{n} f_j \mathbb{1}_{[t_{j-1}, t_j)}(t), \quad t \in \mathbb{R},$$

for some $n \in \mathbb{N}, f_1, \ldots, f_n \in \mathbb{R}$, $t_0, \ldots, t_n, \in \mathbb{R}$, $t_0 < \cdots < t_n$. Define the linear map $i : \mathcal{E}(\mathbb{R}) \to L^2(\Omega; \mathbb{R})$ by

$$i\left(\sum_{j=1}^{n} f_j \mathbb{1}_{[t_{j-1},t_j)}(t)\right) := \sum_{j=1}^{n} f_j \left(b_{t_j} - b_{t_{j-1}}\right).$$

Define a linear operator $\mathcal{K}^* : \mathcal{E}(\mathbb{R}) \to L^2(\mathbb{R}; \mathbb{R})$ by

$$\left(\mathcal{K}^* f\right)(r) := \int_r^{\infty} f(u) \frac{\partial K}{\partial t}(u, r)\, \mathrm{d}u, \quad f \in \mathcal{E}(\mathbb{R}),\, r \in \mathbb{R}.$$

Assume, that $\mathcal{K}^*$ is injective. For $f \in \mathcal{E}(\mathbb{R})$ it holds that

$$\|i(f)\|_{L^2(\Omega;\mathbb{R})} = \|\mathcal{K}^* f\|_{L^2(\mathbb{R};\mathbb{R})}.$$

Indeed, we have that

$$\begin{aligned}
&\|i(f)\|^2_{L^2(\Omega;\mathbb{R})} = \mathbb{E}\left|\sum_{i=1}^{n} f_i (b_{t_i} - b_{t_{i-1}})\right|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} f_i f_j \mathbb{E}(b_{t_i} - b_{t_{i-1}})(b_{t_j} - b_{t_{j-1}}) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} f_j f_i R(t_{i-1}, t_i, t_{j-1}, t_j) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} f_j f_i \int_{\mathbb{R}} \left(K(t_i, r) - K(t_{i-1}, r)\right)\left(K(t_j, r) - K(t_{j-1}, r)\right) \mathrm{d}r \\
&= \int_{\mathbb{R}} \left|\sum_{i=1}^{n} f_i \left(K(t_i, r) - K(t_{i-1}, r)\right)\right|^2 \mathrm{d}r = \int_{\mathbb{R}} \left|\sum_{i=1}^{n} f_i \int_{t_{i-1}}^{t_i} \frac{\partial K}{\partial t}(u, r)\, \mathrm{d}u\right|^2 \mathrm{d}r \\
&= \int_{\mathbb{R}} \left|\sum_{i=1}^{n} \int_{\mathbb{R}} f_i \mathbb{1}_{[t_{i-1},t_i)}(u) \frac{\partial K}{\partial t}(u, r)\, \mathrm{d}u\right|^2 \mathrm{d}r = \int_{\mathbb{R}} \left|\sum_{i=1}^{n} \int_r^{\infty} f_i \mathbb{1}_{[t_{i-1},t_i)}(u) \frac{\partial K}{\partial t}(u, r)\, \mathrm{d}u\right|^2 \mathrm{d}r \\
&= \int_{\mathbb{R}} \left|\int_r^{\infty} f(u) \frac{\partial K}{\partial t}(u, r)\, \mathrm{d}u\right|^2 \mathrm{d}r = \int_{\mathbb{R}} \left(\mathcal{K}^* f\right)^2(r)\, \mathrm{d}r = \|\mathcal{K}^* f\|^2_{L^2(\mathbb{R},\mathbb{R})}.
\end{aligned}$$

On $\mathcal{E}(\mathbb{R})$ we define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ by

$$\langle f, g \rangle_{\mathcal{D}} = \langle \mathcal{K}^* f, \mathcal{K}^* g \rangle_{L^2(\mathbb{R};\mathbb{R})}.$$

Denote $\mathcal{D}(\mathbb{R}; \mathbb{R})$ the completion of $\mathcal{E}(\mathbb{R})$ under $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ and extend $\mathcal{K}^*$ from $\mathcal{E}(\mathbb{R})$ to $\mathcal{D}(\mathbb{R}; \mathbb{R})$. Denote this extension again by $\mathcal{K}^*$. This extends i to a linear isometry between $\mathcal{D}(\mathbb{R}; \mathbb{R})$ and closed linear subspace of $L^2(\Omega; \mathbb{R})$. The set $\mathcal{D}(\mathbb{R}; \mathbb{R})$ is called the set of admissible integrands and, for $f \in \mathcal{D}(\mathbb{R}; \mathbb{R})$, the random variable $i(f)$ is called the stochastic integral of f with respect to the Volterra process b. We use the notation

$$\int f\, \mathrm{d}b := \int f(r)\, \mathrm{d}b_r := i(f).$$

In order to a better specification of admissible integrands we will need the following definition and the theorem which follows it.

Definition. Let $\beta \in (0,1)$, $p \in \left[1, \frac{1}{\beta}\right)$, $f \in L^p(\mathbb{R};\mathbb{R})$. We then define an operator I_+^β by

$$(I_+^\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} f(t)(x-t)^{\beta-1}\,\mathrm{d}t, \quad x \in \mathbb{R}$$

and call it the (left-sided) Riemann-Liouville fractional integral of f of order β on $\mathbb{R}$.

Theorem 1 (Hardy-Littlewood inequality)**.** *Let* $1 \le p \le \infty$, $1 \le q \le \infty$, $\beta \in (0,1)$. *The operator* I_+^β *is a bounded operator from* $L^p(\mathbb{R};\mathbb{R})$ *to* $L^q(\mathbb{R};\mathbb{R})$ *if and only if* $p \in \left(1, \frac{1}{\beta}\right)$ *and* $q = \frac{p}{1-\beta p}$.

Proof. See e.g. [13], Theorem 4. □

The following result gives us a better understanding of $\mathcal{D}(\mathbb{R};\mathbb{R})$.

Theorem 2 ([5], Proposition 1.9)**.** *The space* $L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ *is continuously embedded in* $\mathcal{D}(\mathbb{R};\mathbb{R})$.

Proof. Let $f \in \mathcal{E}(\mathbb{R})$. We show that $\|f\|_{\mathcal{D}} \le C\,\|f\|_{L^{\frac{2}{1+2\alpha}}}$ for a suitable constant $C > 0$ depending only on α. We have

$$\begin{aligned}
\|f\|_{\mathcal{D}}^2 = \|\mathcal{K}^* f\|_{L^2(\mathbb{R};\mathbb{R})}^2 &= \int_{\mathbb{R}} |\mathcal{K}^* f|^2(r)\,\mathrm{d}r = \int_{\mathbb{R}} \left| \int_r^\infty f(u) \frac{\partial K}{\partial t}(u,r)\,\mathrm{d}u \right|^2 \mathrm{d}r \\
&= \int_{\mathbb{R}} \int_r^\infty \int_r^\infty f(u)f(v) \frac{\partial K}{\partial t}(u,r) \frac{\partial K}{\partial t}(v,r)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}r \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{-\infty}^{u\wedge v} f(u)f(v) \frac{\partial K}{\partial t}(u,r) \frac{\partial K}{\partial t}(v,r)\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)f(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Now we use Lemma 1 to get

$$\begin{aligned}
&\le c_\alpha \int_{\mathbb{R}} \int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2\alpha-1}\,\mathrm{d}u\,\mathrm{d}v \\
&= 2c_\alpha \int_{\mathbb{R}} \int_{-\infty}^{v} |f(u)||f(v)||u-v|^{2\alpha-1}\,\mathrm{d}u\,\mathrm{d}v \\
&= 2c_\alpha \int_{\mathbb{R}} |f(v)| \left(\int_{-\infty}^{v} |f(u)|(v-u)^{2\alpha-1} \right) \mathrm{d}v \\
&\le 2c_\alpha \left(\int_{\mathbb{R}} |f(v)|^{\frac{2}{1+2\alpha}}\,\mathrm{d}v \right)^{\frac{1}{2}+\alpha} \left(\int_{\mathbb{R}} \left(\int_{-\infty}^{v} |f(u)|(v-u)^{2\alpha-1}\,\mathrm{d}u \right)^{\frac{2}{1-2\alpha}} \mathrm{d}v \right)^{\frac{1}{2}-\alpha}.
\end{aligned}$$

Finally, we can use the Hardy-Littlewood inequality to get

$$
\begin{aligned}
&= 2c_\alpha \left(\|f\|^2_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})} \right)^{\frac{1}{2}+\alpha} \Gamma(2\alpha) \left(\left\| I^{2\alpha}_+ |f| \right\|^2_{L^{\frac{2}{1-2\alpha}}(\mathbb{R};\mathbb{R})} \right)^{\frac{1}{2}-\alpha} \\
&\le 2c_\alpha \left(\|f\|^2_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})} \right)^{\frac{1}{2}+\alpha} \Gamma(2\alpha) \left(C_\alpha \, \||f|\|^2_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})} \right)^{\frac{1}{2}-\alpha} \\
&= C \, \|f\|^2_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})} ,
\end{aligned}
$$

where C depends only on α. We have shown the desired inequality

$$
\|f\|_{\mathcal{D}} \le C \, \|f\|_{L^{\frac{2}{1+2\alpha}}} . \tag{$*$}
$$

Now, take $f \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$. Then there exists a sequence $\{f_n\}$ such that $f_n \in \mathcal{E}(\mathbb{R})$, $n \in \mathbb{N}$ and $f_n \to f$ in $L^{\frac{2}{1+2\alpha}}$. By the inequality $(*)$, $\{f_n\}$ is Cauchy in $\mathcal{D}(\mathbb{R};\mathbb{R})$ and by completeness thereof there is $\tilde{f} \in \mathcal{D}(\mathbb{R};\mathbb{R})$ such that $\left\|\tilde{f} - f_n\right\|_{\mathcal{D}(\mathbb{R};\mathbb{R})} \to 0$. Now we can identify f with $\tilde{f}$. □

In the sequel we identify $f \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ with its image $\tilde{f}$ in $\mathcal{D}(\mathbb{R};\mathbb{R})$ under the map from the previous theorem. Therefore any $f \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ is now integrable. Also, whenever we have a sequence $f_n \in \mathcal{E}(\mathbb{R})$, $n \in \mathbb{N}$ such that $f_n \to f$ in $L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ we have $i(f_n) \to i(f)$ in $L^2(\Omega;\mathbb{R})$.

In the previous theorem we showed the inequality

$$
\|f\|_{\mathcal{D}} = \|\mathcal{K}^* f\|_{L^2(\mathbb{R};\mathbb{R})} \le C \, \|f\|_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})}
$$

for any $f \in \mathcal{E}(\mathbb{R})$. Now take $f \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ and find a sequence $\{f_n\}$ of functions from $L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ such that $f_n \to f$ in $L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$. Then

$$
\begin{aligned}
\|\mathcal{K}^* f\|_{L^2(\mathbb{R};\mathbb{R})} = \|f\|_{\mathcal{D}(\mathbb{R};\mathbb{R})} &\le \|f - f_n\|_{\mathcal{D}(\mathbb{R};\mathbb{R})} + \|f_n\|_{\mathcal{D}} \\
&\le \|f - f_n\|_{\mathcal{D}(\mathbb{R};\mathbb{R})} + C \, \|f_n\|_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})} \xrightarrow[n\to\infty]{} C \, \|f\|_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})} ,
\end{aligned}
$$

since we identified f and $\tilde{f}$. Therefore, for any $f \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ we have that

$$
\|f\|_{\mathcal{D}} = \|\mathcal{K}^* f\|_{L^2(\mathbb{R};\mathbb{R})} \le C \, \|f\|_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})} . \tag{1.1}
$$

We thus have the same inequality holding for a larger set of functions, i.e. for any $f \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$.

Definition. Let $-\infty < s < t < \infty$. If f is a function defined only on $(s,t]$ we identify f with the function f^*, defined on $\mathbb{R}$, where

$$
f^*(t) = \begin{cases} f(t) : t \in (s,t] \\ 0 \quad\; : \text{elsewhere} \end{cases}
$$

and similarly for functions defined only on (s,t), $[s,t)$ and $[s,t]$. Now, for $f \in L^{\frac{2}{1+2\alpha}}(s,t;\mathbb{R})$ we define the definite integral $i_{s,t}(f)$ by

$$
i_{s,t}(f) := \int_s^t f \, \mathrm{d}b := \int_s^t f(r) \, \mathrm{d}b_r := i(\mathbb{1}_{[s,t)} f).
$$

Remark. Since b is centred, for $f_1 \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ we have $\mathbb{E}\left(i(f_1)\right) = 0$ and for $f_2 \in L^{\frac{2}{1+2\alpha}}(s,t;\mathbb{R})$ we have $\mathbb{E}\left(i_{s,t}(f_2)\right) = 0$. If b is moreover Gaussian, then both $i(f_1)$ and $i_{s,t}(f_2)$ are Gaussian.

In what follows we state some properties of stochastic integrals driven by a Volterra process.

Theorem 3 ([5], Proposition 1.14)**.** *Let $-\infty < a < b < c < \infty$ and $f \in L^{\frac{2}{1+2\alpha}}(a,c;\mathbb{R})$. Then*

$$\int_a^c f(s)\,\mathrm{d}b_s = \int_a^b f(s)\,\mathrm{d}b_s + \int_b^c f(s)\,\mathrm{d}b_s.$$

Proof. We have

$$\begin{aligned}
\int_a^c f(s)\,\mathrm{d}b_s &= \int \mathbb{1}_{[a,c)}(s) f(s)\,\mathrm{d}b_s = \int \left(\mathbb{1}_{[a,b)}(s) + \mathbb{1}_{[b,c)}(s)\right) f(s)\,\mathrm{d}b_s \\
&= \int \mathbb{1}_{[a,b)}(s) f(s)\,\mathrm{d}b_s + \int \mathbb{1}_{[b,c)}(s) f(s)\,\mathrm{d}b_s \\
&= \int_a^b f(s)\,\mathrm{d}b_s + \int_b^c f(s)\,\mathrm{d}b_s.
\end{aligned}$$

□

Lemma 2 ([5], Lemma 1.13)**.** *The extended operator $\mathcal{K}^* : \mathcal{D}(\mathbb{R};\mathbb{R}) \to L^2(\mathbb{R};\mathbb{R})$ satisfies*

$$\left(\mathcal{K}^* f\right)(r) = \int_r^\infty f(u) \frac{\partial K}{\partial t}(u,r)\,\mathrm{d}u$$

for almost every $r \in \mathbb{R}$ and every $f \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$.

