**FACULTY
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BACHELOR THESIS

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Computing and estimating ordered Ramsey numbers

Department of Applied Mathematics

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Title: Computing and estimating ordered Ramsey numbers

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Abstract: We study ordered Ramsey numbers, which are an analogue of the classical Ramsey numbers for ordered graphs. We improve some already obtained results for a special class of ordered matchings and disprove a conjecture of Rohatgi. We expand the classical notion of Ramsey goodness to the ordered case and we attempt to characterize all Ramsey good connected ordered graphs. We outline how Ramsey numbers can be obtained computationally and describe our SAT solver based utility developed to achieve this goal, which might be of use to other researchers studying this topic.

Keywords: ordered graph, ordered Ramsey numbers, Ramsey theory

Contents

1. Introduction

The Ramsey theory is devoted to study the minimum size of a system that guarantees the existence of a highly organized subsystem. We will start with a folklore example to illustrate the topic.

Problem 1.1. *There are six people at a party. Some pairs of these people can be friends, which is a symmetric relation. Prove that we can always find a group of three people such that either*

1. *all of them are friends, or*

2. *no two of them are friends.*

Proof. Let us suppose the statement does not hold and denote by A any of the six people present. We know for sure that A either is friends with at least three other people, or A is not friends with at least three other people. Let us assume the former holds, as the latter can be achieved symmetrically by reversing the relation of friendship. We denote these three people by B, C, D.

If there is any other friendship relation among B, C, D, the two people involved together with A form a group of three people such that all of them are friends. If there is no friendship relation among B, C, D, then B, C, D form a group of three people such that no two of them are friends. Thus, we have reached a contradiction and the statement holds. □

However, if we decrease the number of people at the party to five, the statement no longer holds. This is demonstrated in Figure 1.1, where red edges indicate friendship. Apparently, there does not exist any group of three friends, just as there does not exist any group of three non-friends. Therefore we just found out that six is the least number of people needed at a party for the statement to hold.
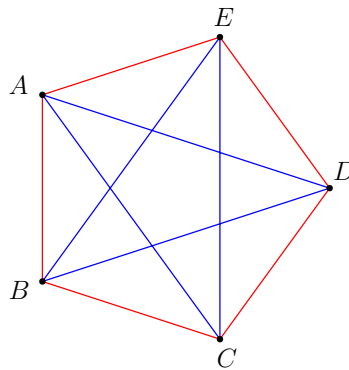

Figure 1.1: An example of a party of five people where neither a group of three friends nor a group of three non-friends exists. The red color represents the friendship relation.

What happens if we look for larger groups of friends/non-friends? Or for even less regular structures? In order to answer these questions, we will get a bit more formal about the problem we are facing.

1.1 Ramsey numbers

In this section, we will state some basic definitions and introduce the Ramsey numbers of graphs.

Definition 1.2. *An **(undirected) graph** G is a pair (V, E) consisting of a set of vertices V and a set of edges E, where each edge is an unordered pair of different vertices. We denote the size of the graph G by $|G| := |V|$.*

An *empty graph* is a graph without any edges. A *complete graph K_N* is a graph on $N \in \mathbb{N}$ vertices such that it contains all possible edges.

Definition 1.3. *A graph $G' = (V', E')$ is a **subgraph** of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If G' contains all the edges among V' present in G, we call G' an **induced subgraph**.*

Definition 1.4. *A **coloring** of a graph $G = (V, E)$ is a mapping $f : E \to \mathcal{C}$ that maps edges into a given set of colors $\mathcal{C}$.*

In this thesis, we will focus mostly on *two-colorings*, which are colorings with $\mathcal{C} = \{red, blue\}$. We shall use the terms *red edges* and *blue edges*. If a vertex v is joined to another vertex u with an edge, we can call u a *red/blue neighbor* of v depending on the color assigned to this edge. We will also use the terms *red subgraph* and *blue subgraph* to express the subgraphs of a colored graph such that all of its edges have this one color. From now on, by a *coloring on N vertices* we mean a two-coloring of K_N.

Definition 1.5. *Given graphs G, H, the **Ramsey number** $r(G, H)$ is the smallest $N \in \mathbb{N}$ such that any two-coloring of K_N contains either G as a red subgraph or H as a blue subgraph.*

The more specific setting when $G = H$ is called the *diagonal* case and the number $r(G, G)$ is often abbreviated with just $r(G)$. The case $G \neq H$ is called the *off-diagonal* case. In this text, we will stick to these conventions.

Taking a look at the motivational party problem, we proved that $r(K_3) = 6$ and the proof consisted of two parts. First, we showed that any coloring on six vertices contains K_3 either as a red or as a blue subgraph. Second, we demonstrated in Figure 1.1 that there exists an *avoiding coloring* on five vertices without K_3 as a monochromatic subgraph.

Ramsey [1] first introduced these numbers and also proved that they are always finite. This result is called the *Ramsey theorem*. This was independently rediscovered by Erdős and Szekeres [2] who also proved the following bounds for all $m, n \in \mathbb{N}$:

$$r(K_m, K_n) \leq \binom{m + n - 2}{n - 1} \quad \text{and} \quad 2^{n/2} \leq r(K_n) \leq 2^{2n}. \qquad (1.1)$$

The lower bound was shown by Erdős [3] using a probabilistic proof.

Surprisingly, despite many efforts, these exponential estimates for Ramsey numbers of complete graphs were only very slightly improved. However, if we impose some additional constraints on the graphs, there are often better estimates of the corresponding Ramsey numbers.

Definition 1.6. *In a graph $G = (V, E)$, the **degree** of a vertex $v \in V$ is the number of other vertices in V connected to v by an edge. The **maximum degree** of a graph is the maximum degree over its vertices.*

A notable result concerning Ramsey theory is that the Ramsey number of a graph with a bounded maximum degree grows at most linearly in the number of its vertices [4].

Theorem 1.7 ([4]). *For each $d \in \mathbb{N}$, there exists some $c > 0$ such that if a graph G on n vertices has maximum degree at most d, then*

$$r(G) \leq cn.$$

1.2 Ordered Ramsey numbers

Various researchers [5, 6] recently started studying Ramsey numbers of graphs with linearly ordered vertex sets. In this thesis, we continue this line of research. First, we give the necessary definitions and preliminaries and state some basic facts about Ramsey numbers of ordered graphs.

Definition 1.8. *An **ordered graph** $G^<$ on N vertices is a graph whose vertex set is $[N] := \{1, \ldots, N\}$ and it is ordered by the standard ordering $<$ of integers.*

Definition 1.9. *An ordered graph $H^<$ on $[n]$ is an **ordered subgraph** of another ordered graph $G^<$ on $[N]$ if there exists a mapping $\phi : [n] \to [N]$ such that $\phi(i) < \phi(j)$ for $1 \leq i < j \leq n$ and also $\{\phi(i), \phi(j)\}$ is an edge of $G^<$ whenever $\{i, j\}$ is an edge of $H^<$; see Figure 1.2.*
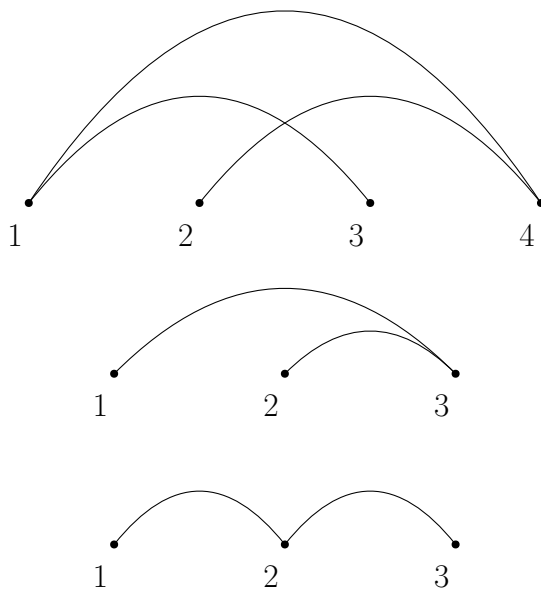
Figure 1.2: Out of these three ordered graphs, the second one is an ordered subgraph of the first one, whereas the third ordered graph is not.

Many definitions that are stated for unordered graphs, such as vertex degrees, colorings and so on, have their natural analogues for ordered graphs. Note that, for every $n \in \mathbb{N}$, there is a unique complete ordered graph $K_n^<$. We are now ready do define an analogous version of the Ramsey theorem for ordered graphs.

Definition 1.10. *Given ordered graphs $G^<, H^<$, the **ordered Ramsey number** $r_<(G^<, H^<)$ is defined as the smallest N such that any two-coloring of $K_N^<$ contains either $G^<$ as a red ordered subgraph or $H^<$ as a blue ordered subgraph.*

Similarly to the unordered Ramsey numbers, we use $r_<(G^<)$ to denote the number $r_<(G^<, G^<)$ in the diagonal case. We note that for any two ordered graph $G^<$ and $H^<$, the number $r_<(G^<, H^<)$ is finite, just as for the unordered Ramsey numbers; see Proposition 1.14.

To illustrate the usefulness of studying this topic, we will highlight a connection to the famous Erdős–Szekeres theorem on monotone subsequences.

Definition 1.11. *A **monotone path** $P_n^<$ is an ordered graph on $n \in \mathbb{N}$ vertices such that any two consecutive vertices are connected by an edge and there are no other edges.*

Lemma 1.12. *For all $r, s \in \mathbb{N}$, any coloring on $(r-1)(s-1)+1$ vertices contains either a red $P_r^<$ or a blue $P_s^<$ as an ordered subgraph.*

We will show a proof of this result along the lines of the proof of the Erdős–Szekeres theorem on monotone subsequences [7].

Proof. For $N = (r-1)(s-1)+1$, we consider a two-coloring of the ordered graph $K_N^<$. Suppose this coloring contains no monotone path $P_r^<$ as an ordered subgraph, otherwise the statement would hold. We label each vertex with the length of the longest monotone red path ending in this vertex and notice that no pair of vertices with the same label can share a red edge, as otherwise we could prolong the path ending in the former vertex to the latter one. Therefore any set of vertices with the same label forms a blue clique.

Since we can only use labels from the set $\{1, \ldots, r-1\}$, by the pigeonhole principle there are at least $\left\lceil \frac{(r-1)(s-1)+1}{r-1} \right\rceil = s$ vertices with the same label. These vertices form an ordered blue clique $K_s^<$ and thus also a blue monotone path $P_s^<$ exists as an ordered subgraph. $\qquad \square$

This lemma is a stronger version of the aforementioned famous theorem.

Theorem 1.13 (Erdős–Szekeres theorem [2])**.** *Assume we have a sequence of $(r-1)(s-1)+1$ distinct real numbers for some $r, s \in \mathbb{N}$. Then there exists either an increasing subsequence of r numbers, or a decreasing subsequence of s numbers.*

Proof. For $N = (r-1)(s-1)+1$, let us assume we have a sequence $s_1, \ldots, s_N$ of distinct real numbers. We create a two-coloring of the ordered graph $K_N^<$, where $\{i, j\}$ with $i < j$ is red if $s_i < s_j$ and blue otherwise.

