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Charles University

DOCTORAL THESIS

Peter Kálnai

Compact modules over nonsingular rings

Department of Algebra

Supervisor of the thesis: doc. Mgr. et Mgr. Jan Žemlička, Ph.D.
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Název práce: Kompaktní moduly nad nesingulárními okruhy

Autor: Peter Kálnai

Katedra: Katedra algebry

Školitel: doc. Mgr. et Mgr. Jan Žemlička, Ph.D., Katedra algebry

Abstrakt: Tato disertace obsahuje několik nových výsledků, ve kterých využíváme vnitřní strukturu nesingulárních, speciálně samoinjektivních von Neumannovsky regulárních okruhů. Nejdříve popíšeme kategoriální a množinově-teoretické podmínky, za kterých všechny součiny kompaktních objektů zůstanou kompaktní, přičemž pojem kompaktnosti je tady vztažen s ohledem na pevnou podtřídu objektů. Speciálními případy, kdy taková uzávěrová vlastnost platí, jsou klasické modulové kategorie nad okruhy našeho zaměření. Navíc ukážeme, že případný protipříklad pro Köetheho hypotézu by mohl mít tvar spočetného lokálního podokruhu vhodného jednoduchého, samoinjektivního, von Neumannovsky regulárního okruhu.

Klíčová slova: nesingulární okruhy, malý modul, kompaktní objekt, samoinjektivní, von Neumannovsky regulární, nilpotentní, projektivní modul

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Author: Peter Kálnai

Department: Department of Algebra

Supervisor: doc. Mgr. et Mgr. Jan Žemlička, Ph.D., Department of Algebra

Abstract: This doctoral thesis provides several new results in which we leverage the inner structure of non-singular rings, in particular of self-injective von Neumann regular rings. First, we describe categorical and set-theoretical conditions under which all products of compact objects remain compact, where the notion of compactness is relativized with respect to a fixed subclass of objects. Special instances when such closure property holds are the classic module categories over rings of our interest. Moreover, we show that a potential counterexample for Köthe's Conjecture might be in the form of a countable local subring of a suitable simple self-injective von Neumann regular ring.

Keywords: non-singular ring, small module, compact object, self-injective, von Neumann regular, nilpotent, projective module

Contents

0. Introduction

This dissertation is based on the following four works (note that the Schwabacher numbering from now on refers to the articles exclusively):

(𝔦) Kálnai P., Žemlička J.: *Products of small modules*, Commentat. Math. Univ. Carol. **55**, No. 1, 2014, pp. 9–16.

(𝔦𝔦) Kálnai P., Žemlička J.: *Compactness in Abelian categories*, J. Algebra **534**, 15 September 2019, pp. 273–288.

DOI: 10.1016/j.jalgebra.2019.05.037

(𝔦𝔦𝔦) Kálnai P., Žemlička J.: *Self-injective von Neumann regular rings and Köthe's Conjecture*, arXiv:1912.12159, 2019.

(𝔦𝔳) Kálnai P.: *Generalizations of projectivity and supplements revisited for superfluous ideals*, Comm. Algebra **47**, No. 1, 2019, pp. 88–100.

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This introduction serves for summarizing the main results of the thesis. For non-explained terminology we refer to standard monographs [1, 2, 6].

The first and the second chapter are tightly connected and deal with the product closure in classic module categories and complete abelian categories, respectively. In the following parts we go through the structure and results of (𝔦) and (𝔦𝔦).

If we start with the definition of a small module M as the functorial compactness condition, namely that the covariant functor $\mathrm{Hom}_R(M, -)$ commutes with all direct sums of modules, then we quickly arrive at the equivalent characterization that such modules are not unions of strictly increasing chains of submodules. Now it is obvious that finitely generated modules are always small. Rings are called right steady if small and finitely generated modules coincide. Regarding the closure properties, homomorphic images of small modules are also small and finite direct sums of small modules remain small, but an infinite direct sum of arbitrary non-zero modules is never small. (𝔦, Lemma 1.2.1) shows there is a correspondence between small modules over a ring and over its ring extension when the extension is not too far, i.e. it is small over the base ring.

In terms of commutativity with any direct sums, the contravariant functor $\mathrm{Hom}_R(-, M)$ behaves a little bit differently, cf. [3], . The equivalent characterizations of compactness split in this dual case into a hierarchy of strict implications dependent on the cardinality of commuting families. The strongest hypothesis assumes arbitrary cardinalities and it leads to the class of so called *slim* modules (also known as *strongly slender*), which is a subclass of the most general class of $\aleph_1$-slim modules (also called as *slender*), which involves only commutativity with countable families. The cardinality of a non-zero slim module is greater than or equal to any measurable cardinal (and the presence of such cardinality is also a sufficient condition for existence of a non-zero slim module) and that the class of slim modules is closed under coproducts. Thus, the absence of a measurable cardinal ensures that there is at least one non-zero slim module and in fact, abundance of them. On the other hand, if there is a proper class of measurable cardinals then there is no such object like a non-zero slim module. This motivated the question in the dual setting, namely if the class of small modules is closed under products, denoted from now on as the DSP property.

The closure properties of the class of non-singular rings satisfying DSP include factor rings and maximal right ring of quotients, (**ı**, Lemma 1.2.2, Proposition 1.2.3). Recall that simple rings are an example of a class of non-singular rings and there is a deep statement saying that the right maximal ring of quotients of a right non-singular ring is a Von Neumann regular ring, shortly a *VNR* ring, and it is right self-injective. Therefore we are able to restrict our attention to the case of simple self-injective VNR rings. Many details about the structure of such ring construction is presented in [6, Chapter XII.].

Corollary 1.2.4. *If R is simple ring satisfying* DSP*, then $Q_{\max}(R)$ satisfies* DSP *as well.*

We say that a (general) ring R is *right purely infinite* if there is a right ideal $K \leq R$ such that $K \simeq R^{(\omega)}$ as right R-modules, i.e., there is an exact sequence $0 \to R^{(\omega)} \to R$ in **Mod**-R. Note that a self-injective VNR ring is defined as *purely infinite* in [2, Definition, p.116] if it contains no nonzero directly finite central idempotents. By [2, Theorem 10.16(a), (d)] it contains itself as a right ideal isomorphic to a countably generated free module, therefore it is purely infinite in the previous general sense.

Several set-theoretical notions and facts are needed in the final part, namely that each Ulam-measurable cardinal is greater or equal to the first measurable cardinal, every measurable cardinal is strongly inaccessible and finally, it is consistent with ZFC that there is no strongly inaccessible cardinal, (**ı**, Fact 1.1.4).

If we assume a potential counterexample of the DSP property for an infinite system of small modules over a right self-injective purely infinite VNR ring, then we infer in (**ı**, Lemma 1.3.3) that the system can not be countable and there exists a subset of the index set on which there is a non-principal prime ideal. This leads immediately to the following theorem.

Theorem 1.3.4. *Let R be a right self-injective right purely infinite VNR ring. Then the following holds:*

(i) A countable product of small R-modules is small.

(ii) If there exists a system $(M_\alpha \mid \alpha < \kappa)$ of small R-modules such that the product $\prod_{\alpha<\kappa} M_\alpha$ is not small, then there exists an uncountable cardinal $\lambda < \kappa$ and a countably complete ultrafilter on λ.

The existence of a counterexample over a uncountable system of small modules now leads to a countably complete ultrafilter. By the facts from set theory, we get that there exists a measurable cardinal which is strongly inaccessible. Plugging in the set-theoretical assumptions yields the desired closure property under arbitrary products.

Corollary 1.3.5. *Let R be a non-Artinian right self-injective, right purely infinite VNR ring. If we assume that there is no strongly inaccessible cardinal, then the class of all small R-modules is closed under direct products.*

An object C of an abelian category closed under coproducts and products is said to be *compact* if the covariant functor $\mathrm{Hom}(C, -)$ commutes with all direct sums i.e. there is a canonical isomorphism between $\mathrm{Hom}(C, \bigoplus \mathcal{D})$ and $\bigoplus \mathrm{Hom}(C, \mathcal{D})$ in the category of abelian groups for every system of objects $\mathcal{D}$. The concept of compactness presents an easy way to replace finitely generated modules in general abelian categories. Nevertheless, the clear form of the categorical definition is a reason why compact objects can be applied as a useful tool also in categories containing finitely generated objects in the standard sense.

For a class $\mathcal{C}$ in a complete abelian category $\mathcal{A}$, we say that $\mathcal{A}$ is $\prod \mathcal{C}$*-compactly generated* if there is a set $\mathcal{G}$ of objects from $\mathcal{A}$ that generates $\mathcal{A}$ and the product of any system of objects in $\mathcal{G}$ is $\mathcal{C}$-compact. Note that $\mathcal{G}$ consists only of $\mathcal{C}$-compact objects. (**ii**, Lemma 2.3.1) then states that a cokernel of the compatible coproduct-to-product morphism is $\mathcal{C}$-compact for countable families of objects. This is a clear analogy of (**i**, Lemma 1.3.2(ii)) stated in the classic module category over a right purely infinite self-injective ring.

If we similarly assume a potential counterexample of the compactness property for a family of $\mathcal{C}$-compact objects in a $\prod \mathcal{C}$-compactly generated category, then we infer in (**ii**, Lemma 2.3.2) that the system can not be countable and there exists a subset of the index set on which there is a non-principal prime ideal:

Corollary 2.3.3. *Let $\mathcal{A}$ be a $\prod \mathcal{C}$-compactly generated category. Then the following holds:*

(i) A product of countably many $\mathcal{C}$-compact objects is $\mathcal{C}$-compact.

(ii) If there exists a system $\mathcal{M}$ of cardinality κ of $\mathcal{C}$-compact objects such that the product $\prod \mathcal{M}$ is not $\mathcal{C}$-compact, then there exists an uncountable cardinal $\lambda < \kappa$ and a countably complete non-principal ultrafilter on λ.

Theorem 2.3.4. *Let $\mathcal{A}$ be a $\prod \mathcal{C}$-compactly generated category, $\mathcal{M}$ a family of $\mathcal{C}$-compact objects of $\mathcal{A}$. If we assume that there is no strongly inaccessible cardinal, then every product of $\mathcal{C}$-compact objects is $\mathcal{C}$-compact.*

Next, let us move to the results of (**iii**), which leverages the structure of right self-injective VNR rings as well. Recall that such ring can be uniquely decomposed as a product of rings of three types, Type I, Type II and Type III with the first two types of two more possible subtypes: directly finite (Types I_f and II_f) or purely infinite ((Types I_∞, II_∞ and III). Moreover, if such VNR ring is prime (simple), then there it could be of exactly one such (sub)type. VNR rings are semiprimitive, which means their Jacobson radical is nil, and therefore there are no nil one-sided ideals. We say that an element is principally nilpotent if the right ideal it generated is nil and obviously this definition is left-right symmetric and the property withstands two-sided multiplication by any element from the ring. Thus, VNR rings contain no non-zero principally nilpotent elements.

The famous Köthe's Conjecture affirmatively assumes that a ring with no non-zero two-sided nil ideal necessarily contains no non-zero one-sided nil ideal. Several important classes of rings are known to satisfy Köthe's Conjecture like all right Noetherian rings, PI-rings, and rings with right Krull dimension. However, the question if the conjecture holds generally is still open. The most famous translations of the conjecture use the language of associative rings without unit and include statements like "the matrix ring $M_n(R)$ is nil for every nil ring R", "$R[x]$ is not left primitive for every nil ring R", "every ring which is a sum of a nilpotent subring and a nil subring is nil" or "$R[x]$ is Jacobson radical for every nil ring R". Over any countable field F, Agata Smoktunowicz proved that there is a nil F-algebra R_1 such that $R_1[x]$ need not to be nil [4, Corollary 13] and a nil F-algebra R_2 such that $R_2[x, y]$ need not to be Jacobson radical [4, Corollary 14]. These two results are therefore approximations of a negative solution for Köthe's Conjecture. The author later improved the result in the sense that there is a nil algebra R_3 with the polynomial ring $R_3[x_1, x_2, x_3, y_1, y_2, y_3]$ containing a non-commutative free F-algebra of rank two [5, Theorem 1.3].

We say that a unital associative ring *satisfies the condition (NK)* if it contains two nil right ideals whose sum is not nil. Then Köthe's Conjecture is equivalent to the property that there exists no ring satisfying the condition (NK) or equivalently if there exist two principally nilpotent elements whose sum is not nilpotent (obviously (NK) is the most negative approximation of the conjecture's solution one could imagine). It is possible to make the following reduction in considering (NK) rings, implying that Köthe's Conjecture holds if and only if there is no countable generated F-algebra satisfying (NK) for either $F = \mathbb{Q}$ or $F = \mathbb{Z}_p$ where p is a prime number (**iii**, Corollary 3.1.9).

Proposition 3.1.8. *Let R be a ring satisfying (NK). Then there exists a subring S of R generated by two elements ξ, υ, an F-algebra A and an epimorphism of rings $\varphi : S \to A$ such that the following conditions hold:*

(K1) either $F = \mathbb{Q}$ or $F = \mathbb{Z}_p$ where p is a prime number,

(K2) $x = \varphi(\xi)$ and $y = \varphi(\upsilon)$ are principally nilpotent generators of A, and $x + y$ is a minimal non-nilpotent element,

(K3) $xA + yA = \mathcal{J}(A)$ is the unique maximal right ideal of A.

Despite the absence of principally nilpotent elements in VNR rings, it is still possible to translate Köthe's Conjecture into a certain structural question about simple right self-injective VNR rings. Indeed, the existence of a countable ring satisfying (K1)–(K3) (and being local by (**iii**, Lemma 3.1.7)) can be considered as a subring of a self-injective simple VNR ring either Type II_f or Type III. So it means that principally nilpotent elements of an (NK) ring lose their property by being embedded into certain VNR rings. In the proof of the main theorem, we first observe that an (NK) ring can not be embedded in a VNR ring Type I_f, because such overrings can be chosen Artinian, leading to a contradiction with (**iii**, Lemma 3.1.13). If we start with a potential overring either Type I_∞ or II_∞, then we will be able to find a desired VNR ring either Type II_f or III, leaving the latter two types only possibilities for such overring.

Theorem 3.2.3. *If there is a ring satisfying (NK) then there exists a countable local subring of a suitable self-injective simple VNR ring of type either* II_f *or* III *that also satisfies (NK).*

Finally, in (**iv**) we provide results about several statements than are weaker than Koethe's Conjecture and that turn out to be equivalent. The statements are related to a mathematical problem that have already enjoyed much interest:

Lazard's Conjecture, a hypothesis denying the existence of a non-finitely generated projective module with the finitely generated radical factor. First proved affirmatively for commutative rings but then Gerasimov and Sakhaev constructed a ground-breaking and in many aspects of ring and module theory a very misbehaved counterexample turning the conjecture negative. We extend results that are true for radical factors to factors by any superfluous ideal, i.e. an ideal contained in the Jacobson radical.

To obtain the goal, four ideal-related generalizations and objects are needed: the *ideal-superfluity*, *projective ideal-covers*, the *ideal-projectivity*, and an *ideal-supplement*. We provide examples of objects demonstrating that these notions are generally not redundant (**iv**, Example 4.1.4), (**iv**, Example 4.1.7), (**iv**, Example 4.1.11) and (**iv**, Example 4.1.15).

Definition 4.1.1. *A submodule N of M* decomposes *M (or shortly is* DM *in M) if there is a summand S of M such that $S \subseteq N$ and $M = S + X$, whenever $N + X = M$ for a submodule X of M. A submodule N of M is called* PDM *in M if there is a projective summand S of M such that $S \subseteq N$ and $M = S + X$, whenever $N + X = M$ for a submodule X of M.*

We say that a submodule N of a right module M is I-superfluous, denoted $N \ll_I M$, if $N \subseteq MI$ and N is PDM *in M.*

Definition 4.1.5. *A pair (P, f) is called* a projective I-semicover *of M if P is projective and $f : P \to M$ is an epimorphism such that* $\ker(f) \subseteq PI$.

A pair (P, f) is called a projective I-cover *of M if it is a projective I-semicover of M and* $\ker(f)$ *is DM in P.*

Definition 4.1.8. *An R-module P is* I-semiprojective *if for every epimorphism $f : X \to Y$ such that $YI = 0$ and every morphism $\varphi : P \to Y$ there is a homomorphism $g : P \to X$ such that $\varphi = f \circ g$. A right R-module P is* I-projective *if for all right R-modules X and Y, every R-epimorphism $f : X \to Y$ and every homomorphism $\varphi : P \to Y$ there exists a homomorphism $g : P \to X$ such that $(f \circ g - \varphi)(P) \ll_I Y$.*

Definition 4.1.13 (Ideal supplements)**.** *We call a submodule G of an R-module M* an I-supplement submodule *if there is a submodule K of M such that $K+G = M$ and $K \cap G \ll_I G$.*

Recall that Köthe's Conjecture could be also expressed in extensibility of nilpotency from the nil radical to the matrix ring over the nil radical. Nil ideals admit idempotent-lifting and so a connection with the condition $(L3')$ of the following theorem starts to appear.

Theorem 4.4.3. *The following conditions are equivalent:*

(L1') for every $n \in \mathbb{N}$, any direct summand of a right R-module $R^{(n)}/I^{(n)}$ has a projective I-cover

(L2') for every $n \in \mathbb{N}$, if P' is a direct summand of $(R/I)^{(n)}$, then there is a direct summand P of $R^{(n)}$ such that $P' = P + I^{(n)}/I^{(n)} (\simeq P/PI)$

(L3') $M_n(I)$ is lifting in $M_n(R)$ for every $n \in \mathbb{N}$

(L4') every direct summand of a finitely generated right R-module with a projective I-cover has a projective I-cover.

Plugging in all the generalized notions we arrive at the following characterization which was previously proved for the edge case $I = \mathcal{J}(R)$.

Theorem 4.5.2. *The following is equivalent:*

(1) every I-supplement in a finitely generated projective R-module P is a direct summand

(2) every finitely generated I-(semi)projective R-module is projective

(3) every finitely generated flat R-module M with projective R/I-module M/MI is projective

(4) for every projective R-module Q, if the ideal factor Q/QI is finitely generated then Q is finitely generated

The statement (4) evokes Lazard's Conjecture parametrized by any superfluous ideal. It is obvious that for the trivial case $I = 0$ each of the four conditions (1), (2), (3), (4) holds true and therefore also their equivalence does. Various authors in the past already observed that (2) is true for any I contained in the Baer radical $\beta(R)$ and (4) is true if I is a nilpotent ideal.

The ideal-projectivity in a finitely generated projective module resp. ideal supplements in a finitely generated module are redundant with respect to the classic definitions if the chosen ideal admits idempotent-lifting up to all its matrix rings.

Proposition 4.5.3. *Let R be a ring satisfying the conditions of Theorem 4.4.3. Then the condition* (4) *in Theorem 4.5.2 holds true.*

It is well known that the Levitzki radical of a ring is an ideal which admits such lifting of idempotents, thus satisfies the equivalent conditions of Theorem 4.5.2. Unfortunately, the question whether the nil radical provides the counterexample for the statements is not resolved yet. However, non-existence of such counterexamples for this ideal would yield an approximation of a positive solution of Köthe's Conjecture. This observation and the complexity of the Gerasim-Sakhaev's construction suggest that finding such counterexample is at least as hard as finding one to the conjecture itself.

Bibliography for Chapter 0

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1. Products of small modules

It is easy to verify that every finitely generated module M satisfies the natural compactness condition that the covariant functor $\mathrm{Hom}_R(M, -)$ commutes with all direct sums of modules. Nevertheless, there are known large classes of infinitely generated modules satisfying that condition, for instance every uncountable union of a chain of finitely generated modules forms an infinitely generated example. Every module M satisfying that functorial compactness condition is called small in this paper.

It is well known that a finite direct sum of small modules is small in general and infinite direct sum of arbitrary nonzero modules is not small. Nevertheless, the case of product of small modules is rather more complicated. In the work [9] it is proved that over each ring which contains a right ideal isomorphic to a 2-generated free module (hence there exists a right ideal isomorphic to infinitely generated free module) every injective module is small. As the class of all injective modules is closed under all direct products, that observation leads to the natural question formulated explicitly in [2, Remark 3.2] whether there exists a ring R such that the class of all small right modules over R is closed under direct products.

The main objective of the present paper is to give a partial answer to that question, dependently on a model of set theory. We use for that purpose mainly tools and methods developed in the works [1, 5, 6, 8, 9, 11, 13], which study properties of classes of all small modules for some particular classes of rings.

Recall that a ring is called *right steady* if every small (right) module is finitely generated. Obviously, a ring over which every product of small modules is small is very far from being steady. Before we start searching rings over which small right modules are closed under direct products, we prove first that we may restrict our consideration to the case of simple self-injective VNR rings (Proposition 1.2.3). Our main result (Theorem 1.3.4) prove necessary condition on set theory which holds true if over a right self-injective right purely infinite ring there exists non-small product o small modules. As that set theoretical condition contradicts to the hypothesis that there is no strongly inaccessible cardinal, which is consistent with ZFC, we can easily see that under the hypothesis of non-existence of a strongly inaccessible cardinal the class of all small module is closed under products (Theorem 1.3.5).

1.1 Preliminaries

Throughout the paper, a ring R means an associative ring with unit, a module is a right R-module and an ideal means a two-sided ideal. We say that $R \subseteq Q$ is a ring extension if R is a subring of Q, note that Q has a natural structure of R-algebra. $E(M)$ denotes an injective envelope of an arbitrary module M. We say that a module M is (less than, at most) *κ-generated* if the least cardinality of any set of generators is (less than, at most) κ and we write $gen(M) = \kappa$ ($< \kappa$, $\leq \kappa$).

