BACHELOR THESIS

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Vector fields on spheres

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Abstract: This thesis deals with partial results concerning the problem of existence of vector fields on spheres. The proof of the Hairy Ball Theorem is given using the tools of the theory of characteristic classes. Basic notions of algebraic topology are stated in order to define the Euler class. Its definition is followed by the computation of the Euler characteristic class for the tangent bundle of even-dimensional sphere. In the rest of the text, the method of construction of vector fields on spheres using the orthogonal multiplication is explained and the Radon-Hurwitz-Eckmann Theorem is proved. A brief historical background of the existence of the finite-dimensional real division algebras is mentioned at the end.

Keywords: vector fields on spheres, Euler characteristic class, Hairy Ball Theorem, Clifford algebras, Radon-Hurwitz numbers
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Introduction

This bachelor thesis aims to approach the problem of existence of vector fields on spheres, which presents a classical problem in differential topology solved by Adams in 1962 using the tools of topological $K$-theory. We will not explain his proof. Nevertheless, we state and prove results that had been known before Adams and revisited by Eckmann in 1950s.

In the beginning of the first chapter we define basic notions concerning vector bundles and give basic examples. Then, we move, briefly, to algebraic topology to define the homology and cohomology functors and state results that we will need in Section (1.3), where we apply the theory of characteristic classes, described in [Milnor et al. 1974], on the problem of existence of nowhere-zero vector field on a sphere. The merit of the first chapter is Section (1.3) about oriented bundles where we explicitly compute the Euler characteristic class of the tangent bundle of even-dimensional sphere of which the so-called Hairy Ball Theorem is a simple corollary. This computation and corresponding lemmas are very detailed unlike the ones that can be usually found in literature.

In the second chapter, we introduce Clifford algebras. Then we explain the connection between the orthogonal multiplication, the structure of ungraded modules over Clifford algebras and the existence of vector fields on spheres. Moreover, using the Radon-Hurwitz numbers we describe the limitations of the method of construction of orthonormal vector fields using orthogonal multiplication. Here we will try to fill in some details in proofs that can be found in [Husemoller 2013] and that the bachelor thesis [Spáčil 2007] omitted.

Finally, in the third chapter we mention how the study of finite-dimensional division algebras is associated to the problem of parallelizable spheres from historical point of view.
1. Euler class

The goal of this chapter is to show that spheres of even dimension are not parallelizable manifolds. It can be even shown that these spheres do not possess any nowhere zero vector field. In order to obtain these results, we will define some characteristic classes, namely the Thom and the Euler classes, state and prove some basic properties of these structures. We assume that the reader is familiar with basics of abstract algebra and differential geometry.

In this whole thesis $n$ will denote an arbitrary non-negative integer and

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

the $n$-dimensional sphere. Furthermore, we will denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$.

1.1 Vector bundles

**Definition 1.1.1.** A real vector bundle $\xi$ is a triple $(E, B, \pi)$ consisting of topological spaces $B$ and $E = E(\xi)$ and a surjective continuous map $\pi : E \to B$, where for each $b \in B$ the structure of a vector space over $\mathbb{R}$ is defined in $\pi^{-1}(b)$.

Moreover, the following condition of local triviality must be satisfied: for each $b \in B$ there exists a neighbourhood $U \subseteq B$ of $b$ in $B$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ together with a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that for each $b' \in U$ the correspondence $x \mapsto h(b', x)$ defines an isomorphism of vector spaces $\mathbb{R}^n$ and $\pi^{-1}(b)$.

The preimage of $b \in B$ with respect to $\pi$ is called a fiber over $b$. The pair $(U, h)$ is called a local coordinate system for $\xi$ about $b$. The spaces $E$ and $B$ are sometimes referred to as the total and the base space respectively.

Note that the dimension of the fibre $n$ needs not to be globally constant. However, it has to be locally constant, that is, constant on every connected component. If $n$ is globally constant, the vector bundle $\xi$ is called an $n$-vector bundle.

We say that a vector bundle is smooth if both the base space and the total space are smooth manifolds, the projection $\pi$ is smooth and for each local coordinate system $(U, h)$ the map $h$ is a diffeomorphism.

For every topological space $B$ and for each $n \in \mathbb{N}$ we are able to construct so called trivial bundle, denoted by $\varepsilon^n_B$, as follows.

Set $\varepsilon^n_B = (B \times \mathbb{R}^n, B, \pi)$ where $\pi : B \times \mathbb{R}^n \to B$ is the projection onto the first factor and $B \times \mathbb{R}^n$ is endowed with the product topology. The linear structure in the fibre $\pi^{-1}(b)$ is given by $t_1(b, v_1) + t_2(b, v_2) = (b, t_1v_1 + t_2v_2)$ for every $t_1, t_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{R}^n$. The condition of local triviality for $\varepsilon^n_B$ to be an $n$-vector bundle follows from the fact that we can choose for every $b \in B$ the local coordinate system with $U = B$ and define $h = \text{id}_{B \times \mathbb{R}^n}$.

Let $M$ be a real smooth manifold of dimension $n$ with smooth atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ where $I$ is an index set. The tangent space of $M$ at $m \in M$, denoted by $T_mM$, is the set of all linear forms $v$ on $C^\infty(M, \mathbb{R}) := \{f : M \to \mathbb{R} : f\text{ smooth}\}$,
called tangent vectors, defined by
\[ v(f) = \frac{d(f \circ \gamma)}{dt}(0) \]
where \( f \in \mathcal{C}^\infty(M, \mathbb{R}) \) and \( \gamma : (-\varepsilon, \varepsilon) \to M \) a smooth curve with \( \gamma(0) = m \) and \( \varepsilon > 0 \). Using a coordinate chart \((U_\alpha, \varphi_\alpha)\) around \( m \) with coordinate functions \( x_1, \ldots, x_n \), it is straightforward to verify that \( T_m M \) is an \( n \)-dimensional vector space with basis \( \{ \frac{\partial}{\partial x_i} \}_{m} \) where \( \frac{\partial}{\partial x_i} \) is the tangent vector induced by the curve \( \gamma_i(t) = \varphi_\alpha^{-1}(\varphi_\alpha(m) + te_i) \) for \( t \in (-\varepsilon, \varepsilon) \). Note that each element \( \frac{\partial}{\partial x_i} \) can be equivalently described by
\[ \frac{\partial}{\partial x_i} \bigg|_m (f) = \frac{\partial(f \circ \varphi_\alpha^{-1})}{\partial x_i}(\varphi_\alpha(m)) \]
on \( f \in \mathcal{C}^\infty(M, \mathbb{R}) \).

Denote by \( TM \) the set of all pairs \((m, v)\) where \( v \in T_m M \) and \( m \in M \). An important example of smooth \( n \)-vector bundle is the tangent bundle \( \tau_M = (TM, M, \pi) \) of the smooth manifold \( M \). The smooth projection map \( \pi : TM \to M \) is given by the correspondence \((m, v) \mapsto m \) where \( m \in M \) and \( v \in T_m M \). The structure of the vector space on fibers is clear. We shall prove that \( TM \) has the structure of a real smooth manifold of dimension \( 2n \) from which the condition of local triviality easily follows.

The smooth coordinate charts on \( TM \) are given as follows. For the coordinate chart \((U_\alpha, \varphi_\alpha)\) around \( m \) as above define the linear map \( O_m^\alpha : T_m M \to \mathbb{R}^n \) assigning to each \( \frac{\partial}{\partial x_i} \) the \( i \)-th element of the canonical basis of \( \mathbb{R}^n \). Now we define \( \psi_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{R}^n \) by \( \psi_\alpha((m, v)) = (\varphi_\alpha(m), O_m^\alpha(v)) \). Let \((U_\alpha, \varphi_\alpha)\) and \((U_\beta, \varphi_\beta)\) be two distinct coordinate charts with non-empty intersection. Then the transition function on \( \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha \cap U_\beta) \) is of the form
\[ (\psi_\beta \circ \psi_\alpha^{-1})(x, v) = (\varphi_\beta \circ \varphi_\alpha^{-1}(x), O_m^\beta \circ (O_m^\alpha)^{-1}(v)) \]
where \( v \in \mathbb{R}^n \), \( x = \varphi_\alpha(m) \) and \( m \in U_\alpha \cap U_\beta \). Since the function in the first component is the transition function between two maps from the smooth atlas on \( M \), \( \varphi_\beta \circ \varphi_\alpha^{-1} \) is a diffeomorphism.

Denote by \( \frac{\partial}{\partial x_i} \) and \( \frac{\partial}{\partial y_j} \) elements from the bases of \( T_m M \) corresponding to the coordinate charts \((U_\alpha, \varphi_\alpha)\) and \((U_\beta, \varphi_\beta)\) respectively.

By chain rule it holds that for \( 1 \leq i \leq n \)
\[ \frac{\partial}{\partial x_i} \bigg|_m (f) = \frac{\partial(f \circ \varphi_\alpha^{-1})}{\partial x_i}(\varphi_\alpha(m)) = \frac{\partial(f \circ \varphi_\alpha^{-1})}{\partial x_i} \circ (\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(m)) = \sum_{j=1}^{n} \frac{\partial}{\partial y_j} \bigg|_m \frac{\partial (\varphi_\beta \circ \varphi_\alpha^{-1})}{\partial x_i}(\varphi_\alpha(m)) \]
where \( \frac{\partial (\varphi_\beta \circ \varphi_\alpha^{-1})}{\partial x_i}(\varphi_\alpha(m)) \) denotes the partial derivative of the \( j \)-th component of \( \varphi_\beta \circ \varphi_\alpha^{-1} \) with respect to \( x_i \) at the point \( \varphi_\alpha(m) \).
It follows that the application of \( O_m^\beta \circ (O_m^\alpha)^{-1} \) or \( O_m^\alpha \circ (O_m^\beta)^{-1} \) (the inverse of \( O_m^\alpha \circ (O_m^\beta)^{-1} \)) is given by multiplication by the Jacobi matrix

\[
\left( \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})}{\partial x_j}(\varphi_\alpha(m)) \right)_{i,j=1}^n \quad \text{or} \quad \left( \frac{\partial(\varphi_\alpha \circ \varphi_\beta^{-1})}{\partial y_j}(\varphi_\beta(m)) \right)_{i,j=1}^n
\]

of the diffeomorphism \( \varphi_\beta \circ \varphi_\alpha^{-1} \) at the point \( \varphi_\alpha(m) \) or of the diffeomorphism \( \varphi_\alpha \circ \varphi_\beta^{-1} \) at the point \( \varphi_\beta(m) \) respectively. The coefficients of these Jacobi matrices are smooth functions of the corresponding coordinate functions. We have for the basic elements that

\[
\frac{\partial}{\partial x_i} \bigg|_m = \sum_{j=1}^n \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})}{\partial x_i}(\varphi_\alpha(m)) \frac{\partial}{\partial y_j} \bigg|_m \quad \text{and} \quad \frac{\partial}{\partial y_j} \bigg|_m = \sum_{i=1}^n \frac{\partial(\varphi_\alpha \circ \varphi_\beta^{-1})}{\partial y_j}(\varphi_\beta(m)) \frac{\partial}{\partial x_i} \bigg|_m.
\]

Thus, both \( O_m^\beta \circ (O_m^\alpha)^{-1} \) and its inverse are linear bijections of \( \mathbb{R}^n \). And so the transition function \( \psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)) \to \psi_\beta(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)) \) is a diffeomorphism. Analogously for \( \psi_\alpha \circ \psi_\beta^{-1} \).

As a result we obtain that \( \{(\pi^{-1}(U_\alpha), \psi_\alpha) : \alpha \in I \} \) forms an atlas on \( TM \).

Now the condition of local triviality for the tangent bundle follows from the fact that for each \( m \in M \) we can choose a coordinate chart \( (U_\alpha, \varphi_\alpha) \) where \( m \in U_\alpha \) and define the diffeomorphism \( h : U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha) = \bigcup_{m \in U_\alpha} T_m M \) by \( (m, v) = (m, (O_m^\alpha)^{-1}(v)) \).

**Definition 1.1.2.** Let \( \xi = (E(\xi), B(\xi), \pi_\xi) \) and \( \nu = (E(\nu), B(\nu), \pi_\nu) \) be two vector bundles. A **vector bundle map** is the pair \( (F, f) \) of continuous maps which is linear in fibres, that is \( F \circ \pi_\xi^{-1}(b) : \pi_\xi^{-1}(b) \to \pi_\nu^{-1}(b) \) is linear, such that the following diagram commutes.

\[
\begin{array}{ccc}
E(\xi) & \xrightarrow{F} & E(\nu) \\
\downarrow{\pi_\xi} & & \downarrow{\pi_\nu} \\
B(\xi) & \xrightarrow{f} & B(\nu)
\end{array}
\]

A vector bundle map is **smooth** if both \( f \) and \( F \) are smooth maps and \( \xi \) and \( \nu \) are smooth bundles.

The abstract definition of the manifold \( T\mathbb{S}^n \) is hard to work with, however, we can identify it with certain \( 2n \)-dimensional submanifold \( M \) of \( \mathbb{R}^{2n+2} \).