Proof. Take $f \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$ and a sequence $\{f_n\}$ of simple functions from $\mathcal{E}(\mathbb{R})$ such that $f_n \to f$ in $L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})$. Then by (1.1) we get

$$\begin{aligned}
&\left\| \mathcal{K}^* f - \int_\cdot^\infty f(u) \frac{\partial K}{\partial t}(u,\cdot)\,\mathrm{d}u \right\|_{L^2(\mathbb{R};\mathbb{R})} \\
&\leq \left\|\mathcal{K}^* f - \mathcal{K}^* f_n\right\|_{L^2(\mathbb{R};\mathbb{R})} + \left\| \int_\cdot^\infty f_n(u) \frac{\partial K}{\partial t}(u,\cdot)\,\mathrm{d}u - \int_\cdot^\infty f(u) \frac{\partial K}{\partial t}(u,\cdot)\,\mathrm{d}u \right\|_{L^2(\mathbb{R};\mathbb{R})} \\
&\leq C \left\| f - f_n \right\|_{L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R})} + \left\| \int_\cdot^\infty (f_n(u) - f(u)) \frac{\partial K}{\partial t}(u,\cdot)\,\mathrm{d}u \right\|_{L^2(\mathbb{R};\mathbb{R})}.
\end{aligned}$$

The first term converges to zero as $n \to +\infty$ trivially. The fact that the second term converges to zero follows from the proof of Theorem 2. □

Theorem 4 ([5], Proposition 1.14)**.** *Let $s_1 < t_1$, $s_2 < t_2$, $f \in L^{\frac{2}{1+2\alpha}}(s_1,t_1;\mathbb{R})$ and $g \in L^{\frac{2}{1+2\alpha}}(s_2,t_2;\mathbb{R})$. Then*

$$\langle i_{s_1,t_1}(f), i_{s_2,t_2}(g) \rangle_{L^2(\Omega;\mathbb{R})} = \int_{s_2}^{t_2} \int_{s_1}^{t_1} f(u) g(v) \varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v,$$

where φ is defined in Lemma 1.

Proof. Denote $\hat{f} = \mathbb{1}_{[s_1,t_1)}f$ and $\hat{g} = \mathbb{1}_{[s_2,t_2)}g$. Then similarly as in Theorem 2 and using the previous lemma we have that

$$\langle i_{s_1,t_1}(f), i_{s_2,t_2}(g)\rangle_{L^2(\Omega;\mathbb{R})} = \left\langle i(\hat{f}), i(\hat{g})\right\rangle_{L^2(\Omega;\mathbb{R})} = \left\langle \mathcal{K}^*\hat{f}, \mathcal{K}^*\hat{g}\right\rangle_{L^2(\mathbb{R};\mathbb{R})}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(u)\hat{g}(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_{s_2}^{t_2}\int_{s_1}^{t_1} f(u)g(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

□

Definition. Let $m \in \mathbb{N}$. An $\mathbb{R}^m$-valued stochastic process $Y = (Y_t, t \in \mathbb{R})$ is said to have

1. stationary increments if for every $n \in \mathbb{N}, s_i, t_i \in \mathbb{R}, s_i < t_i, i = 1, \dots, n$ and for all $h \in \mathbb{R}$ we have

$$\mathrm{Law}(Y_{t_1+h} - Y_{s_1+h}, \dots, Y_{t_n+h} - Y_{s_n+h}) = \mathrm{Law}(Y_{t_1} - Y_{s_1}, \dots, Y_{t_n} - Y_{s_n}).$$

2. reflexive increments if for every $n \in \mathbb{N}, s_i, t_i \in \mathbb{R}, s_i < t_i, i = 1, \dots, n$ we have

$$\mathrm{Law}(Y_{t_1} - Y_{s_1}, \dots, Y_{t_n} - Y_{s_n}) = \mathrm{Law}(Y_{-s_1} - Y_{-t_1}, \dots, Y_{-s_n} - Y_{-t_n}).$$

A weak convergence of probability measures will be denoted by $\xrightarrow{w^*}$.

Theorem 5 ([6], Proposition 2.1)**.** *Let $f \in L^{\frac{2}{1+2\alpha}}_{loc}([0,\infty);\mathbb{R})$ and let b have stationary and reflexive increments. Then $\forall t > 0$ we have*

$$\int_0^t f(t-s)\,\mathrm{d}b_s \overset{\mathcal{D}}{=} \int_0^t f(s)\,\mathrm{d}b_s \overset{\mathcal{D}}{=} \int_{-t}^0 f(-s)\,\mathrm{d}bs.$$

Proof. Take $f = \sum_{j=1}^n f_j \mathbb{1}_{[t_{j-1},t_j)} \in \mathcal{E}(\mathbb{R})$. Then $i_{0,t}(f) = \sum_{j=1}^n f_j \left(b_{t_j} - b_{t_{j-1}}\right)$. Denote $g(s) = f(t-s)$. Then

$$g = \sum_{j=1}^n f_j \mathbb{1}_{(t-t_j, t-t_{j-1}]}$$

and, since we identified function equal almost everywhere,

$$i_{0,t}(g) = \sum_{j=1}^n f_j \left(b_{t-t_{j-1}} - b_{t-t_j}\right).$$

From stationarity and reflexivity of increments we get that

$$\mathrm{Law}\left(b_{t-t_0} - b_{t-t_1}, \dots, b_{t-t_{n-1}} - b_{t-t_n}\right) = \mathrm{Law}\left(b_{-t_0} - b_{-t_1}, \dots, b_{-t_{n-1}} - b_{-t_n}\right)$$
$$= \mathrm{Law}\left(b_{t_1} - b_{t_0}, \dots, b_{t_n} - b_{t_{n-1}}\right)$$

and hence $\mathrm{Law}\,(i_{0,t}(f)) = \mathrm{Law}\,(i_{0,t}(g))$. Now, consider $f \in L^{\frac{2}{1+2\alpha}}_{\mathrm{loc}}([0,\infty);\mathbb{R})$ arbitrary. We fix $t > 0$ and we again denote $g(u) = f(t-u)$. We find a sequence of elementary functions $\{f_n\}$ such that $f_n \to f$ in $L^{\frac{2}{1+2\alpha}}(0,t;\mathbb{R})$. Let $g_n(u) = f_n(t-u)$. Clearly $g_n \to g$ in $L^{\frac{2}{1+2\alpha}}(0,t;\mathbb{R})$. Then

$$\begin{array}{ccc}
\mathrm{Law}\left(\int_0^t f_n(s)\,\mathrm{d}b_s\right) & \xrightarrow[n\to\infty]{w^*} & \mathrm{Law}\,(i_{0,t}(f)) \\
\| & & \\
\mathrm{Law}\left(\int_0^t g_n(s)\,\mathrm{d}b_s\right) & \xrightarrow[n\to\infty]{w^*} & \mathrm{Law}\,(i_{0,t}(g)) .
\end{array}$$

This proves the first equality. The other equalities are proved similarly. □

Definition. Let $B = (B_t, t \in \mathbb{R})$ be an $\mathbb{R}^m$ valued stochastic process, $B = (B^{(i)})_{i=1}^m$ and let $B^{(1)}, \dots, B^{(m)}$ be independent two-sided α-regular Volterra processes, all with the same kernel K. We then call B an m-dimensional α-regular Volterra process.

Define

$$\mathcal{D}(\mathbb{R};\mathbb{R}^{m\times m}) := \left\{ G = (G_{ij})_{i,j=1}^m : \mathbb{R} \to \mathbb{R}^{m\times m} | G_{ij} \in \mathcal{D}(\mathbb{R};\mathbb{R}), i,j = 1,\dots,m \right\}.$$

We again call $\mathcal{D}(\mathbb{R};\mathbb{R}^{m\times m})$ the set of admissible integrands. For $G \in \mathcal{D}(\mathbb{R};\mathbb{R}^{m\times m})$ and B m-dimensional α-regular Volterra process we define

$$i(G) := \int G\,\mathrm{d}B := \int G(r)\,\mathrm{d}B_r := \left(\sum_{j=1}^m \int G_{ij}(r)\,\mathrm{d}B_r^{(j)} \right)_{i=1}^m .$$

Sufficient condition for G to be integrable is $G \in L^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R}^{m\times m})$. Definite integral is defined naturally, i.e. for $G \in L^{\frac{2}{1+2\alpha}}_{loc}(\mathbb{R};\mathbb{R}^{m\times m})$ and $a < b$ we define

$$i_{a,b}(G) := \int_a^b G\,\mathrm{d}B := \int_a^b G(r)\,\mathrm{d}B_r := \left(\sum_{j=1}^m \int_a^b G_{ij}(r)\,\mathrm{d}B_r^{(j)} \right)_{i=1}^m .$$

In what follows we derive a formula for a covariance of two stochastic integrals driven by Volterra processes. Recall the notion of uncorrelated stochastic processes.

Definition. Centred stochastic processes $X = (X_t, t \in \mathbb{R})$ and $Y = (Y_t, t \in \mathbb{R})$ are uncorrelated if $\forall s,t \in \mathbb{R} : \mathbb{E}X_tY_s = 0$.

The following lemma is a slight modification of Proposition 1.18 from [6].

Lemma 3. *For $j = 1,2$ assume that $b^{(j)}$ is a one-dimensional α_j-regular Volterra process. Assume that $b^{(1)}$ and $b^{(2)}$ are uncorrelated. Let $a,b,c,d \in \mathbb{R}, a < b, c < d, f \in L^{\frac{2}{1+2\alpha_1}}(a,b;\mathbb{R}), g \in L^{\frac{2}{1+2\alpha_2}}(c,d;\mathbb{R})$. Then*

$$\mathbb{E}\left[\int_a^b f(u)\,\mathrm{d}b_u^{(1)} \int_c^d g(u)\,\mathrm{d}b_u^{(2)} \right] = 0.$$

Proof. For f, g simple the claim is obvious. Indeed, we have

$$\mathbb{E}\left[\int_a^b f(u)\,\mathrm{d}b_u^{(1)} \int_c^d g(u)\,\mathrm{d}b_u^{(2)}\right] = \mathbb{E}\left[\left(\sum_{i=1}^n f_i(b_{t_i}^{(1)} - b_{t_{i-1}}^{(1)})\right)\left(\sum_{j=1}^k g_j(b_{s_j}^{(2)} - b_{s_{j-1}}^{(2)})\right)\right]$$

$$= \sum_{i=1}^n \sum_{j=1}^k f_i g_j \left(\mathbb{E} b_{t_i}^{(1)} b_{s_j}^{(2)} - \mathbb{E} b_{t_{i-1}}^{(1)} b_{s_j}^{(2)} - \mathbb{E} b_{t_i}^{(1)} b_{s_{j-1}}^{(2)} + \mathbb{E} b_{t_{i-1}}^{(1)} b_{s_{j-1}}^{(2)}\right) = 0.$$

For f, g general, we find simple functions $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ such that $f_n \to f$ in $L^{\frac{2}{1+2\alpha_1}}(a, b; \mathbb{R})$ and $g_n \to g$ in $L^{\frac{2}{1+2\alpha_2}}(c, d; \mathbb{R})$. Denote by i_1 the integral from a to b with respect to $b^{(1)}$ and denote by i_2 the integral from c to d with respect to $b^{(2)}$. Recall equation (1.1) which in our notations states that for $j = 1, 2$ we have that

$$\|f\|_{\mathcal{D}_j} = \left\|\mathcal{K}_j^* f\right\|_{L^2(\mathbb{R};\mathbb{R})} \leq C_j \|f\|_{L^{\frac{2}{1+2\alpha_j}}(\mathbb{R};\mathbb{R})},$$

where $\|\cdot\|_{\mathcal{D}_j}$ and $\mathcal{K}_j^*$ are from the definition of i_j and C_j is some constant (depending only on α_j). Then we have

$$\begin{aligned}
&\mathbb{E}\left[\int_a^b f(u)\,\mathrm{d}b_u^{(1)} \int_c^d g(u)\,\mathrm{d}b_u^{(2)}\right] = \mathbb{E} i_1(f) i_2(g) = \langle i_1(f), i_2(g)\rangle_{L^2(\Omega;\mathbb{R})} \\
&= \langle i_1(f) - i_1(f_n) + i_1(f_n), i_2(g) - i_2(g_n) + i_2(g_n)\rangle_{L^2(\Omega;\mathbb{R})} \\
&\leq \|i_1(f) - i_1(f_n)\|_{L^2(\Omega;\mathbb{R})} \|i_2(g) - i_2(g_n)\|_{L^2(\Omega;\mathbb{R})} \\
&+ \|i_1(f_n)\|_{L^2(\Omega;\mathbb{R})} \|i_2(g) - i_2(g_n)\|_{L^2(\Omega;\mathbb{R})} + \|i_2(g_n)\|_{L^2(\Omega;\mathbb{R})} \|i_1(f) - i_1(f_n)\|_{L^2(\Omega;\mathbb{R})} \\
&+ \langle i_1(f_n), i_2(g_n)\rangle_{L^2(\Omega;\mathbb{R})} \\
&= \|f - f_n\|_{\mathcal{D}_1} \|g - g_n\|_{\mathcal{D}_2} + \|f_n\|_{\mathcal{D}_1} \|g - g_n\|_{\mathcal{D}_2} + \|g_n\|_{\mathcal{D}_2} \|f - f_n\|_{\mathcal{D}_1} + 0 \\
&\leq C_1 \|f - f_n\|_{L^{\frac{2}{1+2\alpha_1}}} C_2 \|g - g_n\|_{L^{\frac{2}{1+2\alpha_2}}} \\
&+ C_1 \|f_n\|_{L^{\frac{2}{1+2\alpha_1}}} C_2 \|g - g_n\|_{L^{\frac{2}{1+2\alpha_2}}} + C_2 \|g_n\|_{L^{\frac{2}{1+2\alpha_2}}} C_1 \|f - f_n\|_{L^{\frac{2}{1+2\alpha_1}}},
\end{aligned}$$

which goes to zero as $n \to +\infty$. □

Now it is easy to show the formula for a covariance of two stochastic integrals driven by Volterra processes.

Theorem 6. *Let $s_1, t_1, s_2, t_2 \in \mathbb{R}$, $s_1 < t_1$, $s_2 < t_2$, $G \in L^{\frac{2}{1+2\alpha}}(s_1, t_1; \mathbb{R}^{m\times m})$ and $H \in L^{\frac{2}{1+2\alpha}}(s_2, t_2; \mathbb{R}^{m\times m})$. Then*

$$\operatorname{Cov}\left(\int_{s_1}^{t_1} G\,\mathrm{d}B, \int_{s_2}^{t_2} H\,\mathrm{d}B\right) = \int_{s_2}^{t_2}\int_{s_1}^{t_1} G(u)H^*(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

Proof. Take $i, j \in \{1, \dots, m\}$. We have

$$\begin{aligned}
\left(\operatorname{Cov}\left(\int_{s_1}^{t_1} G\,\mathrm{d}B, \int_{s_2}^{t_2} H\,\mathrm{d}B\right)\right)_{ij} &= \operatorname{Cov}\left(\left(\int_{s_1}^{t_1} G\,\mathrm{d}B\right)_i, \left(\int_{s_2}^{t_2} H\,\mathrm{d}B\right)_j\right) \\
&= \operatorname{Cov}\left(\sum_{k=1}^m \int_{s_1}^{t_1} G_{ik} B^{(k)}, \sum_{l=1}^m \int_{s_2}^{t_2} H_{jl} B^{(l)}\right) \\
&= \sum_{k=1}^m \sum_{l=1}^m \operatorname{Cov}\left(\int_{s_1}^{t_1} G_{ik} B^{(k)}, \int_{s_2}^{t_2} H_{jl} B^{(l)}\right).
\end{aligned}$$

By Lemma 3 the cross-terms are all zero. Thus

$$= \sum_{k=1}^{m} \operatorname{Cov}\left(\int_{s_1}^{t_1} G_{ik}B^{(k)}, \int_{s_2}^{t_2} H_{jk}B^{(k)}\right)$$

$$= \sum_{k=1}^{m} \int_{s_2}^{t_2}\int_{s_1}^{t_1} G_{ik}(u)H_{jk}(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_{s_2}^{t_2}\int_{s_1}^{t_1} \sum_{k=1}^{m} G_{ik}(u)H_{jk}(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

$$= \int_{s_2}^{t_2}\int_{s_1}^{t_1} \sum_{k=1}^{m} G_{ik}(u)H^*_{kj}(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_{s_2}^{t_2}\int_{s_1}^{t_1} \left(G(u)H^*(v)\right)_{ij}\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

□

Theorem 7. *For $j = 1,2$ assume that $B^{(j)}$ is an m-dimensional α_j-regular Volterra process. Assume that $B^{(1)}$ and $B^{(2)}$ are uncorrelated. Let $s_1, t_1, s_2, t_2 \in \mathbb{R}$, $s_1 < t_1$, $s_2 < t_2$, $G \in L^{\frac{2}{1+2\alpha_1}}(s_1, t_1; \mathbb{R}^{m\times m})$ and $H \in L^{\frac{2}{1+2\alpha_2}}(s_2, t_2; \mathbb{R}^{m\times m})$. Then*

$$\operatorname{Cov}\left(\int_{s_1}^{t_1} G\,\mathrm{d}B^1, \int_{s_2}^{t_2} H\,\mathrm{d}B^2\right) = 0,$$

where 0 is a zero $m \times m$ matrix.

