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REPORT OF THE PHD THESIS:
Compact modules over nonsingular rings
by Peter Kálnai

Compact objects in various additive categories with direct sums play an important role in the study of these categories. They are defined as those objects C such that the covariant functor $\mathrm{Hom}(C, -)$ commutes with respect to direct sums. Hence compact objects can be used to replace the usual finitely presented (generated) modules in categories where the notion "finitely presented (generated)" cannot be considered. Moreover, even in module categories the compact objects (called small modules) can be used to provide information about the ring of scalars or to the whole category (e.g. they can be used to construct equivalences of categories which generalize Morita equivalences).

The thesis is based on four papers written by the author which are presented in distinct chapters.

In the first chapter, entitled *Products of small modules*, some closure properties of the class of small modules are studied. In fact, it is easy to see that this class is not closed under infinite direct sums, and the question is when a direct product of small modules is small. The main results are enunciated for right self-injective rings that are purely infinite. In Theorem 1.3.4 it is proved that for these rings all countable direct products of small modules are small and if there exists a product of small modules that is not small then the number of factors in this product is greater than an uncountable cardinal which has a countably complete ultrafilter.

The second chapter is entitled *Compactness in abelian categories* and presents general properties of $\mathcal{C}$-compact objects in abelian categories, i.e. objects with the property that the induced Hom-covariant functor commutes with direct sums of objects from the class $\mathcal{C}$. The main result is Theorem 2.3.4 which states that for reasonable generated categories the closure of the class of $\mathcal{C}$-compact objects with respect to direct products is true if we assume that there are no strongly inaccessible cardinals.

In Chapter 3, *Self-injective von Neumann regular rings and Köthes Conjecture* are presented some very interesting connections between possible answers of Köthes Conjecture and the existence of some Self-injective von Neumann regular rings that have some very particular structures. More precisely, in Theorem 3.2.3 it is proved that if there is a ring which is a counter-example for Köthes Conjecture then it can be constructed a counter-example which is a countable local subring of a suitable self-injective simple VNR ring of type either II_f or III.

The last chapter, *Generalizations of Projectivity and Supplements Revisited for Superfluous ideals*, has as a main subject the study of a relative version of a conjecture of Lazard (which is known to be false in general but it is true for commutative rings). The main result is Theorem 4.5.3 which gives equivalent conditions to the fact that for every projective R-module Q, if the ideal factor Q/QI is finitely generated then Q is finitely generated. These characterizations can be very useful since they suggest that there exist connections between various research directions in module/ring theory.

With very few exceptions, the thesis is carefully written. However, in the introduction there are some somewhat confusing explanations (e.g. p.2,l.-5–p.3,l.4). I think that this is a consequence of the effort of the author to obtain a very brief presentation of a very complex theory. However, I want to point out that the author used many modern and powerful techniques. The proofs are correct, and many of them are not trivial. All these convinced me that the author acquired very good research skills.

Therefore, taking into account the significance of the mathematical results, **I strongly recommend Mr. Peter Kálnai to be awarded a Ph.D. in Mathematics.**

Simion Breaz

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