For \( f : M \to N \) a smooth map denote by \( Tf : TM \to TN, Tf(m,v) = (f(m), T_m f(v)) \) where

\[
(T_m f(v))(g) = v(g \circ f)
\]

for \( g \in C^\infty(N, \mathbb{R}) \) and \( v \in T_m M \). The map \( Tf \) is called the tangent map for \( f \). The assignment induces a functor on the category of smooth manifolds with smooth maps between them.
Theorem 1.1.3. Let \( i : S^n \to \mathbb{R}^{n+1} \) be the canonical inclusion. Then the image of the tangent map \( Ti : T\mathbb{S}^n \to T\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \) is \( M = \{(x, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} | \langle x, v \rangle = 0\} \) and \( Ti \) is a homeomorphism (even diffeomorphism) between \( T\mathbb{S}^n \) and \( M \).

Proof. As it was mentioned above, the tangent map of a smooth map between two manifolds induces a smooth vector bundle map between tangent bundles.

As \( i \) is a diffeomorphism onto its image, then \( Ti \) is a diffeomorphism onto its image, this is again easily verified in any coordinate chart. It follows that \( Ti(T_i \mathbb{S}^n) \) is an \( n \)-dimensional vector subspace of \( T_x \mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \) where \( x \in \mathbb{S}^n \).

Thus it remains to verify that \( (T_x i(v), x) = 0 \) for every \( v \in T_x \mathbb{S}^n \), \( x \in \mathbb{S}^n \). But by definition \( \mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} \) and \( (T_x i(v))(x) = v(f \circ i)(x) \) for \( f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}) \). If we take \( f(x) = \|x\|^2 \), then \( f \circ i = 1 \) on \( \mathbb{S}^n \) and thus \( v(f \circ i)(x) = (v(1))(x) = \frac{df}{dx}(0) = 0 \). On the other hand

\[
(T_x i(v)) f(x) = \langle T_x i(v), \nabla f(x) \rangle
\]

where \( \nabla f(x) \) denotes the gradient \( (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_{n+1}}(x)) \in \mathbb{R}^{n+1} \). This shows that \( T_x i(v) \perp \nabla f(x) \). Since \( \nabla f(x) = 2x \), we have that \( T_x i(v) \perp x \). \( \Box \)

Definition 1.1.4. Vector bundles \( \xi = (E(\xi), B, \pi_\xi) \) and \( \nu = (E(\nu), B, \pi_\nu) \) are said to be isomorphic, written \( \xi \cong \nu \), if there is a homeomorphism \( f : E(\xi) \to E(\nu) \) for which the restriction \( f \mid \pi_\xi^{-1}(b) \) is an isomorphism of vector spaces \( \pi_\xi^{-1}(b) \) and \( \pi_\nu^{-1}(b) \) for each \( b \in B \).

In some cases the tangent bundle of a manifold and the trivial bundle are isomorphic, these manifolds are called parallelizable. However, this is not necessarily true for all manifolds, as we will see in Section 1.3.

Definition 1.1.5. A cross-section of an \( n \)-vector bundle \( \xi = (E, B, \pi) \) is a continuous function \( s : B \to E \) so that \( \pi \circ s = \text{id}_B \). A cross-section of \( \pi_M \) is called a vector field. The cross-section is nowhere zero if \( s(b) \) is a non-zero vector of \( \pi^{-1}(b) \) for each \( b \in B \). Cross-sections \( s_1, \ldots, s_n \) are nowhere dependent if for each \( b \in B \) the vectors \( s_1(b), \ldots, s_n(b) \) are linearly independent.

Now we will give an example of a vector bundle which is not isomorphic to a trivial one.

Let \( \mathbb{R}P^n \) be the space of lines in \( \mathbb{R}^{n+1} \) passing through the origin. Now there is a canonical projection \( p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n \) which maps a non-zero vector \( x \) to the line \( \{tx : t \in \mathbb{R}\} \) spanned by \( x \). This correspondence induces an equivalence relation \( \sim \) in \( \mathbb{R}^{n+1} \) given by \( [x]_\sim = p(x) \). Denote by \( [x_1 : x_2 : \ldots : x_{n+1}] \) the equivalence class \( [x]_\sim \) of a non-zero element \( x = (x_1, x_2, \ldots, x_{n+1}) \) of \( \mathbb{R}^{n+1} \). For \( 1 \leq i \leq n+1 \) consider the sets \( U_i = \{[x_1 : \ldots : x_{n+1}] | x_i \neq 0\} \) and maps \( \varphi_i : U_i \to \mathbb{R}^n \) given by

\[
\varphi_i([x_1 : \ldots : x_{n+1}]) = \left( \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right).
\]

Obviously, for every equivalence class in \( U_i \) fractions \( \frac{x_j}{x_i} \) does not depend on the choice of representative for \( 1 \leq j \leq n+1 \), \( j \neq i \) and so \( \varphi_i \) is well defined for every \( i = 1, \ldots, n+1 \). Their inverses are given by

\[
\varphi_i^{-1}(x_1, \ldots, x_n) = [x_1 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n].
\]
It is straightforward to verify that the set \( \{(U_i, \varphi_i) : 1 \leq i \leq n + 1 \} \) defines a differential structure on \( \mathbb{R}P^m \) since, for example, the transition map between \( U_i \) and \( U_j \) where \( i < j \) is smooth as it is of the form

\[
\varphi_j \circ \varphi_i^{-1}(x_1, \ldots, x_n) = \left( \frac{x_1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i+1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j} \right).
\]

Now, we shall construct the canonical line bundle \( \gamma_n^1 \) over \( \mathbb{R}P^m \). The total space \( E(\gamma_n^1) \) is defined by \( \{(x)_\sim, v) : [x]_\sim \in \mathbb{R}P^m, v \in [x]_\sim \} \) and it is endowed with the subspace topology of the product \( \mathbb{R}P^m \times \mathbb{R}^{n+1} \). The base space is \( \mathbb{R}P^m \) and the projection \( \pi : E(\gamma_n^1) \rightarrow \mathbb{R}P^m \) is given by \( \pi((x)_\sim, v)) = [x]_\sim \).

Note that the canonical line bundle is a subbundle of the trivial bundle \( \mathbb{R}^{n+1} \).

Now it is easy to verify that the map \( h_i : U_i \times \mathbb{R} \rightarrow \pi^{-1}(U_i), h_i(([x]_\sim, t)) = ([x]_\sim, t\frac{x}{x_i}) \) satisfies the condition of local triviality.

Assume that \( s : \mathbb{R}P^1 \rightarrow E(\gamma_1^1) \) is a nowhere zero cross-section of the canonical line bundle \( \gamma_1^1 \). Now consider the map \( w = \lambda \circ s \circ p |_{S^1} : S^1 \rightarrow \mathbb{R}^2 \), where \( \lambda \) denotes the projection onto the second coordinate of \( ([x]_\sim, v) \) and \( p |_{S^1} \) is the restriction of the canonical projection defined above. Thus \( w(x) = \lambda(x)x \) for a continuous function \( \lambda : S^1 \rightarrow \mathbb{R} \). Note that \( w(x) = w(-x) \) since \( p(x) = p(-x) \), and so \( \lambda(x) = -\lambda(-x) \). Because \( S^1 \) is a connected space the intermediate value theorem holds and so there exists some \( y \in S^1 \) such that \( w(y) = 0 \). We have obtained a contradiction with \( s \) being nowhere zero. As a consequence we have that there is no nowhere zero cross-section of the canonical line bundle \( \gamma_1^1 \).

Finally, for the sake of contradiction suppose that \( \mathbb{R}^{n+1} \cong \gamma_1^1 \). For the trivial bundle there exists \( z \) a nowhere zero cross-section given by \( z([x]_\sim) = ([x]_\sim, e_1) \) for any \( x \in S^1 \) and \( e_1 = 1 \). Let \( f : \mathbb{R}P^1 \times \mathbb{R}^1 \rightarrow E(\gamma_1^1) \) be a homeomorphism from the definition of isomorphism of bundles. Thus the composition \( s = f \circ z \) is a nowhere zero cross-section of \( \gamma_1^1 \) which is a contradiction.

**Theorem 1.1.6.** A real \( n \)-vector bundle \( \xi \) is isomorphic to \( \epsilon_n^1 \) if and only if \( \xi \) admits \( n \) cross-sections \( s_1, \ldots, s_n \) which are nowhere dependent.

We refer to [Milnor et al., 1974, Theorem 2.2.1] for the proof.

Although the notion of \( n \)-vector bundle seems to be an object of pure mathematics, there are applications in physics, for example in the theory of molecular vibrations.

Let \( M \) be a molecule consisting of \( m \) atoms in \( \mathbb{R}^3 \). Assume that there exist some natural state of equilibrium and also forces among the atoms forming \( M \) which force \( M \) to stay in the relatively solid shape. For example, the shape of the molecule of Carbon Tetrachlorine resembles a tetrahedron. Each atom \( x \) can deviate from that state by a vector in \( \mathbb{R}^3 \) and so we assign a 3-dimensional vector space of its deviations \( E_x \) to it. Denote by \( E = \bigcup_{x \in M} E_x \) the disjoint union of \( E_x \) and by \( \pi \) the map from \( E \) to \( M \) such that \( \pi(v) = x \) for \( v \in E_x \). Thus, we obtain the 3-vector bundle \( \delta = (E, M, \pi) \) which is, with a certain group action, used in the description of the molecular vibration. For more details on history and information about the topic of molecular vibrations see [Sternberg, 1995, Section 3].
1.2 Singular Homology and Cohomology

In the next few paragraphs we present some basic notions from algebraic topology and state basic results that we will use in order to define the Euler class.

Let $R$ be a commutative ring with a unit element. The pair $(C_\bullet, \partial_\bullet)$, where $C_\bullet = (C_n : n \in \mathbb{Z})$ is a collection of $R$-modules and $\partial_\bullet = (\partial_n : n \in \mathbb{Z})$ is a collection of $R$-homomorphisms $\partial_n : C_n \to C_{n-1}$, called the boundary operators, such that $\partial_n \circ \partial_{n+1} = 0$, is called a chain complex.

\[ \cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \]

Chain complexes of $R$-modules form a category $\text{Ch}(R)$ with chain maps as morphisms. A chain map between $(C_\bullet, \partial_\bullet)$ and $(D_\bullet, \partial_\bullet')$ is a collection of maps $p_n : C_n \to D_n$ such that for each $n \in \mathbb{N}$ we have $\partial_\bullet' \circ p_n = p_{n-1} \circ \partial_\bullet$.

\[ \cdots \xrightarrow{\partial_{n+2}'} D_{n+1} \xrightarrow{\partial_{n+1}'} D_n \xrightarrow{p_n} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}'} \cdots \]

Denote by $Z_n((C_\bullet, \partial_\bullet); R) = \text{Ker}(\partial_n)$ and by $B_n((C_\bullet, \partial_\bullet); R) = \text{Im}(\partial_{n+1})$. The relation $\partial_n \circ \partial_{n+1} = 0$ implies that $B_n((C_\bullet, \partial_\bullet); R) \subseteq Z_n((C_\bullet, \partial_\bullet); R)$, $n \in \mathbb{Z}$. The $n$-th homology group functor $H_n(\_ ; R)$ from $\text{Ch}(R)$ to $R$-Mod is defined on a chain complex $(C_\bullet, \partial_\bullet)$ as the factor $H_n((C_\bullet, \partial_\bullet); R) = Z_n((C_\bullet, \partial_\bullet); R)/B_n((C_\bullet, \partial_\bullet); R)$ on morphisms it is defined as corresponding maps of factors.

Note that $H_n((C_\bullet, \partial_\bullet); R)$ need not to be a group, however, this abuse of terminology is widely used, see for example in [Hatcher et al., 2002, p.196], [Milnor et al., 1974, p.258].

Consider $C^n = \text{Hom}_R(C_n, R)$ the dual $R$-module for any chain complex $(C_\bullet, \partial_\bullet)$ whose elements are $R$-homomorphisms of left $R$-modules $C_n$ and $R$. Analogously to $\partial_\bullet$ we define a map $\delta^n : C^n \to C^{n+1}$ by the correspondence

$\delta^n(\varphi)(\sigma) = \varphi(\partial_{n+1}(\sigma))$

where $\sigma \in C_{n+1}$ and $\varphi \in C^n$. This definition agrees with the convention used in [Hatcher et al., 2002, p.189], but differs to [Milnor et al., 1974, p.258].

And so we get a co-chain complex $(C^\bullet, \delta^\bullet)$

\[ \cdots \xrightarrow{\delta_{n-2}} C^{n-1} \xrightarrow{\delta_{n-1}} C^n \xrightarrow{\delta_n} C^{n+1} \xrightarrow{\delta_{n+1}} \cdots \]

Denote by $Z^n((C^\bullet, \delta^\bullet); R) = \text{Ker}(\delta^n)$ and $B^n((C^\bullet, \delta^\bullet); R) = \text{Im}(\delta^{n-1})$. The $n$-th cohomology group functor $H^n(\_ ; R)$ is then defined as

$H^n((C^\bullet, \delta^\bullet); R) = Z^n((C^\bullet, \delta^\bullet); R)/B^n((C^\bullet, \delta^\bullet); R)$.

Due to the fact that $\delta^{n+1} \circ \delta^n = 0$, the definition makes sense.

Let $X$ be an arbitrary topological space and $n \in \mathbb{N}_0$. Denote by $\Delta^n$ the convex hull of the canonical basis $e_1, \ldots, e_{n+1}$ of $\mathbb{R}^{n+1}$. 