Proof. Take $i, j \in \{1, \ldots, m\}$. As before we have

$$\left(\operatorname{Cov}\left(\int_{s_1}^{t_1} G\,\mathrm{d}B^1, \int_{s_2}^{t_2} H\,\mathrm{d}B^2\right)\right)_{ij} = \sum_{k=1}^{m}\sum_{l=1}^{m} \operatorname{Cov}\left(\int_{s_1}^{t_1} G_{ik}B^{1^{(k)}}, \int_{s_2}^{t_2} H_{jl}B^{2^{(l)}}\right)$$

and by Lemma 3 the right hand size is zero. □

Lemma 4. *For $a, b \in \mathbb{R}, a < b, G \in L^{\frac{2}{1+2\alpha}}(a, b; \mathbb{R}^{m\times m})$ we have the following equality and inequality:*

$$\left\|\int_a^b G\,\mathrm{d}B\right\|^2_{L^2(\Omega;\mathbb{R}^m)} = \int_a^b\int_a^b \langle G(u), G(v)\rangle_{\mathbb{R}^{m\times m}}\,\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v$$
$$\leq C\,\|G(s)\|^2_{L^{\frac{2}{1+2\alpha}}(a,b;\mathbb{R}^{m\times m})},$$

where C is a positive constant depending only on α, a and b.

Proof.

$$\left\|\int_a^b G\,\mathrm{d}B\right\|^2_{L^2(\Omega;\mathbb{R}^m)} = \mathbb{E}\left\|\int_a^b G\,\mathrm{d}B\right\|^2_{\mathbb{R}^m} = \mathbb{E}\left[\sum_{i=1}^{m}\left(\int_a^b G\,\mathrm{d}B\right)_i^2\right]$$

$$= \sum_{i=1}^{m} \mathbb{E}\left(\sum_{j=1}^{m}\int_a^b G_{ij}\,\mathrm{d}B^{(j)}\right)^2 = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} \mathbb{E}\left(\int_a^b G_{ij}\,\mathrm{d}B^{(j)}\int_a^b G_{ik}\,\mathrm{d}B^{(k)}\right)$$

By Lemma 3 and Theorem 6 we get

$$
\begin{aligned}
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \mathbb{E}\left(\int_a^b G_{ij}\,\mathrm{d}B^{(j)} \int_a^b G_{ij}\,\mathrm{d}B^{(j)}\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \int_a^b\int_a^b G_{ij}(u)G_{ij}(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_a^b\int_a^b \sum_{i=1}^{m}\sum_{j=1}^{m} G_{ij}(u)G^*_{ji}(v)\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_a^b\int_a^b \sum_{i=1}^{m} (G(u)G^*(v))_{ii}\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int_a^b\int_a^b \mathrm{Tr}(G(u)G^*(v))\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_a^b\int_a^b \langle G(u),G(v)\rangle_{\mathbb{R}^{m\times m}}\,\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}
$$

This proves the equality. As for the inequality we can write

$$
\leq \int_a^b\int_a^b \|G(u)\|_{\mathbb{R}^{m\times m}}\,\|G(v)\|_{\mathbb{R}^{m\times m}}\,\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v
$$

and to finish the proof we can follow the exact same steps as in the proof of Theorem 2. □

The final theorem of this chapter is the stochastic version of the theorem by Fubini.

Theorem 8 (Stochastic Fubini's theorem)**.** *Let* (E,μ) *be a measurable space equipped with a finite measure* μ*. Let*

$$
G : (E\times[a,b], \mathcal{B}(E)\otimes\mathcal{B}([a,b])) \to (\mathbb{R}^{m\times m}, \mathcal{B}(\mathbb{R}^{m\times m})
$$

be measurable. Let B *be a two sided* α*-regular* $\mathbb{R}^m$*-valued Volterra process and assume that*

$$
\int_E \left(\int_a^b \|G(x,s)\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}}\,\mathrm{d}s\right)^{\alpha+\frac{1}{2}} \mathrm{d}\mu(x) < +\infty.
$$

Then

$$
\int_E\left(\int_a^b G(x,s)\,\mathrm{d}B_s\right)\mathrm{d}\mu(x) = \int_a^b\left(\int_E G(x,s)\,\mathrm{d}\mu(x)\right)\mathrm{d}B_s,\ \ \mathbb{P}-a.s.
$$

Proof. For the proof of a more general statement see [5], Proposition 2.21. □

2. Stochastic differential equations

In this chapter we will consider the equation

$$\begin{aligned} \mathrm{d}X_t &= AX_t \,\mathrm{d}t + \Phi \,\mathrm{d}B_t, \quad t \geq 0, \\ X_0 &= x_0, \end{aligned} \tag{SDE}$$

where $A, \Phi \in \mathbb{R}^{m\times m}$ are real matrices, x_0 is an $\mathbb{R}^m$-valued random variable (called the initial condition) and B is a two-sided m-dimensional α-regular Volterra process.

Firstly, we will discuss the solution to (SDE) and its properties. Secondly, we will present the result from [6] which gives us a condition under which there exists an initial condition x_∞ such that the solution with x_∞ as an initial condition is a strictly stationary process. Finally, following [19], we will discuss ergodicity.

2.1 Solution and its properties

Firstly, we will define what we mean by a solution to (SDE).

Definition. A continuous $\mathbb{R}^m$-valued stochastic process $X = (X_t, t \geq 0)$ satisfying

$$X_t = x_0 + \int_0^t AX_s \,\mathrm{d}s + \Phi B_t \tag{2.1}$$

for all $t \geq 0$ $\mathbb{P}$-almost surely is called a solution to (SDE).

To show the uniqueness of the solution we will need the following

Theorem 9 (Grönwall's lemma)**.** *Let $I \subset \mathbb{R}$ be an interval, $s \in I$ and $\varepsilon > 0$. Let $\varrho, \xi : I \to [0, +\infty)$ be a non-negative functions. Let ξ be continuous and $\varrho \in L^1_{loc}(I)$ be locally integrable. Assume that*

$$\xi(t) \leq \varepsilon + \left| \int_s^t \xi(r)\varrho(r) \,\mathrm{d}r \right|$$

holds for all $t \in I$. Then for all $t \in I$ it holds that

$$\xi(t) \leq \varepsilon \exp\left(\left| \int_s^t \varrho(r) \,\mathrm{d}r \right| \right).$$

Moreover, if

$$\xi(t) \leq \left| \int_s^t \xi(r)\varrho(r) \,\mathrm{d}r \right|$$

holds for all $t \in I$ we have that $\xi = 0$ on I.

Proof. See [24], Theorem D.1. □

Now we can give an expression for the solution and show its uniqueness.

Theorem 10. *Equation (SDE) admits a unique solution of the form*

$$X_t^{x_0} = e^{At}x_0 + \int_0^t e^{A(t-s)}\Phi \,\mathrm{d}B_s, \;\; t \geq 0.$$

Proof. Firstly we show the uniqueness. Let X^{x_0} and Y^{x_0} be two solutions to (SDE). Then from the definition of the solution we have that for all $t \geq 0$ it $\mathbb{P}$-a.s. holds that

$$X_t^{x_0} - Y_t^{x_0} = \int_0^t A(X_s^{x_0} - Y_s^{x_0})\,\mathrm{d}s.$$

Hence for almost all $\omega \in \Omega$ it holds that

$$\|X^{x_0}(t,\omega) - Y^{x_0}(t,\omega)\|_{\mathbb{R}^m} \leq \int_0^t \|A\|_{\mathbb{R}^{m\times m}} \|X^{x_0}(s,\omega) - Y^{x_0}(s,\omega)\|_{\mathbb{R}^m}\,\mathrm{d}s$$

and the uniqueness follows from Grönwall's lemma, since X^{x_0} and Y^{x_0} are continuous. Define

$$\hat{X}_t^{x_0} := e^{At}x_0 + \int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s, \;\; t \geq 0.$$

Since the map $s \mapsto e^{A(t-s)}\Phi$ is in $L_{\mathrm{loc}}^{\frac{2}{1+2\alpha}}(\mathbb{R};\mathbb{R}^{m\times m})$ the $\hat{X}^{x_0}$ is a well-defined process. We show that $\hat{X}^{x_0}$ is a solution to (SDE). We plug $\hat{X}^{x_0}$ into the right hand side of Equation (2.1). Take $t \geq 0$. We have

$$\begin{aligned} x_0 + \int_0^t A\hat{X}_s^{x_0}\,\mathrm{d}s + \Phi B_t &= x_0 + \int_0^t A\left(e^{As}x_0 + \int_0^s e^{A(s-r)}\Phi\,\mathrm{d}B_r\right)\mathrm{d}s + \Phi B_t \\ &= x_0 + \int_0^t Ae^{As}x_0\,\mathrm{d}s + \int_0^t\int_0^s Ae^{A(s-r)}\Phi\,\mathrm{d}B_r\,\mathrm{d}s + \Phi B_t \\ &= x_0 + (e^{At} - 1)x_0 + \int_0^t\int_0^s Ae^{A(s-r)}\Phi\,\mathrm{d}B_r\,\mathrm{d}s + \Phi B_t. \end{aligned}$$

By the stochastic Fubini's theorem, i.e. Theorem 8, we $\mathbb{P}$-a.s. have that

$$\begin{aligned} \int_0^t\int_0^s Ae^{A(s-r)}\Phi\,\mathrm{d}B_r\,\mathrm{d}s &= \int_0^t\int_r^t Ae^{A(s-r)}\Phi\,\mathrm{d}s\,\mathrm{d}B_r = \int_0^t\int_r^t \frac{\mathrm{d}}{\mathrm{d}s}\left(e^{A(s-r)}\Phi\right)\mathrm{d}s\,\mathrm{d}B_r \\ &= \int_0^t \left[e^{A(s-r)}\Phi\right]_{s=r}^t \mathrm{d}B_r = \int_0^t \left(e^{A(t-r)} - I\right)\Phi\,\mathrm{d}B_r = \int_0^t e^{A(t-r)}\Phi\,\mathrm{d}B_r - \Phi B_t. \end{aligned}$$

Continuing the calculations from before we find out that

$$\begin{aligned} &x_0 + (e^{At} - 1)x_0 + \int_0^t\int_0^s Ae^{A(s-r)}\Phi\,\mathrm{d}B_r\,\mathrm{d}s + \Phi B_t \\ &= e^{At}x_0 + \int_0^t e^{A(t-r)}\Phi\,\mathrm{d}B_r - \Phi B_t + \Phi B_t = \hat{X}_t^{x_0}. \end{aligned}$$

We have thus verified the equation Equation (2.1) with $X^{x_0} = \hat{X}^{x_0}$ and that completes the proof. $\square$

Remark. Assume that B is Gaussian. Assume that either $x_0 \in \mathbb{R}^m$ is deterministic or $x_0 : \Omega \to \mathbb{R}^m$ is Gaussian and independent of B. Then the solution X^{x_0} is also Gaussian.

The following lemma gives an alternative expression for the convolution integral $\int_0^t e^{A(t-s)}\Phi \, \mathrm{d}B_s$. The idea of the lemma and its proof come from [20], Proposition 3.1.

Lemma 5. *Let*

$$Z_t = \int_0^t e^{A(t-s)}\Phi \, \mathrm{d}B_s, \ \ t \geq 0.$$

Then

$$Z_t = \int_0^t Ae^{A(t-s)}\Phi B_s \, \mathrm{d}s + \Phi B_t$$

holds for all $t \geq 0$ $\mathbb{P}$*-a.s.*

Proof. From the previous theorem we see that Z is the solution to (SDE) with the initial condition $x_0 = 0$. Therefore, by the definition of a solution, we have that

$$Z_t = A \int_0^t Z_s \, \mathrm{d}s + \Phi B_t,$$

for all $t \geq 0$ $\mathbb{P}$-a.s. Let $L(t) = \int_0^t Z_s \, \mathrm{d}s$. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} L(t) &= Z_t = A \int_0^t Z_s \, \mathrm{d}s + \Phi B_t = AL(t) + \Phi B_t, \ \ t > 0, \\ L(0) &= 0. \end{aligned}$$

But this is an ordinary differential equation with the (only) solution

$$L(t) = \int_0^t e^{A(t-s)}\Phi B_s \, \mathrm{d}s, \ \ t \geq 0.$$

Therefore we have

$$Z_t = AL(t) + \Phi B_t = \int_0^t Ae^{A(t-s)}\Phi B_s \, \mathrm{d}s + \Phi B_t$$

for all $t \geq 0$ $\mathbb{P}$-a.s. $\square$

2.2 Stationary solution

In this chapter we present a sufficient conditions for the existence of such initial condition x_∞ that the solution X^{x_∞} is a strictly stationary process. We also compute the covariance function of the stationary solution.

Theorem 11 ([6], Proposition 3.6)**.** *Assume that B has stationary and reflexive increments. Assume that*

$$\int_0^\infty \left\| e^{Au}\Phi \right\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}} \mathrm{d}u < \infty. \tag{2.2}$$

Then there exists $x_\infty \in L^2(\Omega;\mathbb{R}^m)$ such that $(X_t^{x_\infty}, t \geq 0)$ is a strictly stationary process.