We can see that a monochromatic path of length $n \in \mathbb{N}$ now corresponds to a monotonically increasing/decreasing subsequence of $s_1, \ldots, s_N$, depending on its color. By Lemma 1.12, we see that the coloring contains either a red $P_r^<$ or a blue $P_s^<$ as an ordered subraph, we are therefore done. $\qquad \square$

Remark. The reason why the Erdős–Szekeres theorem is a weaker version of Lemma 1.12 is that in Lemma 1.12 we consider all possible colorings, whereas in the Erdős–Szekeres theorem we consider only colorings determined by a total order on real numbers.

How do the ordered Ramsey numbers behave in comparison to their unordered counterparts? One observation in [5] relates these two directly. For an ordered graph $G^<$, we use G to denote its unordered counterpart.

Proposition 1.14. *For any two ordered graphs $G_1^<, G_2^<$ and their unordered counterparts G_1, G_2, we have*

$$r(G_1, G_2) \leq r_<(G_1^<, G_2^<) \leq r(K_{|G_1|}, K_{|G_2|}).$$

Proof. The first inequality follows from the fact if a coloring of $K_N^<$ for $N \in \mathbb{N}$ contains an ordered graph $G^<$ as a monochromatic ordered subgraph, then we can just remove the orderings and the coloring of the unordered K_N would still contain G as its monochromatic subgraph.

For the second inequality, we note that $r_<(K_r^<, K_s^<) = r(K_r, K_s)$ for any $r, s \in \mathbb{N}$. As the ordered graphs $G_1^<, G_2^<$ are contained as ordered subgraphs in $K_{|G_1|}^<, K_{|G_2|}^<$, respectively, we also have that $r_<(G_1^<, G_2^<) \leq r_<(K_{|G_1|}^<, K_{|G_2|}^<)$. This together with the previous equality completes the proof. $\square$

Remark. In this Bachelor thesis, we study ordered Ramsey numbers for only two colors. However, some already known results presented here also hold or are originally stated for an arbitrary number of colors. Lemma 1.12 and Proposition 1.14 are examples of this.

Proposition 1.14 gives us an intuition that in general, ordered Ramsey numbers are not smaller than the corresponding unordered Ramsey numbers. In other words, the added ordering makes it somewhat easier for an ordered coloring to avoid these given ordered graphs.

Another intuition we might have (as noted by previous papers on the topic [5, 6]) is that for dense graphs, there is not a huge gap comparing their ordered and unordered Ramsey numbers, since the ordered Ramsey numbers grow at most exponentially in the number of vertices by (1.1) and by Proposition 1.14. One can also show an exponential lower bound on $r(G)$ for every graph G on n vertices with cn^2 edges for any constant $c > 0$. On the other hand, sparse ordered graphs behave very differently from their unordered counterparts.

Definition 1.15. *A **matching** M is a graph such that each vertex has its degree at most one.*

The unordered Ramsey number of matchings is clearly linear in the number of its vertices. This result also follows from Theorem 1.7 as the maximum degree of a matching is one.

However, it was proved independently in [5, 6] that there exist ordered matchings on $n \in \mathbb{N}$ vertices such that their diagonal ordered Ramsey numbers grow superpolynomially, which is in sharp contrast with Theorem 1.7.

Theorem 1.16 ([5]). *There are arbitrarily large ordered matchings $M^<$ on n vertices that satisfy*

$$r_<(M^<) \geq n^{\frac{\log n}{5 \log \log n}}.$$

Definition 1.17. *For a given ordered graph $G^<$, we denote its **interval chromatic number** as the smallest number of contiguous subintervals partitioning its vertex set such that no two vertices from the same subinterval share an edge; see Figure 1.3.*
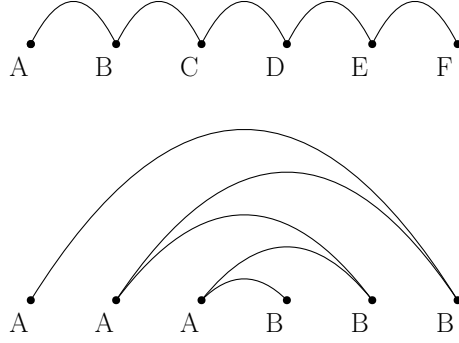
Figure 1.3: Out of these two ordered paths on 6 vertices, the first monotone path has the interval chromatic number 6, whereas the second one (so called *alternating path*) has the interval chromatic number 2, as depicted by the best possible color assignments.

Theorem 1.16 implies that bounding the maximum degree of an ordered graph $G^<$ is not sufficient to obtain a polynomial upper bound on $r_<(G^<)$. However, Conlon, Fox, Lee and Sudakov [6] and Balko, Cibulka, Král and Kynčl [5] proved that if we bound the interval chromatic number as well, we do obtain a polynomial bound on $r_<(G^<)$.

Theorem 1.18 ([6], slightly weaker version)**.** *There exists a constant c such that for any ordered graph $G^<$ on n vertices with maximum degree d and interval chromatic number χ,*

$$r_<(G^<) \le n^{cd \log \chi}.$$

The *length of an edge* $\{i, j\}$ of an ordered graph $G^<$ is the number $|i - j|$. Balko, Cibulka, Král and Kynčl [5] showed that ordered graphs with constant edge lengths have ordered Ramsey numbers at most polynomial in the number of vertices as well.

1.3 Ordered graphs visualization

In order to talk about ordered graphs and their colorings, we will work with two different types of visualizations. In Figure 1.4 there is a visualization of a coloring on 9 vertices that avoids a red $P_4^<$ and a blue $K_4^<$. We already know from Lemma 1.12 and its proof that 9 is the maximum number of vertices to satisfy these conditions.

In all figures, for an ordered graph $G^<$, we order the vertices along a horizontal line from left to right according to the vertex ordering of $G^<$. For a coloring of $K_n^<$ with colors red and blue, we draw the blue edges below the line containing the vertices and the red edges above the line. See Figure 1.4 left.

We also visualize colorings of $K_n^<$ using *coloring matrices*, where the rows and columns are indexed by the vertex set of $K_n^<$ and the entry (i, j) with $i < j$ is red if the corresponding edge $\{i, j\}$ of $K_n^<$ is red and blue otherwise. See Figure 1.4 right.
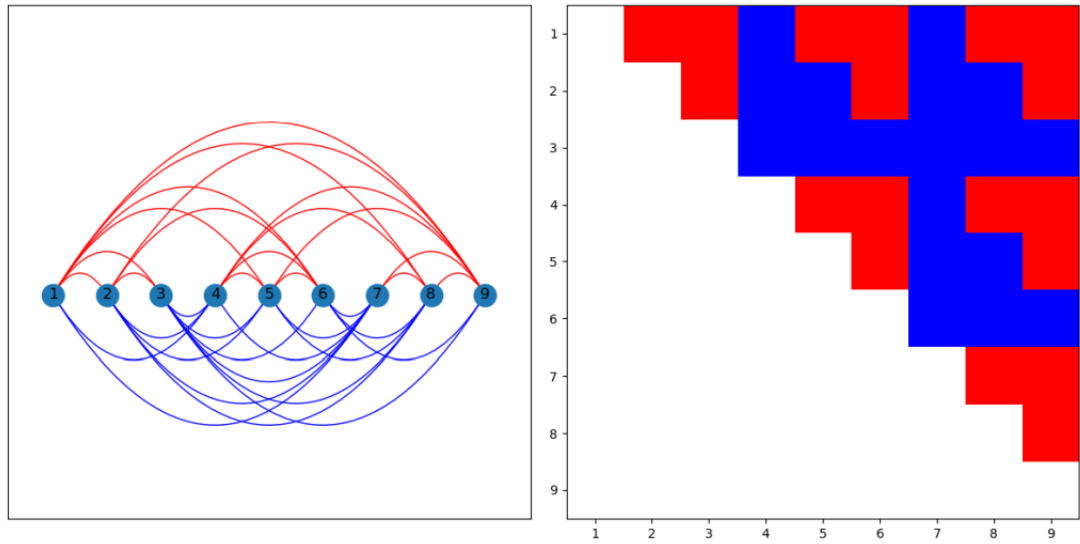
Figure 1.4: Ordered graph coloring visualization example. Graph visualization on the left, coloring matrix of the same coloring on the right. This figure was outputted by our utility described in more detail in Chapter 4.

2. Ordered matchings

In this chapter we focus on those ordered Ramsey numbers that involve ordered matchings. We mostly consider the off-diagonal case "matching vs. clique", but we also mention some known results about the diagonal case. Note that in this chapter we borrow the classical asymptotic notation.

2.1 Known results

Let us have an ordered matching $M^<$ on $n \in \mathbb{N}$ vertices. In their paper, Conlon, Fox, Lee and Sudakov [6] drew attention to the off-diagonal ordered Ramsey number $r_<(M^<, K_3^<)$. They proved that there exist ordered matchings $M^<$ such that $r_<(M^<, K_3^<) = \Omega\left(\left(\frac{n}{\log n}\right)^{\frac{4}{3}}\right)$. On the other hand, there is no better upper bound on $r_<(M^<, K_3^<)$ than

$$ r_<(M^<, K_3^<) \leq r_<(K_n^<, K_3^<) = r(K_n, K_3) = \mathcal{O}\left(\frac{n^2}{\log n}\right), $$

where the last equality is a well-known result from [8]. The first inequality only uses the fact that $M^<$ is an ordered subgraph of $K_n^<$ and does not utilize any special properties of ordered matchings. We note that the upper bound $r(K_n, K_3) = \mathcal{O}\left(\frac{n^2}{\log n}\right)$ is tight [9].

Conlon, Fox, Lee and Sudakov [6] believed that the upper bound on the number $r_<(M^<, K_3^<)$ is subquadratic in $|M^<|$ and asked the following question.

Problem 2.1 ([6]). *Does there exist an $\varepsilon > 0$ such that any ordered matching $M^<$ on $n \in \mathbb{N}$ vertices satisfies $r_<(M^<, K_3^<) = \mathcal{O}\left(n^{2-\varepsilon}\right)$?*

There has been some progress on this problem. Rohatgi [10] resolved two special cases of this problem. He proved that if the edges of an ordered matching $M^<$ do not cross, then the Ramsey number $r_<(M^<, K_3^<)$ is almost linear. Two edges $\{i, j\}$ and $\{k, l\}$ of an ordered graph $G^<$ are *crossing* if $i < k < j < l$.

Theorem 2.2 ([10]). *For any $\varepsilon > 0$, there is a constant c such that any ordered matching $M^<$ on $n \in \mathbb{N}$ vertices without any crossing edges satisfies*

$$ r_<(M^<, K_3^<) \leq cn^{1+\varepsilon}. $$

Rohatgi [10] also proved that the bound from Problem 2.1 holds with high probability for random ordered matchings with interval chromatic number 2.

Theorem 2.3 ([10]). *There is a constant c such that for all $n \in \mathbb{N}$, if a random ordered matching $M^<$ on $2n$ vertices with interval chromatic number 2 is picked uniformly at random, then*

$$ r_<(M^<, K_3^<) \leq cn^{\frac{24}{13}} $$

with high probability.