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A continuous map $\sigma : \Delta^n \to X$ is called a **singular $n$-simplex** in $X$. Then we define $C_n(X; R)$ as the free left $R$-module with basis consisting of all $n$-simplices in $X$. It is clear that $C_n(-; R) : \text{Top} \to R\text{-Mod}$ is a covariant functor since for a continuous map $f : X \to Y$ the map $C_n(f; R) : C_n(X; R) \to C_n(Y; R)$ defined on the basis by $C_n(f; R)(\sigma) = f \circ \sigma$ is a module homomorphism and

$$C_n(g \circ f; R)(\sigma) = g \circ f(\sigma) = C_n(g; R)(f(\sigma)) = (C_n(g; R) \circ C_n(f; R))(\sigma).$$

For $i \in \{1, \ldots, n+1\}$ consider the face maps $d^i : \Delta^{n-1} \to \Delta^n$ induced by the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$. The $n$-th singular boundary operator given by

$$\partial_n(\sigma) = \sum_{i=1}^{n+1} (-1)^{i-1} \sigma \circ d^i$$

is a homomorphism of $R$-modules $\partial_n : C_n(X; R) \to C_{n-1}(X; R)$. The simplices that belong to $\text{Z}_n(X) = \text{Ker}(\partial_n)$ and $\text{B}_n(X) = \text{Im}(\partial_{n+1})$ are called singular $n$-cycles and $n$-boundaries respectively. $\text{Z}_0$ is defined as $C_0(X; R)$.

For any $\sigma \in C_{n+1}(X; R)$ we have

$$\partial_n \circ \partial_{n+1}(\sigma) = \partial_n \left( \sum_{i=1}^{n+2} (-1)^{i-1} \sigma \circ d^i \right) = \sum_{j=1}^{n+1} \sum_{i=1}^{n+2} (-1)^{i+j} \sigma \circ d^i \circ d^j$$

$$= \sum_{j=1}^{n+1} \sum_{i=1}^{j} (-1)^{i+j} \sigma \circ d^i \circ d^j + \sum_{j=1}^{n+1} \sum_{i=j+1}^{n+2} (-1)^{i+j} \sigma \circ d^i \circ d^j$$

$$= \sum_{j=1}^{n+1} \sum_{i=1}^{j} (-1)^{i+j} \sigma \circ d^i \circ d^j + \sum_{j=1}^{n+1} \sum_{i=j+1}^{n+2} (-1)^{i+j+1} \sigma \circ d^i \circ d^j$$

$$= \sum_{j=1}^{n+1} \sum_{i=1}^{j} (-1)^{i+j} \sigma \circ d^i \circ d^j + \sum_{i=1}^{n+1} \sum_{j=i+1}^{n+2} (-1)^{j+i+1} \sigma \circ d^i \circ d^j$$

$$= \sum_{1 \leq i \leq j \leq n+1} \left( (-1)^{i+j} \sigma \circ d^i \circ d^j + (-1)^{j+i+1} \sigma \circ d^i \circ d^j \right)$$

$$= 0,$$

where the forth equality holds since we have

$$d^i \circ d^j(x_1, \ldots, x_{n-1}) = d^i(x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_{n-1})$$

$$= (x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$

$$= d^i(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$

$$= d^i \circ d^{i-1}(x_1, \ldots, x_{n-1}),$$

for $n \geq i > j \geq 1$.

We have shown that the pair $(C_\bullet(X; R), \partial_\bullet)$ forms the **singular chain complex**,

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X; R) \xrightarrow{\partial_n} C_{n-1}(X; R) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(X; R).$$
Now we can finally define the crucial notion of the \( n \)-th singular homology group \( H_n(X; R) \) of \( X \) over \( R \) as the \( R \)-module obtained by applying the homology group functor on the chain complex \( (C_\bullet(X; R), \partial_\bullet) \), that is

\[
H_n(X; R) = H_n((C_\bullet(X; R), \partial_\bullet); R).
\]

Furthermore, we will apply the functor \( H^n(\_; R) \) in the topological setting in the similar way we did the \( n \)-th homology functor. To do so, consider the co-chain complex \( (C^\bullet(X; R), \delta^\bullet) \) dual to the singular chain complex \( (C_\bullet(X; R), \partial_\bullet) \).

\[
C^0(X; R) \xrightarrow{\delta_0} \cdots \xrightarrow{\delta_{n-1}} C^n(X; R) \xrightarrow{\delta_n} C^{n+1}(X; R) \xrightarrow{\delta_{n+1}} \cdots
\]

For \( n = 0 \) define \( B^0(X; R) = \{0\} \). The \( n \)-th singular cohomology group of \( X \) is then defined as the \( R \)-module obtained by applying the \( n \)-th cohomology group functor on \( (C^\bullet(X; R), \delta^\bullet) \)

\[
H^n(X; R) = H^n((C^\bullet(X; R), \delta^\bullet); R).
\]

When examining the construction of the functor \( H^n(\_; R) \) one can ask whether the equality of groups \( H^n(\text{Hom}_R(\_; R); R) = \text{Hom}_R(H_n(\_; R), R) \) holds. The answer is the Universal Coefficient Theorem. Our formulation slightly differs but is equivalent to a weaker form of [Hatcher et al., 2002, Theorem 3.2]. The key difference is that the following formulation does not involve the change of the coefficient ring, however, it explains the obstruction to the isomorphism

\[
H^n(\text{Hom}_R(\_; R); R) \cong \text{Hom}_R(H_n(\_; R), R).
\]

Theorem 1.2.1 (Universal Coefficient Theorem). Let \( R \) be a principal ideal domain. Then the cohomology groups \( H^n((C^\bullet, \delta^\bullet); R) \) of the co-chain complex \( (C^\bullet, \delta^\bullet) = (\text{Hom}_R(C_\bullet, R), \text{Hom}_R(\partial_\bullet, R)) \) are determined by the isomorphism

\[
H^n((C_\bullet, \partial_\bullet); R) \cong \text{Ext}(H_{n-1}((C_\bullet, \partial_\bullet); R), R) \oplus \text{Hom}_R(H_n((C_\bullet, \partial_\bullet); R); R)
\]

From now on let \( R \) be both commutative ring with a unit and a principal ideal domain. The operator \( \text{Ext}(\_; R) \) is called the extension over \( R \) and it is a functor \( R\text{-Mod} \to R\text{-Mod} \). The construction of the extension functor is done in detail in [Hatcher et al., 2002, p.193-5]. It is enough for us to know that for each \( R \)-module \( H \) there exist free \( R \)-modules \( F_0 \) and \( F_1 \) such that the following sequence is exact

\[
\cdots \to 0 \to \cdots \to 0 \to F_1 \to F_0 \to H \to 0.
\]

This exact sequence, called a free resolution of \( H \), is a chain complex \( (F_\bullet^H, \partial_\bullet^H) \) in \( \text{Ch}(R) \). Now define the functor on objects as \( \text{Ext}(H, R) = \text{H}^1((F^H_\bullet, \partial^H_\bullet); R) \). We advise the reader to consult the question of functorial nature of the assignment \( H \mapsto (F^H_\bullet, \partial^H_\bullet) \) with [Hatcher et al., 2002, p.193-5]. It is obvious that

\[
\bigoplus_{x \in A} xR \cong F_0 \xrightarrow{\pi} H \text{ and } F_1 \cong \ker(\pi)
\]

where \( \pi \) is the natural surjective \( R \)-homomorphism and \( A \) is a set of generators of the \( R \)-module \( H \). Here we use the fact that any submodule of a free module over a principal ideal domain is free.
Lemma 1.2.2. If $H$ is a finitely generated free $R$-module, then $\text{Ext}(H, R) = 0$.

Proof. Consider the following free resolution $(F_\bullet, \partial_\bullet)$ of $H$

$$\cdots \to 0 \to 0 \xrightarrow{g^*} 0 \xleftarrow{f^*} \bigoplus_{x \in A} xR \xrightarrow{\pi} H \to 0$$

where $A$ is a smallest finite generating set of $H$. Denote by $n$ the cardinality of $A$. Thus $\pi$ is an isomorphism. Then $F_1 = 0$ and $\text{Hom}_R(0, R) = 0$. Moreover, $\text{Hom}_R(\bigoplus_{x \in A} xR, R) \cong \bigoplus_{x \in A} \text{Hom}_R(xR, R) \cong \bigoplus_{i=1}^n \text{Hom}_R(R, R) \cong \bigoplus_{i=1}^n R$ where the last isomorphism comes from the observation that every $R$-homomorphism is characterised by the image of the unit element in $R$. After applying the functor $\text{Hom}_R(-, R)$ we get a sequence

$$\cdots \leftarrow 0 \xleftarrow{g^*} 0 \xleftarrow{f^*} \bigoplus_{i=1}^n R \leftarrow \bigoplus_{i=1}^n R \leftarrow 0$$

Now it is clear that $\text{Ext}(H, R) = 0$ since $g^* = 0 = f^*$. [Hatcher et al. 2002, Lemma 3.1 b] (the functoriality of $\text{Ext}(-, R)$) completes the proof. \qed

Let $A \subseteq X$ be a subspace of a topological space $X$ with inclusion $i : A \hookrightarrow X$. Then clearly $C_n(i; R) : C_n(A; R) \to C_n(X; R)$ is an injective $R$-homomorphism. Denote by $(D_\bullet, \partial^D_\bullet) = (C_\bullet(X; A; R), \partial_\bullet)$ the chain complex of factors

$$C_n(X, A; R) = C_n(X; R)/C_n(A; R)$$

and $\partial^D_\bullet$ is induced by $\partial_\bullet$ in the obvious way. Applying functors $H_n(-; R)$ and $H^n(-; R)$ to the chain complex $(D_\bullet, \partial^D_\bullet)$ and the associated dual co-chain complex $(D^*_\bullet, \delta^*_\bullet)$, we obtain the $n$-th relative singular homology and cohomology group, respectively, and we denote them by

$$H_n(X, A; R) = H_n((D_\bullet, \partial^D_\bullet); R) \text{ and } H^n(X, A; R) = H^n((D_\bullet, \partial^D_\bullet), R).$$

Theorem 1.2.3 (Excision Theorem for singular cohomology). Let $X$ be a space and let us consider subspaces $X_1, X_2 \subseteq X$ such that $\text{int}(X_1) \cup \text{int}(X_2) = X$, where $\text{int} Y$ denotes the interior of space $Y$. Then the inclusion of pairs $(X_1, X_1 \cap X_2) \to (X, X_2)$ induces isomorphisms on all relative cohomology groups:

$$H^n(X_1, X_1 \cap X_2; \mathbb{Z}) \cong H^n(X, X_2; \mathbb{Z}), \ n \geq 0.$$

Note that the same holds for the singular homology.

The rest of this section is devoted to tools which we use for computations in Section 1.3. For further information about the Mayer-Vietoris sequence and homotopy invariance see [Hatcher et al. 2002].

Definition 1.2.4. Let $X$ and $Y$ be topological spaces and denote by $I$ the closed interval $[0, 1]$. Let $f, g : X \to Y$ be continuous maps. We say that the maps $f, g$ are homotopy equivalent, written $f \simeq g$, if there exist a continuous map $H : X \times I \to Y$, called homotopy, such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

We say that spaces $X$ and $Y$ are homotopy equivalent, denoted by $X \simeq Y$, if there exist maps $i : X \to Y$ and $j : Y \to X$ so that $i \circ j \simeq \text{id}_Y$ and $j \circ i \simeq \text{id}_X$.  

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This notion is very useful as homology and cohomology are homotopy invariant, that is, if \( X \simeq Y \), then \( H_k(X; R) \cong H_k(Y; R) \) and \( H^k(X; R) \cong H^k(Y; R) \).

The notion of homotopy equivalence of spaces has very useful form that we will use frequently in our proofs.

**Definition 1.2.5.** Let \( A \) be a subspace of a topological space \( X \) with inclusion \( i : A \to X \). The subspace \( A \) is called the deformation retract of \( X \) if there exists a continuous map \( r : X \to A \), called retract, such that \( r \circ i = \text{id}_A \) and \( i \circ r \simeq \text{id}_X \).

We shall proceed with remarks concerning singular homology and cohomology of spheres. Denote by \( X_1 = \{ x \in S^n : -\varepsilon < x_{n+1} \leq 1 \} \) and \( X_2 = \{ x \in S^n : -1 \leq x_{n+1} < \varepsilon \} \) where \( \varepsilon > 0 \). Observe that \( X = S^n \) and its subspaces \( X_i \) for \( i = 1, 2 \) satisfy the assumptions of the Excision Theorem (1.2.3). There is a special long exact sequence called the Mayer-Vietoris sequence for singular homology [Hatcher et al., 2002, p.149]

\[
\cdots \to H_n(X_1 \cap X_2; \mathbb{Z}) \to H_n(X_1; \mathbb{Z}) \oplus H_n(X_2; \mathbb{Z}) \to H_n(X; \mathbb{Z}) \to H_{n-1}(X_1 \cap X_2; \mathbb{Z}) \to \cdots
\]

Using the Mayer-Vietoris sequence for singular homology and the facts that \( X_1 \cap X_2 \simeq S^{n-1} \) and \( X_i \) is homotopy equivalent to a point, which implies \( H_n(X_i; \mathbb{Z}) \cong \{ 0 \} \) for \( i = 1, 2 \) and \( n \in \mathbb{N} \), one can compute

**Lemma 1.2.6.** For \( n \geq 1 \) we have
\[
H_k(S^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n, \ 0 & \text{otherwise,} \end{cases} \text{ and } H_k(S^0; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k = 0, \ 0 & \text{otherwise.} \end{cases}
\]

As a consequence of Lemma (1.2.6) and Theorem (1.2.1) and the fact, that \( H^0(S^n; \mathbb{Z}) \cong \mathbb{Z} \), we obtain the following lemma.