As we have remarked, a module M is said to be *small* whenever the natural $\mathbb{Z}$-monomorphism $\bigoplus_{i<\omega} \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \bigoplus_{i<\omega} N_i)$ is surjective for every system of modules $(N_i \mid i < \omega)$. We will usually deal with the following equivalent condition of smallness:

Fact 1.1.1. *[8, Lemma 1.2] A module is small if and only if it is not a union of a strictly increasing infinite countable chain of submodules.*

We will use freely an easy consequence of the Lemma 1.1.1 that any factor of small module is small. Observation that every union of an uncountable strictly increasing chain of finitely generated modules forms an infinitely generated small module naturally leads to the useful definition of a λ-*reducing* module for an infinite cardinal λ as a module M such that every at most λ-generated submodule is contained in some finitely generated submodule of M. Recall that the classes of all small as well as λ-reducing modules are closed under homomorphic images and finite (direct) sums [12, Proposition 1.3].

The following elementary observations about λ-reducing modules are used freely in the sequel.

Lemma 1.1.2. *Let $\lambda \leq \kappa$ be infinite cardinals and M an infinitely generated κ-reducing module. Then:*

(i) M is small and λ-reducing,

(ii) gen$(M) > \kappa$,

(iii) M contains a κ^+-generated κ-reducing submodule,

(iv) M contains an ω_1-generated ω-reducing submodule.

We define the singular submodule $Z(M_R) := \{m \in M \mid \operatorname{rann}_R(m) \subseteq_{\mathrm{e}} R\}$ of a module M, where $\operatorname{rann}_R(-)$ denotes an annihilator and $U \subseteq_{\mathrm{e}} V$ means that U is an *essential* submodule of V, i.e. $U \cap W = 0$ implies $W = 0$ for a submodule W of V.

We say that a ring R is *right non-singular*, if $Z(R_R) = 0$, R is called *(Von Neumann) regular* if for every $x \in R$ there exists $y \in R$ such that $x = xyx$, and R is right self-injective, provided it is injective as a right module over itself. We observe that simple rings form examples of non-singular rings. As a fact we state a deep statement about their maximal right rings of quotients. For a definition of maximal right ring of quotients and other properties of this notion we refer to [7].

Fact 1.1.3 ([7], Proposition XII.2.1)**.** *The maximal right ring of quotients $Q_{\max}(R)$ of a right non-singular ring R is a right self-injective VNR ring and it is injective as a right R-module.*

Description and examples of self-injective VNR rings is given in [3, Chapters 9,10].

Finally, recall several set-theoretical notions and facts which we will need in the final part of this paper. A *filter* on a set X is a non-empty family of non-empty subsets of X closed under finite intersections and supersets. An *ultrafilter* on X is a filter which is not properly contained in any other filter on X. We say that a filter $\mathcal{F}$ is λ-*complete*, if $\bigcap \mathcal{G} \in \mathcal{F}$ for every subsystem $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| < \lambda$ and $\mathcal{F}$ is *countably complete*, if it is ω_1-complete. A cardinal λ is said to be *measurable* if there exists a λ-complete non-principal ultrafilter on λ and it is *Ulam-measurable* if there exists a countably complete non-principal ultrafilter on λ. A regular cardinal κ is *strongly inaccessible* if $2^\lambda < \kappa$ for each $\lambda < \kappa$.

Theorem 1.1.4. *The following holds:*

(i) Every Ulam-measurable cardinal is greater or equal to the first measurable cardinal.

(ii) Every measurable cardinal is strongly inaccessible.

(iii) It is consistent with ZFC that there is no strongly inaccessible cardinal.

Proof. (i) [10, Theorem 2.43.].
(ii) [10, Theorem 2.44.].
(iii) [4, Corollary IV.6.9]. ∎

1.2 Non-singular rings with DSP

We say that a ring R satisfies the condition DSP if every product of an arbitrary family of small R-modules is small. Let us start with an easy observation which states correspondence between small modules over a ring and over its extension.

Lemma 1.2.1. *Let $R \subseteq Q$ be a ring extension, M be a Q-module and Q_R be small as an R-module. Then M is a small Q-module if and only if it is a small R-module.*

Proof. Assume that M is a small Q-module. Let $M = \bigcup_{i<\omega} M_i$ for a countable chain of R-submodules $M_0 \subseteq M_1 \subseteq \dots$. We put $N_i = \{m \in M \mid mQ \subseteq M_i\}$ for each $i < \omega$. Obviously, $N_0 \subseteq N_1 \subseteq \dots$ forms a chain of Q-submodules of M and $N_i \subseteq M_i$ for every $i < \omega$. Let $m \in M$. Since $(mQ)_R$ is a homomorphic image of the small R-module Q_R, there exists n such that $mQ \subseteq M_n$. Thus $M = \bigcup_{i<\omega} N_i$. Now, by the hypothesis there exists $n < \omega$ such that $N_n = M$, hence $M_n = M$ and M is small R-module by Lemma 1.1.1.

The converse is clear, because every Q-module has also a natural structure of an R-module. ∎

The next assertion describes closure properties of the class of all rings satisfying DSP.

Lemma 1.2.2. *Let R satisfy* DSP*.*

(i) Every injective right R-module is small.

(ii) If R is a right non-singular ring with the maximal right ring of quotients Q, then Q satisfies DSP*.*

(iii) Every factor ring of R satisfies DSP*.*

Proof. (i) Let E_R be an injective R-module. Then there exists a cardinal κ and a surjective homomorphism $\pi : R^{(\kappa)} \to E$. Since the canonical embedding $R^{(\kappa)} \to R^\kappa$ is injective, π can be extended to an epimorphism $R^\kappa \to E$ by the injectivity of E. Since $(R_R)^\kappa$ is small by the hypothesis, the module E is a homomorphic image of a small module and therefore small as well.

(ii) By Fact 1.1.3 Q_R is injective, so by (i) it is small as an R-module. Thus every product of small Q-modules is small as an R-module by the hypothesis and Lemma 1.2.1, hence it is a small Q-module.

(iii) Since every (small) module over any factor ring have a natural structure of a (small) R-module, the assertion is clear. ∎

Now, we are able to show that searching of rings satisfying DSP may be restricted to the case of simple self-injective VNR rings.

Proposition 1.2.3. *If a ring R satisfies* DSP *and I is a maximal two-sided ideal, then R/I is (right) non-singular and $Q_{\max}(R/I)$ is a non-artinian right self-injective simple ring satisfying* DSP.

Proof. As R/I is simple, it is (right) non-singular, hence $Q_{\max}(R/I)$ is right self-injective by Fact 1.1.3. By applying Lemma 1.2.2(ii), $Q_{\max}(R/I)$ satisfies DSP, hence it is non-artinian. Finally, let J be a nonzero ideal of $Q_{\max}(R/I)$. Since R is essential in $Q_{\max}(R/I)_R$, the intersection $R/I \cap J$ is a nonzero ideal of R. Thus $1 \in R/I \subseteq J$ and $J = Q_{\max}(R/I)$. ∎

Corollary 1.2.4. *If R is simple ring satisfying* DSP, *then $Q_{\max}(R)$ satisfies* DSP *as well.*

1.3 Self-injective rings

We say that a ring R is *right purely infinite* if there is a right ideal $K \leq R$ such that $K \simeq R^{(\omega)}$ as right R-modules, i.e., there is an exact sequence $0 \to R^{(\omega)} \to R$ in **Mod**-R.

It is easy to see that the endomorphism ring of an infinite-dimensional vector space forms an example of a right purely infinite VNR ring. Recall that there exist right purely infinite, simple, self-injective VNR rings [3, Example 10.11]. Moreover, note that every simple self-injective VNR ring which is not directly finite is purely infinite by [3, Proposition 10.21].

First recall a key fact about the smallness of injective modules.

Fact 1.3.1. *[9, Example 2.8] Every injective module over a right purely infinite ring is small.*

Lemma 1.3.2. *Let κ be an infinite cardinal, R be a right purely infinite self-injective ring and $(M_\alpha \mid \alpha < \kappa)$ be a system of R-modules.*

(i) If M_α is ω_1-reducing for every $\alpha < \kappa$, then $\prod_{\alpha<\kappa} M_\alpha$ is ω_1-reducing as well,

(ii) if $\kappa = \omega$, then $\prod_{\alpha<\omega} M_\alpha / \bigoplus_{\alpha<\omega} M_\alpha$ is ω_1-reducing,

(iii) the product of any system of finitely generated modules is ω_1-reducing.

Proof. Put $M = \prod_{\alpha<\kappa} M_\alpha$. For any product $\prod_{\alpha<\kappa} M_\alpha$ denote by $\nu_\alpha : M_\alpha \to \prod_{\alpha<\kappa} M_\alpha$ the natural embedding and $\pi_\alpha : \prod_{\alpha<\kappa} M_\alpha \to M_\alpha$ the natural projection.

Similarly we define ν_J and π_J for any subset of κ.

(i) Note that $\prod_{\alpha<\kappa} R^{(n_\alpha)} \cong R^\kappa$ is injective for all finite n_α, hence ω_1-reducing by Fact 1.3.1. Fix a countable set $D := \{m_n \mid n < \omega\} \subseteq M$. By hypothesis on M_α, for each $\alpha < \kappa$ there is some finitely generated submodule F_α of M_α such that $\{\pi_\alpha(m_n) \mid n < \omega\} \subseteq F_\alpha$ and there is some n_α such that we can write F_α as a factormodule of a finitely generated free R-module $R^{(n_\alpha)}$. Hence $D \subseteq \prod_{\alpha<\kappa} F_\alpha$ and the exact sequence $\prod_{\alpha<\kappa} R^{(n_\alpha)} \to \prod_{\alpha<\kappa} F_\alpha \to 0$ shows that the middle term is a factor-module of an ω_1-reducing R-module, hence it is itself ω_1-reducing. Then there exists a finitely generated submodule F of $\prod_{\alpha<\kappa} F_\alpha$ such that $D \subseteq F (\subseteq M)$.

(ii) Put $S = \bigoplus_{\alpha<\omega} M_\alpha$. Fix a countable set $D := \{m_n \mid n < \omega\} \subseteq M$ and for each $\alpha < \omega$ define (a finitely generated) R-module $G_\alpha = \sum_{j\leq\alpha} \pi_\alpha(m_j)R$. Observe that $D \subseteq S + \prod_{\alpha<\omega} G_\alpha$. By (i) $\prod_{\alpha<\omega} G_\alpha$ is ω_1-reducing, hence a factor-module $\prod_{\alpha<\omega} G_\alpha + S/S$ is also ω_1-reducing. Then there exists a finitely generated module $F \subseteq \prod_{\alpha<\omega} G_\alpha (\subseteq M)$ such that $m_n + S \in F + S/S$ for all $n < \omega$.

(iii) As finitely generated modules are ω_1-reducing, (iii) is a direct consequence of (ii). ∎

Let $\mathcal{I}$ be a non-empty system of subsets of a set X. We recall that $\mathcal{I}$ is said to be *an ideal* if it is closed under subsets (i.e. if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$) and under finite unions, (i.e. if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$). $\mathcal{I}$ is *a prime ideal* if it is a proper ideal and for all subsets A, B of X, $A \cap B \in \mathcal{I}$ implies $A \in \mathcal{I}$ or $B \in \mathcal{I}$. If $Y \subseteq X$, the system $\mathcal{P}(Y)$ of all subsets of Y forms an ideal on X which is called *principal*. We say that the set $\mathcal{I} \mid Y = \{Y \cap A \mid A \in \mathcal{I}\}$ is *a trace* of $\mathcal{I}$ on Y.

It is easy to see that the trace of an ideal is also an ideal and that $\mathcal{I}$ is a prime ideal if and only if for every $A \subseteq X$, $A \in \mathcal{I}$ or $X \backslash A \in \mathcal{I}$. Moreover, a principal prime ideal on X is of the form $\mathcal{P}(X \setminus \{x\})$ for some $x \in X$. Note that there is a dual one-to-one correspondence between ultrafilters and prime ideals on X defined by $\mathcal{I} \mapsto \mathcal{P}(X) \backslash \mathcal{I}$ for an ideal $\mathcal{I}$.

Lemma 1.3.3. *Let R be a right purely infinite right self-injective ring and let $(M_\alpha \mid \alpha \in I)$ be a family of small modules. Let $M = \prod_{\alpha\in I} M_\alpha$ be the direct product and assume that M is not small, namely $M = \bigcup_{n<\omega} N_n$ for a countable strictly increasing chain of submodules $(N_n \mid n < \omega)$. Denote $\mathcal{A}_n = \{J \subseteq I \mid \prod_{\alpha\in J} M_\alpha \subseteq N_n\}$ and $\mathcal{A} = \bigcup_{n<\omega} \mathcal{A}_n$. Then the following holds:*

(i) $\mathcal{A}_n$ is an ideal for each n,

(ii) $\mathcal{A}$ is closed under countable unions of sets,

(iii) there exists $n < \omega$ for which $\mathcal{A} = \mathcal{A}_n$,

(iv) there exists a subset $I_0 \subseteq I$ such that the trace of $\mathcal{A}$ on I_0 is a non-principal prime ideal.

Proof. (i) Obviously $\emptyset \in \mathcal{A}_n$ and because M is not small, $I \notin \mathcal{A}_n$. The closure of $\mathcal{A}_n$ under subsets is obvious by the definition. The closure of $\mathcal{A}_n$ under finite unions follows from the decomposition $\prod_{\alpha\in J\cup K} M_\alpha = \prod_{\alpha\in J} M_\alpha \oplus \prod_{\alpha\in K\backslash J} M_\alpha \subseteq N_n$.

(ii) First we show that $\mathcal{A}$ is closed under countable unions of pairwise disjoint sets. Let $K_j \in \mathcal{A}$ be pairwise disjoint subsets of I for all $j < \omega$. We show that there exists $k < \omega$ such that $K_j \in \mathcal{A}_k$ for each $j < \omega$. Assume by contradiction that for every $n < \omega$ there exists (possibly distinct) $i(n)$ such that $K_{i(n)} \notin \mathcal{A}_n$. Hence there is $f_n \in \prod_{\alpha \in K_{i(n)}} M_\alpha$ for which $\nu_{K_{i(n)}}(f_n) \notin N_n$. Since $\prod_{j<\omega} f_j R = \bigcup_{n<\omega} \left(\prod_j f_j R \cap N_n\right)$ is small by Lemma 1.3.2(iii) there is $k < \omega$ such that $\nu_{K_{i(k)}}(f_k) \in \prod_{j<\omega} f_j R \subseteq N_k$, a contradiction.

Put $P_j = \prod_{\alpha \in K_j} M_\alpha$ for $j < \omega$. We have proved that there is some $k < \omega$ such that $P_j \subseteq N_k$ and it follows that $\bigoplus_{j<\omega} P_j \subseteq N_k$. Let $P = \prod_{j<\omega} P_j = \prod_j \prod_{\alpha \in K_j} M_\alpha$. As $P/\bigoplus_{j<\omega} P_j$ is small by Lemma 1.3.2(i) there exists some $l \geq k$ such that $P = \bigcup_{j<\omega}(P \cap N_j) \subseteq N_l$.

Now let J_j, $j < \omega$ be any system of subsets of I and put $J_0 = K_0$ and $J_i = K_i \setminus \bigcup_{j<i} K_j$ for $i > 0$. So $\bigcup_{j<\omega} J_j = \bigcup_{j<\omega} K_j$ and by the preceding we get the result.

(iii) Assume that $\mathcal{A} \neq \mathcal{A}_n$ for every n. Then there exists a sequence $(J_j \in \mathcal{A}\setminus\mathcal{A}_j \mid j \in \omega)$. By (ii) $J := \bigcup_{j<\omega} J_j \in \mathcal{A}$ and there is some $n < \omega$ such that $J \in \mathcal{A}_n$. Since $J_j \subseteq J \in \mathcal{A}_n$ for each $j < \omega$, we obtain a contradiction.

(iv) We will show that there exists $I_0 \subseteq I$ such that for every $K \subseteq I_0, K \in \mathcal{A}$ or $I_0\setminus K \in \mathcal{A}$. Assume that such I_0 does not exist. Then we may construct a countably infinite sequence of disjoint sets $(K_i \mid i < \omega)$ where K_i are non-empty for $i > 0$ in the following way: Put $K_0 = \emptyset$ and $J_0 = I_0$. There exist disjoint sets $J_{i+1}, K_{i+1} \subset J_i$ such that $J_i = J_{i+1} \cup K_{i+1}$ where $J_{i+1}, K_{i+1} \notin \mathcal{A}$. Now, for each $n \geq 1$ there exists $g_n \in \prod_{\alpha \in K_n} M_\alpha$ such that $\nu_{K_n}(g_n) \notin N_n$ which contradicts to the fact that $\prod_{n<\omega} g_n R \subseteq N_m$ for some $m < \omega$ (cf. the proof of (ii)).

Finally, assume that the trace of $\mathcal{A}$ on I_0 is principal. Since it is a prime ideal, there exists $i \in I_0$ such that $\mathcal{A} \mid I_0 = P(I_0 \setminus \{i\})$. Thus $I_0 \setminus \{i\} \in \mathcal{A}_n$. Now $\prod_{j \in I_0 \setminus \{i\}} M_j \subseteq N_n$ for some n and $\{i\} \in \mathcal{A}$, so $I_0 \in \mathcal{A} \mid I_0$ a contradiction. ∎

Theorem 1.3.4. *Let R be a right self-injective right purely infinite ring. Then the following holds:*

- *(i) A countable product of small R-modules is small.*
- *(ii) If there exists a system $(M_\alpha \mid \alpha < \kappa)$ of small R-modules such that the product $\prod_{\alpha<\kappa} M_\alpha$ is not small, then there exists an uncountable cardinal $\lambda < \kappa$ and a countably complete ultrafilter on λ.*

Proof. (i) Follows immediately from Lemma 1.3.3(iii).

(ii) Suppose $M = \prod_{\alpha \in I} M_\alpha$ is not a small module. Then by Lemma 1.3.3(iv) there exists $I_0 \subseteq I$ and a non-principal prime ideal $\mathcal{A}_0$ on I_0 which is closed under countable unions of sets by Lemma 1.3.3(ii). If we define $\mathcal{F} = \{I_0 \setminus A \mid A \in \mathcal{A}_0\}$ then $\mathcal{F}$ forms a countably complete non-principal ultrafilter on I_0. ∎

Before we prove our main result, which combines the last theorem and set-theoretical assertions, note that the hypothesis is consistent with ZFC by Theorem 1.1.4(iii).

Theorem 1.3.5. *Let R be a non-artinian right self-injective, right purely infinite ring. If we assume that there is no strongly inaccessible cardinal, then the class of all small R-modules is closed under direct products.*

Proof. If the product of an uncountable system of small modules is not small, then by Theorem 1.3.4(ii) there exists a countably complete ultrafilter on λ. Hence there exists a measurable cardinal $\mu \leq \lambda$ by Theorem 1.1.4(i), which is strongly inaccessible by Theorem 1.1.4(ii). ∎

Bibliography for Chapter 1

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2. Compactness in abelian categories

An object C of an abelian category $\mathcal{A}$ closed under coproducts is said to be *compact* if the covariant functor $\mathcal{A}(C, -)$ commutes with all coproducts, i.e. there is a canonical isomorphism between $\mathcal{A}(C, \bigoplus \mathcal{D})$ and $\bigoplus \mathcal{A}(C, \mathcal{D})$ in the category of abelian groups for every system of objects $\mathcal{D}$. The foundations for a systematic study of compact objects in the context of module categories were laid in 60's by Hyman Bass [1, p.54]. The introductory work on the theory of dually slender modules goes back to Rudolf Rentschler [13] and further research of compact objects has been motivated by progress in various branches of algebra such as the theory of representable equivalences of module categories [2, 3], the structure theory of graded rings [9], and almost free modules [14].

From the categorically dual point of view discussed in [7], commutativity of the contravariant functor on full module categories behaves a little bit differently. The equivalent characterizations of compactness split in this dual case into a hierarchy of strict implications dependent on the cardinality of commuting families. The strongest hypothesis assumes arbitrary cardinalities and it leads to the class of so called *slim* modules (also known as *strongly slender*), which is a subclass of the most general class of $\aleph_1$-slim modules (also called as *slender*), which involves only commutativity with countable families. It is proved in [7] that the cardinality of a non-zero slim module is greater than or equal to any measurable cardinal (and the presence of such cardinality is also a sufficient condition for existence of a non-zero slim module) and that the class of slim modules is closed under coproducts. Thus, the absence of a measurable cardinal ensures that there is at least one non-zero slim module and in fact, abundance of them. On the other hand, if there is a proper class of measurable cardinals then there is no such object like a non-zero slim module. This motivated the question in the dual setting, namely if the class of compact objects in full module categories (termed also as *dually slender* modules) is closed under products. Offering no surprise, set-theoretical assumptions have helped to establish the conclusion also in this case.

The main objective of this paper is to refine several results on compactness. The obtained improvement comes from transferring behavior of modules to the context of general abelian categories. In particular we provide a generalized description of classes of compact objects closed under products that was initially exposed for dually slender modules in [10]. Our main result shows that the class of all $\mathcal{C}$-compact objects of a reasonably generated category is closed under suitable set-theoretical assumption:

Theorem 2.3.4. *Let $\mathcal{A}$ be a $\prod\mathcal{C}$-compactly generated category, $\mathcal{M}$ a family of $\mathcal{C}$-compact objects of $\mathcal{A}$. If we assume that there is no strongly inaccessible cardinal, then every product of $\mathcal{C}$-compact objects is $\mathcal{C}$-compact.*

Note that this outcome is essentially based on the characterization of non-$\mathcal{C}$-compactness formulated in Theorem 2.1.5. Dually slender and self-small modules (which may be identically translated as self-dually slender) form naturally available instances of compact and self-compact objects (see e.g. [5] and [4]). For unexplained terminology we refer to [8, 12].

2.1 Compact objects in abelian categories

Let us recall basic categorical notions. A category with a zero object is called *additive* if for every finite system of objects there exist the product and coproduct which are canonically isomorphic, every Hom-set has a structure of abelian groups and the composition of morphisms is bilinear. An additive category is *abelian* if there exist a kernel and a cokernel for each morphism, monomorphisms are exactly kernels of some morphisms and epimorphisms are cokernels. A category is said to be *complete (cocomplete)* whenever it has all limits (colimits) of small diagrams. Finally, a cocomplete abelian category where all filtered colimits of exact sequences preserve exactness is *Ab5*. For further details on abelian category see e.g. [12].