Proof. Let $x_n = \int_{-n}^{0} e^{-As}\Phi \, \mathrm{d}B_s$ for all $n \in \mathbb{N}$. We show that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^2(\Omega;\mathbb{R}^m)$. Take $n, k \in \mathbb{N}$, $n > k$. Using the Lemma 4 we find out that

$$\begin{aligned}
\|x_n - x_k\|^2_{L^2(\Omega;\mathbb{R}^m)} &= \left\| \int_{-n}^{0} e^{-As}\Phi \, \mathrm{d}B_s - \int_{-k}^{0} e^{-As}\Phi \, \mathrm{d}B_s \right\|^2_{L^2(\Omega;\mathbb{R}^m)} \\
&= \left\| \int_{-n}^{-k} e^{-As}\Phi \, \mathrm{d}B_s \right\|^2_{L^2(\Omega;\mathbb{R}^m)} \leq C \left(\int_{-n}^{-k} \left\| e^{-Au}\Phi \right\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}} \mathrm{d}u \right)^{1+2\alpha} \\
&= C \left(\int_{k}^{n} \left\| e^{Au}\Phi \right\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}} \mathrm{d}u \right)^{1+2\alpha} \leq C \left(\int_{k}^{\infty} \left\| e^{Au}\Phi \right\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}} \mathrm{d}u \right)^{1+2\alpha},
\end{aligned}$$

which under our assumption tends to zero as $n, k \to \infty, n > k$. Therefore there exists $x_\infty \in L^2(\Omega;\mathbb{R}^m)$ such that $x_n \to x_\infty$ in $L^2(\Omega;\mathbb{R}^m)$. Take $k \in \mathbb{N}$ and $h \in \mathbb{R}$. Denote by $\underset{n\to\infty}{\text{l.i.m.}}$ (limit in the mean) the limit in $L^2(\Omega;\mathbb{R}^m)$. We then have

$$\begin{aligned}
X^{x_\infty}_{t+h} &= \underset{n\to\infty}{\text{l.i.m.}} \left(e^{A(t+h)} x_n + \int_0^{t+h} e^{A(t+h-s)}\Phi \, \mathrm{d}B_s \right) \\
&= \underset{n\to\infty}{\text{l.i.m.}} \left(e^{A(t+h)} \int_{-n}^{0} e^{-As}\Phi \, \mathrm{d}B_s + \int_0^{t+h} e^{A(t+h-s)}\Phi \, \mathrm{d}B_s \right) \\
&= \underset{n\to\infty}{\text{l.i.m.}} \left(\int_{-n}^{t+h} e^{A(t+h-s)}\Phi \, \mathrm{d}B_s \right).
\end{aligned}$$

Denote by $B^h_s = B_{h-s}$, $s \in \mathbb{R}$ the process B shifted in time by h. Then

$$\begin{aligned}
&\mathrm{Law}\left(X^{x_\infty}_{t_1+h}, \dots, X^{x_\infty}_{t_k+h} \right) \\
&= w^*\text{-}\lim_{n\to\infty} \mathrm{Law} \left(\int_{-n}^{t_1+h} e^{A(t_1+h-s)}\Phi \, \mathrm{d}B_s, \dots, \int_{-n}^{t_k+h} e^{A(t_k+h-s)}\Phi \, \mathrm{d}B_s \right) \\
&= w^*\text{-}\lim_{n\to\infty} \mathrm{Law} \left(\int_{-n-h}^{t_1} e^{A(t_1-s)}\Phi \, \mathrm{d}B^h_s, \dots, \int_{-n-h}^{t_k} e^{A(t_k-s)}\Phi \, \mathrm{d}B^h_s \right).
\end{aligned}$$

Since for a fixed $h \in \mathbb{R}$ it holds that $n \to \infty \iff n+h \to \infty$ we have

$$= w^*\text{-}\lim_{n\to\infty} \operatorname{Law}\left(\int_{-n}^{t_1} e^{A(t_1-s)}\Phi \, \mathrm{d}B_s^h, \dots, \int_{-n}^{t_k} e^{A(t_k-s)}\Phi \, \mathrm{d}B_s^h \right)$$

and using a stationarity of increments we get

$$\begin{aligned} &= w^*\text{-}\lim_{n\to\infty} \operatorname{Law}\left(\int_{-n}^{t_1} e^{A(t_1-s)}\Phi \, \mathrm{d}B_s, \dots, \int_{-n}^{t_k} e^{A(t_k-s)}\Phi \, \mathrm{d}B_s \right) \\ &= \operatorname{Law}\left(X_{t_1}^{x_\infty}, \dots, X_{t_k}^{x_\infty} \right), \end{aligned}$$

which concludes the proof. □

Before we state some remarks concerning previous theorem we introduce the concept of exponential stability.

Definition. We say that $\left(e^{At}, t \geq 0\right)$ is exponentially stable if there exist constants $M, a > 0$ such that the estimate

$$\left\| e^{At} \right\|_{\mathbb{R}^{m\times m}} \leq M e^{-at}$$

holds for any $t \geq 0$.

We follow up with two remarks concerning Theorem 11.

Remark. Assume that $\left(e^{At}, t \geq 0\right)$ is exponentially stable. Then

$$\int_0^\infty \left\| e^{Au}\Phi \right\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}} \mathrm{d}u \leq \|\Phi\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}} M^{\frac{2}{1+2\alpha}} \int_0^\infty e^{-\frac{2a}{1+2\alpha}u} \, \mathrm{d}u < +\infty.$$

Therefore, exponential stability is a sufficient condition for (2.2) to hold.

Remark. Assume that B is Gaussian. Denote by $\operatorname*{l.i.m.}_{n\to\infty}$ (limit in the mean) the limit in $L^2(\Omega; \mathbb{R}^m)$. By construction from Theorem 11, the stationary solution X^{x_∞} can be written as

$$\begin{aligned} X_t^{x_\infty} &= e^{At}x_\infty + \int_0^t e^{A(t-s)}\Phi \, \mathrm{d}B_s = \operatorname*{l.i.m.}_{n\to\infty} \left(e^{At} \int_{-n}^0 e^{-As}\Phi \, \mathrm{d}B_s + \int_0^t e^{A(t-s)}\Phi \, \mathrm{d}B_s \right) \\ &= \operatorname*{l.i.m.}_{n\to\infty} \int_{-n}^t e^{A(t-s)} \, \mathrm{d}B_s \end{aligned}$$

for any $t \geq 0$ $\mathbb{P}$-a.s. Since $\int_{-n}^{t} e^{A(t-s)} \, \mathrm{d}B_s$ is Gaussian for all $n \in \mathbb{N}$ it follows that the stationary solution X^{x_∞} is Gaussian .

The following theorem gives us the formula for covariance of the stationary solution. The idea for the theorem comes from [16], Lemma 5.1.

Theorem 12. *Under the assumptions of the previous theorem we have that*

$$\mathrm{Cov}\,(X_t^{x_\infty}, X_0^{x_\infty}) = e^{At}\mathrm{Cov}\,(x_\infty, x_\infty) + \int_{-\infty}^{0}\int_{0}^{t} e^{A(t-u)}\Phi\Phi^* e^{-A^* v}\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

holds for any $t \geq 0$.

Proof. Denote by $\underset{n\to\infty}{\mathrm{l.i.m.}}$ (limit in the mean) the limit in $L^2(\Omega;\mathbb{R}^m)$. We have that

$$\begin{aligned}
\mathrm{Cov}\,(X_t^{x_\infty}, X_0^{x_\infty}) &= \mathrm{Cov}\left(e^{At}x_\infty + \int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s, \int_{-\infty}^0 e^{-As}\Phi\,\mathrm{d}B_s\right)\\
&= \mathrm{Cov}\left(e^{At}x_\infty, x_\infty\right) + \mathrm{Cov}\left(\int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s,\ \underset{n\to\infty}{\mathrm{l.i.m.}}\int_{-n}^0 e^{-As}\Phi\,\mathrm{d}B_s\right)\\
&= e^{At}\mathrm{Cov}\,(x_\infty, x_\infty) + \lim_{n\to\infty}\mathrm{Cov}\left(\int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s, \int_{-n}^0 e^{-As}\Phi\,\mathrm{d}B_s\right).
\end{aligned}$$

Using Theorem 6 we can continue with

$$\begin{aligned}
&= e^{At}\mathrm{Cov}\,(x_\infty, x_\infty) + \lim_{n\to\infty}\int_{-n}^{0}\int_{0}^{t} e^{A(t-u)}\Phi\Phi^* e^{-A^* v}\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v\\
&= e^{At}\mathrm{Cov}\,(x_\infty, x_\infty) + \int_{-\infty}^{0}\int_{0}^{t} e^{A(t-u)}\Phi\Phi^* e^{-A^* v}\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v,
\end{aligned}$$

which completes the proof. □

2.3 Ergodicity

In this chapter we will deal with the notion of ergodicity. The results of this part of the Thesis are crucial in deriving strongly consistent parameter estimators which will be the content of the last chapter. In this chapter we will show that under certain conditions the strictly stationary solution, constructed in Theorem 11, is ergodic. As a corollary we will obtain obtain similar result for any initial condition. This strongly follows [19], where the fractional Brownian motion as a driving process is considered.

Firstly, recall the famous Birkhoff's theorem and the definition of ergodic process.

Theorem 13 (Birkhoff's theorem)**.** *Let* $(X_t, t \geq 0)$ *be an* $\mathbb{R}^m$*-valued strictly stationary stochastic process on* $(\Omega, \mathcal{F}, \mathbb{P})$*. Then for every measurable function* $f : \mathbb{R}^m \to \mathbb{R}$ *such that* $\mathbb{E}|f(X_0)| < +\infty$ *there exists a measurable function* $\xi : \Omega \to \mathbb{R}$ *such that*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X_t)\,\mathrm{d}t = \xi,\quad \mathbb{P}\text{-}a.s.$$

Definition. An $\mathbb{R}^m$-valued strictly stationary stochastic process $(X_t, t \geq 0)$ is said to be ergodic, if ξ from the previous theorem satisfies

$$\xi = \mathbb{E}\left[f(X_0)\right], \;\; \mathbb{P}\text{-}a.s.$$

To show that a given stochastic process is ergodic is in general not easy, but for the Gaussian processes we have the following sufficient condition.

Theorem 14. *Let $Y = (Y_t, t \geq 0)$ be an $\mathbb{R}^m$-valued strictly stationary Gaussian stochastic process. Let $R(t) := \mathrm{Cov}(Y_t, Y_0)$. If $\|R(t)\|_{\mathbb{R}^{m\times m}} \to 0$ as $t \to +\infty$, then Y is ergodic.*

Proof. See e.g. [23]. □

The following theorem gives us ergodicity for a strictly stationary solution.

Theorem 15. *Assume that B is a Gaussian process with stationary and reflexive increments. Assume that $(e^{At}, t \geq 0)$ is exponentially stable. Then the strictly stationary solution $(X_t^{x_\infty}, t \geq 0)$ is ergodic.*

Proof. Our assumptions imply that Theorem 11 and Theorem 12 hold. Strictly stationary solution X^{x_∞} is therefore well-defined (by Theorem 11) and we have a formula for its covariance (by Theorem 12). Because B is Gaussian we have that X^{x_∞} is Gaussian and we can use Theorem 14. It therefore suffices to show that

$$\lim_{t\to+\infty} \|\mathrm{Cov}\left(X_t^{x_\infty}, X_0^{x_\infty}\right)\| = 0.$$

We use the formula for the covariance matrix from Theorem 12. For $t \geq 0$ we have

$$\begin{aligned}
&\|\mathrm{Cov}\left(X_t^{x_\infty}, X_0^{x_\infty}\right)\| = \\
&= \left\| e^{At}\mathrm{Cov}\left(x_\infty, x_\infty\right) + \int_{-\infty}^{0}\int_{0}^{t} e^{A(t-u)}\Phi\Phi^* e^{-A^*v}\varphi(u,v)\,\mathrm{d}u\,\mathrm{d}v \right\| \\
&\leq \left\|e^{At}\right\| \|\mathrm{Cov}\left(x_\infty, x_\infty\right)\| + \int_{-\infty}^{0}\int_{0}^{t} \left\| e^{A(t-u)}\Phi\Phi^* e^{-A^*v}\varphi(u,v)\right\| \mathrm{d}u\,\mathrm{d}v \\
&\leq Me^{-at}\|\mathrm{Cov}\left(x_\infty, x_\infty\right)\| + \int_{-\infty}^{0}\int_{0}^{t} |\varphi(u,v)| \left\|e^{A(t-u)}\right\| \|\Phi\| \|\Phi^*\| \left\|e^{-A^*v}\right\| \mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

The first term on the right hand side converges to 0 as $t \to +\infty$. Recall that for any $m \times m$ matrix B we have $\|B^*\| = \|B\|$ and also $\left(e^B\right)^* = e^{B^*}$. Thus

$$\left\|e^{-A^*v}\right\| = \left\|e^{(-vA)^*}\right\| = \left\|\left(e^{-vA}\right)^*\right\| = \left\|e^{-vA}\right\| = \left\|e^{-Av}\right\|.$$

Using again the inequality from Lemma 1 we can estimate the second term further. We get

$$\begin{aligned}
&\int_{-\infty}^{0}\int_{0}^{t} |\varphi(u,v)| \left\|e^{A(t-u)}\right\| \|\Phi\| \|\Phi^*\| \left\|e^{-A^*v}\right\| \mathrm{d}u\,\mathrm{d}v \\
&\le c_\alpha \|\Phi\|^2 \int_{-\infty}^{0}\int_{0}^{t} |u-v|^{2\alpha-1} \left\|e^{A(t-u)}\right\| \left\|e^{-Av}\right\| \mathrm{d}u\,\mathrm{d}v \\
&\le c_\alpha \|\Phi\|^2 M^2 \int_{-\infty}^{0}\int_{0}^{t} (u-v)^{2\alpha-1} e^{-a(t-u)} e^{-a(-v)} \,\mathrm{d}u\,\mathrm{d}v \\
&= c_\alpha \|\Phi\|^2 M^2 e^{-at} \int_{-\infty}^{0}\int_{0}^{t} (u-v)^{2\alpha-1} e^{a(u+v)} \,\mathrm{d}u\,\mathrm{d}v \\
&= c_\alpha \|\Phi\|^2 M^2 e^{-at} \int_{0}^{\infty}\int_{0}^{t} (u+v)^{2\alpha-1} e^{a(u-v)} \,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Next we use the Fubini's theorem and the fact that for $u, v > 0$ we have $(u+v)^{2\alpha-1} \le u^{2\alpha-1}$. We get

$$\begin{aligned}
&e^{-at}\int_{0}^{\infty}\int_{0}^{t} (u+v)^{2\alpha-1} e^{a(u-v)} \,\mathrm{d}u\,\mathrm{d}v = e^{-at}\int_{0}^{t}\int_{0}^{\infty} (u+v)^{2\alpha-1} e^{a(u-v)} \,\mathrm{d}v\,\mathrm{d}u \\
&= e^{-at}\int_{0}^{t} e^{au}\int_{0}^{\infty} (u+v)^{2\alpha-1} e^{-av} \,\mathrm{d}v\,\mathrm{d}u \le e^{-at}\int_{0}^{t} e^{au}u^{2\alpha-1}\int_{0}^{\infty} e^{-av} \,\mathrm{d}v\,\mathrm{d}u \\
&= \frac{1}{a}e^{-at}\int_{0}^{t} e^{au}u^{2\alpha-1} \,\mathrm{d}u.
\end{aligned}$$

It remains to show that

$$e^{-at}\int_{0}^{t} e^{au}u^{2\alpha-1} \,\mathrm{d}u \to 0, \;\; t \to +\infty.$$

Clearly

$$e^{-at}\int_{0}^{t} e^{au}u^{2\alpha-1} \,\mathrm{d}u = e^{-at}\int_{0}^{1} e^{au}u^{2\alpha-1} \,\mathrm{d}u + e^{-at}\int_{1}^{t} e^{au}u^{2\alpha-1} \,\mathrm{d}u$$

and the first term on the right hand side converges to 0 as $t \to +\infty$. As for the second term we can use the L'Hôpital's rule to get

$$\lim_{t\to+\infty} \frac{\int_{1}^{t} e^{au}u^{2\alpha-1} \,\mathrm{d}u}{e^{at}} = \lim_{t\to+\infty} \frac{e^{at}t^{2\alpha-1}}{ae^{at}} = \lim_{t\to+\infty} \frac{t^{2\alpha-1}}{a} = 0$$

which completes the proof. □

Before we proceed to the main theorem of this chapter we will need the following estimate.

Lemma 6. *For almost all $\omega \in \Omega$ and all $\varepsilon > 0$ there exists a constant $K(\varepsilon, \omega)$ such that*

$$\|B(t,\omega)\|_{\mathbb{R}^m} \leq \varepsilon t^2 + K(\varepsilon,\omega)$$

holds for all $t \geq 0$.

