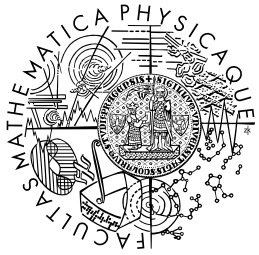

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MASTER THESIS

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Fixed point theorems in the theory of differential equations

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Abstract:

This thesis is devoted to show various applications of fixed point theorems on differential equations. In the beginning we use a notion of topological degree to derive several fixed points theorems, primarily Brouwer, Schauder and Kakutani-Ky Fan theorem. Then we apply them on a wide range of relatively simple problems from ordinary and partial differential equations (ode and pde). Finally, we take a look on a few more complex problems. First is an existence of a solution to the model of mechanical oscillator with non-monotone dependence of both displacement and velocity. Second is a solution to so called Gause predator-prey model with a refuge. The last one is certain partial differential equation with a constraint which determines maximal monotone graph.

Keywords: fixed point, differential equation, compactness, topological degree

Contents

Introduction

> The essence of mathematics lies in its freedom.
>
> *Georg Cantor*

After several years of learning mathematics we know that there is a wide spectrum of applications for some verison of fixed point theorem. It means existence of point x from space X which is mapped to itself under mapping $f : X \to X$. We saw it in the well-known Peano existence theorem or as a part of method of Galerkin approximation in pde's. Someone could see them also in the proof of existence of the Nash equlibrium in game theory or in popular hairy ball theorem, which is colloquially formulated as "you can't comb a hairy ball flat without creating a cowlick".

Our major motivation is folowing. The equations governing the vibratory motion of a spring, dashpot and mass, represented as a lumped parameter system take the form

$$mx''(t) = F(t) - F_s - F_d,$$

where x is the displacement, m the mass, F the externally applied force on the mass and F_s and F_d denote the forces in the spring and dashpot, respectively. If we have $F_s = \hat{f}(x)$ and $F_d = \hat{g}(x')$ then the system is trivially solvable if we endow the system with initial conditions $x(0) = x_0, x'(0) = x_1$.

Interesting situation is when we cannot express functions F_s, F_d in terms of functions x, x', i. e. we may assume that our constitutive relations are given by $x = f(F_s)$ and $x' = g(F_d)$. If f and g are not invertible then we cannot simply substituti those relations into the equation. In paper [27] is a proof for this situation if both f and g are non-decreasing functions.

It is reasonable to expect that similar result should holds also if f and g are "deformed" non-decreasing fucntions, i. e. they are non-decreasing for big values of their argument but for small ones they might have wild behaviour. To achieve this kind of result it could be effective to apply fixed point theorem for some compact mapping.

This approach could be possibly applicable to more complex pde's problem, for example some flow of non-newtonian fluids characterised by a non-monotone relation.

Main purpose of this thesis is try to use suitable fixed point theorem to solve problem sketched above or similar ones, i. e. show an existence of a solution to certain differential equation with some kind of perturbation. At first we will develop the theory of topological degree to derive classical fixed point theorems as Brouwer or Schauder. In progress we will see that the degree itself is very powerful tool and has much more deep consequences. We do not intend to spend too much time on this theoretical level.

Then we show some simple, but certainly nontrival, applications in ode's, pde's and in functional equations (integro-differential ones). For example elliptic operator with bounded countinuous perturabtion depending on the solution.

Probably the most interesting abstract result will be Kakutani-Ky Fan theorem (Theorem 22). It will show an existence of fixed point not to a function but to a multifunction, i. e. value of f is not a single point but some set and by this theorem we find x such that $x \in f(x)$. We will apply this tool on two problems. One of them is generalized predator-prey model, so called Gause model with a refuge. Second one is an interesting alternative to a standard Galerkin method, we solve some partial differential equation with a constraint in the form

$$\frac{(|g - f(u)| - 1)^+}{|g - f(u)|}(g - f(u)) = u.$$

1. Preliminaries

> It's fine to work on any problem, so long as it generates interesting mathematics along the way - even if you don't solve it at the end of the day.
>
> *Andrew Wiles*

In this chapter we will briefly summarize basic knowledge and introduce simultaneously our notation which we will use throughout the whole text.

1.1 Symbolism in use

The following notation is for the most part not explained elsewhere and as such it is usually taken for granted. Let $\Omega \subset \mathbb{R}^d$ be an open subset for $d \in \mathbb{N}$.

$A \subset B$	a set A is a subset of a set B
$\mathbb{N}, \mathbb{N}_0$	the set of positive and non-negative integers respectively
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
f_{-1}	the preimage of a mapping f
$\Re, \Im$	the real and imaginary part of a complex number
M^c or $X \setminus M$	the complement of a subset M in X
$\overline{M}^X$ or $\overline{M}$	the closure of a subset M in a topological space X
$\overline{M}^w$	the weak closure of a subset M in a topological space X
∂M	the boundary of a subset M in a topological space
$\mathrm{conv}(M)$ or $\mathrm{conv}M$	the (open) convex hull of M
2^X	the power set of a set X
$B(x,r)$ or $B_r(x)$	the open ball in a given metric space X with center at x and radius $r > 0$
$B(0,r)$ or B_r	as above for $x = 0$
$\mathcal{C}^k(\Omega)$	the space of all $k-$ times differentiable functions on $\Omega, k \in \mathbb{N}_0$
$\mathcal{C}_0^k(\Omega)$	the space of functions from $\mathcal{C}^k(\Omega)$ with a compact support
$\mathcal{C}^k(\Omega, X)$	the space of all $k-$ times differentiable functions on Ω with values in the space X, where $k \in \mathbb{N}_0$
$\mathcal{C}(\Omega, X)$, $\mathcal{C}(\Omega)$ and $\mathcal{C}_0(\Omega)$	spaces as above in the case $k = 0$
$L^p(I, X)$	the standard Bochner space for Banach space X, interval I and $p \in [1, \infty]$
$L^p(\Omega)$	the standard Lebesgue space, where $p \in [1, \infty]$ (or the space of $p-$ integrable functions)
$L^p_{loc}(\Omega)$	the space of locally $p-$ integrable functions $p \in [1, \infty]$

$\|\|\cdot\|\|_{L^p(\Omega)}$ or $\|\|\cdot\|\|_p$	the standard norm on $L^p(\Omega)$
$\ell^p(\mathbb{R})$	the sequence space of real numbers with $p \in [1, \infty)$
$W^{k,p}(\Omega)$	the standard Sobolev space, where $p \in [1, \infty]$ and $k \in \mathbb{N}$
$W_0^{k,p}(\Omega)$	the standard Sobolev space with zero boundary values, where $p \in [1, \infty]$ and $k \in \mathbb{N}$
$\|\|\cdot\|\|_{W^{k,p}(\Omega)}$ or $\|\|\cdot\|\|_{k,p}$	the standard norm on $W^{k,p}(\Omega)$
$AC(I)$	the space of absolutely continuous functions on an interval I
$\mathcal{C}_y^\alpha(U, X)$	see the beginning of Chapter 2
$\mathcal{K}(X, Y)$	the set of all both continuous and compact operators from normed space X into normed space Y (see section 1.2)
$\deg(f, U, y)$	the topological degree, see Definition 2 and 4 in Chapter 2
$\hookrightarrow$	continuous embedding
$\hookrightarrow\hookrightarrow$	compact embedding
$\rightharpoonup$	weak convergence
$\rightrightarrows$	uniform convergence
$J_f(x)$	the Jacobian of a mapping f at a point x
$\Omega \in C^{0,1}$	the Lipschitz domain, see Definition 1
id	the identity mapping
X^*	the continuous dual space of X
$S_f, RV(f)$	singular and regular values of f (see remark after Claim 1)

Let us make some remarks about notation. If we do not say otherwise we work over the field $\mathbb{R}^d$, where the dimension d is positive integer. Most of the time we will work on some Banach space $(X, \|\|\cdot\|\|)$ which we usually denote just by X. Similarily we abbreviate metric, normed and Hilbert spaces.

If we will have some sequence of functions, which we denote as $(u_n)_{n\in\mathbb{N}}$ or $(u_n)_{n=1}^\infty$ and it converges e. g. in $L^2(\Omega)$ we write just $u_n \to u$ in $L^2(\Omega)$, i. e. we usually omit phrase "as $n \to \infty$". We will specify it only in the case of more indices.

If it is clear on which set we work we may omit to write it down, e. g. instead of $f \in \mathcal{C}(\Omega)$ we may write just $f \in \mathcal{C}$. In the case of (1-dimensional) interval we may write e. g. $AC[0, T]$ or $L^1(0, T)$ instead of precise $AC([0, T])$ and $L^1((0, 1))$.

Let us recall a couple of well-known facts. Lebesgue spaces are complete for any p, moreover for $p \neq \infty$ they are separable and for $p \in (1, \infty)$ they are reflexive. Same properties (for same exponents p) have also Sobolev spaces ($W_0^{k,p}(\Omega)$ too).

Notion of equicontinuity will be important to apply famous Arzelà–Ascoli theorem. We say that family F of functions mapping one metric space (X, d_X) into another metric space (Y, d_Y) is equicontinuous at a point $x_0 \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ for all $f \in F$ and all $x \in X$ satysfying $d_X(x, x_0) < \delta$. It is equicontinuous on the set M if it is equicontinuous at each point of M.

We remind the reader that a connected space is a topological space that cannot be represented as the union of two or more disjoint non-empty open subsets. And, as usual, a subset of a topological space X is a connected set if it is a connected space when viewed as a subspace of X. Finally a maximal connected subsets (ordered by inclusion) of a non-empty topological space are called the connected components.

We also assume that Hölder, Young and Cauchy–Bunyakovsky–Schwarz inequalities are known, but we will try to underline moment when used.

1.2 Compactness

In the most of our work we will be dealing with compact sets or operators. So let us take a look at those notions more closely. By the definition X is compact space, if each of its open covers has a finite subcover. Subset is compact if it is compact as a subspace (in the topology of subspace). The space X is locally compact if any point has some compact neighbourhood. Relatively compact (or precompact) subspace Y of X is a subset whose closure is compact.

It is purely topological point of view. If we have some better strucutre we are able to describe it in terms of finite nets. Metric space X is be totally bounded, if for any $\varepsilon > 0$ there exists finite colection of open balls with center in X and radius ε whose union contains the set X. A subset of complete metric space is totally bounded if and only if it is relatively compact.

If we have linear operator $T : X \to Y$, between two normed spaces, we say that it is compact if the image of any bounded set from X is relatively compact in Y. Alternatively that sequence of images of bounded sequence in X has converging subsequence. If Y is the Banach space we can write totally bounded instead of relatively compact.

Remark. Note that any compact (linear) opeator is automatically continuous. In other words also bounded, because of linearity.

If we have an operator $T : X \to Y$, not necesarilly linear, we say that it is compact, if it is continuous and the image of any bounded set from X is relatively compact in Y. Continuity means that image of convergeing sequence converge to the image of the limit. The set of all continuous and compact operators (i. e. both linear and non-linear) from X to Y, where X, Y are normed space, we denote by $\mathcal{K}(X, Y)$.

Example. There exists an operator which is compact, but not continuous. Consider the real function $f(x) = \begin{cases} \arctan x, & x < 0 \\ c + \arctan x, & x \geq 0 \end{cases}$ for some non-zero constant c.

There are probably two most classical examples of compact operators.

Example (Hilbert-Schmidt integral operators)**.** Let us denote by $k(x, y) \in L^2((0, 1)^2)$ the so called Hilbert-Schmidt kernel and introduce an operator $T : L^2((0, 1)^2) \to L^2((0, 1)^2)$, which is compact, by the formula

$$Tf(x) = \int_0^1 k(x, y) f(y) \mathrm{d}y.$$

Example (Compact operator as an approximation by finite dimensional operators)**.** Let $p \in [1, \infty)$ and denote $X = \ell^1 = \ell^1(\mathbb{R}), Y = L^p(0, 1)$. Then the operator defined by

$$T((x_n)_n)(t) = \sum_{n=1}^{\infty} x_n t^n$$

is compact as an operator $T : X \to Y$. It holds because of approximation $P_k((x_n)_n) = (x_1, x_2, \ldots, x_k, 0, \ldots)$ and the fact that $||T - T(P_k)||_p \leq (1 + p(n + 1))^{-\frac{1}{p}}$ and so our operator is limit of operators which have finite rank and so it is compact.

Remark. Above we used the fact that the operator of finite rank is automatically compact: we have that image of T has finite dimension and T is bounded, we use then Bolzano-Weierstrass theorem to find some converging subsequence. Second used fact was that the limit of totally bounded operators is also totally bounded.

Remark. Let us note that compact operator in general Banach space does not need to be a limit of some sequence of operators with finite rank. See e. g. [11].

Example (Compactness heavily depends on the spaces)**.** Let us consider two spaces $X = \mathcal{C}[0,1], Y = \mathcal{C}^1[0,1]$ and an operator $Tf(x) = \int_0^x f(s)\mathrm{d}s$. If $T : X \to X$ then it is compact due to the Arzelà–Ascoli theorem. On the other hand if $T : X \to Y$ then it can be shown to be a non-compact operator.

1.3 Fundamental results

In this passage we state several nontrivial result which will be needed later on.

1.3.1 Measure theory

Claim 1 (Sard theorem)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $\mathcal{C}^k$ mapping, where $k \geq \max\{n - m + 1, 1\}$. Denote as S_f the set of points at which the Jacobian matrix has rank less than m. Then the $f(S_f)$ has Lebesgue measure in $\mathbb{R}^m$ equal to 0.

Remark. Symbol S_f stand for singular values of f. Complement of S_f we denote as $RV(f)$, i. e. regular values of function f.

Proof. See Theorem 3.10 and following remark in [13]. ■

Claim 2 (Lusin theorem)**.** Let $f : \mathbb{R}^d \to \mathbb{R}^m$ be a measurable function and $M \subset \mathbb{R}^d$ be bounded set. Fix $\varepsilon > 0$. Then there exists a compact subset K of M such that $|M \setminus K| < \varepsilon$ and f is continuous on $M \setminus K$.

Proof. See Theorem 1.14 in [13]. ■

1.3.2 Sobolev spaces

Claim 3 (Testing by min/max)**.** Assume that $p \in [1, \infty)$ and Ω is open, bounded set in $\mathbb{R}^d$. Suppose that $u \in W^{1,p}(\Omega)$ then $\min\{0, u\} \in W^{1,p}(\Omega)$.

Proof. Theorem 4.4 in [13] says that $u^+ = \max\{0, u\}, u^- = \max\{0, -u\} \in W^{1,p}(\Omega)$ and we see that $\max\{0, -u\} = -\min\{0, u\}$. ■

Definition 1 (Lipschitz boundary)**.** We say that an open, bounded $\Omega \subset \mathbb{R}^d$ has Lipschitz boundary ($\Omega \in C^{0,1}$) if for each point $x \in \partial\Omega$, there exist $r > 0$ and a Lipschitz continuous mapping $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, upon our rotating and relabeling the coordinate axes if necessary, we have

$$\Omega \cap Q(x, r) = \{y \in \mathbb{R}^d; \gamma(y_1, \ldots, y_{d-1}) < y_d\} \cap Q(x, r),$$

where $Q(x, r) = \{y \in \mathbb{R}^d; |y_k - x_k| < r, k = 1, \ldots, d\}$.

Claim 4 (Lax-Milgram theorem)**.** Let H be a Hilbert space and $B : H \times H \to \mathbb{R}$ is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

(i) $|B(u, v)| \leq \alpha||u||_H||v||_H, \forall u, v \in H$ and

(ii) $\beta||u||_H^2 \leq B(u, u), \forall u \in H$.

Finally, let $f : H \to \mathbb{R}$ be a bounded and linear functional on H. Then there exists unique $u \in H$ such that $B(u, v) = \langle f, v\rangle_H, \forall v \in H$, where $\langle f, v\rangle_H$ is duality pairing between f and v.

Proof. See section 6.2.1 in [12]. ■

Claim 5 (Poincaré inequality)**.** Assume that $p \in [1, \infty)$ and Ω is a bounded, open set in $\mathbb{R}^d$. Then there exists a constant $c_p > 0$ depending only on p and Ω such that

$$||u||_p \leq c_p ||\nabla u||_p, \forall u \in W_0^{1,p}(\Omega).$$

Proof. See corollary 9.19 in [4]. ■

Claim 6 (Nemytskii operator)**.** Let $\Omega \subset \mathbb{R}^d$ be measurable, we say that mapping $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfy Carathéodory conditions if

(i) for a. a. $x \in \Omega$ is $f(x, \cdot)$ continuous and

(ii) for all $y \in \mathbb{R}^m$ is $f(\cdot, y)$ measurable.

Then $f(x, u(x))$ is measurable provided that $u : \Omega \to \mathbb{R}^m$ is measurable.

Moreover let $p \in [1, \infty)$ and suppose that $g \in L^p(\Omega)$ and exists a constant $C \geq 0$ such that

$$|f(x, y)| \leq g(x) + C|y|^{\frac{p-1}{p}}, \text{ for a. a. } x \in \Omega \text{ and all } y \in \mathbb{R}^m.$$

Then the mapping $u \mapsto f(x, u(x))$ is continuous as a mapping from $L^{\frac{p-1}{p}}(\Omega)$ into $L^p(\Omega)$.

Proof. See chapter IV, page 166 in [24]. ■

Claim 7 (Compact embedding)**.** If X, Y are Banach spaces and $X \subset Y$ we say that X is compactly embedded in Y, written $X \hookrightarrow\hookrightarrow Y$, provided both

(i) $||x||_Y \leq C||x||_X, \forall x \in X$ and some fixed $C > 0$ and

(ii) each bounded sequence in X is relatively compact in Y.

Assume that $\Omega \in C^{0,1}$.

(i) If $p \in [1, d)$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in \left[1, \frac{dp}{d-p}\right)$.

(ii) If $p \in [1, \infty]$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$.

Proof. See section 5.7 in [12]. ■

1.3.3 General topology

Claim 8 (Tietze theorem)**.** A topological space X is normal if and only if whenever A is a closed set in X and $f : A \to \mathbb{R}$ is continuous, there exists continuous extension F of f on the whole space X, i. e. $F : X \to \mathbb{R}$ and $F = f$ on A.

Proof. See Theorem 15.8 in [36]. ■

Remark. Every metric space (so Banach too) is normal. See exercise 15.3 in [36].

Claim 9 (Partition of unity)**.** Let $U_1, U_2, \ldots, U_n$ are open subsets of a locally compact Hausdorff space X, $K \subset X$ compact and

$$K \subset U_1 \cup U_2 \cup \cdots \cup U_n.$$

For any $k = 1, 2, \ldots, n$ there exist continuous function $h_k : X \to \mathbb{R}$ with a compact support in U_k such that

$$h_1(x) + h_2(x) + \cdots + h_n(x) = 1, \forall x \in K.$$

Proof. See Theorem 2.13 in [30]. ■

Claim 10 (Separation axiom)**.** A topological space X is normal if and only if whenever A and B are disjoint closed subsets in X, there are disjoint open sets U_1, U_2 with $A \subset U_1$ and $B \subset U_2$.

Proof. See definition 15.1 in [36]. ■

Claim 11 (Finite intersection property)**.** Let X be a topological space. We say that a family $\mathcal{F}$ of subsets of X has the finite intersection property if the intersection of any finite subfamily of $\mathcal{F}$ is non-empty.

A topological space X is compact if and only if each family $\mathcal{F}$ of closed subsets of X with the finite intersection property has non-empty intersection.

Proof. See Theorem 17.4 in [36]. ■

Remark. In the same book as above can be found also following assertions. Every closed subset of a compact space is compact (see Theorem 17.5). Product of non-empty compact spaces is compact (see Theorem 17.8).

1.3.4 Functional analysis

Claim 12 (Hahn-Banach separation theorem)**.** Suppose A, B are non-empty, disjoint, convex sets in a topological vector space X. If A is compact, B closed and X is locally convex, then there exist $\Lambda \in X^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\Re\Lambda(a) < \gamma_1 < \gamma_2 < \Re\Lambda(b), \forall a \in A, \forall b \in B.$$

Proof. See Theorem 3.4 in [31]. ■

Claim 13 (Mazur theorem)**.** Let X be a normed linear space and let K be a convex subset of X. Then K is norm closed if and only if K is weakly closed.

Proof. See Theorem 3.7 in [4]. ■

Claim 14 (Eberlein-Šmulian theorem)**.** Let X be a real Banach space and $B := \{x \in X; ||x|| \leq c\}$ for $c > 0$. Then X is reflexive if and only if B is weakly compact.

Proof. See corollary 3.22 in [4]. ■

Claim 15 (Arzelà–Ascoli theorem)**.** Let X be a compact Hausdorff space and F be a subset of $\mathcal{C}(X)$. Then F is relatively compact in the topology induced by uniform norm if and only if it is equicontinuous and pointwise bounded.

Proof. See Appendix A - Compactness and Continuity in [31]. ■

Claim 16 (Kolmogorov-Fréchet-Riesz theorem)**.** Let $p \in [1, \infty), T > 0$ and M is subset of $L^p((0,T), \mathbb{R}^k)$, where $L^p((0,T), \mathbb{R}^k) = \underbrace{L^p(0,T) \times \cdots \times L^p(0,T)}_{\text{k-times}}$. Then the M is relatively compact in $L^p((0,T), \mathbb{R}^k)$ if

(i) $\left\{\int_{t_1}^{t_2} f(t)\mathrm{d}t; f \in M\right\}$ is bounded in $\mathbb{R}^k$ for all $0 < t_1 < t_2 < T$ and

(ii) $\lim_{a \to 0} \int_a^T ||\tau_a f - f||_{\mathbb{R}^k} = 0$ uniformly on M,

where $\tau_a f(t) = f(t-a)$ is a translation of f by a.

Proof. See Theorem 1 in the paper [33]. This is special case for $p \neq \infty$ and $B = \mathbb{R}^k$. ■

1.3.5 Complex analysis

Claim 17 (Cauchy's argument principle)**.** Suppose Γ is a closed path in a set Ω, which is open, connected and non-empty subset of $\mathbb{C}$. Suppose that $\mathrm{Ind}_\Gamma(a) = 0$ or 1 for all $a \in \Omega \setminus \langle\Gamma\rangle$. If $f \in \mathcal{C}^1(\Omega)$ (or holomorphic) and f has no zeros on $\langle\Gamma\rangle$ then

$$\mathrm{Ind}_{f(\Gamma)}(0) = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\mathrm{d}z.$$

Proof. See Theorem 10.43 in [30]. ■

Remark. As usual the symbol $\langle\Gamma\rangle$ means image of a path Γ and $\mathrm{Ind}_\Gamma(a)$ is the index (or winding number) of a with respect to Γ. We assume that Cauchy's integral theorem is well-known.

1.3.6 Ordinary differential equations

Claim 18 (Grönwall inequalities)**.** Let $I = [a, b]$ be an interval with $-\infty < a < b < \infty$. Let u be real-valued non-negative and integrable function on $[a, b]$. Suppose that there holds

$$u(t) \le C_1 + C_2\int_a^t u(s)\mathrm{d}s$$

for a. a. $t \in [a, b]$ and some constants $C_1, C_2 \ge 0$. Then, for a. a. $t \in [a, b]$,

$$u(t) \le C_1(1 + C_2 t e^{C_2 t}).$$

In particular , if

$$u(t) \le C\int_a^t u(s)\mathrm{d}s$$

for a. a. $t \in [a, b]$ and some fixed constant $C > 0$ then

$$u(t) = 0$$

a. e. in $[a, b]$.

Proof. See appendix B. 2 in [12]. ■

Claim 19 (Carathéodory existence theorem)**.** Let $I = [0, T]$ for $T > 0$ and $x_0 \in \mathbb{R}^d$ is given. Suppose that $f : I \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

(i) for any $x \in B(x_0, \Delta)$ the function $f(\cdot, x)$ is measurable in $B(0, \delta)$,

(ii) for almost every $t \in B(0, \delta)$ the function $f(t, \cdot)$ is continuous in $B(x_0, \Delta)$,

(iii) $|f(t, x)| \le m(t)$ for a. a. t and for all x such that $(t, x) \in B(0, \Delta) \times B(x_0, \delta)$,

where $\delta, \Delta > 0$ are some numbers and $m \in L^1((0, \delta))$. Then there exists $x \in AC([0, \tau])$ for some $\tau > 0$ such that $x'(t) = f(t, x(t))$ holds for a. a. $t \in (0, \tau)$ and $x(0) = x_0$.

Proof. See Theorem 5.1 in [15]. ■

As a final note: This summary is almost surely not exhaustive, but it is the list of things which autor finds to be nontrivial or good to recall.

2. Topological methods

> Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.
>
> *Joseph Fourier*

There are several ways how to get the fundamental results about fixed point theorem. Theorems like Banach contraction principle can be proven even elementarily, more sofisticated as Brouwer or Schauder theorems are different stories.

In the case of Brouwer fixed point theorem we have many kind of proofs. Nice historical and summarizing articles are [23] and [2]. One aproach is combinatorial via so called Sperner lemma, see for example [16]. Another one use famous Stoke theorem [17]. They are based on the proof of the fact that there is no continuous function mapping the unit ball into its boundary, such that it is the identity on the boundary (such functions are called retractions). Then there is a quite general result - Lefschetz fixed point theorem, which is interesting because does not contain any convexity condition. We skip details, but we state it for delight of the soul.

Claim 20 (Lefschetz fixed-point theorem)**.** Let X be a compact, trianguable space and $f : X \to X$ be a continuous mapping. Define so called Lefschetz number of f by $\Lambda_f = \sum_{k\geq 1}(-1)^k \mathrm{trace}(f_*|H_k(X,\mathbb{Q}))$. If $\Lambda_f \neq 0$ then f has a fixed point.

Proof. See for example articles [5] and [20]. ■

In our work, we choose an approach of using poweful notion of the topological degree. There are several reasons for it. The topological degree itself is quite useful in proving of some intuitively obvious results. Some of them we prove in this chapter, at this point we may mention for example popular "Hair-dressing phenomenon" or "obvious" Jordan curve theorem. Next thing is that if once we have this tool and its properties then it is very simple to prove many fixed point theorems. After all, we do not need to introduce many other things, which could be quite abstract and far away from view of mathematical analysis.

To introduce the notion of topological degree we choose more or less axiomatic definiton. By this we cut corners because of more simple proofs, on the other hand we may lose intuition behind it. To catch intuition again it is good to see example in the middle of this chapter, which shows that our "degree" extends a well-known notion of winding number in complex analysis.

More intuitive and comprehensive approach can be found in [9], where is used construction using simplices, or in [22]. Different way, via Kronecker index and k-forms (i. e. exterior derivatives), can be found in a text [10].

We define the number deg of some function, with respect to a given set at some point for first just in Euclidean spaces. Then we extend this definition for an arbitrary Banach space of finite dimension by taking of isomorphism between the space and suitaible Euclidean space. Then we cover the infinite dimensional case using of certain approximation by compact operators.

We introduce the symbol $\mathcal{C}^\alpha_y(U,X)$ defined for normed space X and $\alpha \in \mathbb{N}$ as

$$\mathcal{C}^\alpha_y(U,X) = \{f \in \mathcal{C}^\alpha(\overline{U},X); y \notin f(\partial U)\}.$$

If $X = \mathbb{R}^d$, then we write just $\mathcal{C}^\alpha_y(U)$ and if $\alpha = 0$ then we omit a superscript and sometimes we can omit also the set U if it is clear from a context.

2.1 Topological degree in Euclidean space

In this section we define the degree for functions defined in $\mathbb{R}^d$.