From now on, we suppose that $\mathcal{A}$ is an abelian category closed under arbitrary coproducts and products. We shall use the terms *family* or *system* for any discrete diagram, which can be formally described as a mapping from a set of indices to a set of objects. Assume $\mathcal{M}$ is a family of objects in $\mathcal{A}$. Throughout the paper, the corresponding coproduct is designated $(\bigoplus \mathcal{M}, (\nu_M \mid M \in \mathcal{M}))$ and the product $(\prod \mathcal{M}, (\pi_M \mid M \in \mathcal{M}))$. We call ν_M and π_M as the *structural morphisms* of the coproduct and the product, respectively.

Suppose that $\mathcal{N}$ is a subfamily of $\mathcal{M}$. We call the coproduct $(\bigoplus \mathcal{N}, (\overline{\nu}_N \mid N \in \mathcal{N}))$ in $\mathcal{A}$ as the *subcoproduct* and dually the product $(\prod \mathcal{N}, (\overline{\pi}_N \mid N \in \mathcal{N}))$ as the *subproduct*. Note that there exist the unique canonical morphisms $\nu_{\mathcal{N}} \in \mathcal{A}(\bigoplus \mathcal{N}, \bigoplus \mathcal{M})$ and $\pi_{\mathcal{N}} \in \mathcal{A}(\prod \mathcal{M}, \prod \mathcal{N})$ given by the universal property of the colimit $\bigoplus \mathcal{N}$ and the limit $\prod \mathcal{N}$ satisfying $\nu_N = \nu_{\mathcal{N}} \circ \overline{\nu}_N$ and $\pi_N = \overline{\pi}_N \circ \pi_{\mathcal{N}}$ for each $N \in \mathcal{N}$, to which we refer as the *structural morphisms* of the subcoproduct and the subproduct over a subfamily $\mathcal{N}$ of $\mathcal{M}$, respectively. The symbol 1_M is used for the identity morphism of an object M.

We start with formulation of two introductory lemmas which collects several basic but important properties of the category $\mathcal{A}$. The lemmas express relations between the coproduct and product over a family using their structural morphisms.

Lemma 2.1.1. *Let $\mathcal{A}$ be a complete abelian category, $\mathcal{M}$ a family of objects of $\mathcal{A}$ with all coproducts and $\mathcal{N} \subseteq \mathcal{M}$. Then*

(i) There exist unique morphisms $\rho_{\mathcal{N}} \in \mathcal{A}(\bigoplus \mathcal{M}, \bigoplus \mathcal{N})$, $\mu_{\mathcal{N}} \in \mathcal{A}(\prod \mathcal{N}, \prod \mathcal{M})$ such that $\rho_{\mathcal{N}} \circ \nu_M = \overline{\nu}_M$, $\pi_M \circ \mu_{\mathcal{N}} = \overline{\nu}_M$ if $M \in \mathcal{N}$ and $\rho_{\mathcal{N}} \circ \nu_M = 0$, $\pi_M \circ \mu_{\mathcal{N}} = 0$ if $M \notin \mathcal{N}$.

(ii) For each $M \in \mathcal{M}$ there exist unique morphisms $\rho_M \in \mathcal{A}(\bigoplus \mathcal{M}, M)$ and $\mu_M \in \mathcal{A}(M, \prod \mathcal{M})$ such that $\rho_M \circ \nu_M = 1_M$, $\pi_M \circ \mu_M = 1_M$ and $\rho_M \circ \nu_N = 0$, $\pi_N \circ \mu_M = 0$ whenever $N \neq M$. If $\overline{\rho}_M$ and $\overline{\mu}_M$ denote the corresponding morphisms for $M \in \mathcal{N}$, then $\mu_{\mathcal{N}} \circ \overline{\mu}_N = \mu_N$ and $\rho_{\mathcal{N}} \circ \overline{\rho}_N = \rho_N$ for all $N \in \mathcal{N}$.

(iii) There exists a unique morphism $t \in \mathcal{A}(\bigoplus \mathcal{M}, \prod \mathcal{M})$ such that $\pi_M \circ t = \rho_M$ and $t \circ \nu_M = \mu_M$ for each $M \in \mathcal{M}$.

Proof. (i) It suffices to prove the existence and uniqueness of $\rho_{\mathcal{N}}$, the second claim has a dual proof.

Consider the diagram $(M \mid M \in \mathcal{M})$ with morphisms $(\widetilde{\nu}_M \mid M \in \mathcal{M}) \in \mathcal{A}(M, \bigoplus \mathcal{N})$ where $\widetilde{\nu}_M = \nu_M$ for $M \in \mathcal{N}$ and $\widetilde{\nu}_M = 0$ otherwise. Then the claim follows from the universal property of the coproduct $(\bigoplus \mathcal{M}, (\nu_M \mid M \in \mathcal{M}))$.

(ii) Note that for the choice $\mathcal{N} := \bigoplus(M) \simeq M$ we have $\overline{\nu}_M = 1_M$ and the claim follows from (i).

(iii) We obtain the requested morphism by the universal property of the product $(\prod \mathcal{M}, (\pi_M \mid M \in \mathcal{M}))$ applying on the cone $(\bigoplus \mathcal{M}, (\rho_M \mid M \in \mathcal{M}))$ that is provided by (ii). Dually, there exists a unique $t' \in \mathcal{A}(\bigoplus \mathcal{M}, \prod \mathcal{M})$ with $t' \circ \nu_M = \mu_M$. Then

$$\pi_M \circ (t \circ \nu_M) = \rho_M \circ \nu_M = 1_M = \pi_M \circ \mu_M = \pi_M \circ (t' \circ \nu_M),$$

hence $t \circ \nu_M = \mu_M$ by the uniqueness of the associated morphism μ_M and $t = t'$ because t' is the only one satisfying the condition for all $M \in \mathcal{M}$. ∎

We call the morphism $\rho_\mathcal{N}$ $(\mu_\mathcal{N})$ from (i) as the *associated morphism* to the structural morphism $\nu_\mathcal{M}$ $(\pi_\mathcal{M})$ over the subcoproduct (the subproduct) over $\mathcal{N}$. For the special case in (ii), the morphisms ρ_M (μ_M) from (ii) as the *associated morphism* to the structural morphism ν_M (π_M). Let the unique morphism t be called the *compatible coproduct-to-product* morphism over a family $\mathcal{M}$. Note that this morphism need not be a monomorphism, but it is in case $\mathcal{A}$ is an Ab5-category [12, Chapter 2, Corollary 8.10]. Moreover, t is an isomorphism if the family $\mathcal{M}$ is finite.

Lemma 2.1.2. *Let us use the notation from the previous lemma.*

(i) For the subcoproduct over $\mathcal{N}$, the composition of the structural morphism of the subcoproduct and its associated morphism is the identity. Dually for the subproduct over $\mathcal{N}$, the composition of the associated morphism of the subproduct and its structural morphism is the identity, i.e. $\rho_\mathcal{N} \circ \nu_\mathcal{N} = 1_{\bigoplus \mathcal{N}}$ and $\pi_\mathcal{N} \circ \mu_\mathcal{N} = 1_{\prod \mathcal{N}}$, respectively.

(ii) If $\bar{t} \in \mathcal{A}(\bigoplus \mathcal{N}, \prod \mathcal{N})$ and $t \in \mathcal{A}(\bigoplus \mathcal{M}, \prod \mathcal{M})$ denote the compatible coproduct
-to-product morphisms over $\mathcal{N}$ and $\mathcal{M}$ respectively, then the following diagram commutes:

$$\begin{array}{ccccc} \bigoplus\mathcal{N} & \xrightarrow{\nu_\mathcal{N}} & \bigoplus\mathcal{M} & \xrightarrow{\rho_\mathcal{N}} & \bigoplus\mathcal{N} \\ {\scriptstyle \bar{t}}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \bar{t}} \\ \prod\mathcal{N} & \xrightarrow{\mu_\mathcal{N}} & \prod\mathcal{M} & \xrightarrow{\pi_\mathcal{N}} & \prod\mathcal{N} \end{array}$$

(iii) Let κ be an ordinal, $(\mathcal{N}_\alpha \mid \alpha < \kappa)$ be a disjoint partition of $\mathcal{M}$ and for $\alpha < \kappa$ let $S_\alpha := \bigoplus \mathcal{N}_\alpha$, $P_\alpha := \prod \mathcal{N}_\alpha$. Denote the families of the limits and colimits like $\mathcal{S} := (S_\alpha \mid \alpha < \kappa)$, $\mathcal{P} := (P_\alpha \mid \alpha < \kappa)$. Then $\bigoplus \mathcal{M} \simeq \bigoplus \mathcal{S}$ and $\prod \mathcal{M} \simeq \prod \mathcal{P}$ where both isomorphisms are canonical, i.e. for every object $M \in \mathcal{M}$ the diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\nu_M} & S_\alpha \\ {\scriptstyle \nu_M}\downarrow & & \downarrow{\scriptstyle \nu_{S_\alpha}} \\ \bigoplus\mathcal{M} & \overset{\simeq}{\rightsquigarrow} & \bigoplus\mathcal{S} \end{array} \qquad \begin{array}{ccc} \prod\mathcal{P} & \overset{\simeq}{\rightsquigarrow} & \prod\mathcal{M} \\ {\scriptstyle \pi_{P_\alpha}}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ P_\alpha & \xrightarrow{\pi_M} & M \end{array}$$

Proof. (i) The equality $\rho_{\mathcal{N}} \circ \nu_{\mathcal{N}} = 1_{\bigoplus \mathcal{N}}$ is implied by the uniqueness of the universal morphism and the equalities $(\rho_{\mathcal{N}} \circ \nu_{\mathcal{N}}) \circ \overline{\nu}_N = \rho_{\mathcal{N}} \circ \nu_N = \overline{\nu}_N$ and $1_{\bigoplus \mathcal{N}} \circ \overline{\nu}_N = \overline{\nu}_N$ for all $N \in \mathcal{N}$. The equality $\pi_{\mathcal{N}} \circ \mu_{\mathcal{N}} = 1_{\prod \mathcal{N}}$ is dual.

(ii) We need to show that $t \circ \nu_{\mathcal{N}} = \mu_{\mathcal{N}} \circ \overline{t}$. For all $N \in \mathcal{N}$, $(\pi_N \circ t) \circ \nu_N = \rho_N \circ \nu_N = 1_N$ by Lemma 2.1.1(iii), (ii). But $\pi_N \circ \mu_N = 1_N$, hence $\mu_N = t \circ \nu_N$ by the uniqueness of μ_N. If $\overline{\mu}_N \in \mathcal{A}(N, \prod \mathcal{N})$ denotes the unique homomorphism ensured by Lemma 2.1.1(ii), then the last argument proves that $\overline{\mu}_N = \overline{t} \circ \overline{\nu}_N$. Thus

$$
\begin{aligned}
(t \circ \nu_{\mathcal{N}}) \circ \overline{\nu}_N = t \circ (\nu_{\mathcal{N}} \circ \overline{\nu}_N) = t \circ \nu_N = \mu_N = \mu_{\mathcal{N}} \circ \overline{\mu}_N = \mu_{\mathcal{N}} \circ (\overline{t} \circ \overline{\nu}_N) = \\
= (\mu_{\mathcal{N}} \circ \overline{t}) \circ \overline{\nu}_N
\end{aligned}
$$

and the claim follows from the universal property of the coproduct $(\bigoplus \mathcal{N}, (\overline{\nu}_N \mid N \in \mathcal{N}))$. The dual argument proves that $\pi_{\mathcal{N}} \circ t = \overline{t} \circ \rho_{\mathcal{N}}$.

(iii) A straightforward consequence of the universal properties of the coproducts and products. ∎

Let us suppose that M is an object in $\mathcal{A}$ and $\mathcal{N}$ is a system of objects of $\mathcal{A}$. As the functor $\mathcal{A}(M, -)$ on any additive category maps into Hom-sets with a structure of abelian groups we can define a mapping

$$
\Psi_{\mathcal{N}} : \bigoplus (\mathcal{A}(M, N) \mid N \in \mathcal{N}\ ) \to \mathcal{A}(M, \bigoplus \mathcal{N})
$$

in the following way:

For a family of mappings $\varphi = (\varphi_N \mid N \in \mathcal{N})$ from $\bigoplus (\mathcal{A}(M, N) \mid N \in \mathcal{N})$ let us denote by $\mathcal{F}$ a finite subfamily such that $\varphi_N = 0$ whenever $N \notin \mathcal{F}$ and let $\tau \in \mathcal{A}(M, \prod \mathcal{N})$ be the unique morphism given by the universal property of the product $(\prod \mathcal{N}, (\pi_N \mid N \in \mathcal{F}))$ applied on the cone $(M, (\varphi_N \mid N \in \mathcal{N}))$, i.e. $\pi_N \circ \tau = \varphi_N$ for every $N \in \mathcal{N}$. Then

$$
\Psi_{\mathcal{N}}(\varphi) = \nu_{\mathcal{F}} \circ \nu^{-1} \circ \pi_{\mathcal{F}} \circ \tau
$$

where $\nu \in \mathcal{A}(\bigoplus \mathcal{F}, \prod \mathcal{F})$ denotes the canonical isomorphism.

Note that the definition $\Psi_{\mathcal{N}}(\varphi)$ does not depend on the choice of $\mathcal{F}$. Recall an elementary observation which plays a key role in the definition of a compact object.

Lemma 2.1.3. *The mapping $\Psi_{\mathcal{N}}$ is a monomorphism in the category of abelian groups for every family of objects $\mathcal{N}$.*

Proof. If $\Psi_{\mathcal{N}}(\sigma) = 0$, then $\sigma = (\rho_N \circ \sigma)_N = (0)_N$, hence $\ker(\Psi_{\mathcal{N}}) = 0$. ∎

Applying the currently introduced categorical tools we are ready to present the central notion of the paper. Let $\mathcal{C}$ be a subclass of objects of $\mathcal{A}$. An object M is said to be $\mathcal{C}$-*compact* if $\Psi_{\mathcal{N}}$ is an isomorphism for every family $\mathcal{N} \subseteq \mathcal{C}$, M is *compact* in the category $\mathcal{A}$ if it is $\mathcal{A}^o$-compact for the class of all objects $\mathcal{A}^o$, and M is *self-compact* if it is $\{M\}$-compact. Note that every object is $\{0\}$-compact.

First we formulate an elementary criterion of identifying $\mathcal{C}$-compact object.

Lemma 2.1.4. *If M is an object and $\mathcal{C}$ a class of objects in $\mathcal{A}$, then the following are equivalent:*

(1) M is $\mathcal{C}$-compact,

(2) for every $\mathcal{N} \subseteq \mathcal{C}$ and $f \in \mathcal{A}(M, \bigoplus \mathcal{N})$ there exists a finite subsystem $\mathcal{F} \subseteq \mathcal{N}$ and a morphism $f' \in \mathcal{A}(M, \bigoplus \mathcal{F})$ such that $f = \nu_{\mathcal{F}} \circ f'$.

(3) for every $\mathcal{N} \subseteq \mathcal{C}$ and every $f \in \mathcal{A}(M, \bigoplus \mathcal{N})$ there exists a finite subsystem $\mathcal{F}$ contained in $\mathcal{N}$ such that $f = \sum_{F \in \mathcal{F}} \nu_F \circ \rho_F \circ f$.

Proof. (1) → (2): Let $\mathcal{N} \subseteq \mathcal{C}$ and $f \in \mathcal{A}(M, \bigoplus \mathcal{N})$. Then there exists a $\Psi_{\mathcal{N}}$-preimage φ of f, hence there can be chosen a finite subsystem $\mathcal{F} \subseteq \mathcal{N}$ such that

$$f = \Psi_{\mathcal{N}}(\varphi) = \nu_{\mathcal{F}} \circ \nu^{-1} \circ \pi_{\mathcal{F}} \circ \tau,$$

where we use the notation from the definition of the mapping $\Psi_{\mathcal{N}}$. Now it remains to put $f' = \rho_{\mathcal{F}} \circ f$ and utilize Lemma 2.1.1(ii) to verify that

$$\nu_{\mathcal{F}} \circ f' = \nu_{\mathcal{F}} \circ \rho_{\mathcal{F}} \circ f = \nu_{\mathcal{F}} \circ \rho_{\mathcal{F}} \circ \nu_{\mathcal{F}} \circ 1_{\oplus \mathcal{F}} \circ \nu^{-1} \circ \pi_{\mathcal{F}} \circ \tau = f.$$

(2) → (3): Since $\rho_{\mathcal{F}} \circ \nu_{\mathcal{F}} = 1_{\bigoplus \mathcal{F}}$ by Lemma 2.1.1(ii), we obtain that

$$\nu_{\mathcal{F}} \circ \rho_{\mathcal{F}} \circ f = \nu_{\mathcal{F}} \circ \rho_{\mathcal{F}} \circ \nu_{\mathcal{F}} \circ f' = \nu_{\mathcal{F}} \circ f' = f.$$

Moreover, $\nu_{\mathcal{F}} \circ \rho_{\mathcal{F}} = \sum_{F \in \mathcal{F}} \nu_F \circ \rho_F$, hence

$$f = \nu_{\mathcal{F}} \circ \rho_{\mathcal{F}} \circ f = \sum_{F \in \mathcal{F}} \nu_F \circ \rho_F \circ f.$$

(3) → (1): If we put $\varphi_F := \rho_F \circ f$ for $F \in \mathcal{F}$ and $\varphi_N := 0$ for $N \notin \mathcal{F}$ and take $\varphi := (\varphi_N \mid N \in \mathcal{N})$, then it is easy to see that $f = \Psi_{\mathcal{N}}(\varphi)$ hence $\Psi_{\mathcal{N}}$ is surjective. ∎

Now, we can prove a characterization, which generalizes equivalent conditions well known for the categories of modules. Note that it will play similarly important role for the categorical approach to compactness as in the special case of module categories.

Theorem 2.1.5. *The following conditions are equivalent for an object M and a class of objects $\mathcal{C}$:*

(1) M is not $\mathcal{C}$-compact,

(2) there exists a countably infinite system $\mathcal{N}_\omega$ of objects from $\mathcal{C}$ and $\varphi \in \mathcal{A}(M, \bigoplus \mathcal{N}_\omega)$ such that $\rho_N \circ \varphi \neq 0$ for every $N \in \mathcal{N}_\omega$,

(3) for every system $\mathcal{G}$ of $\mathcal{C}$-compact objects and every epimorphism $e \in \mathcal{A}(\bigoplus \mathcal{G}, M)$ there exists a countable subsystem $\mathcal{G}_\omega \subseteq \mathcal{G}$ such that $f^c \circ e \circ \nu_{\mathcal{G}_\omega} \neq 0$ for the cokernel f^c of every morphism $f \in \mathcal{A}(F, M)$ where F is a $\mathcal{C}$-compact object.

Proof. (1) → (2): Let $\mathcal{N}$ be a system of objects from $\mathcal{C}$ for which there exists a morphism $\varphi \in \mathcal{A}(M, \bigoplus \mathcal{N}) \setminus \operatorname{im}(\Psi_{\mathcal{N}})$. Then it is enough to take $\mathcal{N}_\omega$ as any countable subsystem of the infinite system $(N \in \mathcal{N} \mid \rho_N \circ \varphi \neq 0)$.

(2) $\to$ (3) Let $\mathcal{G}$ be a family of $\mathcal{C}$-compact objects and $e \in \mathcal{A}(\bigoplus \mathcal{G}, M)$ an epimorphism. If $N \in \mathcal{N}_\omega$, then $(\rho_N \circ \varphi) \circ e \neq 0$, hence by the universal property of the coproduct $\bigoplus \mathcal{G}$ applied on the cone $(N, (\rho_N \circ \varphi \circ e \circ \nu_G \mid G \in \mathcal{G}))$ there exists $G_N \in \mathcal{G}$ such that $\mathcal{A}(G_N, N) \ni \rho_N \circ \varphi \circ e \circ \nu_{G_N} \neq 0$. Put $\mathcal{G}_\omega = (G_N \mid N \in \mathcal{N}_\omega)$, where every object from the system $\mathcal{G}$ is taken at most once, i.e. we have a canonical monomorphism $\nu_{\mathcal{G}_\omega} \in \mathcal{A}(\bigoplus \mathcal{G}_\omega, \bigoplus \mathcal{G})$.

Assume to the contrary that there exist a $\mathcal{C}$-compact object F and a morphism $f \in \mathcal{A}(F, M)$ such that $f^c \circ e \circ \nu_{\mathcal{G}_\omega} = 0$ where $f^c \in \mathcal{A}(M, \mathrm{cok(f)})$ is the cokernel of f. Let $N \in \mathcal{N}_\omega$ and, furthermore, assume that $\rho_N \circ \varphi \circ f = 0$. Then the universal property of the cokernel ensures the existence of a morphism $\alpha \in \mathcal{A}(\mathrm{cok(f)}, \mathrm{N})$ such that $\alpha \circ f^c = \rho_N \circ \varphi$, i.e. that commutes the diagram:

$$\begin{array}{ccccccc} & & & & F & & \\ & & & & \downarrow{\scriptstyle f} & & \\ \bigoplus\mathcal{G}_\omega & \xrightarrow{\nu_{\mathcal{G}_\omega}} & \bigoplus\mathcal{G} & \xrightarrow{e} & M & \xrightarrow{\varphi} & \bigoplus\mathcal{N}_\omega \\ & & & & \downarrow{\scriptstyle f^c} & & \downarrow{\scriptstyle \rho_N} \\ & & & & \mathrm{cok(f)} & \xrightarrow{\alpha} & N \end{array}$$

Thus $(\rho_N \circ \varphi) \circ e \circ \nu_{\mathcal{G}_\omega} = (\alpha \circ f^c) \circ e \circ \nu_{\mathcal{G}_\omega} = 0$, which contradicts the construction of $\mathcal{G}_\omega$. We have proved that $\rho_N \circ (\varphi \circ f) \neq 0$ for each $N \in \mathcal{N}_\omega$, hence $\varphi \circ f \in \mathcal{A}(F, \bigoplus \mathcal{N}) \setminus \mathrm{im}(\Psi_{\mathcal{N}_\omega})$. We get the contradiction with the assumption that F is $\mathcal{C}$-compact, thus $f^c \circ e \circ \nu_{\mathcal{G}_\omega} \neq 0$.