Proof. The proof of a spacial case of B being the fractional Brownian motion can be found in [20], Lemma 2.6. The proof of the general case can be done by mimicking the cited proof. □

We set $\mu_\infty =$ Law(x_∞). We can thus write

$$\mathbb{E}\left[f(X_0^{x_\infty})\right] = \mathbb{E}[f(x_\infty)] = \int\limits_{\mathbb{R}^m} f(y)\,\mathrm{d}\mu_\infty(y).$$

Under the assumptions of Theorem 15 we have an ergodic stationary solution. That means that

$$\lim_{T\to+\infty} \frac{1}{T}\int\limits_0^T f(X_t^{x_\infty})\,\mathrm{d}t = \int\limits_{\mathbb{R}^m} f(y)\,\mathrm{d}\mu_\infty(y), \ \ \mathbb{P}-a.s.,$$

holds for every measurable function $f : \mathbb{R}^m \to \mathbb{R}$ such that $\mathbb{E}|f(x_\infty)| < +\infty$. We will now show the same type of convergence for any initial condition x_0 under a suitable Lipschitz-like condition on f.

Theorem 16. *Let the assumptions of Theorem 15 hold. Let $f : \mathbb{R}^m \to \mathbb{R}$ be masurable and such that $\mathbb{E}|f(x_\infty)| < +\infty$. Moreover, let f satisfy the following condition: there exist constants $L > 0$ and $d \in \mathbb{Z}, d \geq 0$ such that*

$$|f(x) - f(y)| \leq L\,\|x-y\|_{\mathbb{R}^m}\left(1 + \|x\|_{\mathbb{R}^m}^d + \|y\|_{\mathbb{R}^m}^d\right), \ \ x,y \in \mathbb{R}^m.$$

Then

$$\lim_{T\to+\infty} \frac{1}{T}\int\limits_0^T f(X_t^{x_0})\,\mathrm{d}t = \int\limits_{\mathbb{R}^m} f(y)\,\mathrm{d}\mu_\infty(y), \ \ \mathbb{P}-a.s.$$

holds for any initial condition x_0.

Proof. Throughout this proof a norm without a subscript is understood as a norm in $\mathbb{R}^m$, i.e. $\|\cdot\| = \|\cdot\|_{\mathbb{R}^m}$. The desired convergence is equivalent to

$$\left|\frac{1}{T}\int\limits_0^T f(X_t^{x_0})\,\mathrm{d}t - \int\limits_{\mathbb{R}^m} f(y)\,\mathrm{d}\mu_\infty(y)\right| \to 0$$

as $T \to +\infty$ $\mathbb{P}$-a.s. Clearly

$$\begin{aligned}&\left|\frac{1}{T}\int\limits_0^T f(X_t^{x_0})\,\mathrm{d}t - \int\limits_{\mathbb{R}^m} f(y)\,\mathrm{d}\mu_\infty(y)\right| \\ &\leq \left|\frac{1}{T}\int\limits_0^T f(X_t^{x_0})\,\mathrm{d}t - \frac{1}{T}\int\limits_0^T f(X_t^{x_\infty})\,\mathrm{d}t\right| + \left|\frac{1}{T}\int\limits_0^T f(X_t^{x_\infty})\,\mathrm{d}t - \int\limits_{\mathbb{R}^m} f(y)\,\mathrm{d}\mu_\infty(y)\right|\end{aligned}$$

and the second term on the right hand side converges to 0 as $T \to +\infty$ $\mathbb{P}$-a.s thanks to Birkhoff's theorem and Theorem 15. As for the first term we have

$$\left|\frac{1}{T}\int_0^T f(X_t^{x_0})\,\mathrm{d}t - \frac{1}{T}\int_0^T f(X_t^{x_\infty})\,\mathrm{d}t\right| \le \frac{1}{T}\int_0^T |f(X_t^{x_0}) - f(X_t^{x_\infty})|\,\mathrm{d}t$$

$$\le \frac{L}{T}\int_0^T \|X_t^{x_0} - X_t^{x_\infty}\| \left(1 + \|X_t^{x_0}\|^d + \|X_t^{x_\infty}\|^d\right) \mathrm{d}t$$

$$\le \frac{L}{T}\int_0^T \left\|e^{At}\right\|_{\mathbb{R}^{m\times m}} \|x_0 - x_\infty\| \left(1 + \|X_t^{x_0}\|^d + \|X_t^{x_\infty}\|^d\right) \mathrm{d}t$$

$$\le \frac{L}{T}\|x_0 - x_\infty\| M \int_0^T e^{-at}\,\mathrm{d}t + \frac{L}{T}\|x_0 - x_\infty\| M \int_0^T e^{-at}\left(\|X_t^{x_0}\|^d + \|X_t^{x_\infty}\|^d\right) \mathrm{d}t$$

and the first term on the right hand side converges to 0 as $T \to +\infty$ $\mathbb{P}$-a.s. It remains to show that

$$\frac{1}{T}\int_0^T e^{-at}\left(\|X_t^{x_0}\|^d + \|X_t^{x_\infty}\|^d\right) \mathrm{d}t \to 0$$

as $T \to +\infty$ $\mathbb{P}$-a.s. We have

$$\begin{aligned}
&\|X_t^{x_0}\|^d + \|X_t^{x_\infty}\|^d \\
&\le \left(\left\|e^{At}x_0\right\| + \left\|\int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s\right\|\right)^d + \left(\left\|e^{At}x_\infty\right\| + \left\|\int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s\right\|\right)^d \\
&= \sum_{k=0}^d \binom{d}{k}\left(\left\|e^{At}x_0\right\|^{d-k} + \left\|e^{At}x_\infty\right\|^{d-k}\right)\left\|\int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s\right\|^k \\
&\le \sum_{k=0}^d \binom{d}{k} M^{d-k}e^{-a(d-k)t}\left(\|x_0\|^{d-k} + \|x_\infty\|^{d-k}\right)\left\|\int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s\right\|^k.
\end{aligned}$$

Furthermore, using Lemma 5 we have

$$\left\|\int_0^t e^{A(t-s)}\Phi\,\mathrm{d}B_s\right\|^k = \left\|\int_0^t Ae^{A(t-s)}\Phi B_s\,\mathrm{d}s + \Phi B_t\right\|^k \le \left(\int_0^t \left\|Ae^{A(t-s)}\Phi B_s\right\|\mathrm{d}s + \|\Phi B_t\|\right)^k$$

$$= \sum_{l=0}^k \binom{k}{l}\left(\int_0^t \left\|Ae^{A(t-s)}\Phi B_s\right\|\mathrm{d}s\right)^l \|\Phi B_t\|^{k-l}$$

$$= \sum_{l=1}^k \binom{k}{l}\left(\int_0^t \left\|Ae^{A(t-s)}\Phi B_s\right\|\mathrm{d}s\right)^l \|\Phi B_t\|^{k-l} + \|\Phi B_t\|^k,$$

$\mathbb{P}$-a.s. By Hölder inequality (with $p = l$) the right hand side may be estimated by

$$\leq \sum_{l=1}^{k}\binom{k}{l}t^{l-1}\int_0^t \left\|Ae^{A(t-s)}\Phi B_s\right\|^l \mathrm{d}s \left\|\Phi B_t\right\|^{k-l} + \left\|\Phi B_t\right\|^k$$

$$\leq \sum_{l=1}^{k}\binom{k}{l} \|A\|^l_{\mathbb{R}^{m\times m}} \|\Phi\|^k_{\mathbb{R}^{m\times m}} \|B_t\|^{k-l} t^{l-1}\int_0^t \left\|e^{A(t-s)}\right\|^l \|B_s\|^l \mathrm{d}s + \|\Phi B_t\|^k$$

$$\leq \sum_{l=1}^{k}\binom{k}{l} \|A\|^l_{\mathbb{R}^{m\times m}} \|\Phi\|^k_{\mathbb{R}^{m\times m}} \|B_t\|^{k-l} t^{l-1} M^l\int_0^t e^{-al(t-s)} \|B_s\|^l \mathrm{d}s + \|\Phi B_t\|^k .$$

Setting

$$C_{k,l} := M^{d-k}\left(\|x_0\|^{d-k} + \|x_\infty\|^{d-k}\right) \|A\|^l_{\mathbb{R}^{m\times m}} \|\Phi\|^k_{\mathbb{R}^{m\times m}} M^l$$

we have that

$$\frac{1}{T}\int_0^T e^{-at}\left(\|X_t^{x_0}\|^d + \|X_t^{x_\infty}\|^d\right)\mathrm{d}t$$

$$\leq \frac{1}{T}\int_0^T e^{-at}\sum_{k=0}^{d}\sum_{l=1}^{k}\binom{d}{k}\binom{k}{l}C_{k,l}e^{-a(d-k)t}t^{l-1}\|B_t\|^{k-l}\int_0^t e^{-al(t-s)}\|B_s\|^l \mathrm{d}s\,\mathrm{d}t$$

$$+\frac{1}{T}\int_0^T e^{-at}\sum_{k=0}^{d}\binom{d}{k}C_{k,0}e^{-a(d-k)t}\|B_t\|^k \mathrm{d}t$$

$$= \sum_{k=0}^{d}\sum_{l=1}^{k}\binom{d}{k}\binom{k}{l}C_{k,l}\frac{1}{T}\int_0^T e^{-a(1+d-k)t}t^{l-1}\|B_t\|^{k-l}\int_0^t e^{-al(t-s)}\|B_s\|^l \mathrm{d}s\,\mathrm{d}t$$

$$+\sum_{k=0}^{d}\binom{d}{k}C_{k,0}\frac{1}{T}\int_0^T e^{-a(1+d-k)t}\|B_t\|^k \mathrm{d}t.$$

What remains to show is that

$$\frac{1}{T}\int_0^T e^{-a(1+d-k)t}t^{l-1}\|B_t\|^{k-l}\int_0^t e^{-al(t-s)}\|B_s\|^l \mathrm{d}s\,\mathrm{d}t \to 0$$

as $T \to +\infty$ $\mathbb{P}$-a.s. and that

$$\frac{1}{T}\int_0^T e^{-a(1+d-k)t}\|B_t\|^k \mathrm{d}t \to 0$$

as $T \to +\infty$ $\mathbb{P}$-a.s. We will show only the first convergence as the second one is analogous and simpler. To show this we are going to work pathwise and utilize the previous Lemma 6. Using this lemma we take $\omega \in \Omega$ such that

$$\|B(t,\omega)\| \leq t^2 + K(1,\omega).$$

Then

$$\int_0^t e^{-al(t-s)}\|B(s,\omega)\|^l \mathrm{d}s \leq \int_0^t e^{-al(t-s)}(s^2+K(1,\omega))^l \mathrm{d}s$$

$$= \int_0^t e^{-al(t-s)}\sum_{i=0}^{l}\binom{l}{i}s^{2i}K(1,\omega)^{l-i}\mathrm{d}s = \sum_{i=0}^{l}\binom{l}{i}K(1,\omega)^{l-i}\int_0^t e^{-al(t-s)}s^{2i}\mathrm{d}s$$

and similarly

$$\|B(t,\omega)\|^{k-l} \leq \left(t^2 + K(1,\omega)\right)^{k-l} = \sum_{j=0}^{k-l} \binom{k-l}{j} t^{2j} K(1,\omega)^{k-l-j}.$$

Therefore

$$\frac{1}{T}\int_0^T e^{-a(1+d-k)t} t^{l-1} \|B(t,\omega)\|^{k-l} \int_0^t e^{-al(t-s)} \|B(s,\omega)\|^{l} \,\mathrm{d}s\,\mathrm{d}t$$

$$\leq \sum_{i=0}^{l}\sum_{j=0}^{k-l}\binom{l}{i}K(1,\omega)^{l-i}\binom{k-l}{j}K(1,\omega)^{k-l-j}\frac{1}{T}\int_0^T e^{-a(1+d-k)t}t^{l-1}t^{2j}\int_0^t e^{-al(t-s)}s^{2i}\mathrm{d}s\mathrm{d}t.$$

Lastly, we will use twice the L'Hôpital's rule to calculate the limit:

$$\begin{aligned}
&\lim_{T\to+\infty}\frac{\int_0^T e^{-a(1+d-k)t}t^{l-1+2j}\int_0^t e^{-al(t-s)}s^{2i}\,\mathrm{d}s\,\mathrm{d}t}{T} \\
&= \lim_{T\to+\infty}\frac{e^{-a(1+d-k)T}T^{l-1+2j}\int_0^T e^{-al(T-s)}s^{2i}\,\mathrm{d}s}{1} \\
&= \lim_{T\to+\infty}\frac{\int_0^T e^{als}s^{2i}\,\mathrm{d}s}{e^{a(1+d-k+l)T}T^{1-l-2j}} \\
&= \lim_{T\to+\infty}\frac{e^{alT}T^{2i}}{a(1+d-k+l)e^{a(1+d-k+l)T}T^{1-l-2j}+(1-l-2j)T^{-l-2j}e^{a(1+d-k+l)T}} \\
&= \lim_{T\to+\infty}e^{-a(1+d-k)T}\frac{T^{2i}}{a(1+d-k+l)T^{1-l-2j}+(1-l-2j)T^{-l-2j}} \\
&= 0,
\end{aligned}$$

because $0 \leq k \leq d$. In the view of Lemma 6 we have shown the desired convergence for almost all trajectories and thus finished the proof. □

3. SDEs with a mixed noise

Results in this chapter are a straightforward generalization of results from the previous chapter. The reason we do not omit the previous chapter is for clarity reasons. Throughout this chapter we will consider the equation

$$\begin{aligned} \mathrm{d}X_t &= AX_t\,\mathrm{d}t + \sum_{i=1}^{p} \Phi^i\,\mathrm{d}B_t^i, \quad t \geq 0, \\ X_0 &= x_0, \end{aligned} \tag{3.1}$$

where $p \in \mathbb{N}$, $A \in \mathbb{R}^{m\times m}$ is a real matrix, x_0 is an $\mathbb{R}^m$-valued random variable and for each $i \in \{1, \dots, p\}$ we have that $\Phi^i \in \mathbb{R}^{m\times m}$ is a real matrix and B^i is a two-sided m-dimensional α_i-regular Volterra process. Furthermore, we assume that $B^1, \dots, B^p$ are independent.

3.1 Solution, its properties and the strictly stationary solution

Firstly we define what we mean by a solution to equation (3.1).

Definition. A continuous $\mathbb{R}^m$-valued stochastic process $X = (X_t, t \geq 0)$ satisfying

$$X_t = x_0 + \int_0^t AX_s\,\mathrm{d}s + \sum_{i=1}^{p} \Phi^i B_t^i$$

for all $t \geq 0$ $\mathbb{P}$-almost surely is called a solution to equation (3.1).

We have the following formula for a solution.

Theorem 17. *Equation (3.1) admits a unique solution of the form*

$$X_t^{x_0} = e^{At}x_0 + \sum_{i=1}^{p} \int_0^t e^{A(t-s)}\Phi^i\,\mathrm{d}B_s^i, \quad t \geq 0.$$

Proof. This can be shown in the exactly the same way as in the proof of Theorem 10. □

Remark. Assume that for each $i \in \{1, \dots, p\}$ we have that B^i is Gaussian. Assume that either $x_0 \in \mathbb{R}^m$ is deterministic or $x_0 : \Omega \to \mathbb{R}^m$ is Gaussian and independent of all B^i. Then the solution X^{x_0} is also Gaussian.