Definition 2. Let $f \in \mathcal{C}_y(U), y \in \mathbb{R}^d$. Real number $\deg(f, U, y)$ will be called degree if it has all following properties:

(d-1) $\deg(f, U, y) = \deg(f - y, U, 0)$.

(d-2) $\deg(\mathrm{id}, U, y) = 1$ if $y \in U$.

(d-3) If we have open, disjoint sets $U_i \subset U$ for $i \in \{1, 2\}$ and $y \notin f(\overline{U} \setminus (U_1 \cup U_2))$ then

$$\deg(f, U, y) = \deg(f, U_1, y) + \deg(f, U_2, y) \ .$$

(d-4) If we denote $H(t) = (1 - t)f + tg \in \mathcal{C}_y(U)$ for all $t \in [0, 1]$ then

$$\deg(f, U, y) = \deg(g, U, y)$$

for any $g \in \mathcal{C}_y(U)$.

Remark. The second property will be used quite often together with the fourth one, where one of functions will be the identity mapping. The function H in the definition is usually called homotopy. It means that one function is "continuously deformed" into the other.

As almost immediate consequence of the definition we have the following assertions.

Theorem 1. Let us have $f, g \in \mathcal{C}_y(U), y \in \mathbb{R}^d$. Then the number $\deg(f, U, y)$ satisfies

(i) $\deg(f, \emptyset, y) = 0$.

(ii) If U_i are disjoint, open subsets of U for $i = 1, 2, \ldots, n \in \mathbb{N}$ and $y \notin f(\overline{U} \setminus \bigcup_{i=1}^{n} U_i)$ then

$$\deg(f, U, y) = \sum_{i=1}^{n} \deg(f, U_i, y) \ .$$

(iii) If $y \notin f(U)$ then $\deg(f, U, y) = 0$.

(iv) If $\deg(f, U, y) \neq 0$, then there exists an $x \in U$ such that $f(x) = y$.

(v) If $|f(x) - g(x)| < \mathrm{dist}\ (y, f(\partial U))$ hold for all $x \in \partial U$ then

$$\deg(f, U, y) = \deg(g, U, y) \ .$$

Proof. See Theorem 2.1 in [35]. ■

This theorem imply a few nice properties.

Theorem 2. Mapping $f \mapsto \deg(f, U, y)$ is continuous for fixed $y \in \mathbb{R}^d$ and $y \mapsto \deg(f, U, y)$ is continuous for fixed $f \in \mathcal{C}_y(U)$. Moreover

(i) $\deg(\cdot, U, y)$ is constant on each component of $\mathcal{C}_y(U)$,

(ii) $\deg(f, U, \cdot)$ is constant on each component of $\mathbb{R}^d \setminus f(\partial U)$.

Proof. See Theorem 2.2 in [35]. ■

As an illustration of use of both homotopy and definition above we solve one simple problem.

Example. Let $f : \overline{B_1} \to \mathbb{R}^d$ be a continuous function which is not identically zero, where B_1 is in $\mathbb{R}^d$. If $f(x) \neq \lambda x, \forall \lambda \geq 0, \forall x \in \partial B_1$ then $f(x) = 0$ has a solution in the B_1.

Solution. We see that there cannot be a solution on the boundary of B_1, it follows from the assumption used with $\lambda = 0$. Let us suppose that f has not a solution in B_1. For $\lambda > 0$ we will consider the homotopy $H(t, x) = \lambda(1 - t)x + tf(x)$ for $t \in [0, 1]$ and $x \in \partial B_1$. We intend to show that $H(t, x) \neq 0$.

If $t = 0$, then the equality would mean that $x = 0$ which cannot holds. Similarly $t = 1$ would imply that $f(x)$ is zero for some point on the boundary. So we work just with $t \in (0, 1)$. At least for large λ's we obtain

$$\begin{aligned} |H(t, x)| &= |\lambda x - t(\lambda x - f(x))| \geq \lambda|x| - t|\lambda x - f(x)| \\ &\geq \lambda - t(\lambda|x| - |f(x)|) = (1 - t)\lambda - t|f(x)|. \end{aligned}$$

But the quantity on the right hand side is strictly possitive for big values of λ, because f is continuous on the whole $\overline{B_1}$ and so is bounded. But then by (d-4) we know that

$$\deg(f, B_1, 0) = \deg(\lambda \mathrm{id}, B_1, 0).$$

And again, by the homotopy between id and λid we can show that $\deg(f, B_1, 0) = \deg(\mathrm{id}, B_1, 0)$ which is equal to 1 by (d-2). Because of Theorem 1 (iv) there is some solution of $f(x) = 0$ in B_1. Alternatively it follows by the determinant characterizetion of the degree which will be given below.

$\bigtriangledown$

Now we can prove important representation of our degree. In the Sard theorem (Claim 1) we defined set of singular values of a mapping and denoted it by S_f. Its complement we usually denote as $\mathrm{RV}(f)$.

Theorem 3 (Determinant representation for smooth mapping)**.** Let $f \in \mathcal{C}^1_y(U), y \in \mathrm{RV}(f)$ then the number $\deg(f, U, y)$ satisfies

$$\deg(f, U, y) = \sum_{x \in f_{-1}(y)} \text{sign } J_f(x) \ .$$

Proof. See Theorem 2.4 in [35]. ■

There are some simple applications and consequences.

Example. Calculate the degree of a mapping $f(x, y) = (x^3 - 3xy^2, -y^3 + 3x^2y)$ at $(1, 0)$ with respect to the $B_r, r > 0$.

Solution. We intend to use the determinant formula above. Let us find the preimage of the point $(1, 0)$. We need to solve the following system

$$\begin{aligned} 1 &= x^3 - 3xy^2 \\ 0 &= -y^3 + 3x^2y. \end{aligned}$$

From the last equation we immediately get one solution $[1, 0]$. If $y \neq 0$ we can divide by y and conlude, from the second equation, that $3x^2 = y^2$ and using of first one we obtain $x^3 = -\frac{1}{8}$ and so there are two more solutions $\left[-\frac{1}{2}, \frac{\sqrt{3}}{2}\right], \left[-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right]$.

Wee see that f is smooth and $(1, 0) \in f(\partial B_r)$ only for several values of r. If $r < \frac{\sqrt{3}}{2}$ then the degree is 0, if it is bigger than 1 we use Theorem 3 to find

$$\deg(f, B_r, [1, 0]) = \text{sign } J_f(1, 0) + \text{sign } J_f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + \text{sign } J_f\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = 3.$$

For r's between those critical values we have only two terms and so the degree is 2. $\bigtriangledown$

Example. Let $f \in \mathcal{C}^1(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^d$ is an open, bounded set. If f is bijection then $\deg(f, \Omega, p) = \pm 1$, for those p's where the deg is defined.

Solution. Because f is 1-1 on the whole set, we have, that $J_f(x) \neq 0, \forall x \in \Omega$. So it is either positive or negative. Without loss of generality we may assume that it is possitive, otherwise we would work with $-f$. Because of Theorem 3 there holds

$$\deg(f, \Omega, p) = \sum_{x \in f_{-1}(p)} \text{sign } J_f(x) = 1,$$

because there is only one preimage of p. $\triangledown$

Example. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set and $u, v : \overline{\Omega} \to \mathbb{R}$ are $\mathcal{C}^1$ functions such that $u_x = v_y, u_y = -v_x$. Suppose that a function $f(x, y) = (u(x, y), v(x, y))$ has k roots in $\Omega \setminus \{x \in \Omega; J_f(x) = 0\}$. Then $\deg(f, \Omega, 0) = k$.

Solution. At the beginning we notice that deg is well defined object. We use our determinant representation again to obtain

$$\deg(f, \Omega, 0) = \underbrace{\sum_{x \in f^{-1}(0)} \text{sign } J_f(x)}_{\text{exactly } k \text{ terms}}.$$

Jacobian is simply

$$J_f(x) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ -u_y & u_x \end{vmatrix},$$

which imply that Jacobian is positive and so $\deg(f, \Omega, 0) = k$. $\triangledown$

Remark. We will work almost exclusively with real functions, but previous example shows that we can extend whole theory also for complex functions, because of well-known Cauchy-Riemann equations.

Our next goal is to extend this determinant formula for functions from $\mathcal{C}_y(U)$.

Theorem 4 (Determinant representation)**.** There exists unique degree satisfying properties (d-1) – (d-4). The mapping $f \mapsto \deg(f, U, y)$ is constant on each component and for any given function $f \in \mathcal{C}_y(U)$ we have the formula

$$\deg(f, U, y) = \sum_{x \in g_{-1}(y)} \text{sign } J_g(x) \ ,$$

where $g \in \mathcal{C}_y^2(U)$ is in the same component of $\mathcal{C}_y(U)$ such that $|f - g| < \text{dist } (y, f(\partial U))$ and $y \in \text{RV}(g)$.

Proof. See Theorem 2.11 in [35]. ■

Let us use this theorem in one direct application.

Example. Investigate the zeros of a function

$$f(x, y) = \left(x - 2y + \cos\left(\frac{x}{2} + y\right), y + 2x + \sin\left(\frac{x}{2} + y\right) \right).$$

Solution. Denote by g linear part of f, i. e. $g(x, y) = (x - 2y, y + 2x)$ and take ussual homotopy $H(t, (x, y)) = tf(x, y) + (1 - t)g(x, y)$. We need that $0 \notin H(t, (x, y)), \forall t \in [0, 1]$ and all (x, y) from the boundary of some suitable ball. By reverse triangle inequality we get

$$|H(t, \cdot)| \geq |g| - t|f - g|.$$

Observe that $|f(x,y)-g(x,y)| = \left|\left(\cos\left(\frac{x}{2}+y\right), \sin\left(\frac{x}{2}+y\right)\right)\right| = 1$ and so it suffices to have $|g|>1$. But it works for $|(x,y)| > \frac{1}{\sqrt{5}}$ because

$$|g(x,y)|^2 = x^2+4y^2-4xy+y^2+4x^2+4xy = 5(x^2+y^2) = 5|(x,y)|^2.$$

And so we use crucial property of homotopy (d-4) to conclude that

$$\deg(f, B_5, (0,0)) = \deg(g, B_5, (0,0)),$$

but Theorem 4 says that $\deg(g, B_5, (0,0)) \neq 1$, because g is a linear mapping and Jacobian is thus constant.

We could do it also by direct application of previous theorem. We can also observe that Jacobi matrix is

$$\begin{pmatrix} 1-\frac{1}{2}\sin\left(\frac{x}{2}+y\right) & -2-\sin\left(\frac{x}{2}+y\right) \\ 2+\frac{1}{2}\cos\left(\frac{x}{2}+y\right) & 1+\cos\left(\frac{x}{2}+y\right) \end{pmatrix}$$

and its determinant is bigger than 1 and so it is positive definite matrix (because all leading principal minors are possitive). From this we get injectivity of f, and therefore $f(x,y)=0$ has at most one solution. Together with the fact that there actually is a solution we observe that there exists exactly one point $(x_0,y_0) \in \mathbb{R}^2$ which solves $f(x,y)=0$. Moreover, since

$$1 = |f(x_0,y_0)-g(x_0,y_0)| = |g(x_0,y_0)| = \sqrt{5}|(x,y)|,$$

we see that the point (x_0,y_0) lies on the boundary of $B\left(0,\frac{1}{\sqrt{5}}\right)$. ▽

Now we are able to prove several immediate results of our theory. First one is quite classical and can be found bassicaly everywhere, but there are two good ideas.

Example. Let us suppose that a set $U \subset \mathbb{R}^d$ contains the origin and there is a continuous function $f : \partial U \to \mathbb{R}^d \setminus \{0\}$. If d is an odd number then there exists an $x \in \partial U$ and real $\lambda \neq 0$ such that $f(x) = \lambda x$.

Solution. By the Tietze theorem (Claim 8) we can extend our function to be continuous on the whole $\overline{U}$. Next observation is that for odd dimension we have

$$\deg(-\mathrm{id}, U, 0) = -1,$$

this follows easily from the determinant formula (Theorem 3 or 4), note that $0 \in U$.

Case 1: $\deg(f,U,0) = -1$. Let us consider the homotopy $H(t,x) = (1-t)f(x)+tx$. If there does not exist any $(t,x) \in [0,1]\times\partial U$ which is a root of H then, by (d-4) and (d-2), we conclude $\deg(f,U,0) = \deg(\mathrm{id},U,0) = 1$. It is contradiction with assumption of case 1. So there exists such a point (t_0,x_0). Obviously $t_0 \notin \{0,1\}$ (f is not mapped into 0). We have the equality

$$0 = (1-t_0)f(x_0)+t_0x_0$$

which imply that $f(x_0) = -\frac{t_0}{1-t_0}x_0$ and we are done.

Case 2: $\deg(f,U,0) \neq -1$. Now we will consider little bit different homotopy $H(t,x) = (1-t)f(x)-tx$. As above, if there is no root of H we get $\deg(f,U,0) = \deg(-\mathrm{id},U,0) = -1$, contradiction. And final relation would be $f(x_0) = \frac{t_0}{1-t_0}x_0$.

In both (disjoint) cases we found the λ. ▽

Remark. If d is even then the conclusion does not need to hold. Consider a function

$$f(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$$

as a mapping $\partial B_1 \to \mathbb{R}^4 \setminus \{0\}$. Then the x_1, x_3 would have equal to 0, and thus x_2, x_4 would be zero too, which is not possible.

Example. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function and $\Omega \subset \mathbb{R}^d$ be an open bounded set which contains 0. If $(f(x), x) > 0$ for all $x \in \partial\Omega$ then $\deg(f, \Omega, 0) = 1$ and so f has root in Ω. In particular we can apply it for functions defined on some ball with center at origin.

Solution. We will consider function $H(t, x) = tf(x) + (1-t)x$. Then, for any $x \in \partial\Omega$,

$$(H(t, x), x) = t(f(x), x) + (1-t)(x, x) > t(f(x), x) + (1-t)|x|^2 > 0, \forall t \in [0, 1].$$

It imply that $0 \notin H(t, x)$, because x lies on the boundary. As usual we get $\deg(f, \Omega, 0) = \deg(\mathrm{id}, \Omega, 0) = 1$. ▽

We will not give a proog to next two examples, because they can be found in many books, but the results are very interesting and will be nedeed later on.

Example. (Borsuk theorem) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and symetric set with $0 \in \Omega$. If $f \in \mathcal{C}(\overline{\Omega})$ is odd function such that $0 \notin f(\partial\Omega)$ then $\deg(f, \Omega, 0)$ is odd.

Solution. It can be found for example as Theorem 20.1.1 in [10]. ▽

Example. (Borsuk-Ulam theorem) Let $\Omega \subset \mathbb{R}^d$ be open, bounded and symetric set with $0 \in \Omega$. If $f \in \mathcal{C}(\partial\Omega, \mathbb{R}^{d'})$ for $d' < d$, then there exists $x_0 \in \partial\Omega$ such that $f(x_0) = f(-x_0)$.

Solution. It can be found as Theorem 20.2.1 in [10]. ▽

Remark. One well-known consequence of Borsuk-Ulam theorem is that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for $n > m$.

Following theorem is also quite standard, we will show it in same way as in Theorem 5.6 from [22]. We will later prove more general version of it.

Example. (Open mapping) Let $\Omega \subset \mathbb{R}^d$ be an open subset and $f : \Omega \to \mathbb{R}^d$ be continuous and locally 1-1. Then f is an open mapping.

Solution. We need to prove that for arbitrary $x_0 \in \Omega$ we are able to find radius $r > 0$ in such a way that $f(B(x_0, r))$ contains ball centered at $f(x_0)$. Wlog $x_0 = 0$, because otherwise we translate our set by x_0 and put $\tilde{f}(x) = f(x + x_0) - f(x_0)$.

Because f is locally injective, we can take $r > 0$ that f is 1-1 on whole $\overline{B_{r-\delta}}$, where $\delta > 0$ is arbitrary small number. In this ball we define the homotopy by the formula

$$H(t, x) = f\left(\frac{x}{1+t}\right) - f\left(-\frac{tx}{1+t}\right).$$

If $H(t, x) = 0$ for some $t \in [0, 1]$ and $x \in \partial B_{r-\delta}$, then there holds $f\left(\frac{x}{1+t}\right) = f\left(-\frac{tx}{1+t}\right)$. But f is injective up to the boundary and so we have equality of arguments. But it clearly imply that $x = 0$, which is a contradiction. Hence $H(t, x) \neq 0$ on $[0, 1] \times \partial B_{r-\delta}$ and we obtain the equality (use (d-4) and (d-2))

$$\deg(f, B_{r-\delta}, 0) = \deg(H(0, x), B_{r-\delta}, 0) = \deg(H(1, x), B_{r-\delta}, 0) =$$
$$= \deg\left(f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right), B_{r-\delta}, 0\right).$$

By the Borsuk theorem we know (0 surely is not in image of the boundary of our ball) that last quantity is odd, and therefore non-zero. Now we choose $s > 0$ so small that

$$s < \inf\{|f(x)|; x \in \partial B_{r-\delta}\}.$$

Because of Theorem 2(ii) it follows that

$$\deg(f, B_{r-\delta}, y) = \deg(f, B_{r-\delta}, 0) \neq 0$$

for all $y \in B_s$. For any $y \in B_s$ it gives to us, by Theorem 1(iv), an existence of $x \in B_{r-\delta}$ such that $y = f(x)$. In other words $B_t \subset f(B_{r-\delta})$, which was our goal. $\bigtriangledown$

Because of this result we are able to make a proof of Fundamental theorem of algebra.

Example. (Fundamental theorem of algebra) Let $p(z) = a_k z^k + \sum_{j=1}^{k-1} a_j z^j$ be a polynomial over $\mathbb{C}$ and $a_k \neq 0, k > 0$. Then p has a root.

Solution. At the beginning we notice that $p(C)$ is a closed set for any closed $C \subset \mathbb{C}$. If we consider sequence $p(z_j) \to w$ as $j \to \infty$ then we need that $p(z) = w$ for $z_j \to z \in C$ as $j \to \infty$. Because $p(z_j)$ converges to something, we know that it is bounded sequence. But then the $(z_j)_{j\in\mathbb{N}}$ is bounded too, because otherwise we obtain

$$\limsup_{j\to\infty} |p(z_j)| = \limsup_{j\to\infty} |z_j^k| \cdot \left| a_k + \sum_{i=1}^{k-1} a_i \frac{z_j^i}{z_j^k} \right| = \infty$$

which is a contradiction with boundedness of $(p(z_j))_j$. Hence, the whole sequence $(z_j)_j$ is in bounded subset of closed set C, and thus in some compact. Thanks to the Bolzano-Weierstrass theorem we know that $z_{j_i} \to z$ as $i \to \infty$ for some $z \in C$ and some subsequence $(z_{j_i})_i$ of $(z_j)_j$. Due to the continuity of p we get $p(z_{j_i}) \to p(z) = w$ as $i \to \infty$. As ussual it follows that even $p(z_j) \to z$ as $j \to \infty$. In other words p is a closed mapping.

Assume that p is locally 1-1. Then it is also an open mapping and so we may write

$$\mathbb{C} = p(\mathbb{C}) \cup (p(\mathbb{C}))^c,$$

where $p(\mathbb{C})$ is both open and closed and moreover non-empty. Because complex plane is connected space we immediately get $p(\mathbb{C}) = \mathbb{C}$, in particular $0 \in p(\mathbb{C})$.

Now we suppose that p is not locally injective. Clearly $k > 1$. We denote by S_p the set of points where p' vanishes (we know that injective function has non-zero derivative, so it is natural). We see that $|S_p| \leq k-1$, because p' is a polynomial of the order $k-1$ and it cannot have more roots than its order, because otherwise we would get decomposition which would lead to a polynomial of higher order. For the same reason the set $p_{-1}(p(S_p))$ is finite too. Therefore both $T = \mathbb{C} \setminus p(S_p)$ and $S = \mathbb{C} \setminus p_{-1}(p(S_p))$ are open.

Because of standard result of complex analysis we know that p is locally 1-1 on S, because its derivative nowhere vanishes. Therefore $p(S)$ is open. From the above we know that $p(\mathbb{C})$ is closed and so $T \cap p(\mathbb{C})$ is closed set in T. From the definiton of our sets we clearly have $p(S) = T \cap p(\mathbb{C})$. Altogether the set $p(S)$ is both closed and open in T. Thanks to finiteness of $p(S_p)$ we have that T is connected set in $\mathbb{C}$. As in the previous case we conlude that $T = p(S)$. And finally observe

$$p(\mathbb{C}) = p(S \cup p_{-1}(p(S_p)) = p(S) \cup p((p_{-1}(p(S_p))) = T \cup p(S_p) = \mathbb{C}.$$

$\bigtriangledown$

Remark. Key moment was that if we exclude finitely many points from $\mathbb{C}$, the resulting set will still be connected. This would not holds for $\mathbb{R}$.

Theorem 5 (Degree of a composition)**.** Let $U \subset \mathbb{R}^d$ be an open, bounded set and by G_j denote the connected components of $\mathbb{R}^d \setminus f(\partial U)$. If $f \circ g \in \mathcal{C}_y(U)$, then

$$\deg(f \circ g, U, y) = \sum_j \deg(g, U, G_j) \deg(f, G_j, y),$$

where object $\deg(g, U, G_j)$ makes sense due to Theorem 2(ii).

Proof. See Theorem 2.19 in [35]. ■

Example. (Degree of a power) Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that $f(\partial B_1) = \partial B_1$, where $B_1 \subset \mathbb{R}^d$. Then

$$\deg(f^n, B_1, 0) = (\deg(f, B_1, 0))^n, \forall n \in \mathbb{N}.$$

Solution. Because f is continuous and the boundary is mapped to the boundary, we know that $f(\overline{B_1}) \subset \overline{B_1}$. So there are only two components of $f(\partial B_1)$. Of course zero lies only in the bounded one, denote it by G_1. And so, for $n = 2$,

$$\begin{aligned}\deg(f^2, B_1, 0) &= \sum_j \deg(f, B_1, G_j) \deg(f, G_j, 0) = \\ &= \underbrace{\deg(f, B_1, G_1)}_{=\deg(f, B_1, 0)} \cdot \underbrace{\deg(f, G_1, 0)}_{=\deg(f, B_1, 0)} + \deg(f, B_1, G_2) \cdot \underbrace{\deg(f, G_2, 0)}_{=0} = \\ &= (\deg(f, B_1, 0))^2,\end{aligned}$$

where we used previous theorem about composition and then Theorems 2 and 1. Similarly (and by induction) we prove the general statesment.

▽

2.2 Topological degree in Banach spaces

Until now we have defined notion of degree just on sets in $\mathbb{R}^d$. Our next goal is to extend its definition to a general Banach space.

2.2.1 Finitely dimensional case

We start with a quite natural extension of degree to arbitrary Banach space with finite dimension.

Definition 3. Let X be a real Banach space with dimension $d < \infty$. For $f \in \mathcal{C}_y(U, X)$ where $U \subset X$ is an open set, $y \in X$ we define degree of f on U with respect to y as

$$\deg(f, U, y) = \deg(\varphi \circ f \circ \varphi^{-1}, \varphi(U), \varphi(y)) \ ,$$

where φ is an isomorphism from X onto $\mathbb{R}^d$.

Remark. Of course any Banach space X over $\mathbb{R}$ of dimension d is isometrically isomorphic to $(\mathbb{R}^d, ||\cdot||_\infty)$. Consider the map $\varphi(x) = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $x = \sum_{k=1}^d \alpha_k e_k \in X$ for basis $(e_1, e_2, \ldots, e_d)$ in X. It is clearly an isomorphism from X onto $\mathbb{R}^d$. It is even isometry because we may assume (because all norms are equivalent) that on X we have supremum norm of x defined as $\max_{k=1,2,\ldots,d} |\alpha_k|$.

Theorem 6. Definition above is meaningful.

Proof. Let us consider the second isomorphism ψ. Then $\Lambda := \psi \circ \varphi^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ is again isomorphism. It thus means that it is linear mapping and so can be represeneted by regular matrix of constants. Now we will pick $\hat{f} \in \mathcal{C}^1_y(\varphi(\overline{U}))$ in the same component of $\mathcal{C}_y(\varphi(\overline{U}))$ as $\varphi \circ f \circ \varphi^{-1}$ such that $\varphi(y)$ is regular value of $\varphi \circ f \circ \varphi^{-1}$. Then $\Lambda \circ \hat{f} \circ \Lambda^{-1} \in \mathcal{C}^1_y(\psi(\overline{U}))$ is in the same component of $\mathcal{C}_y(\psi(\overline{U}))$ as

$$\Lambda \circ \varphi \circ f \circ \varphi^{-1} \circ \Lambda^{-1} = \psi \circ f \circ \psi^{-1} \ .$$

By chain rule we can compute sign of the Jacobian as follows

$$\begin{aligned} \text{sign } J_{\Lambda\circ\hat{f}\circ\Lambda^{-1}}(\Lambda\varphi(y)) &= \text{sign } (J_\Lambda(\hat{f}(\varphi(y)) \cdot J_{\hat{f}}(\varphi(y)) \cdot J_{\Lambda^{-1}}(\Lambda\varphi(y))) = \\ &= \text{sign } J_{\hat{f}}(\varphi(y)), \end{aligned}$$

where we used the regularity of $\varphi(y)$ and fact that continuous deformations of regular mapping preserve sign of determinant (or just that Λ is a constant matrix).

From our determinant formula (Theorem 3) follows equality

$$\deg(\varphi \circ f \circ \varphi^{-1}, \varphi(U), \varphi(y)) = \deg(\psi \circ f \circ \psi^{-1}, \psi(U), \psi(y))$$

and so definition is independent on choice of isomorphism (i. e. choice of basis).

■

Theorem 7 (Reduction property)**.** Let $f \in \mathcal{C}(\overline{U}, \mathbb{R}^{d'}), U \subset \mathbb{R}^d, d' < d$ and $y \in \mathbb{R}^{d'} \setminus (\mathrm{id} + f)(\partial U)$, then

$$\deg(\mathrm{id} + f, U, y) = \deg(\mathrm{id} + f|_{d'}, U_{d'}, y),$$

where $f|_{d'}$ is restriction of f on $U_{d'} = \{x = (x_1, \ldots, x_{d'}, \ldots, x_d) \in U; x_{d'}, \ldots, x_d = 0\}$.

Proof. See Theorem 2.18 in [35]. ■

Corollary. Degree in finitely dimensional Banach space has exactly same properties as just in $\mathbb{R}^d$. Moreover reduction formula, which has been proven before, holds also if $\mathbb{R}^{d'}$ is replaced by any subspace X_0, since we can choose the isomorphism $\varphi : X_0 \to \mathbb{R}^d$ with $\varphi(X_0) = \mathbb{R}^{d'}$.

We borrow the following example from [22], section IV. 4. Winding number or also [8], theorem 1.2.18.

Example. (Winding number) Let $\overline{B}_1 \subset G \subset \mathbb{C}$ be an open set and $\Gamma = \partial B_1$ be a positively oriented curve and $f : G \to \mathbb{C}$ be a $\mathcal{C}^1$ function. Assume that $a \notin f(\Gamma)$ then

$$\deg(f, B_1, a) = \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{1}{z - a} \mathrm{d}z.$$

Solution. We will prove it for a which is regular value of f. If it would be singular, then we would approximate f by some function for which it is regular (see Theorem 4). It is clear that $f_{-1}(a)$ cannot be infinite. Otherwise we have sequence $(z_n)_n$ satisfying $f(z_n) = a$ for all $n \in \mathbb{N}$. Because all z_n lies in $\overline{B_1}$, which is compact set, we get $z_{n_k} \to z \in \overline{B_1}$. From the continuity we obtain $f(z) = a$ and because a is regular value, f is injective on some neigbourhood of z. But it cannot be satisfied if we consider indices n big enough.