**Lemma 1.2.7.** For \( n \geq 1 \) we have \( H^k(S^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n, \ 0 & \text{otherwise.} \end{cases} \)

### 1.3 Oriented bundles

In this section, we define the notion of orientation of an \( n \)-vector bundle and state the Thom Isomorphism Theorem. Using these results we introduce the Euler class, which is essential for our proof of non-existence of nowhere zero cross-section on even-dimensional sphere. This result is given at the end of the section.

Let \( V \) be a real vector space of finite dimension \( n \). Recall that an orientation of \( V \) is an equivalence class of ordered bases of \( V \) given by the sign of the determinant of their transition matrix. We will denote by \( V_0 \) the set of all non-zero vectors in \( V \).

**Lemma 1.3.1.** The choice of orientation of \( \mathbb{R}^n \) corresponds to the choice of one of the generators of the group \( H_n(\mathbb{R}^n, \mathbb{R}^n_0; \mathbb{Z}) \) which is isomorphic to \( \mathbb{Z} \) for \( n \in \mathbb{N} \).

**Proof.** From the construction of connecting homomorphism in terms of the reduced homology \( \tilde{H}_k(\cdot; \mathbb{Z}) \) (see [Hatcher et al., 2002, p.110]) we obtain the following long exact sequence of groups
\[ \cdots \rightarrow H_{n+1}(\mathbb{R}^n, \mathbb{R}_0^n; \mathbb{Z}) \rightarrow \hat{H}_{n}(\mathbb{R}_0^n; \mathbb{Z}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}_0^n; \mathbb{Z}) \rightarrow \hat{H}_{n-1}(\mathbb{R}_0^n; \mathbb{Z}) \rightarrow \cdots \]

As \( S^{n-1} \) is a deformation retract of \( \mathbb{R}_0^n \) with the retract \( r : \mathbb{R}_0^n \rightarrow S^{n-1}, r(x) = \frac{x}{\|x\|} \) it holds that \( H_k(\mathbb{R}_0^n; \mathbb{Z}) \cong H_k(S^{n-1}; \mathbb{Z}) \). The space \( \mathbb{R}^n \) is clearly contractible and so its \( k \)-th reduced homology group vanishes for all \( k \geq 0 \).

Note that we used the notion of reduced homology because for the choice \( n = 1 \) the long exact sequence in homology

\[ \cdots \rightarrow H_1(\mathbb{R}^1; \mathbb{Z}) \rightarrow H_n(\mathbb{R}^1, \mathbb{R}_0^1; \mathbb{Z}) \rightarrow H_0(\mathbb{R}_0^1; \mathbb{Z}) \rightarrow H_0(\mathbb{R}^1; \mathbb{Z}) \rightarrow \cdots \]

transforms to

\[ \cdots \rightarrow \hat{H}_1(\mathbb{R}^1; \mathbb{Z}) \rightarrow H_1(\mathbb{R}^1, \mathbb{R}_0^1; \mathbb{Z}) \rightarrow \hat{H}_0(\mathbb{R}_0^1; \mathbb{Z}) \rightarrow \hat{H}_0(\mathbb{R}^1; \mathbb{Z}) \rightarrow \cdots \]

And thus by Lemma 1.2.6, \( \hat{H}_n(\mathbb{R}_0^n; \mathbb{Z}) \cong Z \) and the following commutative ladder with long exact rows

\[ \cdots \rightarrow \hat{H}_n(\mathbb{R}^n; \mathbb{Z}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}_0^n; \mathbb{Z}) \rightarrow \hat{H}_{n-1}(\mathbb{R}_0^n; \mathbb{Z}) \rightarrow \hat{H}_n(\mathbb{R}^n; \mathbb{Z}) \rightarrow \cdots \]

implies that \( H_n(\mathbb{R}^n, \mathbb{R}_0^n; \mathbb{Z}) \cong \mathbb{Z} \).

The canonical choice of orientation \( \{e_1, \ldots, e_n\} \) and the choice of the orientation \( \{-e_1, \ldots, -e_n\} \) yield topological subspaces \( X_1 \) and \( X_{-1} \) of \( \mathbb{R}^n \) consisting of all convex combinations of vectors \( -f_1, e_1, \ldots, e_n \) and \( -f_{-1}, e_1, \ldots, -e_n \) in \( \mathbb{R}^n \) respectively where \( f_i = e_1 + \ldots + e_{n-1} + ie_n \), respectively, for \( i = -1, 1 \).

Furthermore, let \( \rho_i \) be the linear embedding \( \Delta^n \rightarrow X_i \) that is determined by \( e_1 \mapsto f_i, e_j \mapsto e_{j+1} \) for \( i = -1, 1 \).

Since the origin lies in the interior of both \( X_i \), the simplex \( \rho_i \in C_n(\mathbb{R}^n, \mathbb{R}_0^n; \mathbb{Z}) \) and so we define \( \sigma_i = \partial \rho_i \in C_{n-1}(\mathbb{R}^n, \mathbb{R}_0^n; \mathbb{Z}) \). Obviously the homology class of \( \sigma_i \) in \( H_n(\mathbb{R}^n, \mathbb{R}_0^n; \mathbb{Z}) \) can be identified with certain element in \( H_{n-1}(S^{n-1}; \mathbb{Z}) \).

The map \( r \circ \sigma_i \) for \( i = -1, 1 \) defines a simplicial structure on \( S^{n-1} \). Let \( \mathbb{H}^{n-1} \) denote the boundary of \( \Delta^n \), then \( \sigma_i \) also gives rise to the isomorphism between the simplicial complex \( \mathbb{H}^{n-1} \) and the one on \( S^{n-1} \). The simplicial homology (see Hatcher et al. [2002] p.102) \( H_k(\mathbb{H}^{n-1}; \mathbb{Z}) \) and the reduced singular homology \( H_k(S^{n-1}; \mathbb{Z}) \) are isomorphic for every \( k \in \mathbb{N}_0 \) by Hatcher et al. [2002] Theorem 2.27 (this holds for any simplicial complex).

The simplex \( \Delta^n \) is contractible and so its the \( k \)-th simplicial homology vanishes for \( 1 \leq k \). Then the homology of the simplicial complex \( \mathbb{H}^{n-1} \) in dimension \( n-1 \) is isomorphic to \( \mathbb{Z} \) with a generator \( \mathbb{H}^{n-1} \) as the alternating sum of \( n-1 \) dimensional faces.

Since \( S^{n-1} \) and \( \mathbb{H}^{n-1} \) are isomorphic as simplicial complexes a generator of the \((n-1)\)-th simplicial homology \( \mathbb{H}^{n-1} \) gets mapped onto a generator of the simplicial homology of \( S^{n-1} \). Thus, \( \sigma_i \) is a generator of this simplicial homology.
Let $T$ be the reflection of $\mathbb{R}^n$ with respect to the plane given by vectors $e_1, \ldots, e_{n-1}$ that maps vector $e_n$ to $-e_n$. Clearly $T(X_i) = X_{-i}$ for $i = -1, 1$, in effect $\sigma_i \circ T = \sigma_{-i}$. The reflection $T$ acts as multiplication by $-1$ in homology (see [Hatcher et al. 2002, P.134]), in effect $\sigma_i = -\sigma_{-i}$. The fact that $H_{n-1}(S^{n-1}; \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n_0; \mathbb{Z})$ completes the proof.

**Corollary.** A choice of orientation of $V$ corresponds to a choice of one of the generators of the group $H_n(V, V_0; \mathbb{Z})$.

Note that $H_{n-1}(\mathbb{R}^n, \mathbb{R}^n_0; \mathbb{Z})$ is a finitely generated free group, since it is zero by Lemma (1.2.6). Thus, by Lemma (1.2.2) and Theorem (1.2.1) (Universal Coefficient Theorem) we can conclude that $H^n(V, V_0; \mathbb{Z}) \cong \text{Hom}_\mathbb{Z}(H_n(V, V_0; \mathbb{Z}), \mathbb{Z})$. This isomorphism implies that our choice of orientation of the vector space $V$ which yields the choice of generator $u_V$ of $H_n(V, V_0; \mathbb{Z})$ induces a unique element $m_V \in H^n(V, V_0; \mathbb{Z})$ such that $m_V$ is a generator and $m_V(u_V) = +1$.

For the rest of this section consider the preferred generator $m_V$ to be the one coming from the choice of canonical basis $\{e_1, \ldots, e_n\}$ of $V = \mathbb{R}^n$.

**Definition 1.3.2.** Let $\xi = (E, B, \pi)$ be an $n$-vector bundle. An **orientation** is a function which assigns an orientation to each fiber $\pi^{-1}(b)$ satisfying the following local compatibility condition.

For every $b_0 \in B$ there exists a local coordinate system $(N, h)$, with $b_0 \in N$ and $h : N \times \mathbb{R}^n \to \pi^{-1}(N)$, so that for each fiber $\pi^{-1}(b)$ over $b \in N$ the linear homomorphism $x \mapsto h(b, x)$ from $\mathbb{R}^n$ to $\pi^{-1}(b)$ is orientation preserving.

Since it is well known that any sphere $S^n$ is an orientable manifold, which means that there is an atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ such that the Jacobian matrix of the transition function $\varphi_{\alpha \beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ has positive determinant for any two coordinate charts $(U_\alpha, \varphi_\alpha)$ and $(U_\beta, \varphi_\beta)$ in the atlas. It follows that the tangent bundle $\tau_{S^n}$ is also orientable.

So far, we described the orientation of a $n$-vector bundle just in terms of local coordinate systems. Next result can be found in [Milnor et al. 1974, p.96-97] and is crucial for the preferred orientation $m_V$ to be, in a certain way, extended globally from fibres to the whole bundle.

**Theorem 1.3.3 (Thom Isomorphism Theorem).** Let $\xi = (E, B, \pi)$ be an oriented $n$-vector bundle with total space $E$ and let $E_0 = E \setminus s_0(B)$ be complement of the image of zero section $s_0 : B \to E$ in $E$. Then the cohomology group $H^n(E, E_0; \mathbb{Z})$ is zero for $i < n$, and $H^n(E, E_0; \mathbb{Z})$ contains one and only one cohomology class $u$ whose restriction $u| (F, F_0) \in H^n(F, F_0; \mathbb{Z})$ is equal to the preferred generator $u_F$ for every fiber $F$ of $\xi$. Furthermore the correspondence $y \mapsto y \circ u$ maps $H^k(E, \mathbb{Z})$ isomorphically onto $H^{k+n}(E, E_0; \mathbb{Z})$ for every integer $k$.

The unique cohomology class $u$ is called the **Thom class** and from now will be denoted by $u(\xi)$. The operation $\circ$ in the Thom Isomorphism theorem is the cup product.

**Definition 1.3.4.** Let $n \geq k$ define linear embeddings $\iota_{(1,k)} : \Delta^k \hookrightarrow \Delta^{n+k}$, $\iota_{(1,k)}(e_i) = e_i$ for $i = 1, \ldots, k + 1$ and $\iota_{(k+1,n)} : \Delta^{n-k} \hookrightarrow \Delta^{n+k}$, $\iota_{(k+1,n)}(e_j) = e_{j+k}$ for $j = 1, \ldots, n - k + 1$. We define the **cup product** $v \cup w \in H^{k+n}(X; \mathbb{Z})$ of $v \in H^k(X; \mathbb{Z})$ and $w \in H^n(X; \mathbb{Z})$ so that for each singular $(n + k)$-simplex $\sigma \in X$

$$(v \cup w)(\sigma) = v(\sigma \circ \iota_{(1,k)}) \cup w(\sigma \circ \iota_{(k+1,n)}) \in \mathbb{Z}.$$
Note that the same formula defines the cup product for singular relative cohomology and induces a $\mathbb{Z}$-homomorphism $H^k(X;\mathbb{Z}) \times H^l(X;\mathbb{A};\mathbb{Z}) \to H^{k+l}(X;\mathbb{A};\mathbb{R})$.

The set
\[
H^*(X;\mathbb{Z}) = \bigoplus_{n \in \mathbb{N}_0} H^n(X;\mathbb{Z})
\]

together with the cup product operation is a graded ring, where the element $y$ is said to have degree $k$ if it is an element of $H^k(X;\mathbb{Z})$. Since $\mathbb{Z}$ is a commutative ring with a unit 1, the ring $H^*(X;\mathbb{Z})$ has a unit 1 such that $1(\sigma) = 1$, where $\sigma$ is a singular 0-simplex of $X$. Similarly, we obtain that $H^*(X;\mathbb{A};\mathbb{Z})$ is a graded ring for $\mathbb{A} \subseteq X$.

**Theorem 1.3.5.** If $R$ is a commutative ring and $A$ is a subset of the space $X$, then $v \circ w = (-1)^{kl}w \circ v$ for $v \in H^k(X;\mathbb{A};R)$ and $w \in H^l(X;\mathbb{A};R)$.

For proof see Theorem 3.11 in [Hatcher et al., 2002, p.210].

Considering the ring $H^*(X;\mathbb{Z})$ the Thom Isomorphism theorem can be restated by saying that $H^*(E,E_0;\mathbb{Z})$ is a free $H^*(E;\mathbb{Z})$-module on one generator $u(\xi)$ of degree $n$.

