FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Dalimil Peša

Amalgam Spaces

Department of Mathematical Analysis

Supervisor of the master thesis:	prof. RNDr. Luboš Pick, CSc., DSc.
Study programme:	Mathematics
Study branch:	Mathematical Analysis

Prague 2019

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date .... signature of the author

My foremost thanks go to professor Pick, my supervisor, for his invaluable guidance and support throughout the completion of this thesis and, most of all, for the careful review and constructive critique he provided.

I would also like to extend my deep gratitude to professor Feichtinger for introducing me to the concept of Wiener amalgams and providing many valuable insights as well as sources for further study.

Title: Amalgam Spaces

Author: Dalimil Peša

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Luboš Pick, CSc., DSc., Department of Mathematical Analysis

Abstract: In this thesis we introduce the concept of Wiener–Luxemburg amalgam spaces which are a modification of the more classical Wiener amalgam spaces intended to address some of the shortcomings the latter face in the context of rearrangement invariant Banach function spaces.

We first provide some new results concerning quasinormed spaces. Then we illustrate the asserted shortcomings of Wiener amalgam spaces by providing counterexamples to certain properties of Banach function spaces as well as rearrangement invariance. We introduce the Wiener–Luxemburg amalgam spaces and study their properties, including (but nor limited to) their normability, embeddings between them and their associate spaces. Finally we provide some applications of this general theory.

Keywords: Amalgam space Banach function space functional properties

Contents

1 Introduction

In this thesis we introduce the concept of Wiener–Luxemburg amalgam spaces which are a modification of the more classical Wiener amalgam spaces. The principal idea of both kinds of amalgam spaces is to treat separately the local and global behaviour of a given function, in the sense that said function is required to be locally in one space and globally in a different space. The exact meaning of being locally and globally in a space varies depending on the desired generality and author's preference.

The classical Wiener amalgams approach this issue in a very general, albeit quite non-trivial, manner. They were, in their general form, first introduced by Feichtinger in [4], although the less general cases were studied earlier, see for example the paper [9] due to Holland, and some special cases date as far back as 1926 when the first example of such a space was introduced by Wiener in [17]. The different versions of these spaces saw many applications in the last decades, great surveys of which have been conducted, concerning a somewhat restricted version, by Fournier and Stewart in [7] and, concerning the more general versions, by Feichtinger in [5] and [6]. Probably the most famous example is the Tauberian theorem for the Fourier transform on the real line due to Wiener (see [18] and [19]).

One unfortunate property of Wiener amalgams is that their construction does not preserve the properties of Banach function spaces, nor does it preserve rearrangement invariance. This approach is therefore unsuitable when one wishes to work in this context. But this often is the case, since there are many situations when the need arises naturally to prescribe separately the conditions on local and on global behaviour of a function. One such situation is the study of optimal Sobolev type embedding over the entire Euclidean space in the context of rearrangement invariant Banach function spaces as performed by Alberico, Cianchi, Pick and Slavíková in [1]. A very natural example is the optimal target space, in the context of rearrangement invariant Banach function spaces, for the limiting case of the classical Sobolev embedding over the entire Euclidean space, which has been found by Vybíral in [16]. Another such situation arised during the study of generalised Lorentz–Zygmund spaces which led to the introduction of broken logarithmic functions to allow separate treatment of local and global properties of functions in this context. For further details and a comprehensive study of generalised Lorentz–Zygmund spaces we refer the reader to [14].

This led us to develop the theory of Wiener–Luxemburg amalgam spaces, which aims to eliminate these limitations and to provide a general framework for separate prescription of local and global conditions in the context fo rearrangement invariant Banach function spaces. The starting point is provided by the non-increasing rearrangement, which is the crucial element in the theory of said spaces and which naturally separates the local behaviour of a function from its global behaviour, at least in the sense of size. This allows us to define Wiener–Luxemburg amalgam spaces in a very easy and straightforward manner.