For the sake of simplicity assume $a = 0$ and let us denote $f_{-1}(a) = \{z_1, \ldots, z_n\}$. We intend to prove

$$\frac{1}{2\pi i} \int_{f(\Gamma)} \frac{1}{z} \mathrm{d}z = \sum_{k=1}^{n} \text{sign } J_f(z_k).$$

We take neigbourhoods of each z_k so small that f is bijective on that set, i. e. we have $U_k = B(z_k, 2\varepsilon)$ so small that sign $J_f(z) = \text{sign } J_f(z_k)$ on the whole U_k, for all $k = 1, \ldots, n$ and moreover let $\overline{U_k} \subset B_1$ for all k's.

Let us define $\Omega = G \setminus \bigcup_{k=1}^{n} \overline{B(z_k, \varepsilon)}$, it is non-empty, open and connected set. Now we denote as γ_k positively oriented circle around z_k and radius ε. Without loss of generality we may assume that all balls U_k are pairwise disjoint. We see that Ω contains curves $\Gamma, \gamma_k, k = 1, 2, \ldots, n$.

Because of Cauchy integral theorem we have (note that integrand is holomorphic on Ω) the equality

$$\int_\Gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \sum_{k=1}^{n} \int_{\gamma_k} \frac{f'(z)}{f(z)} \mathrm{d}z.$$

Thanks to Claim 17 we may conclude

$$\int_{f(\Gamma)} \frac{1}{z} \mathrm{d}z = \sum_{k=1}^{n} \int_{f(\gamma_k)} \frac{1}{z} \mathrm{d}z.$$

We see that closed curve $f(\gamma_k)$ winds exactly ones around 0 and so the integral $\frac{1}{2\pi i} \int_{f(\gamma_k)} \frac{1}{z} \mathrm{d}z$ is equal either 1 or -1 (it is well-known with some basics about winding numbers). It is "obvious" that the sign is determined by sign $J_f(z_k)$, i. e. if $J_f(z_k) > 0$ then $\frac{1}{2\pi i} \int_{f(\gamma_k)} \frac{1}{z} \mathrm{d}z = 1$ and if Jacobian is negative then the value is -1. It is the desired relation written above.

That Jacobian determines "orientation" of our curves is clear from geometrical point of view, detailed proof would be more involved.

$\bigtriangledown$

Remark. If the reader is familiar with complex analysis, then the previous example shows that our notion of the topological degree is good generalization of winding number, which determines number of times that curve travels (counterclockwise) around the point.

It also gives the reason why we really needed (see Definition 2) the assumption saying that the point y does not belong into image of the boundary (it does not make sense to speak about winding around such a point).

2.2.2 Infinitely dimensional case

At now we are prepared to define degree also for infinitely dimensional Banach space. Recall that by the symbol $\mathcal{K}(X, Y)$ we mean the set of both continuous and compact mappings between two normed spaces X and Y.

Lemma 8. Let X be a real Banach space.

(i) Suppose that Ω is an open bounded subset of X and $T \in \mathcal{K}(\overline{\Omega}, X)$. Then for any $\varepsilon > 0$ there exist a finite dimensional subspace Y of X and a continuous operator $T_\varepsilon : \Omega \to Y_\varepsilon$ such that

$$||Tx - T_\varepsilon x||_X \leq \varepsilon, \forall x \in \overline{\Omega}.$$

(ii) Suppose that $F \subset X$ is a closed, bounded set and $T \in \mathcal{K}(F, X)$. Suppose that $(\mathrm{id} - T)x \neq 0$ for all $x \in F$. Then there is an $\varepsilon > 0$ such that $x \neq \lambda T_{\varepsilon_1} x + (1 - \lambda) T_{\varepsilon_2}$ for all $\lambda \in [0, 1]$ and any $x \in F$, where $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon)$ and $T_{\varepsilon_1}, T_{\varepsilon_2}$ are approximations from (i) with ranges in $Y_{\varepsilon_1}, Y_{\varepsilon_2}$.

Proof. See Lemma 3.1 in [35] and following discussion or Lemma 2.2.1 and 2.2.2 in [8]. Note that in this proof is needed so called Reduction property mentioned above. ■

Definition 4. Let X be a real Banach space, $\Omega \subset X$ be an open, bounded subset and $T \in \mathcal{K}(\overline{\Omega}, X)$. Suppose that $0 \notin (\mathrm{id} - T)(\partial\Omega)$, due to the second part of previous lemma we have an existence of positive ε and two operators $T_{\varepsilon_k} : \overline{\Omega} \to Y_k$ satisfying

$$x \neq \lambda T_{\varepsilon_1} x + (1-\lambda) T_{\varepsilon_2}, \forall \lambda \in [0,1], \forall x \in \partial\Omega,$$

where $\varepsilon_k \in (0, \varepsilon)$ and Y_k are finite dimensional subspaces of X for $k = 1, 2$. Now, thanks to the first part of same lemma, we can define so called Leray-Schauder degree as

$$\deg(\mathrm{id} - T, \Omega, 0) = \deg(\mathrm{id} - T_{\overline{\varepsilon}}, \Omega \cap Y_{\overline{\varepsilon}}, 0), \text{ for } \overline{\varepsilon} \in (0, \varepsilon).$$

If $y \notin (\mathrm{id} - T)(\partial\Omega)$ we also define

$$\deg(\mathrm{id} - T, \Omega, y) = \deg(\mathrm{id} - T - y, \Omega, 0).$$

As before we get same result about properties of our (now quite general) degree.

Theorem 9. Let us consider situation as in the definition above with $y \notin (\mathrm{id} - T)(\partial\Omega)$. Then Leray-Schauder degree satisfies

(dg-1) $\deg(\mathrm{id} - T, \Omega, y) = \deg(\mathrm{id} - T - y, \Omega, 0)$.

(dg-2) $\deg(\mathrm{id}, \Omega, y) = 1$ if and only if $y \in \Omega$.

(dg-3) Let Ω_1, Ω_2 be open, disjoint subsets of Ω and $y \notin (\mathrm{id} - T)(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then

$$\deg(\mathrm{id} - T, \Omega, y) = \deg(\mathrm{id} - T, \Omega_1, y) + \deg(\mathrm{id} - T, \Omega_2, y) \ .$$

(dg-4) If $\deg(\mathrm{id} - T, \Omega, y) \neq 0$, then $x - Tx - y = x$ has a solution in Ω.

(dg-5) If $H_\lambda \in \mathcal{K}([0,1] \times \overline{\Omega}, X)$ and $H_\lambda x \neq x$ in $[0,1] \times \partial\Omega$, then $\deg(\mathrm{id} - H_\lambda, \Omega, y)$ does not depend on λ.

Proof. See Theorem 3.4 in [35]. ■

Theorem 10. Let X be a real Banach space, Ω its open, bounded subset and $T, G \in \mathcal{K}(\overline{\Omega}, X)$. For $y \notin (\mathrm{id} - T)(\partial\Omega)$ there also holds

(i) Let $\Omega_1, \cdots, \Omega_n$ be open, disjoint subsets of Ω and $y \notin (\mathrm{id} - T)(\overline{\Omega} \setminus (\bigcup_{i=1}^n \Omega_i)$, then

$$\deg(\mathrm{id} - T, \Omega, y) = \sum_{i=1}^{n} \deg(\mathrm{id} - T, \Omega_i, y) \ .$$

(ii) If $||T(x) - G(x)|| < \mathrm{dist}\,(y, T(\partial\Omega))$ for all $x \in \partial\Omega$ then $\deg(\mathrm{id} - T, \Omega, y) = \deg(\mathrm{id} - G, \Omega, y)$.

(iii) Function $y \mapsto \deg(\mathrm{id} - T, \Omega, y)$ is constant on each component of $X \setminus T(\partial\Omega)$.

Proof. See Theorem 3.5 in [35]. ■

In the following example we prove the analogical open mapping theorem as in the finite dimensional case.

Example. Let X be a Banach space, Ω be its open subset and $T : \Omega \to X$ be a continuous, compact operator. If $\mathrm{id} - T$ is locally 1-1 then $\mathrm{id} - T$ is an open mapping.

Solution. We will start in the same fashion as in the finite dimension. Hence, we know that it is enough to find $r > 0$ such that $(\mathrm{id} - T)(B_r)$ contains the ball centered at $-T(0) = 0$, where we without loss of generality assume that $0 \in \Omega$ and $T(0) = 0$. At first, we find $r > 0$ such that $\mathrm{id} - T$ is 1-1 on the whole $\overline{B_r}$. Now we introduce the homotopy defined as

$$H(t, x) = T\left(\frac{x}{1+t}\right) - T\left(-\frac{tx}{1+t}\right).$$

Thus, we have $H(1, x) = T\left(\frac{x}{2}\right) - T\left(-\frac{x}{2}\right)$ and $H(0, x) = T(x)$, moreover we see $H(t, x) \neq x$ on $[0, 1] \times \partial B_r$, because otherwise

$$\begin{aligned} T\left(\frac{x}{1+t}\right) - T\left(-\frac{tx}{1+t}\right) &= x \\ -\frac{tx}{1+t} - T\left(-\frac{tx}{1+t}\right) &= \frac{x}{1+t} - T\left(\frac{x}{1+t}\right) \\ (\mathrm{id} - T)\left(-\frac{tx}{1+t}\right) &= (\mathrm{id} - T)\left(\frac{x}{1+t}\right) \end{aligned}$$

which would, altogether with the fact that $\mathrm{id} - T$ is 1-1 in the ball, imply $x = 0$.

By Theorem 9(v) there holds

$$\deg(x - T, B_r, 0) = \deg\left(x - \left(T\left(\frac{x}{2}\right) - T\left(-\frac{x}{2}\right)\right), B_r, 0\right).$$

Mapping $x - \left(T\left(\frac{x}{2}\right) - T\left(-\frac{x}{2}\right)\right)$ is obviously "odd", i. e. the value at x is opposite to the value at $-x$. We want to show that the degree on the right hand side is an odd number. If it is true, then we continue as in finite dimension. We would find some ball B_t such that $\deg(\mathrm{id} - T, B_r, y) = \deg(\mathrm{id} - T, B_r, 0) = 2k + 1$, for some fixed $k \in \mathbb{N}$ and for all $y \in B_t$, and thus $B_t \subset (\mathrm{id} - T)(B_r)$.

Now we need to find an "odd" approximation. From the Lemma 8 we know that for any $\varepsilon > 0$ there exists Y subspace of X of finite dimension and an operator $F_\varepsilon : \overline{B_r} \to Y$ such that $||F(x) - F_\varepsilon(x)|| \leq \varepsilon, x \in \overline{B_r}$, where $F(x) = T\left(\frac{x}{2}\right) - T\left(-\frac{x}{2}\right)$.

Observe that

$$||F(x) - \frac{1}{2}(F_\varepsilon(x) - F_\varepsilon(-x))|| = \frac{1}{2}||F(x) - F_\varepsilon(x) - F(-x) - F_\varepsilon(-x)|| \leq \varepsilon.$$

So if we define $G_\varepsilon(x) = \frac{1}{2}(F_\varepsilon(x) - F_\varepsilon(-x))$, then we get the "odd" finitely dimensional approximation of F, because $||F(x) - G_\varepsilon(x)|| \leq \varepsilon$. By the definition above

$$\deg\left(x - \left(T\left(\frac{x}{2}\right) - T\left(-\frac{x}{2}\right)\right), B_r, 0\right) = \deg(x - G_\varepsilon(x), B_r \cap Y, 0)$$

for $\varepsilon > 0$ small enough. Now we would identify finitely dimensional Banach space $B_r \cap Y$ with appropriate $\mathbb{R}^d$ via some isomorphism, but for sake of clarity we will omit such a notation. So we want to apply Borsuk theorem. Obviously $f(x) = (\mathrm{id} - G_\varepsilon)(x)$ is $\mathcal{C}^1$ up to the boundary and it is an odd function. Remains to show that $0 \notin f(\partial(B_r \cap Y))$. But it is almost clear, because

$$||x - G_\varepsilon(x)|| = ||(x - F(x)) + (-G_\varepsilon(x) + F(x))|| \geq ||x - F(x)|| - \varepsilon.$$

Above we showed that $H(1, x) = F(x)$ is not equal to x on the boundary and because F is contiunous and the boudnary is closed, we know that there exists some positive $d = \mathrm{dist}\ (\mathrm{id} - F, f(\partial(B_r \cap Y))$ and this number will be fixed, so for small ε we get that quantity above is strictly positive. Therefore the proof is complete now. $\bigtriangledown$

Example. Let X be a reflexive Banach space of infinite dimension, $T : X \to X$ be a continous and compact mapping. Suppose that $\mathrm{id} - T$ is locally 1-1 and let there exists the limit $\lim_{||x||\to\infty} \frac{||Tx||}{||x||}$ and is not equal to 1. Then $\mathrm{id} - T$ is surjective.

Solution. Because T cannot be equal to the identity mapping (would not be compact) we see that $(\mathrm{id} - T)(X)$ is not empty and thanks to the previous example we have that it is an open set. On the other hand it is also closed. Let us assume that we have $y \in X$ and a sequence $(x_n)_n \subset X$ for which $(x_n - T(x_n)) \to y$. We need to show that $x - Tx = y$ where x would be limit of $(x_n)_n$. So the sequence $(x_n - T(x_n))_n$ is bounded and thanks to our assumption we have

$$1 \neq \lim_{||x_n||\to\infty} \frac{||Tx_n||}{||x_n||} \leq \lim_{||x_n||\to\infty} \frac{||Tx_n - x_n|| + ||x_n||}{||x_n||} = 1$$

and so $(x_n)_n$ must be also bounded. Thanks to the reflexivity of X we get an existence of $x \in X$ for which $x_{n_k} \overset{k}{\rightharpoonup} x$ where x_{n_k} is some subsequence of (x_n). Because of compactness we have that $T(x_{n_k})$ converge to Tx strongly. Hence $x_n - Tx_n \to x - Tx$.

Altogether $(\mathrm{id} - T)(X)$ is clopen set in Banach space X, which is of course connected and so

$$X = (\mathrm{id} - T)(X) \cup ((\mathrm{id} - T)(X))^c,$$

which leads to the desired equality $(\mathrm{id} - T)(X) = X$. ▽

Example. Let X be a Banach space and $L : \overline{B_R} \to X$ be a linear and compact mapping for $R > 0$. If $\lambda \neq 0$ and $\frac{1}{\lambda} \notin \sigma(L)$ then

$$\deg(\mathrm{id} - \lambda L, B_R, 0) = (-1)^{m(\lambda)},$$

where $m(\lambda)$ is the sum of algebraic multiplities of the eigenvalues μ satysfying $\mu > \frac{1}{\lambda}$ (it is zero if there is no such a number).

Solution. See Theorem 2.2.16 in [8]. ▽

Example. Let X be a Banach space. Let $T : \overline{B_R} \to X$ be a continuous and compact mapping for some $R > 0$ and $L : X \to X$ be a linear and compact mapping such that $||Tx - Lx|| < ||x - Lx||$ for all $x \in \partial B_R$. Then $\deg(\mathrm{id} - T, B_R, 0)$ is an odd number.

Solution. We define the homotopy as $H(t, x) = Lx + (1 - t)(T - L)(x)$ for $t \in [0, 1]$ and $x \in \partial B_R$. Then $H(0, x) = Tx$ and $H(1, x) = Lx$. Observe that $x \neq H(t, x)$ for any t and any x from the boundary, it follows from our inequality, because

$$\begin{aligned} ||(Lx - x) - (1 - t)(T - L)x|| &\geq ||Lx - x|| - (1 - t)||Tx - Lx|| \\ &> (1 - 1 + t)||Lx - x|| \geq 0. \end{aligned}$$

By Theorem 9(v) we get an equality

$$\deg(\mathrm{id} - T, B_R, 0) = \deg(\mathrm{id} - L, B_R, 0).$$

We see that $1 \notin \sigma(L)$, because otherwise we would get some element $x' = R\frac{x}{||x||} \in \partial B_R$ such that $L(x') = x'$ which would contradict the assumed inequality. By the previous example we get the conclusion. ▽

3. Fixed point theorems

> Mathematics is the most beautiful and most powerful creation of the human spirit.
>
> *Stefan Banach*

In this chapter we show several theorems about existence of fixed points. Most of them we prove and discuss whether are their assumptions really necessary. For sake of completness we start with results about contraction mappings, which are quite classical. Then we continue with famous Brouwer and Schauder theorem and some of their consequences. At the end, we take a look on fixed points of multivalued mappings.

3.1 Theorems with contraction like mappings

Let us recall two definitions and well-known result of Banach.

Definition 5. Let (X, d) be a metric space and $f : X \to X$ is a mapping. We say that f is contraction if there exist $r \in [0, 1)$ such that for all $x, y \in X$ holds $d(f(x), g(x)) \leq rd(x, y)$.

We say that f is contractive if for all $x \neq y \in X$ there holds $d(f(x), g(x)) < d(x, y)$.

Remark. Clearly any contraction is automaticaly contractive too.

Theorem 11 (Banach contraction principle / Banach fixed point theorem)**.** Let (X, d) be a non-empty complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point.

Proof. It is a standard result, for the proof see [29]. ■

Example. (Completness is needed in general) Let $f : (0, 1) \to (0, 1)$ be given by $f(x) = \frac{x}{2}$. Then it is clearly contraction, but cannot have a fixed point.

In this example we somehow evaded the point, which would be a fixed one. But we can take nicer example.

Let us consider $1 > a > 0$ and define $A = [-a, a] \cap \mathbb{Q}$ with function f given by $f(x) = x^2 - \frac{1}{2}a$. Then f maps A onto $[-\frac{1}{2}a, a^2 - \frac{1}{2}a] \cap \mathbb{Q}$ which is subset of A for our a's. However this function has only two fixed points, namely $x_{1,2} = \frac{1}{2}(1 \pm \sqrt{1 + 2a})$. We see that positive choice leads to the point outside of $[-a, a]$. And for a good choice of a the negative one will be in our interval but irrational. ▽

On the other hand, completness is not necessary all the time.

Example. Let us consider the set given by $M = \left\{\left(x, \sin\frac{1}{x}\right); x \in (0, 1]\right\} \subset \mathbb{R}^2$. Clearly it is not complete (otherwise it would be closed, but it is not). Now we take arbitrary contraction $f : M \to M$ (there obviously exist at least one) with constant of contractivity $r \in [0, 1)$ and small $\varepsilon > 0$, where its smallness will be specified later.

Now we decompose $M = L_\varepsilon \cup R_\varepsilon$, where $L_\varepsilon := M \cap \left\{\left(x, \sin\frac{1}{x}\right); x \in (0, \varepsilon)\right\}$ and similarly $R_\varepsilon := M \cap \left\{\left(x, \sin\frac{1}{x}\right); x \in [\varepsilon, 1]\right\}$. Denote as $m := \inf\{x; (x, y) \in f(M)\}$.

Let us consider $m = 0$. Then $0 = \min\{\inf\{x; (x, y) \in f(L_\varepsilon)\}, \inf\{x; (x, y) \in f(R_\varepsilon)\}\}$. Thanks to the contractivity f is also continuous and because R_ε is a compact set in M, and thus we obtain $0 = \inf\{x; (x, y) \in f(L_\varepsilon)\}$. From this we can find two

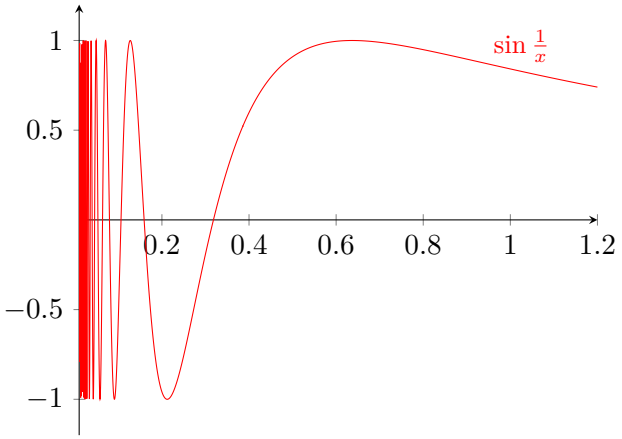

Figure 3.1: The set M.

positive numbers $x_a < x_b < \varepsilon$ (i. e. close to zero) so that $a := \left(x_a, \sin\frac{1}{x_a}\right), b := \left(x_b, \sin\frac{1}{x_b}\right) \in f(L_\varepsilon) \subset M$ and $\sin\frac{1}{x_a} > 1-\varepsilon, \sin\frac{1}{x_b} < -1+\varepsilon$. There obviously holds

$$|a-b| = \sqrt{|x_a - x_b|^2 + \left|\sin\frac{1}{x_a} - \sin\frac{1}{x_b}\right|^2} > 2 - 2\varepsilon.$$

There exist preimages of a, b, i. e. some points $a', b' \in L_\varepsilon$ for which $f(a') = a, f(b') = b$. From the definition of L_ε follows that

$$|a'-b'|^2 \leq |2\varepsilon|^2 + \left|\sin\frac{1}{x_{a'}} - \sin\frac{1}{x_{b'}}\right|^2 \leq 4\varepsilon^2 + 4.$$

Altogether we get

$$2(1-\varepsilon) < |a-b| = |f(a') - f(b')| \leq r|a'-b'| \leq 2r\sqrt{\varepsilon^2+1}.$$

So if we choose ε at least so small that $r < \frac{1-\varepsilon}{\sqrt{\varepsilon^2+1}}$, which we may because r is fixed number strictly smaller than 1, we get a desired contradiction with $m = 0$. So it remains to work with $m > 0$. Then

$$f(R_m) \subset f(M) \subset M \cap \{(x,y) \in \mathbb{R}^2; x \in [m,1]\} \subset R_m.$$

In other words $f : R_m \to R_m$, moreover R_m is clearly complete space as a subspace of $\mathbb{R}^2$. Now we invoke Banach theorem above to get the fixed point in $R_m \subset M$.

Remark. See [34] for example of the non-complete space, where every continuous function has a fixed point.

Example. (We need contraction) Let $f : [1,\infty) \to [1,\infty)$ is given by $f(x) = x + \frac{1}{x}$. Then f clearly does not have any fixed point on $[1,\infty)$. The interval $[1,\infty)$ is complete, because it is closed subset of $\mathbb{R}$. But f is just contrative. It holds because we have

$$\begin{aligned} |f(x) - f(y)| &= \left|x - y + \left(\frac{1}{x} - \frac{1}{y}\right)\right| = \left|(x-y) + \frac{y-x}{xy}\right| \\ &= |x-y|\left(1 - \frac{1}{xy}\right), \end{aligned}$$

where the last equality holds if $x \neq y$. Because f does not have a fixed point we see that it is not a contraction (or we can see it directly - there is no uniform number r from the definition above). ▽

Contractive mapping can be useful too. We have following theorem, which gives us sufficient condition on the existence of fixed point. See [25] section 1.7 for a different proof.

Theorem 12 (Edelstein fixed point theorem)**.** Let (X, d) be a non-empty, compact, metric space and $f : X \to X$ be a contractive. Then f has a unique fixed point.

Proof. Let us take the function $g(x) = d(x, f(x))$. Then, for any x, y, we have

$$\begin{aligned} g(x) - g(y) &= d(x, f(x)) - d(y, f(y)) \\ &< d(x, y) + d(y, f(y)) + d(f(y), f(x)) - d(y, f(y)) \\ &< 2d(x, y), \end{aligned}$$

where last inequality follows from the fact that f is contractive. Therefore g is continuous. Thanks to compactness of X there exists point x_0 which is a minimum of g. Suppose that $g(x_0) \neq 0$. Then $d(x_0, f(x_0)) > 0$ i. e. $x_0 \neq f(x_0)$. But then

$$g(f(x_0)) = d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = g(x_0).$$

It is a contradiction with x_0 being the minimum. Therefore $g(x_0) = 0$ which means that there is a fixed point x_0. Uniqueness follows immediately from the fact that f is contractive. ■

Remark. For more results of same kind see [18].

Important application of Banach theorem above is of course standard result about existence of a solution to the given ordinary differential equation. And not only for classical functions (smooth,...) but also for just integrable ones. For details see any book about ordinary differential equations, e. g. [15].

3.2 Theorems with compact mappings

Because of our notion of the degree we are fully prepared to prove following famous result.

Theorem 13 (Brouwer fixed point theorem)**.** Let K be a topological space homeomorphic to a closed ball B_r, for some, $r > 0$ in $\mathbb{R}^d$ and $f \in \mathcal{C}(K, K)$. Then there exists at least one $x \in K$ such that $f(x) = x$.

Proof. Because of homoeorphism we can assume that $K \subset \mathbb{R}^d$. More elaborately, suppose that it holds for $K \subset \mathbb{R}^d$. Then we have a homeomorphism $h : K \to \mathbb{R}^d$, where $h(K) =: \tilde{K}$ is a convex and compact set. Clearly $\tilde{f} := h \circ f \circ h^{-1} \in \mathcal{C}(\tilde{K}, \tilde{K})$. Now we can find $\tilde{x} \in \tilde{K}$ such that $\tilde{f}(\tilde{x}) = h \circ f \circ h^{-1}(\tilde{x}) = \tilde{x}$. And so we also have wanted fixed point $h^{-1}(\tilde{x}) \in K$ for function f.

Now consider that $K = B_r$. We may assume $x \neq f(x)$ on the whole boundary, otherwise we are done. Set $H(t, x) = x - tf(x)$ for $t \in [0, 1], x \in \overline{B_r}$. But then $H(t, x) \neq 0$ on $[0, 1] \times \overline{B_r}$ because for $t \in [0, 1)$

$$|H(t, x)| \geq |x| - t|f(x)| \geq (1 - t)r > 0$$

and for $t = 1$ we have our assumption. Therefore

$$\deg(\mathrm{id} - f, B_r, 0) = \deg(\mathrm{id}, B_r, 0) = 1$$

and we are done again. ■

It is possible to prove stronger version of previous theorem. For this we need following lemma. It is just a technical one, so we will skip its proof.

Lemma 14. Let X, Y be two Banach spaces, $K \subset X, K = \overline{K}$. Then any $f \in \mathcal{C}(K, Y)$ has continuous extension $f \in \mathcal{C}(X, Y)$ and $f(x) \subset \text{conv} f(K)$.

Proof. See Theorem 2.15 in [35]. ■

Theorem 15. Let K be a topological space homeomorphic to a compact convex subset of $\mathbb{R}^d$ and $f \in \mathcal{C}(K, K)$. Then there exists at least one $x \in K$ such that $f(x) = x$.

Proof. As in the proof of Brouwer theorem it suffice to work with $K \subset \mathbb{R}^d$. So let $K \subset \mathbb{R}^d$ be any convex and compact set. We can find B_r which contains whole K and thanks to the previous lemma we can find a continuous extension $F : \overline{B_r} \to K \subset \overline{B_r}$. By Brouwer theorem there exists $x_0 \in \overline{B_r} : x_0 = F(x_0) \in \text{conv} f(K) \subset K$. And so $x_0 = F(x_0) = f(x_0)$. ■

Now we take a look on assumptions of Brouwer theorem.

Example (Assumptions are necessary)**.** Continuity is clearly nedeed, for example we may consider $f(x) = \begin{cases} \arctan x, & x \in [-1, 0) \cup (0, 1] \\ \pi, & x = 0 \end{cases}$, there hold everything except of continuity and f clearly does not have any fixed point.

Counterexample for the convexity is also simple. For f we can take rotation by $\alpha \in (0, 2\pi)$ in the plane. We can take annulus as our set K, i. e. $K = \{x \in \mathbb{R}^2, 1 \leq r \leq 2\}$, which is compact but not convex. Rotation of cource does not have any fixed point in this set.