The following theorem gives us a strictly stationary solution.

Theorem 18. *Assume that for each $i \in \{1, \dots, p\}$ we have that B^i has stationary and reflexive increments. Assume that for all $i \in \{1, \dots, p\}$ we moreover have that*

$$\int_0^\infty \left\| e^{Au}\Phi^i \right\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}} \mathrm{d}u < \infty. \tag{3.2}$$

Then there exists $x_\infty \in L^2(\Omega; \mathbb{R}^m)$ such that $(X_t^{x_\infty}, t \geq 0)$ is a strictly stationary process.

Proof. Following the lines of the proof of Theorem 11 we let

$$x_n = \sum_{i=1}^{p} \int_{-n}^{0} e^{-As} \Phi^i \, \mathrm{d}B_s^i$$

for all $n \in \mathbb{N}$. We show that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^2(\Omega; \mathbb{R}^m)$. Take $n, k \in \mathbb{N}$, $n > k$. We again find out that

$$\begin{aligned} \|x_n - x_k\|_{L^2(\Omega;\mathbb{R}^m)}^2 &= \left\| \sum_{i=1}^{p} \int_{-n}^{-k} e^{-As} \Phi^i \, \mathrm{d}B_s^i \right\|_{L^2(\Omega;\mathbb{R}^m)}^2 \leq \sum_{i=1}^{p} \left\| \int_{-n}^{-k} e^{-As} \Phi^i \, \mathrm{d}B_s^i \right\|_{L^2(\Omega;\mathbb{R}^m)}^2 \\ &\leq \sum_{i=1}^{p} C_i \left(\int_{k}^{\infty} \left\| e^{Au} \Phi^i \right\|_{\mathbb{R}^{m\times m}}^{\frac{2}{1+2\alpha}} \mathrm{d}u \right)^{1+2\alpha}, \end{aligned}$$

which under our assumption tends to zero as $n, k \to \infty, n > k$. The rest of the proof can be carried in the same way as the proof of Theorem 11. □

The same remarks as in the previous chapter apply.

Remark. Firstly, note that the exponential stability of $\left(e^{At}, t \geq 0\right)$ is again sufficient for (3.2) to hold. Secondly, assume that for every $i \in \{1, \ldots, p\}$ we have that B^i is Gaussian. By construction from Theorem 18 we have that

$$X_t^{x_\infty} = \underset{n\to\infty}{\text{l.i.m.}} \left(\sum_{i=1}^{p} \int_{-n}^{t} e^{A(t-s)} \Phi^i \, \mathrm{d}B_s^i \right),$$

holds for any $t \geq 0$ and it follows that X^{x_∞} is Gaussian.

We again have a formula for the covariance of a strictly stationary solution:

Theorem 19. *Under the assumptions of the previous theorem we have that*

$$\mathrm{Cov}\left(X_t^{x_\infty}, X_0^{x_\infty}\right) = e^{At} \mathrm{Cov}\left(x_\infty, x_\infty\right) + \sum_{i=1}^{p} \int_{-\infty}^{0} \int_{0}^{t} e^{A(t-u)} \Phi^i \left(\Phi^i\right)^* e^{-A^* v} \varphi_i(u, v) \, \mathrm{d}u \, \mathrm{d}v.$$

Proof. Denote by $\underset{n\to\infty}{\text{l.i.m.}}$ (limit in the mean) the limit in $L^2(\Omega; \mathbb{R}^m)$. We have that

$$\begin{aligned} \mathrm{Cov}\left(X_t^{x_\infty}, X_0^{x_\infty}\right) &= \mathrm{Cov}\left(e^{At} x_\infty + \sum_{i=1}^{p} \int_{0}^{t} e^{A(t-s)} \Phi^i \, \mathrm{d}B_s^i, \sum_{j=1}^{p} \int_{-\infty}^{0} e^{-As} \Phi^j \, \mathrm{d}B_s^j \right) \\ &= \mathrm{Cov}\left(e^{At} x_\infty, x_\infty\right) + \mathrm{Cov}\left(\sum_{i=1}^{p} \int_{0}^{t} e^{A(t-s)} \Phi^i \, \mathrm{d}B_s^i, \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=1}^{p} \int_{-n}^{0} e^{-As} \Phi^j \, \mathrm{d}B_s^j \right) \\ &= e^{At} \mathrm{Cov}\left(x_\infty, x_\infty\right) + \lim_{n\to\infty} \mathrm{Cov}\left(\sum_{i=1}^{p} \int_{0}^{t} e^{A(t-s)} \Phi^i \, \mathrm{d}B_s^i, \sum_{j=1}^{p} \int_{-n}^{0} e^{-As} \Phi^j \, \mathrm{d}B_s^j \right) \\ &= e^{At} \mathrm{Cov}\left(x_\infty, x_\infty\right) + \lim_{n\to\infty} \sum_{i=1}^{p} \sum_{j=1}^{p} \mathrm{Cov}\left(\int_{0}^{t} e^{A(t-s)} \Phi^i \, \mathrm{d}B_s^i, \int_{-n}^{0} e^{-As} \Phi^j \, \mathrm{d}B_s^j \right). \end{aligned}$$

Theorem 7 implies that

$$= e^{At}\mathrm{Cov}\left(x_\infty, x_\infty\right) + \lim_{n\to\infty}\sum_{i=1}^{p}\mathrm{Cov}\left(\int_0^t e^{A(t-s)}\Phi^i \,\mathrm{d}B_s^i, \int_{-n}^0 e^{-As}\Phi^i \,\mathrm{d}B_s^i\right).$$

Finally, Theorem 6 gives us

$$= e^{At}\mathrm{Cov}\left(x_\infty, x_\infty\right) + \lim_{n\to\infty}\sum_{i=1}^{p}\int_{-n}^{0}\int_0^t e^{A(t-u)}\Phi^i\left(\Phi^i\right)^* e^{-A^*v}\varphi_i(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

$$= e^{At}\mathrm{Cov}\left(x_\infty, x_\infty\right) + \sum_{i=1}^{p}\int_{-\infty}^{0}\int_0^t e^{A(t-u)}\Phi^i\left(\Phi^i\right)^* e^{-A^*v}\varphi_i(u,v)\,\mathrm{d}u\,\mathrm{d}v.$$

□

3.2 Ergodicity

We have the following ergodic theorems.

Theorem 20. *Assume that for each $i \in \{1,\dots,p\}$ we have that B^i is a Gaussian process with stationary and reflexive increments. Assume that $(e^{At}, t \geq 0)$ is exponentially stable. Then the strictly stationary solution $(X_t^{x_\infty}, t \geq 0)$ is ergodic.*

Proof. Throughout this proof all norms $\|\cdot\|$ are understood as norms in $\mathbb{R}^{m\times m}$. Using Theorem 14 it suffices to show that

$$\lim_{t\to+\infty}\left\|\mathrm{Cov}\left(X_t^{x_\infty}, X_0^{x_\infty}\right)\right\| = 0.$$

We use the expression for the covariance matrix from Theorem 12. For $t \geq 0$ we have

$$\left\|\mathrm{Cov}\left(X_t^{x_\infty}, X_0^{x_\infty}\right)\right\| =$$

$$= \left\|e^{At}\mathrm{Cov}\left(x_\infty, x_\infty\right) + \sum_{i=1}^{p}\int_{-\infty}^{0}\int_0^t e^{A(t-u)}\Phi^i\left(\Phi^i\right)^* e^{-A^*v}\varphi_i(u,v)\,\mathrm{d}u\,\mathrm{d}v\right\|$$

$$\leq \left\|e^{At}\right\|\left\|\mathrm{Cov}\left(x_\infty, x_\infty\right)\right\| + \sum_{i=1}^{p}\int_{-\infty}^{0}\int_0^t \left\|e^{A(t-u)}\Phi^i\left(\Phi^i\right)^* e^{-A^*v}\varphi_i(u,v)\right\|\mathrm{d}u\,\mathrm{d}v.$$

The first term converges to 0 as $t \to +\infty$ due to exponential stability. The fact that the second term converges to 0 as $t \to +\infty$ have been shown in the proof of Theorem 15. □

Theorem 21. *Let the assumptions of Theorem 20 hold. Let $f : \mathbb{R}^m \to \mathbb{R}$ be masurable and such that $\mathbb{E}|f(x_\infty)| < +\infty$. Moreover, let f satisfy the following condition: there exist constants $L > 0$ and $d \in \mathbb{Z}, d \geq 0$ such that*

$$|f(x) - f(y)| \leq L\,\|x-y\|_{\mathbb{R}^m}\left(1 + \|x\|_{\mathbb{R}^m}^d + \|y\|_{\mathbb{R}^m}^d\right),\ \ x, y \in \mathbb{R}^m.$$

Then

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(X_t^{x_0})\,\mathrm{d}t = \int_{\mathbb{R}^m} f(y)\,\mathrm{d}\mu_\infty(y),\ \ \mathbb{P}-a.s.$$

for all $x_0 \in \mathbb{R}^m$.

Proof. We can mimic the proof of Theorem 16 until the point where it remains to show that

$$\frac{1}{T}\int_0^T e^{-at}\left(\|X_t^{x_0}\|^d + \|X_t^{x_\infty}\|^d\right) \mathrm{d}t \to 0$$

as $T \to +\infty$ $\mathbb{P}$-a.s. Now we have that

$$\begin{aligned}
&\|X_t^{x_0}\|^d + \|X_t^{x_\infty}\|^d \\
&\leq \left(\left\|e^{At}x_0\right\| + \left\|\sum_{i=1}^{p}\int_0^t e^{A(t-s)}\Phi^i \,\mathrm{d}B_s^i\right\|\right)^d + \left(\left\|e^{At}x_\infty\right\| + \left\|\sum_{i=1}^{p}\int_0^t e^{A(t-s)}\Phi^i \,\mathrm{d}B_s^i\right\|\right)^d \\
&= \sum_{k=0}^{d}\binom{d}{k}\left(\left\|e^{At}x_0\right\|^{d-k} + \left\|e^{At}x_\infty\right\|^{d-k}\right)\left\|\sum_{i=1}^{p}\int_0^t e^{A(t-s)}\Phi^i \,\mathrm{d}B_s^i\right\|^k \\
&\leq \sum_{i=1}^{p}\sum_{k=0}^{d}\binom{d}{k} M^{d-k}e^{-a(d-k)t}\left(\|x_0\|^{d-k} + \|x_\infty\|^{d-k}\right)\left\|\int_0^t e^{A(t-s)}\Phi^i \,\mathrm{d}B_s^i\right\|^k.
\end{aligned}$$

The rest of the proof can be completed in the same way as the proof of Theorem 16. □

4. Parameter estimation

In this chapter we consider the stochastic differential equation from the previous section with added multiplicative parameter in the drift. We use the results from previous chapters in order to derive two strongly consistent estimators of the unknown parameter. As the main tool we employ the ergodic results from the previous section. This mimics the approach from [19] where the authors considered an infinite-dimensional analogy to our equation and a fractional Brownian motion as a noise. We will specify the differences between the above article and this Thesis later in this chapter.

Consider the equation

$$\begin{aligned} \mathrm{d}X_t &= \gamma A X_t \,\mathrm{d}t + \sum_{i=1}^{p} \Phi^i \,\mathrm{d}B_t^i, \quad t \geq 0, \\ X_0 &= x_0, \end{aligned} \tag{4.1}$$

where $\gamma > 0$ is the unknown scalar parameter, $p \in \mathbb{N}$, $A \in \mathbb{R}^{m\times m}$ is a real matrix, x_0 is an $\mathbb{R}^m$-valued random variable and for each $i \in \{1, \ldots, p\}$ we have that $\Phi^i \in \mathbb{R}^{m\times m}$ is a real matrix. As in the beginning of the previous chapter we start with the following basic assumption about the noise:

0. Assume that for $i \in \{1, \ldots, p\}$ we have that B^i is a two-sided m-dimensional α_i-regular Volterra process. Furthermore, assume that $B^1, \ldots, B^p$ are independent.

These assumptions are the same as those at the begining of the previous chapter. In order for the results of Theorem 18, Theorem 19, Theorem 20 and Theorem 21 to be valid we, as in the previous chapter, add the following assumptions:

1. For all $i \in \{1, \ldots, p\}$ we have that B^i has stationary and reflexive increments.
2. We assume the exponential stability, i.e.
$$\left\| e^{At} \right\| \leq M e^{-at}, \quad t \geq 0,$$
holds for some $a, M > 0$.
3. We assume that B^i is a Gaussian process for all $i \in \{1, \ldots, p\}$.

Under all these assumptions we have the following results from the previous chapter. The solution to (4.1) is

$$X_t^{x_0} = e^{\gamma A t} x_0 + \sum_{i=1}^{p} \int_0^t e^{\gamma A (t-s)} \Phi^i \,\mathrm{d}B_s^i, \quad t \geq 0.$$

For $x_0 \in \mathbb{R}^m$ deterministic the solution X^{x_0} is a Gaussian process. For the initial condition

$$x_\infty^\gamma = \operatorname*{l.i.m.}_{n\to\infty} \sum_{i=1}^{p} \int_{-n}^{0} e^{-\gamma A s} \Phi^i \,\mathrm{d}B_s^i$$

the solution $X^{x^{\gamma}_{\infty}}$ is a strictly stationary Gaussian process and most importantly the $X^{x^{\gamma}_{\infty}}$ is ergodic.

In order to estimate γ we will utilize the following simple lemma.