Let $\xi = (E,B,\pi)$ be an oriented $n$-vector bundle. Denote by $\pi^* = H^k(\pi;\mathbb{Z}) : H^k(B;\mathbb{Z}) \to H^k(E;\mathbb{Z})$ and by $s_0$ the zero cross-section, that is, for each $b \in B$, $s_0(b)$ is the zero element in $\pi^{-1}(b)$. Now observe that $s_0(B)$ is a deformation retract of $E$ and thus $\pi^* : H^n(E;\mathbb{Z}) \to H^n(B;\mathbb{Z})$ is an isomorphism. The composition of isomorphisms $\phi = (\circ \quad u(\xi)) \circ \pi^* : H^k(B;\mathbb{Z}) \to H^{k+n}(E,E_0;\mathbb{Z})$ is called the **Thom isomorphism**.

With almost all necessary tools and notation established we can proceed towards the definition of the Euler class.

Consider the inclusion of pairs of spaces $i : (E,\emptyset) \hookrightarrow (E,E_0)$. The induced mapping $i^* : H^n(E,E_0;\mathbb{Z}) \to H^n(E;\mathbb{Z})$ in cohomology is the restriction homomorphism $i^*(u) = u\upharpoonright E$.

**Definition 1.3.6.** The **Euler class** of an oriented $n$-vector bundle $\xi = (E,B,\pi)$ is the cohomology class $e(\xi) = (\pi^*)^{-1}(i^*(u(\xi))) \in H^n(B;\mathbb{Z})$.

The following Theorem ([1.3.7] and its proof can be found in [Milnor et al., 1974, Property 9.7].

**Theorem 1.3.7.** If the oriented $n$-vector bundle $\xi$ possesses a nowhere zero cross-section $s$, then $e(\xi)$ must be zero.

**Proof.** It is clear that for the inclusion $i : E_0 \hookrightarrow E$ the composition $\pi \circ i \circ s = id_B$ by the definition of section. By functoriality of $H^n(\_,\mathbb{Z})$ we get $s^* \circ i^* \circ \pi^* = id_{H^n(B;\mathbb{Z})}$. Furthermore, $\pi^*(e(\xi)) = i^*(u(\xi))$. For every $k \in \mathbb{N}$ we have that the composition
\[
C_k(E_0;\mathbb{Z}) \xrightarrow{C_k(i;\mathbb{Z})} C_k(E;\mathbb{Z}) \xrightarrow{C_k(\pi;\mathbb{Z})} C_k(E,E_0;\mathbb{Z})
\]
is the zero homomorphism. Thus, $H^n(E,E_0;\mathbb{Z}) \xrightarrow{i^*} H^n(E;\mathbb{Z}) \xrightarrow{s^*} H^n(E_0;\mathbb{Z})$ is also zero. And so $e(\xi) = id_{H^n(B;\mathbb{Z})}(e(\xi)) = s^* \circ i^* \circ \pi^*(u(\xi)) = s^*(0) = 0$. \qed

The rest of this section is devoted to the solution of [Milnor et al., 1974, Problem 9-C], of which the fact that $n$-spheres with $n$ even are **not** parallelizable manifolds is a simple corollary.
Denote by $\sigma_y : S^n \setminus \{−y\} \to (−y)^\perp$ the stereographic projection from the point $−y \in S^n$ onto the orthogonal complement of $−y$ in $\mathbb{R}^{n+1}$ given by
\[
\sigma_y(x) = \frac{x + y}{\langle x, y \rangle + 1} − y.
\]

**Lemma 1.3.8.** Let $\tau$ be the tangent bundle of the $n$-sphere, and let $A, D \subset S^n \times S^n$ be the sets $A = \{(x, −x) : x \in S^n\}$ and $D = \{(x, x) : x \in S^n\}$. The map $\psi : S^n \times S^n \setminus A \to M$ given by $(x, y) \mapsto (x, \sigma_y(x))$ is a diffeomorphism (and thus a homeomorphism).

Let $E = E(\tau_{\mathbb{S}^n}) = TS^n$ then, in view of Theorem (1.1.3), there is a homeomorphism $E \to S^n \times S^n \setminus A$ which restricts to homeomorphism $E_0 \to S^n \times S^n \setminus (A \cup D)$. Moreover, the projection $\pi : E \to B$ is equivalent to the map $S^n \times S^n \setminus A \to S^n$, $(x, y) \mapsto x$.

**Proof.** It is clear that the map $\psi$ is well-defined and smooth. Moreover, the tangent map of $\psi$ is in every point of the form
\[
\begin{pmatrix}
Id & 0 \\
0 & J(y, x)
\end{pmatrix}
\]
(1.1)
where $J(y, x)$ is the tangent map of $x \mapsto \sigma_y(x)$ represented by the corresponding Jacobi matrix.

Consider $\sigma_{−e_n+1} : S^n \setminus \{-e_{n+1}\} \to (-e_{n+1})^\perp$ given by
\[
\sigma_{−e_n+1}(x) = \frac{x + e_{n+1}}{\langle x, e_{n+1} \rangle + 1} − e_{n+1}
\]
and its smooth inverse $(-e_{n+1})^\perp \to S^n \setminus \{-e_{n+1}\}$
\[
\rho_{−e_n+1}(u) = \frac{2(u + e_{n+1})}{\|u\|^2 + 1} − e_{n+1}.
\]

Thus, the stereographic projection from $−e_{n+1}$ is a diffeomorphism $S^n \setminus \{-e_{n+1}\} \to (-e_{n+1})^\perp$.

Let $U : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an orthonormal transformation which sends the vector $−e_{n+1}$ to $−y$. It is well known that the tangent map of unitary transformations are invertible which yields that $\sigma_y : S^n \to (−y)^\perp$ is a diffeomorphism for arbitrary $y \in S^n$ since we have $\sigma_y = U\sigma_{−e_1}U^{-1}$.

It follows that the matrix (1.1) has full rank. This shows that $\psi$ is a diffeomorphism.

Observe that the element $y$ is mapped to zero by $\sigma_{−y}$ and so the second claim follows.

The last claim is clear. \[\square\]

**Claim 1.3.9.** Let $E = E(\tau_{\mathbb{S}^n})$. Then for $k \in \mathbb{N}$
\[
H^k(E, E_0) \cong H^k(S^n \times S^n, S^n \times S^n \setminus D) \cong H^k(S^n \times S^n, A)
\]

**Proof.** Denote by $X = S^n \times S^n$, $X_1 = X \setminus A$ and $X_2 = X \setminus D$. Then clearly $X_1 \cap X_2 = X \setminus (A \cup D)$ and both $X_1, X_2$ are open subsets of $X$ such that $X = X_1 \cup X_2$. Now consider inclusion $\iota : (X_1, X_1 \cap X_2) \to (X, X_2)$. Since we are in
Lemma (1.3.8) implies that \( H^k(E, E_0) \cong H^k(X_1, X_1 \cap X_2) \), because \( E \) and \( E_0 \) are homeomorphic to \( X_1 \) and \( X_1 \cap X_2 \) respectively.

The last isomorphism follows from the fact that pairs \( (X, X \setminus D) \) and \( (X, A) \) are homotopy equivalent since \( A \) is the deformation retract of \( X \setminus D \). That continuous retract is given by fixing \( x \) in the first coordinate and contracting \( X \setminus \{x\} \) to \(-x\) in the second coordinate. \( \square \)

**Lemma 1.3.10.** For \( n \geq 1 \) we have \( H^k(S^n \times S^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k = n, \\ \mathbb{Z} & k = 0, 2n, \\ 0 & \text{otherwise.} \end{cases} \)

**Proof.** We use the Kunneth formula [Hatcher et al., 2002, p. 273]

\[
H^k(S^n \times S^n; \mathbb{Z}) \cong \bigoplus_{i=1}^{k} H^i(S^n; \mathbb{Z}) \otimes \mathbb{Z} H^{k-i}(S^n; \mathbb{Z})
\]

and Lemma (1.2.7) to prove the statement.

For the case \( k = n \) we obtain

\[
H^n(S^n \times S^n; \mathbb{Z}) \cong \bigoplus_{i=0}^{n} H^i(S^n; \mathbb{Z}) \otimes \mathbb{Z} H^{n-i}(S^n; \mathbb{Z})
\]

\[
\cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus (0 \oplus \mathbb{Z}) \oplus \cdots \oplus (0 \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z})
\]

\[
\cong \mathbb{Z} \oplus \mathbb{Z}.
\]

The cases \( k = 2n, 0 \) imply that the only non-zero summand in the Kunneth formula is \( H^n(S^n; \mathbb{Z}) \otimes \mathbb{Z} H^{2n-n}(S^n; \mathbb{Z}) \cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \) and \( H^0(S^n; \mathbb{Z}) \otimes \mathbb{Z} H^0(S^n; \mathbb{Z}) \cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \) respectively. \( \square \)

**Claim 1.3.11.** The Euler class \( e(\tau) = \phi^{-1}(u(\tau) \cup u(\tau)) \) is twice a generator of \( H^n(S^n; \mathbb{Z}) \) where \( \tau = (TS^n, S^n, \pi) \) is the tangent vector bundle of the \( n \)-sphere with \( n \)-even.

**Proof.** Both \( (\pi^*)^{-1} \) and \( \phi \) are isomorphisms and the following diagram commute by the definition of \( \phi \)

\[
\begin{array}{ccc}
H^n(S^n; \mathbb{Z}) & \xrightarrow{(\pi^*)^{-1}} & H^n(TS^n; \mathbb{Z}) \\
& \searrow \phi & \downarrow -u(\tau) \\
& & H^{2n}(TS^n, TS^n_0; \mathbb{Z})
\end{array}
\]

and so \( e(\tau) = \phi^{-1}(u(\tau) \cup u(\tau)) \) and \(- \cup u(\tau)\) is an isomorphism. Thus, it is enough to prove that \( \phi(e(\tau)) = u(\tau) \cup u(\tau) \) is twice a generator of \( H^{2n}(E, E_0; \mathbb{Z}) \).

For our computations we will use the following long exact sequence in cohomology

\[
\cdots \rightarrow H^n(S^n \times S^n, A; \mathbb{Z}) \rightarrow H^n(S^n \times S^n; \mathbb{Z}) \xrightarrow{f} H^n(A; \mathbb{Z}) \rightarrow H^{n+1}(S^n \times S^n, A; \mathbb{Z}) \rightarrow \cdots
\]
induced by the short exact sequence of complexes corresponding to level-wise short exact sequences for the triple \((S^n \times S^n, A, \emptyset)\)

\[
0 \leftarrow C^n(A, \emptyset; \mathbb{Z}) \leftarrow C^n(S^n \times S^n, \emptyset; \mathbb{Z}) \leftarrow C^n(S^n \times S^n, A; \mathbb{Z}) \leftarrow 0
\]

with \(f = i^* = H^n(i; \mathbb{Z})\) where \(i : A \to S^n \times S^n\) is the inclusion. For the construction of this exact sequence see [Hatcher et al., 2002, p.200].

Recall Claim (1.3.9) and consider the following commutative ladder with exact rows

\[
\begin{array}{ccccccccc}
\cdots & \to & H^{n-1}(A; \mathbb{Z}) & \to & H^n(S^n \times S^n, A; \mathbb{Z}) & \to & H^n(S^n \times S^n; \mathbb{Z}) & \to & H^n(A; \mathbb{Z}) & \to & \cdots \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \\
\cdots & \to & 0 & \to & H^n(S^n \times S^n, A; \mathbb{Z}) & \overset{g}{\to} & Z \oplus Z & \overset{f}{\to} & Z & \to & \cdots \\
\end{array}
\]

(1.3)

Observe that \(\psi : S^n \to A\) which maps \(x\) to \((x, -x)\) is a homeomorphism and so we get \(H^{n-1}(A; \mathbb{Z}) \cong H^{n-1}(S^n; \mathbb{Z}) \cong 0\) and \(H^n(A; \mathbb{Z}) \cong \mathbb{Z}\). Denote by \(u\) a generator of \(H^n(S^n; \mathbb{Z})\) and by \(p_1, p_2 : S^n \times S^n \to S^n\) the projections onto the first and second coordinate, respectively. By the Kunneth formula, the elements \(1 \otimes u = p_1^*(1) \sim p_2^*(u)\) and \(u \otimes 1 = p_1^*(u) \sim p_2^*(1)\) are generators of \(H^n(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\).