Rohatgi [10] also considered the following special family of ordered matchings.
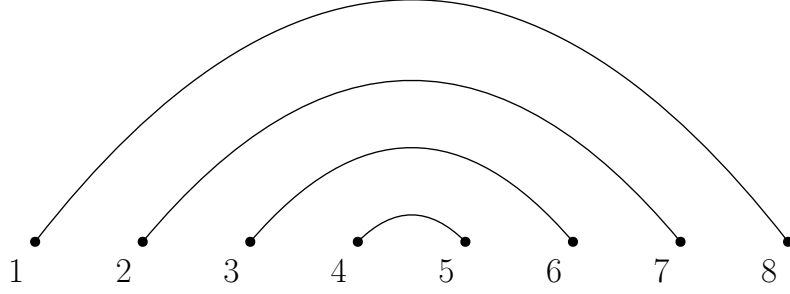
Figure 2.1: The nested matching $NM_4^<$.

Definition 2.4. *A **nested matching** $NM_n^<$ is the ordered matching on $2n$ vertices for $n \in \mathbb{N}$ with the edges $\{i, 2n - i + 1\}$ for $1 \leq i \leq n$; see Figure 2.1.*

Rohatgi [10] proved the following bounds for nested matchings.

Proposition 2.5 ([10]). *For any $n \in \mathbb{N}$, we have*

$$4n - 1 \leq r_<(NM_n^<, K_3^<) \leq 6n.$$

He believed that the upper bound is far from optimal and also posed the following conjecture.

Conjecture 2.6 ([10]). *For any $n \in \mathbb{N}$, we have*

$$r_<(NM_n^<, K_3^<) = 4n - 1.$$

2.2 Nested matchings and triangles

In this section, we improve the upper bound from Proposition 2.5 and disprove Conjecture 2.6 by stating some counterexamples found by a computer-assisted proof based on SAT solvers. For more details about the use of SAT solvers for finding avoiding colorings computationally, we refer the reader to Chapter 4.

We start by improving the upper bound from Proposition 2.5. We first state a helpful lemma about the structure of edges in an ordered graph that avoids $NM_n^<$ for some $n \in \mathbb{N}$.

Lemma 2.7. *For $n \in \mathbb{N}$, if an ordered graph $G^<$ on N vertices with $N \geq 2n$ does not contain $NM_n^<$ as an ordered subgraph, then the number of edges in $G^<$ is at most $(n - 1)(2N - 2n + 1)$. This upper bound is tight.*

Proof. Let $G^<$ be an ordered graph on the vertex set $\{1, \ldots, N\}$ such that $G^<$ does not contain $NM_n^<$ as an ordered subgraph. Let A be an $N \times N$ $\{0, 1\}$-matrix such that the (i, j) entry of A is 1 if and only if $\{i, j\}$ is an edge of $G^<$. For $k \in \{3, \ldots, 2N - 1\}$, we define the kth *anti-diagonal* of A to be the set of (i, j) entries such that $i + j = k$ and $i < j$; see Figure 2.2. Note that there are exactly $2N - 3$ anti-diagonals and each one of them contains at most $n - 1$ entries with 1 as otherwise these 1-entries form the nested matching $NM_n^<$ we are trying to avoid.

It follows from these observations that there can be at most $(n - 1)(2N - 3)$ edges in $G^<$. To obtain a stronger estimate, we take into account the fact that, for
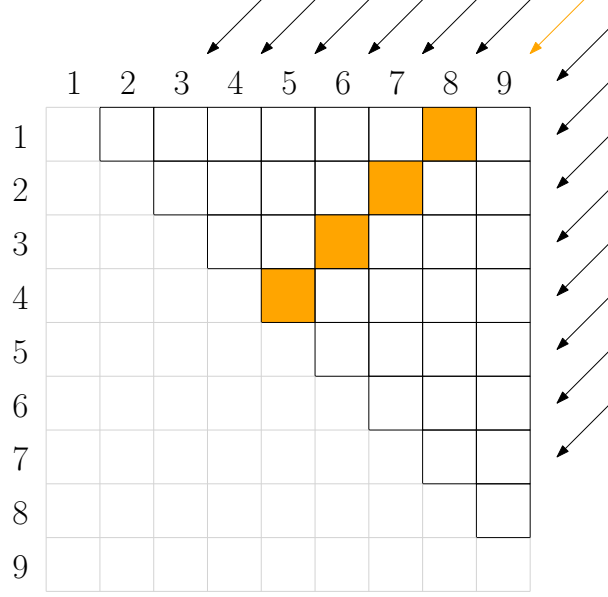
Figure 2.2: An example of an anti-diagonal in the matrix A for $N = 9$.

each $k < 2n - 1$, the kth anti-diagonal contains only at most $\lfloor \frac{k-1}{2} \rfloor$ 1-entries. A similar estimate holds for the kth anti-diagonals with $k > 2N - 2n + 3$. Altogether, for $N \geq 2n$, by summing the estimates for all anti-diagonals, we have an upper bound of

$$\sum_{k=3}^{2n-2} \left\lfloor \frac{k-1}{2} \right\rfloor + \sum_{k=2n-1}^{2N-2n+3} (n-1) + \sum_{k=2N-2n+4}^{2N-1} \left\lfloor \frac{2N-k+1}{2} \right\rfloor$$
$$= 2 \sum_{k=1}^{n-2} k + (2N - 4n + 5)(n-1) + 2 \sum_{k=1}^{n-2} k$$
$$= (n-2)(n-1) + (2N - 4n + 5)(n-1) + (n-2)(n-1)$$
$$= (n-1)(2N - 2n + 1)$$

on the number of edges of $G^<$.

This upper bound is tight, which can be seen by considering the ordered graph $G^<$ on N vertices with all edges of length at most $2n - 2$. Summing over the lengths k of the edges of $G^<$, we see that $G^<$ has

$$\sum_{k=1}^{2n-2} (N - k) = (n-1)(2N - 2n + 1)$$

edges. By construction, $G^<$ does not contain $NM_n^<$ as an ordered subgraph. $\square$

A general construction to achieve this maximum number of edges is to lead disjoint paths in the coloring matrix as illustrated in Figure 2.3. For such a construction, all the estimates in the previous proof are tight, therefore we also achieve the maximum number of edges.

It is easy to see that such a construction with $n - 1$ disjoint matrix paths does not contain $NM_n^<$ as an ordered subgraph. If it did, every edge of this copy of $NM_n^<$ would have to belong to one of the disjoint matrix paths. Note that no
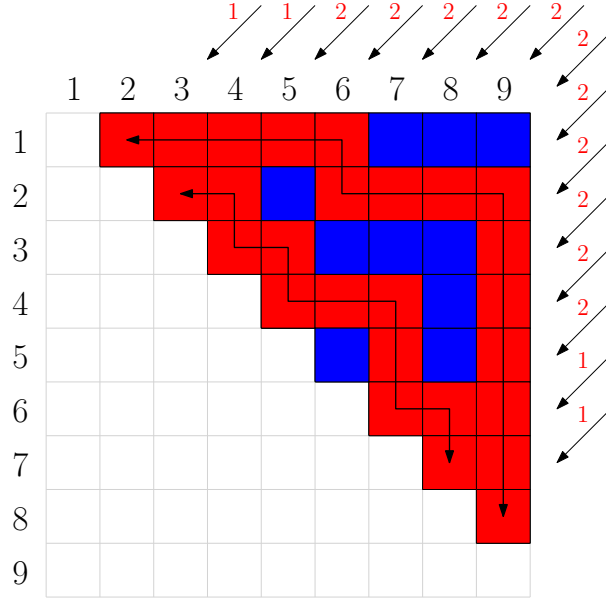
Figure 2.3: Construction of disjoint matrix paths in A for $N = 9$ and $n = 3$ where red represents 1-entries. We can see that each anti-diagonal achieves the maximum possible number of 1-entries.

two different edges of a nested matching can belong to the same red matrix path, since a copy of $NM_2^<$ corresponds to two entries (i, j) and (k, l) of the matrix A such that $i < k < l < j$, whereas any two entries (i, j) and (k, l) of a red matrix path satisfy $i \leq k$ and $j \leq l$. However, by the pigeonhole principle, there would be at least one such matrix path with at least two edges of $NM_n^<$, a contradiction.

By the depiction in Figure 2.3 we naturally moved towards colorings and gained some insight about the structure of colorings avoiding a red $NM_n^<$. Note that even if some of the entries of the $n - 1$ disjoint red matrix paths of the coloring matrix are colored blue, we do not create a red copy of $NM_n^<$.

Remark. In Figure 2.3 and other examples where we use the disjoint matrix paths construction, we can also notice the behavior of the blue 0-entries. If all the blue cells "moved" in the north-east direction, a blue triangle would be formed in the upper right corner of the matrix. Therefore they form an interesting "tetris-like" structure.

We are now ready to improve the upper bound from Proposition 2.5.

Proposition 2.8. *For every $n \in \mathbb{N}$,*

$$r_<(NM_n^<, K_3^<) \leq \left(3 + \sqrt{5}\right) n < 5.3n.$$

Proof. Let us assume we have an avoiding coloring on N vertices. In order to avoid a blue $K_3^<$, there cannot be any vertex with at least $2n$ blue neighbors, as this would imply the existence of a red $K_{2n}^<$ and thus also $NM_n^<$.

Assuming that every vertex has less than $2n$ blue neighbors, there can be at most $2nN/2 = nN$ blue edges. Therefore the avoiding coloring needs to have at least $\frac{N(N-1)}{2} - nN$ red edges. However, by Lemma 2.7, there can be at most $(n-1)(2N - 2n + 1)$ red edges, as otherwise the existence of $NM_n^<$ is guaranteed.

Therefore we have
$$\frac{N(N-1)}{2} - nN \le (n-1)(2N - 2n + 1),$$
which can be rewritten as
$$N^2 + 3N(1 - 2n) + (4n^2 - 6n + 2) \le 0.$$

By solving the last quadratic inequality for $N \in \mathbb{N}$ we arrive at a condition
$$N \le \frac{1}{2}\left(\sqrt{20n^2 - 12n + 1} + 6n - 3\right).$$

For $n \ge \frac{1}{2}$, we have $\frac{1}{2}\left(\sqrt{20n^2 - 12n + 1} + 6n - 3\right) \le \left(3 + \sqrt{5}\right)n$, which concludes the proof. $\square$

We thus see that the upper bound from Proposition 2.5 is indeed not tight. We believe that the bound from Proposition 2.8 can be improved as well, for example by using a better upper bound on the number of blue edges. Next we show that the lower bound from Proposition 2.5 is also not tight for some values of n. This disproves Conjecture 2.6 by Rothagi [10].

Theorem 2.9. *We have*
$$r_<(NM_4^<, K_3^<) > 15 \quad \text{and} \quad r_<(NM_5^<, K_3^<) > 19.$$

That is, the equality $r_<(NM_n^<, K_3^<) = 4n - 1$ from Conjecture 2.6 is not true for $n = 4, 5$.

Proof. It is enough to show that there exist colorings on 15 and 19 vertices, which avoid a blue triangle and red $NM_4^<$ and $NM_5^<$, respectively. We present computer generated counterexamples for each case in Figures 2.4 and 2.5. We prove the latter case by hand.

We call a coloring on N vertices *symmetric* if, for all $i, j \in \mathbb{N}$ with $1 \le i < j \le N$, the edges $\{i, j\}$ and $\{N - j + 1, N - i + 1\}$ have the same color. Let χ be the symmetric coloring on 19 vertices with the coloring matrix from Figure 2.5.

The coloring matrix of χ consists of 4 disjoint red matrix paths. It follows from the remark after Lemma 2.7 that the ordered graph formed by red edges does not contain $NM_5^<$ as an ordered subgraph.