In the Euclidean space any compact set is bounded and closed (and vice versa). Closedness is clear, because we may consider $f(x) = \frac{1}{x}$ for $x \in \left[\frac{1}{2}, 1\right)$. And finally boundedness may be contradicted by $f(x) = x + \frac{1}{x}$ in $[1, \infty)$. The function is clearly continuous and in the set $[1, \infty)$ does not have any fixed point, althought the set is convex, but not bounded in the standard topology of $\mathbb{R}$. ▽

Both following simple applications of Brouwer theorem are borrowed from [25].

Example (Fundamental theorem of algebra)**.** Every polynomial of degree n with complex coefficients has at least one complex root.

Solution (Sketch)**.** For a polynomial p we define the auxiliary function

$$g(z) = \begin{cases} z - \frac{p(z)}{2+|a_0|+\cdots+|a_{n-1}|} e^{i(1-n)\varphi}, & \text{if } |z| \leq 1 \\ z - \frac{p(z)}{2+|a_0|+\cdots+|a_{n-1}|} e^{1-n}, & \text{if } |z| > 1 \end{cases}$$

where we work in polar coordinates, i. e. $z = \rho e^{i\varphi}$. Now define clever set K and show that $g(K) \subset K$ and so it has a fixed point in it, and therefore there exists root of p. ▽

Example (Frobenius)**.** Let $M \in {\mathbb{R}_+}^{d \times d}$ be a matrix. Then M has at least one eigenvalue, which is strictly positive. Moreover all components of corresponding eigenvector are non-negative.

Solution. Let us define the set

$$K = \{x \in \mathbb{R}^d; \sum_{k=1}^{d} x_k = 1, x_k \geq 0, k = 1, \ldots, d\}$$

and the mapping $f(x) = \frac{1}{||Mx||_1} Mx$, where $||x||_1 = \sum_k |x_k|$. We see that if x lies in K then all its coordinates are non-negative and at least one of them is stricly positive. Because all entries of M are non-egative, the product Mx consist of positive numbers. Altogether with definition of f we have that $f : K \to K$ and as a linear mapping is also continous. Therefore there exists fixed point $x_0 \in K$, i. e. $Mx_0 = ||Mx_0||_1 x_0$. $\bigtriangledown$

In the article [1] can be found much more examples and applications. Following theorem is relatively easy generalization (in some way) of previous theorem. It is possible to prove it at now by Brouwer theorem with help of barycentric coordinates.

Theorem 16 (Kakutani)**.** Suppose that we have non-empty, compact and convex set K in $\mathbb{R}^d$ and $f : K \to 2^K$. If the set

$$\Gamma = \{(x, y); y \in f(x)\} \subset K^2$$

is closed and $\emptyset \neq f(x)$ is closed and convex for all $x \in K$ then there exists $x \in K$ such that $x \in f(x)$.

Proof. Follows from Theorem 22. ■

Remark. Previous theorem may be used to show an existence of Nash equilibrium in game theory, see Theorem 2.23 in [25]. There is even whole book about fixed points theorems in economics and game theory, see [3].

Now we will prove generalization of Brouwer theorem for Banach spaces.

Theorem 17 (Schauder fixed point theorem)**.** Let K be homeomorphic to a closed, convex and bounded subset of a Banach space X. If $T \in \mathcal{K}(K, K)$, then T has a fixed point.

Proof. We can basically prove it by using of Brouwer theorem and by some approximation. But we can use also our strong theory of the degree and use same idea as in the proof of Brouwer.

So we will suppose that T does not have a fixed point in the K. Wlog $0 \in K$ and then define $H(t, x) = tT(x)$. But obviously $tTx \neq x$ on the boudary (for $t = 1$ it is the contradiction with previous assumption and for $t < 1$ we have that $tTx \in \text{int}K$). Therefore $\deg(\text{id} - H(0, x), K, 0) = \deg(\text{id} - H(1, x), K, 0)$ and we are done. ■

Remark. Note that in the case of infinitely dimensional Banach spaces the compactness is not same as closedness and boundedness together (as was in the finite case). We can find closed and bounded set even in Hilbert space which is not compact. Consider $M = \{e_k = (0, \dots, 1, 0, \dots); k \in \mathbb{N}\} \subset \ell^2$, then the M cannot be compact, because $||e_k - e_l||_2^2 = 2$ and so it cannot be totally bounded. Or more generally we can even take unit ball in any Banach space (of infinite dimension).

On the other hand any compact set, in any metric space, is always bounded and closed.

Example (Assumptions are necessary)**.** Interesting are just assumptions which we added. Firstly we can observe that Brouwer theorem does not hold in infinite dimension. For this we may consider (the most beautiful) Hilbert space ℓ^2 and a mapping T, which sends $x \in \ell^2$ to $y = ((1 - ||x||_2^2)^{\frac{1}{2}}, x_1, x_2, \dots) \in \ell^2$. Clearly the unit ball is mapped into the unit ball and T is continuous, because

$$||T(x^k) - T(x)||_2^2 = \sum_{n=1}^{\infty} |x_n^k - x_n|^2 + (1 + ||x^k||_2^2) - (1 + ||x||_2^2),$$

so if $x^k \to x$ as $k \to \infty$ in ℓ^2 we are done (of course we simply obtain $||x^k||_2 \to ||x||_2$ as $k \to \infty$, which is nedeed too). So assumptions of Brouwer are satisfied, but T does not have any fixed point by trivial comparison x with y.

On the other hand if we have just compact operator even in finite dimension which is not continuous (non-linear) then we also do not need to find fixed point. We can just take the real function $f(x) = \begin{cases} \arctan x, & x < 0 \\ \pi + \arctan x, & x \geq 0 \end{cases}$ on $[-1, 1]$. $\triangledown$

For the illustration we use Schauder theorem to show one abstract result about mappings.

Example. Let X be a Banach space, $T : X \to X$ is a continuous and compact mapping such that $\lim_{||x|| \to \infty} \frac{||Tx||}{||x||} = 0$. Then $\mathrm{id} - T$ is surjective mapping.

Solution. We are in the Banach space and so the compactness is characterized by the total boundedness of image of arbitrary bounded set.

We need to show that for any $y \in X$ there exists some $x \in X$ such that $x - T(x) = y$. We may assume that $y = 0$, because otherwise we will work with $\tilde{T}x = Tx + y$ instead of T. This opearor is clearly continuous and compact and moreover

$$\frac{||\tilde{T}x||}{||x||} \leq \frac{||Tx|| + ||y||}{||x||} \to 0, \text{as } ||x|| \to \infty.$$

Now we would find $x \in X$ solving $x - \tilde{T}x = 0$ which imply $x - Tx = y$. So we work with $y = 0$ and it is enough to show that $Tx = x$, i. e. T has a fixed point in X. We know that $T \in \mathcal{K}(X, X)$.

From the assumption $\lim_{||x|| \to \infty} \frac{||Tx||}{||x||} = 0$ we know that there exists some big radius $R > 0$ such that $||Tx|| < \frac{1}{2}||x||, \forall x \notin B_R$. Due to the compactness we have that $T(\overline{B_R})$ is totally bounded, and therefore also bounded. It means that there is some $M > 0$ such that $T(\overline{B_R}) \subset B_M$. We may assume $M > R$. But now

$$T(\overline{B_M}) \subset T(\overline{B_R} \cup (\overline{B_M} \setminus B_R)) = T(\overline{B_R}) \cup T(\overline{B_M} \setminus B_R) \subset B_M \cup B_{\frac{1}{2}M} \subset \overline{B_M}.$$

We now invoke the Schauder theorem and we are done. $\triangledown$

Theorem 18 (Leray-Schauder principle / Schaefer)**.** Let $T \in \mathcal{K}(X, X)$ where X is a Banach space. Suppose that any solution x of $\lambda T(x) = x, \lambda \in [0, 1]$, satisfies apriori bound $|x| \leq C$ for some $C > 0$. Then T has a fixed point.

Proof. Thanks to the existence of bound, we can introduce $R > \sup\{||x||; x = \lambda T(x) \in X, \text{ for some } \lambda \in [0, 1]\}$ and

$$Sx = \begin{cases} Tx & \text{if } ||Tx|| \leq 2R \\ 2R\frac{Tx}{||Tx||} & \text{if } ||Tx|| > 2R \end{cases}.$$

Then S is the mapping from $\overline{B_{2R}}$ into itself and both continuity and compactness is preserved. By Schauder theorem we find a point x_0 from that ball satisfying $Sx_0 = x_0$. If there holds first row from the definition of S we get the desired fixed point, i. e. $Sx_0 = x_0 = Tx_0$. But if $||Tx_0|| > 2R$, then

$$Sx_0 = x_0 = \frac{2R}{||Tx_0||} Tx_0$$

and so $x_0 \in \{x \in X; x = \lambda T(x), \text{ for some } \lambda \in [0, 1]\}$. Therefore $R > ||x_0||$, but it is a contradiction with the equality above, because it imply $||x_0|| = 2R$. ■

On applications of Schauder and Schaefer theorems we will look in much more details in the following chapter.

Theorem 19. Let X be a Banach space, U its open, bounded subset and $T \in \mathcal{K}(\overline{U}, X)$. Suppose that some point $x_0 \in U$ satisfy

$$T(x) - x_0 \neq \lambda(x - x_0), \forall x \in \partial U, \forall \lambda > 1 \ .$$

Then T has a fixed point.

Proof. To prove this result we define the homotopy by $H(t, x) = x_0 + t(Tx - x_0)$. Then $H(t, x) - x$ cannot be equal to zero for $(t, x) \in [0, 1] \times \partial U$ due to our assumption. As usual we obtain

$$\deg(\mathrm{id} - T, U, 0) = \deg(\mathrm{id} - x_0, U, 0) = \deg(\mathrm{id}, U, x_0) = 1,$$

and thus T has a fixed point. ■

From previous theorem we can get several relatively known fixed point theorems.

Example. Let X be a Banach space and $T \in \mathcal{K}(\overline{B_r}, X)$ for $r > 0$. Let us suppose that hold at least one of following statements:

(Rothe) $T(\partial B_r) \subset \overline{B_r}$.

(Altman) $||T(x) - x||^2 \geq ||Tx||^2 - ||x||^2$ on the boundary of B_r.

(Krasnoselskii) X is Hilbert and $(T(x), x) \leq ||x||^2$ on the boundary of B_r.

Then T has a fixed point.

Solution. Ad Rothe. Let us take $x_0 = 0$ and assume that $Tx = \lambda x$ for some $\lambda > 0$ and $x \in \partial B_r$. We know that $||Tx|| \leq \rho$ and simultaneously $||\lambda x|| = \lambda\rho$, and hence $\lambda \leq 1$. By the previous theorem we are done.

Ad Altman. As above consider $Tx = \lambda x$. Now we get

$$(\lambda - 1)^2\rho^2 = (\lambda - 1)^2||x||^2 \geq \lambda^2\rho^2 - \rho^2 = (\lambda^2 - 1)\rho^2,$$

so we need $(\lambda - 1)^2 \geq \lambda^2 - 1$ which imply that $\lambda < 1$, and thus it suffices to use the theorem again.

Ad Krasnoselskii. This case follows from previous one, because there holds

$$||T(x) - x||^2 = (Tx, Tx) + (x, x) - 2(Tx, x) \geq (Tx, Tx) - (x, x).$$

▽

Remark. Names of individual claims in the exercise above are from [26]. There are also some more general results for so called P-compact mappings and moreover application to approximate solution of certain equation.

3.3 Theorems with differential inclusions

So far we looked for an element x of Banach space X such that $T(x) = x$ for some mapping $T : X \to X$. In this section we will go to another setting, namely we will look for a point x in some space X which will be contained in the set $T(x)$, i. e. $x \in T(x)$, where $T : X \to 2^X$. By 2^X we mean set of all subsets of X. In other words we will not work with equations but inclusions.

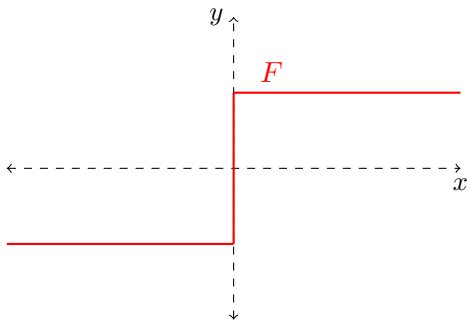

Figure 3.2: Monotone graph $\mathcal{C}$.

Such an abstraction may be quite useful. In the next chapter we will apply theorem, which we obtain in the moment, to two nontrivial examples. Now we briefly sketch what kind of problem we would like to solve.

Consider following simple model of mechanical oscillator, see paper [28]. We are looking for unknown function x of time which solves

$$\begin{aligned} mx'' + F &= h(t) \\ g(x', F) &= 0, \end{aligned}$$

where $m > 0$ is the mass, $k > 0$ is the spring constant, $h(t)$ is given external force and F is the damping force. Interesting is the case, when we are not able to rewrite $g(x', F) = 0$ neither as $x' = \tilde{g}(F)$ nor as $F = \tilde{g}(x')$. Probably the most simple (nontrivial) example is Coulomb law for friction. For this we suppose that

$$F = \begin{cases} \{F_c\}, & \text{if } x' > 0 \\ [-F_c, F_c], & \text{if } x' = 0 \\ \{-F_c\}, & \text{if } x' < 0 \end{cases},$$

where F_c is some real number.

Therefore the Coulomb problem can be formulated as

$$\begin{aligned} mx'' + F &= h(t) \\ (x', F) &\in \mathcal{C}, \end{aligned}$$

where $\mathcal{C} = \{(x, -F_c), x < 0\} \cup \{(0, y), y \in [-F_c, F_c]\} \cup \{(x, -F_c), x > 0\} \subset \mathbb{R}^2$. In this problem we are looking for a couple (x, F), where both relations above hold almost everywhere on some interval $[0, T]$ and x, x' are absolutely continuous and F is square integrable. We are able to show that for any $T > 0$ and any $x(0), x'(0)$ there exist a solution to the problem above, i. e. a couple (x, F), which satisfies given relations almost everywhere.

Key ingredient is a notion of the maximal monotone graph.

Definition 6. Let H be a Hilbert space. Then $F : H \to 2^H$ is called the multifunction. Its graph is the set $G_F = \{(x, y) \in H \times H; y \in F(x)\}$.

We say that G_F is monotone graph, if for all couples $(x_1, y_1), (x_2, y_2) \in G_F$ holds $(y_2 - y_1)(x_2 - x_1) \geq 0$, where $(\cdot)$ is the scalar product on H.

Moreover if for the given element $(x, y) \in H \times H$ and for all $(x_0, y_0) \in G_F$ holds that $(y - y_0)(x - x_0) \geq 0$ imply $(x, y) \in G_F$, then G_F is called maximal monotone graph, in other words there is no strict extension of G_F which is monotone.

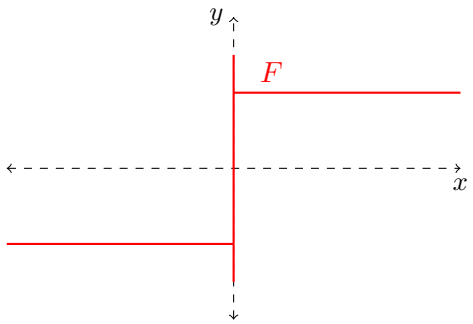

Figure 3.3: Non-monotone graph.

Remark. Note that for a given multifunction there is only one corresponding graph and vice versa. So it suffice to write what is a graph.

Two examples below we left as an exercise, it is straightforward use of a definition. Two other situations we will study in great detail in the section 4.5 as a part of two problems.

Example. Show that the "Coulomb" graph $\mathcal{C} = \{(x, -F_c), x < 0\} \cup \{(0, y), y \in [-F_c, F_c]\} \cup \{(x, -F_c), x > 0\}$ is maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

Example. Show that the mapping $\varphi : y \mapsto \{F; (y(t), F(t)) \in \mathcal{C} \text{ a. e.}\}$ (i. e. set of admissible forces) is "maximally monotone" as the mapping from $L^1(0,T)$ to $L^\infty(0,T)$ with respect to the duality

$$\langle F, y \rangle = \int_0^T F(t)y(t)\mathrm{d}y \ .$$

Remark. We thus see that in the definition we do not really need strucutre of Hilbert space, what we need is some duality pairing.

Now we will work on the proof of fixed point theorem for multifunctions. It can be used to prove the same result as in the cited article (they use approximation and variant of Banach contraction theorem). Note that it is also possible to develop theory of topological degree for multifunctions (see section 2.3 in [8]), but we choose the method as in [25].

Definition 7. Let X be a topological space, then a function $f : X \to [-\infty, \infty)$ is said to be an upper semicontinuous if the set $f^{-1}([-\infty, \alpha))$ is open for any $\alpha \in \mathbb{R}$. Similarly a $g : X \to (-\infty, \infty]$ is said to be lower semicontinuous if the set $g^{-1}((\alpha, \infty])$ is open for any $\alpha \in \mathbb{R}$.

Lemma 20 (Ky Fan inequality)**.** Let X be a locally convex topological space and $K \neq \emptyset$ its convex, compact subset. Let $f : K \times K \to \mathbb{R}$ be a function such that

(i) $f(\cdot, y)$ is lower semicontinuous for all $y \in K$,

(ii) $f(x, \cdot)$ is concave for all $x \in K$.

Then there is some $x_0 \in K$ satisfying

$$\sup_{y \in K} f(x_0, y) \leq \sup_{y \in K} f(y, y) \ .$$

Proof. Let $\varepsilon > 0$ be arbitrary. For any $x \in K$ we are able to find point $y_x \in K$ such that $f(x, y_x) > \sup_{y\in K} f(x, y) - \varepsilon$, because of definition of suprema. Thaks to part (i) we find small open set U_x containing x such that

$$f(z, y_x) > \sup_{y\in K} f(x, y) - \varepsilon \text{ for all } z \in K \cap U_x.$$

Because $K \subset \cup_{x\in K} U_x$ and K is compact we find points $x_1, \dots, x_n \in K$ such that $K \subset \bigcup_{k=1}^n U_{x_k}$.

By the Claim 9 we find $\varphi_1, \dots, \varphi_n \in \mathcal{C}(K)$, now we define a function $g(x) = \sum_{k=1}^n \varphi_k(x) y_{x_k}$ on K. Then g is obviously continuous mapping and

$$g(\mathrm{conv}(\{y_{x_1}, \dots, y_{x_n}\})) \subset \mathrm{conv}(\{y_{x_1}, \dots, y_{x_n}\}) \subset K,$$

because of $\sum_{k=1}^n \varphi_k(x) = 1$ on K. By Theorem 15 we find $\bar{x} \in K$ satisfying $g(\bar{x}) = \bar{x}$. Hence

$$\sup_{y\in K} f(y, y) \geq f(\bar{x}, \bar{x}) \overset{\text{(ii)}}{\geq} \sum_{k=1}^n \varphi_k(\bar{x}) f(\bar{x}, y_{x_k}) \geq \sum_{k=1}^n \varphi_k(\bar{x})(\sup_{y\in K} f(x_k, y) - \varepsilon) \geq$$
$$\geq \inf_{x\in K} \sup_{y\in K} f(x, y) - \varepsilon = \sup_{y\in K} f(x_0, y) - \varepsilon,$$

where the last equality holds because it is well-known that lower semicontinuous function attains its minimum on a compact set. Letting $\varepsilon \to 0_+$ we obtain the desired inequality. ■

Definition 8. Let X be a topological space and K its non-empty subset. Then a mapping (or a multifunction) $F : K \to 2^K$ is said to be an upper semicontinuous if for any $x_0 \in K$ and any open $G \supset F(x_0)$ there is an open neighbourhood $U(x_0)$ such that $F(x) \subset G$ for all $x \in U(x_0)$.

Lemma 21. Let X be a topological space and $K \neq \emptyset$ its compact subset. Suppose that $F : K \to 2^K$ has $F(x)$ closed for all $x \in K$. Then following is equivalent

(i) F is an upper semicontinuous.

(ii) Graph $G_F = \{(x, y) \in K \times K; y \in F(x)\}$ is closed in $K \times K$.

Proof. Ad (i) $\implies$ (ii). Let F be an upper semicontinuous mapping and $(x_0, y_0) \in K \times K \setminus G_F$, and therefore $y_0 \notin F(x_0)$. Since K is compact set we are able to find (see Claim 10) two disjoint open sets U_1, U_2 such that $y_0 \in U_1$ and $F(x_0) \subset U_2$. Because of upper semicontinuity of F we are able to find an open set V which contains x_0 such that $F(x) \subset U_2, \forall x \in V$. Therefore the open set $V \times U_1$, which contains (x_0, y_0), does not intersect G_F and so, by definition of closed set, G_F is closed.

Ad (ii) $\implies$ (i). Let G_F be a closed set. Let $x \in K$ be an arbitrary point and U be an arbitrary open set which contains $F(x_0)$. Suppose that F is not upper semicontinous at x. Because of it for any V neigbourhood of x we are able to find $y \in V$ such that $F(y) \not\subset V$. Because $(y, f(y)) \in G_F$ we obtain $G_F \cap (\overline{V} \times U^c) \neq \emptyset$. Because both G_F and $\overline{V} \times U^c$ are closed they are also compact and so there exists point $(x_0, y_0) \in K \times K$ such that

$$(x_0, y_0) \in \bigcap_V \left(G_F \cap (\overline{V} \times U^c) \right),$$

it is due to the Claim 11 (and remark). Because the intersection is over all open sets containing x we obtain that $x = x_0$ (all sets in the intersection are compact), and since $y_0 \in F(x_0) = F(x)$ we get $y_0 \in U$. But it is desired contradiction because $y_0 \in U^c$. ■

Theorem 22 (Kakutaini - Ky Fan fixed point theorem)**.** Let X be a locally convex topological space and $K \neq \emptyset$ its convex, compact subset. Let $F : K \to 2^K$ be an upper semicontinuous mapping such that each $F(x)$ is non-void, convex and closed set. Then F has a fixed point, i. e. $x_0 \in F(x_0)$ for some $x_0 \in K$.

Proof. Suppose that the assertion is not true. Then we have disjoint sets $\{x\}$ and $F(x)$, for every $x \in K$. Both of them are non-empty, closed (and thus also compact, because K is compact) and whole space X is locally convex. So we invoke Claim 12 to obtain an existence of $\Lambda_x \in X^*, \gamma_x \in \mathbb{R}$ such that

$$\sup_{z \in F(x)} \Re\Lambda_x(z) < \gamma_x < \Re\Lambda_x(x).$$

The set $U_x = \{z \in X; \Re\Lambda_x(z) < \gamma_x\}$ is open from continuity of Λ_x and contains $F(x)$. From the definition of upper semicontinuous multifunction there exists an open set V_x containing x such that $F(y) \subset U_x, \forall y \in V_x \cap K$.

Again from continuity we may found W_x neigbourhood of x so small that $\Re\Lambda_x(y) > \gamma_x, \forall y \in W_x \cap K$, and simultaneously $\Re\Lambda_x(z) < \gamma_x, \forall y \in W_x \cap K, \forall z \in F(y)$. Because $K \subset \bigcup_{x \in K} W_x$ and K is a compact set, we are able to find points $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{k=1}^n W_{x_k}$. We use Claim 9 to find continuous functions $\varphi_1, \ldots, \varphi_n$ and define the mapping $f : K \times K \to \mathbb{R}$ by

$$f(x, z) = \sum_{k=1}^n \varphi_k(x) \Re\Lambda_{x_k} x - \sum_{k=1}^n \varphi_k(x) \Re\Lambda_{x_k} z.$$

Clearly f is lower semicontinuous in the first variable and concave in the second one and so, by Lemma 20, there exists point $x_0 \in K$ such that

$$\sup_{z \in K} f(x_0, z) \leq \sup_{z \in K} f(z, z) = 0.$$

Therefore, for $x \in F(x_0)$, we obtain

$$\sum_{k=1}^n \varphi_k(x_0) \Re\Lambda_{x_k}(x) \leq \sum_{k=1}^n \varphi_k(x_0) \Re\Lambda_{x_k}(z)$$
$$\sum_{k=1}^n \varphi_k(x_0) \gamma_{x_k} < \sum_{k=1}^n \varphi_k(x_0) \gamma_{x_k},$$

which is clerly contradiction. And so the original statement holds. ■

Remark. If X is a normed space the proof can be done more quickly, but we need the statement in this generality, because we will use $X = L^2$ with the weak topology.

Remark. Once more time we repeat that lemmata 20, 21 and Theorem 22 are borrowed from [25].

4. Applications

Nature is written in mathematical language.

Galileo Galilei

4.1 Ordinary differential equations

It is natural to try to use our tools on the most basic problem, i. e. local existence of a solution to the given ordinary differential equation with some initial condition.

Example (Peano theorem)**.** Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. Then there exists some time $\tau > 0$ such that there is a solution to the problem

$$\begin{aligned} x'(t) &= f(t, x(t)) \text{ for } t \in (0, \tau), \\ x(0) &= x_0. \end{aligned}$$

Solution. Thanks to the continuity of f there are some (small enough) $t_1 > 0, r > 0$ such that

$$|f(t, x(t))| \leq M, \forall (t, x) \in [-t_1, t_1] \times \overline{B_r(x_0)},$$

for some $M > 0$.

Take $\tau \in (0, t_1)$ such that $Mt_0 < r$ and denote

$$K = \{x(\cdot) \in \mathcal{C}([0, \tau], \mathbb{R}^d); x(0) = x_0, |x(t) - x_0| \leq r, \text{ for } t \in [0, \tau]\} .$$

Clearly K is a closed, convex and bounded subset of Banach space $\mathcal{C}([0, \tau], \mathbb{R}^d)$. Now we define an operator $T : \mathcal{C}([0, \tau], \mathbb{R}^d) \to \mathcal{C}([0, \tau], \mathbb{R}^d)$ by the formula

$$Tx(t) = x_0 + \int_0^t f(s, x(s))\mathrm{d}s, x(\cdot) \in \mathcal{C}([0, \tau], \mathbb{R}^d)).$$

By differentiation follows that if we find fixed point of T we get the solution to the original differential equation. From the continuity of f in the second variable it easy follows also continuity of T. Due to our choice of τ we have that $T : K \to K$ and by the Arzelà–Ascoli theorem (Claim 15) we obtain also compactness of T. It is due to the estimate

$$|Tx(t) - Tx(t')| \leq \text{sign } (t - t') \int_{t'}^t |f(s, x(s))|\mathrm{d}s \leq M|t - t'|$$

which imply equicontinuity. Now we invoke Schauder theorem and get an existence of $x(t)$ satisfying

$$x(t) = x_0 + \int_0^t f(s, x(s))\mathrm{d}s, \forall t \in (0, \tau).$$

And our result follows by differentiating. ▽

We will show an existence of periodic solution to a certain problem.

Example. Let $T \geq 0$ be fixed and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that $f(t + T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. Moreover suppose that the following hold:

(i) $\forall x \in \mathbb{R}^d \exists r_x > 0, L_x > 0 : |f(t, y) - f(t, x)| \leq L_x|y - z|, \forall t \in \mathbb{R}, \forall y, z \in B_{r_x}(x),$

(ii) $\exists r > 0 : f(t,x) \cdot x < 0, \forall t \in [0,T], \forall x \in \mathbb{R}^d$ with $|x| = r$.

Then there exists solution to the equation

$$\begin{aligned} x'(t) &= f(t, x(t)) \text{ for } t \in [0, \infty), \\ x(0) &= x(T). \end{aligned}$$

Solution. By above result we know that for any fixed $x_0 \in \overline{B_r}$ there exists some solution to the problem

$$\begin{aligned} x'(t) &= f(t, x(t)), \\ x(0) &= x_0, \end{aligned}$$

at least for $t \in (0, t_0)$, where $t_0 > 0$.