For the inclusion \(i : A \to S^n \times S^n\) we have the following diagram

\[
\begin{array}{ccc}
S^n & \xleftarrow{p_1} & S^n \times S^n \\
\psi \downarrow & & \downarrow \psi \\
A & \xleftarrow{i} & S^n \\
\end{array}
\]

For \(x \in S^n\) we obtain

\[
\begin{align*}
p_1 \circ i \circ \psi(x) &= p_1(x, -x) = x = \text{id}_{S^n}(x), \\
p_2 \circ i \circ \psi(x) &= p_2(x, -x) = -x = T(x)
\end{align*}
\]

(1.4)

where \(T : S^n \to S^n\) is the antipodal map. Because \(\psi\) is a homeomorphism we can reverse it and then apply the \(n\)-th cohomology group functor.

\[
\begin{array}{ccc}
H^n(S^n; \mathbb{Z}) & \xrightarrow{p_2^*} & H^n(S^n \times S^n; \mathbb{Z}) \\
\downarrow \psi^{-1} & & \downarrow f \\
H^n(A; \mathbb{Z}) & \xleftarrow{p_1^*} & H^n(S^n; \mathbb{Z}) \\
\end{array}
\]

Denote by \(v = p_1^*(u) \sim p_2^*(1)\) and \(w = p_1^*(1) \sim p_2^*(u)\) the generators of \(H^n(S^n \times S^n; \mathbb{Z})\). By (1.4) we have for any \(n\)-simplex \(\sigma \in H_n(S^n; \mathbb{Z})\) that

\[
f(v)(\psi \circ \sigma) = f(p_1^*(u) \circ p_2^*(1))(\psi \circ \sigma) = i^* \circ p_1^*(u)(\psi \circ \sigma) = (p_1 \circ i)^*(u)(\psi \circ \sigma) = u(p_1 \circ i \circ \psi \circ \sigma) = u(\sigma) = u(\psi^{-1} \circ \psi \circ \sigma) = (\psi^{-1})^*(u)(\psi \circ \sigma).
\]

Since, \(T^*(\alpha) = (-1)^{n+1}\alpha\) for \(\alpha \in H_n(S^n; \mathbb{Z})\) (see [Hatcher et al., 2002, p.134]) and \(n+1\) is odd, for \(w\) we have that

\[
f(w)(\psi \circ \sigma) = f(p_1^*(1) \sim p_2^*(u))(\psi \circ \sigma) = i^* \circ p_2^*(u)(\psi \circ \sigma) = (p_2 \circ i)^*(u)(\psi \circ \sigma) = -u(\sigma) = -u(\psi^{-1} \circ \psi \circ \sigma) = -(\psi^{-1})^*(u)(\psi \circ \sigma)
\]
That completes the proof. That yields $f(v + w) = f(v) + f(w) = (\psi^{-1})^*(u) - (\psi^{-1})^*(u) = 0$, and thus, $v + w$ is a generator of $\text{Ker}(f) \cong \mathbb{Z}$. By the exactness of the commutative ladder (1.3) we have that $H^n(S^n \times S^n; A; \mathbb{Z}) \cong \text{Ker}(f)$. Thus $H^n(S^n \times S^n; A; \mathbb{Z}) \subseteq H^n(S^n \times S^n; \mathbb{Z})$ and we can view $v + w$ as a generator of $H^n(S^n \times S^n; A; \mathbb{Z})$.

For arbitrary $x \in S^n$ and its fibre $x^{-1}(x) = \{x\} \times (S^n \setminus \{x, -x\})$ and $x^{-1}(x)_0 = \{x\} \times (S^n \setminus \{x, -x\})$ we have the canonical inclusion $j_x : (x^{-1}(x), x^{-1}(x)_0) \to (S^n \times S^n \setminus A, S^n \times S^n \setminus D)$. The restriction $j_x^*(v + w)$ of $v + w$ to $H^n(x^{-1}(x), x^{-1}(x)_0; \mathbb{Z}) \approx \mathbb{Z}$ is obviously a preferred generator and so $v + w$ is by uniqueness the Thom class $u(\tau)$. We have

$$(v + w) \sim (v + w) = (v \sim v) + (v \sim w) + (w \sim v) + (w \sim w).$$

We know that

$$v \sim v = p_1^*(u) \sim p_1^*(u) = p_1^*(u \sim u) = p_1^*(0) = 0,$$

$$w \sim w = p_2^*(u) \sim p_2^*(u) = p_2^*(u \sim u) = p_2^*(0) = 0,$$

since $u \sim u \in H^2n(S^n; \mathbb{Z}) \cong \{0\}$ by Lemma (1.2.6). By the Kunneth theorem, the group $H^2n(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z}$ is generated by $u \otimes u$ and this class gets identified with $p_1^*(u) \sim p_2^*(u) = v \sim w$.

Now we use the parity of $n$ to find

$$(v \sim w) = (-1)^n(v \sim v) = (w \sim v).$$

Thus we get $(v + w) \sim (v + w) = 2(w \sim v)$. It is clear that $(w \sim v)$ is also a generator of $H^2n(S^n \times S^n; A; \mathbb{Z})$ since we have the following commutative ladders with exact rows taken from the long exact sequence (1.2) and Lemma (1.2.6).

$$
\begin{array}{cccc}
H^{2n-1}(A; \mathbb{Z}) & \to & H^{2n}(S^n \times S^n; A; \mathbb{Z}) & \to \ H^{2n}(S^n \times S^n; \mathbb{Z}) & \to \ H^{2n}(A; \mathbb{Z}) \\
\uparrow^{\cong} & & \uparrow^{\text{id}} & & \uparrow^{\cong} & & \uparrow^{\cong} \\
0 & \to & H^{2n}(S^n \times S^n; A; \mathbb{Z}) & \to \ H^{2n}(S^n \times S^n; \mathbb{Z}) & \to & \mathbb{Z} & \to 0
\end{array}
$$

That completes the proof.

As an immediate consequence of previous claim, Theorems (1.3.7) and (1.1.6) we obtain the following corollary.

**Corollary.** For $n$ even the bundle $\tau_{S^n}$ does not possess any nowhere zero cross-section $s$ and so $S^n$ is not a parallelizable manifold.

This result is sometimes referred to as the Hairy Ball Theorem.
2. Clifford algebras

2.1 Clifford algebras

Definition 2.1.1. A ring $C$ with a unit is called a (real unital) algebra if $C$ has the structure of a vector space over $\mathbb{R}$ and for each $x, y \in C$ and $s \in \mathbb{R}$, $s(x \cdot y) = (sx) \cdot y = x \cdot (sy)$, where $s$ acts by the scalar multiplication and $\cdot$ is the ring multiplication.

Let $G$ be an abelian group. We say that the algebra $C$ is $G$-graded if $C = \bigoplus_{g \in G} C^g$ and $C^g \cdot C^{g'} \subseteq C^{g+g'}$. If $c \in C^g$, then we say that $c$ has degree $g$.

Let $C, D$ be $G$-graded algebras. A homomorphism of algebras $f : C \rightarrow D$ is called a (real unital) algebra if $f(C^g) \subseteq D^g$ for $g \in G$.

For the rest of this section we will denote by $V$ a real finite-dimensional vector space. Let $f : V \times V \rightarrow \mathbb{R}$ be a symmetric bilinear form on $V$ and $q$ be the associated quadratic form, that is $q(x) = f(x, x)$.

Definition 2.1.2. Let $V, f$ and $q$ be as above. The Clifford algebra associated to $(V, f)$ is the pair $(C(f), \theta)$ where $C(f)$ is a real unital algebra with unit $1$ and $\theta : V \rightarrow C(f)$ is a linear map such that, for each $x \in V$, $(\theta(x))^2 = q(x)1$ which satisfies the following universal condition: for every real unital algebra $A$ and a linear map $\alpha : V \rightarrow A$ such that $(\alpha(x))^2 = q(x)1$ there exist a unique algebra homomorphism $\alpha' : C(f) \rightarrow A$ so that $\alpha' \circ \theta = \alpha$.

Let $T^k(V)$ be the $k$-fold tensor product of $V$. The tensor algebra $T(V)$ is defined as

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$$

where $T^0(V) = \mathbb{R}$ and $T^1(V) = V$. The tensor algebra $T(V)$ has natural structure of an associative algebra where the multiplication is given by $x \otimes y \in T^{k+l}(V)$ for $x \in T^k(V)$ and $y \in T^l(V)$. The $k$-fold tensor product $T^k(V)$ satisfies that for the $k$-fold Cartesian product $V^k = V \times \ldots \times V$ of the vector space $V$, a real vector space $W$ and a multilinear map $\psi : V^k \rightarrow W$ there exists a unique $\tilde{\psi}$ such that the following diagram commutes

\[
\begin{array}{ccc}
V^k & \otimes^k & T^k(V) \\
\downarrow{\psi} & & \downarrow{\tilde{\psi}} \\
& W & \\
\end{array}
\]

This universal condition determines $T^k(V)$ uniquely up to an isomorphism for each $k \in \mathbb{N}$, and thus, so is $T(V)$.

Denote by $I_V$ the ideal in $T(V)$ generated by the set $\{ x \otimes x - q(x)1 : x \in V \}$. Then $C(f) \cong T(V)/I_V$ and $\theta$ is the composition of the canonical inclusion $\iota : V \cong T^1(V) \rightarrow T(V)$ and the canonical projection $p : T(V) \rightarrow C(f)$. It is clear that the image of $V$ with respect to $\theta$ generates $C(f)$ since the algebra $T(V)$ is generated by $T^1(V)$. From this it is relatively straightforward to show that the following Theorem 2.1.3 holds. The proof can be found in [Husemoller 2013 Theorem 4.2] or [Spáčil 2007 Věta 1.3].
Theorem 2.1.3. For every real finite-dimensional vector space $V$ with a symmetric bilinear form $f$ there exists a Clifford algebra $(C(f), \theta)$ associated to $(V, f)$ and it is unique up to an isomorphism.

Let $f = -\langle , \rangle$ and $f' = \langle , \rangle$ be the standard negative and positive definite inner product on $\mathbb{R}^n$, respectively, and $q(x) = -q'(x)$ and $q'(x) = x_1 + \cdots + x_n$ be the induced quadratic forms. We will denote by $C_n = C(f)$ and $C'_n = C(f')$ the corresponding Clifford algebras. Therefore, $C_n$ and $C'_n$ are subjects of the following theorem.

Theorem 2.1.4. Let $(V, f)$ be a real vector space with basis $v_1, \ldots, v_n$ and a symmetric bilinear form $f$ so that $f(v_i, v_j) = 0$ and $f(v_i, v_i) = a_i$ for $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Let $(C(f), \theta)$ be the Clifford algebra associated to $(V, f)$. Then $(C(f), \theta)$ is generated by $\theta(v_1), \ldots, \theta(v_n)$ satisfying

$$\theta(v_i)^2 = a_i 1 \text{ and } \theta(v_i) \cdot \theta(v_j) + \theta(v_j) \cdot \theta(v_i) = 0.$$

Moreover, the algebra $C(f)$ has a basis consisting of all $\theta(v_I)$ where

$$\theta(v_I) = \theta(v_{i_1}) \cdots \theta(v_{i_k})$$

for $I = (i_1, \ldots, i_k)$ a multiindex of length $|I| = k$, $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq k \leq n$ and for $I = \emptyset$, that is $k = 0$, we define $\theta(v_I) = 1$. So the dimension of $C(f)$ over $\mathbb{R}$ is equal to $2^n$.

We refer to [Husemöller 2013 Theorem 5.2] for the proof.

Remark. Let $(V, f)$ be a real finite-dimensional space and a symmetric bilinear form. Let $(C(f), \theta)$ be the Clifford algebra associated to $(V, f)$. Denote by $\alpha : C(f) \to C(f)$ the prolongation of $-\theta : V \to C(f)$ obtained from the universal property of $C(f)$. Since $\alpha^2 = \text{id}_{C(f)}$ and $\alpha$ is a homomorphism of algebras, then $C(f)$ gains the structure of $\mathbb{Z}_2$-graded algebra given by

$$C(f)^0 = \{c \in C(f) : \alpha(c) = c\} \text{ and } C(f)^1 = \{c \in C(f) : \alpha(c) = -c\}.$$

Proof of this remark is straightforward and follows from the discussion below.

The tensor algebra $T(V)$ has $\mathbb{Z}_2$-grading given by

$$T(V) = (T(V))^0 \oplus (T(V))^1$$

where

$$(T(V))^i = \bigoplus_{k=0}^{\infty} T^{2k+i}(V)$$

for $i = 0, 1$. Furthermore, ideal $I_V$ in $T(V)$ splits into $I_V \cong I^0_V \oplus I^1_V$ where $I^i_V \cong I_V \cap (T(V))^i$ for $i = 0, 1$. It can be verified that $C(f)^g = (T(V))^g/I^g_V$. It is clear, that both $C(f)^0$ and $C(f)^1$ are subspaces of $C(f)$. Moreover, $C(f) = C(f)^0 \oplus C(f)^1$ and $C(f)^0$ is a Clifford subalgebra of $C(f)$.

Remark. Using Theorem [2.1.4] we obtain that the elements $\theta(v_I)$, where the length of $I$ is even, form a basis of $C_n^0$. Similarly for the odd length and $C_n^1$.

The graded structure of $C_n$ will play an important role in later parts of this chapter.
Lemma 2.1.5. The map $\phi : C_n \rightarrow C_{n+1}^0$ defined by $\phi(x_0 + x_1) = x_0 + \theta_{n+1}(e_{n+1})\cdot x_1$ for $x = x_0 + x_1$ with $x_0 \in C_n^0$ and $x_1 \in C_n^1$ is an (ungraded) isomorphism of algebras $C_n$ and $C_{n+1}^0$.

Proof. First, observe that from the construction of the Clifford algebra $(C_n, \theta_n)$ each $\theta(e_i)$ is an element of $C_n^1$ for $1 \leq i \leq n$ since $\theta(e_i) \in T^1(V)$. Similarly for $\theta_{n+1}(e_i)$ with $1 \leq i \leq n+1$ we obtain $\theta_{n+1}(e_i) \in C_{n+1}^1$. Clearly, by the universal property of Clifford algebra, the assignment $\theta_n(e_i) \mapsto \theta_{n+1}(e_i)$ for $1 \leq i \leq n$ induces an injective algebra homomorphism and so $C_n \subseteq C_{n+1}$. The linearity of $\phi$ is obvious. Let $x, y \in C_n$ with $x = x_0 + x_1$ and $y = y_0 + y_1$ where $x_i, y_i \in C_n^i$ for $i = 0, 1$. And so

$$x_0 = \sum_{2 \mid |I|} x_i^I \theta_{n+1}(e_I) \text{ and } x_1 = \sum_{2 \mid |I|} x_i^I \theta_{n+1}(e_I).$$

where $x_i^I$ are the coordinates of $x_i$ with respect to the basis of $C_n^i$ given above.