(3) $\to$ (1): If M is $\mathcal{C}$-compact itself, then the system $\mathcal{G} = (M)$ and the identity map e on M are the counterexamples for the condition (3). ∎

Corollary 2.1.6. *If $\mathcal{A}$ contains injective envelopes $E(U)$ for all objects $U \in \mathcal{C}$, then an object M is not compact if and only if there exists a (countable) system of injective envelopes $\mathcal{E}$ in $\mathcal{A}$ of objects of $\mathcal{C}$ for which $\Psi_{\mathcal{N}}$ is not surjective for some subsystem $\mathcal{N}$ of $\mathcal{C}$.*

Proof. By the previous proposition, it suffices to consider the composition of $\varphi \in \mathcal{A}(M, \bigoplus \mathcal{N}_\omega) \setminus \mathrm{im}\, \Psi_{\mathcal{N}_\omega}$ where $\mathcal{N}_\omega$ implies that M is not $\mathcal{C}$-compact together with the canonical morphism $\iota \in \mathcal{A}(\bigoplus \mathcal{N}_\omega, \bigoplus \mathcal{E})$, where we put $\mathcal{E} := (E(N) \mid N \in \mathcal{N}_\omega)$. ∎

2.2 Classes of compact objects

Let us denote by $\mathcal{A}$ a complete abelian category and $\mathcal{C}$ a class of some objects of $\mathcal{A}$. First, notice that several closure properties of the class of $\mathcal{C}$-compact objects are identical to the closure properties of classes of dually slender modules since these follow from the fact that the contravariant functor $\mathcal{A}(-, \bigoplus \mathcal{N})$ commutes with finite coproducts and it is left exact. We present a detailed proof of the fact that the class of all $\mathcal{C}$-compact objects is closed under finite coproducts and cokernels using Theorem 2.1.5.

Lemma 2.2.1. *The class of all $\mathcal{C}$-compact objects is closed under finite coproducts and all cokernels of morphisms $\alpha \in \mathcal{A}(M, C)$ where C is $\mathcal{C}$-compact and M is arbitrary.*

Proof. Suppose that $\bigoplus_{i=1}^{n} M_i$ is not $\mathcal{C}$-compact. Then by Theorem 2.1.5 there exist a sequence $(N_i \mid i < \omega)$ of objects and a morphism $\varphi \in \mathcal{A}(\bigoplus_{i=1}^{n} M_i, \bigoplus_{j<\omega} N_j)$ such that $\rho_j \circ \varphi \neq 0$ for each $j < \omega$. Since $\omega = \bigcup_{i=1}^{n}\{j < \omega \mid \rho_j \circ \varphi \circ \nu_i \neq 0\}$ there exists i for which the set $\{j < \omega \mid \rho_j \circ \varphi \circ \nu_i \neq 0\}$ is infinite, hence M_i is not $\mathcal{C}$-compact by applying Theorem 2.1.5.

Similarly, suppose that α^c is the cokernel of $\alpha \in \mathcal{A}(M, C)$, where $\mathrm{cok}(\alpha)$ is not $\mathcal{C}$-compact, and $\varphi \in \mathcal{A}(\mathrm{cok}(\alpha), \bigoplus_{\mathrm{j}<\omega} \mathrm{N_j})$ for $(N_i \mid i < \omega)$ satisfies $\rho_j \circ \varphi \neq 0$ for every $j < \omega$. Then, obviously, $\rho_j \circ \varphi \circ \pi \neq 0$ for each $j < \omega$ and so C is not $\mathcal{C}$-compact again by Theorem 2.1.5. ∎

Lemma 2.2.2. *If $\mathcal{M}$ is an infinite system of objects in $\mathcal{A}$ satisfying that for each $M \in \mathcal{M}$ there exists $C \in \mathcal{C}$ such that $\mathcal{A}(M, C) \neq 0$, then $\bigoplus \mathcal{M}$ is not $\mathcal{C}$-compact.*

Proof. It is enough to take $\mathcal{N} = (C_M \mid M \in \mathcal{M})$ where $\mathcal{A}(M, C_M) \neq 0$ and apply Theorem 2.1.5(2)→(1). ∎

We obtain the following consequence:

Corollary 2.2.3. *Let $\mathcal{M}$ be a system of objects of $\mathcal{A}$. Then $\bigoplus \mathcal{M}$ is $\mathcal{C}$-compact if and only if the system $\{M \in \mathcal{M} \mid \exists C \in \mathcal{C} : \mathcal{A}(M, C) \neq 0\}$ is finite.*

Proof. Put $\mathcal{K} = \{M \in \mathcal{M} \mid \exists C \in \mathcal{C} : \mathcal{A}(M, C) \neq 0\}$. Then we have the canonical isomorphism $\mathcal{A}(\bigoplus \mathcal{K}, \bigoplus \mathcal{N}) \cong \mathcal{A}(\bigoplus \mathcal{M}, \bigoplus \mathcal{N})$ for every system $\mathcal{N}$ of objects of $\mathcal{C}$, hence $\bigoplus \mathcal{K}$ is $\mathcal{C}$-compact if and only if $\bigoplus \mathcal{M}$ is so. Furthermore, $\bigoplus \mathcal{K}$ is not $\mathcal{C}$-compact by Lemma 2.2.2 whenever $\mathcal{K}$ is infinite.

If $\bigoplus \mathcal{M}$ is $\mathcal{C}$-compact, then $\bigoplus \mathcal{K}$ and every $M \in \mathcal{M}$ is $\mathcal{C}$-compact by Lemma 2.2.1, hence $\mathcal{K}$ is finite. On the other hand, if $\mathcal{K}$ is finite and all objects $M \in \mathcal{M}$ are $\mathcal{C}$-compact, then $\bigoplus \mathcal{K}$ is $\mathcal{C}$-compact by Lemma 2.2.1, hence $\bigoplus \mathcal{M}$ is $\mathcal{C}$-compact as well. ∎

Let us confirm that relativized compactness behaves well under taking finite unions of classes and verify with an example that this closure property can not be extended to an infinite case.

Lemma 2.2.4. *Let $\mathcal{C}_1, \dots, \mathcal{C}_n$ be a finite number of classes of objects and let $C \in \mathcal{A}$. Then C is $\bigcup_{i=1}^{n} \mathcal{C}_i$-compact if and only it C is $\mathcal{C}_i$-compact for every $i \leq n$.*

Proof. The direct implication is trivial. If C is not $\bigcup_{i=1}^{n} \mathcal{C}_i$-compact, there exists a sequence $(C_i \mid i < \omega)$ of objects of $\bigcup_{i=1}^{n} \mathcal{C}_i$ with a morphism $\varphi \in \mathcal{A}(C, \bigoplus_{j<\omega} C_j)$ such that $\rho_j \circ \varphi \neq 0$ for every $j < \omega$ by Theorem 2.1.5(1)→(2). Since there exists $k \leq n$ for which infinitely many C_i's belong to $\mathcal{C}_j$ we can see that C is not $\mathcal{C}_k$-compact by Theorem 2.1.5(2)→(1). ∎

Example 2.2.5. *Let R be a ring over which there is an infinite set of pairwise non-isomorphic simple right modules. Any non-Artinian VNR ring serves as an example where the property holds. Suppose that $\mathcal{A}$ is the full subcategory of category consisting of all semisimple right modules, which is generated by all simple modules. Fix a countable sequence S_i, $i < \omega$, of pairwise non-isomorphic simple modules. Then the module $\bigoplus_{i<\omega} S_i$ is $\{S_i\}$-compact for each i but it is not $\bigcup_{i<\omega}\{S_i\}$-compact.*

Recall that an object A is *cogenerated* by $\mathcal{C}$ it there exist a system $\mathcal{N}$ of objects of $\mathcal{C}$ and a monomorphism in $\mathcal{A}A(A, \prod \mathcal{N})$. Relative compactness of an object is preserved if we close the class under all cogenerated objects.

Lemma 2.2.6. *Let $Cog(\mathcal{C})$ be the class of all objects cogenerated by $\mathcal{C}$. Then every $\mathcal{C}$-compact object is $Cog(\mathcal{C})$-compact.*

Proof. Let us suppose that an object C is not $Cog(\mathcal{C})$-compact and fix a sequence $\mathcal{B} := (B_i \mid i < \omega)$ of objects of $Cog(\mathcal{C})$ and a morphism $\varphi \in \mathcal{A}(C, \bigoplus \mathcal{B})$ such that $\rho_j \circ \varphi \neq 0$ for each $j < \omega$ which exist by Theorem 2.1.5(1)→(2). Since $Cog(\mathcal{C})$ is closed under subobjects we may suppose that $\rho_j \circ \varphi$ are epimorphisms. Furthermore, for every $j < \omega$ there exists a non-zero morphism $\tau_j \in \mathcal{A}(B_j, T_j)$ with $T_j \in \mathcal{C}$. Form the sequence $\mathcal{T} := (T_i \mid i < \omega)$. Let τ be the uniquely defined morphism from $\mathcal{A}(\bigoplus \mathcal{B}, \bigoplus \mathcal{T})$ satisfying $\tau \circ \nu_j = \overline{\nu}_j \circ \tau_j$. Then $\overline{\rho}_j \circ \tau \circ \nu_i = \overline{\rho}_j \circ \overline{\nu}_i \circ \tau_i$ which is equal to τ_i whenever $i = j$ and it is zero otherwise, hence $\overline{\rho}_i \circ \tau \circ \nu_i \circ \rho_i = \overline{\rho}_i \circ \tau$ by the universal property of $\bigoplus \mathcal{B}$. Finally, since $\rho_i \circ \varphi$ is an epimorphism and τ_i is non-zero, $\tau_i \circ \rho_i \circ \varphi \neq 0$ and so

$$\overline{\rho}_j \circ \tau \circ \varphi = \overline{\rho}_i \circ \tau \circ \nu_i \circ \rho_i \circ \varphi = \overline{\rho}_i \circ \overline{\nu}_i \circ \tau_i \circ \rho_i \circ \varphi = \tau_i \circ \rho_i \circ \varphi \neq 0$$

for every $i < \omega$. Thus the composition $\tau \circ \varphi$ implies that C is not $\mathcal{C}$-compact again by Theorem 2.1.5(2)→(1). ∎

A complete abelian category $\mathcal{A}$ is *$\mathcal{C}$-steady*, if there exists an $\mathcal{A}$-projective $\mathcal{C}$-compact object G which finitely generates the class of all $\mathcal{C}$-compact objects, i.e. for every $\mathcal{C}$-compact object F there exists $n \in \mathbb{N}$ and an epimorphism $h \in \mathcal{A}(G^{(n)}, F)$. $\mathcal{A}$ is said to be *steady* whenever it is an $\mathcal{A}^o$-steady category for the class $\mathcal{A}^o$ of all objects of $\mathcal{A}$.

Example 2.2.7. *Let R be a ring and let $\mathcal{A} = \mathbf{Mod}$-$R$ denote the category of all right R-modules. Recall that a module $M \in \mathcal{A}$ is called small if it is compact in the category $\mathcal{A}$. If R is a right steady ring, i.e. a ring over which every small module is finitely generated (for details see e.g. [5]), then $\mathcal{A}$ is a steady category.*

Furthermore, in [9, Theorem 1.7] it was proved that a locally Noetherian Grothendieck category is steady.

Recall that an object A is *simple* if for every $B \in \mathcal{A}^o$, any non-zero morphism from $\mathcal{A}(A, B)$ is a monomorphism and an object is *semisimple* if it is isomorphic to a coproduct of simple objects. A category is called *semisimple* if all its objects are semisimple. We characterize steadiness of semisimple categories.

Lemma 2.2.8. *Let $\mathcal{A}$ be a semisimple category, $\mathcal{S}$ be a representative class of simple objects and suppose that every object $S \in \mathcal{S}$ is compact. Then $\mathcal{A}$ is steady if and only if $\mathcal{S}$ is finite.*

Proof. Note that all objects of $\mathcal{A}$ are projective and if and any nonzero $\varphi \in \mathcal{A}(S, T)$ for $S, T \in \mathcal{S}$ is an isomorphism. Moreover, if $\mathcal{S}'$ is a subsystem of $\mathcal{S}$ then $\bigoplus \mathcal{S}'$ is compact if and only if $\mathcal{S}'$ is finite.

Suppose that $\mathcal{A}$ is steady. Then there exists a compact object A isomorphic to $\bigoplus \mathcal{S}'$ for a finite system of simple objects $\mathcal{S}'$, which finitely generates the class of all compact objects, in particular all simple objects. Since $\mathcal{A}(A, S) \neq 0$, there exists $i \in I$ such that $S_i \cong S$ for each $S \in \mathcal{S}$, hence I is finite. If $\mathcal{S}$ is, on the other hand, finite, it is easy to see that $A = \bigoplus \mathcal{S}$ finitely generates $\mathcal{A}$, and so $\mathcal{A}$ is steady. ∎

Example 2.2.9. *Let $\mathcal{A}$ be a category of semisimple right modules over a ring with an infinite set of pairwise non-isomorphic simple right modules as in Example 2.2.5. Then $\mathcal{A}$ is a semisimple category which is not steady by Lemma 2.2.8. On the other hand, if the ring R is right steady, which is true for example for each countable commutative VNR ring, then the category of all right R-modules is* **Mod**-R *steady.*

We say that a complete abelian category $\mathcal{A}$ is $\prod\mathcal{C}$-*compactly generated* if there is a set $\mathcal{G}$ of objects of $\mathcal{A}$ that generates $\mathcal{A}$ and the product of any system of objects in $\mathcal{G}$ is $\mathcal{C}$-compact. Note that $\mathcal{G}$ consists only of $\mathcal{C}$-compact objects.

Lemma 2.2.10. *If E is a $\mathcal{C}$-compact injective generator of $\mathcal{A}$ such that there exists a monomorphism $m \in \mathcal{A}(E^{(\omega)}, E)$, then $\mathcal{A}$ is $\prod\mathcal{C}$-compactly generated.*

Proof. It follows immediately from Theorem 2.1.5(3)→(1). ∎

Example 2.2.11. *Let R be a right self-injective, purely infinite ring. Then $E := R$ is an injective generator and there is an embedding $0 \to R^{(\omega)} \to R$. By the previous lemma, the category* **Mod**-R *is $\prod\mathcal{C}$-generated.*

2.3 Products of compact objects

We start the section by an observation that the cokernel of the compatible coproduct-to-product morphism over a countable family is $\mathcal{C}$-compact where $\mathcal{C}$ is a class of objects in an abelian category $\mathcal{A}$. This initial step will be later extended to families regardless of their cardinality.

Lemma 2.3.1. *Let $\mathcal{A}$ be $\prod\mathcal{C}$-compactly generated and let $\mathcal{M}$ be a countable family of objects in $\mathcal{A}$. If $t \in \mathcal{A}\left(\bigoplus\mathcal{M}, \prod\mathcal{M}\right)$ is the compatible coproduct-to-product morphism, then* $\mathrm{cok(t)}$ *is $\mathcal{C}$-compact.*

Proof. As for a finite $\mathcal{M}$ there is nothing to prove, suppose that $\mathcal{M} = (M_n \mid n < \omega)$. Let $\mathcal{G}$ be a family of objects of $\mathcal{A}$ such that every product of a system of objects in $\mathcal{G}$ is $\mathcal{C}$-compact and let $e \in \mathcal{A}(\bigoplus\mathcal{G}, \prod\mathcal{M})$ be an epimorphism, which exists by the hypothesis. Let t^c be the cokernel of t. Then both t^c and $e' := t^c \circ e$ are epimorphisms and $t^c \circ t = 0$. We will show that for every countable subsystem $\mathcal{G}_\omega$ of $\mathcal{G}$ there exists a $\mathcal{C}$-compact object F and a morphism $f \in \mathcal{A}(F, \mathrm{cok(t)})$ such that $\mathcal{A}\left(\bigoplus\mathcal{G}_\omega, \mathrm{cok(f)}\right) \ni f^c \circ e' \circ \nu_{\mathcal{G}_\omega} = 0$ for the cokernel $f^c \in \mathcal{A}(\mathrm{cok(t)}, \mathrm{cok(f)})$. By Theorem 2.1.5 this yields that $\mathrm{cok(t)}$ is $\mathcal{C}$-compact.

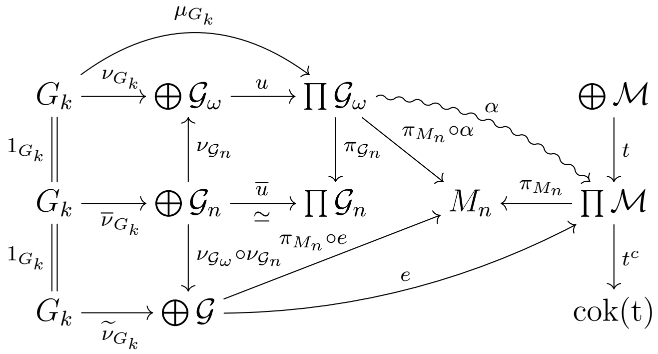
Since for any finite $\mathcal{G}_\omega \subseteq \mathcal{G}$ it is enough to take $F := \bigoplus\mathcal{G}_\omega$ and $f := e' \circ \nu_{\mathcal{G}_\omega}$, we may fix a countably infinite family $\mathcal{G}_\omega = (G_n \mid n < \omega) \subseteq \mathcal{G}$. For each $n < \omega$ put $\mathcal{G}_n = (G_i \mid i \leq n)$ and let $\pi_{\mathcal{G}_n} \in \mathcal{A}\left(\prod\mathcal{G}_\omega, \prod\mathcal{G}_n\right)$ and $\pi_{M_n} \in \mathcal{A}\left(\prod\mathcal{M}, M_n\right)$ denote the structural morphisms, and let $\overline{u}^{-1} \in \mathcal{A}\left(\prod\mathcal{G}_n, \bigoplus\mathcal{G}_n\right)$ be the inverse of the compatible coproduct-to-product morphism $\overline{u} \in \mathcal{A}\left(\bigoplus\mathcal{G}_n, \prod\mathcal{G}_n\right)$ that exists for finite families.

First, let us fix $n \in \omega$ and we prove that $\nu_{G_k} = \nu_{\mathcal{G}_n} \circ \overline{u}^{-1} \circ \pi_{\mathcal{G}_n} \circ \mu_{G_k}$ for each $k \leq n$. Let $\overline{\nu}_{G_k} \in \mathcal{A}(G_k, \bigoplus\mathcal{G}_n)$ be the structural morphism of the coproduct $\bigoplus\mathcal{G}_n$, $u \in \mathcal{A}(\bigoplus\mathcal{G}_\omega, \prod\mathcal{G}_\omega)$ the canonical coproduct-to-product morphism, and $\overline{\mu}_{G_k} \in \mathcal{A}(G_k, \prod\mathcal{G}_n)$ the associated morphism to the product $\prod\mathcal{G}_n$. Since $\nu_{\mathcal{G}_n} \circ \overline{\nu}_{G_k} = \nu_{G_k}$

and $\mu_{G_k} = u \circ \nu_{G_k}$ (by Lemma 2.1.1(iii)), then we immediately infer the following equalities from Lemma 2.1.2(ii) :

$$\begin{aligned}\nu_{G_k} &= \nu_{\mathcal{G}_n} \circ \overline{\nu}_{G_k} = \nu_{\mathcal{G}_n} \circ (\overline{u}^{-1} \circ \overline{u} \circ \overline{\nu}_{G_k}) = (\nu_{\mathcal{G}_n} \circ \overline{u}^{-1}) \circ \overline{u} \circ \overline{\nu}_{G_k} = \\ &= (\nu_{\mathcal{G}_n} \circ \overline{u}^{-1}) \circ (\pi_{\mathcal{G}_n} \circ u \circ \nu_{\mathcal{G}_n}) \circ \overline{\nu}_{G_k} = (\nu_{\mathcal{G}_n} \circ \overline{u}^{-1}) \circ \pi_{\mathcal{G}_n} \circ u \circ \nu_{G_k} = \\ &= (\nu_{\mathcal{G}_n} \circ \overline{u}^{-1}) \circ \pi_{\mathcal{G}_n} \circ \mu_{G_k}\end{aligned}$$

Now, if we employ the universal property of the product $(\prod \mathcal{M}, (\pi_{M_n} \mid n < \omega))$ with respect to the cone $(\prod \mathcal{G}_\omega, (\pi_{M_n} \circ e \circ \nu_{\mathcal{G}_n} \circ \overline{u}^{-1} \circ \pi_{\mathcal{G}_n} \mid n < \omega))$, then there exists a unique morphism $\alpha \in \mathcal{A}(\prod \mathcal{G}_\omega, \prod \mathcal{M})$ such that the middle non-convex pentagon in the following diagram commutes :



Then for each $k \leq n$ we deduce that

$$\begin{aligned}\pi_{M_n} \circ (\alpha \circ \mu_{G_k} - e \circ \widetilde{\nu}_{G_k}) &= \pi_{M_n} \circ (\alpha \circ \mu_{G_k} - e \circ \nu_{\mathcal{G}_\omega} \circ \nu_{\mathcal{G}_n} \circ \overline{u}^{-1} \circ \pi_{\mathcal{G}_n} \circ \mu_{G_k}) = \\ &= (\pi_{M_n} \circ \alpha - \pi_{M_n} \circ e \circ \nu_{\mathcal{G}_\omega} \circ \nu_{\mathcal{G}_n} \circ \overline{u}^{-1} \circ \pi_{\mathcal{G}_n}) \circ \mu_{G_k} = 0\end{aligned}$$

and $\alpha \circ \mu_{G_n} = e \circ \widetilde{\nu}_{G_n}$ for every $n < \omega$ is yielded as the number n was fixed. Note that $\prod \mathcal{G}_\omega$ is $\mathcal{C}$-compact by the hypothesis. Now, consider f^c the cokernel of the morphism $f = t^c \circ \alpha \in \mathcal{A}(\prod \mathcal{G}_\omega, \text{cok(t)})$. Then

$$\begin{aligned}0 &= f^c \circ t^c \circ (e \circ \widetilde{\nu}_{G_n} - \alpha \circ \mu_{G_n}) = \\ &= f^c \circ t^c \circ e \circ \widetilde{\nu}_{G_n} - f^c \circ t^c \circ \alpha \circ \mu_{G_n} = f^c \circ e' \circ \widetilde{\nu}_{G_n}\end{aligned}$$

hence $0 = f^c \circ e' \circ \widetilde{\nu}_{\mathcal{G}_n} = f^c \circ e' \circ \nu_{\mathcal{G}_\omega} \circ \nu_{\mathcal{G}_n}$ for every $n < \omega$, which finishes the proof. ∎

Let $\mathcal{I}$ be a non-empty subset of $\mathcal{P}(X)$, the power set of a set X. We recall that $\mathcal{I}$ is said to be

- *an ideal* if it is closed under subsets (i.e. if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$) and under finite unions, (i.e. if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$),
- *a prime ideal* if it is a proper ideal and for all subsets A, B of X, $A \cap B \in \mathcal{I}$ implies that $A \in \mathcal{I}$ or $B \in \mathcal{I}$,
- *a principal ideal* if there exists a set $Y \subseteq X$ such that $\mathcal{I} = \mathcal{P}(Y)$, the power set of Y.