This solution is already unique, because if there are two such solutions then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|x(t) - y(t)|^2 &= 2(x'(t) - y'(t), x(t) - y(t)) = 2(f(t, x(t)) - f(t, y(t)), x(t) - y(t)) \\ &\leq L_{x_0}|x(t) - y(t)|^2, t \in (0, t_1), t_1 \leq t_0, \end{aligned}$$

where we used the equation and well-known Cauchy–Bunyakovsky–Schwarz inequality $|(u,v)|^2 \leq |(u,u)||(v,v)|$ for $u, v \in \mathbb{R}^d$. Thanks to the Grönwall inequality (Claim 18) we have $x(t) = y(t)$ for $t \in (0, t_1)$. Hence, uniqueness of the whole solution follows.

Due to (ii) we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 = 2(x'(t), x(t)) = 2(f(t, x(t)), x(t)) < 0, \text{ if } x(t) = r \ .$$

So solution x must stay in ball $\overline{B_r}$ for all $t \in [0, t_0]$, and thus we can extend this solution up to infinity (in time) and x will be in the same ball for $t \geq 0$.

Let's define operator $T : \overline{B_r} \to \overline{B_r}$ by $Ty = x(y, T), \forall y \in \overline{B_r}$, and $x(y,t)$ is, of course, our (unique) solution of the problem above, where $x_0 = y$. Similarily as above we are able to show continuous dependence of the solution on data of problem and from it follows continuity of the operator T, and thus, by Brouwer fixed point theorem, there exists some solution to our periodic problem. $\bigtriangledown$

Now we will consider different condition on f and add one more term to the equation.

Example. Let $F : \mathbb{R}^d \to \mathbb{R}$ be an even, $\mathcal{C}^1$ function such that its gradient is lipschitz and $f : \mathbb{R} \to \mathbb{R}^d$ be a continous function with antiperiodic property, i. e. $f(t+T) = -f(t), \forall t \in \mathbb{R}$, and some fixed number T. Then the problem

$$\begin{aligned} x'(t) &= \nabla F x(t) + f(t), \\ x(t+T) &= -x(t), \end{aligned}$$

has a solution for $t \in \mathbb{R}$.

Solution. If we have such a solution, then for all times we have

$$|x'(t)|^2 = (\nabla F x(t), x'(t)) + (f(t), x'(t)).$$

But

$$\int_0^T (\nabla F x(t), x'(t)) \mathrm{d}t = Fx(T) - Fx(0) = F(-x(0)) - Fx(0) = 0$$

and by Hölder inequality and dividing (if we cannot divide, it holds trivially) we obtain

$$\left(\int_0^T |x'(t)|^2 \mathrm{d}t\right)^{1/2} \leq \left(\int_0^T |f(t)|^2 \mathrm{d}t\right)^{1/2}.$$

Notice that

$$\max_{t\in[0,T]} |x(t)| = \frac{1}{2} \max_{t\in[0,T]} \left(\int_0^t x' + \int_T^t x'\right) \leq \frac{\sqrt{T}}{2} \left(\int_0^T |f(t)|^2 \mathrm{d}t\right)^{1/2} =: M,$$

where the first equality holds thanks to differentiability of x and the assumption $x(t+T) = -x(t)$. The inequality is an application of Hölder inequality again and of the estimate above. Now let's consider even, smooth function φ such that $\varphi(x) = 1, |x| \leq M$ and $\nabla\varphi(x) = 0, |x| > 2M$.

Let's define a new problem

$$x'(t) + x(t) = \nabla\left[\varphi(x(t))\left(Fx(t) + \frac{1}{2}|x(t)|^2\right)\right] + f(t) \text{ for } t \in \mathbb{R}$$

with initial condition $x(0) = y \in \mathbb{R}^d$. For each solution to this problem we have, due to the boundedness of $\nabla\varphi$, following estimate (for some $L > 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 + |x(t)|^2 \leq 2L|x(t)| + 2|f(t)||x(t)|.$$

Thus if we know that $|x(0)| \leq C$, for some $C > 0$, then by integration and using part of inequality above we obtain also $|x(T)| \leq C$.

Similarly as in the previous example we define an operator $S : \mathbb{R}^d \to \mathbb{R}^d$ by $Sy = -x(T)$, where x is the unique solution (follows from classical theory) of our new problem with initial condition $x(0) = y$. This operator is clearly continuous and again by Brouwer there exists some $y \in \mathbb{R}^d$ such that $Sy = y$. In other words $x(T) = -y = -x(0)$, thus $|x(T)| \leq M, \forall t \in \mathbb{R}$, and so x is the solution to the given problem, because

$$x'(t) - f(t) + x(t) = \nabla\left[\varphi(x(t))\left(Fx(t) + \frac{1}{2}|x(t)|^2\right)\right] = \nabla Fx(t) + x(t).$$

$\bigtriangledown$

Corollary. If A is an $n \times n$ symmetric matrix, then by choice $Fx = \frac{1}{2}(Ax, x)$ we have $\nabla F = A$, and therefore there exists a solution to the problem

$$\begin{aligned} x'(t) &= Ax(t) + f(t) \text{ for } t \in \mathbb{R}, \\ x(t+T) &= -x(t). \end{aligned}$$

In the chapter II in [9] reader may found much more wide discussion of a periodic solutions of certains ode's. In the end we solve one simple ordinary differential equation of higher order.

Example. Show that there exists a function $x \in \mathcal{C}([0,1])$ solving

$$\begin{aligned} x''(t) &= f(t, x(t)), \\ x(0) &= x(1) = 0, \end{aligned}$$

where $f(t,\cdot)$ is continuous for a. a. $t \in [0,1]$ and $f(\cdot, y)$ is measurable for all $y \in \mathbb{R}$ and $|f(t,y)| \leq m(y)$ for a. a. $y \in \mathbb{R}$ and all $t \in [0,1]$, where $m > 0$ is an integrable function.

Solution. Its is well-known (if not then it is an exercise) that x solves given equation if and only if x solves equation

$$x(t) = \int_0^1 k(t,s) f(s, x(s)) \mathrm{d}s,$$

where $k(t,s) = \begin{cases} s(1-t), & \text{if } 0 < s \leq t < 1 \\ t(1-s), & \text{if } 0 < t < s < 1 \end{cases}$.

Now we consider an operator $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ which maps any x to the function y given by the formula

$$y(t) = \int_0^1 k(t,s) f(s, x(s)) \mathrm{d}s.$$

At first we see that it is well defined object. We also see that T is a continuous opearator, it follows from continuity of f in the second variable. Now we show that it is compact, because we use Arzelà–Ascoli theorem (Claim 15) again. We take two points $t, t' \in [0,1]$ and arbitrary $\varepsilon > 0$. We obtain an estimate

$$|y(t) - y(t')| \leq \int_0^1 |(k(t,s) - k(t',s)) f(s, x(s))| \mathrm{d}s.$$

Because k is continuous in the first variable we are able to find δ so small that

$$|k(t,s) - k(t',s)| \leq \delta < \frac{\varepsilon}{||m||_1}.$$

With this δ and assumption on f we get the equicontinuity. And, of course, if $||x|| < M$ for some $M > 0$ we get $||y|| < M$ for M big enough. And so, by Schauder theorem, we find desired fixed point of T, and thus the solution to the given differential equation. $\bigtriangledown$

4.2 Partial differential equations

In this section we solve two simple partial differential equations (pde) by methods of fixed point theorems. Despite the fact that those equations are rather simple its take some time to solve them. Moreover we suppose that reader is more or less familiar with standard notions and methods of pde's. One more pde will be solved in section 4.5 devoted to the differential inclusion.

Example. Let Ω be an open, bounded Lipschitz subset of $\mathbb{R}^d$, $b : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded mapping and $f \in L^2(\Omega)$. Then there exists $u \in W_0^{1,2}(\Omega)$ such that for any $\varphi \in W_0^{1,2}(\Omega)$ there holds

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega b(\nabla u)\varphi = \int_\Omega f\varphi,$$

i. e. u weakly solves the problem

$$\begin{aligned} -\triangle u + b(\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Solution. Let us take arbitrary $v \in W_0^{1,2}(\Omega)$ and consider problem

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi - \int_\Omega b(\nabla v)\varphi.$$

We obtain the existence of single $u \in W_0^{1,2}(\Omega)$ by standard use of Lax-Milgram theorem (Claim 4). To find the solution of original problem we introduce a mapping

$$T : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$$
$$v \mapsto u$$

which is well defined, thanks to the existence of u above. We want to show an existence of fixed point of T via Schauder theorem. At first we need continuity. To show this we consider $v_n \to v$ in $W_0^{1,2}(\Omega)$, i. e. we have equations

$$\int_\Omega \nabla u_n \cdot \nabla \varphi = \int_\Omega f\varphi - \int_\Omega b(\nabla v_n)\varphi$$

for arbitrary $\varphi \in W_0^{1,2}(\Omega)$. In particular we have $\nabla v_n \to \nabla v$ in $L^2(\Omega)$ and due to the Nemytskii theorem (Claim 6) and properites of b we obtain $b(\nabla v_n) \to b(\nabla v)$ in $L^2(\Omega)$. Thanks to the estimate (where $\eta > 0$ is arbitrarily small)

$$||\nabla u_n||_2^2 \leq ||f||_2||u_n||_2 + ||b(\nabla v_n)||_2||u_n||_2 \leq \frac{1}{\eta}||u_n||_2^2 + c(||f||_2, M, \Omega, \eta)$$

where we used Young inequality and boundedness of b. Because Ω is Lipschitz we can use Poincare inequality (Claim 5) to get the estimate

$$||u_n||_{1,2}^2 \leq C$$

where constant C depends only on our data. Thanks to the relfexivity of $W_0^{1,2}(\Omega)$ we find subsequence of $(u_n)_n$ which weakly converges to u. In other words $\int_\Omega \nabla u_n \cdot \nabla\varphi \to \int_\Omega \nabla u \cdot \nabla\varphi$. We already said that $\int_\Omega f\varphi - \int_\Omega b(\nabla v_n)\varphi \to \int_\Omega f\varphi - \int_\Omega b(\nabla v)\varphi$. Those facts shows us that T is continuous operator from $W_0^{1,2}(\Omega)$ into itself.

Now we find big ball B_R in $W_0^{1,2}(\Omega)$ such that $T : B_R \to B_R$. This ball is bounded, convex and closed set, which we use in Schahuder theorem. But this part we already did, because we are able to find constant C which does not depend on choice of v such that $||u||_{1,2} \leq C$. So it suffice to take $R = C$.

It remains to show compactness of T. We will prove it from the definition. So we consider bounded set $\{v_n\}_n \subset W_0^{1,2}(\Omega)$ and we need to show that its image is totally bounded. We have our formulation for v_n, v_m and take the difference of formulations for those two functions. We obtain

$$\int_\Omega \nabla(u_n - u_m) \cdot \nabla\varphi = -\int_\Omega (b(\nabla v_n) - b(\nabla v_m))\varphi$$

for all $\varphi \in W_0^{1,2}(\Omega)$. We choose $\varphi = u_n - u_m \in W_0^{1,2}(\Omega)$, which give us the estimate

$$||\nabla(u_n - u_m)||_2 \leq c||u_n - u_m||_1,$$

where we used again the boundedness of b.

This estimate is useful, because we now invoke compact embedding theorem (Claim 7), which in particular says $W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$. Therefore $T(\{v_n\})$ is totally bounded in $L^1(\Omega)$ and so the sequence $(u_n)_n$ is Cauchy in $L^1(\Omega)$. But the estimate above show that this sequence is Cauchy even in $W_0^{1,2}(\Omega)$. Because it is complete space, we have some limit u. Therefore $\{u_n\}$ is totally bounded in $W_0^{1,2}(\Omega)$ and so T is compact. Now we apply Schauder theorem to get the existence of solution to the given problem. $\triangledown$

Example. Let Ω be an open, bounded Lipschitz subset of $\mathbb{R}^d$ and $f \in L^2(\Omega)$ is a non-negative fucntion. Then there exists unique and non-egative $u \in W_0^{1,2}(\Omega)$ such that for any $\varphi \in W_0^{1,2}(\Omega)$ there holds

$$\int_\Omega \nabla u \cdot \nabla\varphi - \int_\Omega \frac{1}{1+u}\varphi = \int_\Omega f\varphi.$$

Solution. Similarly as in example above we solve auxiliary problem

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi + \int_\Omega \frac{1}{1+v}\varphi,$$

where $v \in L^2(\Omega)$ is fixed and $v \geq 0$. We are able to estimate

$$||u||_{1,2} \leq C,$$

where C depends on f and Ω. Note that $0 \leq \frac{1}{1+v} \leq 1$. So there exist unique $u \in W_0^{1,2}(\Omega)$ solving auxiliary problem for all $\varphi \in W_0^{1,2}(\Omega)$. Define $X = \{f \in L^2(\Omega); f \geq 0 \text{ a. e. in } \Omega\}$, it is obviously closed and convex subset of $L^2(\Omega)$.

Now we consider an operator $T : X \to X$ which maps v to u. We are again looking for a fixed point. Because we know that $u \in W_0^{1,2}(\Omega)$ and because of Claim 7, which says $W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ we immediately have compactness of T into L^2 (we do not know that $u \geq 0$ yet). Thanks to the estimate above we also see that there exists ball $B_R \subset L^2(\Omega)$ for $R > 0$ big enough such that $T : B_R \to B_R$. Continuity of T is simple. On the left hand side of equality above we use weak convergence, which follows again from the estimate. On the right hand side we use that $\frac{1}{1+v_n} \to \frac{1}{1+v}$ in $L^2(\Omega)$ if $v_n \to v$ in $L^2(\Omega)$.

But we assumed that $v \geq 0$ so we need to show that $u \geq 0$ too. It is not hard, we take as a test function (it is possible due to Claim 3) $\min\{0, u\}$, so

$$\int_\Omega \nabla u \cdot \nabla(\min\{0,u\}) = \int_\Omega \left(f + \frac{1}{1+v}\right) \min\{0,u\} \leq 0$$

where the inequality holds thanks to our assumption that $f \geq 0$. But we see that

$$\int_\Omega |\nabla(\min\{0,u\})|^2 = \int_\Omega \nabla u \cdot \nabla(\min\{0,u\}) \leq 0$$

which implies that the function $\min\{0, u\}$ is a constant. Because u has zero trace, the constant must be zero, and therefore $u \geq 0$. Now we use Schauder theorem to get an existence of non-negative u solving the original problem.

Remains to show uniqueness. As ussual we consider that we have two solutions u_1, u_2, take the diference and test by $u_1 - u_2$ (we see that it is in $W_0^{1,2}(\Omega)$). We get the equality

$$\int_\Omega |\nabla(u_1 - u_2)| = \int_\Omega \frac{u_2 - u_2}{(1+u_1)(1+u_2)}(u_1 - u_2).$$

Its right hand side is obviously non-positive and so $u_1 - u_2$ must be a constant. Because all solutions lie in the space $W_0^{1,2}(\Omega)$ we see that the constant must be zero and the solution is therefore unique. $\triangledown$

Following example we left as an exercise for the reader. It can be solved in the same way as both examples above.

Example. Let Ω be an open, bounded and Lipschitz subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Then there exists $u \in W_0^{1,2}(\Omega)$ weakly solving

$$\begin{aligned} -\triangle u + g(u) &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

4.3 Integral equations

In the section about applications in ode's we saw that it is useful to transform ordinary differential equation into some integral form. Because then we have a good chance to obtain compactness of corresponding integral operator.

Nevertheless there exist many physical situations which are naturally described by integral equation or by some system of equations, where at least one of them is in this form. Great book which deals with integral equations is [14]. As the motivation we briefly desribe one simple model.

Following system of equations desribes a nuclear reactor with linear feedback

$$\begin{aligned}\frac{\mathrm{d}p(t)}{\mathrm{d}t} &= \frac{k(t)-\beta}{l}p(t) + \sum_{i=1}^{m}\lambda_i c_i(t),\\ \frac{\mathrm{d}c_i(t)}{\mathrm{d}t} &= \frac{\beta_i}{l}p(t) - \lambda_i c_i(t), i = 1,2,\ldots,m,\\ k(t) &= -\int_0^t K(t-s)(p(s)-p_0)\mathrm{d}s + f(t),\end{aligned}$$

for $t \geq 0$. Above p denotes the neutron density, k is reactivity, B_i are the delayed neutron fractions, $\beta = \sum_{i=1}^{m}\beta_i$, l is the neutron average lifetime and λ_i are decay constants of the delayed neutrons. Finally functions c_i are delayed neutrons densities, K is the reactivity feedback kernel, f is the external control and p_0 denotes the equlibrium power level.

In the mentioned book we find much more examples and also plenty of references. Now we show a few simple results. They are natural examples of applications of fixed point theorems, but deep investitagation, which they truly deserve, reaches far beyond our text.

Example. Let $g : I \times I \times \mathbb{R}^d \to \mathbb{R}^d$, where $I = [0,T]$, be a continuous function. Let $\lambda \in \mathbb{R}$ and $f \in \mathcal{C}(I,\mathbb{R}^d)$ are given. If λ is appropriately small, then there exists continuous solution x to the equation

$$x(t) - \lambda T(x)(t) = f(t),$$

where $T(x)(t) = \int_0^t g(s,t,x(s))\mathrm{d}s$.

Solution. We will use Schauder theorem. We will work with the operator $x \to \lambda Tx - f$ Firstly we need continuity, but it is clear from the continuity of $y \to g(s,t,y)$ and the fact that f does not depend on x. Now we find ball $B_r \subset \mathcal{C}(I)$ such that $(\lambda T - f)(B_r) \subset B_r$. Take $r > 0$ arbitrary and denote $M = 2\max\{\max\{|g(s,t,x(s))|; (s,t,x(s)) \in I \times I \times \overline{B_r}\}; \max\{|f(t)|; t \in I\}\}$. For $|\lambda| < \frac{1}{T}$ we now obtain

$$|\lambda Tx(t) - f| = \left|\lambda\int_0^t g(s,t,x(s))\mathrm{d}s - f\right| \leq \frac{\lambda TM}{2} + \frac{M}{2} < M.$$

Remains to show its compactness. But it is simple with aid of Arzelà–Ascoli theorem (Claim 15), i. e. we show equicontinuity. Let us take $t, t' \in [0,T]$ and assume (for sake of simplicity) that $t > t'$, then we get

$$\begin{aligned}|Tx(t) - Tx(t')| &= \left|\int_0^t g(s,t,x(s))\mathrm{d}s - \int_0^{t'} g(s,t',x(s))\mathrm{d}s\right| \leq\\ &\leq \int_0^{t'} |g(s,t,x(s)) - g(s,t',x(s))|\mathrm{d}s + \int_{t'}^t |g(s,t,x(s))|\mathrm{d}s \leq\\ &\leq \int_0^{t'} |g(s,t,x(s)) - g(s,t',x(s))|\mathrm{d}s + |t-t'|M.\end{aligned}$$

Now we use the continuity of g in the second variable and altogether we see that the right hand side can be made arbitrarily small. Therefore we get

$$|\lambda Tx(t) - f(t) - (\lambda Tx(t') - f(t'))| \leq |f(t) - f(t')| + \lambda |Tx(t) - Tx(t')|$$

which is also arbitrarily small thanks to the continuity of f and so desired equicontinuity follows. By Schauder theorem we get an existence of some continuous function x solving $x - \lambda Tx = f$. $\bigtriangledown$

Example. Let us consider the equation

$$x(t) = f(t) + \int_0^t a(t-s)g(x(s))\mathrm{d}s, t \geq 0,$$

where $a \in L^1_{loc}([0,\infty), \mathbb{R}^{d\times d}), g \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ and $f \in \mathcal{C}([0,\infty), \mathbb{R}^d)$. Then there exists a continuous function $x(t) : [0,\infty) \to \mathbb{R}^d$ which satisfies equation above.

Solution. Firstly we will show a local existence. So, let us choose positive $M > 2\sup_{t\in[0,1]} |f(t)|$ and a number ε ("small time") for which

$$\sup_{|v|\leq M} |g(v)| \cdot \int_0^\varepsilon |a(s)|\mathrm{d}s < \frac{M}{2} \text{ and } \varepsilon < 1,$$

where we used a local integrability and of course continuity.

Let's define the operator

$$T(\varphi)(t) = f(t) + \int_0^t a(t-s)g(\varphi(s))\mathrm{d}s \text{ for } t \in [0,\varepsilon].$$

As a domain of this operator we will consider the set

$$K = \{\varphi \in \mathcal{C}([0,\infty), \mathbb{R}^d); |\varphi(t)| \leq M, \forall t \in [0,\varepsilon])\} .$$

We intend to show that $T(K) \subset K$ for later use of Schauder theorem. Because of continuity of g we immediately get also continuity of $T(\varphi)(\cdot)$ on our small interval and for $\varphi \in K$ we have

$$|T(\varphi)(t)| \leq \sup_{t\in[0,\varepsilon]} |f(t)| + \sup_{|v|\leq M} |g(v)| \cdot \int_0^\varepsilon |a(t)|\mathrm{d}t \leq 2 \cdot \frac{M}{2}$$

and so T maps K into K.

Of course K is a bounded, convex and closed set in the Banach space of continuous functions. We need to show compactness of T. For this we use Arzelà–Ascoli theorem (Claim 15), i. e. we want to show equicontinuity.

Let $\eta > 0$ and take $t_1 < t_2$ so that $|t_1 - t_2| \leq \delta$, where δ will be specified later (for $t_1 > t_2$ we would work similarly). We thus obtain an estimate

$$|T\varphi(t_1) - T\varphi(t_2)| \leq |f(t_1) - f(t_2)| + \left|\int_0^{t_1} (a(t_1-s) - a(t_2-s))g(\varphi(s))\mathrm{d}s\right| + \\ + \left|\int_{t_1}^{t_2} a(t_2-s)g(\varphi(s))\mathrm{d}s\right| .$$

Thanks to the well-known absolute continuity of Lebesgue integral and integrability of a on $(0,\varepsilon)$ we find a constant $C > 0$ so big that

$$\int_0^\varepsilon a(t)\chi_{\{a>C\}} < \eta.$$

Now we choose δ so small that $|f(t_1) - f(t_2)| < \eta$ and estimate the second integral

$$\int_{t_1}^{t_2} a(t_2 - s)g(\varphi(s))\mathrm{d}s \leq \int_{t_1}^{t_2} |(a(t_2 - s)\chi_{\{a>C\}} + a(t_2 - s)\chi_{\{a\leq C\}})g(\varphi(s))|\mathrm{d}s \leq$$
$$\leq \eta \sup_{|v|\leq M} |g(v)| + C|t_1 - t_2| \sup_{|v|\leq M} |g(v)|.$$

Therefore we also want δ so small that $|t_1 - t_2| < \eta$. Now we estimate the first integral. On the beginning we decompose as above

$$\int_0^{t_1} \Big|(a(t_1 - s) - a(t_2 - s))(\chi_{\{a>C\}} + \chi_{\{a\leq C\}})g(\varphi(s))\Big| \mathrm{d}s \leq$$
$$\leq 2\eta \sup_{|v|\leq M} |g(v)| + \int_0^{t_1} \Big|(a(t_1 - s) - a(t_2 - s))\chi_{\{a\leq C\}}g(\varphi(s))\Big| \mathrm{d}s.$$

We invoke Lusin theorem (Claim 2) to find $E \subset (0, t_1)$ such that $|E| < \eta$ and $a(t_1 - s) - a(t_2 - s)$ is continuous on $(0, t_1)\setminus E$. Last condition on δ is that $|(a(t_1 - s) - a(t_2 - s)| < \eta$ for $s \in (0, t_1) \setminus E$, it is possible due to the continuity. We obtain

$$\int_0^{t_1} \Big|(a(t_1 - s) - a(t_2 - s))(\chi_{\{a>C\}} + \chi_{\{a\leq C\}})g(\varphi(s))\Big| \mathrm{d}s \leq$$
$$\leq 2\eta \sup_{|v|\leq M} |g(v)| + \int_E \Big|(a(t_1 - s) - a(t_2 - s))\chi_{\{a\leq C\}}g(\varphi(s))\Big| \mathrm{d}s +$$
$$+ \int_{(0,t_1)\setminus E} \Big|(a(t_1 - s) - a(t_2 - s))\chi_{\{a\leq C\}}g(\varphi(s))\Big| \mathrm{d}s \leq$$
$$\leq 2\eta \sup_{|v|\leq M} |g(v)| + 2C\eta \sup_{|v|\leq M} |g(v)| + t_1\eta \sup_{|v|\leq M} |g(v)|.$$

Altogether

$$|T\varphi(t_1) - T\varphi(t_2)| \leq \eta + (1 + C + 2 + 2C + t_1)\eta \sup_{|v|\leq M} |g(v)|.$$

It proves the equicontinuity and the compactness follows. Now we apply Schauder theorem to find an $x \in K$ which is a fixed point of T and so

$$x(t) = T(x)(t) = f(t) + \int_0^t a(t - s)g(x(s))\mathrm{d}s \text{ for } t \in [0, \varepsilon] \ .$$

Now we will extend our local solution for all positive times t. Define a function

$$\tilde{f}(t) = f(t + \varepsilon) + \int_0^{\varepsilon} a(t + \varepsilon - s)g(x(s))\mathrm{d}s, t \geq 0.$$

By repeating the procedure above we can find a solution $\tilde{x}$ on $[0, \tilde{\varepsilon}]$ solving an equation

$$\tilde{x}(t) = \tilde{f}(t) + \int_0^t a(t - s)g(\tilde{x}(s))\mathrm{d}s.$$

We extend our solution $x(t) := \tilde{x}(t - \varepsilon)$ for $(\varepsilon, \varepsilon + \tilde{\varepsilon}]$. Thanks to continuity of our data (and use of both equations) we have continuity of $x(\cdot)$ at the point ε. By this way we can extend our local solution x as long as it is bounded. By ussual application (as in standard ode's) of Zorn lemma we can find at least one maximal solution. $\bigtriangledown$

In closing we state one abstract theorem about integral equations as an illustration of results which may be obtained.

Example. Let X be the Banach space of functions defined on the interval $I \subset \mathbb{R}$ with values in $\mathbb{R}^d$ and K its closed, bounded and convex subset. Let there be an operator $F \in \mathcal{K}(K, X)$. Let $k : I \times I \to \mathbb{C}^{n\times n}$ be a measurable function (called Fredholm kernel) and denote by $r : I \times I \to \mathbb{C}^{n\times n}$ so called Fredholm resolvent of k on I if

$$r + \int_I r(t,u)k(u,s)\mathrm{d}u = r + \int_I k(t,u)r(u,s)\mathrm{d}u = r \text{ on } I \times I.$$

Suppose that

(i) the mapping $x \mapsto \int_I r(t,s)Gx(s)\mathrm{d}s$ is continuous and compact from K into X,

(ii) the mapping $x \mapsto F(x) + \int_I r(t,s)Gx(s)\mathrm{d}s$ goes from K into K.

Then the problem

$$x(t) = F(x)(t) + \int_I r(t,s)Gx(s)\mathrm{d}s, t \in I,$$

has at least one solution $x \in K$.