The upside of Wiener–Luxemburg amalgam spaces is that they retain most of the properties of Banach function spaces as well as rearrangement invariance. The downside is that they lose one property (which Wiener amalgams do not loose), namely, while they are always quasinormed, they may be not normable.

The thesis is structured as follows. In Chapter 2 we present the basic theoretical background needed in order to build the theory in later chapters.

In Chapter 3 we show some abstract results concerning quasinormed spaces which we will need later in the thesis and which we believe to be both new and of independent interest.

In Chapter 4 we present some counterexamples which show that our claim that Wiener amalgams are usually neither Banach function spaces nor rearrangement invariant is justified. Those results are quite simple and are certainly known to the experts in the field of Wiener amalgams but we present them here for the reader's convenience since we were unable to find them in literature.

The Chapter 5 is the main part of the thesis where the abstract theory of Wiener–Luxemburg amalgam spaces is developed in some detail. We show that they are quasinormed linear spaces and that they satisfy every axiom of rearrangement invariant Banach function spaces, except normability, then we provide (partial) characterisation of their associate spaces, full characterisation of their embeddings and put them in relation with the concepts of sum and intersection of Banach spaces. Furthermore we generalise the well known classical result that

$$L^1 \cap L^\infty \hookrightarrow A \hookrightarrow L^1 + L^\infty$$

by showing that L^1 is the locally weakest and globally strongest rearrangement invariant Banach function space, while L^∞ is, in the same setting, the locally strongest and globally weakest space. Needless to say, our definition of Wiener–Luxemburg amalgam spaces is general enough to cover all the spaces obtained in the applications outlined above.

Finally, in Chapter 6 we apply this theory to two classical classes of spaces, namely to that of Lebesgue and Orlicz spaces. By doing so, we provide a concrete examples of application of the general theory and hopefully provide some insight into why we believe our approach to be the right one.

Our main contribution is Chapter 5 which consists entirely of new results (or, in one case, a new alternative proof of a classical result). Chapter 3 is also new and due to the author, although the method at the heart of the proofs has been obtained by Nekvinda in [13]. The counterexamples in Chapter 4 cannot be found in literature and their proofs are due to the author, however these results are not new as they are known by experts in the field of Wiener amalgams. The results in Chapter 6 are either classical or known but the proofs are new and, as we believe, provide more insight into the matter than the classical ones.

2 Preliminaries

This chapter serves to establish the basic theoretical background upon which we will build our theory of Wiener–Luxemburg amalgam spaces. The definitions and notation is intended to be as standard as possible. The usual reference for most of this theory is [2].

Throughout this thesis we will denote by (R, μ), and occasionally by (S, ν), some arbitrary (totally) sigma-finite measure space. Provided a μ-measurable set $E \subseteq R$ we will denote its characteristic function by χ_E. By $M(R, \mu)$ we will denote the set of all extended complex-valued μ-measurable functions defined on R. As is customary, we will identify functions that coincide μ-almost everywhere. We will further denote by $M_0(R, \mu)$ and $M_+(R, \mu)$ the subsets of $M(R, \mu)$ containing, respectively, the functions finite μ-almost everywhere and the non-negative functions.

For brevity, we will abbreviate μ-almost everywhere, $M(R, \mu)$, $M_0(R, \mu)$ and $M_+(R, \mu)$ to μ-a.e., M, M_0 and M_+, respectively, when there is no risk of confusing the reader.

When X, Y are two topological linear spaces, we will denote by $Y \hookrightarrow X$ that $Y \subseteq X$ and that the identity mapping $I : Y \to X$ is continuous.

As for some special cases, we will denote by λ^n the classical n-dimensional Lebesgue measure, with the exception of the 1-dimensional case in which we will simply write λ. We will further denote by m the counting measure over $\mathbb{N}$. When $p \in [1, \infty]$ we will denote by L^p the classical Lebesgue space (of functions in $M(R, \mu)$) defined by

$$L^p = \left\{ f \in M(R, \mu); \int_R |f|^p \, d\mu < \infty \right\}$$

equipped with the usual norm

$$\|f\|_p = \left(\int_R |f|^p \, d\mu \right)^{\frac{1}{p}},$$

with the usual modifications when $p = \infty$. In the special case when $(R, \mu) = (\mathbb{N}, m)$ we will denote this space by l^p.