The set $\mathcal{I} \mid Y = \{Y \cap A \mid A \in \mathcal{I}\}$ is called *a trace* of an ideal $\mathcal{I}$ on Y.

Note that the trace of an ideal is also an ideal and that $\mathcal{I}$ is a prime ideal if and only if for every $A \subseteq X$, $A \in \mathcal{I}$ or $X \setminus A \in \mathcal{I}$. Moreover, a principal prime ideal on X is of the form $\mathcal{P}(X \setminus \{x\})$ for some $x \in X$.

Dually, a set $\mathcal{F} \neq \emptyset$ of non-empty subsets of X is said to be

- *a filter* if it is closed under finite intersections and supersets,
- *an ultrafilter* if it is a filter which is not properly contained in any other filter on X,

We say that a filter $\mathcal{F}$ is λ*-complete*, if $\bigcap \mathcal{G} \in \mathcal{F}$ for every subset $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| < \lambda$ and $\mathcal{F}$ is *countably complete*, if it is ω_1-complete.

Note that there is a one-to-one correspondence between ultrafilters and prime ideals on X defined by $\mathcal{I} \mapsto \mathcal{P}(X) \setminus \mathcal{I}$ for an ideal $\mathcal{I}$.

Let $\mathcal{M}$ be a family of objects. Then there exits a set of indices I such that $\mathcal{M} = (M_i \mid i \in I)$, i.e. there exists a bijection between objects of the family $\mathcal{M}$ and the set I. Since families of objects seem to be more convenient for a reader than using indexed sets, we will keep the notation. Thus in the sequel, we will understand families as sets in the described sense since we need to apply set-theoretical properties.

Now, we are able to generalize [10, Lemma 3.3].

Proposition 2.3.2. *Let $\mathcal{A}$ be a $\prod \mathcal{C}$-compactly generated category, $\mathcal{M}$ a family of $\mathcal{C}$-compact objects of $\mathcal{A}$ and $\mathcal{N} = (N_n \mid n < \omega)$ a countable family of objects of $\mathcal{C}$. Suppose that $\Psi_{\mathcal{N}}$ is not surjective and fix $\varphi \in \mathcal{A}(\prod \mathcal{M}, \bigoplus \mathcal{N}) \setminus \operatorname{im} \Psi_{\mathcal{N}}$. If we denote $\mathcal{I}_n = \{\mathcal{J} \subseteq \mathcal{M} \mid \rho_{N_k} \circ \varphi \circ \mu_{\mathcal{J}} = 0 \ \forall k \geq n\}$ and $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n \subseteq \mathcal{P}(\mathcal{M})$, then the following holds:*

(i) $\mathcal{I}_n$ is an ideal for each n,

(ii) $\mathcal{I}$ is closed under countable unions of subfamilies,

(iii) there exists $n < \omega$ for which $\mathcal{I} = \mathcal{I}_n$,

(iv) there exists a subfamily $\mathcal{U} \subseteq \mathcal{M}$ such that the trace of $\mathcal{I}$ on $\mathcal{U}$ forms a non-principal prime ideal.

Proof. Let $\mathcal{G}$ be a set of $\mathcal{C}$-compact objects satisfying that every product of a system of objects in $\mathcal{G}$ is $\mathcal{C}$-compact, which is guaranteed by the hypothesis.

(i) Obviously, $\emptyset \in \mathcal{I}_n$ and $\mathcal{I}_n$ is closed under subsets. The closure of $\mathcal{I}_n$ under finite unions follows from Lemma 2.1.2(iii) applied on the disjoint decomposition $\mathcal{J} \cup \mathcal{K} = \mathcal{J} \cup (\mathcal{K} \setminus \mathcal{J})$, i.e. from the canonical isomorphism $\prod \mathcal{J} \cup \mathcal{K} \cong \prod \mathcal{J} \times \prod \mathcal{K} \setminus \mathcal{J}$.

(ii) First we show that $\mathcal{I}$ is closed under countable unions of pairwise disjoint sets. Let $\mathcal{K}_j, j < \omega$ be pairwise disjoint subfamilies of $\mathcal{I}$ and put $\mathcal{K} = \dot{\bigcup}_{j<\omega} \mathcal{K}_j$. Let $K_i := \prod \mathcal{K}_i$. We show that there exists $k < \omega$ such that $\mathcal{K}_j \in \mathcal{I}_k$ for each $j < \omega$. Assume that for all $n < \omega$ there exist possibly distinct $i(n)$ such that $\mathcal{K}_{i(n)} \notin \mathcal{I}_n$. Hence $\rho_{N_{l(n)}} \circ \varphi \circ \mu_{\mathcal{K}_{i(n)}} \neq 0$ for some $l(n) \geq n$ and there is a $\mathcal{C}$-compact generator $G_n \in \mathcal{G}$ and a morphism $f_n \in \mathcal{A}(G_n, K_{i(n)})$ with $\rho_{N_{l(n)}} \circ \varphi \circ \mu_{\mathcal{K}_{i(n)}} \circ f_n \neq 0$. Set $\mathcal{K}' := (K_{i(n)} \mid n < \omega)$.

Put $\mathcal{G}_\omega := (G_j \mid j < \omega)$ and denote by $(\prod \mathcal{G}_\omega, (\pi_{G_j} \mid j < \omega))$ the product of $\mathcal{G}_\omega$ and by $\mu_{G_j} \in \mathcal{A}(G_j, \prod \mathcal{G}_\omega)$, $j < \omega$, the associated morphisms given by Lemma 2.1.1(i). Then the universal property of the product $\prod \mathcal{K}'$ applied to the constructed cone gives us a morphism $f \in \mathcal{A}(\prod \mathcal{G}_\omega, \prod \mathcal{K}')$ such that $f_n \circ \pi_{G_n} = \pi_{K_{i(n)}} \circ f$, hence

$$f_n = f_n \circ \pi_{G_n} \circ \mu_{G_n} = \pi_{K_{i(n)}} \circ f \circ \mu_{G_n} = \pi_{K_{i(n)}} \circ \mu_{K_{i(n)}} \circ f_n$$

Since $\prod \mathcal{G}_\omega$ is $\mathcal{C}$-compact by the hypothesis there exists arbitrarily large $m < \omega$ such that $\rho_{N_{l(m)}} \circ \varphi \circ \mu_{\mathcal{K}'} \circ f = 0$ where $\mu_{\mathcal{K}'} \in \mathcal{A}(\prod \mathcal{K}', \prod \mathcal{M})$ is the associated morphism to $\pi_{\mathcal{K}'} \in \mathcal{A}(\prod \mathcal{M}, \prod \mathcal{K}')$ over the subcoproduct of $\mathcal{K}'$. Hence

$$\rho_{N_{l(m)}} \circ \varphi \circ (\mu_{K_{i(m)}} \circ f_m) = \rho_{N_{l(m)}} \circ \varphi \circ \mu_{\mathcal{K}'} \circ f \circ \mu_{G_m} = 0,$$

a contradiction.

We have proved that there is some $n < \omega$ such that $\rho_{N_k} \circ \varphi \circ \mu_{\mathcal{K}_j} = 0$ for each $k \geq n$ and $j < \omega$, and without loss of generality we may suppose that $n = 0$. Denote by t^c the cokernel of the compatible coproduct-to-product morphism $t \in \mathcal{A}(\bigoplus \mathcal{K}, \prod \mathcal{K})$. As $\varphi \circ \mu_{\mathcal{K}} \circ t = 0$, the universal property of the cokernel ensures the existence of the morphism $\tau \in \mathcal{A}(\mathrm{cok(t)}, \bigoplus \mathcal{N})$ such that $\varphi \circ \mu_{\mathcal{K}} = \tau \circ t^c$. Hence there exists $n < \omega$ such that $\rho_{N_k} \circ \varphi \circ \mu_{\mathcal{K}} = 0$ for each $k \geq n$ since cok(t) is $\mathcal{C}$-compact by Lemma 2.3.1, which proves that $\mathcal{K} \subseteq \mathcal{I}_n$.

To prove the claim for whatever system $(\mathcal{J}_j \mid j < \omega)$ in $\mathcal{I}$ is chosen, it remains to put $\mathcal{J}_0 = \mathcal{K}_0$ and $\mathcal{J}_i = \mathcal{K}_i \setminus \bigcup_{j<i} \mathcal{K}_j$ for $i > 0$.

(iii) Assume that $\mathcal{I} \neq \mathcal{I}_j$ for every $j < \omega$. Then there exists a countable sequence of families of objects $(\mathcal{J}_j \in \mathcal{I} \setminus \mathcal{I}_j \mid j \in \omega)$. By (ii) we get $\mathcal{J} := \bigcup_{j<\omega} \mathcal{J}_j \in \mathcal{I}$ and there is some $n < \omega$ such that $\mathcal{J} \in \mathcal{I}_n$. Having $\mathcal{J}_n \subseteq \mathcal{J} \in \mathcal{I}_n$ leads us to a contradiction.

(iv) We will show that there exists a family $\mathcal{U} \subseteq \mathcal{M}$ such that for every $\mathcal{K} \subseteq \mathcal{U}, \mathcal{K} \in \mathcal{I}$ or $\mathcal{U} \setminus \mathcal{K} \in \mathcal{I}$. Assume that such $\mathcal{U}$ does not exist. Then we may construct a countably infinite sequence of disjoint families $(\mathcal{K}_i \mid i < \omega)$ where $\mathcal{K}_i$ are non-empty for $i > 0$ in the following way: Put $\mathcal{K}_0 = \emptyset$ and $\mathcal{J}_0 = \mathcal{M}$. There exist disjoint sets $\mathcal{J}_{i+1}, \mathcal{K}_{i+1} \subset \mathcal{J}_i$ such that $\mathcal{J}_i = \mathcal{J}_{i+1} \cup \mathcal{K}_{i+1}$ where $\mathcal{J}_{i+1}, \mathcal{K}_{i+1} \notin \mathcal{I}$. Now, for each $n \geq 1$ there exists a compact generator $G_n \in \mathcal{G}$ and a morphism $f_n \in \mathcal{A}(G_n, \prod \mathcal{K}_n)$ such that $\rho_{N_k} \circ \varphi \circ \mu_{\mathcal{K}_n} \circ f_n \neq 0$ for some $k > n$. This contradicts to the fact that $\prod_{n<\omega} G_n$ is $\mathcal{C}$-compact (hence $\rho_{N_k} \circ \varphi \circ \mu_{\mathcal{K}_n} \circ f_n \circ \pi_n = 0$ starting from some large enough $k < \omega$).

The trace of $\mathcal{I}$ on $\mathcal{U}$ is a prime ideal and assume that it is principal, i.e. it consists of all subfamilies of $\mathcal{U}$ excluding one particular index $U \in \mathcal{U}$, so $\mathcal{I} \mid \mathcal{U} = \mathcal{P}(\mathcal{U} \backslash \{U\}) \in \mathcal{I}$. On the other hand, U is $\mathcal{C}$-compact itself, which implies $\{U\} \in \mathcal{I}$. This yields $\mathcal{I} \mid \mathcal{U}$ containing $\mathcal{U}$, a contradiction. ∎

As a consequence of Proposition 2.3.2 we can formulate a generalization of [10, Theorem 3.4]:

Corollary 2.3.3. *Let $\mathcal{A}$ be a $\prod \mathcal{C}$-compactly generated category. Then the following holds:*

- *(i) A product of countably many $\mathcal{C}$-compact objects is $\mathcal{C}$-compact.*
- *(ii) If there exists a system $\mathcal{M}$ of cardinality κ of $\mathcal{C}$-compact objects such that the product $\prod \mathcal{M}$ is not $\mathcal{C}$-compact, then there exists an uncountable cardinal $\lambda < \kappa$ and a countably complete nonprincipal ultrafilter on λ.*

Proof. (i) An immediate consequence of Proposition 2.3.2(iii).

(ii) Let $\mathcal{M}$ be a system of cardinality κ of $\mathcal{C}$-compact objects and suppose that $\prod \mathcal{M}$ is not a $\mathcal{C}$-compact object. Then there exists a countable family $\mathcal{N}$ such that $\Psi_{\mathcal{N}}$ is not surjective. By Lemma 2.3.2(iv) there exists a subfamily $\mathcal{U} \subseteq \mathcal{M}$ such that the trace of $\mathcal{I}$ on $\mathcal{U}$ forms a non-principal prime ideal which is closed under countable unions of families by Lemma 2.3.2(ii). If we define $\mathcal{V} = \mathcal{P}(\mathcal{U}) \setminus (\mathcal{I} \mid \mathcal{U})$ then $\mathcal{V}$ forms a countably complete non-principal ultrafilter on $\mathcal{U}$. It is uncountable by applying (i). ∎

Before we formulate the main result of this section which answers the question from [6] for abelian categories, let us list several set-theoretical notions and their properties guaranteeing that the hypothesis of the theorem is consistent with ZFC.

A cardinal number λ is said to be *measurable* if there exists a λ-complete non-principal ultrafilter on λ and it is *Ulam-measurable* if there exists a countably complete non-principal ultrafilter on λ. A regular cardinal κ is *strongly inaccessible* if $2^\lambda < \kappa$ for each $\lambda < \kappa$. Recall that

- [15, Theorem 2.43.] every Ulam-measurable cardinal is greater or equal to the first measurable cardinal;
- [15, Theorem 2.44.] every measurable cardinal is strongly inaccessible;
- [11, Corollary IV.6.9] it is consistent with ZFC that there is no strongly inaccessible cardinal.

Theorem 2.3.4. *Let $\mathcal{A}$ be a $\prod\mathcal{C}$-compactly generated category, $\mathcal{M}$ a family of $\mathcal{C}$-compact objects of $\mathcal{A}$. If we assume that there is no strongly inaccessible cardinal, then every product of $\mathcal{C}$-compact objects is $\mathcal{C}$-compact.*

Proof. Suppose that the product of an uncountable system of $\mathcal{C}$-compact objects is not $\mathcal{C}$-compact. Then Corollary 2.3.3(ii) ensures the existence of a countable complete ultrafilter on λ. Thus there exists a measurable cardinal $\mu \leq \lambda$, which is necessarily strongly inaccessible. ∎

Bibliography for Chapter 2

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3. Self-injective von Neumann regular rings and Köthe's Conjecture

In the famous paper [8], Gottfried Köthe asked whether a ring with no non-zero two-sided nil ideal necessarily contains no non-zero one-sided nil ideal. The affirmative answer to this question is usually referred as Köthe's Conjecture. Despite of plenty established equivalent reformulations of it and many known particular classes of rings satisfying it, a general answer on this question is not known. The original formulation and most famous translations of Köthe's Conjecture use the language of associative rings without unit. Recall several of them [10, 16, 5] (see also [14, 15, 17]):

- the matrix ring $M_n(R)$ is nil for every nil ring R,
- $R[x]$ is Jacobson radical for every nil ring R,
- $R[x]$ is not left primitive for every nil ring R,
- every ring which is a sum of a nilpotent subring and a nil subring is nil.

The present paper deals with its characterization in terms of unital rings, namely, we say that a (unital associative) ring *satisfies the condition (NK)* if it contains two nil right ideals whose sum is not nil. Köthe's Conjecture is then equivalent to the property that there exists no ring satisfying the condition (NK) [11, 10.28]. Recall that the conjecture holds for all right Noetherian rings [11, 10.30], PI-rings [3] and rings with right Krull dimension [12].

The main goal of this text is to translate properties of a potential counterexample for Köthe's Conjecture to a certain structural question about simple self-injective Von Neumann regular rings. Our main result, Theorem 3.2.3, shows that existence of a ring satisfying (NK) implies existence a countable local ring satisfying (NK) which is a subring of a self-injective simple VNR ring either Type II_f or Type III.

As it was said, all rings in this paper are supposed to be associative with unit. An ideal means a two-sided ideal and C-algebra is any ring R with a subring C contained in the center of R. A (right) ideal I is called *nil* whenever all elements $a \in I$ are nilpotent, i.e. there exists n such that $a^n = 0$. A ring R is said to be *Von Neumann regular* (VNR) if for every $x \in R$ there exists $y \in R$ such that $x = xyx$. A VNR ring is called *abelian regular* provided all its idempotents are central. For non-explain terminology we refer to [6] and [18].

3.1 Algebras over $\mathbb{Q}$ and $\mathbb{Z}_p$

Let R be a ring, $a \in R$, and I an ideal. The element a is said to be *nilpotent modulo I* provided there exists n such that $a^n \in I$, and a is *principally nilpotent* provided that the right ideal aR is nil. Note that then also the left ideal generated by a is nil, $a \in \mathcal{J}(R)$ and each $\alpha a \beta \in R$ is principally nilpotent as well for every

$\alpha, \beta \in R$. We say that a is a *minimal non-nilpotent* element of R if a is not nilpotent and a is nilpotent modulo J for every non-zero ideal J. It is easy to see that a ring satisfies (NK) if there exist two principally nilpotent elements whose sum is not nilpotent.

Example 3.1.1. *Every non-zero element of any commutative domain is minimal non-nilpotent.*

First, we make two elementary observations on generators of algebras satisfying (NK) and on the existence of minimal non-nilpotent elements:

Lemma 3.1.2. *Let R be a C-algebra satisfying the condition (NK). Then there exists a C-subalgebra S of R generated by two elements x, y such that both x and y principally nilpotent in S, but $x + y$ is not nilpotent.*

Proof. Since there exist two nil right ideals K and L of R such that $K + L$ is not nil, there exists elements $x \in K$ and $y \in L$ such that $x + y$ is not nilpotent. If S denotes a C-subalgebra of R generated by $\{x, y\}$, then xS and yS are nil right ideals, as $xS \subseteq xR$ and $yS \subseteq yR$. Clearly, every principal nilpotent element belongs to the Jacobson radical. ∎

Lemma 3.1.3. *Let R be a ring and $a \in R$. If a is not nilpotent, then there exists a prime ideal I such that $a + I$ is a minimal non-nilpotent element of the factor ring R/I.*

Proof. Let us take a maximal ideal I that does not contain any power of a, which exists by Zorn's Lemma. Obviously a is not nilpotent modulo I and it is nilpotent modulo J for every ideal $J \supset I$. Since for any ideals U and V such that $I \subset U, V$ there exist $m, n \in \mathbb{N}$ satisfying $a^m \in U$ and $a^n \in V$, hence $a^{m+n} \in UV$. Now, it is clear that $UV \not\subseteq I$, thus I is a prime ideal. ∎

As the consequence we can easily see that, if the class of all F-algebras satisfying the (NK) condition is non-empty, then we can choose countably-dimensional one:

Corollary 3.1.4. *If F is a field and R is an F-algebra which satisfies (NK), then there exists an F-subalgebra A of R satisfying (NK) such that $\dim_F(A) \leq \omega$.*

Recall that the fact that the Jacobson radical of a countably-dimensional algebra over an uncountable field is nil implies that there is no F-algebra satisfying (NK) over uncountable field F [1, Corollary 4]. It implies that for search of algebras over a field satisfying (NK) we can restrict our attention just to countable fields and to algebras of countable cardinality. Nevertheless, we will show below that it is enough to research existence of general rings satisfying (NK) in the class of countable generated algebras either over the field of rational numbers or over field $\mathbb{Z}_p$ for a prime number p.

Lemma 3.1.5. *Let R be a C-algebra generated by two principally nilpotent elements x, y such that $x + y$ is not nilpotent. Then there exits a prime ring S satisfying (NK) and a surjective ring homomorphism $\pi : R \to S$ such that $\pi(C)$ is a commutative domain, S is $\pi(C)$-algebra generated by principally nilpotent elements $\pi(x), \pi(y)$ contained in $\mathcal{J}(S)$, and $\pi(x) + \pi(y)$ is a minimal non-nilpotent element.*

Proof. Let I is a prime ideal such that $x+y+I$ is a minimal non-nilpotent element of the factor ring R/I which exists by Lemma 3.1.3. Put $S = R/I$ and denote by π the canonical projection $R \to S$. Then $\pi(C) \cong C/C \cap I$ is a commutative domain and $\pi(x), \pi(y)$ generates S as a $\pi(C)$-algebra. Obviously, $\pi(x) + \pi(y) = x + y + I$ is a minimal non-nilpotent by the construction and homomorphic images of nil right ideals $\pi(xR) = \pi(x)S$, $\pi(yR) = \pi(y)S$ are nil. ∎

Given a ring R, we denote by C_R the subring generated by the unit of R. Since C_R is isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}_n$ for an integer $n \in \mathbb{N}$ and R has the natural structure of C_R-algebra, we obtain the following immediate consequence of the Lemmas 3.1.2, 3.1.5:

Corollary 3.1.6. *If there exists a ring satisfying (NK), then there exists an algebra satisfying (NK) over either $\mathbb{Z}$ or $\mathbb{Z}_p$ for some prime number $p \in \mathbb{N}$ generated by two principally nilpotent elements.*

Before we formulate claim that we can deal with $\mathbb{Q}$-algebras instead of $\mathbb{Z}$-algebras we make the following easy observation:

Lemma 3.1.7. *Let F be a field and R be an F-algebra generated by two principally nilpotent elements x, y such that $x + y$ is not nilpotent.*

Then R is a local ring satisfying (NK) with $\mathcal{J}(R) = xR + yR = Rx + Ry$.