Solution. See Theorem 11.2.2 in [14]. ▽

4.4 Model with non-monotone dependence

4.4.1 The problem

In the introduction to the whole thesis we mentioned problem of a certain mechanical oscillator. We will give a promised proof to more general situation than in [27]. We will deal with the following system

$$\begin{aligned} x''(t) + F(t) + G(t) &= \mathcal{F}(t), \\ x'(t) &= g(G(t)), \\ x(t) &= f(F(t)), \\ x(0) = x_0,\ x'(0) &= x_1, \end{aligned} \tag{P}$$

where $t \in [0,T]$ for given $T \geq 0$. We suppose that $\mathcal{F} \in L^1(0,T)$, $f, g : \mathbb{R} \to \mathbb{R}$ are given functions and x, F, G are unknown real functions of time.

Let us make small remark for a slight insight into background of this model. Those equations describe the vibratory motion of a spring, dashpot and mass. Symbol x is the displacement, $\mathcal{F}$ the externally applied force on the mass and F and G denote the forces in the spring and dashpot, respectively. We are interested in the situation when both functions f and g cannot be inverted to obtain some (non-implicit) expression for G and F and thus whole solution of (P) as in [27].

Remark. For a sake of simplicity we suppressed mass from our equations by assuming that it is a constant number 1.

Consider functions $f, g \in \mathcal{C}^1(\mathbb{R})$ in the following form

$$\begin{aligned} f &= f_{\mathrm{mon}} + f_0, \\ g &= g_{\mathrm{mon}} + g_0, \end{aligned}$$

where f_0, g_0 are at least $\mathcal{C}^1$ with compact support and $f_{\mathrm{mon}}, g_{\mathrm{mon}}$ are monotone functions with $f_{\mathrm{mon}}(0) = 0 = g_{\mathrm{mon}}(0)$ such that

$$\begin{aligned} c_1|\xi|^\alpha - c_2 \leq |f(\xi)|, c_1, c_2 > 0, \\ c_3|\xi|^\alpha - c_4 \leq |g(\xi)|, c_3, c_4 > 0, \end{aligned}$$

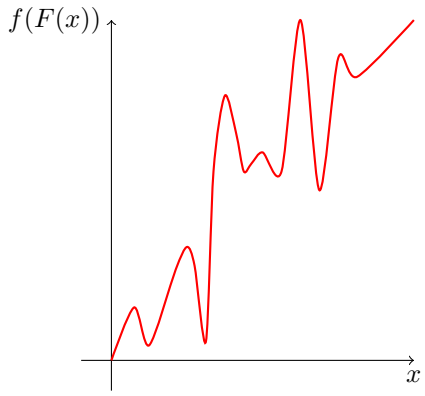

Figure 4.1: Typical function f.

where number $\alpha \geq 1$ will be specified later. Finally we will need that both f and g have up to finitely many local extrama (to obtain reasonable inverse). Thus we forbid to work with somehow wild perturbations.

We will aim to use Schauder theorem. In the following paragraphs we solve modified problem by construction of an "inverse" to both F and G. To obtain compactnees we will need Kolmogorov-Frechét-Riesz theorem and then we find a solution to the original problem as a fixed point of an appropriate mapping.

4.4.2 Auxiliary equation

Let $u, v \in L^1(0,T)$ are arbitrary and let us consider system

$$\begin{aligned} x'' + u + v &= \mathcal{F}, \\ x' &= g(G), \\ x &= f(F), \end{aligned}$$

with same initial conditions as in (P). From the first equation (and both conditions) we can determine (even unique) solution x, which satisfies first equation almost everywhere, $x, x' \in AC[0,T]$ and $x'' \in L^1(0,T)$. Our dream is to somehow "invert" function f. Of course it is not monotone function, and thus it canot be done directly, so we will choose "correct" parts of it, which will be increasing (and smooth) and invert these parts.

Note that solution x and everything which we derive in a moment depend heavily on our initial choice of u and v, but for sake of readability we will not write things like x''_u. But we should keep it in mind.

Existence of a preimage

We will work with the last equation $x = f(F) = (f_{\text{mon}} + f_0)(F)$ for times $t \in [0,T]$. We mainly need $J_f(F(t')) \neq 0$ for $t' \in [0,T]$. Because f_0 is compactly supported we have, for $r > 0$ large enough, $f_{\text{mon}} + f_0 = f = f_{\text{mon}}$ on ∂B_r. By Theorem 1(v) we know that

$$\deg(f_{\text{mon}}, B_r, y) = \deg(f_{\text{mon}} + f_0, B_r, y)$$

for $y \in B_{r-1}(0)$ (even in bigger balls, but this is enough). Later we show, that for u, v bounded in some norm we get also boundedness of F, G and this kind of relation will be enough for our purposes.

For $f_{\text{mon}} = \text{id}$ we immediately get

$$1 = \deg(\text{id} + f_0, B_r, y), y \in B_{r-1}(0).$$

If f_{mon} is general (monotone) we take the folowing homotopy function

$$H_\lambda = (1-\lambda)\text{id} + \lambda f_{\text{mon}}, \text{for } \lambda \in [0,1].$$

Now we want to show that for any $\lambda \in [0,1]$ and any $z \in \partial B_r$ there holds $H_\lambda \neq y$. This will implies, thanks to the Definition 2(d-4), that

$$\deg(\underbrace{\text{id}}_{=H_0}, B_r, y) = \deg(\underbrace{f_{\text{mon}}}_{=H_1}, B_r, y).$$

Case $\lambda = 0$: we want $z \neq y$, because $y \in B_{r-1}(0)$ we are done immediately.

Case $\lambda = 1$: we want $f_{\text{mon}}(z) \neq y$. Because the number $\alpha \geq 1$ (in the growth condition) we are also done, because $|f_{\text{mon}}(z)| > |y|$, for sufficiently big r's.

Case $\lambda \in (0,1)$: we observe the relation

$$y = (1-\lambda)z + \lambda f_{\text{mon}}(z).$$

If $\alpha > 1$ we sligtly rearange the equation above and due to the growing estimate of f_{mon} we can divide and obtain

$$\lambda = \frac{y-z}{f_{\text{mon}}(z) - z}\ .$$

Either $z > 0$ or $z < 0$. In the first case numerator is negative and denominator possitive. In the second case vice versa. Because λ is a non-egative number we are done. In the case $\alpha = 1$ it is obviously satisfied, either as above or by observing what happens if the denominator above would be equal to 0.

Remark. We see that similar procedure should work also in $\mathbb{R}^d$.

Therefore we know that

$$1 = \deg(f_{\text{mon}} + f_0, B_r, y), y \in B_{r-1}(0).$$

By the determinant characterization (Theorem 3) we know that

$$1 = \sum_{z \in f_{-1}(y)} \text{sign } J_{f_{\text{mon}}+f_0}(z).$$

which imply that there is at least one point $z \in (f_{\text{mon}} + f_0)_{-1}(y)$ such that $J_f(z) > 0$, i. e. non-zero and f is an increasing function at the point.

Definition of F

In this crucial part we construct function $F : [0,T] \to \mathbb{R}$. At first we know that f has finitely many extrema points, clearly the first one must be a local maximum and the last one local minimum (because of monotonocity of f_{mon}). Moreover there is an even number of extrema points, denote this number by $C_e + 1$. Let us denote, successively, all local extrema by $y_k, k = 1, 2, \ldots, C_e + 1$, i. e. y_{2k+1} are local maxima and y_{2k} are local minima. Furthermore set $y_0 = 0$ and $y_{C_{e+2}} = \infty$.

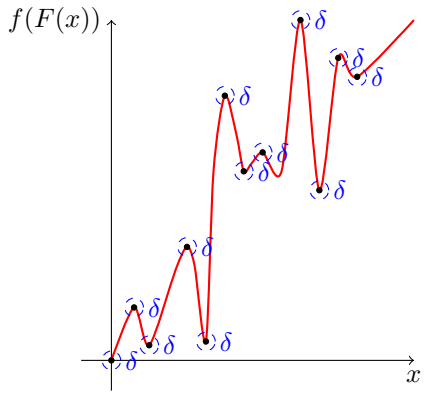

Figure 4.2: Choice of branches and appropriate δ.

Now we find $\delta > 0$ so small that $f(y_{2k+1}) > \max\{f(y_{2k}), f(y_{2k+2})\} + \delta$ for $2k+1 = 1, 3, \ldots, C_e$. At this point we will cut the graph of f into several branches. Denote

$$\begin{aligned}
B_1 &= \left\{[y, f(y)]; y \in [y_0, y_1), f(y_0) + \frac{\delta}{4} \leq f(y) \leq f(y_1) - \frac{\delta}{4}\right\}, \\
B_3 &= \left\{[y, f(y)]; y \in [y_2, y_3), f(y_2) + \frac{\delta}{4} \leq f(y) \leq f(y_3) - \frac{\delta}{4}\right\}, \\
&\vdots \quad \vdots \\
B_{C_e} &= \left\{[y, f(y)]; y \in [y_{C_e-1}, y_{C_e}), f(y_{C_e-1}) + \frac{\delta}{4} \leq f(y) \leq f(y_{C_e}) - \frac{\delta}{4}\right\}, \\
B_{C_e+2} &= \left\{[y, f(y)]; y \in [y_{C_e+1}, y_{C_e+2}), f(y_{C_e+1}) + \frac{\delta}{4} \leq f(y) \leq f(y_{C_e+2}) - \frac{\delta}{4}\right\}.
\end{aligned}$$

Those are exactly all increasing parts of f which are cut near to all extrema points, so derivative is bounded from below on the union of all B_{2k+1}. Moreover we see that the maximal value in some branch is bigger, at least by $\frac{\delta}{2}$, then the minimal value in subsequent branch. In the formal language

$$\max_{y \in B_{2k+1}} f(y) - \min_{y \in B_{2k+3}} f(y) = f(y_{2k+1}) - \frac{\delta}{4} - \left(f(y_{2k+2}) + \frac{\delta}{4}\right) > \frac{\delta}{2}.$$

Now we consider some branch B^0 with maximal value M_0. Then we have finitely many subsequent branches $B^1, B^2, \ldots, B^r$ with maximal values $M_1, M_2, \ldots, M_r$ and minimal values $m_1, m_2, \ldots, m_r$. There exists some first $0 \leq s \leq r$ for which $M^0 < M^{s+1}$, this holds because the last branch is unbounded. Now we see that $M^0 \geq \frac{\delta}{2} + m_t$ for $t = 0, 1, \ldots, s$. It follows from the previous paragraph, because $M^0 \geq M^t$ and $M^t \geq \frac{\delta}{2} + m_{t+1}$. This paragraph will be needed in the moment to show finitness of number of points of discontinuity.

Now we finally assign some value to each $F(t), t \in [0, T]$. We start with $x(0) = x_0$ and choose first (i. e. with the smallest index) branch $B_{2\theta_1+1}$ of branches introduced above for which $x_0 = f(y)$ for some $y \in \mathbb{R}$. It can be done because of the part "Existence of a preimage", from which we know that there exists some point from f_{-1} where f grows. Naturally we identify $F(0)$ with this single y.

Now we go throught the whole time interval $[0, T]$ and look on values of $x(t)$. Up to the some critical point $t_{cric} \in [0, T]$ we have $[y(t), x(t)] \in B_{2\theta_1+1}$ for all $t \in [0, t_{cric})$, and thus we define $F(t) = y(t), t \in (0, t_{cric})$. Clearly F is uniquelly determined and is $\mathcal{C}^1[0, t_0)$, because any branch of $f \in \mathcal{C}^1$ has continuously differentiable inverse in that branch of f.

We decide what will be a value of F at the critical point t_0. There are two possibilities. Either we are leaving branch from "above" or from "below".

Case "above": Assume that $x(t_{cric}) > \max_{y\in B_{2\theta_1+1}} f(y)$. Thanks to the monotonocity we can take first branch with bigger index (so part of the graph lies on the right) such that $[y, x(t_{cric})] \in B_{2\theta_1+1+\eta}$ for some $\eta \in \mathbb{N}$.

Case "below": Assume that $x(t_{cric}) < \min_{y\in B_{2\theta+1}} f(y)$. We use again monotonocity to find first branch on the left, i. e. find y so that $[y, x(t_{cric})] \in B_{2\theta_1+1+\eta}$ for $-\eta \in \mathbb{N}$.

In both cases there could be appropriate point also on the other side, but not in general. In any case we may denote $B_{2\theta_1+1+\eta}$ by $B_{2\theta_2+1}$.

We continue as above, instead of $x(0)$ we have $x(t_{cric})$ and instead of $B_{2\theta_1+1}$ we work with $B_{2\theta_2+1}$. And so one. Hence, we are able to guarantee an existence of F at least on some (nontrivial) subinterval of $[0, T]$. But if there will only finite number of choices θ_n (i. e. only finitely many jumps of F) then it will be much more better, because we find F on the the whole $[0, T]$.

Note that if F would not have any jump, then everythig is trivial, because f would be monotone. And so we would use the original result from the article [27].

Properties of F

Function F is obviously $\mathcal{C}^1$ between any two succeeding critical points. Therefore F will be bounded, and thus L^1 on the whole interval of existence.

So why has function F only finitely many jumps? Let us for a while assume that $||u||_1, ||v||_1 \leq M$ for some constant $M > 0$ then we show that there exists some uniform bound K_{jumps} on the maximal number of jumps of F. For a contradiction, let us assume that we have some u, v for which we find F on some nontrivial subinterval of $[0, T]$ with $K_{jumps} > T\frac{4C(N+1)}{\delta}$, where $C = |x_1| + ||\mathcal{F}||_1 + ||u||_1 + ||v||_1$.

In the previous part we constructed $N = \frac{C_e+1}{2}$ branches of f, so if F has $N + 1$ jumps then there was at least one "change of direction". If F, in the construction above, starts with value in some branch and then there are jumps in direction "above" (i. e. it leaves each branch and jumps to some on the right) then we would occur on the last unbounded branch, so if there is more than N jumps then F has to leave at least of branches from "below". Of course the same holds if we change words "above" and "below", because the whole situation is symmetrical.

Let us thus assume that on some time interval $(\tilde{t}, \tau_j)$, maybe arbitrarily small one, was F on branch B, where τ_j is the critical point. Then there was jump and F took values as inverse of f for values corresponding to some other branch B' and we leave it from "below" at second critical point τ_{j+1}. Then F jumps to some of preceding branches. Crucial point is that (τ_j, τ_{j+1}) cannot be in fact arbitrarily small.

We know, from the observation in the previous subsection, that the maximal value on B is bigger than minimal value on B' at least by $\frac{\delta}{2}$, because otherwise B' would not be on the right of B. So the solution x needs to overcome this distance of $\frac{\delta}{2}$ and it takes some time, which can be estimated as

$$\frac{\delta}{2} \leq |x(\tau_j + t) - x(\tau_j)| \leq C|\tau_j + t - \tau_j| = Ct,$$

and thus $t \geq \frac{\delta}{2C}$. This estimetate holds due to the relation which arise from integration of the first equation. Specifically we advance as below

$$\begin{aligned} x''(t) + u(t) + v(t) &= \mathcal{F}(t) \\ x'(\tau) &= x_1 + \int_0^\tau \mathcal{F}\mathrm{d}s - \int_0^\tau (u+v)\mathrm{d}s \\ x(t) &= tx_1 + x_0 + \int_0^t\int_0^\tau \mathcal{F}\mathrm{d}s\mathrm{d}\tau - \int_0^t\int_0^\tau (u+v)\mathrm{d}s\mathrm{d}\tau, \end{aligned}$$

and thus

$$|x(t_1) - x(t_2)| \leq |t_1 - t_2||x_1| + \int_{t_1}^{t_2} \int_0^{\tau} (|\mathcal{F}| + |u| + |v|)\,\mathrm{d}s\mathrm{d}\tau.$$

So we know that the solution x needs at least $\frac{\delta}{2C}$ to pass throught $N+1$ critical points at which F would have jump. But if it has K_{jumps} defined above, then altogether

$$\frac{K_{jumps}}{N+1} \cdot \frac{\delta}{2C} > \frac{4TC}{\delta} \cdot \frac{\delta}{2C} = 2T,$$

which is the contradiction because F is defined at most on the interval $[0,T]$.

Altogether we found function F defined on the whole interval $[0,T]$, it is bounded and integrable. If there is $M > 0$ such that $||u||_1, ||v||_1 \leq M$ and we write $C = |x_1| + ||\mathcal{F}||_1 + ||u||_1 + ||v||_1$ then there exist at most $K_{jumps} = \frac{8CT}{\delta}$ critical points $\tau_k \in [0,T]$ where F has jump. From the construction we also know that F is $\mathcal{C}^1[\tau_k, \tau_{k+1}]$ for all $k = 0, 1, \ldots, K_{jumps} - 1$, where we moreover define $\tau_0 = 0$. Therefore F is also Lipschitz on these segments and the Lipschitz constant is value of its derivative at some point. But $F'(t)$ can be estimated by $\frac{1}{f'(\xi)}$ on appropriate branch for some ξ. But at the beginning we cut f close to the all extrema points and so $|f'(\xi)| \geq \beta > 0, \forall \xi \in \bigcup_k B_{2k+1}$, for some constant β. It holds even without an aboslute value, because f is incereasing on all branches. Therefore F is Lipschitz at worst with constant $\frac{1}{\beta}$ for some fixed $\beta > 0$, which is independent on the choice of x.

We define also function G in the same way as F, i. e. we would find some branches of G and make an "inverse", then we would derive analogous estimates and conlude that G has at most finitely many jumps. For the sake of readibility we skip further details.

What we got up to now: For given $u, v \in L^1(0,T)$ we found $x, x' \in AC[0,T]$ and $x'', F, G \in L^1(0,T)$ such that the system

$$\begin{aligned} x'' + u + v &= \mathcal{F} \\ x' &= g(G) \\ x &= f(F) \\ x(0) = x_0,\ x'(0) &= x_1, \end{aligned}$$

holds for a. a. $t \in [0,T]$.

4.4.3 Schauder theorem

At this part we will consider an operator

$$\begin{aligned} \mathcal{T} : L^1(0,T) \times L^1(0,T) &\to L^1(0,T) \times L^1(0,T) \\ (u,v) &\mapsto (F,G) \end{aligned}$$

and our goal is to show that it has a fixed point. If it holds, we will be done, because we find the solution of problem (P), because then we obtain $u = F, v = G$ in the auxuliary system from the previous section.

First and foremost we need the compactness of $\mathcal{T}$. By the definition, we need to show, that for any bounded set $\mathcal{M} \subset L^1 \times L^1$, of couples (u,v), its image under $\mathcal{T}$ is precompact. Because we work on Banach space it is enough to show total boundedness of $\mathcal{T}(\mathcal{M})$. To prove this we will use Kolmogorov-Fréchet-Riesz theorem (Claim 16) for $p = 1$.

Assume that $\mathcal{M}$ is arbitrary set as stated above. Suppose that L^1 norms of all u, v (from considered set) are bounded by some number $M > 0$.

Boundedness of $\mathcal{T}$

We want to show that $\mathcal{T}(\mathcal{M})$ is bounded set in $L^1(0,T)\times L^1(0,T)$. We will show that for M big enough there also holds $||F||_1 \leq M$. In the same way will holds $||G||_1 \leq M$.

Let's take the growth assumption of f, i. e.

$$c_1|F|^\alpha - c_2 \leq |f(F)| = |x|.$$

Now we will go back to the system in previous section and integrate first equation (we obviously can). As before we successively get

$$\begin{aligned} x'(\tau) &= x_1 + \int_0^\tau \mathcal{F}\mathrm{d}s - \int_0^\tau (u+v)\mathrm{d}s \\ x(t) &= tx_1 + x_0 + \int_0^t\int_0^\tau \mathcal{F}\mathrm{d}s\mathrm{d}\tau - \int_0^t\int_0^\tau (u+v)\mathrm{d}s\mathrm{d}\tau \\ ||x||_\infty &\leq T|x_1| + |x_0| + T||\mathcal{F}||_1 + T(||u||_1 + ||v||_1). \end{aligned}$$

Therefore we can estimate

$$\begin{aligned} c_1|F|^\alpha - c_2 &\leq T|x_1| + |x_0| + T||\mathcal{F}||_1 + T(||u||_1 + ||v||_1) \\ |F|^\alpha &\leq \frac{c_2}{c_1} + \frac{1}{c_1}(T|x_1| + |x_0| + T||\mathcal{F}||_1) + \frac{T}{c_1}(||u||_1 + ||v||_1). \end{aligned}$$

Let us take M so big that

$$|F|^\alpha \leq \frac{2T}{c_1}M + \frac{2T}{c_1}M = \frac{4T}{c_1}M.$$

Hence we obtain two estimates

$$||F||_1 \leq T\left(\frac{4T}{c_1}M\right)^{\frac{1}{\alpha}} \quad \& \quad ||F||_\infty \leq \left(\frac{4T}{c_1}M\right)^{\frac{1}{\alpha}}.$$

We will need to estimate those norms in fact as

$$||F||_1 \leq M \quad \& \quad ||F||_\infty \leq M.$$

In the case $\alpha = 1$ we get it on some sufficiently short time interval $[0,T]$, for $\alpha > 1$ it is satisfied on time interval of an arbitrary (finite) length by appropriate choose of number M. Similarly we would estimate norm of G (we would obtain better constant, because of just one integration instead of two).

Conditions of Kolmogorov-Frechét-Riesz

We can finally start with verifying conditions of Kolmogorov-Frechét-Riesz theorem. We naturally extend functions F, G outside of $[0,T]$ by zero. Thanks to the boundedness of $\mathcal{T}(\mathcal{M})$ the condition

$$\left\{\int_{t_1}^{t_2} F(s)\mathrm{d}s; F\in\mathcal{T}(\mathcal{M})\right\} \text{ is bounded in } \mathbb{R}^2 \text{ for all } 0<t_1<t_2<T,$$

holds trivially. So there remains to show

$$\lim_{a\to 0}\int_a^T |\tau_a F(t) - F(t)|\mathrm{d}t = 0, \text{ uniformly on } \mathcal{T}(\mathcal{M}),$$

where τ_a is a translation, i. e. $\tau_a F(z) = F(z-a)$. Equivalently we can show that

$$\forall\varepsilon>0\exists\Delta>0\forall F\in\mathcal{T}(\mathcal{M})\forall|a|<\delta : ||\tau_a F - F||_1 < \varepsilon.$$

Then we would use analogous argument to conclude that

$$\lim_{a\to 0}\int_a^T \left(|\tau_a F(t) - F(t)| + |\tau_a G(t) - G(t)|\right) \mathrm{d}t = 0, \text{ uniformly on } \mathcal{T}(\mathcal{M}),$$

which is desired condition of mentioned theorem for $k = 2$. It means that we take any admissible F and try to obtain uniform estimate, i. e. independent on the initial choice of u, v. So let $\varepsilon > 0$ be given. At first we know that F has some jumps in points $\tau_1, \tau_2, \ldots, \tau_K \in [0, T]$, where $K \leq K_{jumps}$. The value $F(\tau_j)$ is defined as one of $f_{-1}(x(\tau_k))$, where appropriate branch of f was chosen so that $[F(\tau_j), x(\tau_j)]$ lies inside of the branch. We take small neighbourhoods around each of τ_j, i. e. we have balls $B_\varepsilon(\tau_j)$ for $j = 1, 2, \ldots, K$.

So we can decompose time interval $[0, T + |a|]$ in the following way

$$[0, T + |a|] = N \cup \left(\bigcup_{j=1}^{K} (\tau_j - \varepsilon, \tau_j + \varepsilon)\right) \cup [0, |a|) \cup (T, T + |a|] \cup S,$$

where N is some set of zero measure (we got everything almost everywhere) and S is complement of other (known) sets in $[0, T + |a|]$.

At this moment we can finally show desired conditions of Kolmogorov-Frechét-Riesz. For $a > 0$ let us rewrite

$$\begin{aligned}
||\tau_a F - F||_1 &= \int_{\mathbb{R}} |F(t-a) - F(t)| \mathrm{d}t = \int_0^{T+a} |F(t-a) - F(z)| \mathrm{d}t = \\
&= \int_0^a |F(t)| + \int_T^{T+a} |F(t-a)| + \sum_{j=1}^{K} \int_{\tau_j-\varepsilon}^{\tau_j+\varepsilon} |F(t-a) - F(t)| + \\
&\quad + \int_S |F(t-a) - F(t)|.
\end{aligned}$$

Due to the uniform boundedness of F we obtain

$$\begin{aligned}
||\tau_a F - F||_1 &\leq 2aM + 2K_{\text{jumps}}\varepsilon \cdot 2M + \int_S |F(t-a) - F(t)| \leq \\
&\leq 2aM + 4K_{\text{jumps}} M\varepsilon + \int_S |F(t-a) - F(t)|.
\end{aligned}$$

Now we use that F is Lipschitz on all branchech, with constant $\frac{1}{\beta}$, and number of branches is determined by the number of jumps, which is bounded. We thus obtain

$$\begin{aligned}
||\tau_a F - F||_1 &\leq 2aM + 4K_{\text{jumps}} M\varepsilon + \frac{|t - a - t|}{\beta} \int_0^T 1 \leq \\
&\leq a\left(2M + \frac{T}{\beta}\right) + 4K_{\text{jumps}} M\varepsilon.
\end{aligned}$$

So we want to choose a so that

$$|a| < \frac{\varepsilon}{2M + \frac{T}{\beta}}$$

for given $\varepsilon > 0$.

Therefore we get the final estimate

$$||\tau_a F - F||_1 \leq \varepsilon \left(1 + 4K_{\text{jumps}} M\right),$$

which is desired inequality, because booth M and K_{jumps} are given on the beginning. For $a < 0$ it holds from analogous reasoning, or we can use substitution and convert this problem to the situation above.

Existence of a fixed point of $\mathcal{T}$

By the Kolmogorov-Frechét-Riesz theorem (Claim 16) we know that $\mathcal{T}$ is a compact mapping. From previous paragraphs we know that there is a closed, convex, bounded ball B_M in $L^1(0,T) \times L^1(0,T)$ such that the operator goes from the ball into itself. Remains to show that the operator is also continuous.

Let us have some sequence of functions $u_n, v_n \in L^1(0,T)$ which converges to u and v respectively $L^1(0,T)$. Because of convergence they are bounded by some big constant M. We want to show that $F_n \to F$ in $L^1(0,T)$, where F would be found from the equation for u. We take a look on the "limit system"

$$\begin{aligned} x'' + u + v &= \mathcal{F}, \\ x' &= g(G), \\ x &= f(F), \end{aligned}$$

we are able (because both u and v are bounded) to find $x \in AC[0,T]$, which satisfies $x' \in AC[0,T], x \in L^1(0,T)$ and F from the construction above.

In the same fashion we may find, for any function u_n and v_n, the function $x_n \in AC[0,T]$ such that $x_n' \in AC[0,T], x_n'' \in L^1(0,T)$ and corresponding function $F_n \in L^1(0,T)$.

Because $u_n \to u$ and $v_n \to v$ both in $L^1(0,T)$ we see from the equation

$$x_n'' + u_n + v_n = \mathcal{F},$$

that $x_n'' \to x$ in $L^1(0,T)$. Thanks to the paragraph "Boundedness of $\mathcal{T}$" and the fact that norms of u, u_n, v, v_n are bounded by M we see that x_n are uniformly bounded on $[0,T]$, and thus we may invoke Arzelà–Ascoli theorem (Claim 15) to conclude that $x_n \rightrightarrows x$ on $[0,T]$. Similarly also $x_n' \rightrightarrows x'$ on $[0,T]$.

We have same bound on both F and F_n in norms of L^1 and L^∞, and therefore those functions are in $L^2(0,T)$, which is reflexive space. From the boundedness we know that F_n weakly converges to some $\overline{F}$ in $L^2(0,T)$. We show that $F = \overline{F}$.