It suffices to show that the coloring χ contains no blue triangle $K_3^<$. Let us denote by $S = \{1, \ldots, 10\}$ and suppose there exists a blue copy of $K_3^<$ as an ordered subgraph in χ. It follows from the symmetry of χ that there is a blue ordered subgraph $K_3^<$ formed by vertices $v_1 < v_2 < v_3$ such that $v_1, v_2 \in S$.

Vertices $1, 2, 4$ do not have any blue neighbors in S. The only blue neighbors of 3 in S are $8, 9, 10$ and the only blue neighbors of 3 outside of S are $18, 19$. But there is no blue edge among $8, 9, 10, 18, 19$, hence 3 also cannot be a part of any blue triangle. Thus $v_1, v_2 \in \{5, 6, \ldots, 10\}$.

Blue neighbors of 8 are $3, 5, 11, 12$ and these vertices form a red clique. Blue neighbors of 9 are $3, 5, 6, 12, 13$ and also form a red clique. Blue neighbors of 10 are two symmetric sets of vertices $3, 6, 7$ and $13, 14, 17$. These six vertices also form a red clique.

The only option left is $v_1, v_2 \in \{5, 6, 7\}$, which is impossible, as there are no blue edges between these vertices. We have reached a contradiction and thus the coloring χ contains no blue triangle $K_3^<$ as an ordered subgraph. $\square$
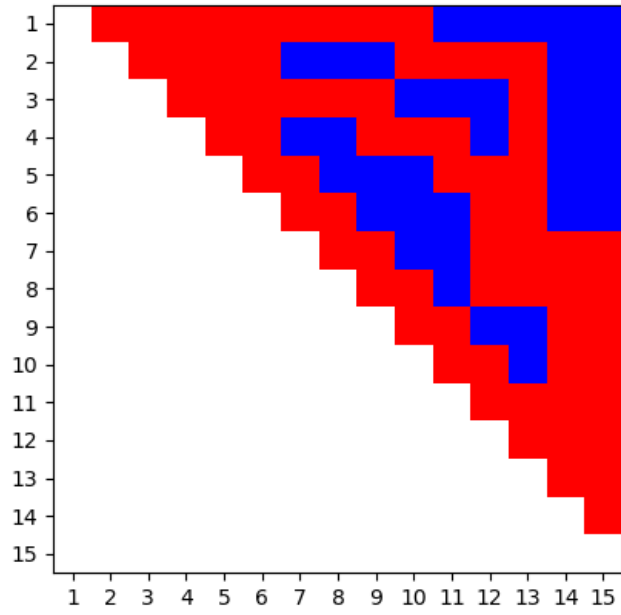
Figure 2.4: A coloring on 15 vertices avoiding red $NM_4^<$ and blue $K_3^<$.
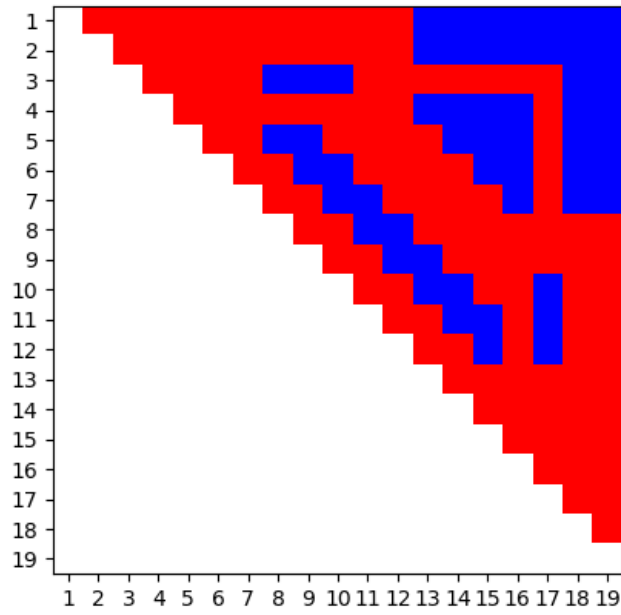

Figure 2.5: A symmetric coloring on 19 vertices avoiding red $NM_5^<$ and blue $K_3^<$.

We have done an exhaustive computer search (more details in Chapter 4) and proved that $r_<(NM_4^<, K_3^<) = 16$. The interesting part is that for $N = 15$, there were only 326 avoiding colorings with the exact same core structure except for up to 6 red edges additionally switched to blue while not introducing a blue triangle $K_3^<$. Using the same computer search we also proved that $r_<(NM_5^<, K_3^<) = 20$. In this case, however, the structure was a bit more "relieved" as for $N = 19$ we were able to find many and even symmetric colorings while for $r_<(NM_4^<, K_3^<)$ there were no symmetric colorings on 15 vertices. The main difficulty of the problem therefore seems to be "leading the disjoint matrix paths in a smart way" in order to avoid the blue triangle $K_3^<$ as an ordered subgraph.

2.3   Nested matchings and larger cliques

In the previous section we worked mostly with the ordered Ramsey numbers of the form $r_<(NM_n^<, K_3^<)$. Here, we consider larger cliques and we estimate the ordered Ramsey numbers $r_<(NM_m^<, K_n^<)$ for arbitrary $m, n \in \mathbb{N}$. First, we consider the case $m = 2$ and we prove the exact formula for $r_<(NM_2^<, K_n^<)$. To do so, we need the following auxiliary lemma.

Definition 2.10. *A **proper coloring** of a graph $G = (V, E)$ is a function which assigns a color to each vertex from V such no two adjacent vertices have the same color. A graph G is k-**colorable** if there exists a proper coloring of G with k colors.*

Although this definition is stated for unordered graphs, it naturally extends to ordered graphs as well.

Lemma 2.11. *Every ordered graph $G^<$ that does not contain $NM_2^<$ as an ordered subgraph is 3-colorable.*

Proof. We show that $G^<$ is 3-colorable with an algorithm. We denote the vertices of this ordered graph by $1, \ldots, N$ in their respective order and we let $\mathcal{C} = \{A, B, C\}$ be the set of three colors. We define a function $RMN(S)$, which, for a given set S of consecutive vertices from $G^<$, returns their rightmost neighbor, if there is one. We also define a function $NV(S)$, which, for a given set of consecutive vertices S from $G^<$, returns the immediate successor of S in the vertex ordering $<$, if there is such a vertex.

We can assume all vertices of $G^<$ have degree at least 3 as any vertex with at most 2 neighbors can be easily colored by taking the color not used by its neighbors, not influencing the rest of the graph.

Let I, J be two consecutive contiguous intervals of vertices of $G^<$ such that I precedes J. We call such I, J *separating*, if any vertex to the left of I and any vertex to the right of J do not share an edge. Therefore, if we assume that all the vertices of $G^<$ to the left of I are properly colored, then, for coloring vertices to the right of J, it is enough to take into account only vertices from $I \cup J$.

Now, let us color the ordered graph $G^<$ from left to right by the following algorithm. First we color the vertex 1 by color A, the vertex $v = RMN(\{1\})$ by B and all vertices from the set $S = \{2, \ldots, v-1\}$ by C. From our previous assumption about the degrees of $G^<$ we know that v exists and $v - 1 \geq 3$, thus
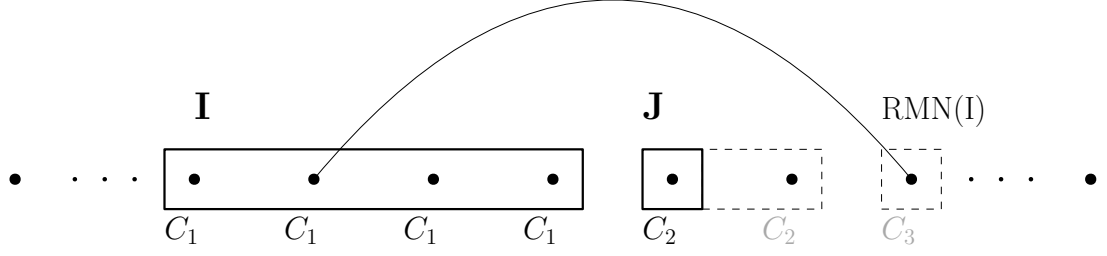
Figure 2.6: A step in the algorithm for 3-coloring an ordered graph without $NM_2^<$.

S is non-empty. Furthermore, no two vertices from S are adjacent in $G^<$, as any edge between these vertices would form a copy of $NM_2^<$ together with the edge $\{1, v\}$. Thus we have two nonempty contiguous intervals $I := \{2, \ldots, v - 1\}$ and $J := \{v\}$, which are properly colored and also separating by the choice of v.

From now on, we can continue inductively; see Figure 2.6. Assume we have two separating intervals I and J such that all vertices of I are colored by $C_1 \in \mathcal{C}$, all vertices of J are colored by $C_2 \in \mathcal{C}, C_2 \neq C_1$, and the induced ordered subgraph of $G^<$ on vertices $1, \ldots, max(J)$ is properly colored.

Let us denote by C_3 the last color not used within I, J. We distinguish a few cases based on $v := RMN(I)$, in all of which we either finish the coloring, or update the values of I, J in order to continue with the algorithm.

1. If $v \in J$, then the vertex $u := NV(J)$ can be colored by C_3, as the intervals I, J are separating. We then set $I := J$ and $J := \{u\}$, as these intervals are again monochromatic with different colors and separating by the choice of v.

2. If v is to the right of J, then we can color v by C_3, as I, J are separating. We again set $u := NV(J)$ and denote a new interval of vertices by $S := \{u, \ldots, v - 1\}$ if $u > v$ or $S := \{\}$, if $u = v$.

   We color all vertices in S by C_2, the same color as used in the current J. This preserves the proper coloring property, because none of the vertices in $J \cup S$ can be adjacent, as this would contradict the non-existence of $NM_2^<$ as an ordered subgraph in $G^<$, as there already exists an edge from I to v.

   Now, we set $I := J \cup S$ and $J := \{v\}$. Again, these new sets I, J are monochromatic with different colors and separating by the choice of v. This is the case illustrated in Figure 2.6.

3. Assume v is in I or it does not exist. We know that $u := NV(J)$ exists, as otherwise we are done. Since I, J are separating, there are no edges going from u to the left of J. Thus we can color u by C_3 and set $I := J, J := \{u\}$, which are again separating.

The algorithm is finite as in each step we color at least one vertex, thus this algorithm colors any $G^<$ without $NM_2^<$ as an ordered subgraph by three colors. □

Proposition 2.12. *For every $n \in \mathbb{N}$, we have $r_<(NM_2^<, K_n^<) = 3n - 2$.*

Proof. The lower bound follows from a simple coloring that consists of $n - 1$ consecutive red triangles connected by blue edges; see Figure 3.1.

It remains to show the upper bound. Let χ be a coloring on $3n - 2$ vertices. Assume that it does not contain a red copy of $NM_2^<$. By Lemma 2.11, the graph formed by the red edges in χ is 3-colorable. Therefore we can partition its vertex set into 3 disjoint sets such that no two vertices from the same set are connected by a red edge.