Lemma 7. *Let $Y = (Y_1, \ldots, Y_m)^* : \Omega \to \mathbb{R}^m$ be a random variable with $\mathbb{E}[Y] = 0$ and $\mu = \mathrm{Law}\,(Y)$ its probability law. Let $z \in \mathbb{R}^m$. Then*

$$\int\limits_{\mathbb{R}^m} \|y\|^2_{\mathbb{R}^m} \,\mathrm{d}\mu(y) = \mathrm{Tr}\,\mathrm{Cov}(Y, Y)$$

and

$$\int\limits_{\mathbb{R}^m} \langle y, z\rangle^2_{\mathbb{R}^m} \,\mathrm{d}\mu(y) = \langle \mathrm{Cov}(Y, Y)z, z\rangle_{\mathbb{R}^m}\,.$$

Proof. We have that

$$\int\limits_{\mathbb{R}^m} \|y\|^2_{\mathbb{R}^m} \,\mathrm{d}\mu(y) = \mathbb{E}\,\|Y\|^2_{\mathbb{R}^m} = \sum_{i=1}^{m} \mathbb{E}Y_i^2 = \sum_{i=1}^{m} \mathrm{Var}(Y_i) = \mathrm{Tr}\,\mathrm{Cov}(Y, Y)$$

and similarly for the second equality. □

Set $\mu^{\gamma}_{\infty} = \mathrm{Law}(x^{\gamma}_{\infty})$ and $f(x) = \|x\|^2_{\mathbb{R}^m}\,, x \in \mathbb{R}^m$. Then we can use Theorem 21 to get

$$\lim_{T\to+\infty} \frac{1}{T} \int\limits_0^T \|X^{x_0}_t\|^2_{\mathbb{R}^m} \,\mathrm{d}t = \int\limits_{\mathbb{R}^m} \|y\|^2_{\mathbb{R}^m} \,\mathrm{d}\mu^{\gamma}_{\infty}(y) = \mathrm{Tr}\,\mathrm{Cov}(x^{\gamma}_{\infty}, x^{\gamma}_{\infty}), \ \ \mathbb{P}\text{-a.s.}$$

Moreover, we have

$$\begin{aligned}
\mathrm{Cov}(x^{\gamma}_{\infty}, x^{\gamma}_{\infty}) &= \mathrm{Cov}\left(\underset{n_1\to\infty}{\mathrm{l.i.m.}} \sum_{i=1}^{p} \int\limits_{-n_1}^{0} e^{-\gamma A s}\Phi^i \,\mathrm{d}B^i_s, \underset{n_2\to\infty}{\mathrm{l.i.m.}} \sum_{j=1}^{p} \int\limits_{-n_2}^{0} e^{-\gamma A s}\Phi^j \,\mathrm{d}B^j_s\right) \\
&= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \sum_{i=1}^{p}\sum_{j=1}^{p} \mathrm{Cov}\left(\int\limits_{-n_1}^{0} e^{-\gamma A s}\Phi^i \,\mathrm{d}B^i_s, \int\limits_{-n_2}^{0} e^{-\gamma A s}\Phi^j \,\mathrm{d}B^j_s\right) \\
&= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \sum_{i=1}^{p} \mathrm{Cov}\left(\int\limits_{-n_1}^{0} e^{-\gamma A s}\Phi^i \,\mathrm{d}B^i_s, \int\limits_{-n_2}^{0} e^{-\gamma A s}\Phi^i \,\mathrm{d}B^i_s\right).
\end{aligned}$$

Since B^i has stationary and reflexive increments we can use Theorem 5 to get

$$\begin{aligned}
&= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \sum_{i=1}^{p} \mathrm{Cov}\left(\int\limits_{0}^{n_1} e^{\gamma A s}\Phi^i \,\mathrm{d}B^i_s, \int\limits_{0}^{n_2} e^{\gamma A s}\Phi^i \,\mathrm{d}B^i_s\right) \\
&= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \sum_{i=1}^{p} \int\limits_0^{n_2}\int\limits_0^{n_1} e^{\gamma A u}\Phi^i \left(\Phi^i\right)^* e^{\gamma A^* v} \varphi_i(u, v) \,\mathrm{d}u \,\mathrm{d}v \\
&= \sum_{i=1}^{p} \int\limits_0^{\infty}\int\limits_0^{\infty} e^{\gamma A u}\Phi^i \left(\Phi^i\right)^* e^{\gamma A^* v} \varphi_i(u, v) \,\mathrm{d}u \,\mathrm{d}v \\
&= \frac{1}{\gamma^2} \sum_{i=1}^{p} \int\limits_0^{\infty}\int\limits_0^{\infty} e^{A u}\Phi^i \left(\Phi^i\right)^* e^{A^* v} \varphi_i\left(\frac{u}{\gamma}, \frac{v}{\gamma}\right) \mathrm{d}u \,\mathrm{d}v.
\end{aligned}$$

Now, suppose we have

$$\varphi_i\left(\frac{u}{\gamma}, \frac{v}{\gamma}\right) = \psi_i(\gamma)\varphi_i(u,v) \tag{4.2}$$

for some function $\psi_i : (0, +\infty) \to (0, +\infty)$. It follows that

$$\mathrm{Cov}(x_\infty^\gamma, x_\infty^\gamma) = \frac{1}{\gamma^2}\sum_{i=1}^{p} \psi_i(\gamma) Q_i,$$

where

$$Q_i = \int_0^\infty \int_0^\infty e^{Au}\Phi^i \left(\Phi^i\right)^* e^{A^* v}\varphi_i\left(u,v\right) \mathrm{d}u \, \mathrm{d}v$$

does not depend on the unknown γ. This gives us

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \|X_t^{x_0}\|_{\mathbb{R}^m}^2 \, \mathrm{d}t = \sum_{i=1}^{p} \frac{\psi_i(\gamma)}{\gamma^2} \mathrm{Tr}\, Q_i, \quad \mathbb{P}\text{-a.s.}$$

Provided the function

$$\gamma \mapsto \sum_{i=1}^{p} \frac{\psi_i(\gamma)}{\gamma^2} \mathrm{Tr}\, Q_i$$

has a continuous inverse, call it $\widetilde{\psi}$, we get

$$\widetilde{\psi}\left(\frac{1}{T}\int_0^T \|X_t^{x_0}\|_{\mathbb{R}^m}^2 \, \mathrm{d}t\right) \to \gamma$$

as $T \to +\infty$ $\mathbb{P}$-a.s. and thus we get a strongly consistant estimator of γ. Let us return to equation (4.2). Under our assumptions, it turns out equation (4.2) implies that B^i is self similar.

Definition. Let $X = (X_t, t \in \mathbb{R})$ be a stochastic process. We say that X is self-similar, if there exists a function g such that $\forall t \in \mathbb{R}$ and $\forall a > 0$ we have that

$$X_{at} \overset{\mathcal{D}}{=} g(a) X_t \; .$$

We assumed that

$$B_t^i \sim \mathrm{N}(0, \mathrm{Var}(B_t^i)), \;\; t \in \mathbb{R}.$$

For simplicity assume now that B^i is one dimensional and $t \geq 0$. Assume that equation (4.2) holds. Recalling the definition of Volterra process we have

$$\mathrm{Var}(B_t^i) = \mathbb{E}\left[B_t^i B_t^i\right] = R(0,t,0,t) = \int_0^t \int_0^t \varphi_i(u,v)\, \mathrm{d}u \, \mathrm{d}v.$$

Then

$$\begin{aligned}\mathrm{Var}(B_{at}^i) &= \int_0^{at}\int_0^{at} \varphi_i(u,v)\, \mathrm{d}u \, \mathrm{d}v = a^2 \int_0^t \int_0^t \varphi_i(au,av)\, \mathrm{d}u \, \mathrm{d}v \\ &= \frac{a^2}{\psi_i(a)} \int_0^t \int_0^t \varphi_i(u,v)\, \mathrm{d}u \, \mathrm{d}v = \frac{a^2}{\psi_i(a)} \mathrm{Var}(B_t^i).\end{aligned}$$

Therefore

$$B^i_{at} \overset{\mathcal{D}}{=} \frac{a}{\sqrt{\psi_i(a)}} B^i_t$$

and B^i is self-similar. If B^i is m-dimensional, it is composed of m independent one-dimensional self-similar Volterra processes, all with the same function φ_i, and it follows it is self-similar. To summarize, to obtain an estimator of γ we assume that (4.2) holds. This, combined with the previous assumptions on B^i, implies that each B^i is self-similar. Therefore, we assume that each B^i is centred continuous Gaussian process starting from zero with stationary (and reflexive) increments and that B^i is self-similar. Recall the following well-known characterization of a (two-sided) fractional Brownian motion.

Theorem 22. *Let $X = (X_t, t \in \mathbb{R})$ be a continuous centred self-similar Gaussian process with stationary increments and with $X_0 = 0$, $\mathbb{P}$-a.s. Then X is a fractional Brownian motion.*

Proof. See e.g. [17], Proposition 3.8. □

In the rest of this chapter we will therefore assume that for each $i \in \{1, \dots, p\}$ the B^i is a two-sided Fractional Brownian motion with the Hurst parameter $H_i \in \left(\frac{1}{2}, 1\right)$. To emphasize this we will write B^{H_i} in place of B^i. For B^{H_i} we have

$$\varphi_{H_i}(u, v) = H_i(2H_i - 1)|u - v|^{2H_i - 2}$$

and thus

$$\varphi_{H_i}\left(\frac{u}{\gamma}, \frac{v}{\gamma}\right) = H_i(2H_i - 1)\left|\frac{u}{\gamma} - \frac{v}{\gamma}\right|^{2H_i - 2} = \frac{1}{\gamma^{2H_i - 2}} \varphi_{H_i}(u, v).$$

The following simple lemma will be used later.

Lemma 8. *Let $Q \in \mathbb{R}^{m \times m}$ be a non-zero, symmetric and positive semidefinite matrix. Then $\mathrm{Tr}(Q) > 0$.*

Proof. Assume that $\mathrm{Tr}(Q) = 0$. For $z \in \mathbb{R}^m$ we have

$$\langle Qz, z \rangle_{\mathbb{R}^m} = \sum_{i=1}^m \sum_{j=1}^m Q_{ij} z_i z_j = 2 \sum_{1 \le i < j \le m} Q_{ij} z_i z_j.$$

Since $Q \neq 0$ we have that $Q_{kl} \neq 0$ for some indexes $k < l$. Now take $\tilde{z} := e_k - \mathrm{sgn}(Q_{kl}) e_l$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)^*$ and the 1 is the i-th coordinate. Then

$$\langle Q\tilde{z}, \tilde{z} \rangle_{\mathbb{R}^m} = 2 Q_{kl} \tilde{z}_k \tilde{z}_l = -2\, \mathrm{sgn}(Q_{kl}) Q_{kl} < 0.$$

Therefore Q is not a positive semidefinite matrix. □

Before we state the two main theorems of this Thesis we for clarity repeat our final set of assumptions. We consider the equation

$$\begin{aligned} \mathrm{d}X_t &= \gamma A X_t \,\mathrm{d}t + \sum_{i=1}^p \Phi^i \,\mathrm{d}B^{H_i}_t, \quad t \ge 0, \\ X_0 &= x_0, \end{aligned} \tag{4.3}$$

where $\gamma > 0$ is the unknown scalar parameter, $p \in \mathbb{N}$, $A \in \mathbb{R}^{m \times m}$ is a real matrix, x_0 is an $\mathbb{R}^m$-valued random variable and for each $i \in \{1, \dots, p\}$ we have that $\Phi^i \in \mathbb{R}^{m \times m}$ is a real matrix.

(A1) For each $i \in \{1, \dots, p\}$ the B^{H_i} is a two-sided fractional Brownian motion with the Hurst parameter $H_i \in \left(\frac{1}{2}, 1\right)$ and $B^{H_1}, \dots, B^{H_p}$ are independent.

(A2) The $(e^{At}, t \geq 0)$ is exponentially stable.

Let us return to the article [19] which this chapter mimics. In this article the authors considered the Equation 4.3 with $p = 1$ and within an infinite-dimensional setting. In case of $p = 1$ the following two theorems from this Thesis are a finite-dimensional versions of Theorem 5.2 and Theorem 5.1 from [19]. In case of $p \geq 2$ the following two theorem provide a partial extension of the results from [19].

Theorem 23. *Let X^{x_0} be a solution to (4.3). Assume that* (A1) *and* (A2) *hold. Assume that there is $i \in \{1, \dots, p\}$ such that $\Phi^i \neq 0$. Then*

$$\hat{\gamma}_T = \widetilde{\psi}\left(\frac{1}{T}\int_0^T \|X_t^{x_0}\|_{\mathbb{R}^m}^2 \,\mathrm{d}t\right) \to \gamma$$

as $T \to +\infty$ $\mathbb{P}$-a.s., (i.e., the estimator $\hat{\gamma}_T$ is strongly consistent,) where $\widetilde{\psi}$ is the inverse function (which exists) to

$$\gamma \mapsto \sum_{i=1}^{p} \frac{Tr\, Q_i}{\gamma^{2H_i}}, \quad \gamma > 0,$$

where

$$Q_i = \int_0^\infty \int_0^\infty e^{Au}\Phi^i \left(\Phi^i\right)^* e^{A^*v} \varphi_{H_i}(u, v) \,\mathrm{d}u \,\mathrm{d}v.$$

Proof. Firstly, note that $\Phi^i \neq 0$ implies that $Q_i \neq 0$. Indeed, we find $x \in \mathbb{R}^m$ such that $\left\|(\Phi^i)^* x\right\|_{\mathbb{R}^m}^2 > 0$. We have

$$\langle Q_i x, x\rangle_{\mathbb{R}^m} = \int_0^{+\infty}\int_0^{+\infty} \left\langle \left(\Phi^i\right)^* e^{A^*v}x, \left(\Phi^i\right)^* e^{A^*u}x\right\rangle_{\mathbb{R}^m} \varphi_{H_i}(u, v) \,\mathrm{d}u \,\mathrm{d}v.$$

We also have that

$$\lim_{[u,v]\to[0,0]} \left\langle \left(\Phi^i\right)^* e^{A^*v}x, \left(\Phi^i\right)^* e^{A^*u}x\right\rangle_{\mathbb{R}^m} = \left\|\left(\Phi^i\right)^* x\right\|_{\mathbb{R}^m}^2 > 0. \qquad (**)$$

Take $\varepsilon \in (0, \left\|(\Phi^i)^* x\right\|_{\mathbb{R}^m}^2)$. Thanks to $(**)$ we can find $\delta > 0$ such that

$$\left\langle \left(\Phi^i\right)^* e^{A^*v}x, \left(\Phi^i\right)^* e^{A^*u}x\right\rangle_{\mathbb{R}^m} > \varepsilon$$

holds for all $u, v \in (0, \delta]$. Then

$$\begin{aligned}\langle Q_i x, x\rangle_{\mathbb{R}^m} \geq\ & \varepsilon \int_0^\delta \int_0^\delta \varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ & + \int_0^\delta \int_\delta^{+\infty} \left\langle \left(\Phi^i\right)^* e^{A^*v}x, \left(\Phi^i\right)^* e^{A^*u}x\right\rangle_{\mathbb{R}^m} \varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ & + \int_\delta^{+\infty} \int_0^\delta \left\langle \left(\Phi^i\right)^* e^{A^*v}x, \left(\Phi^i\right)^* e^{A^*u}x\right\rangle_{\mathbb{R}^m} \varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ & + \int_\delta^{+\infty} \int_\delta^{+\infty} \left\langle \left(\Phi^i\right)^* e^{A^*v}x, \left(\Phi^i\right)^* e^{A^*u}x\right\rangle_{\mathbb{R}^m} \varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

The last three term on the right hand side are all non-negative thanks to the fact that any covariance matrix is positive semidefinite. To be more precise we for instance have that

$$\begin{aligned}&\int_0^\delta \int_\delta^{+\infty} \left\langle \left(\Phi^i\right)^* e^{A^*v}x, \left(\Phi^i\right)^* e^{A^*u}x\right\rangle_{\mathbb{R}^m} \varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ &\qquad = \left\langle \left(\int_0^\delta \int_\delta^{+\infty} e^{Au}\Phi^i \left(\Phi^i\right)^* e^{A^*v}\varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)x, x\right\rangle_{\mathbb{R}^m} \\ &\qquad = \lim_{n\to\infty} \left\langle \left(\int_0^\delta \int_\delta^{n} e^{Au}\Phi^i \left(\Phi^i\right)^* e^{A^*v}\varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)x, x\right\rangle_{\mathbb{R}^m}\end{aligned}$$

and by Theorem 6

$$= \lim_{n\to\infty} \left\langle \mathrm{Cov}\left(\int_\delta^n e^{As}\Phi^i\,\mathrm{d}B_s^{H_i}, \int_0^\delta e^{As}\Phi^i\,\mathrm{d}B_s^{H_i}\right)x, x\right\rangle_{\mathbb{R}^m} \geq 0$$

since any covariance matrix is positive semidefinite. Thanks to $\int_0^\delta \int_0^\delta \varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v > 0$ we have $\langle Q_i x, x\rangle_{\mathbb{R}^m} > 0$ which implies that $Q_i \neq 0$. Using Lemma 8 we get $\mathrm{Tr}(Q_i) > 0$. Now, as before we utilize Theorem 21 with $f(y) = \|y\|^2_{\mathbb{R}^m}$ and we use Lemma 7 to get

$$\begin{aligned}&\lim_{T\to+\infty} \frac{1}{T}\int_0^T \|X_t^{x_0}\|^2_{\mathbb{R}^m}\,\mathrm{d}t = \frac{1}{\gamma^2}\sum_{i=1}^p \mathrm{Tr}\left(\int_0^\infty \int_0^\infty e^{Au}\Phi^i \left(\Phi^i\right)^* e^{A^*v}\varphi_{H_i}\left(\frac{u}{\gamma}, \frac{v}{\gamma}\right)\mathrm{d}u\,\mathrm{d}v\right) \\ &= \frac{1}{\gamma^2}\sum_{i=1}^p \frac{1}{\gamma^{2H_i-2}}\mathrm{Tr}\left(\int_0^\infty \int_0^\infty e^{Au}\Phi^i \left(\Phi^i\right)^* e^{A^*v}\varphi_{H_i}(u,v)\,\mathrm{d}u\,\mathrm{d}v\right) = \sum_{i=1}^p \frac{\mathrm{Tr}\, Q_i}{\gamma^{2H_i}}.\end{aligned}$$