Set of all points where F or any of F_n has jump has zero measure. It follows from σ-additivity and the fact that each of those single set is finite. Denote this set of jumps by $\mathcal{J}$. If we fix $t \in [0,T] \setminus \mathcal{J}$ then for indices big enough (i. e. $n \geq n_0$) and for any $\varepsilon > 0$ there holds

$$|x_n(t) - x(t)| < \varepsilon,$$

thanks to the uniform continuity. Because $x(t)$ is not equal to any of boundary values of first coordinate of B_{2k+1}, for all k, we see that for ε small enough are both $[F(t), x(t)]$ and $[F(t), x_n(t)], n \geq n_0$, in one branch B_{2k+1}. But for all y's from corresponding interval we have that $f(y)$ has $\mathcal{C}^1$ inverse and so

$$\begin{aligned} f^{-1}(x_n(t)) &= F_n(t), \\ f^{-1}(x(t)) &= F(t). \end{aligned}$$

From the first equation and thanks to the continuity mentioned above we obtain

$$F_n(t) = f^{-1}(x_n(t)) \to f^{-1}(x(t)) = F(t).$$

Therefore the sequence $(F_n)_n$, which has some limit in L^2, converges pointwisely to the F for almost all times. So $F = \overline{F}$ and because of pointwise convergence we are completely done. It follows from boundedness of $||F_n||_\infty$ and Lebesgue dominated convergence theorem, because then $\int_0^T |F_n - F| \to 0$ as $n \to \infty$. Which is exactly needed convergence, and therefore $\mathcal{T}$ is a continuous operator.

Due to the Schauder theorem we have an existence of at least one fixed point of the operator $\mathcal{T}$. Thus we solved the problem (P).

Theorem 23. (Non-monotone right hand side of the problem (P))

Let $T \geq 0, x_0, x_1 \in \mathbb{R}$ and $\mathcal{F} \in L^1(0,T)$ are arbitrary. Suppose that there are functions $f, g : \mathbb{R} \to \mathbb{R}$ with at most finitely many local extrema such that

$$c_1|\xi|^\alpha - c_2 \leq |f(\xi)| \quad \& \quad c_3|\xi|^\alpha - c_4 \leq |g(\xi)|$$

for some constants c_1, c_2, c_3, c_4 and $\alpha \geq 1$. Moreover assume that there is a decomposition

$$f = f_{\text{mon}} + f_0 \quad \& \quad f = g_{\text{mon}} + g_0,$$

where f_0, g_0 are $\mathcal{C}_0^1(\mathbb{R})$ and $f_{\text{mon}}, g_{\text{mon}}$ are monotone $\mathcal{C}^1(\mathbb{R})$ functions with $f_{\text{mon}}(0) = g_{\text{mon}}(0) = 0$. Then there exists the solution of problem (P) on the whole $[0,T]$ if $\alpha > 1$. In the case $\alpha = 1$ there is also solution on the whole interval, but just for T which are small (and necessary smallness depends on coefficients c_1, c_2).

Remark. The most important part of the proof was construction of unknown function F. More precisely choice of "good" monotone parts of f where we took inverse. We could consider all monotone parts, but then we would not be able to guarantee finitness of number of jumps of resulting function F, which was crucial for compactness.

4.5 Differential inclusion

Let us remind the reader that we considered multifunction $F : K \to 2^K$, where $K \neq \emptyset$ was compact and convex subset of a locally convex space X. This object has graph $G_F = \{(x,y) \in K \times K; y \in F(x)\}$.

Before we start we state one key lemma on which we will depend several times later.

Lemma 24. Let G_F be a maximal monotone graph in the sense of Definition 6 (section 3.3). Assume that $x_n \to x$ in H, $y_n \rightharpoonup y$ in H and $(x_n, y_n) \in G_F$. Then $(x,y) \in G_F$.

Proof. Let us take arbitrary $(x', y') \in G_F$, thanks to the monotonocity we have

$$(x_n - x', y_n - y') \geq 0.$$

Thanks to assumed convergences we conclude that $(x - x', y - y') \geq 0$ and due to the maximality we obtain $(x,y) \in G_F$. ■

4.5.1 Predator - Prey model

The predator–prey equations, are a pair of differential equations, used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The simplest example is Lotka-Volterra model, it is given by the system

$$\begin{aligned} x' &= (r - ky)x, \\ y' &= (-h + px)y, \end{aligned}$$

where x is population of prey, y is population of predator and $r, k, h, p > 0$ are given constants. These numbers describe how the one of populations would prosper without effect of the second one and how effective are predators in this system, see for example section 4.4 in [32] or chapter 3 in [21] for details and further discussion.

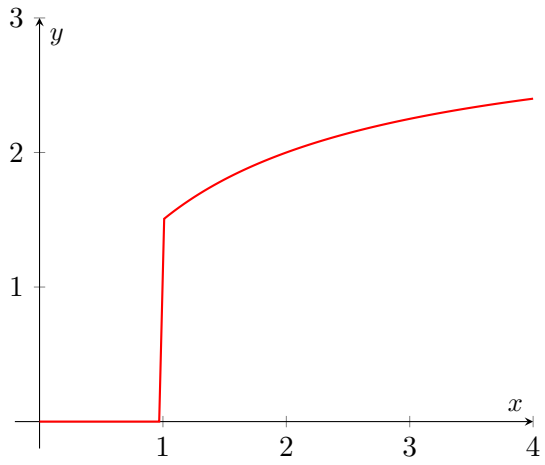

Figure 4.3: Graph $\mathcal{A}$ of Gause problem.

Problem

In this section we will show both existence and uniqueness of the solution to the so called Gause predator-prey model with a refuge, see [19] for historical informations and brief qualitative analysis.

Let us consider the problem

$$\begin{aligned} x' &= rx - y\varphi(x), \\ y' &= (e\varphi(x) - h)y, \end{aligned} \tag{P}$$

where x is population of prey, y is population of predators, $r, e, h > 0$ are given constants and $\varphi : \mathbb{R} \to \mathbb{R}$ is rate of killed prey by predators. The multifunction is given by

$$\varphi(x) = \begin{cases} \{0\}, & \text{if } x < x_s \\ \left[0, \frac{mx_s}{a+x_s}\right], & \text{if } x = x_s \\ \left\{\frac{mx}{a+x}\right\}, & \text{if } x > x_s \end{cases},$$

where x_s is capacity of the refuge, $m, a > 0$ are again some constants (how effective predators are).

Definition 9. Let us suppose that $x_0, y_0 \geq 0$ are given numbers and $T > 0$. By the solution of (P) we will understand the triple (x, y, φ), where $x, y : [0, T] \to \mathbb{R}$ are absolutely continuous and $\varphi : \mathbb{R} \to \mathbb{R}$ is measurable function such that (P) holds for a. a. $t \in [0, T]$ and $(x(t), \varphi(t)) \in \mathcal{A} := G_\varphi$ also for a. e. t's, where

$$\mathcal{A} = \{(x, 0); x < x_s\} \cup \left\{(x_s, y); y \in \left[0, \frac{mx_s}{a + x_s}\right]\right\} \cup \left\{\left(x, \frac{mx}{a + x}\right); x > x_s\right\} \subset \mathbb{R} \times \mathbb{R}.$$

The plan is following. In order to use Kakutani-Ky Fan fixed point theorem it is convenient to show that $\mathcal{A}$ is maximal monotone graph (see Definition 6, section 3.3). Then we construct some bounded, convex and weakly closed set and find apropriate upper semicontinuous multifunction F. Finally we will show uniqueness of the solution, it is not as straightforward as usual, but not hard.

Maximal monotone graph

We need to check two properties. At first assume that we have two elements of the $\mathcal{A}$, denote them $(x_1, \varphi_1), (x_2, \varphi_2)$, we want that $(x_1 - x_2)(\varphi_1 - \varphi_2) \geq 0$. Without loss of generality we may assume that $x_1 > x_2$ and now we have several cases.

If $x_s \geq x_1 > x_2$ it is trivialy satisfied.

If $x_1 > x_2 \geq x_s$ then conclusion holds due to the monotonocity of a function $\frac{mx}{a+x}$.

If $x_1 \geq x_s \geq x_2$ then we have again monotonocity and positivity of same function.

Second property is that if $(x, \varphi) \subset \mathbb{R} \times \mathbb{R}$ and for all $(x_1, \varphi_1) \in \mathcal{A}$ hold $(x - x_1)(\varphi - \varphi_1) \geq 0$ then $(x, \varphi) \in \mathcal{A}$.

Consider that $x > x_s$. Let us take $x_1 = x - \varepsilon y$, where $y \in \mathbb{R}$ is arbitrary and $\varepsilon > 0$ is small as much as we want. Then we find corresponding φ_1 and suppose that inequality above is satisfied, in other words

$$\varepsilon y \left(\varphi - \frac{m(x - \varepsilon y)}{a + (x - \varepsilon y)} \right) \geq 0.$$

Now we divide by $\varepsilon > 0$ and pass to the limit as $\varepsilon \to 0_+$ to obtain

$$y \left(\varphi - \frac{mx}{a + x} \right) \geq 0.$$

It holds for all y's, and therefore whole bracket is equal to zero.

Similarly if $x < x_s$ we obtain $y \cdot (\varphi) \geq 0$ which implies that $\varphi = 0$. Finally if $x = x_s$ we see that φ may be equal to anything from the segment $\left[0, \frac{mx_s}{a+x_s}\right]$. We conclude that $(x, \varphi) \in \mathcal{A}$.

Existence

We will look for a fixed point of the mapping $F = \beta \circ \alpha$, where

$$\varphi \overset{\alpha}{\mapsto} \text{ first component of a solution of the system } \overset{\beta}{\mapsto} \text{ admissible functions } \varphi.$$

Let us introduce following sets

$$\begin{aligned} X &= L^2(0,T) \text{ with the weak topology,} \\ K &= \left\{ \varphi \in L^2(0,T); \operatorname{esssup}_{t \in [0,T]} |\varphi(t)| \leq \lim_{x \to \infty} \frac{mx}{a+x} = m \right\}, \\ Y &= AC([0,T]), \end{aligned}$$

and mappings $\alpha : K \to Y, \beta : Y \to 2^K$, where $\alpha(\varphi)$ is a solution x of (P) for the given φ (it exists by Claim 19), which exists thanks to the well-known theory of ode and the fact that $\varphi \in L^\infty(0,T)$ and

$$\beta(x) = \{\tilde{\varphi} \in K; (x(t), \tilde{\varphi}(t)) \in \mathcal{A} \text{ for a. a. } t \in [0,T]\} \subset 2^K.$$

Of course $F = \beta \circ \alpha : K \to 2^K$ will be our upper semicontinuous multifunction. Obviously if it has a fixed point we find "correct" φ and then also functions x, y which will be absolutely continuous and solve the problem. Moreover the couple $(x, \varphi) \in \mathcal{A}$ due to the deifinition of β.

We see that $K \neq \emptyset$, it is also bounded and convex set. Moreover this set is weakly closed, which is same as strongly closed, thanks to the Mazur theorem (Claim 13). By Eberlein-Smulian theorem (Claim 14) we obtain weak compactness of K, i. e. compactness of K in X. Of course X is a locally convex space, so we have already half of assumptions of Kakutani-Ky Fan theorem.

Now we observe properties of $F(\varphi)$ for fixed $\varphi \in K$. Because we find x and this prescribes function $\tilde{\varphi}$ which belongs to the graph we obtain that $F(\varphi)$ cannot be empty.

Now we show convexity. Suppose that $\tilde{\varphi}_1, \tilde{\varphi}_2 \in F(\varphi)$ and $\lambda \in [0,1]$, we want $\lambda\tilde{\varphi}_1 + (1-\lambda)\tilde{\varphi}_2 \in F(\varphi)$. It is simple. We know that $(x, \tilde{\varphi}_1), (x, \tilde{\varphi}_2) \in \mathcal{A}$ for some absolutely continuous function x and we consider couple $(x_0, \tilde{\varphi}_0) \in \mathcal{A}$. Then

$$\begin{aligned}(x_0 - x, \tilde{\varphi}_0 - (\lambda\tilde{\varphi}_1 + (1-\lambda)\tilde{\varphi}_2) &= (x_0 - x, \lambda(\tilde{\varphi}_0 - \tilde{\varphi}_1) + (1-\lambda)(\tilde{\varphi}_0 - \tilde{\varphi}_2)) = \\ &= \lambda(x_0 - x, \tilde{\varphi}_0 - \tilde{\varphi}_1) + (1-\lambda)(x_0 - x, \tilde{\varphi}_0 - \tilde{\varphi}_2) \geq 0\end{aligned}$$

where the inequality holds thansk to the monotonocity of $\mathcal{A}$. Because of maximality of $\mathcal{A}$ we see that $(x, \lambda\tilde{\varphi}_1 + (1-\lambda)\tilde{\varphi}_2) \in \mathcal{A}$.

Similarly we show weak closedness of $F(\varphi)$. Let us assume that $\tilde{\varphi}_n \in F(\varphi)$ weakly converges to $\tilde{\varphi}$ in $L^2(0,T)$. Then we know that for any $(x_0, \tilde{\varphi}_0) \in \mathcal{A}$ we have

$$(x_0 - x)(\tilde{\varphi}_0 - \tilde{\varphi}_n) \geq 0, \forall n \in \mathbb{N}.$$

Because $x_0 - x$ is absolutely continuous function we may pass to the limit to obtain same inequality with $\tilde{\varphi}$ instead of $\tilde{\varphi}_n$. Because of maximality of $\mathcal{A}$ we get that $(x, \tilde{\varphi}) \in \mathcal{A}$.

If we show that graph of F is closed, then we are done, because we invoke Lemma 21 to obtain upper semicontinuity of F (we already know that F has non-empty, convex and closed values, so we may use it) and the rest of work follows from Theorem 22, because by it we obtain fixed point of F.

Let us suppose that

$$\begin{aligned}\varphi_n &\rightharpoonup \varphi \text{ in } L^2(0,T), \\ \tilde{\varphi}_n &\rightharpoonup \tilde{\varphi} \text{ in } L^2(0,T), \\ \tilde{\varphi}_n &\in F(\varphi_n), \forall n \in \mathbb{N},\end{aligned}$$

we aim to show that $\tilde{\varphi} \in F(\varphi)$. Last two conditions we may rewrite in the following way

$$\begin{aligned}\tilde{\varphi}_n &\rightharpoonup \tilde{\varphi} \text{ in } L^2(0,T), \\ \tilde{\varphi}_n &\in \beta(\alpha(\varphi_n)), \forall n \in \mathbb{N}.\end{aligned}$$

If we moreover show that

$$\alpha(\varphi_n) \to \alpha(\varphi) \text{ in } L^2(0,T)$$

we will be done, because β obviously has maximal monotone graph (just by its definition) and we use Lemma 24 to conclude that $\tilde{\varphi} \in \beta(\alpha(\varphi))$.

The convergence follows in routine way from apriori estimates. We have $\varphi_n \rightharpoonup \varphi$ in $L^2(0,T)$, for each of φ_n we may find absolutely continuous functios x_n, y_n solving (P), i. e.

$$\begin{aligned}x_n' &= rx_n - y_n\varphi_n, \\ y_n' &= (e\varphi_n - h)y_n.\end{aligned}$$

We multiply first equation by $2x_n$ and second one by $2y_n$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x_n^2 &= 2rx_n^2 - 2y_nx_n\varphi_n, \\ \frac{\mathrm{d}}{\mathrm{d}t}y_n^2 &= 2(e\varphi_n - h)y_n^2.\end{aligned}$$

From the second equation we obtain an estimate $\frac{\mathrm{d}}{\mathrm{d}t}y_n^2 \leq 2(2em + h)y_n^2$ and because of Grönwall inequality (Claim 18) we obtain $|y_n| \leq C$, where $C \geq 0$ is constant independent of n.

By the Young inequality and boundedness of y_n we obtain also boundedness of $|x_n|$. From the both equations above we get uniform boundedness of derivatives of x_n, y_n. It immediately imply equicontinuity which is needed in Arzelà–Ascoli theorem (Claim 15) to obtain that $x_n \rightrightarrows x$ and $y_n \rightrightarrows y$ on $[0,T]$.

Therefore (because φ_n converges weakly)

$$\begin{aligned} x' &= rx - y\varphi, \\ y' &= (e\varphi - h)y. \end{aligned}$$

For the limit function φ we also find solution, i. e. there holds

$$\begin{aligned} \bar{x}' &= r\bar{x} - \bar{y}\varphi, \\ \bar{y}' &= (e\varphi - h)\bar{y}. \end{aligned}$$

Question is whether $x = \bar{x}$ and $y = \bar{y}$. Then trivially $\alpha(\varphi_n) \to \alpha(\varphi)$ in $L^2(0,T)$. This fact follows from the following paragraph.

Uniqueness

If we try to directly substract two solutions, it will not lead immediately to the desired result (it is caused by φ in second equation). So let us for a start rewrite our system as following

$$\begin{aligned} x' &= rx - y\varphi \\ y' &= (e\varphi - h)y \\ \hline ex' + ey\varphi &= rex \\ y' + hy &= e\varphi y \\ \hline ex' + ey\varphi &= rex \\ ex' + y' + hy &= rex \\ \hline x' + y\varphi &= rx \\ (ex + y)' + h(ex + y) &= e(r + h)x. \end{aligned}$$

Now we will suppose that there are two solutions $(x_1, \varphi_1), (x_2, \varphi_2) \in \mathcal{A}$ and y_1, y_2, denote $\bar{x} = x_1 - x_2, \bar{y} = y_1 - y_2$. We use the system above to get

$$\begin{aligned} \bar{x}' + y_1\varphi_1 - y_2\varphi_2 &= r\bar{x}, \\ (ex + \bar{y})' + h(e\bar{x} + \bar{y}) &= e(r + h)\bar{x}, \end{aligned}$$

and by multiplication we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^2 + 2\bar{x}y_1\varphi_1 - 2\bar{x}y_2\varphi_2 = 2r\bar{x}^2, \\ &\frac{\mathrm{d}}{\mathrm{d}t}(e\bar{x} + \bar{y})^2 + h(e\bar{x} + \bar{y})^2 = e(r + h)\bar{x}(e\bar{x} + \bar{y}) \leq c_1(e\bar{x} + \bar{y})^2 + c_2\bar{x}^2, \end{aligned}$$

where we used Young inequality.

The first equality we may rewrite

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^2 + 2\bar{x}y_1\varphi_1 - 2\bar{x}y_2\varphi_2 = 2r\bar{x}^2 \\ &\quad \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^2 + 2y_1\bar{x}(\varphi_1 - \varphi_2) = 2r\bar{x}^2 - 2\bar{x}\bar{y}\varphi_2. \end{aligned}$$

The term $2y_1\bar{x}(\varphi_1 - \varphi_2)$ is non-negative, because of the fact $(x_1, \varphi_1), (x_2, \varphi_2) \in \mathcal{A}$ and $\mathcal{A}$ is monotone (and observation that $y_1, y_2 \geq 0$). Now we use Young inequality to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^2 &\leq 2r\bar{x}^2 - 2\bar{x}(e\bar{x} + \bar{y} - e\bar{x})\varphi_2 \\
\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^2 &\leq (2r + 2e)\bar{x}^2 - 2\bar{x}(e\bar{x} + \bar{y})\varphi_2 \\
\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^2 &\leq (2r + 2e)\bar{x}^2 + \bar{x}^2\varphi_2 + (e\bar{x} + \bar{y})^2.
\end{aligned}$$

We may add estimates for derivatives of both $\bar{x}^2$ and $(e\bar{x} + \bar{y})^2$ to find

$$\frac{\mathrm{d}}{\mathrm{d}t}((e\bar{x} + \bar{y})^2 + \bar{x}^2) \leq c(\bar{x}^2 + (e\bar{x} + \bar{y})^2)$$

and now as usual use Grönwall inequality (Claim 18) to find that $(ex+y)^2+x^2)(t) \leq 0$ for a. a. $t \in [0, T]$. Which means that $\bar{x} = x_1$, and thus also $\bar{y} = y_1$ almost everywhere. Therefore uniqueness follows and as we said above also the existence. We can formulate what we actually proved.

Theorem 25 (Predator-prey model with a refuge)**.** Let $T > 0$ and $x_0, y_0 \geq 0$ are given values. If positive parameters r, e, h are fixed and $\varphi : \mathbb{R} \to \mathbb{R}$ is a function given by formula

$$\varphi(x) = \begin{cases} \{0\}, & \text{if } x < x_s \\ \left[0, \frac{mx_s}{a+x_s}\right], & \text{if } x = x_s \\ \left\{\frac{mx}{a+x}\right\}, & \text{if } x > x_s \end{cases},$$

then the problem (P) has unique solution in the sense of Definition 9 above.

Remark. Proof of uniqueness was done in the lecture NMMA574 in summer 2019.

4.5.2 Partial differential equations with a constraint

We will use a stronger notion of maximal monotone graph than before. See articles [6] and [7] for further details. We write just a definition and one necessary lemma.

Definition 10. We say that set $\mathcal{A} \subset \mathbb{R}\times\mathbb{R}$ is maximal monotone 2-graph (we abbreviate as MM2G) if all following conditions hold:

(A1) $(0, 0) \in \mathcal{A}$.

(A2) For all $(S_1, D_1), (S_2, D_2) \in \mathcal{A}$ hold $(S_1 - S_2) \cdot (D_1 - D_2) \geq 0$.

(A3) If $(S, D) \in \mathbb{R} \times \mathbb{R}$ satisfies for all $(S_1, D_1) \in \mathcal{A}$ that $(S - S_1) \cdot (D - D_1) \geq 0$, then also $(S, D) \in \mathcal{A}$.

(A4) There exists measurable selection, i. e. there is a measurable $S^* : \mathbb{R} \to \mathbb{R}$ such that $(S^*(D), D) \in \mathcal{A}, \forall D \in \mathbb{R}$.

(A5) There exists constant $c > 0$ such that $SD \geq \frac{1}{c}(|S|^2 + |D|^2) - c$ for all $(S, D) \in \mathcal{A}$.

Remark. Let us note that number 2 in the name of MM2G corresponds to the exponent in condition (A5). In general there are of course MMrG for $r \geq 1$.

Lemma 26 (Convergence tool)**.** Let $\mathcal{A}$ be MM2G. Assume that there are sequences $(S^n)_{n=1}^{\infty}, (D^n)_{n=1}^{\infty}$ defined on a measurable set $G \subset \mathbb{R}^d$ such that the following conditions hold:

(i) $(S^n, D^n) \in \mathcal{A}$ a. e. in G.

(ii) $D^n \rightharpoonup D$ in $L^2(G)$.

(iii) $S^n \rightharpoonup S$ in $L^2(G)$.

(iv) $\limsup_{n\to\infty} \int_G S^n D^n \leq \int_G SD$.

Then $(S, D) \in \mathcal{A}$ a. e. in G.

Proof. See [6] for the proof of even more general statement. We write it down for the sake of completeness in our situation of MM2G.

We see that

$$\limsup_{n\to\infty} \int_G (S^n - S^*(D)) \cdot (D^n - D) \leq 0.$$

Thanks to the (A2), i. e. monotonocity, it is in fact equivalent to

$$\limsup_{n\to\infty} \int_G |(S^n - S^*(D)) \cdot (D^n - D)| = 0.$$

Therefore, integrand converges (strongly) in $L^1(G)$, and thus also weakly and so, for all $\varphi \in L^\infty(G)$,

$$\lim_{n\to\infty} \int_G (S^n - S^*(D)) \cdot (D^n - D)\varphi = 0$$

and so

$$\lim_{n\to\infty} \int_G [(S^n \cdot D^n - S^n \cdot D) - S^*(D) \cdot (D^n - D)]\varphi = 0.$$

Due to (ii) and (iii) we conclude that

$$\lim_{n\to\infty} \int_G S^n \cdot D^n \varphi = \lim_{n\to\infty} \int_G S^n \cdot D\varphi = \int_G S \cdot D\varphi.$$

Now we use (A3) to conclude the assertion. We consider any $B \in \mathbb{R}^{d\times d}$, thanks to (A4) and (A2) we obtain

$$0 \leq \lim_{n\to\infty} \int_G (S^n - S^*(B)) \cdot (D^n - B)\varphi = \int_G (S - S^*(B)) \cdot (D - B)\varphi,$$

for any non-negative $\varphi \in L^\infty(G)$. Consequently, since φ is arbitrary, we get

$$(S - S^*(B)) \cdot (D - B) \geq 0 \text{ a. e. in } G$$

and by (A3) follows that $(S, D) \in \mathcal{A}$ a. e. in G. ■

Following problem was assigned to solve as an exam to subject Partial differential equations 3. Here we modify its solution by use of fixed point theory.

Problem

Let us consider bounded function $f \in \mathcal{C}(\mathbb{R})$ and $\Omega \in C^{0,1}$ in $\mathbb{R}^d$. Let $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function given by formula

$$H(g, u) = \frac{(|g - f(u)| - 1)^+}{|g - f(u)|}(g - f(u)) - u.$$

We want to solve, i. e. show when there is a (unique) solution, problem

$$\begin{aligned} -\triangle u(x) + g(x) &= h(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ H(g, u) &= 0 \text{ in } \Omega, \end{aligned} \tag{P}$$

for a given function $h \in L^2(\Omega)$ and unknown functions $u, g : \Omega \to \mathbb{R}$.

Remark. We may observe that if $h \equiv 1$ and $0 \leq f \leq 2$ then we can take $g \equiv 1, u \equiv 0$. Moreover u is uniquely determined by this choice of g.

Now we define what is the notion of a solution. Of course it is standard way of introducing of weak solutions, important is how to deal with the third equality.

Definition 11. Solution to the problem (P) is the couple (S, u), where $u \in W_0^{1,2}(\Omega)$ and $S \in L^2(\Omega)$, satifying (weak formulation)

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega f(u)\varphi + \int_\Omega S\varphi = \int_\Omega h\varphi, \forall \varphi \in W_0^{1,2}(\Omega), \tag{WF}$$

moreover there hold that S, u are elements of maximal monotone 2-graph, i. e.

$$(S(x), u(x)) \in \mathcal{A} \text{ for a. a. } x \in \Omega, \tag{G}$$

where $(S, D) \in \mathcal{A} \iff \frac{(|S|-1)^+}{|S|} S = D$ for $S \neq 0$ and if $S = 0$ we take $D = 0$.

Remark. We could define it directly for couple $(g, u) \in \mathcal{A}$, but there would be a problem with first property of MM2G.

Remark. In the (WF) we "lost" the function g, but we obtain it simply as $g = S + f(u) \in L^2(\Omega)$.

Our strategy is the following. In the next section we prove that $\mathcal{A}$ is really MM2G. Then we show that we obtain uniqueness of u under some conditions on f. We cannot show an existence directly and so we will introduce some kind of approximation. On the level of approximation we find an appropriate set K from Kakutani - Ky Fan fixed point theorem (Theorem 22). Using it we obtain existence of a solution to approximated problem. By passing to the limit we get existence of a solution to the original problem.

MM2G

Now we show that $\mathcal{A}$ is in fact maximal monotone 2-graph. We need to verify all properties.

Ad (A1). This point is simple, because for $D = 0$ we get $0 = \left|\frac{(|S|-1)^+}{|S|} S\right| = (|S|-1)^+$, it means that we can take any element with property $|S| \leq 1$, in particular also point $S = 0$.