Using the pigeonhole principle, at least one of these sets contains at least $\left\lceil \frac{3n-2}{3} \right\rceil = n$ vertices. Therefore, these n vertices form a blue clique $K_n^<$ and we are finished. $\qquad\square$

We note there exists a notion of *k-queue graphs*. These graphs are proven to be equivalent to the graphs without NM_{k+1} for a given $k \in \mathbb{N}$ [11]. Specifically, 1-queue graphs are also known to correspond to so called *arched-leveled-planar graphs* [11].

Next, we generalize Proposition 2.5 and Proposition 2.8 to larger cliques using well-known Turán's theorem combined with Lemma 2.7.

Theorem 2.13 (Turán's theorem [12])**.** *For some $n, N \in \mathbb{N}$, let G be any graph with N vertices such that it does not contain K_{n+1} as a subgraph. Then the number of edges in G is at most*

$$\left(1 - \frac{1}{n}\right) \frac{N^2}{2}.$$

Theorem 2.14. *For every $m, n \in \mathbb{N}$, we have*

$$r_<(NM_m^<, K_{n+1}^<) = \Theta(mn).$$

Proof. Let us assume we have a coloring χ on N vertices, which avoids red $NM_m^<$ and blue $K_{n+1}^<$ as ordered subgraphs. We proceed along the lines of our proof for Proposition 2.8. By Lemma 2.7 that there can be at most

$$(m - 1)(2N - 2m + 1)$$

red edges in χ. At the same time, Theorem 2.13 implies that there can be at most

$$\left(1 - \frac{1}{n}\right) \frac{N^2}{2}$$

blue edges in χ. Thus it needs to hold that

$$(m - 1)(2N - 2m + 1) + \left(1 - \frac{1}{n}\right) \frac{N^2}{2} \geq \frac{N(N - 1)}{2},$$

which can be rewritten as

$$N^2 - Nn(4m - 3) + n(4m^2 - 6m + 2) \leq 0.$$

By solving the quadratic inequality we get

$$N \leq \frac{1}{2}\left(4mn - 3n + \sqrt{(4m - 3)^2 n^2 - 16m^2 n + 24mn - 8n}\right),$$

thus $N = \mathcal{O}(mn)$. The lower bound $N = \Omega(mn)$ can be obtained from a coloring on $(2m - 1)(n - 1)$ vertices formed by $n - 1$ red cliques of size $2m - 1$ such that every two vertices from different cliques form a blue edge. This coloring certainly contains no red $NM_m^<$ and no blue $K_n^<$ as ordered subgraphs. $\qquad\square$

We note that the coloring we used to show lower bounds for Proposition 2.12 and Theorem 2.14 is a well-known construction, which is described in more detail in the following chapter.

3. Ramsey goodness

In this chapter we will introduce the notion of *Ramsey goodness*, which has been extensively studied for unordered graphs. We extend Ramsey goodness to ordered graphs and we prove some new results.

3.1 Unordered case

There exists an easy construction for the lower bound on $r(G, K_n)$ if the graph G is connected.

Proposition 3.1. *For any connected graph G on m vertices, we have*

$$r(G, K_n) \geq (m-1)(n-1) + 1.$$

Proof. We will prove that there exists an avoiding coloring on $N = (m-1)(n-1)$ vertices for any such graph G. The coloring consists of $n-1$ red cliques of $m-1$ vertices. Vertices in different cliques are connected by blue edges. See Figure 3.1 for an example. There cannot be any red G, because $|G| > m-1$ implies that two of its vertices would belong to different cliques, which is impossible, as there are no red edges between these cliques and the graph G is connected. There also cannot be a blue K_n, as, by the pigeonhole principle, at least two of its vertices belong to the same red clique, hence the edge between them is not blue. $\square$
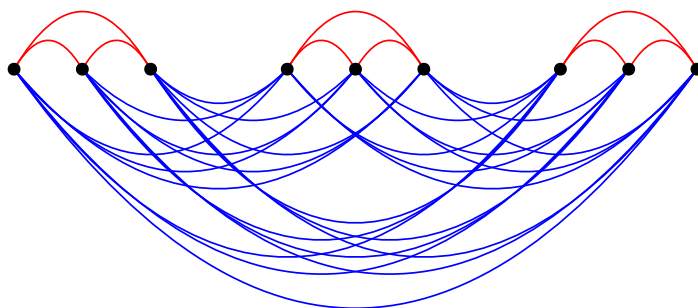

Figure 3.1: A coloring on 9 vertices avoiding any connected G on 4 vertices as a red subgraph and K_4 as a blue subgraph. Note that the vertex ordering is not necessary for this construction to work.

There is now an interesting question awaiting. For which graphs is this type of coloring the best possible? Since these graphs have been studied for the unordered case, they have a name.

Definition 3.2. *For $n \in \mathbb{N}$, a graph G is n-good if $r(G, K_n) = (|G|-1)(n-1)+1$. We call a graph (Ramsey) good if it is n-good for all $n \in \mathbb{N}$.*

Remark. Although we do not need it in this thesis, there is a more general construction which does not necessarily involve a clique as the second graph. Interested reader may consult the corresponding section in [13].

We mention a famous result of Chvátal [14] which predates this definition of Ramsey goodness.

Definition 3.3. *A **tree** is a graph that is both acyclic and connected.*

Theorem 3.4 ([14]). *Every unordered tree is good.*

In an effort to determine what properties contribute to Ramsey goodness, Burr and Erdős [15, 16] conjectured that, for given $n \in \mathbb{N}$, every sufficiently large graph with a fixed maximum degree is n-good. This has been disproved by Brandt [17].

However, fixing another parameter of the graph is enough to prove goodness for sufficiently large graphs, as Burr and Erdős [15] showed that sufficiently large graphs with bounded bandwidth are n-good.

Definition 3.5. *A **bandwidth** of a graph G is the smallest number $l \in \mathbb{N}$ such that G can be ordered so that every edge has length at most l.*

Theorem 3.6 ([15]). *Let $n, l \in \mathbb{N}$ be fixed. Then all sufficiently large connected graphs with bandwidth at most l are n-good.*

Another interesting result uses the notion of some forbidden substructures in the graph.

Definition 3.7. *A graph H is a **minor** of another graph G if H can be obtained from a subgraph of G by contracting edges.*

Conlon, Fox, Lee and Sudakov [18] showed that graphs without a fixed minor are n-good.

Theorem 3.8 ([18]). *For every $n \in \mathbb{N}, n \geq 3$ and fixed graph H, every sufficiently large connected graph that does not contain H as a minor is n-good.*

Since the family of planar graphs consists precisely of those graphs that do not contain K_5 and $K_{3,3}$ as a minor, Theorem 3.8 implies the following.

Corollary 3.9 ([18]). *For any $n \in \mathbb{N}$, every sufficiently large connected planar graph is n-good.*

Conlon, Lee, Fox and Sudakov [18] also investigated Ramsey goodness of hypercube graphs.

Definition 3.10. *The **hypercube** Q_k is the graph on the vertex set $\{0, 1\}^k$ where two vertices are connected by an edge if and only if they differ in exactly one coordinate.*

Building upon their work, the following theorem was recently proved by Fiz Pontiveros, Griffiths, Morris, Saxton and Skokan [19, 20].

Theorem 3.11 ([20]). *For a fixed $n \in \mathbb{N}$ and every sufficiently large $k \in \mathbb{N}$, the hypercube graph Q_k is n-good.*

3.2 Ordered case

From what we know, Ramsey goodness has not been studied for ordered Ramsey numbers at all. We therefore naturally extend the definitions from the previous section to ordered graphs and we prove some basic properties, followed by more interesting results in an attempt to characterize all good connected ordered graphs.

Remark. Note that Proposition 3.1 holds trivially for ordered graphs as the proof does not need the ordering restriction, which is apparent from Figure 3.1. For ordered graphs there can exist other constructions with more red edges which also produce the same bound; see Figure 3.2.
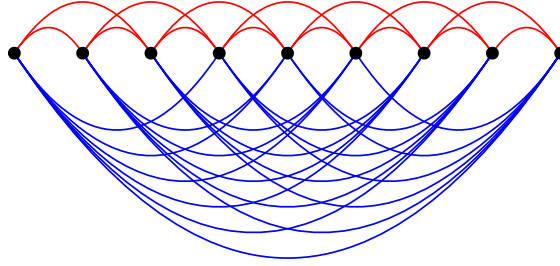

Figure 3.2: Example of an ordered coloring on 9 vertices avoiding any ordered $G^<$ with at least one edge of length at least 3 as a red ordered subgraph and $K_4^<$ as a blue ordered subgraph.

To illustrate, the proof of Lemma 1.12 yields a stronger result $r_<(P_r^<, K_n^<) = (r-1)(n-1)+1$ for any $r, n \in \mathbb{N}$ and the monotone path $P_r^<$, therefore any monotone path is good. However, in contrast with Theorem 3.4, not all ordered trees are good. Not even all ordered paths are good, which is shown by a counterexample from Figure 3.3.

For disconnected ordered graphs, the situation is somehow wilder as the constructions from Figure 3.1 no longer provides the lower bound. In fact, some sparse disconnected ordered graphs $G^<$ can be "better" than good, as it holds that $r_<(G^<, K_n^<) < (|G^<|-1)(n-1)+1$ for some $n \in \mathbb{N}$, for example the empty ordered graph. These "better" graphs can be possibly joined with another graph in order to make them good, thus characterization for these ordered graphs can be messy.

In the previous chapter we proved that $NM_4^<$ and $NM_5^<$ are not 3-good in Proposition 2.9, therefore they are not good. Somewhat surprisingly, we then proved in Proposition 2.12 that $NM_2^<$ is good.

3.3 Our results

Focusing on connected graphs from now on, we move into a more general setting and prove some basic properties, some of them resembling their unordered counterparts.

Theorem 3.12. *An ordered graph $G^<$ is not good if it has at least as many edges as vertices.*
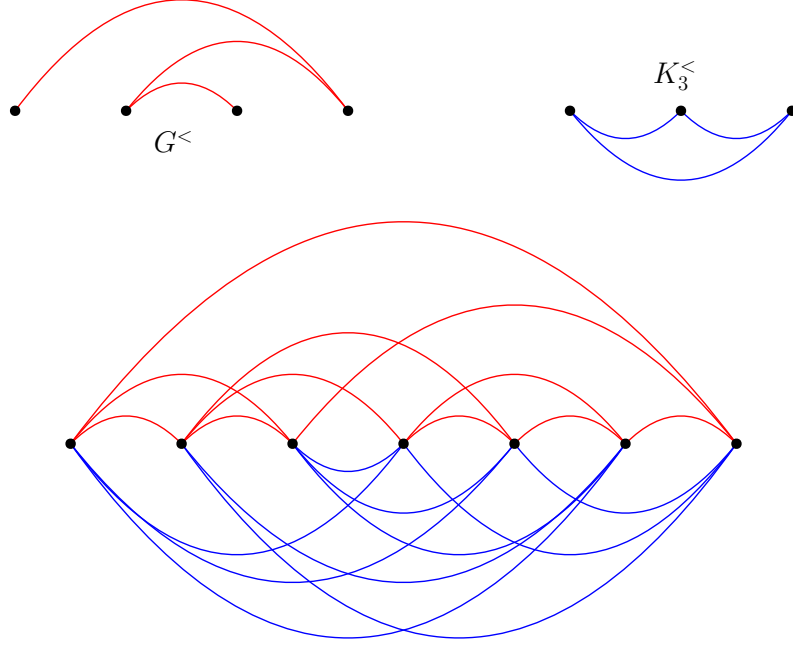
Figure 3.3: An example of a coloring on 7 vertices that contains no red $G^<$ and no blue $K_3^<$, thus showing that not all ordered paths are good.