Proof. Since x and y are principally nilpotent, $xR + yR, Rx + Ry \subseteq \mathcal{J}(R)$. Observe that $R = F + xR + yR = F + Rx + Ry$ which implies $R/(xR+yR) \cong F$, hence $\mathcal{J}(R) \subseteq xR + yR, Rx + Ry$. ∎

Proposition 3.1.8. *Let R be a ring satisfying (NK). Then there exists a subring S of R generated by two elements ξ, υ, an F-algebra A and an epimorphism of rings $\varphi : S \to A$ such that the following conditions hold:*

(K1) either $F = \mathbb{Q}$ or $F = \mathbb{Z}_p$ where p is a prime number,

(K2) $x = \varphi(\xi)$ and $y = \varphi(\upsilon)$ are principally nilpotent generators of A, and $x + y$ is a minimal non-nilpotent element,

(K3) $xA + yA = \mathcal{J}(A)$ is the unique maximal right ideal of A.

The F-algebra A satisfies (NK) and if R is a PI-algebra then A can be taken as a PI-algebra.

Proof. Since R has a structure of C_R-algebra, we can take a subring S generated by principally nilpotent elements ξ, υ such that $\xi + \upsilon$ is not nilpotent which is ensured by Lemma 3.1.2. Moreover, by Lemma 3.1.7 we may suppose that $\xi + \upsilon$ is a minimal non-nilpotent element and $C_S \cong \mathbb{Z}$ or $C_S \cong \mathbb{Z}_p$. For each integer $p \in \mathbb{N}$ let $\mu_p : S \to S$ be the S-endomorphism induced by multiplication by p and note that $\ker(\mu_p)$ is an ideal for an arbitrary p.

First, suppose that there exists a prime number p such that $\ker(\mu_p) \neq 0$. Since $\xi + \upsilon$ is minimal non-nilpotent, it follows by the hypothesis that there exists $n \in \mathbb{N}$ for which $(\xi + \upsilon)^n \in \ker(\mu_p)$, i.e. $p(\xi + \upsilon)^n = 0$. Put

$$x' = \xi(\xi + \upsilon)^n, y' = \upsilon(\xi + \upsilon)^n \in \ker(\mu_p),$$

and denote by S' a C_S-subalgebra of S generated by $\{x', y'\}$. Since $px' = 0 = py'$ and p is not an invertible element, we get that $pS' = pC \neq C$, hence $C/pC \cong \mathbb{Z}_p$ and we may identify C/pC and $\mathbb{Z}_p$. Furthermore, assume that $(x' + y')^k \in pS'$ for some k. Then there exists $c \in C$ such that $(x' + y')^k = pc$ because $pS' = pC$, and so

$$0 \neq (\xi + \upsilon)^{2k(n+1)} = (x' + y')^{2k} = (pc)^2 = p(x' + y')^k c = 0,$$

a contradiction. Hence $(x'+y')^k \notin pS'$ for all k. Note that S'/pS' is $(\mathbb{Z}_p \cong)C/pC$-algebra generated by principally nilpotent elements $x' + pS'$ and $y' + pS'$ with non-nilpotent $x' + y' + pS'$. So there exists a prime factor of S'/pS' which is $\mathbb{Z}_p$-algebra satisfying (K2) by Lemma 3.1.5, which satisfies also (K3) by Lemma 3.1.7. Because the constructed homomorphism $\varphi : S \to A$ is a composition of surjective homomorphisms, it is an epimorphism.

Now, suppose that $\ker(\mu_p) = 0$ for all primes p, hence $S_{\mathbb{Z}}$ is a torsion-free abelian group. Denote by $E(S_{\mathbb{Z}})$ the injective envelope of $S_{\mathbb{Z}}$ as a $\mathbb{Z}$-module. Since every endomorphism of $S_{\mathbb{Z}}$ can be extended to an endomorphism of $E(S_{\mathbb{Z}})$ and multiplication by each element of S can be viewed as an endomorphism of $S_{\mathbb{Z}}$, there exist $x, y \in \operatorname{End}_{\mathbb{Z}}(E(S_{\mathbb{Z}}))$ such that $x(s) = \xi \cdot s$ and $y(s) = \upsilon \cdot s$ for every $s \in S$. As $S_{\mathbb{Z}}$ is torsion-free, $\operatorname{End}_{\mathbb{Z}}(E(S_{\mathbb{Z}}))$ has a natural structure of a $\mathbb{Q}$-algebra. Let us denote by $\overline{A}$ its $\mathbb{Z}$-subalgebra generated by x and y, by A its $\mathbb{Q}$-subalgebra generated by x and y and by $i : \overline{A} \to A$ the inclusion homomorphism. Note that a map $\psi : \overline{A} \to S$ defined by the rule $\psi(\alpha) = \alpha(1)$ provides correctly defined isomorphism of $\mathbb{Z}$-algebras such that $\psi(x) = \xi$ and $\psi(y) = \upsilon$. Furthermore, it is easy to see that $i \circ \psi^{-1} : S \to A$ is a ring epimorphism.

Finally, if $r \in A$, there exists $m \in \mathbb{Z} \setminus \{0\}$ for which $rm \in \overline{A}$ which implies that there is $k \in \mathbb{N}$ such that $m^k(xr)^k = (xrm)^k = 0$. As $\operatorname{End}_{\mathbb{Z}}(E(S_{\mathbb{Z}}))$ is a torsion-free abelian group, $(xr)^k = 0$ which shows that x is a principal nilpotent element. The same argument applied on y proves that y is principal nilpotent as well. Now A is $\mathbb{Q}$-algebra satisfying (K2) and (K3) again by Lemmas 3.1.5 and 3.1.7.

To prove the addendum, suppose that R is a PI-ring and note that the class of PI-algebras is closed under taking subrings and factor rings. Thus F-algebras A are PI-algebras for finite fields F. If $F = \mathbb{Q}$ we can see that $A \cong \mathbb{Q} \otimes \overline{A}$ which is polynomial by [19, Theorem 6.1]. ∎

The previous claim easily allows to restrict reformulation of Köthe's Conjecture due to Krempa [10, Theorem 6] just to fields $\mathbb{Q}$ and $\mathbb{Z}_p$:

Corollary 3.1.9. *Köthe's Conjecture holds if and only if there is no countable generated F-algebra satisfying (NK) for either $F = \mathbb{Q}$ or $F = \mathbb{Z}_p$ where p is a prime number.*

Applying an old Amitsur's result we can reprove a well-known fact that PI-rings satisfies Köthe's Conjecture. Let us first state a general result by Braun extending previous works in [13, 9].

Fact 3.1.10. *[3, Theorem 5] The Jacobson radical of a finitely generated PI-algebra over a Noetherian commutative ring is nilpotent.*

Proposition 3.1.11. *There is no PI-ring satisfying (NK).*

Proof. Let us assume that R is a PI-ring satisfying (NK). Then by Proposition 3.1.8 there exists a 2-generated PI-algebra A over a field F with the Jacobson radical of A not nil, hence we get a contradiction with Fact 3.1.10. ∎

Since an algebra over a field is easily embeddable into a VNR ring, we are able to find a prime VNR extension of algebras constructed in Proposition 3.1.8.

Lemma 3.1.12. *An algebra A satisfying the conditions (K1)–(K3) from Proposition 3.1.8 is embeddable into a countable prime VNR ring R such that $J \cap A \neq 0$ for each nonzero ideal J of R.*

Proof. As A is an algebra over a field F, there exists a canonical embedding ν of A into the endomorphism ring $\mathrm{End}_F(A)$. Note that $\mathrm{End}_F(A)$ is a VNR F-algebra and for each $x \in \mathrm{End}_F(A)$ fix an element y_x such that $xy_xx = x$. Now we will construct by induction a chain of subrings $R'_i \subset R'_{i+1} \subset \cdots \subset \mathrm{End}_F(A)$. We put $R'_1 = \nu(A)$ and if R'_i is defined, R'_{i+1} is an F-subalgebra of $\mathrm{End}_F(A)$ generated by the set $R'_i \cup \{y_x \mid x \in R'_i\}$. Then $R' = \bigcup_{i\in\mathbb{N}} R'_i$ is a countable VNR F-algebra containing $\nu(A) \simeq A$ as an F-subalgebra.

Fix a maximal ideal M of R' such that $M \cap A = 0$. Then the map $a \to a + M, a \in A$ induces an embedding of A into a countable VNR ring R'/M. We may identify A with the image of the embedding. Note that if $\overline{I}, \overline{J}$ are two nonzero ideals of R'/M, then the intersection $\overline{I} \cap A$ resp. $\overline{J} \cap A$ forms a nonzero ideal of A. Since A is prime, $0 \neq (\overline{I} \cap A)(\overline{J} \cap A) \subset \overline{IJ}$, hence $R := R'/M$ is prime too. ∎

We finish the section with a technical lemma due to [7]:

Lemma 3.1.13. *Let R be a subring of a ring Q. If Q satisfies (ACC) on right annihilators, then every non-zero nil right ideal of R contains a non-zero nilpotent right ideal.*

Proof. As R also satisfies (ACC) on right annihilators, the result is proved in [7, Lemma 1]. ∎

3.2 Self-injective VNR rings

Before we present a construction of self-injective VNR rings containing rings satisfying (NK), we need to recall several notions and structural results concerning self-injective VNR rings. Let R be a VNR ring. An idempotent e is called *abelian* if the ring eRe is abelian regular and e is *directly finite* if the ring eRe is directly finite [6, p.110], i.e. $xy = 1$ implies $yx = 1$ for all $x, y \in eRe$. A self-injective VNR ring R is *purely infinite* if it contains no nonzero directly finite central idempotent and it is

- *Type I* if every nonzero right ideal contains a nonzero abelian idempotent [6, 10.4],
- *Type II* if every nonzero right ideal contains a nonzero directly finite idempotent [6, 10.8],
- *Type III* if it contains no nonzero directly finite idempotent.

Moreover, R is Type I_f (Type II_f) provided it is Type I (Type II) and directly finite. Next, R is Type I_∞ (Type II_∞) if it is Type I (Type II) and purely infinite. Recall the structural decomposition of self-injective VNR rings:

Fact 3.2.1. *[6, Theorem 10.22] Every self-injective VNR ring is uniquely a direct product of rings Type $\mathcal{T} = \mathrm{I}_f, \mathrm{I}_\infty, \mathrm{II}_f, \mathrm{II}_\infty$, and* III. *In particular, each prime self-injective VNR ring is exactly one type from $\mathcal{T}$.*

Let R be a right self-injective VNR ring. Define a function μ on non-singular injective right R-modules as follows: for an injective non-singular module M, if there exists a non-zero central idempotent e such that $Me = 0$, then $\mu(M) = 0$. Otherwise set $\mu(M)$ to be the smallest infinite cardinal α such that Mf does not contain a direct sum of α nonzero pairwise isomorphic submodules for some non-zero central idempotent f [6, p.143]. If R is moreover prime and α is a cardinal, then we set

$$H(\alpha) := \{r \in R \mid \mu(rR) \leq \alpha\}$$

Fact 3.2.2. *[6, Corollary 12.22] Let R be a prime, right self-injective VNR ring either Type I or Type II. Then the rule $\alpha \to H(\alpha)$ defines a lattice isomorphism between the lattice of non-zero two-sided ideals and the cardinal interval $[\omega, \mu(R)]$.*

Theorem 3.2.3. *If there is a ring satisfying (NK) then there exists a countable local subring of a suitable self-injective simple VNR ring of type either* II_f *or* III *that also satisfies (NK).*

Proof. Let A be a prime F-algebra over a field F with two principally nilpotent generators x and y satisfying the conditions (K1)–(K3), which exists by Proposition 3.1.8 ($x = \varphi(\xi)$ and $y = \varphi(\upsilon)$ are images of the constructed epimorphism φ from the proposition and $x + y$ is a minimal non-nilpotent element). Let R be a countable prime VNR ring extension of A ensured by Lemma 3.1.12. Denote by $Q := Q_{max}(R)$ the maximal right ring of quotients of R. Let $I, J \subseteq Q$ be two non-zero ideals of Q. Since R is essential in Q_R by [18, Corollary 2.3] and both I, J are right R-submodules of Q, then $0 \neq (I \cap R)(J \cap R) \subseteq IJ$, hence Q is prime. By Fact 3.2.1 we get that Q is exactly one of type from $\mathcal{T}$.

Assume that Q is Type I_f. Then it is Artinian by [6, Corollary 10.3], hence Q satisfies (ACC) on right annihilators. By applying Lemma 3.1.13 we obtain a non-zero nilpotent right ideal $K' \subseteq K$ in any nil right ideal K of R, which is in contradiction to the fact that R is prime.

Suppose that Q is of type I_∞ or II_∞. Note that Q does not contain any uncountable direct sum of right ideals since Q_R is an essential extension of a countable ring R, i.e. $\mu(Q_Q) \leq \omega_1$. Moreover, it is purely infinite, so by [6, Theorem 10.16](a)→(d), Q_Q contains a countably generated free submodule which yields $\mu(Q_Q) \neq \omega$, thus $\mu(Q_Q) = \omega_1$. By Fact 3.2.2 (cf. also [4, Theorem 7.3]), Q contains exactly one nontrivial ideal $H(\omega)$. As $I = H(\omega) \cap A$ is a nonzero ideal of A by Lemma 3.1.12 and $x + y$ is a minimal non-nilpotent element of A by the condition (K2) in Proposition 3.1.8, there exists $n \in \mathbb{N}$ such that $(x + y)^n \in I$. Also $(x+y)^n x, (x+y)^n y \in (x+y)^n Q = eQ \cap I$ for an idempotent $e \in H(\omega)$. Put $\xi := (x + y)^n x$, $\upsilon := (x + y)^n y$ and let B denote the F-subalgebra of eQe generated by elements ξe and υe and $\widetilde{B}$ denote the F-subalgebra of Q generated by elements ξ and υ. We claim that the elements ξe and υe are principally nilpotent in B and that $\xi e + \upsilon e$ is not nilpotent, hence B satisfies (NK).

Indeed, since $\widetilde{B} \subseteq eQ$, we can see that $e\widetilde{B} = \widetilde{B}$ and so $B = \widetilde{B}e = e\widetilde{B}e$. Hence every element from the right ideal $(\xi e)B$ of B has the form $\xi\widetilde{b}e$ for some element $\widetilde{b} \in \widetilde{B}$. Because ξ and υ are principally nilpotent in A, so in $\widetilde{B}$ as well, there exists $k \in \mathbb{N}$ such that $\left(\xi\widetilde{b}\right)^k = 0$. As $e\xi e = \xi e$ we get $\left(\xi\widetilde{b}e\right)^k = \left(\xi\widetilde{b}\right)^k e = 0$. This proves that ξe is a principally nilpotent element. The argument for υe is the same. Finally, $(\xi e + \upsilon e)^k = (x+y)^{k(n+1)}e^k \neq 0$ because $(x+y)^{k(n+1)}e^k(x+y) = (x+y)^{k(n+1)+1} \neq 0$.

Because xQ is a direct summand of an injective module Q_Q, it is directly finite for any $x \in H(\omega)$ by [6, Proposition 5.7]. Thus $eQe \simeq End_Q(eQ)$ is directly finite by [6, Lemma 5.1] and right self-injective by [6, Corollary 9.3]. Clearly, eQe is prime and it contains the subalgebra B which is local and satisfies the condition (NK) by Lemma 3.1.7.

Recall from the initial part of the proof that eQe can not be Type I_f. So we have proved that there exists a prime right self-injective VNR ring eQe being Type either II_f or III which contains a local ring satisfying (NK). Finally note that if Q is Type II_f, then it is simple by [6, Proposition 9.26] and, if Q is of Type III then it is simple by [6, Theorem 12.21]. ∎

Observe that a non-Artinian self-injective simple VNR ring Type II_f or Type III is necessarily uncountable because it contains an infinite set of orthogonal idempotents.

We conclude the paper by examples of self-injective simple VNR rings Type II_f or III:

Example 3.2.4. *Let F be a field and $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Put $R = \prod_{n\in\mathbb{N}} M_n(F)$ and $I = \{r \in R \mid (\exists U \in \mathcal{U})(\forall i \in U)\pi_i(r) = 0\}$ where $\pi_i : R \to M_i(F)$ denotes the natural projection. Then I is a maximal ideal of R, hence R/I forms a simple self-injective VNR ring Type* II_f *by [6, Theorem 10.27].*

Example 3.2.5. *[6, Example 10.11] Let F be a field, $Q = \mathrm{End}_F(F^{(\omega)})$ and $M = \{x \in Q \mid \dim_F(xF^{(\omega)}) < \omega\}$. Then Q/M is a simple VNR ring. If R is the maximal right ring of quotients of Q/M then R is a simple self-injective VNR ring Type* III.

Bibliography for Chapter 3

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4. Generalizations of Projectivity and Supplements Revisited for Superfluous ideals

This study is motivated by two mathematical problems in ring theory that have already enjoyed interest and a detailed treatment. The first one is so called Lazard's Conjecture originated in [11]. The conjecture was denying existence of a non-finitely generated projective module with the finitely generated radical factor and it was supported affirmatively by commutative rings. In [6], Gerasimov and Sakhaev constructed a breaking counterexample of a ring, here and subsequently denoted R_Σ, that proved the opposite. Even more characteristics of the ring and its module category were revealed later in [4]. The ring itself is the universal localization of a 2-generated monomial algebra $k\langle x, y\rangle$ over a field with the single relation $yx = 0$ factored out. The counterexample is very far from being well-behaved in many more aspects of ring and module theory. It leads to additional anomalies, like a projective module with a $\mathcal{J}(R)$-supplement that is not a direct summand; a $\mathcal{J}(R)$-projective module being non-projective; or a projective $\mathcal{J}(R)$-semicover that is not a projective cover.

The proofs are technical and elementary in nature. Most of the time, well-known techniques that work for the Jacobson radical are adapted to any superfluous ideal (i.e. an ideal contained in the Jacobson radical). To achieve the goal, we reintroduce four ideal-related generalizations and objects that extend classic properties and objects from the module theory. We start with defining *the ideal-superfluity* that extends the familiar property of being a superfluous submodule in a module. Recall that a submodule N of a module M is said to be *superfluous* in M, denoted $N \ll M$, if $N + L \neq M$ for any proper submodule L of M. After that, we continue with a generalized version of projective covers, so called *projective ideal-covers*, that were established in [1]. The authors proved that projective covers are in agreement with projective $\mathcal{J}(R)$-covers. One immediately gets that projective ideal-covers are characterized by the ideal-superfluity of their kernels. Therefore, these two generalizations work together similarly like their classic counterparts do. Next, we recall the definition of *the ideal-projectivity* which makes good sense for arbitrary ideals (and was mentioned in [13] for the first time). Finally, we shall introduce an ideal-related version of a supplement, *an ideal-supplement*. The classic definition says that a submodule G of a module M is *a supplement* in M if there is a submodule K of M with $K + G = M$ and G is minimal in the set $\{G' \leq M \mid K + G' = M\}$, see [8, Introduction]. This holds if and only if there exists a submodule K with $K + G = M$ and $K \cap G \ll G$. The supplement generalization is based on the ideal-superfluity of the latter intersection.

An extensive account of material concerning this topic and the proof of the main theorem for $I = \mathcal{J}(R)$ can be found in [16], [13] and [8]. Addressing the details below, we will be able to prove the equivalence of the following four conditions:

(1) every I-supplement in a finitely generated projective R-module is a direct

summand,

(2) every finitely generated I-(semi)projective R-module is projective,

(3) every finitely generated flat R-module M with projective R/I-module M/MI is projective,

(4) for every projective R-module Q, if Q/QI is finitely generated then Q is finitely generated.

The result is an extension of the well-known fact proved previously for the edge case $I = \mathcal{J}(R)$. The statement (4) evokes Lazard's Conjecture parametrized by any superfluous ideal. Considering the trivial case $I = 0$, it is obvious that each of the four conditions (1), (2), (3), (4) holds true and therefore also their equivalence does. In [13, Corollary 3.5] it was proved that (2) is true for any I contained in the Baer radical $\beta(R)$. It is easy to see that (4) is true if I is a nilpotent ideal, since the canonical projection $Q \to Q/QI$ is then a projective cover for any projective module Q. It will turn out that ideal projectivity in a finitely generated projective module resp. ideal supplements in a finitely generated module are redundant with respect to the classic definitions if the chosen ideal admits idempotent-lifting up to all its matrix rings. The Levitzki radical $\mathcal{L}$ of a ring is an ideal that does so.

This brings us to the second mathematical problem - the famous Köthe's Conjecture. One of its notable characterization is expressed in extensibility of nilpotency to elements of matrix rings over the nil radical, [10]. Unfortunately, we have not resolved the question yet whether the nil radical provides counterexamples of modules for (1)–(4). However, non-existence of such counterexamples for this ideal would yield an approximation of a positive solution of Köthe's Conjecture. This suggests that finding such counterexample is at least as hard as finding one to the conjecture itself. One is left to definitely avoid classes of rings where this is not achievable, e.g. right noetherian rings, algebras over uncountable fields, monomial algebras, right Goldie rings, rings with right Krull dimension etc. Also the complexity of the R_Σ construction furthermore emphasizes the difficulties to accomplish the task.

4.1 Preliminaries

We assume through the whole paper that R is an associative ring with unity and I denotes a superfluous ideal of R, i.e. I is contained in the Jacobson radical $\mathcal{J}(R)$. For a right module M over the ring R we call the right module M/MI ($M/M\mathcal{J}(R)$) *the ideal* (*the radical*) *factor* of M.