Ad (A2). By the definition of $\mathcal{A}$ we need to show that

$$(S_1 - S_2)\left(\frac{(|S_1|-1)^+}{|S_1|} S_1 - \frac{(|S_2|-1)^+}{|S_2|} S_2\right) \geq 0.$$

We estimate

$$\begin{aligned}
&|S_1|(|S_1|-1)^+ + |S_2|(|S_2|-1)^+ - S_1 S_2 \left(\frac{(|S_1|-1)^+}{|S_1|} + \frac{(|S_2|-1)^+}{|S_2|}\right) \geq \\
&\quad \geq |S_1|(|S_1|-1)^+ + |S_2|(|S_2|-1)^+ - |S_1||S_2| \left(\frac{(|S_1|-1)^+}{|S_1|} + \frac{(|S_2|-1)^+}{|S_2|}\right) = \\
&\quad = |S_1|(|S_1|-1)^+ + |S_2|(|S_2|-1)^+ - \left(|S_2|(|S_1|-1)^+ + |S_1|(|S_2|-1)^+\right) = \\
&\quad = (|S_1| - |S_2|)((|S_1|-1)^+ - (|S_2|-1)^+).
\end{aligned}$$

We see that the function $\alpha(|x|) = (|x|-1)^+$ has non-negative derivative with respect to $|x|$ a. e. in $\mathbb{R}$. Consequently, the whole expression above is non-egative, because α is non-decreasing function of $|x|$.

Ad (A3). We take special choice of $S_1 = S - \varepsilon W$ for $\varepsilon > 0$ and W arbitrary number. We take corresponding D_1 for which a couple (S_1, D_1) is in the graph. Let us consider only $S_1 \neq 0$ (we take smaller ε) and assume that

$$0 \leq (S - S_1)(D - D_1) = \varepsilon W \left(D - \frac{(|S - \varepsilon W| - 1)^+}{|S - \varepsilon W|}(S - \varepsilon W) \right).$$

Now we divide by $\varepsilon > 0$ and pass to the limit

$$\begin{aligned} 0 \leq \lim_{\varepsilon \to 0^+} W \left(D - \frac{(|S - \varepsilon W| - 1)^+}{|S - \varepsilon W|}(S - \varepsilon W) \right) \leq \\ \leq W \left(D - \frac{(|S| - 1)^+}{|S|} S \right). \end{aligned}$$

Number W is arbitrary, but whole product is non-negative, and therefore the quantity inside of brackets is zero, and thus $(S, D) \in \mathcal{A}$. Of course if $S = 0$ then the limiting process above is not valid, but then the assertion is trivial and we obtain $D = 0$, and thus the conclusion still holds.

Ad (A4). We know that for $D = 0$ is $|S| \leq 1$, we choose $S^*(0) = 0$. If $D > 0$, we need also $S > 0$, because otherwise the $\frac{(|S|-1)^+}{|S|} S$ woould not be positive. It is natural to take $S^*(D) = 1 + D, D > 0$. The case of $D < 0$ is analogous and for such a D we choose $S^*(D) = -1 + D$. Altogether we obtain function

$$S^*(D) = \begin{cases} 0, \text{ for } D = 0 \\ \frac{D}{|D|} + D, \text{ for } D \neq 0 \end{cases}.$$

It is of course measurable (in fact it suffice to know it for a function sign x).

Ad (A5). The last property is coercivity. We get estimate

$$\begin{aligned} SD = \frac{S^2}{|S|}(|S| - 1)^+ = |S|(|S| - 1)^+ \geq (|S| - 1)(|S| - 1)^+ \geq \\ \geq ((|S| - 1)^+)^2 \geq \frac{1}{2}|S|^2 - 2, \end{aligned}$$

but simultaneously $|D| = (|S| - 1)^+$, and therefore

$$\frac{1}{2} SD \geq \frac{1}{2}|D|^2.$$

We add them together and obtain desired inequality for $c = 3$

$$\begin{aligned} \frac{3}{2} SD &\geq \frac{1}{2}(|S|^2 + |D|^2) - 2 \\ SD &\geq \frac{1}{3}(|S|^2 + |D|^2) - \frac{4}{3} \\ SD &\geq \frac{1}{3}(|S|^2 + |D|^2) - 3. \end{aligned}$$

Uniqueness

Suppose that we have two solutions $(u_1, S_1), (u_2, S_2) \in \mathcal{A}$ of the problem (P). Then both of them satisfy (WF) and so we substract those two equlities to find that

$$\int_\Omega \nabla(u_1 - u_2)\varphi + \int_\Omega (f(u_1) - f(u_2))\varphi + \int_\Omega (S_1 - S_2)\varphi = 0, \forall \varphi \in W_0^{1,2}(\Omega).$$

We know that $\varphi = u_1 - u_2 \in W_0^{1,2}(\Omega)$, and thus

$$||\nabla(u_1 - u_2)||_2^2 + \int_\Omega (f(u_1) - f(u_2))(u_1 - u_2) + \int_\Omega (S_1 - S_2)(u_1 - u_2) = 0.$$

Because couples $(u_1, S_1), (u_2, S_2)$ are in our monotone graph, we have, by (A2), $\int_\Omega (S_1 - S_2)(u_1 - u_2) \geq 0$ and it leads us to the inequality

$$||\nabla(u_1 - u_2)||_2^2 + \int_\Omega (f(u_1) - f(u_2))(u_1 - u_2) \leq 0.$$

By Poincaré inequality (Claim 5) we get

$$\frac{1}{c_p}||u_1 - u_2||_2^2 + \int_\Omega (f(u_1) - f(u_2))(u_1 - u_2) \leq 0.$$

We see that if the f is monotone function, we get

$$||u_1 - u_2||_2^2 \leq 0,$$

which implies $u_1 = u_2$ a. e. in Ω.

Remark. Instead of monotonocity we may demand that differential quotients of f are bounded from below, more precisely that

$$\inf \left\{ \left|\left| \frac{f(u_1) - f(u_2)}{u_1 - u_2} \right|\right|_2 ; u_1, u_2 \in W_0^{1,2}, u_1 \neq u_2 \right\} > -\frac{1}{c_p} \ .$$

If we know that $\frac{1}{c_p} = \lambda_{\min}$, where $\lambda_{\min}$ is the smallest eigenvalue of negative Laplacian, i. e. $-\triangle u = \lambda_{\min} u$, we obtain nice and precise condition for uniqueness of u (at least for smooth boundary).

Auxiliary problem

We slightly modify our graph and solve different problem. In the process we underline moment which is crucial and cannot be done in the original setting.

Let $n \in \mathbb{N}$. We will look for a couple of functions $u_n \in W_0^{1,2}(\Omega), S_n \in L^2(\Omega)$ such that $(S_n(x), u_n(x)) \in \mathcal{A}^n$ for a. a. $x \in \Omega$, where

$$(S_n, u_n) \in \mathcal{A}^n \iff \frac{S}{|S|}[(|S| - 1)^+]^{\frac{2n+3}{2n+1}} = D \text{ for } S \neq 0 \text{ and} \quad S = 0 \text{ for } D = 0, \tag{Gn}$$

moreover there holds

$$\int_\Omega \nabla u_n \cdot \nabla \varphi + \int_\Omega f(u_n)\varphi + \int_\Omega S_n \varphi = \int_\Omega h\varphi, \forall \varphi \in W_0^{1,2}(\Omega). \tag{WFn}$$

Observe that $\mathcal{A}^n$ is again MM2G. It can be done by repeating same procedure as above. Let us just mention that measurable selection is

$$S^*(D) = \begin{cases} 0, \text{ for } D = 0 \\ \frac{D}{|D|} + D^{\frac{2n+1}{2n+3}}, \text{ for } D \neq 0 \end{cases}$$

and the coercivity follows from the estimate

$$((|S| - 1)^+)^{2+\frac{4}{2n+1}} \geq \frac{1}{2}|S|^2 - 2, \forall n \in \mathbb{N}.$$

Remark. Let us note that this form of exponent is nice, because we do not obtain something like $\sqrt{u}$, and therefore signs are not a problem.

Let us suppose that there exists $K_n > 0$ (we will specify this number later) and that we already have S_n for which $||S_n||_2 \leq K_n$. Then, by third example from the section 4.2, there exists some $u_n \in W_0^{1,2}(\Omega)$ which solves (WFn) in the weak sense. Then we can test the equation by the solution and due to Hölder, Young and Poincaré (Claim 5) inequalities we obtain following chain of estimates

$$\begin{aligned}
\int_\Omega |\nabla u_n|^2 &= -\int_\Omega f(u_n)u_n - \int_\Omega S_n u_n + \int_\Omega h u_n \\
||\nabla u_n||_2^2 &\leq \frac{\varepsilon}{2}||u_n||_2^2 + \frac{1}{2\varepsilon}||h||_2^2 + \frac{\varepsilon}{2}||u_n||_2^2 + \frac{1}{2\varepsilon}||f||_2^2 + \frac{\varepsilon}{2}||u_n||_2^2 + \frac{1}{2\varepsilon}||S_n||_2^2 \\
\frac{1}{c_p^2}||u_n||_2^2 &\leq \frac{3}{2}\varepsilon||u_n||_2^2 + \frac{1}{2\varepsilon}(||h||_2^2 + ||f||_2^2 + ||S_n||_2^2).
\end{aligned}$$

Let us set $\varepsilon = \frac{1}{3c_p^2}$ and plug it into the relation above to get

$$||u_n||_2^2 \leq 3c_p^4(||h||_2^2 + ||f||_2^2 + ||S_n||_2^2) \leq c_0 K_n^2,$$

where $c_0 > 0$ is some constant which depend on c_p and norms of h and f.

Remark. Note that this estimate is nedeed also in the proof of existence of u_n.

Existence of solution to the auxiliary problem

Now we tend to apply Kaukutani - Ky Fan theorem, so we need some sets and multifunctions. It is reasonable to define sets

$$\begin{aligned}
X &= L^2(\Omega) \text{ with the weak topology,} \\
\mathcal{K} &= \{S \in L^2(\Omega); ||S||_2 \leq K_n\}, \forall n \in \mathbb{N}, \\
Y &= W_0^{1,2}(\Omega) \cap \{u \in L^2(\Omega); ||u||_2 \leq \sqrt{c_0}K_n\}.
\end{aligned}$$

We should denoted it as $\mathcal{K}_n, Y_n$, but for the sake of readibility we will omit those indices. Now we may introduce a mapping $\alpha : \mathcal{K} \to Y$ as $\alpha S_n \mapsto u_n$. It makes sense, because from previous paragraph we know that the solution exists. Moreover, by Claim 7, we know that it is a compact mapping from $\mathcal{K}$ to $\{u \in L^2(\Omega); ||u||_2 \leq \sqrt{c_0}K_n\}$.

We know that X is a locally convex space, we see that $\mathcal{K}$ is non-empty, bounded and convex set. Moreover $\mathcal{K}$ is closed both weakly and strongly, it is true because of Mazur theorem (Claim 13) and convexity of $\mathcal{K}$. By Eberlein-Šmulian theorem (Claim 14) we obtain weak compactness of $\mathcal{K}$.

Now we need to find upper semicontinuous mapping $F : \mathcal{K} \to 2^{\mathcal{K}}$ such that $F(S) \neq \emptyset$ is convex and (weakly) closed set for all $S \in \mathcal{K}$.

For the solution u_n we find a set $\{\tilde{S} \in \mathcal{K}; (\tilde{S}(x), u(x)) \in \mathcal{A} \text{ a. e. in } \Omega\}$. It will be our set of admissible functions S and also the image of our multifunction, which we do not have yet. But it is not clear that the set is non-empty.

Of course we may define at least one of them by the formula

$$\tilde{S} = \begin{cases} \frac{u_n}{|u_n|} + u^{\frac{2n+1}{2n+3}}, & \text{if } u_n \neq 0 \\ 0, & \text{if } u_n = 0 \end{cases},$$

which is integrable, but we need also same bound as initial function S_n, which is K_n.

Remark. This is the moment when we would be lost in the original problem, because we would define $\tilde{S}$ by almost same formula, but without exponent $\frac{2n+1}{2n+3}$. But the fact that this exponent is less than 1 is crucial.

So let us make the following estimates

$$||\tilde{S}||_2^2 = \int_\Omega \left| \frac{u_n}{|u_n|} + u_n^{\frac{2n+1}{2n+3}} \right|^2 \leq 4 \int_\Omega \left(1 + |u_n|^{2\cdot\frac{2n+1}{2n+3}}\right) \leq$$
$$\leq 4|\Omega| + 4\left(\int_\Omega |u_n|^2\right)^{\frac{2n+1}{2n+3}} |\Omega|^{\frac{2}{2n+3}} = 4|\Omega| + 4|\Omega|^{\frac{2}{2n+3}} \left(||u_n||_2^2\right)^{\frac{2n+1}{2n+3}} \leq$$
$$\leq 4|\Omega| + 4|\Omega|^{\frac{2}{2n+3}} \left(c_0 K_n^2\right)^{\frac{2n+1}{2n+3}} = c_1 K_n^{2\frac{2n+1}{2n+3}},$$

where c_1 is positive constant depending only on Ω, n and c_0, i. e. not on K_n. By the choice $K_n = c_1^{\frac{2n+3}{4}}$ we obtain $||\tilde{S}||_2^2 \leq K_n^2$.

Hence, we may define a multifunciton $\beta : \{u \in L^2(\Omega); ||u||_2 \leq \sqrt{c_0} K_n\} \to 2^{\mathcal{K}}$ as

$$u_n \mapsto \{\tilde{S} \in \mathcal{K}; (\tilde{S}(x), u_n(x)) \in \mathcal{A}^n \text{ a. e. in } \Omega\}.$$

We see that graph of β (i. e. G_β) is monotone in the sense of Definition 6 (section 3.3), because of monotonocity of $\mathcal{A}^n$. But it is also maximal monotone graph. To see this we consider $(u, S) \in \mathbb{R} \times \mathbb{R}$ and suppose that for all $(\tilde{u}, \tilde{S}) \in G_\beta$ holds $(u - \tilde{u})(S - \tilde{S}) \geq 0$. By the definition of G_β we know that $(\tilde{S}, \tilde{u}) \in \mathcal{A}^n$ and so $(S, u) \in \mathcal{A}^n$ from maximality of $\mathcal{A}^n$. We know that $||u||_2 \leq \sqrt{c_0} K_n$. Because $(S, u) \in \mathcal{A}^n$ we have $|u|^2 = [(|S| - 1)^+]^{2\frac{2n+3}{2n+1}}$. By trivial estimate we get

$$|u|^{2\frac{2n+1}{2n+3}} = [(|S| - 1)^+]^2 \geq \frac{1}{2}|S|^2 - 2$$

and by Hölder inequality we obtain

$$||S||_2^2 \leq \int_\Omega \left(4 + 2|u|^{2\frac{2n+1}{2n+3}}\right) \leq 4|\Omega| + 2\left(\int_\Omega |u|^2\right)^{\frac{2n+1}{2n+3}} |\Omega|^{\frac{2}{2n+3}} < K_n^2,$$

where the last estimate is true due to our choice of K_n above. Therefore $S \in \mathcal{K}$ and so $(u, S) \in G_\beta$.

We finally introduce $F : \mathcal{K} \to 2^{\mathcal{K}}$ as $F = \beta \circ \alpha$. Right now we proved that $F(S) \neq \emptyset$ for all $S \in \mathcal{K}$. Our next task is to observe other properties of $F(S)$ for arbitrary $S \in \mathcal{K}$.

Ad convexity. We take $\lambda \in [0, 1]$ and two (not necesarilly distinct) elements $\tilde{S}_1, \tilde{S}_2 \in F(S)$. We want to show that $\lambda\tilde{S}_1 + (1 - \lambda)\tilde{S}_2$ belongs also into $F(S)$. Let us take any $(\tilde{S}, \tilde{u}) \in \mathcal{A}^n$ and compute

$$(\tilde{S} - (\lambda\tilde{S}_1 + (1 - \lambda)\tilde{S}_2), \tilde{u} - u) = (\lambda\tilde{S} - \lambda\tilde{S}_1 + (1 - \lambda)\tilde{S} - (1 - \lambda)\tilde{S}_2), \tilde{u} - u) =$$
$$= \lambda(\tilde{S} - \tilde{S}_1, \tilde{u} - u) + (1 - \lambda)(\tilde{S} - \tilde{S}_2, \tilde{u} - u).$$

We know that $(\tilde{S}, \tilde{u}), (\tilde{S}_1, u)$ and $(\tilde{S}, \tilde{u}), (\tilde{S}_2, u)$ belong to the graph $\mathcal{A}^n$. By (A2) we thus obtain

$$(\tilde{S} - (\lambda\tilde{S}_1 + (1 - \lambda)\tilde{S}_2), \tilde{u} - u) \geq 0.$$

Consequently, by (A3), we get $(\lambda\tilde{S}_1 + (1 - \lambda)\tilde{S}_2, u) \in \mathcal{A}^n$ and by the definition of F we finally have $\lambda\tilde{S}_1 + (1 - \lambda)\tilde{S}_2 \in F(S)$.

Remark. Note that exactly same procedure was done already in section 4.5.1.

Ad closedness. Let us take any sequence $(\tilde{S}^k)_{k=1}^\infty \in \mathcal{K}$ such that $(\tilde{S}^k(x), u_n(x)) \in \mathcal{A}^n$ a. e. in Ω for all $k \in \mathbb{N}$ (so they belong to $F(S)$) and $\tilde{S}^k \rightharpoonup \tilde{S}$ in $L^2(\Omega)$. We want to show that $(\tilde{S}, u_n) \in \mathcal{A}^n$ too and moreover $\tilde{S} \in L^2(\Omega)$. The fact that $\tilde{S} \in L^2(\Omega)$ is trivial thanks to the weak convergence. We immediately have the inequality $\limsup_{k\to\infty} \int_\Omega \tilde{S}^k u_n \leq \int_\Omega \tilde{S} u$ and so, by Lemma 26, we obtain $(\tilde{S}, u_n) \in \mathcal{A}^n$. Weak

closedness (which is again same as the strong one, due to the convexity) of $F(S)$ follows.

Alternatively we may apply Lemma 24 to obtain same conclusion.

Now we show that F is in fact upper semicontinuous mapping. Thanks to the Lemma 21 it suffice to show that its graph is closed. Let us suppose that

$$S^k \rightharpoonup S \text{ in } L^2(\Omega),$$
$$\tilde{S}^k \rightharpoonup \tilde{S} \text{ in } L^2(\Omega)$$

and $\tilde{S}^k \in F(S^k)$ for all $k \in \mathbb{N}$, we need to show that $\tilde{S} \in F(S)$ too. There are two options: either we use Lemma 24 or Lemma 26.

If we try Lemma 26, we will find that we really need uniqueness of the solution (to obtain that limit of $(u^k)_{n=1}^{\infty}$ and u corresponding to function S are same). So we use Lemma 24. We can rewrite our assumption as $\tilde{S}^k \in \beta(\alpha(S^k))$. Recall that β has maximal monotone graph and α is compact mapping from K to $L^2(\Omega)$. We thus use Lemma 24 on the sequences

$$\alpha(S^k) \to \alpha(S) \text{ in } L^2(\Omega),$$
$$\tilde{S}^k \rightharpoonup \tilde{S} \text{ in } L^2(\Omega)$$

with the assumption $\tilde{S}^k \in \beta(\alpha(S^k))$ which implies that $(\alpha(S^k(x)), \tilde{S}^k(x)) \in G_\beta$ for a. a. $x \in \Omega$, where G_β is (maximal monotone) graph of β. Therefore we obtain $(\alpha(S(x)), \tilde{S}(x)) \in G_\beta$ for a. a. $x \in \Omega$. Which can be rewritten as $\tilde{S} \in \beta(\alpha(S)) = F(S)$. This completes the proof of upper semicontinuity of F and by Theorem 22 we obtain existence of fixed point $S \in F(S), S \in K$. For this S we have u, such that (S, u) solves (WFn) and moreover this couple is element of $\mathcal{A}^n$. In other words we proved an existence of solution to the our approximation.

Solution to the original problem

Now we need to pass to the limit as $n \to \infty$. We know that for any n there holds

$$\int_\Omega \nabla u_n \cdot \nabla \varphi + \int_\Omega f(u_n)\varphi + \int_\Omega S_n \varphi = \int_\Omega h\varphi, \forall \varphi \in W_0^{1,2}(\Omega).$$

We now test by the solution, because $(u_n, S_n) \in \mathcal{A}^n$ we may use the fact that $S_n u_n \geq \frac{1}{2}|S_n|^2 - 2$ and again by Hölder, Young and Poincaré inequalities we obtain

$$\int_\Omega |\nabla u_n|^2 \varphi + \int_\Omega f(u_n)u_n + \int_\Omega S_n u_n = \int_\Omega h u_n$$
$$\frac{1}{c_p^2}||u_n||_2^2 + \frac{1}{2}||S_n||_2^2 - 2|\Omega| \leq \varepsilon ||u_n||_2^2 + \frac{1}{2\varepsilon}(||f||_2^2 + ||h||_2^2),$$

from which we are able to deduce

$$||u_n||_{1,2} + ||S_n||_2 \leq C(f, h, \Omega),$$

where $C = C(f, h, \Omega) > 0$ is unifrom constant as usual. This is big improvement of estimates which we had up to now. Because of reflexivity we have existence of weak limits, namely

$$u_n \rightharpoonup u \text{ in } W_0^{1,2}(\Omega),$$
$$S_n \rightharpoonup S \text{ in } L^2(\Omega)$$

and we know that $||u||_{1,2} \leq C, ||S||_2 \leq C$. Because of those limits and with help of Nemytskii opearator (Claim 6) we can pass to the limit in the equation to obtain (WF), i. e.

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega f(u)\varphi + \int_\Omega S\varphi = \int_\Omega h\varphi, \forall \varphi \in W_0^{1,2}(\Omega).$$

To conclude the proof it is enough to show that $(S, u) \in \mathcal{A}$.

We have couples $(S_n, u_n) \in \mathcal{A}^n$, it means that $u_n = \frac{S_n}{|S_n|}[(|S_n| - 1)^+]^{\frac{2n+3}{2n+1}}$, or it is equal to zero, but then everything is simple. Let us consider mapping $T : \mathcal{A}^n \to \mathcal{A}$ given by formula

$$T(S, u) = (T_1(S, u), T_2(S, u)) = \left(S, \frac{S}{|S|}(|S| - 1)^+\right) \text{ for } (S, u) \in \mathcal{A}^n.$$

We claim that $T_2(S_n, u_n) \rightharpoonup u$ in $L^2(\Omega)$. Because $u_n \rightharpoonup u$ and

$$\int_\Omega (T_2(S_n, u_n) - u)\varphi = \int_\Omega (T_2(S_n, u_n) - u_n)\varphi + \int_\Omega (u_n - u)\varphi, \forall \varphi \in L^2(\Omega,)$$

it suffice to prove that $T_2(S_n, u_n) - u_n \to 0$ in $L^\infty(\Omega)$. Let us estimate

$$\begin{aligned} |T_2(S_n, u_n) - u_n| &= \left| \frac{S_n}{|S_n|}(|S_n| - 1)^+ - \frac{S_n}{|S_n|}[(|S_n| - 1)^+]^{\frac{2n+3}{2n+1}} \right| \\ &= (|S_n| - 1)^+ \left| 1 - [(|S_n| - 1)^+]^{\frac{2}{2n+1}} \right| \leq \\ &\leq ||(|S_n| - 1)^+||_2^2 \cdot \left|\left| 1 - [(|S_n| - 1)^+]^{\frac{2}{2n+1}} \right|\right|_2^2. \end{aligned}$$

As we see the first term is uniformly bounded (due to uniform bound of S_n) and second one is arbitrarily small. It holds because of same boundedness and by application of Lebesgue dominated convergence theorem. Hence $T_2(S_n, u_n) \rightharpoonup u$ in $L^2(\Omega)$.

Now we aim to show that $\limsup_{n\to\infty} \int_\Omega S_n T_2(S_n, u_n) \leq \int_\Omega Su$. It is simple, we may write

$$\limsup_{n\to\infty} \int_\Omega S_n T_2(S_n, u_n) = \limsup_{n\to\infty} \int_\Omega S_n(T_2(S_n, u_n) - u_n) + \limsup_{n\to\infty} \int_\Omega S_n u_n.$$

From the strong convergence above we know that $\lim \int_\Omega S_n(T_2(S_n, u_n) - u_n) = 0$ and because $u_n \to u$ in $L^2(\Omega)$, which as usual follows from the Claim 7, we see that the desired inequality holds.

What remains is to invoke Lemma 26 for sequences $(S_n)_n, (T_2(S_n, u_n))_n$ to conclude that $(S, u) \in \mathcal{A}$. It completes the whole proof and we may formulate what we achieved.

Theorem 27. Let $\Omega \in C^{0,1}$ be a subset of $\mathbb{R}^d$, $h \in L^2(\Omega)$, $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function and $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function given by the formula

$$H(g, u) = \frac{(|g - f(u)| - 1)^+}{|g - f(u)|}(g - f(u)) - u.$$

Then there exist functions $u, g : \Omega \to \mathbb{R}$, such that $u \in W_0^{1,2}(\Omega), g \in L^2(\Omega)$, weakly solving problem

$$\begin{aligned} -\triangle u(x) + g(x) &= h(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ H(g, u) &= 0 \text{ in } \Omega. \end{aligned}$$

Moreover if f is monotone function, then u is uniquely determined.

We left the following example as an amusing exercise for the reader.

Example. Show the same result by use of Galerkin approximation instead of Kakutani-Ky Fan theorem.

Remark. In the light of the approach demnostrated above and the preceding example we may informally conclude that both methods are in fact equivalent in the sense that we really need to apply same estimates and inequalities. But Galerkin method is more straightforward and even shorter to do.

Epilogue

> It is impossible to be a mathematician without being a poet in soul.
>
> *Sofia Kovalevskaya*

We introduced sucesfully notion of the topological degree. We initially intended to write down all, or at least most, of the proofs, but the whole work would become too extensive, and therefore we reduced this part into statements alone. Among other things there was another proof of fundamental theorem of algebra and some interesting results about mappings in the infinitely dimensional Banach spaces.

Afterwards, we proved Brouwer and Schauder theorems and some of their immediate consequences as Schaefer or Altman theorems. We made simple examples to ilustrate way how to apply them and moreover we discussed assumptions of both Brouwer and Schauder theorems. For the sake of completeness we thought about assumptions of Banach contraction principle and proved an existence of non-complete space, where any contraction has at least one fixed point.

As the consequence of Brouwer theorem we showed important Kakutani-Ky Fan fixed point theorem. At this point we would like to mention that it is interesting to introduce notion of the degree for multivalued mappings too, but we choose more straightforward way how to prove it.

In the first half of fourth chapter we made a whole range of illustrative applications in the theory of ode's, pde's and integro-differential equations. In its second half we solved three problems, which we recall below.

First one was solution to a more general equation than in known article [27], where we moreover considered perturbation with a compact support and not just a monotone dependence. We guess that after some work on generalization it could be applied also for pde's problems. Such a result would be very nice, because we are quite capable to deal with a monotone terms by standard methods of pde's, but with non-monotone ones there are big problems in general. But such a generalization might not be as straightforward as we would wish.

Both second and third problem use Kakutani-Ky Fan theorem. Second was concerned by the solution to predaor-prey model with a refuge, which is definitely nice application. Last one worked with a certain pde with constraint which determined maximal monotone graph. Its proof is more complicated than more standard approach by Galerkin, but we saw that both approaches are somehow equivalent, beacuse in our approach we find certain moment when we need the same tool (estimate or convergence) as in the classical one and vice versa. It certainly gives us nice perspective on the whole problem.

Besides points mentioned above, there are definitely many other ways how we could extend this work. Firstly it is possible to work more on the theoretical level and try to find new fixed point theorems. We also might focus on narrower application, e. g. periodical solutions to ode's or perturation theory (we did not mention it, but it is also natural area for use of fixed point theorems). But there are also applications out of sphere of influence of differential equation, e. g. game theory.

We see that issues of fixed point theory are very wide, interesting and there is still plenty of work to do.

List of Figures

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