Corresponding to the element $\theta(e_I) \in C_n^I$ from the basis of $C_n(f)$ for multiindex $I$. Put $e = \theta_{n+1}(e_{n+1})$. Again, Theorem (2.1.4) implies

$$e \cdot \theta_{n+1}(e_I) = (-1)^{|I|}\theta_{n+1}(e_I) \cdot e$$

since $n + 1 \notin I$. Thus, $ex_0 = xe_0$ and $-ex_1 = x_1e$. And so using $e^2 = -1$ we can compute that

$$\phi(x_0 + x_1)\phi(y_0 + y_1) = (x_0 + ex_1)(y_0 + ey_1)$$

$$= x_0y_0 + x_0ey_1 + ex_1y_0 + ex_1ey_1$$

$$= (x_0y_0 + x_1y_1) + e(x_0y_1 + x_1y_0)$$

and also

$$\phi((x_0 + x_1)(y_0 + y_1)) = \phi((x_0y_0 + x_1y_1) + (x_0y_1 + x_1y_0))$$

$$= (x_0y_0 + x_1y_1) + e(x_0y_1 + x_1y_0).$$

The algebra homomorphism $\phi$ is injective since if $n + 1 \notin I$ then the elements of the form $\theta_{n+1}(e_I) \cdot e$ are linearly independent by Theorem (2.1.4). Because both $C_n$ and $C_{n+1}^0$ are finite-dimensional with the same dimension the map $\phi$ is an algebra isomorphism.

Lemma 2.1.6. There are isomorphisms $C_1 \cong \mathbb{C}$ and $C_1^0 \cong \mathbb{R}$.

Proof. The Clifford algebra $C_1$ is generated by the element $\theta(e_1)$ such that $\theta(e_1)^2 = -1$ and it is of dimension 2 over $\mathbb{R}$ by Theorem (2.1.4). Thus, we obtain that the map $C_1 \rightarrow \mathbb{C}$, $x + y\theta(e_1) \mapsto x + iy$ is an isomorphism of algebras. Now certainly $C_1^0 \cong \mathbb{R}$.

Furthermore, since it holds that $C_{n+8} \cong C_n \otimes \mathbb{R}(16)$ and $C_{n+8}^0 \cong C_n^0 \oplus \mathbb{R}(16)$, we have classified all Clifford algebras $C_n$ and $C_n^0$. Proofs of this facts are beyond the scope of this thesis, however, they can be found in [Husemoller, 2013, p.158-161].

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Table 2.1: Clifford algebras \(C_n\) and \(C'_n\) for \(1 \leq n \leq 8\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_n)</td>
<td>(\mathbb{C})</td>
<td>(\mathbb{H})</td>
<td>(\mathbb{H} \oplus \mathbb{H})</td>
<td>(\mathbb{H}(2))</td>
<td>(\mathbb{C}(4))</td>
<td>(\mathbb{R}(8))</td>
<td>(\mathbb{R}(8) \oplus \mathbb{R}(8))</td>
<td>(\mathbb{R}(16))</td>
</tr>
<tr>
<td>(C'_n)</td>
<td>(\mathbb{R} \oplus \mathbb{R})</td>
<td>(\mathbb{R}(2))</td>
<td>(\mathbb{C}(2))</td>
<td>(\mathbb{H}(2))</td>
<td>(\mathbb{H}(2) \oplus \mathbb{H}(2))</td>
<td>(\mathbb{H}(4))</td>
<td>(\mathbb{C}(8))</td>
<td>(\mathbb{R}(16))</td>
</tr>
</tbody>
</table>

### 2.2 Vector fields on spheres and \(C_k\)-modules

In this section we will consider Euclidean vector spaces \(\mathbb{R}^k\) and \(\mathbb{R}^n\) with standard inner products \(\langle -, - \rangle\) and Euclidean norm \(\| - \|\). Recall the following parallelogram law

\[
\langle x, y \rangle = \frac{1}{2}(\|x + y\|^2 - \|x\|^2 + \|y\|^2).
\]

An orthogonal transformation of \(\mathbb{R}^n\) is a linear map \(\mathbb{R}^n \to \mathbb{R}^n\) which preserves the inner product of \(\mathbb{R}^n\). Moreover, orthogonal transformation is invertible with transpose as its inverse. Thus, the set of all orthonormal transformations forms a group called the orthogonal group. We call a linear map \(T : \mathbb{R}^k \to \mathbb{R}^n\) isometry if it preserves the norm.

**Definition 2.2.1.** A bilinear map \(\mu : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n\) is called an orthogonal multiplication if \(\|\mu(x, y)\| = \|x\|\|y\|\) for all \(y \in \mathbb{R}^k\) and \(x \in \mathbb{R}^n\). An orthogonal multiplication \(\mu\) is normalised if \(\mu(e_k, y) = y\) for all \(y \in \mathbb{R}^n\) where \(e_k = (0, \ldots, 0, 1) \in \mathbb{R}^k\).

For \(y \in \mathbb{S}^{n-1}\) we have that \(T(x) = \mu(x, y) : \mathbb{R}^k \to \mathbb{R}^n\) is an isometry because \(\|\mu(x, y)\| = \|x\|\|y\| = \|x\|\) by the definition of the orthogonal multiplication. By the parallelogram law we can compute for fixed \(x \in \mathbb{S}^{k-1}\)

\[
\langle \mu(x, y), \mu(x, z) \rangle = \frac{1}{2}((\|\mu(x, y) + \mu(x, z)\|^2 - \|\mu(x, y)\|^2 + \|\mu(x, z)\|^2)
\]

\[
= \frac{1}{2}(\|\mu(x, y + z)\|^2 - \|\mu(x, y)\|^2 + \|\mu(x, z)\|^2)
\]

\[
= \frac{1}{2}(\|y + z\|^2 - \|y\|^2 + \|z\|^2)
\]

\[
= \langle y, z \rangle.
\]

And so \(\mu(x, -) : \mathbb{R}^n \to \mathbb{R}^n\) is an orthogonal transformation.

Assume that orthogonal multiplication \(\mu\) is not normalised, in effect, the map \(\mu(e_k, y) = T(y)\) is not the identity orthogonal transformation. The map \(T = \mu(x, -)\) is invertible since it is an orthogonal transformation as we have mentioned above. Then the bilinear map \(\tilde{\mu}(x, y) = \mu(x, T^{-1}(y))\) satisfies \(\tilde{\mu}(e_k, y) = \mu(x, T^{-1}(y)) = TT^{-1}(y) = y\) and \(\|\tilde{\mu}(x, y)\| = \|x\|\|T^{-1}(y)\| = \|x\|\|y\|\). And so we can assume that \(\mu\) is normalised without loss of generality.

**Theorem 2.2.2.** There exists a bijection between the set of all normalised orthogonal multiplications \(\mu : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n\) and the set consisting of all sequences \((U_1, \ldots, U_{k-1})\) such that each \(U_i\) is an orthogonal transformation of \(\mathbb{R}^n\),

\[
U_i^2 = -I \text{ and } U_iU_j + U_jU_i = 0
\]

for \(i, j \in \{1, \ldots, k - 1\}\) where \(i \neq j\).
Proof. Assume that $(U_1, \ldots, U_{k-1})$ satisfy the relations above. Denote by $U_k$ the identity and define $\mu(x, y)$ by $\mu(x, y) = \sum_{i=1}^k x_i U_i(y)$ for $x = \sum_{i=1}^k x_i e_i$.

The map $\mu$ is clearly bilinear and normalised orthogonal multiplication because

$$\langle \mu(x, y), \mu(x, y) \rangle = \sum_{i=1}^k \sum_{j=1}^k x_i x_j \langle U_i(y), U_j(y) \rangle = \sum_{i=1}^k x_i^2 \|y\|^2 = \|x\|^2 \|y\|^2$$

where we use that

$$\langle U_j(y), U_i(y) \rangle = \langle -y, U_j U_i(y) \rangle = \langle y, U_i U_j(y) \rangle = -\langle U_i(y), U_j(y) \rangle$$

for $1 \leq i < j \leq k - 1$ and $-\langle y, U_i(y) \rangle = \langle U_i(y), y \rangle$ and for $1 \leq i \leq k - 1$.

Conversely, let $\mu$ be the normalised orthogonal multiplication. Define $U_i(y) = \mu(e_i, y)$ for $1 \leq i \leq k$. Choose an arbitrary element $x = \sum_{i=0}^k x_i e_i$ of $\mathbb{S}^{k-1}$ and so $\mu(x, -) = \sum_{i=0}^k x_i U_i$ is an orthogonal transformation of $\mathbb{R}^n$. Denote by $I$ the identity mapping on $\mathbb{R}^n$. Thus,

$$I = \sum_{i=1}^k \sum_{j=1}^k x_i x_j U_i U_j^T = \left( \sum_{i=1}^k x_i^2 \right) I + \sum_{1 \leq i < j \leq k} x_i x_j (U_i U_j^T + U_j U_i^T).$$

In particular, for $x = \frac{1}{\sqrt{2}} e_i + \frac{1}{\sqrt{2}} e_j$ we obtain $U_i U_j^T + U_j U_i^T = 0$ for $1 \leq i < j \leq k$. For $1 \leq i < k$ and $j = k$ we get that $U_i = -U_j^T = -U_i^{-1}$ and so $U_i U_i = -U_i^{-1} U_i = -I$. Furthermore, $0 = U_i U_j^T + U_j U_i^T = -U_i U_j - U_j U_i$ for $1 \leq i < j \leq k - 1$ as these correspondences are inverse to each other, the proof is complete.

Definition 2.2.3. Let $k \in \mathbb{N}$ and $C_k$ be the corresponding Clifford algebra. We define $C_{k-1}$-module as an ungraded (left) $C_k^0$-module that is a pair $(M, \psi)$ where $M$ is an abelian group and $\psi : C_k \to \text{End}(M)$ denotes a ungraded ring homomorphism.

Recall that every $C_k^0$ is a finite-dimensional vector space, that means there is the canonical ring homomorphism $\nu : \mathbb{R} \to \text{End}(C_k^0)$, $\nu(r)(c) = rc$ for $r \in \mathbb{R}$ and $c \in C_k^0$, that makes the ring $C_k^0$ into a finite-dimensional vector space. And so the composition $\psi(\nu(-)(1)) : \mathbb{R} \to \text{End}(M)$, makes the abelian group $M$ into a real vector space.

Now recall Lemma 2.1.5 $C_k^0 \cong C_{k-1}$ for $k \geq 2$. The point of definition 2.2.3 is that we obtain a $C_{k-1}$-module also for $k = 1$ even though $C_0$ is not defined.

Theorem 2.2.4. There exists a normalised orthogonal multiplication $\mu : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ if and only if $\mathbb{R}^n$ is a $C_{k-1}$-module.

Proof. Assume that there is a normalised orthogonal multiplication $\mu$.

By Theorem 2.1.4 we obtain that the set of generators $\{\theta(e_1), \ldots, \theta(e_{k-1})\}$ of $(C_{k-1}, \theta)$ with $\theta(x)^2 = -1$ satisfying the same relations as $(U_1, \ldots, U_{k-1})$ obtained from Theorem 2.2.2. Denote by $A$ the real unital algebra generated by the $U_i$ for $1 \leq i \leq k - 1$. Thus, the map $\alpha : \mathbb{R}^{k-1} \to A$, $x \mapsto \sum_{i=1}^{k-1} x_i U_i$ for $x = (x_i)_{i=1}^{k-1} \in \mathbb{R}^{k-1}$, satisfies the relation $(\alpha(x))^2 = -\|x\|^2$ and so it extends to a homomorphism $C_{k-1} \to \text{End}(\mathbb{R}^n)$. Thus, $\mathbb{R}^n$ is a $C_{k-1}$-module.

On the other hand, if $\mathbb{R}^n$ is a $C_{k-1}$-module, then there are linear maps $T_i$ for $i \in I = \{1, \ldots, k - 1\}$ satisfying the relations from Theorem 2.1.4 and so they
are invertible since $T_i(T_i)^3 = I$. Let $\Gamma$ be the subgroup of all invertible linear transformations of $\mathbb{R}^n$ generated by $\{T_i : i \in \{1, \ldots, k-1\}\}$. Using the relations from (2.2.2) we get that $\Gamma$ is the set

$$\{\pm I, \pm T_{j_1} \circ \cdots \circ T_{j_r} : 1 \leq j_1 < \cdots < j_r \leq n, \ 1 \leq r \leq k-1\}.$$  

Then the order of $\Gamma$ is at most twice the sum of $r$-tuples of $T_i$ for $0 \leq r \leq k-1$, where $0$-tuple corresponds to the identity $I$, since we have to pick a sign for each element. By the binomial theorem

$$|\Gamma| \leq 2 \sum_{r=0}^{k-1} \binom{k-1}{r} = 2^k.$$  

We define a new inner product on $\mathbb{R}^n$ by

$$(x, y) = \frac{1}{2^k} \sum_{T \in \Gamma} (T(x), T(y)).$$  

where $x, y \in \mathbb{R}^n$. Each $T \in \Gamma$ preserves $(-, -)$ because the post-composition by $T$ acts as a permutation on the set $\Gamma$. Let $f_1, \ldots, f_n$ be an orthonormal basis of $\mathbb{R}^n$ with respect to $(-, -)$. The assignment

$$v = \sum_{i=1}^{n} (v, f_i) f_i \mapsto \sum_{i=1}^{n} (v, f_i) e_i$$  

induces an isomorphism of vector spaces $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Since

$$(v, v) = (\sum_{i=1}^{n} (v, f_i) f_i, \sum_{j=1}^{n} (v, f_j) f_j) = \|v\|^2$$

$G$ is an isometry of $(\mathbb{R}^n, (-, -))$ and $(\mathbb{R}^n, \langle - , - \rangle)$ and by the parallelogram law it also preserves the inner product. Define $U_i = G \circ T_i \circ G^{-1}$. It is clear that $U_i$ satisfy the relations in (2.2.2) and orthogonality follows from $\langle U_i(x), U_i(y) \rangle = (T_i \circ G^{-1}(x), T_i \circ G^{-1}(y)) = (G^{-1}(x), G^{-1}(y)) = \langle x, y \rangle$. The application of Theorem (2.2.2) finishes the proof.  