Definition 4.1.1 (Ideal-supefluity)**.** *A submodule N of M* decomposes *M (or shortly is* DM *in M) if there is a summand S of M such that $S \subseteq N$ and $M = S + X$, whenever $N + X = M$ for a submodule X of M.*

A submodule N of M is called PDM *in M if there is a projective summand S of M such that $S \subseteq N$ and $M = S + X$, whenever $N + X = M$ for a submodule X of M.*

We say that a submodule N of a right module M is I-superfluous, denoted $N \ll_I M$, if $N \subseteq MI$ and N is PDM *in M.*

The definition of a submodule that decomposes a module comes from [1, Definition 3.1] and it is also called *a partial summand* of the module in [2, p.1882]. Obviously, every submodule that is PDM in a module also decomposes the module and the notions coincide for submodules of projective modules. The zero submodule witnesses the PDM property for every superfluous submodule. It is straightforward that $N \ll_I M$ implies $N \ll_{\mathcal{J}(R)} M$ for any submodule N of M.

The next lemma shows basic properties of the ideal-superfluity. For the case $I = \mathcal{J}(R)$, it gives nothing new than the classic definition at least within the scope of projective modules. Moreover, we will freely use the following intuitive fact through the text: if G, H are submodules of M, then $G \ll_I M$ together with $H \subseteq G$ imply $H \ll_I M$.

Lemma 4.1.2. *Let M be a right R-module and G be a submodule of M. Then*

(i) $G \ll_{\mathcal{J}(R)} M$ implies $G \ll M$.

(ii) $G \ll M$ and $G \subseteq MI$ if and only if $G \ll_I M$.

(iii) if M satisfies $Rad(M) = M\mathcal{J}(R)$, then $G \ll M$ implies $G \ll_{\mathcal{J}(R)} M$.

(iv) if M is finitely generated, then $G \subseteq MI$ implies $G \ll_I M$.

Proof. (i) Let $G \ll_{\mathcal{J}(R)} M$ and let X be any submodule of M with $X + G = M$. Then there is a witnessing projective summand S of M such that $M = S + X$ and $S \subseteq G$ ($\subseteq Rad(M)$). Since $M = S \oplus Y$ for some submodule Y of M, we get

$$Rad(M) = Rad(S \oplus Y) = Rad(S) \oplus Rad(Y) \subseteq S \oplus Rad(Y) \subseteq Rad(M)$$

and $Rad(S) \oplus Rad(Y) = S \oplus Rad(Y)$. By the modularity of the lattice of submodules of M we get

$$S = (Rad(S) + Rad(Y)) \cap S = Rad(S) + (Rad(Y) \cap S) = Rad(S)$$

and by projectivity of S, $S = 0$ and $M = X$.

(ii) If G is superfluous in M, then the zero submodule witnesses that G is PDM in M. The assumption on the inclusion then implies the conclusion.

(iii) Follows from (ii) for $I = \mathcal{J}(R)$.

(iv) Follows from (ii) by the fact that $Rad(M) \ll M$ for a finitely generated M. ∎

Lemma 4.1.3. *The relation $\ll_I$ on submodules is:*

(i) preserved under sums, i.e. $N_1, N_2 \ll_I M$ implies $N_1 + N_2 \ll_I M$.

(ii) preserved under taking homomorphic images, cf. [9, Lemma 1.1].

Proof. The relation $\ll$ on submodules satisfies the conditions, therefore by the characterization in Lemma 4.1.2(ii) also $\ll_I$ satisfies them. ∎

Here is a basic example that $\ll$ and $\ll_{\mathcal{J}(R)}$ do not coincide.

Example 4.1.4. *Consider $\mathbb{Z}$ the ring of integers and $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then $Rad(\mathbb{Q}) = \mathbb{Q}$ as $\mathbb{Q}$ does not have any maximal submodules, but $\mathbb{Z} \not\subseteq \mathbb{Q}\mathcal{J}(\mathbb{Z}) = 0$. Therefore $\mathbb{Z} \ll \mathbb{Q}$ but $\mathbb{Z} \not\ll_{\mathcal{J}(R)} \mathbb{Q}$.*

The generalization of projective covers using ideals contained in the Jacobson radical of a ring are provided as they appeared in [1, Definition 3.5]. The definition of a projective $\mathcal{J}(R)$-semicover gives the same as the familiar notion if the projective module is finitely generated.

Definition 4.1.5. *(Projective ideal-(semi)covers) A pair* (P, f) *is called* a projective I-semicover *of* M *if* P *is projective and* $f : P \to M$ *is an epimorphism such that* $\ker(f) \subseteq PI$.

A pair (P, f) *is called* a projective I-cover *of* M *if it is a projective* I*-semicover of* M *and* $\ker(f)$ *is DM in* P.

Proposition 4.1.6. *(i) A right module* M *has a projective* $\mathcal{J}(R)$*-cover if and only if* M *has a projective cover.*

(ii) Let M *be a module. A projective module* P *with a homomorphism* $f : P \to M$ *is a projective* I*-cover of* M *if and only if* f *in an epimorphism and* $\ker(f)$ *is* I*-superfluous in* P.

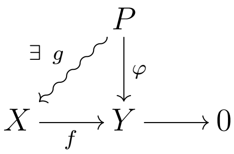
Proof. (i) Proved in [1, Proposition 3.6].

(ii) Note that PDM and DM submodules coincide in any projective module P, in particular it holds for its submodule $\ker(f)$. ∎

Example 4.1.7. *Let* R_Σ *be the Gerasimov-Sakhaev counterexample ([6], [4]) and let* P *be a non-finitely generated projective module such that* $P/P\mathcal{J}(R_\Sigma)$ *is finitely generated. Then* $P\mathcal{J}(R_\Sigma) = Rad(P) \not\ll P$ *and* $\pi_{P\mathcal{J}(R_\Sigma)} : P \to P/P\mathcal{J}(R_\Sigma)$ *is a projective* $\mathcal{J}(R_\Sigma)$*-semicover that is not a projective cover. In particular,* $P\mathcal{J}(R_\Sigma)$ *is neither PDM or DM in* P.

The ideal-projectivity was initially introduced in [13, Definition 3.1] and later reused in [8]. In these original papers, an ideal-semiprojective module was called just ideal-projective. However, we prefer to add the prefix *semi-* to suggest the relation in the similar fashion like projective ideal-semicovers have towards projective ideal-covers. By the same token, we add the prefix in the definition of radical-semiprojective modules. They are a weaker version of radical-projective modules introduced in [8, Definition 2.1].

Definition 4.1.8 (Ideal-(semi)projectivity)**.** *An* R*-module* P *is* I-semiprojective *if for every epimorphism* $f : X \to Y$ *such that* $YI = 0$ *and every morphism* $\varphi : P \to Y$ *there is a homomorphism* $g : P \to X$ *such that* $\varphi = f \circ g$:



A right R*-module* P *is* I-projective *if for all right* R*-modules* X *and* Y*, every* R*-epimorphism* $f : X \to Y$ *and every homomorphism* $\varphi : P \to Y$ *there exists a homomorphism* $g : P \to X$ *such that* $(f \circ g - \varphi)(P) \ll_I Y$.

Lemma 4.1.9. *Let* M *be a module. Then* M *is* I*-semiprojective if and only if for every epimorphism* $f : X \to Y$ *and every homomorphism* $\varphi : M \to Y$ *there exists a homomorphism* $g : M \to X$ *such that* $(\varphi - f \circ g)(M) \subseteq YI$.

Proof. Let M be I-semiprojective and let $f : X \to Y$ be an epimorphism and $\varphi : M \to Y$ a homomorphism. Denote by $\pi : Y \to Y/YI$ the canonical projection. Because $(Y/YI)I = 0$, by I-semiprojectivity of M there exists a homomorphism $g : M \to X$ such that $\pi \circ \varphi = (\pi \circ f) \circ g$, and so $\pi \circ (\varphi - f \circ g) = 0$. Then $(\varphi - f \circ g)(M) \subseteq \ker(\pi) = YI$ as required.

Let $f : X \to Y$ be an epimorphism with $YI = 0$ and let $\varphi : M \to Y$ be arbitrary. By the assumption there is a homomorphism $g : M \to X$ such that $(f \circ g - \varphi)(M) \subseteq YI$. But $YI = 0$, so g also witnesses the I-semiprojectivity of M. ∎

Definition 4.1.10. *An R-module P is called* radical-semiprojective *or* radical-projective *if for all right R-modules X and Y, every R-epimorphism $f : X \to Y$ and every homomorphism $\varphi : P \to Y$ there exists a homomorphism $g : P \to X$ such that $(f \circ g - \varphi)(P) \subseteq Rad(Y)$ or $(f \circ g - \varphi)(P) \ll Y$ respectively.*

Obviously, all ideal- and radical-projective modules are ideal- and radical-semiprojective, respectively. Ideal-projective modules are radical-projective. Projective modules are exactly 0-projective modules. In [8, Proposition 2.8] it was proved that a finitely generated module is $\mathcal{J}(R)$-semiprojective if and only if M is radical-projective. It is false for modules that are not finitely generated.

Example 4.1.11. *Let $F[[x]]$ be the ring of formal power series over a field F. Then there exists a $\mathcal{J}(F[[x]])$-semiprojective module that is not radical-projective (in particular not $\mathcal{J}(F[[x]])$-projective) [8, Example 3.11].*

For the ring R_Σ, the right ideal xR_Σ is a (finitely generated) radical-projective module that is not projective [8, Example 3.6].

Proposition 4.1.12. *Let M be a finitely generated right R-module.*

(i) M is I-semiprojective if and only if M is I-projective.

(ii) M is I-(semi)projective if and only if for the canonical projection $\pi : M \to M/MI$ there exists a finitely generated module F and a pair of homomorphisms $\alpha : M \to F$ and $\beta : F \to M$ such that $\pi = \pi \circ \beta \circ \alpha$.

Proof. (i) Let $f : X \to Y$ be an epimorphism and let $\varphi : M \to Y$. The goal is to find some $g : M \to X$ with $(\varphi - f \circ g)(M) \ll_I M$. Because $\varphi(M)(\subseteq Y)$ is finitely generated, by Lemma 4.1.2(iv) it is enough to proof that $(\varphi - f \circ g)(M) \subseteq YI$. Let us consider the epimorphism $\pi \circ f : X \to Y/YI$ and the homomorphism $\pi \circ \varphi : M \to Y/YI$. Because $(Y/YI)I = 0$, there is a homomorphism $g : M \to X$ with $\pi \circ \varphi = \pi \circ f \circ g$. Then $(\varphi - f \circ g)(M) \subseteq \ker(\pi \circ \varphi) = \varphi(M)I$.

(ii) Let M be I-semiprojective. Because M is finitely generated, there exists a finitely generated free module F and a surjective homomorphism $\beta : F \to M$. From $(M/MI)I = 0$ we get a homomorphism $\alpha : M \to F$ with $(\pi \circ \beta) \circ \alpha = \pi$.

On the other hand, let $f : X \to Y$ be an epimorphism with $YI = 0$ and let $\varphi : M \to Y$ be arbitrary. Let $\pi^k : MI \to M$ be the inclusion homomorphism, $\pi \circ \pi^k = 0$. Then $\varphi(MI) = \varphi(M)I \subseteq YI = 0$ and $\varphi \circ \pi^k = 0$. From the universal property of the cokernel M/MI there exists $\psi : M/MI \to Y$ such that $\psi \circ \pi = \varphi$. The assumption supplies a finitely generated free module F and a pair of homomorphisms $\beta : F \to M$ and $\alpha : M \to F$ with $\pi = \pi \circ (\beta \circ \alpha)$. Applying the projectivity of F for the diagram consisting of the epimorphism $f : X \to Y$ and a homomorphism $\psi \circ \pi \circ \beta : F \to M/MI$ it follows that there is a homomorphism $\rho : F \to M/MI$ such that the triangle commutes. Then

$$(f \circ \rho) \circ \alpha = (\psi \circ \pi \circ \beta) \circ \alpha = \psi \circ \pi = \varphi$$

and $g := \rho \circ \alpha$ is witnessing the I-semiprojectivity of M. ∎

We now provide a formulation of an ideal-supplement that works well in the edge case: a $\mathcal{J}(R)$-supplement coincides with a supplement within projective modules. Note that direct summands are exactly 0-supplements.

Definition 4.1.13 (Ideal supplements)**.** *We call a submodule G of an R-module M* an I-supplement submodule *if there is a submodule K of M such that $K+G = M$ and $K \cap G \ll_I G$.*

Lemma 4.1.14. *Let M be a module and G be a submodule of M. If G is a $\mathcal{J}(R)$-supplement then G is a supplement. Moreover, if G satisfies $Rad(G) = G\mathcal{J}(R)$, then also the reverse implication holds.*

Proof. The first part follows by Lemma 4.1.2(i).

Assume that G is a supplement of a submodule K in M. Then $K + G = M$ with $K \cap G \ll G$ and applying the assumption on G, Lemma 4.1.2(ii) gives $K \cap G \ll_{\mathcal{J}(R)} G$. We conclude that G is a $\mathcal{J}(R)$-supplement. ∎

Example 4.1.15. *(i) The Weyl algebra W over the complex numbers is a simple domain, therefore $\mathcal{J}(W) = 0$. In [12, Thorem 6.2], an indecomposable non-uniserial module Z of length 3 was constructed with a minimal submodule T and two intermediate, $\subseteq$-incomparable submodules X,Y. Then $Rad(Z) = T \neq 0$ and X is a supplement in Z, that is not a $\mathcal{J}(W)$-supplement. Clearly, $X + Y = Z$, $X \cap Y = T$ and $T \ll X$.*

(ii) Consider the ring R_Σ. Then xR_Σ is a supplement but not a direct summand [8, Example 3.6].

4.2 Relations between ideal generalizations

In this section we prove the equivalence of (1) and (2). In fact, we pass even further and prove it for arbitrarily generated modules. In [8, Corollary 3.4], the author managed to get the same for the special case $I = \mathcal{J}(R)$. Again, recall that I is any two-sided ideal of a ring R that is contained in $\mathcal{J}(R)$. All ideal-dependent notions in the sequel refer to this ideal implicitly.

Lemma 4.2.1. *Let M be an I-projective right R-module and G an I-supplement in M. Then*

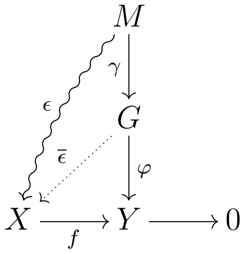
(i) there exists an endomorphism γ of M with $\gamma(M) = G$,

(ii) G is I-projective

Proof. (i) By the hypothesis, there is a submodule K of M such that $K+G = M$ and $K \cap G \ll_I G \simeq M/K$. Denote $\overline{G} := G/G \cap K$ and let $\rho : G \to \overline{G}$ and $\pi : M \to \overline{G}$ be the natural projections. By the ideal-projectivity of M there is γ :

$M \to G$ with $(\pi - \rho\circ\gamma)(M) \ll_I \overline{G}$. Now $(\rho\circ\gamma)(M) + (\pi - \rho\circ\gamma)(M) = \pi(M) = \overline{G}$ implies $(\rho\circ\gamma)(M) = \overline{G}$. Then $\gamma(M) + \ker(\rho) = G$. But $\ker(\rho) = G \cap K \ll_I G$, which by Lemma 4.1.2(i) means also $\ker(\rho) \ll G$. It follows that $\gamma(M) = G$.

(ii) By (i) there is an epimorphism $\gamma : M \to G$. Since M is I-projective, there is ϵ with $(f \circ \epsilon - \varphi \circ \gamma)(M) \ll_I Y$ and we put $\overline{\epsilon} = \epsilon \restriction G$:



First we observe that

$$\begin{aligned}(f \circ \overline{\epsilon}) \circ (1_G - \gamma)(G) &= (f \circ \epsilon) \circ (1_G - \gamma)(\gamma(G)) = \\ &= (f \circ \overline{\epsilon}) \circ (\gamma - \gamma^2)(G) \subseteq (f \circ \overline{\epsilon})(K \cap G)\end{aligned}$$

Then we have:

$$\begin{aligned}(\varphi - f \circ \overline{\epsilon})(G) &= (\varphi - f \circ \overline{\epsilon})(\gamma(G)) = \\ &= (f \circ \epsilon - f \circ \overline{\epsilon} \circ \gamma + \varphi \circ \gamma - f \circ \epsilon)(G) \subseteq \\ &\subseteq f \circ (\epsilon - \overline{\epsilon} \circ \gamma)(G) + (\varphi \circ \gamma - f \circ \epsilon)(G) = \\ &\subseteq (f \circ \overline{\epsilon})(K \cap G) + (\varphi \circ \gamma - f \circ \epsilon)(G)\end{aligned}$$

because the first summand is I-superfluous in $(f \circ \overline{\epsilon})(G)$ and the second is contained in the I-superfluous submodule of Y. Lemma 4.1.3(i) is now applied, $(\varphi - f \circ \overline{\epsilon})(G) \ll_I Y$. ∎

Recall that [8, Corollary 2.7(i)] states this radical equality $Rad(M) = M\mathcal{J}(R)$ for any radical-projective module M. But any supplement submodule G of a radical-projective module is radical-projective too by [8, Corollary 2.7(ii)]. In particular, radical-projective modules can not provide an example of a supplement submodule that is not a $\mathcal{J}(R)$-supplement, cf. Lemma 4.1.14.

Lemma 4.2.2. *If a right R-module M is I-projective then there is an I-supplement submodule G of a free right R-module F and an epimorphism $\psi : M \to G$ with $\ker(\psi) \ll_I M$.*

Proof. Let M be I-projective. There is a free module F and an epimorphism $\alpha : F \to M$ and we can find $\beta : M \to F$ with $(1_M - \alpha \circ \beta)(M) \ll_I M$. Set $G := (\beta\circ\alpha)(F)$ and $\psi := \beta$. The inclusions $\ker(\beta) \subseteq \ker(\alpha\circ\beta) \subseteq (1_M - \alpha\circ\beta)(M) \ll_I M$ imply $\ker(\beta) \ll_I M$.

We show that G is an I-supplement of $\ker(\alpha)$ in F. First observe that $F = G + \ker(\alpha)$ because $\alpha \circ \beta$ is surjective. To conclude this part, it is enough to show that $\ker \alpha \restriction G \ll_I G$. Note that $\beta(\ker \alpha \circ \beta) = \ker(\alpha \restriction G)$. Indeed, let $g \in \beta(\ker(\alpha\circ\beta))$, then obviously $\alpha(g) = 0$. On the other hand, let $g' \in G \cap \ker(\alpha)$. Then there is $m \in M$ with $\beta(m) = g'$. But $(\alpha \circ \beta)(m) = \alpha(g') = 0$ and this yields $m \in \ker(\alpha \circ \beta)$. Now apply the homomorphism β on $\ker(\alpha \circ \beta) \ll_I M$ and use Lemma 4.1.3(ii). ∎

Corollary 4.2.3. *The following is equivalent:*

(1') every I-supplement in a projective R-module P is a direct summand

(2') every I-projective R-module is projective

Proof. $(2') \to (1')$: Let G be an I-supplement of P. By Lemma 4.2.1(i), there exists an epimorphism $\gamma : P \to G$ and by Lemma 4.2.1(ii), G is I-projective. But (2') implies that G is projective. This yields G is a direct summand of P.

$(1') \to (2')$: Let M be I-projective. By Lemma 4.2.2 there exists an I-supplement G of a free module F and an epimorphism $\psi : M \to G$ with $\ker(\psi) \ll_I M$. By (1'), G is a direct summand of F, therefore projective itself and ψ splits. Then $\ker(\psi)$ is a direct summand and, at the same time, it is superfluous. This yields $\ker(\psi) = 0$ and $M(\simeq G)$ is projective. ∎

4.3 Lifting projectives modulo superfluous ideals

In this section we prove the equivalence between (3) and (4). The techniques used here heavily rely on those in [5] where just the case $I = \mathcal{J}(R)$ was discussed. For two projective modules P and Q, if there exists a homomorphism $\alpha : P/PI \to Q/QI$ then there exists a homomorphism $f : P \to Q$ such that $\pi_Q \circ f = \alpha \circ \pi_P$ where $\pi_Q : Q \to Q/QI$, $\pi_P : P \to P/PI$ are the canonical projections. We say that f is *a lift* of α.

Proposition 4.3.1. *The following holds:*

(i) Let P, Q be projective right R-modules and let $\alpha : P/PI \to Q/QI$ be an R/I-homomorphism. Let f be a lift of α. If α is a pure monomorphism, then f is a pure monomorphism.

(ii) Let $P' \oplus Q' \simeq (R/I)^{(X)}$ be a decomposition of a free R/I-module into two projective summands and denote $\pi : (R/I)^{(X)} \to P'$ the canonical projection. Then there exists a flat M_R of projective dimension ≤ 1 with the ideal factor isomorphic to P' via α, an epimorphism $\psi : R^{(X)} \to M$ such that $\alpha \circ (\psi \otimes_R R/I) = \pi$ if and only if there exists a projective Q with the ideal factor isomorphic to Q'.

Proof. Adapting the proof of [5, Proposition 6.1] and [5, Proposition 6.3]. ∎

Fact 4.3.2. *A finitely generated, countably presented flat right R-module M has projective dimension ≤ 1.*

Proof. Proved in [11, Théorème 3.2]. ∎

Proposition 4.3.3. *If $(R/I)^{(n)} = P' \oplus Q'$, then the following is equivalent:*

(L1) there exists a finitely generated flat module M_R such that the ideal factor of M is isomorphic to P'

(L2) there exists a finitely generated, countably presented flat module M_R such that the ideal factor of M is isomorphic to P'

(L3) there exists a projective Q_R such that the ideal factor of Q is isomorphic to Q'

(L4) there exists a finitely generated flat ${}_RN$ such that the ideal factor of N is isomorphic to $Hom_R(Q', R/I)$

(L5) there exists a finitely generated, countably presented flat ${}_RN$ such that the ideal factor of N is isomorphic to $Hom_R(Q', R/I)$

(L6) there exists a projective ${}_RP$ such that the ideal factor of P is isomorphic to $Hom_R(P', R/\mathcal{J}(R))$

Proof. The equivalence of $(L3)$ and $(L6)$ is proved in [7, Theorem 2.9]. The equivalence of $(L3)$ and $(L1)$ is a special case of Proposition 4.3.1(ii) together with Fact 4.3.2. The rest of equivalences are acquired by applying the previous two to the opposite ring R^{op}. ∎

Lemma 4.3.4. *Let M be a finitely generated flat right R-module and let P be projective. If $\gamma : P \to M$ is a projective I-cover of M, then γ is an isomorphism. Moreover, if P is finitely generated and the ideal factor of P isomorphic to the ideal factor of M, then $M \simeq P$.*

Proof. The first part follows by adapting the proof of [5, Lemma 7.2].