Note that in this thesis we consider 0 to be an element of $\mathbb{N}$.

2.1 Non-increasing rearrangement

In this section we present the concept of the non-increasing rearrangement of a function and state some of its properties that will be important later in the thesis. We proceed in accordance with [2, Chapter 2].

The first step is to introduce the distribution function which is defined as follows.

Definition 2.1. *The distribution function μ_f of a function $f \in M$ is defined for $s \in [0, \infty)$ by*

$$\mu_f(s) = \mu(\{t \in R; f(t) > s\}).$$

The non-increasing rearrangement is then defined as the generalised inverse of the distribution function.

Definition 2.2. *The non-increasing rearrangement* f^* *of function* $f \in M$ *is defined for* $t \in [0, \infty)$ *by*

$$f^*(t) = \inf\{s \in [0, \infty); \mu_f(s) \leq t\}.$$

For the basic properties of the distribution function and the non-increasing rearrangement, with proofs, see [2, Chapter 2, Propositin 1.3] and [2, Chapter 2, Propositin 1.7] respectively. We consider those basic properties to be classical and well known and we will be using them without further explicit reference.

An important concept used in the thesis is that of equimeasurability defined below.

Definition 2.3. *We say that the functions* $f \in M(R, \mu)$ *and* $g \in M(S, \nu)$ *are equimeasurable if* $\mu_f = \nu_g$.

It is not hard to show that two functions are equimeasurable if and only if their non-increasing rearrangements coincide too.

A very important classical result is the Hardy–Littlewood inequality which we will use extensively in the thesis. For proof, see for example [2, Chapter 2, Theorem 2.2].

Theorem 2.4. *It holds for all* $f, g \in M$ *that*

$$\int_R |fg| \, d\mu \leq \int_0^\infty f^* g^* \, d\lambda.$$

It follows directly from this result that it holds for every $f, g \in M$ that

$$\sup_{\substack{\tilde{g} \in M \\ \tilde{g}^* = g^*}} \int_R |f\tilde{g}| \, d\mu \leq \int_0^\infty f^* g^* \, d\lambda.$$

This motivates the definition of resonant measure spaces.

Definition 2.5. *A sigma-finite measure space* (R, μ) *is said to be resonant if it holds for all* $f, g \in M(R, \mu)$ *that*

$$\sup_{\substack{\tilde{g} \in M \\ \tilde{g}^* = g^*}} \int_R |f\tilde{g}| \, d\mu = \int_0^\infty f^* g^* \, d\lambda.$$

The property of being resonant is an important one. Luckily there is a straightforward characterisation of resonant measure spaces which we list below. For proof and further details see [2, Chapter 2, Theorem 2.7].

Theorem 2.6. *A sigma-finite measure space is resonant if and only if it is either non-atomic or completely atomic with all atoms having equal measure.*

2.2 Norms and quasinorms

In this and the following section we provide the definitions for several classes of functionals we will study in the thesis. All definitions should be standard or at least straightforward generalisations of standard ones.

The starting point shall be the class of norms.

Definition 2.7. *Let X be a complex linear space. A functional $\|\cdot\| : X \to [0, \infty)$ will be called a norm if it satisfies the following conditions:*

1. *it is positively homogeneous, i.e.* $\forall a \in \mathbb{C}\ \forall x \in X : \|ax\| = |a|\|x\|$,
2. *it satisfies* $\|x\| = 0 \Leftrightarrow x = 0$ *in* X,
3. *it is subadditive, i.e.* $\forall x, y \in X : \|x + y\| \leq \|x\| + \|y\|$.