Theorem 2.2.5. If $\mathbb{R}^n$ has the structure of a $C_{k-1}$-module then there is at least $k-1$ nowhere dependent vector fields for $\mathbb{S}^{n-1}$.  

Proof. By Theorem (2.2.4) there exists a normalised orthogonal multiplication $\mu : \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let $i, j \in \{1, \ldots, k\}$, $x \in \mathbb{S}^n$ and compute

$$\langle \mu(e_i, x), \mu(e_j, x) \rangle = \frac{1}{2} (\|\mu(e_i, x) + \mu(e_j, x)\|^2 - \|\mu(e_i, x)\|^2 - \|\mu(e_j, x)\|^2)$$

$$= \frac{1}{2} (\|\mu(e_i + e_j, x)\|^2 - \|\mu(e_i, x)\|^2 - \|\mu(e_j, x)\|^2)$$

$$= \frac{1}{2} (\|e_i + e_j\|^2 - \|e_i\|^2 - \|e_j\|^2)$$

$$= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Since $\mu$ is normalised we obtain that $\mu(e_i, x) \in T_x \mathbb{S}^{n-1}$ for $i \in \{1, \ldots, k-1\}$. Thus, we can define $k-1$ vector fields $s_i(x) = \mu(e_i, x)$ that are orthonormal by the previous calculation. \hfill $\square$
2.3 Radon-Hurwitz numbers

Lemma 2.3.1. Let $F$ be a field (not necessarily a commutative one), then the ring of $n \times n$ matrices $F(n)$ over $F$ has no non-zero proper two-sided ideal. The ring $F(n) \oplus F(n)$ has two maximal ideals $F(n) \oplus 0$ and $0 \oplus F(n)$.

Proof. Assume that $J \neq \{0\}$ is a both-sided ideal in $F(n)$. Let $A = (a_{i,j})_{i,j=1}^{n} \in J$ be a non-zero matrix such that $a_{k,l} \neq 0$ for some $1 \leq k, l \leq n$. Denote by $E_{i,j}$ a matrix unit, i.e. the matrix where the only non-zero element is in the $(i,j)$-th entry and it is equal to 1. It is clear that $a_{k,l}^{-1}E_{k,l}AE_{k,l} = E_{k,l} \in J$. Then $E_{k,l}E_{k,k} = E_{k,k} \in J$ and $E_{j,k}E_{k,k} = E_{j,k} \in J$ for each $1 \leq j, k, l \leq n$ and thus also we have that $E_{j,k}E_{k,l}E_{l,j} = E_{j,l} \in J$. The ideal $J$ is closed under addition and so $I = E_{11} + E_{22} + \ldots + E_{nn} \in J$, i.e. $J = F(n)$. The second statement is a simple consequence of the first statement. □

Lemma 2.3.2. The Clifford algebra $C_k$ is a semisimple algebra for every $k \in \mathbb{N}$.

Proof. Clearly $C_k$ is semisimple for $k = 1, \ldots, 8$ by Table [2.1] and $\mathbb{R}(16)$ is an irreducible algebra by Lemma (2.3.1). Now consider semisimple algebra $C$ over $\mathbb{R}$ with decomposition $C = \bigoplus_{i \in I} C_i$ into irreducible subalgebras $C_i$. The tensor product $C \otimes \mathbb{R}(16)$ preserves the direct sum and so $C \otimes \mathbb{R}(16) = \bigoplus_{i \in I} C_i \otimes \mathbb{R}(16)$. Let $J$ be a both-sided non-zero ideal generated by $\sum a_i \otimes b_i \in C_i \otimes \mathbb{R}(16)$. The multiplication in the tensor product of algebras over $\mathbb{R}$ yields that $(1 \otimes e)(a_i \otimes b)(1 \otimes f) = a_i \otimes eb_1f \in J$ and $(x \otimes 1)(a_i \otimes b_i)(y \otimes 1) = xa_iy \otimes b_i \in J$, in effect $J = C_i \otimes \mathbb{R}(16)$ since both $C_i$ and $\mathbb{R}(16)$ are irreducible. □

Lemma (2.3.2) implies that every $C_k$-module is a direct summand of simple modules.

In Theorem (2.2.5) we have reduced the question of existence of nowhere-dependent vector fields on sphere $S^{n-1}$ to the problem whether $\mathbb{R}^n$ can be equipped with the structure of $C_{k-1}$-module.

As we have already discussed after Definition (2.2.3), every $C_k$-module has the structure of real-dimensional vector space.

Theorem 2.3.3. For the ring $F(n)$ as above $F^n$ is the only irreducible left $F(n)$-module up to isomorphisms. For $F(n) \oplus F(n)$ there are two classes of irreducible left $F(n) \oplus F(n)$-modules up to isomorphisms corresponding to projections onto the first and the second coordinate in $F(n) \oplus F(n)$ followed by the action of $F(n)$ on $F^n$.

For detailed proof see [Spáčil 2007, Véty 3.7-8].

As we have shown in Table [2.1], the Clifford algebra $C_k$ for $1 \leq k \leq 8$ is isomorphic to $F(n)$ or $F(n) \oplus F(n)$, where $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Let $F^n$ be the irreducible $C_k$-module. Since $C_{k+8} \cong C_k \otimes \mathbb{R}(16)$ the vector space $F^n \otimes \mathbb{R}(16)$ is an irreducible $C_{k+8}$-module with $(A \otimes B) \cdot (x \otimes y) = Ax \otimes By$, analogously for $(F(n) \oplus F(n)) \otimes \mathbb{R}(16)$. Thus, any irreducible $C_k$-module is isomorphic to $F^n$.

Definition 2.3.4. For $k \in \mathbb{N}$ denote by $b_k$ the smallest natural number such that $\mathbb{R}^{b_k}$ can be equipped with the structure of an irreducible left $C_k^0$-module.

By Table [2.2] and Lemmas [2.1.5] and [2.1.6] we get that $b_{k+8} = 16b_k$. 

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Table 2.2: Values of $b_k$ for $1 \leq k \leq 8$

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_k^0$</td>
<td>$\mathbb{R}$</td>
<td>$C$</td>
<td>$H$</td>
<td>$H \oplus H$</td>
<td>$H(2)$</td>
<td>$\mathbb{C}(4)$</td>
<td>$\mathbb{R}(8)$</td>
<td>$\mathbb{R}(8) \oplus \mathbb{R}(8)$</td>
</tr>
<tr>
<td>$b_k$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

Definition 2.3.5. Let $n$ be a natural number of the form $n = 2^b a$ with $b \in \mathbb{N}_0$ arbitrary and $a \in \mathbb{N}$ odd. Decompose $b$ into the sum of integers $b = c + 4d$ such that $0 \leq c \leq 3$. Define the $n$-th **Radon-Hurwitz number** $\rho(n)$ by $\rho(n) = 2^c + 8d$.

For $n = 2^b a$ an odd number we immediately get $\rho(n) - 1 = 2^0 + 8 \cdot 0 - 1 = 0$. For $n$ even one can simply compute the following table. Where the first row counts the dimension of $S^{n-1}$ and in the second row are numbers of vector fields constructed by our method which will be proven in Theorem (2.3.6). Observe that $S^1, S^3$ and $S^7$ are parallelizable.

Table 2.3: Values of $\rho(n) - 1$ for $n$ odd

<table>
<thead>
<tr>
<th>$n-1$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(n)$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>8</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Theorem 2.3.6 (Hurwitz, Radon, Eckmann). Let $S^{n-1} \subseteq \mathbb{R}^n$ be $(n-1)$-sphere, then we can construct at most $\rho(n) - 1$ nowhere-dependent vector fields on $S^{n-1}$ by the method using the orthogonal multiplication.

**Proof.** For a fixed $n$ denote by $m$ the maximal number such that $\mathbb{R}^n$ is a $C_{m-1}$-module, in effect, $m-1$ is maximal number of nowhere dependent vector fields coming from the method used in Section (2.2) by Theorem (2.2.4). The algebra $C_{m-1}$ is semisimple by Lemma (2.3.2) and so $\mathbb{R}^n$ is a direct sum of irreducible $C_{m-1}$-modules. As we have discussed, every irreducible $C_{m-1}$-module is isomorphic to $\mathbb{R}^k$ for some $k \in \mathbb{N}$ as a vector space by Theorem (?). Clearly $b_m \leq k$ by the definition of $b_m$. If $b_m < k$, then $\mathbb{R}^{b_m} \subsetneq \mathbb{R}^k$ a non-trivial submodule, which is a contradiction with $\mathbb{R}^n$ being irreducible. And so $\mathbb{R}^n$ decomposes into the following sum

$$
\mathbb{R}^n \cong \mathbb{R}^{b_m} \oplus \mathbb{R}^{b_m} \oplus \ldots \oplus \mathbb{R}^{b_m}
$$

of irreducible $C_{m-1}$-modules. Then $m$ is the greatest number so that $b_m \mid n$ as $(b_i)_{i \in \mathbb{N}}$ is a non-decreasing sequence of natural numbers.

By inspection of Table (2.2) note that for $0 \leq c \leq 3$ the number $k = 2^c$ is the greatest index such that $b_k = 2^c$. Now write $n = 2^{c+4d} a$ as in Definition (2.3.5). Clearly, $b_{k+8d} = 16^d b_k$, thus,

$$
n = 2^{c+4d} a = 2^c 16^d a = 16^d b_{2^c} a = b_{(2^c+8d)} a = b_{\rho(n)} a.
$$

And so $\rho(n)$ is maximal index such that $b_{\rho(n)} \mid n$. This yields $\rho(n) = m$ which finishes the proof.

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Note that this result does not mention the upper bound for the number of nowhere-dependent vector fields. It only describes the limitations of the method of construction described in the previous section.

The task to show that there are no more vector fields on $S^{n-1}$ than $\rho(n) - 1$ is challenging. It is due to Adams, who proved this statement using $K$-theory. The proof is beyond the scope of this thesis, however, it can be found in Adams [1962].
3. Division algebras

**Definition 3.0.1.** A real unital finite-dimensional algebra $A$ is called a division algebra if for any non-zero $x \in A$ there exist unique $y, z \in A$ such that $y \cdot x = 1$ and $x \cdot z = 1$.

The problem whether $\mathbb{R}^n$ can be endowed with the structure of division algebra is a classical one. First result was due to Frobenius in 1877. It states that $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ are the only real finite-dimensional division algebras. Then, in 1898, Hurwitz proved that 1, 2, 4 and 8 are the only admissible dimensions for which there is multiplication on $\mathbb{R}^n$ that makes $\mathbb{R}^n$ into a division algebra such that $\|x \cdot y\| = \|x\|\|y\|$.

The existence of $k-1$ vector fields on $S^{n-1}$ is tightly connected to the existence of an orthogonal multiplication $\mu: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ as we have already seen. Clearly, by Theorem (2.2.4) and Theorem (2.2.5), it holds that $S^{n-1}$ is parallelizable if and only if there is an orthogonal multiplication $\mu: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

**Theorem 3.0.2 (Adam's).** The $(n-1)$-sphere $S^{n-1}$ is parallelizable if and only if $n = 2, 4, 8$.

As a corollary we obtain the original Hurwitz’s conclusion that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only real finite-dimensional division algebras.

Surprisingly, the fact that the dimension must be a power of two can be proven using Stiefel-Whitney classes considering these characteristic classed for the projective space see [Milnor et al., 1974, Theorem 4.7].
Conclusion

Since the existence of vector fields on spheres is a solved classical problem, unfortunately, there are not any new results in this thesis. Main author’s contribution lies in the computation of the Euler characteristic class of the tangent bundle of even-dimensional spheres (see Claim 1.3.11) in the first chapter which is greatly inspired by Milnor et al. [1974]. Relevant notions and theorems from algebraic topology were taken from Hatcher et al. [2002].

The author took pleasure especially in learning basics of the theory of characteristic classes, that resulted in the first chapter, where the Hairy Ball Theorem can be proven in slightly easier way using homotopy and antipodal mapping.

The second chapter mainly follows the structure of Chapter 12 in Husemoller [2013]. There can be found some minor contributions in details of proofs. However, technical results including the construction and the periodicity of certain Clifford algebras was left for reader to find out for sake of brevity. This task is done in great detail in the bachelor thesis Spáčil [2007], though, it is accessible only for Czech speaking readers. In Section 2.3.5 concerning Radon-Hurwitz numbers the proof of Theorem 2.3.6 is done in great detail.

The last chapter about historical background of the parallelizable spheres in the view of the existence of division algebras was inspired by Hatcher et al. [2002] and the unfinished book Hatcher.
Bibliography