Proof. It is easy to see that such $G^<$ contains at least one cycle $C^<$ as an ordered subgraph. In [21] it is proven that for $l, t \in \mathbb{N}$ with $l \geq 4$, any unordered cycle C_l on l vertices and for n going to infinity, we have

$$r(C_l, K_n) = \Omega\left(n^{\frac{l-1}{l-2}}/\log n\right).$$

In other words, the Ramsey number $r(C, K_n)$ grows superlinearly with n. Since $r_<(G^<, K_n^<) \geq r_<(C^<, K_n^<) \geq r(C, K_n)$, the number $r_<(G^<, K_n^<)$ is superlinear in n and thus the graph $G^<$ cannot be good. □

Note that this proof holds for unordered graphs as well. Thus by Theorem 3.4 the class of good connected unordered graphs consists exactly of trees. If we also require the ordered graph $G^<$ to be connected, Theorem 3.12 implies that any good connected ordered graph has to be an ordered tree.

Proposition 3.13. *If a connected ordered graph $G^<$ is $(n+1)$-good, then it is n-good.*

Proof. Suppose for contradiction that an ordered graph $G^<$ is $(n+1)$-good, but not n-good. Then there exists a coloring on $N = (|G^<| - 1)(n-1) + 1$ vertices avoiding red $G^<$ and blue $K_n^<$ as ordered subgraphs. This coloring can be easily extended by adding a red clique of $|G^<| - 1$ new vertices to the right of the original coloring. Edges between the new vertices and the original ordered graph shall be colored blue. This way we ensure that the resulting coloring on $n(|G^<| - 1) + 1$ vertices contains no red $G^<$ and no blue $K_{n+1}^<$. Therefore the graph $G^<$ is not $(n+1)$-good, a contradiction. □

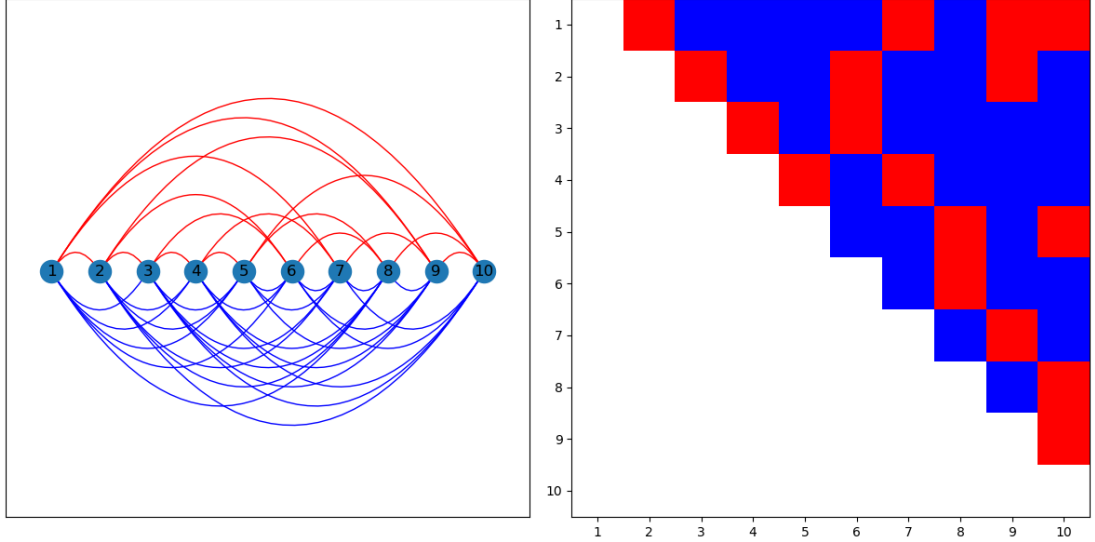Note that the contrary does not hold and an n-good ordered graph is not necessarily $(n+1)$-good; see Figure 3.4.

Figure 3.4: Visualization of a coloring on 10 vertices disproving that the ordered graph $G^<$ on 4 vertices with three edges $\{1,3\}$, $\{2,3\}$ and $\{2,4\}$ is 4-good. However, $G^<$ is 3-good, as was proven by exhaustive computer search.

Definition 3.14. *An **ordered star graph** $S_{l,r}^<$ is an ordered graph on $r + l - 1$ vertices such that the lth vertex in the vertex ordering is adjacent to all other vertices and there are no other edges; see Figure 3.5.*
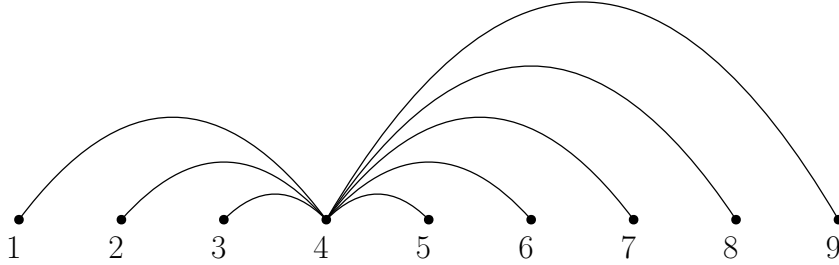

Figure 3.5: The ordered star graph $S_{4,6}$.

Lemma 3.15. *Let $G^<$ be a connected ordered graph such that $r_<(G^<, K_n^<) = N$ and let χ be a coloring on $M \geq N$ vertices. Then either a blue $K_n^<$ exists in χ or we can choose $M - N + 1$ vertices such that each one of them is the rightmost vertex of some red copy of $G^<$ contained in χ as an ordered subgraph.*

Proof. By the definition of M, any coloring on M vertices contains either a red copy of $G^<$ or a blue copy of $K_n^<$ as an ordered subgraph. In the latter case we are finished. In the former case, we can find a red copy of $G^<$ in χ and delete its rightmost vertex from the coloring, obtaining a new coloring on $M - 1$ vertices. We can iterate this approach as we are sure to find another red copy of $G^<$ as long as the number of vertices is at least N. Thus we removed at least $M - N + 1$ vertices and these are the distinct vertices stated by the lemma. $\square$

Definition 3.16. *For any two ordered graphs $G^<$ and $H^<$, we define $F^< = G^< + H^<$ as an ordered graph on $|G^<| + |H^<| - 1$ vertices constructed by joining the two graphs by setting the leftmost vertex of $H^<$ as the rightmost vertex of $G^<$.*

The operation "+" is illustrated in Figure 3.6. Obviously, it is associative and it preserves the connectivity of the two ordered graphs.
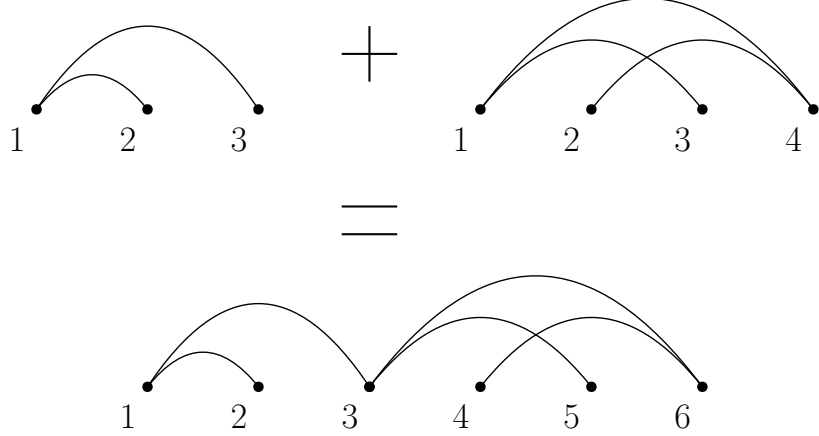

Figure 3.6: An example of joining two ordered graphs.

Theorem 3.17. *If a connected ordered graph $G^<$ is n-good, then, for all $r, l \in \mathbb{N}$, the ordered graphs $G^< + S_{1,r}^<$, $G^< + S_{l,1}^<$, $S_{l,1}^< + G^<$ and $S_{1,r}^< + G^<$ are also n-good.*

Proof. We will prove only the first two cases, since the latter two are symmetric. Let us assume we have a connected n-good ordered graph $G^<$ with $k+1$ vertices. Let us denote $F_1^< = G^< + S_{1,m-k}^<$ and $F_2^< = G^< + S_{m-k,1}^<$ for some $m \in \mathbb{N}, m > k$. Note that $|F_1^<| = |F_2^<| = m$.

Let us consider a coloring χ on $N = (m-1)(n-1)+1$ vertices. We shall prove that χ contains either red copies of both $F_1^<$ and $F_2^<$, or a blue copy of $K_n^<$ as an ordered subgraph. Note that from our assumptions we have $r_<(G^<, K_n^<) = k(n-1)+1$ for every $n \in \mathbb{N}$ and we can use Lemma 3.15 to choose a set W of

$$[(m-1)(n-1)+1] - [k(n-1)+1] + 1 = (n-1)(m-k-1)+1$$

vertices such that each one of them is the rightmost vertex of at least one red copy of $G^<$ in χ.

1) Suppose the coloring χ contains no red copy of $F_1^<$. We then set $W_0 = W$ and define W_i for $i = 1, \ldots, n$ as the set of all right blue neighbors of the leftmost vertex w_i from W_{i-1}. See Figure 3.7 for an illustration of the process. We know that each w_i has at most $(m-k-2)$ red neighbors to the right, otherwise we would obtain a copy of $F_1^<$ in χ and we would be done. Therefore, in each transition from W_{i-1} to W_i, at most $(m-k-1)$ vertices are removed. Since W_0 consisted of $(n-1)(m-k-1)+1$ vertices, the set W_{n-1} has at least one vertex which can be chosen as w_n, because

$$(n-1)(m-k-1)+1 - (n-1)(m-k-1) = 1.$$

We know that vertices $w_1, \ldots, w_n$ have no red edges among them, therefore they form a blue $K_n^<$.