The function

$$\gamma \mapsto \sum_{i=1}^p \frac{\mathrm{Tr}\, Q_i}{\gamma^{2H_i}}, \quad \gamma > 0,$$

is strictly monotone, continuous and non-zero. It therefore has a continuous inverse. Call this inverse $\widetilde{\psi}$. Then

$$\hat{\gamma}_T = \widetilde{\psi}\left(\frac{1}{T}\int\limits_0^T \|X_t^{x_0}\|_{\mathbb{R}^m}^2 \,\mathrm{d}t\right) \to \gamma$$

as $T \to +\infty$ $\mathbb{P}$-a.s. □

In case of $p = 1$ the function $\widetilde{\psi}$ can be expressed and the estimator has the form

$$\hat{\gamma}_T = \left(\frac{\operatorname{Tr} Q_1}{\frac{1}{T}\int_0^T \|X_t^{x_0}\|_{\mathbb{R}^m}^2 \,\mathrm{d}t}\right)^{\frac{1}{2H}}.$$

Theorem 24. *Let X^{x_0} be a solution to (4.3). Assume that* (A1) *and* (A2) *hold. Let $z \in \mathbb{R}^m$ be such that for some $i \in \{1, \dots, p\}$ we have $\langle Q_i z, z\rangle_{\mathbb{R}^m} > 0$, where Q_i is as in the previous theorem. Then*

$$\widetilde{\gamma}_T = \widetilde{\psi}\left(\frac{1}{T}\int_0^T \langle X_t^{x_0}, z\rangle_{\mathbb{R}^m}^2 \,\mathrm{d}t\right) \to \gamma$$

as $T \to +\infty$ $\mathbb{P}$-a.s., (i.e., the estimator $\hat{\gamma}_T$ is strongly consistent,) where $\widetilde{\psi}$ is the inverse (which exists) to

$$\gamma \mapsto \sum_{i=1}^p \frac{\langle Q_i z, z\rangle_{\mathbb{R}^m}}{\gamma^{2H_i}}, \quad \gamma > 0.$$

Proof. The proof is very similar to the proof of the previous theorem. We again use Theorem21 with $f(x) = \langle x, z\rangle_{\mathbb{R}^m}^2$ and Lemma 7 to get

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \langle X_t^{x_0}, z\rangle_{\mathbb{R}^m}^2 \,\mathrm{d}t = \langle \operatorname{Cov}(x_\infty^\gamma, x_\infty^\gamma) z, z\rangle_{\mathbb{R}^m} = \left\langle \left(\sum_{i=1}^p \frac{1}{\gamma^{2H_i}} Q_i\right) z, z\right\rangle_{\mathbb{R}^m}$$
$$= \sum_{i=1}^p \frac{1}{\gamma^{2H_i}} \langle Q_i z, z\rangle_{\mathbb{R}^m},$$

$\mathbb{P}$-a.s. Since all $Q_1, \dots, Q_p$ are positive semidefinite matrices and for some $i \in \{1, \dots, p\}$ we have $\langle Q_i z, z\rangle_{\mathbb{R}^m} > 0$, the function

$$\gamma \mapsto \sum_{i=1}^p \frac{\langle Q_i z, z\rangle_{\mathbb{R}^m}}{\gamma^{2H_i}}, \quad \gamma > 0,$$

is again strictly monotone, continuous and non-zero and thus it again has a continuous inverse. This finishes the proof. □

In case of $p = 1$ the function $\widetilde{\psi}$ can be expressed and the estimator has the form

$$\widetilde{\gamma}_T = \left(\frac{\langle Q_1 z, z\rangle_{\mathbb{R}^m}}{\frac{1}{T}\int_0^T \langle X_t^{x_0}, z\rangle_{\mathbb{R}^m}^2 \,\mathrm{d}t}\right)^{\frac{1}{2H}}.$$

Remark. The difference between estimators $\hat{\gamma}_T$ and $\widetilde{\gamma}_T$ is as follows. In order to employ the estimator $\hat{\gamma}_T$ we usually need to observe the whole m-dimensional trajectory of X^{x_0} for we need to know its norm. On the other hand, if we for instance only observe the k-th coordinate of a trajectory, $k \in \{1, \dots, m\}$, then (provided that for some $i \in \{1, \dots, p\}$ we have $\langle Q_i e_k, e_k\rangle_{\mathbb{R}^m} > 0$) we can still use the estimator $\widetilde{\gamma}_T$.

Conclusion

We opened the Thesis with the definition and basic properties of α-regular Volterra processes and presented the construction of Wiener-like integral of deterministic function with respect to an α-regular Volterra process. We also presented various properties of this integral. In Section 2 we considered a linear stochastic differential equation driven by an α-regular Volterra process B. We started with no restrictions on B. In order to obtain the existence of a strictly stationary solution we assumed B to have stationary and reflexive increments. We assumed that B is Gaussian and showed the ergodicity of the strictly stationary solution and we obtained a similar result for any solution. In Section 3 we generalized the results for equations with a mixed noise. In Section 4 we added a parameter to the stochastic differential equation considered in Section 3. In order to utilize the results of the first three chapters we assumed our noise to be self-similar. The assumptions we employed on the noise throughout the Thesis implied that it must be a fractional Brownian motion. Under the assumption that the noise is a fBm we derived two strongly consistent estimators. The strong consistency of these estimators have already been shown in [19] under slightly different assumptions. However, our setting of a mixed noise at least partially extends these results.

List of symbols

$\mathbb{N}$	$=\{1,2,\dots\}$, set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^m$	set of m-dimensional real vectors
$\mathbb{R}^{m\times n}$	set of $m\times n$ dimensional matrices
A^*	(conjugate) transpose of the matrix A, see 2. below
$\mathrm{Tr}(A)$	trace of square matrix A, see 3. below
I	identity matrix
$\langle\cdot,\cdot\rangle_{\mathbb{R}^m}$	standard (Euclidean) inner product in $\mathbb{R}^m$, see 4. below
$\lVert\cdot\rVert_{\mathbb{R}^m}$	Euclidean norm in $\mathbb{R}^m$, see 4. below
$\lVert\cdot\rVert_{\mathbb{R}^{m\times n}}$	Hilbert-Schmidt norm in $\mathbb{R}^{m\times n}$, see 5. below
$\mathcal{B}(X)$	Borel σ-algebra of matrix space X
$L^p(U;V)$	see 6. and 7. below
$L^p_{loc}(U;V)$	see 6. and 7. below
$\underset{n\to\infty}{\text{l.i.m.}}$	limit in $L^2(\Omega;\mathbb{R}^m)$
$\xrightarrow{w^*}$	weak convergence of probability measures

Further notes

1. We identify $\mathbb{R}^m$ with $\mathbb{R}^{m\times 1}$ i.e. vectors are understood as column vectors.
2. $A^* = (A_{ji})_{j=1,i=1}^{n\ \ m} \in \mathbb{R}^{n\times m}$ for $A=(A_{ij})_{i=1,j=1}^{m\ \ n}\in\mathbb{R}^{m\times n}$.
3. $\mathrm{Tr}(A)=\sum\limits_{i=1}^{m} A_{ii}$ for $A=(A_{ij})_{i,j=1}^{m}\in\mathbb{R}^{m\times m}$.
4. For $x=(x_1,\dots,x_m)^*, y=(y_1,\dots,y_m)^*\in\mathbb{R}^m$ we have
$$\langle x,y\rangle_{\mathbb{R}^m}=\sum_{i=1}^{m} x_i y_j \quad\text{and}\quad \lVert x\rVert_{\mathbb{R}^m}=\sqrt{\langle x,x\rangle_{\mathbb{R}^m}}=\sqrt{\sum_{i=1}^{m} x_i^2}.$$
5. $\lVert A\rVert_{\mathbb{R}^{m\times n}}=\sqrt{\mathrm{Tr}(AA^*)}=\sqrt{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}A_{ij}^2}$ for $A=(A_{ij})_{i=1,j=1}^{m\ \ n}\in\mathbb{R}^{m\times n}$.
6. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $1\le p<+\infty$. The $L^p(\Omega;\mathbb{R}^m)$ is the space of equivalence classes of $\mathbb{R}^m$-valued random variables $X:\Omega\to\mathbb{R}^m$ such that
$$\lVert X\rVert_{L^p(\Omega;\mathbb{R}^m)} := \left(\int_\Omega \lVert X\rVert^p_{\mathbb{R}^m}\,\mathrm{d}\mathbb{P}\right)^{\frac{1}{p}} < +\infty.$$
7. Let $-\infty\le a<b\le+\infty$ and $1\le p<+\infty$. The space $L^p(a,b;\mathbb{R}^{m\times n})$ is the space of equivalence classes of measurable functions $f:[a,b]\to\mathbb{R}^m$ such that
$$\lVert f\rVert_{L^p(a,b;\mathbb{R}^m)} := \left(\int_{[a,b]} \lVert f\rVert^p_{\mathbb{R}^{m\times n}}\right)^{\frac{1}{p}} < +\infty.$$

The space $L^p_{loc}(a, b; \mathbb{R}^{m\times n})$ is the space of equivalence classes of measurable functions $f : [a, b] \to \mathbb{R}^m$ such that for any K, K being a compact subset of $[a, b]$, we have

$$\left(\int_K \|f\|^p_{\mathbb{R}^{m\times n}}\right)^{\frac{1}{p}} < +\infty.$$

We set

$$L^p(\mathbb{R}; \mathbb{R}^{m\times n}) := L^p(-\infty, +\infty; \mathbb{R}^{m\times n})$$

and

$$L^p_{loc}(\mathbb{R}; \mathbb{R}^{m\times n}) := L^p_{loc}(-\infty, +\infty; \mathbb{R}^{m\times n}).$$

Bibliography

[1] Alòs E., Mazet O., Nualart D., *Stochastic calculus with respect to Gaussian processes*, Ann. Probab. 29, pp. 766–801, 2001.

[2] Balde M.F., Es-Sebaiy K., Tudor C.A., *Ergodicity and drift parameter estimation for infinite-dimensional fractional Ornstein–Uhlenbeck process of the second kind*, Appl. math. Optim. 81, pp. 785–814, 2020.

[3] Bonaccorsi S., Tudor C. A., *Dissipative stochastic evolution equations driven by general Gaussian and non-Gaussian noise.*, J. Dyn. Diff. Equat. 23, pp. 791–816, 2011.

[4] Cialenco I., Lototsky S.V., Pospíšil J., *Asymptotic properties of the maximum likelihood estimator for stochastic parabolic equations with additive fractional Brownian motion*, Stoch. Dyn. 9, pp. 169-185, 2009.

[5] Čoupek P., *Stochastic Evolution Equations*, Doctoral Thesis, Charles University in Prague, 2017.

[6] Čoupek P., *Limiting measure and stationarity of solutions to stochastic evolution equations with Volterra noise*, Stochastic Analysis and Applications, 2018.

[7] Čoupek P., Maslowski B., *Stochastic Evolution Equations with Volterra noise*, Stoch. Proc. Appl. 127, pp. 877-900, 2017.

[8] Čoupek P., Maslowski B., Ondreját, M., *L^p-valued stochastic convolution integral driven by Volterra noise*, Stoch. Dyn., 2018.

[9] Čoupek P., Maslowski B., Šnupárková J., *SPDEs with Volterra noise*, Stochastic Partial Differential Equations and Related Fields: In Honor of Michael Röckner, SPDERF, Bielefeld, Germany, 2016.

[10] Duncan T. E., Maslowski B., Pasik-Duncan, B., *Fractional Brownian motion and stochastic equations in Hilbert spaces*, Stoch. Dyn. 2, pp. 225–250, 2002.

[11] Duncan T. E., Maslowski B., Pasik-Duncan, B., *Stochastic equations in Hilbert space with a multiplicative fractional Gaussian noise*, Stoch. Proc. Appl. 115, pp. 1357–1383, 2005.

[12] Duncan T. E., Maslowski B., Pasik-Duncan, B., *Linear stochastic equations in a Hilbert space with a fractional Brownian motion*, International Series in Operations Research & Management Science, Springer Verlag, 94, pp. 201-222, 2006.

[13] Hardy G. H., Littlewood J. E., *Some properties of fractional integrals I.*, Mathematische Zeitschrift 27, pp. 565–606, 1928.

[14] Huebner M., Rozovskii B.L., *On asymptotic properties of maximum likelihood estimators for parabolic stochastic PDE's*, Probab. Th. Rel. Fields 103, pp. 143–163, 1995.

[15] Koski T., Akademi Å., Loges W., *On identification for distributed parameter systems*, Stochastic Processes - Mathematics and Physics II, Albeverio S., Blanchard P. and Streit L., eds., Lecture Notes in Mathematics, Vol. 1250, Springer, Berlin, Heidelberg, pp. 152-159, 1987.

[16] Kříž P., Maslowski B., *Central limit theorems and minimum-contrast estimators for linear stochastic evolution equations*, Stochastics 91, pp. 1109-1140, 2019

[17] Mandelbrot B.B., Van Ness, J.W., *Fractional Brownian motions, fractional noises and applications*, SIAM Rev. 10, pp. 422-437, 1968

[18] Maslowski B., Nualart D., *Evolution equations driven by a fractional Brownian motion*, J. Funct. Anal. 202, pp. 277–305, 2003.

[19] Maslowski B., Pospíšil J., *Ergodicity and parameter estimates for infinite-dimensional fractional Ornstein-Uhlenbeck process*, Applied Mathematics and Optimization. 57, pp. 401-429., 2008

[20] Maslowski B., Schmalfuß B., *Random dynamical systems and stationary solutions of differential equations driven by the fractional Brownian motion*, Stochastic Anal. Appl. 22, pp. 1577-1607, 2004.

[21] Maslowski B., Šnupárková J., *Stochastic Affine Evolution Equations with Multiplicative Fractional Noise*, Appl. Math. 63, pp. 7 - 35, 2018.

[22] Maslowski B., Tudor C.A., *Drift parameter estimation for infinite-dimensional fractional Ornstein–Uhlenbeck process*, Bull. Sci. Math. 137, pp. 880-901, 2013.

[23] Rozanov Yu. A., *Stationary random processes*, Holden-Day, San Francisco, 1966.

[24] Seidler J., *Vybrané kapitoly ze stochastické analysy*, Matfyzpress, Praha, 2011.

[25] Šnupárková, J., *Stochastic bilinear equations with fractional Gaussian noise in Hilbert space*, Acta Univ. Carolin. Math. Phys. 51, pp. 49–68, 2010.