Because the definition of norms is sometimes too strong we will need a class of weaker functionals, namely quasinorms.

Definition 2.8. *Let X be a complex linear space. A functional $\|\cdot\| : X \to [0, \infty)$ will be called a quasinorm if it satisfies the following conditions:*

1. *it is positively homogeneous, i.e.* $\forall a \in \mathbb{C}\ \forall x \in X : \|ax\| = |a|\|x\|$,
2. *it satisfies* $\|x\| = 0 \Leftrightarrow x = 0$ *in* X,
3. *there is a constant $C > 0$, called the modulus of concavity of $\|\cdot\|$, such that it is subadditive up to this constant, i.e.* $\forall x, y \in X : \|x + y\| \leq C(\|x\| + \|y\|)$.

It is obvious that every norm is also a quasinorm with the modulus of concavity equal to 1 and that every quasinorm with the modulus of concavity less than or equal to 1 is also a norm.

We will, for technical reasons, always assume that the modulus of concavity of any quasinorm is at least 1. This does not lessen the generality in any way, since it is obvious that if $C > 0$ is a modulus of concavity of $\|\cdot\|$ then so is any $C' > C$.

It is a well-known fact that every norm defines a metrisable topology on X and that it is continuous with respect to that topology. This is not true for quasinorms, but this can be remedied thanks to the Aoki–Rolewitz theorem which we list below. Further details can be found for example in [11] or in [3, Appendix H].

Theorem 2.9. *Let $\|\cdot\|_X$ be a quasinorm over the linear space X. Then there is a quasinorm $\|\cdot\|_{\tilde{X}}$ such that*

1. *there is a finite constant $C_0 > 0$ such that it holds for all $x \in X$ that*
$$C_0^{-1}\|x\|_X \leq \|x\|_{\tilde{X}} \leq C_0\|x\|_X,$$
2. *there is an $r \in (0, 1]$ such that it holds for all $x, y \in X$ that*
$$\|x + y\|_{\tilde{X}}^r \leq \|x\|_{\tilde{X}}^r + \|y\|_{\tilde{X}}^r.$$

The direct consequence of this result is that every quasinorm defines a metrisable topology on X and that the convergence in said topology is equivalent to the convergence with respect to the original quasinorm, in the sense that $x_n \to x$ in the induced topology if and only if $\lim_{n\to\infty}\|x_n - x\| = 0$.

Natural question to ask is when do different quasinorms define equivalent topologies. It is an easy exercise to show that the answer is the same as in the case of norms, that is that two quasinorms are topologically equivalent if and only if they are equivalent in the following sense.

Definition 2.10. *Let* $\|\cdot\|_X$ *and* $\|\cdot\|_{\tilde{X}}$ *be quasinorms over the linear space* X*. We say that* $\|\cdot\|_X$ *and* $\|\cdot\|_{\tilde{X}}$ *are equivalent if there is some* $C_0 > 0$ *such that it holds for all* $x \in X$ *that*

$$C_0^{-1}\|x\|_X \leq \|x\|_{\tilde{X}} \leq C_0\|x\|_X.$$

In the last part of this section, we recall the concepts of sum and intersection of normed spaces.

Definition 2.11. *Let* X *and* Y *be normed linear spaces equipped with the norms* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$ *respectively. Suppose that there is a Hausdorff topological linear space* Z *into which* X *and* Y *are continuously embedded. We then define the spaces* $X + Y$ *and* $X \cap Y$ *as*

$$\begin{aligned} X + Y &= \{z \in Z; \exists x \in X \; \exists y \in Y : z = x + y\}, \\ X \cap Y &= \{z \in Z; z \in X, \; z \in Y\}, \end{aligned}$$

equipped with the norms

$$\begin{aligned} \|z\|_{X+Y} &= \inf\{\|x\|_X + \|y\|_Y; x \in X, y \in Y, x + y = z\}, \\ \|z\|_{X\cap Y} &= \max\{\|z\|_X, \|z\|_Y\}, \end{aligned}$$

respectively.