2) Suppose the coloring χ contains no red copy of $F_2^<$. We continue similarly, we set $W_0 = W$ and define W_i for $i = 1, \ldots, n$ as the set of all *left* blue neighbors of the *rightmost* vertex w_i from W_{i-1}. We know that each w_i has at most $(m-k-2)$
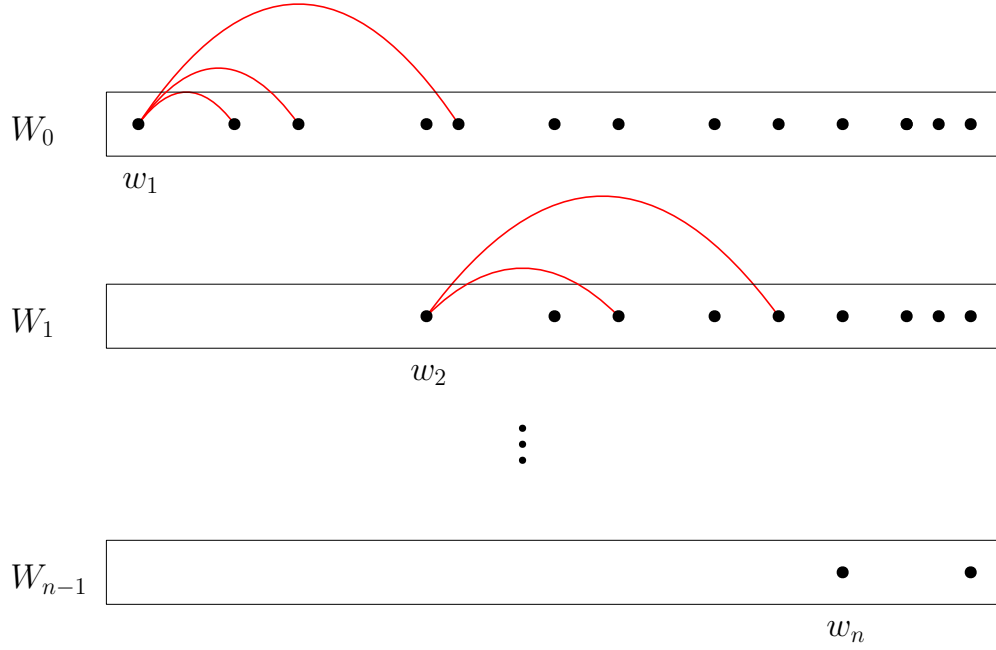
Figure 3.7: Visualization of our proof of Theorem 3.17. By the construction, vertices $w_1, \ldots, w_n$ form a blue clique $K_n^<$.

red neighbors to the left, otherwise we would obtain a copy of $F_2^<$ in χ and we would be done. Following the same calculation, W_{n-1} has at least one vertex which can be chosen as w_n and thus vertices $w_1, \ldots, w_n$ form a blue $K_n^<$ and we are finished. $\qquad\square$

Theorem 3.17 immediately implies that every ordered star graph is good.

Corollary 3.18. *All ordered star graphs are good.*

Theorem 3.17 gives more than the previous corollary though. It gives rise to an interesting set of ordered graphs created by repeating the operation of appending left or right stars. We will define these ordered graphs.

Definition 3.19. *We call an ordered graph $G^<$ a* **monotone caterpillar graph** *if there exist $l_1, \ldots, l_n, r_1, \ldots, r_n \in \mathbb{N}$ for some $n \in \mathbb{N}$ such that $l_i = 1$ or $r_i = 1$ for each $1 \leq i \leq n$ and*

$$G^< = S_{l_1, r_1} + \cdots + S_{l_n, r_n}$$

holds. In other words, if $G^<$ can be obtained by performing joins on one-sided ordered star graphs; see Figure 3.8.

It follows from Theorem 3.17 that every monotone caterpillar graph is good.

Corollary 3.20. *All monotone caterpillar graphs are good.*

We conjecture that there are no other good connected ordered graphs.

Conjecture 3.21. *A connected ordered graph $G^<$ is good if and only if it is a monotone caterpillar graph.*
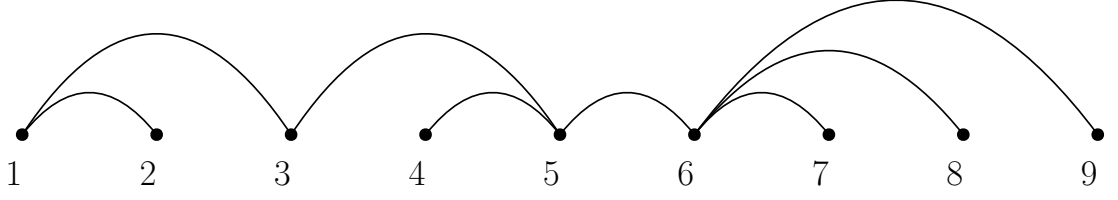
Figure 3.8: An example of a monotone caterpillar graph $S_{1,3} + S_{3,1} + S_{1,2} + S_{1,4}$.

For ordered graphs up to 6 vertices, Conjecture 3.21 is verified by the exhaustive computer search performed by our SAT solver based utility, which is described in Chapter 4.

In an attempt to get a better hold of our conjecture, we will prove an alternative characterization for monotone caterpillar graphs using some forbidden ordered subgraphs.

Proposition 3.22. *A connected ordered graph $G^<$ is a monotone caterpillar graph if and only if it does not contain any of the four ordered graphs from Figure 3.9 as ordered subgraphs.*
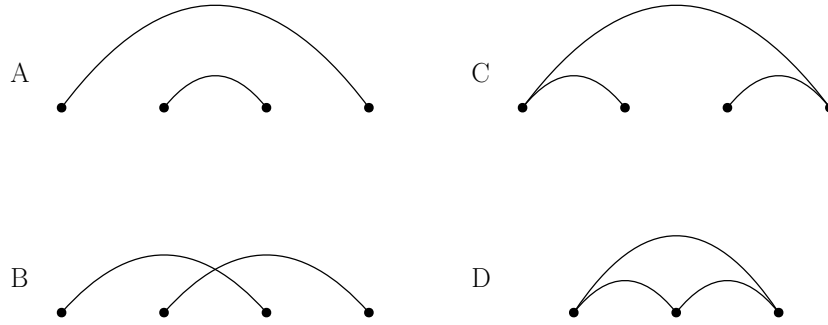

Figure 3.9: Any ordered graph that does not contain any of these four ordered graphs as ordered subgraphs is a monotone caterpillar graph.

Proof. One implication is trivial, because by the definition of a monotone caterpillar graph, it cannot contain any of the ordered graphs from Figure 3.9 as an ordered subgraph.

It remains to prove that if an ordered graph $G^<$ does not contain any of these ordered graphs as an ordered subgraph, then it is a monotone caterpillar graph. We will prove this by induction. For graphs with less than 4 vertices it is trivial and for graphs with 4 vertices it can be easily checked by hand. Assume it is true for all connected ordered graphs with at most n vertices and let us have a connected ordered graph $G^<$ with vertices $1, \ldots, n+1$ such that it avoids all ordered graphs from Figure 3.9 as ordered subgraphs.

Since $G^<$ is connected, the last vertex $n+1$ certainly has at least one neighbor, we can thus denote by v its leftmost neighbor. The ordered subgraph $H^<$ induced by the vertices $1, \ldots, v$ of $G^<$ is a monotone caterpillar graph by induction. If $v = n$, we are done, as $G^< = H^< + S_{1,2}$. If $v < n$, we let $I = \{v+1, \ldots, n\}$. Note that I is non-empty.

There can be no edges between vertices in I as this would imply the existence of A as an ordered subgraph. There also cannot be any edges from I to the left of v as this would imply B as an ordered subgraph. As the graph is connected, all vertices from I thus have to be adjacent either to v or $n+1$, but not both, as this would imply the existence of D as an ordered subgraph of $G^<$. If $v_1 \in I$ was adjacent to v and some other $v_2 \in I$ to $n+1$, it would imply either the existence of B (for $v_1 > v_2$), or the existence of C (for $v_1 < v_2$). Thus, vertices from I are adjacent either all to $n+1$, or all to v. Therefore $G^<$ is a monotone caterpillar graph, as either $G^< = H^< + S_{1,n-v+2}$ in the former case, or $G^< = H^< + S_{n-v+2,1}$ in the latter case. $\qquad\square$

Remark. If we assume $G^<$ to be an ordered tree, we can leave out D in Figure 3.9 and the characterization still holds.

We can extend this characterization of monotone caterpillar graphs with these "forbidden" ordered subgraphs. The following corollary follows immediately from Proposition 3.22.

Corollary 3.23. *An ordered tree $G^<$ is a monotone caterpillar graph if and only if it does not contain any ordered graph from Figure 3.10 as an ordered subgraph.*
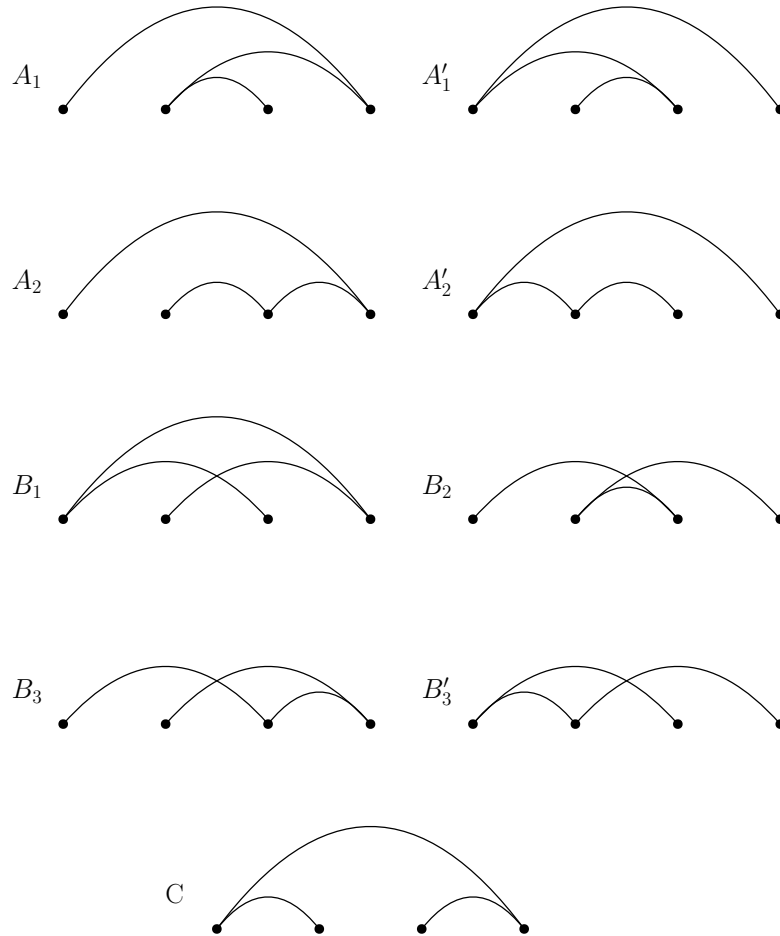

Figure 3.10: Forbidden subgraphs of size 4 for a monotone caterpillar graph. Also all ordered trees on four vertices that are not monotone caterpillar graphs.

The difference between Corollary 3.23 and Proposition 3.22 is that all the new ordered graphs in Figure 3.10 are obtained by extending the two disconnected ordered graphs A, B from Figure 3.9 into connected ordered graphs.

Also, none of the ordered graphs from Figure 3.10 is good. Goodness of A_1 and its reflection A_1' is disproved by the coloring in Figure 3.3. The ordered graph B_2 is shown not to be 4-good in Figure 3.4, although it is 3-good. By exhaustive search we found out that none of the other ordered graphs is 3-good.

Encouraged by our characterization of monotone caterpillar graphs from Corollary 3.23 and supported by our computed results for graphs with up to 6 vertices, we ask this question.

Problem 3.24. *Does it hold that a connected ordered graph is n-good if and only if each of its connected ordered subgraphs is n-good?*

A positive answer to Problem 3.24 would together with Corollary 3.23 verify Conjecture 3.21 and also give the complete characterization of all Ramsey good connected ordered graphs. However, this seems unlikely considering the known results about Ramsey goodness, as a graph usually becomes n-good for some $n \in \mathbb{N}$ only when the graph is sufficiently large with respect to n.

4. Computing ordered Ramsey numbers

In this chapter, we showcase how to computationally deal with searching for avoiding colorings and thus calculating ordered Ramsey numbers for given ordered graphs.