To proof the addendum, we follow the proof of [5, Proposition 7.3]. Let $\alpha : P/PI \simeq M/MI$ be the isomorphism. Because P is projective, there exists a homomorphism $f : P \to M$ such that $\pi_M \circ f = \alpha \circ \pi_P$ where $\pi_M : M \to M/MI$, $\pi_P : P \to P/PI$ are the canonical projections. Then $\ker(f) \subseteq PI$. Since P is finitely generated, Lemma 4.1.2(iv) implies $\ker(f) \ll_I P$. Now (P, f) is a projective I-cover of M. By the previous part, f is an isomorphism and M is projective. ∎

Now we employ the previous pieces to provide the equivalence of (3) and (4).

Corollary 4.3.5. *Every finitely generated flat R-module M with projective R/I-module M/MI is projective if and only if for every projective R-module Q, the finitely generated ideal factor Q/QI implies Q is finitely generated itself.*

Proof. Let Q be projective and the ideal factor Q/QI be finitely generated. There is a split short exact sequence $0 \to Q/QI \xrightarrow{\alpha} (R/I)^{(n)} \to C \to 0$ for some $n \in \mathbb{N}$. Let $f : Q \to R^{(n)}$ be a lift of α. By Proposition 4.3.1(i), f is a pure monomorphism. Denote by M the cokernel of f. Then M is a finitely generated flat R-module with $M \otimes_R R/I$ isomorphic to C which is a direct summand of $(R/I)^{(n)}$. By the assumption, M is projective. We have that f is a split monomorphism and Q is finitely generated.

Let M be a finitely generated flat module such that the ideal factor of M is projective. So $M/MI \oplus Q' \simeq (R/I)^{(k)}$ for some $k \in \mathbb{N}$ and a projective right R/I-module Q'. By Proposition 4.3.3, $(L1) \to (L3)$, there exists a projective module Q_R such that $Q/QI \simeq Q'$. Because Q' is finitely generated, the assumption yields that Q is finitely generated. Then $M \oplus Q$ is a finitely generated flat module such that its ideal factor is isomorphic to $(R/I)^{(k)}$. By the addendum of Proposition 4.3.4, $M \oplus Q$ is isomorphic to $R^{(k)}$, which leads to the projectivity of M. ∎

4.4 Lifting idempotents in matrix rings

In this part we aim for a characterization that relates idempotent lifting in matrix rings and existence of projective ideal-covers of direct summands. It will turn out later that the equivalent conditions in the characterization, denoted $(L1')$–$(L4')$, are sufficient for (1)–(4) to hold, but not necessary. Recall that a two-sided ideal I is *lifting* or *idempotents lift modulo* I if for any element $a \in R$ with $a^2 - a \in I$, there exists an idempotent $e \in R$ satisfying $a - e \in I$. The idempotent e is also called *a lift* of a. If a lift of a can be chosen from the one-sided ideal generated by a, then the ideal I is said to be *strongly lifting*. The notions of "strongly lifting" and "lifting" coincide for superfluous ideals, cf. [2, Theorem 2 + Proposition 5].

Fact 4.4.1. *The following holds:*

(i) if I is lifting in R then every right ideal $B \subseteq I$ is lifting in R.

(ii) I is lifting in R if and only if every direct summand of the right R-module R/I has a projective cover.

Proof. (i) Proved in [14, Lemma 5].
(ii) Proved in [9, Corollary 1.8]. ∎

Example 4.4.2. *(i) Consider the subring $\mathbb{Z}_{(6)}$ of the field $\mathbb{Q}$ of rational numbers. Then $\mathbb{Z}_{(6)}/\mathcal{J}(\mathbb{Z}_{(6)}) \simeq \mathbb{Z}/6\mathbb{Z}$ and idempotents do not lift modulo the Jacobson radical because the factor ring contains four idempotents while $\mathbb{Z}$ only two.*

(ii) As shown recently [3, Theorem 2.1], there exists a counterexample of a (commutative) ring when idempotents lift modulo the Jacobson radical but this property does not pass to its ring of square matrices.

The proof of how $(L3')$ implies $(L4')$ is an adaptation of [9, Theorem 2.9]. Note the expected transmission from Fact 4.4.1(ii) to the equivalence of $(L3')$ and $(L4')$.

Theorem 4.4.3. *The following conditions are equivalent:*

(L1') for every $n \in \mathbb{N}$, any direct summand of a right R-module $R^{(n)}/I^{(n)}$ has a projective I-cover

(L2') for every $n \in \mathbb{N}$, if P' is a direct summand of $(R/I)^{(n)}$, then there is a direct summand P of $R^{(n)}$ such that $P' = P + I^{(n)}/I^{(n)} (\simeq P/PI)$

(L3') $M_n(I)$ is lifting in $M_n(R)$ for every $n \in \mathbb{N}$

(L4') every direct summand of a finitely generated right R-module with a projective I-cover has a projective I-cover.

Proof. $(L4') \to (L1')$: A special case when the finitely generated module is $R^{(n)}/I^{(n)}$, because the canonical projection $R^{(n)} \to R^{(n)}/I^{(n)}$ forms its projective I-cover.

$(L1') \to (L2')$: Let P' be a direct summand of $(R/I)^{(n)} \simeq R^{(n)}/I^{(n)}$, i.e. there are $j : P' \to R^{(n)}/I^{(n)}$, $\theta : R^{(n)}/I^{(n)} \to P'$ with $\theta \circ j = 1_{P'}$ and let $\psi : Q \to P'$ be a projective I-cover of P'. We know that the canonical projection $\pi : R^{(n)} \to R^{(n)}/I^{(n)}$ is a projective I-cover. The free module R^n is obviously projective, hence we have a homomorphism $\rho : R^n \to Q$ such that $\varphi \circ \rho = \theta \circ \pi$. We have a commutative diagram:

$$\begin{array}{ccccc} R^{(n)} & \xrightarrow{\pi} & R^{(n)}/I^{(n)} & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\theta} & & \\ Q & \xrightarrow[\psi]{} & P' & \longrightarrow & 0 \end{array}$$

But $\theta \circ \pi$ is an epimorphism and the kernel of φ is superfluous (since it is even I-superfluous), thus ρ is an epimorphism. By the projectivity of Q, there is $i : Q \to R^{(n)}$ with $\rho \circ i = 1_Q$. It follows that $\theta(i(Q) + I^{(n)}/I^{(n)}) = (\theta \circ \pi \circ i)(Q) = (\psi \circ \rho \circ i)(Q) = \psi(Q) = P' = (\theta \circ j)(P')$ and $(\pi \circ i)(Q) = P + I^{(n)}/I^{(n)} = j(P') = P'$. We put $P := i(Q)$.

$(L2') \to (L3')$: First, there is an isomorphism $M_n(R/I) \simeq M_n(R)/M_n(I)$ and denote $\overline{\mathbb{A}} \in M_n(R/I)$ for a matrix $\mathbb{A} \in M_n(R)$ (then $\overline{\mathbb{A}} = \mathbb{A} + M_n(I)$). Let $\mathbb{X}^2 - \mathbb{X} \in M_n(I)$. Since $P' := \overline{\mathbb{X}}(R/I)^{(n)}$ is a direct summand of $(R/I)^{(n)}$, by $(L2')$ there is a direct summand P of $R^{(n)}$ with the ideal factor equal to P'. Let $\mathbb{Y} = \mathbb{Y}^2$ be an idempotent matrix with $P = \mathbb{Y}R^{(n)}$. Then $P' = P + I^{(n)}/I^{(n)} = \overline{\mathbb{Y}}(R/I)^{(n)}$ implying $\overline{\mathbb{X}} = \overline{\mathbb{Y}}$. Thus $\mathbb{X} - \mathbb{Y} \in M_n(I)$ and $\mathbb{Y}$ is a lift of $\mathbb{X}$.

$(L3') \to (L4')$: The trivial case $M = 0$ is obvious. Let M be a non-zero finitely generated right R-module with a projective I-cover. Similarly as in [9, Lemma 2.8], we infer that M is of the form $\mathbb{E}R^{(n)}/\mathbb{E}B^{(n)}$ for a non-zero idempotent matrix $\mathbb{E}$ and a right ideal $B \subseteq I$. Then $\psi : \mathbb{E}R^{(n)} \to \mathbb{E}R^{(n)}/\mathbb{E}B^{(n)}$ is a projective I-cover of M with the kernel $\mathbb{E}B^{(n)}$. Let

$$\begin{aligned} \widetilde{S} &:= \{f \in End_R(R^{(n)}) \mid f(\mathbb{E}B^{(n)}) \subseteq \mathbb{E}B^{(n)}\} \\ \widetilde{T} &:= \{g \in End_R(R^{(n)}) \mid g(R^{(n)}) \subseteq \mathbb{E}B^{(n)}\} \\ S &:= \{h \in End_R(\mathbb{E}R^{(n)}) \mid h(\mathbb{E}B^{(n)}) \subseteq \mathbb{E}B^{(n)}\} \\ T &:= \{k \in End_R(\mathbb{E}R^{(n)}) \mid k(R^{(n)}) \subseteq \mathbb{E}B^{(n)}\} \end{aligned}$$

and as in [9, Theorem 2.9] we conclude that $End_R(M) = S/T = \mathbb{E}\widetilde{S}\mathbb{E}/\mathbb{E}\widetilde{T}\mathbb{E}$. Note that $\widetilde{T}$ is a right ideal in $End_R(R^{(n)}) \simeq M_n(R)$ that is canonically isomorphic to a right ideal contained in $M_n(I)$. Since $M_n(I)$ is lifting by $(L3')$, $\widetilde{T}$ is lifting in $M_n(R)$ by Proposition 4.4.1(i). Then $\mathbb{E}\widetilde{T}\mathbb{E}$ is lifting in $\mathbb{E}\widetilde{S}\mathbb{E}$ by [9, Lemma 1.5], and, in particular, T is lifting in S.

Let G be a direct summand of M. Then there is an idempotent endomorphism $\overline{f} = \overline{f}^2 \in End_R(M)$ with $\overline{f}(M) = G$. Because T is lifting in S, choose $f \in S$ to be a lift of $\overline{f}$. Now $\psi(f(p)) = \overline{f}(\psi(p))$ for each $p \in \mathbb{E}R^{(n)}$ and $G = \overline{f}(M) = \overline{f}(\psi(\mathbb{E}R^{(n)})) = \psi(f(\mathbb{E}R^{(n)}))$. By defining $Q := f(\mathbb{E}R^{(n)})$ we acquire a projective direct summand of $\mathbb{E}R^{(n)}$, because f was idempotent. Then ψ restricted to Q is a projective I-cover of G. Indeed, $\ker(\psi) \ll_I \mathbb{E}R^{(n)}$ implies $\ker(\psi \restriction Q) = \ker(\psi) \cap Q = \pi_Q(\ker \psi) \ll_I \pi_Q(\mathbb{E}R^{(n)}) = Q$ where we used Lemma 4.1.3(ii) on the canonical projection $\pi_Q : \mathbb{E}R^{(n)} \to Q$. ∎

4.5 Main Result

Let us start first with a technical lemma that is used in the proof of the main theorem. It is based on [16, Lemma 2.1(b)]. We state and prove it for the sake of completeness.

Lemma 4.5.1. *Let G be an I-supplement submodule of a projective module P. Then there is a pure submodule K of P such that G is an I-supplement of K. Moreover, if K is finitely generated then $K \oplus G = P$.*

Proof. There is a submodule K' of P with $K' + G = P$ and $G \cap K' \ll_I G$. By the projectivity of P there is some $\gamma \in End_R(P)$ such that $\gamma(P) \subseteq G$ and $(1-\gamma)(P) \subseteq K'$. By Lemma 4.1.14, G is also a supplement of K' in P, especially of $(1-\gamma)(P)$ in P. By the minimality of G in the set $\{G' \leq M \mid (1-\gamma)(P)+G' = M\}$, $\gamma(P) = G$. Also $\gamma^2(P) + (\gamma \circ (1-\gamma))(P) = \gamma(P) = G$. Since $G \cap K'$ is superfluous in G, $\gamma^2(P) = \gamma(P)$. Define $K := \sum_{i=1}^{\infty} \ker(\gamma^i)$. For any $i \in \mathbb{N}$, $\ker(\gamma^i) \subseteq (1-\gamma^i)(P) = (1-\gamma)(P) \subseteq K'$, hence $K \subseteq K'$. We have proved $G \cap K \ll_I G$.

By the projectivity of P there is $\delta \in End_R(P)$ with $\gamma^2 \circ \delta = \gamma$. Applying induction, we obtain the equation $\gamma^{n+1} \circ \delta^n = \gamma$ for any $n \in \mathbb{N}$. Define $\alpha_n := 1_P - \delta^n \circ \gamma^n \in End_R(P)$, $n \in \mathbb{N}$. All the inclusions $\ker(\gamma^n) \subseteq \alpha_n(P) \subseteq \ker(\gamma^{n+1})$ hold, because $\gamma^{n+1} \circ (1_P - \delta^n \circ \gamma^n) = \gamma^{n+1} - \gamma \circ \gamma^n = 0$. Hence $K = \sum_{i=1}^{\infty} \alpha_i(P)$. Obviously $(1_P - \gamma \circ \delta)(P) + (\gamma \circ \delta)(P) = P$. The inclusions $(1_P - \gamma \circ \delta)(P) \subseteq \ker(\gamma) \subseteq K$, $(\gamma \circ \delta)(P) \subseteq G$ imply $K + G = P$. Now G is an I-supplement of K in P.

We prove that K is pure in P by testing positive for the flatness of P/K in the short exact sequence $0 \to K \oplus C \to P \oplus C \to P/K \to 0$, where C is a complement of P in a free module F. For any $(x, c) \in K \oplus C$, let $m \in \mathbb{N}$ be the appropriate index for which $x \in \ker(\gamma^m)$. Then $\alpha_m(P) \subseteq K$ and $\alpha_m(x) = x$. To get the desired homomorphism, we put $\widetilde{\alpha}_m := \alpha_m \oplus 1_C \in \mathrm{Hom}_R(F, K \oplus C)$.

Finally, assume $K = \ker(\gamma^m)$ for some $m \in \mathbb{N}$. Let $y \in K \cap G$. Then $y = \gamma^m(z)$ for some $z \in P$ and $\gamma^m(y) = 0$. It follows that $z \in \ker(\gamma^{2m}) = \ker(\gamma^m)$ and so $y = 0$. We have shown that G is a direct summand of P with the complement K. ∎

Theorem 4.5.2. *The following is equivalent:*

(1) every I-supplement in a finitely generated projective R-module P is a direct summand

(2) every finitely generated I-(semi)projective R-module is projective

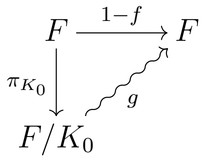
(3) every finitely generated flat R-module M with projective R/I-module M/MI is projective

(4) for every projective R-module Q, if the ideal factor Q/QI is finitely generated then Q is finitely generated

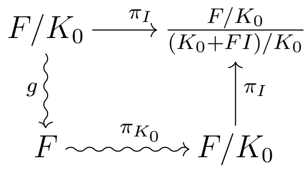
Proof. (1) $\leftrightarrow$ (2): By Theorem 4.2.3, this holds true even for a non-finitely generated projective modules.

(2) $\to$ (3): Let M be a finitely generated flat right R-module. Let $\alpha : F \to M$ be an epimorphism from a finitely generated free module F. Then there is a short exact sequence $0 \to \ker(\alpha) \to F \overset{\alpha}{\to} M \to 0$. Denote $K := \ker(\alpha)$. By the assumption on M, the induced short exact sequence $0 \to K/KI \to F/FI \to M/MI \to 0$ of R/I-modules splits. Then K/KI is finitely generated and so $K = K_0 + KI$ for some finitely generated submodule K_0 of K. By the well known Villamayor's flatness test applied on M, there is a homomorphism $f : F \to K$

that is identical on generators of K_0 and therefore on the whole K_0. The factor-module F/K_0 is finitely presented. Because $K_0 \subseteq \ker(1_F - f)$ we have some homomorphism g that makes the diagram commuting:



We want to show that F/K_0 is I-semiprojective. By Proposition 4.1.12(ii) it is enough to show that the square commutes:



From the inclusions $f(F) \subseteq K \subseteq K_0 + FI$ we infer the following identity:

$$\pi_I \circ \pi_{K_0} \circ (g \circ \pi_{K_0}) = (\pi_I \circ \pi_{K_0}) \circ (1 - f) = \pi_I \circ \pi_{K_0},$$

Because π_{K_0} is an epimorphism, the square indeed commutes.

By (2), F/K_0 is projective and K_0 is a direct summand of F. Denote by C a complement of K_0 in F, which is obviously projective and finitely generated, because it is also a direct summand of F. Then the factor-module $C/(C \cap K) \simeq (C + K)/K = F/K$ is isomorphic to M and therefore it is flat. Note that we have the inclusions

$$C \cap K \subseteq C \cap (K_0 + (K_0 + C)I) = C \cap (K_0 + CI) = (C \cap K_0) + CI = CI$$

which means that $C \cap K$ is I-superfluous in C by Lemma 4.1.2(iv), i.e the canonical projection $\pi_{C \cap K} : C \to C/C \cap K$ is a projective I-cover of $C/(C \cap K)$. By Lemma 4.3.4 $\pi_{C \cap K}$ is an isomorphism and we infer that the module $M \simeq C/C \cap K$ is projective.

(3) $\leftrightarrow$ (4): By Corollary 4.3.5.

(3) $\to$ (1): Let P be a finitely generated projective R-module and G be a submodule of P such that it is an I-supplement. By Lemma 4.5.1 there is a pure submodule K of P that has an I-supplement G.

Observe that K/KI is finitely generated. Indeed, from $G \cap K \subseteq GI \subseteq PI$ we get

$$\frac{G + PI}{PI} \oplus \frac{K + PI}{PI} = P/PI$$

Because K is pure in P we have $K \cap PI = KI$. Then

$$K + PI/PI \simeq K/K \cap PI = K/KI$$

and K/KI is isomorphic to a direct summand of a finitely generated R-module P/PI, in particular it is finitely generated.

The right R-module P/K is countably presented, finitely generated and flat with the ideal factor $(P/K)/(PI+K/K)) \simeq P/(PI+K) \simeq G+PI/PI$ projective as a right R/I-module. By the condition (3), P/K projective. This yields K is finitely generated and by addendum of Lemma 4.5.1 we accomplish what was required. ∎

By Example 4.1.7, the condition "every projective $\mathcal{J}(R)$-semicover is a projective cover" is stronger than (4) with $I = \mathcal{J}(R)$.

Using terminology from radical theory of rings, a (Kurosh-Amitsur) radical γ is *matrix extensible* if it satisfies the equation $M_n(\gamma(R)) = \gamma(M_n(R))$, $n \in \mathbb{N}$ for any ring R. As an application of the previous theorem, we show now that (4), and hence all of (1)–(4), are true for the Levitzki radical $\mathcal{L}(R)$, i.e. the condition $(L3')$ of Theorem 4.4.3 holds. This is ensured when the ideal I consists of nilpotent elements and is matrix extensible, since then also the ideal $M_n(I)$ would consist only of nilpotent elements and idempotents lift modulo these ideals. It is well known that both β and $\mathcal{L}$ are matrix extensible and the nil radical $\mathcal{N}$ is matrix extensible if the famous Köthe's Conjecture holds true.

By Example 4.4.2, we were afforded examples of rings that do not satisfy $(L3')$, nevertheless the condition (4) still holds (and in fact it does for any commutative ring).

Proposition 4.5.3. *Let R be a ring satisfying the conditions of Theorem 4.4.3. Then the condition* (4) *in Theorem 4.5.2 holds true.*

Proof. Let P be projective with the ideal factor P/PI finitely generated. Then $Q' := P/PI$ is a direct summand of a finitely generated free R/I-module isomorphic to $(R/I)^{(m)}$ for some $m \in \mathbb{N}$.

By the assumption, idempotents lift modulo $M_n(I)$ in $M_n(R)$ for all $n \in \mathbb{N}$. By Theorem 4.4.3$(L2')$, there is a (finitely generated) direct summand Q of $R^{(m)}$ such that $Q/QI \simeq Q'$. From the second isomorphism theorem for modules $\frac{P/PI}{P\mathcal{J}(R)/PN} \simeq P/P\mathcal{J}(R)$ and $\frac{Q/QI}{Q\mathcal{J}(R)/QN} \simeq Q/Q\mathcal{J}(R)$. Projective modules are determined by their radical factors [15, Theorem 2.3] so $P/P\mathcal{J}(R) \simeq Q/Q\mathcal{J}(R)$ implies $P \simeq Q$ and P is finitely generated. ∎

Example 4.5.4. *The conditions* (1)–(4) *of Theorem 4.5.2 hold for $I = \beta(R)$ and $I = \mathcal{L}(R)$. They also hold for $I = \mathcal{N}(R)$ if Köthe's Conjecture is true.*

Bibliography for Chapter 4

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5. Open Problems

The following questions of interest have remained still unanswered:

Question 1. *Does the Corollary 1.3.5 hold without the additional set-theoretic axiom?*

Question 2. *Does the Theorem 2.3.4 hold without the additional set-theoretic axiom?*

Question 3. *Is it possible to exclude in the Theorem 3.2.3 the Type* II_f *or Type* III *or both?*

Question 4. *Does the equivalent conditions in Theorem 4.5.2 hold for* $I = \mathcal{N}(R)$ *without the assumption that Köthe's Conjecture is true?*