Referee's report on doctoral thesis
Compact modules over nonsingular rings
by Peter Kálnai

The thesis consists of 3 published papers, 1 preprint and an introduction. In this report I will briefly summarize and comment each of these papers separately.

Chapter 1 is the paper *Products of small modules* by P. Kálnai and J. Žemlička published in CMUC in 2014. Recall that an R-module is *small* provided $\mathrm{Hom}_R(M, -)$ canonically commutes with direct sums. Inspired by results of T. Kepka and R. El Bashir on slim modules, the question when (every) product of small R-modules is small is studied. A ring R satisfies *DSP* if the class of all small R-modules is closed under products. Proposition 1.2.3 says that if there exists a nonzero ring with DSP, the there exists a non-artinian self injective (von Neumann-regular) simple ring satisfying DSP. The main result (Theorem 1.3.4) says that if R is a right self injective ring containing a right ideal isomorphic to $R^{(\omega)}$ then every countable product of small modules is small and if there is a system of small modules of size κ such that the product of the system is not small, then there exists $I \subseteq \kappa$ and a non-principal countably complete ultrafilter on I.

Comments and questions on Chapter 1:

(i) The main result Theorem 1.3.4 is quite nontrivial and very nice. I would slightly reformulate the first sentence in the proof of Lemma 1.3.3, part (iv). 'We will show that there exists nonempty $I_0 \subseteq I, I_0 \notin \mathcal{A} \ldots$'

(ii) Theorem 1.3.4 (ii) suggests, that one can find I_0 satisfying $|I_0| = \lambda < \kappa$ but it is not discussed in the proof.

Chapter 2 *Compactness in abelian categories* by the same authors appeared in Journal of Algebra in 2019. This chapter is a continuation of the previous one in the context of abelian categories with products and coproducts. Section 2.1 gives a detailed study of products and coproducts in abelian categories, key notion of a compact (resp. $\mathcal{C}$-compact) object is introduced. Theorem 2.1.5 is a characterization of $\mathcal{C}$-compact object generalizing the classical description of a small module via its lattice of submodules. The main result of the chapter, Corollary 2.3.3 generalizes Theorem 1.3.4 for products of $\mathcal{C}$-compact objects of $\Pi\mathcal{C}$-compactly generated category.

Comments and questions on Chapter 2

(i) This chapter is rather technical, the key idea in Proposition 2.3.2 has already appeared in the proof of Lemma 1.3.3. In my opinion the main point of this article is the categorical setup where a generalization of Theorem 1.3.4 could be done.

(ii) In Corollary 2.3.3 the isomorphism $\mathcal{A}(\oplus\mathcal{K}, \oplus\mathcal{N}) \simeq \mathcal{A}(\oplus\mathcal{M}, \oplus\mathcal{N})$ seems to use that if $\mathcal{A}(M, N) = 0$ for every $N \in \mathcal{N}$, then also $\mathcal{A}(M, \oplus\mathcal{N}) = 0$. I can see that for Ab-5 categories but is it true in Abelian categories in general?

The third chapter is again a join work *Self-injective von Neumann regular rings and Koethe's conjecture* of the author and his supervisor. As the title indicates the paper is related to the conjecture by G. Koethe formulated in 1930. There are many equivalent formulations of this conjecture, the one used here is that the sum of two nil right ideals in a ring is nil. This paper gives several results of the form 'if a counterexample to the Koethe conjecture exists, then there exists a counterexample of a special form'. The main Theorem 3.2.3 says that if Koethe's conjecture is not true then a counterexample can be found as a countable local subring of a suitable self-injective von Neumann regular ring of type II_f or III.

Comments and questions on Chapter 3

(i) As far as I know considered candidates for the counterexample of Koethe's conjecture formulated in this form were usually suitable factors of free algebras. Then to verify that an element is (not) nilpotent is a non-commutative variant of a radical membership problem which is already (computationally) difficult for commutative rings. On the other hand, the class of von Neumann regular rings is widely studied. Therefore I think this result could be relevant for the future study of Koethe's conjecture.

(ii) Example 3.1.1 is not correct - consider for example a commutative domain of polynomials in 2 variables over a field.

The final chapter is the paper *Generalizations of projectivity and supplements revisited for superfluous ideals* the author published in Communications in Algebra in 2019. The paper contains various technical results, the main result is the following: Let I be an ideal of R contained in $J(R)$. The following are equivalent:
1. If P is a finitely generated module containing submodules G, K such that $G + K = P$ and $G \cap K$ is a superfluous submodule of G contained in GI, then G is a direct summand of G.
2. Every finitely generated I-(semi)projective module is projective. (A module M is I-semiprojective if $\mathrm{Hom}_R(M, -)$ is exact on short exact sequences of the form $0 \to A \to B \to C \to 0$ with $CI = 0$).
3. If M is a finitely generated flat module such that M/MI is a projective R/I-module then M is projective.
4. If Q is a projective module with Q/QI finitely generated then Q is finitely generated.

Apart from this several interesting results are proved. For example, Theorem 4.4.3 gives a characterization when lifting of idempotents modulo I transfers to matrix rings. These problems are also related to Koethe's conjecture as explained in Example 4.5.4.

Comments and questions on Chapter 4

(i) This paper connects various concepts studied in the literature, maybe the presentation of the results could be a bit better. At this moment it is hard to say whether these results could be applied in future research.

(ii) In my opinion the proof of Theorem 4.4.3 should be revised. For example, I think the module B^n should be a submodule of R^n contained in I^n

Summary: The submitted work contains interesting results related to the recent research in the theory of rings and modules. The author showed his ability to do a creative research and I recommend this work to be *accepted* as a doctoral dissertation.

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