Problem 4.1. *We are given ordered graphs $G_1^<$ and $G_2^<$ and a number $N \in \mathbb{N}$.*
  Does there exist a coloring on N vertices that contains neither a red $G_1^<$ nor a blue $G_2^<$ as an ordered subgraph?

We will solve this problem constructively by an algorithm so that we can find the avoiding coloring if we prove there is one.

4.1 Reduction to SAT

We will reduce Problem 4.1 to a problem of finding a satisfiable assignment of a formula in the conjunctive normal form, the so-called *SAT problem*. The reason it helps is that although SAT problems are generally NP-hard, there already exist many heavily optimized SAT solvers, which can often handle the problem quickly for sufficiently small numbers $|G_1^<|, |G_2^<|, N$.

Let us have a coloring of an ordered graph $K_N^<$ with vertices $V = \{1, \ldots, N\}$ and let $r := |G_1^<|$ and $b := |G_2^<|$. For each edge $\{i, j\}$ of $K_N^<$ we create a boolean variable $x_{i,j}$ set to *true* if the edge $\{i, j\}$ is red and *false* if it is blue. We will devise a SAT formula φ_r whose satisfiability is equivalent to the non-existence of a red $G_1^<$ as an ordered subgraph in our coloring of $K_N^<$.

For the case $r > N$, any coloring of $K_N^<$ certainly does not contain $G_1^<$ as an ordered subgraph, we can therefore set $\varphi_r = true$. Let us assume $r \leq N$ and let PS_r be the set of all subsets of V of size r. If we fix any subset $S = \{v_1, \ldots, v_r\} \in PS_r$, the satisfiability of the following SAT clause

$$C_S = \bigvee_{\{i,j\} \in E(G_1^<)} \neg x_{v_i, v_j}$$

is equivalent to the non-existence of $G_1^<$ as a red ordered subgraph on these k vertices as the edge locations are uniquely determined.

Now we can easily express φ_r as a conjunction of these clauses:

$$\varphi_r = \bigwedge_{S \in PS_r} C_S = \bigwedge_{\substack{v_1, \ldots, v_r \in [N] \\ v_1 < \cdots < v_r}} \bigvee_{\{i,j\} \in E(G_1^<)} \neg x_{v_i, v_j}.$$

Note that the resulting expression is in the conjunctive normal form.

We can analogically obtain a SAT formula S_b whose satisfiability is equivalent to the non-existence of a blue $G_2^<$ as a blue ordered subgraph of our colored $K_N^<$:

$$\varphi_b = \bigwedge_{\substack{v_1, \ldots, v_b \in [N] \\ v_1 < \cdots < v_b}} \bigvee_{\{i,j\} \in E(G_2^<)} x_{v_i, v_j}.$$

By performing the conjunction of these formulas we get our desired formula $\varphi = \varphi_r \wedge \varphi_b$, whose satisfiability is equivalent to the non-existence of both red $G_1^<$ and blue $G_2^<$ as ordered subgraphs of $K_N^<$. Note that the resulting SAT formula φ is in the conjunctive normal form.

If φ is not satisfied for any choice of the variables $x_{i,j}$, then every coloring of $K_N^<$ contains either a red copy of $G_1^<$ or a blue copy of $G_2^<$ as an ordered subgraph. On the other hand, if we do find values $x_{i,j}$ which satisfy φ, we can treat them as a coloring that can be visualized.

4.2 Our SAT-based utility

We combined the SAT reduction with a graphical interface in order to create a working utility, which greatly helped with obtaining many previous results; see Figure 4.1. We include a brief description as we believe this tool, which is based on the SAT solver Minisat [22], can be used by other researchers.
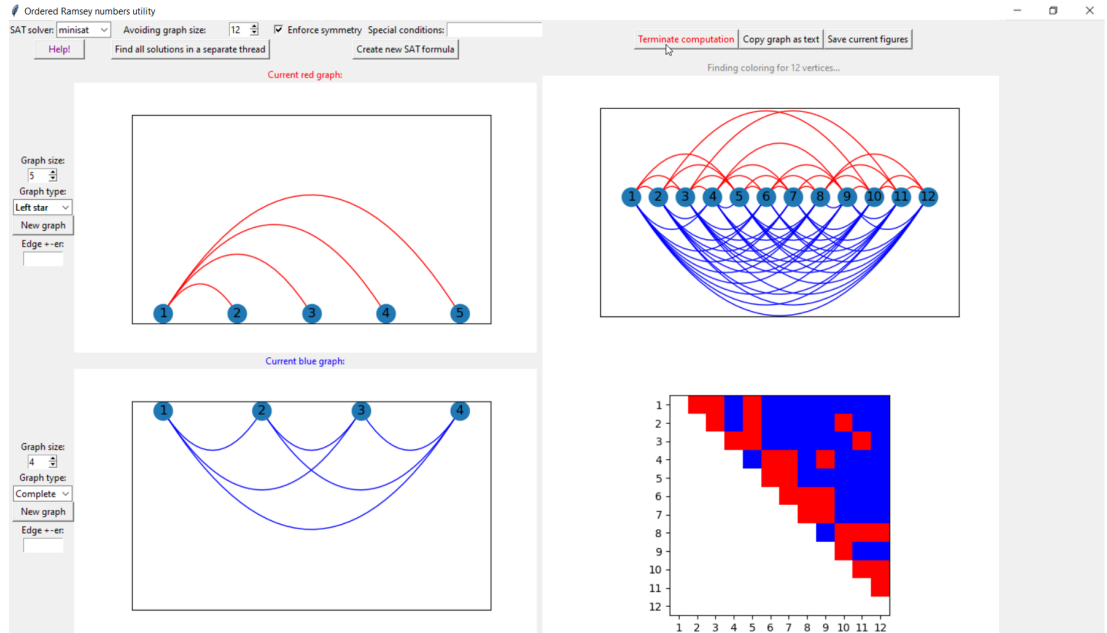

Figure 4.1: Our utility used to compute ordered Ramsey numbers and to search for avoiding colorings. One can set the desired size of an avoiding coloring and customize red and blue graphs this coloring is supposed to avoid.

The utility provides an option to build and customize a "red" ordered graph $G_1^<$ and a "blue" ordered graph $G_2^<$. Then we set N, the size of a coloring avoiding $G_1^<$ as a red subgraph and $G_2^<$ as a blue subgraph. We can also force some edges of the coloring to be red or blue, or we can force the whole coloring to be symmetric, both of which is implemented by a simple extension of the SAT formula.

After setting all these parameters, we can set the SAT formula and either exhaustively search for all possible colorings while saving them to a folder, or interactively click through the solutions in the GUI while always searching for the next one. The outputs are graph colorings and coloring matrices if a satisfying

coloring exists, or a statement that the SAT formula is not satisfiable and thus no coloring of the chosen size exists.

This utility can reasonably well handle colorings with up to 17 vertices, however, this depends on the computational power and on the ordered graphs $G_1^<$ and $G_2^<$. To be precise, the SAT formula φ has $\binom{N}{|G_1^<|}$ clauses with $|G_1^<|$ literals each and $\binom{N}{|G_2^<|}$ clauses with $|G_2^<|$ literals each, thus it can quickly get very large. It can get even larger when we search for multiple colorings, as after finding a satisfying assignment for φ, we simply append a new clause forbidding this solution, so that we find other solutions when we feed the modified SAT formula to the SAT solver. This new appended clause has $\binom{N}{2}$ literals.

The code of our utility is available on GitHub; see [23]. It should be noted that the utility will not work without a supported SAT solver installed and included in the system path.

4.3 Applications

Here we sum up how we used the utility to obtain some supporting results mentioned in the same order as stated in this thesis. Note that we have also used a slight modification of our utility where we ran experiments on some specific families of ordered graphs like ordered star graphs, ordered trees, ordered paths etc.

In Chapter 2 we used the utility to give us an idea about the structure of colorings avoiding $NM_n^<$ as an ordered subgraph, which was then introduced in Proposition 2.8. We also used it to obtain counterexamples for Conjecture 2.6 stated in the proof of Theorem 2.9. The proof of Proposition 2.12 stating that $NM_2^<$ is good has also been preceded by showing that $NM_2^<$ is n-good for n up to 8.

In Chapter 3 we used the utility to exhaustively search for n-good graphs for small values of n and small graphs with size up to 8. This gave us the idea about monotone caterpillar graphs and encouraged us to state Conjecture 3.21 and Problem 3.24 at the end of the chapter.

The utility has more applications apart from the topics discussed in this thesis. We show one such example.

Definition 4.2. An **alternating path** $Alt_n^<$ is the ordered path on $n \in \mathbb{N}$ vertices with edges $\{i, j\}$ such that $i + j \in \{n + 1, n + 2\}$; see Figure 4.2.
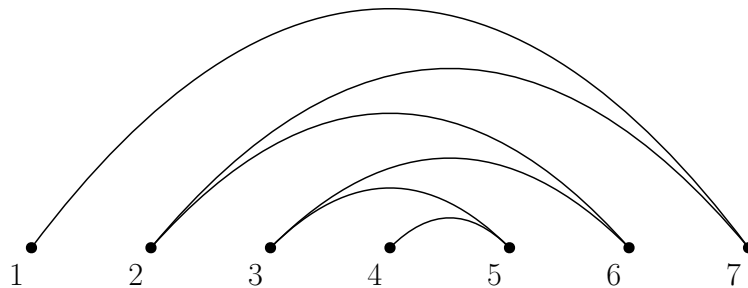

Figure 4.2: The alternating path $Alt_7^<$.

Balko, Cibulka, Král and Kynčl [5] proved that the diagonal ordered Ramsey number $r_<(Alt_n^<)$ is linear in n. They also asked whether alternating paths have the smallest ordered Ramsey number $r_<(G_n^<)$ over all ordered paths $G_n^<$ for a given $n \in \mathbb{N}$. Exhaustive search over all ordered paths on at most 7 vertices offers a positive answer to this question for $n \leq 7$.

5. Conclusion

We left a couple of open problems. In Chapter 2, we investigated nested matchings and improved or generalized some already known bounds. Using definitions from Chapter 3, we proved that $NM_2^<$ is good, whereas $NM_4^<$ and $NM_5^<$ are not even 3-good. For larger nested matchings, it seems likely that they are also not 3-good, though we lack a general construction. For $NM_3^<$ we have computationally proved that it is n-good for n up to 5.

Problem 5.1. *Is the ordered graph $NM_3^<$ good?*

These results seem to be related to proper colorings of ordered graphs that do not contain nested matchings as ordered subgraphs. For example, similarly as in the proof of Proposition 2.12, the ordered graph $NM_3^<$ would be good if every ordered graph avoiding $NM_3^<$ as an ordered subgraph was 5-colorable.

In Chapter 3 we investigated Ramsey goodness of ordered graphs. While we proved Ramsey goodness for a wide class of graphs, we still lack complete characterization of all ordered Ramsey good graphs.

Problem 5.2. *Are there any other connected good ordered graphs except for monotone caterpillar graphs?*

We also restate the briefly discussed question of Balko, Cibulka, Král and Kynčl [5], which we were able to prove computationally for $n \leq 7$.

Problem 5.3. *Does $r_<(Alt_n^<) \leq r_<(G^<)$ hold for any ordered path $G^<$ on $n \in \mathbb{N}$ vertices?*

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