The concepts presented above play a crucial role in the theory of interpolation. For further details, we refer the reader to [2, Chapter 3], where one can also find the following result (as [2, Chapter 3, Theorem 1.3]).

Theorem 2.12. *Let* X *and* Y *be as above. Then* $X+Y$ *and* $X\cap Y$*, when equipped with their respective norms, are normed linear spaces. Furthermore, if* X *and* Y *are Banach spaces, then so are* $X + Y$ *and* $X \cap Y$*.*

2.3 Banach function norms and quasinorms

In this section we turn our attention to the case which we are most interested in, that is the case of norms and quasinorms acting on spaces of functions. The approach taken here is the same as in [2, Chapter 1, Section 1], which means that it differs, at least formally, from that in Section 2.2.

The major definitions are of course those of Banach function norm and the corresponding Banach function space.

Definition 2.13. *Let* $\|\cdot\| : M(R, \mu) \to [0, \infty]$ *be a mapping satisfying* $\||f|\| = \|f\|$ *for all* $f \in M$*. We say that* $\|\cdot\|$ *is a Banach function norm if its restriction to* M_+ *satisfies the following axioms:*

(P1) *it is a norm, in the sense that it satisfies the following three conditions:*

- *(a) it is positively homogeneous, i.e.* $\forall a \in \mathbb{C} \; \forall f \in M_+ : \|af\| = |a|\|f\|$,
- *(b) it satisfies* $\|f\| = 0 \Leftrightarrow f = 0$ *μ-a.e.,*
- *(c) it is subadditive, i.e.* $\forall f, g \in M_+ \; : \; \|f + g\| \leq \|f\| + \|g\|$,

(P2) *it has the lattice property, i.e. if some* $f, g \in M_+$ *satisfy* $f \leq g$ *μ-a.e., then also* $\|f\| \leq \|g\|$, (P3) *it has the Fatou property, i.e. if some* $f_n, f \in M_+$ *satisfy* $f_n \uparrow f$ μ*-a.e., then also* $\|f_n\| \uparrow \|f\|$*,*

(P4) $\|\chi_E\| < \infty$ *for all* $E \subseteq R$ *satisfying* $\mu(E) < \infty$*,*

(P5) *for every* $E \subseteq R$ *satisfying* $\mu(E) < \infty$ *there exists some finite constant* C_E*, dependent only on* E*, such that the inequality* $\int_E f \, d\mu \leq C_E \|f\|$ *is true for all* $f \in M_+$*.*

Definition 2.14. *Let* $\|\cdot\|_X$ *be a Banach function norm. We then define the corresponding Banach function space* X *as the set*

$$X = \{f \in M; \|f\|_X < \infty\}.$$

It is easy to see that a Banach function norm, when restricted to the space it defines, is indeed a norm in the sense of Definition 2.7 and therefore Banach function spaces, when equipped with their defining norm, are normed linear spaces. Detailed study of these spaces can be found in [2]. Let us just list here their most important properties, proofs of which can be found in [2, Chapter 1, Section 1]

Theorem 2.15. *Let* $\|\cdot\|_X$ *be a Banach function norm and let* X *be the corresponding Banach function space. Then* X *is a Banach space.*

Theorem 2.16. *Let* $\|\cdot\|_X$ *and* $\|\cdot\|_Y$ *be Banach function norms and let* X *and* Y *be the corresponding Banach function spaces. If* $Y \subseteq X$ *then also* $Y \hookrightarrow X$*.*

The last result concerning Banach function spaces we want to list at this point concerns the properties of the intersection of two Banach function spaces. The proof is an easy exercise.

Proposition 2.17. *Let* X *and* Y *be two Banach function spaces. Then* $X \cap Y$ *is also a Banach function space.*

Just as with general norms, the triangle inequality is sometimes too strong a condition to require. We therefore introduce the notions of Banach function quasinorms and of the corresponding quasi-Banach function spaces.

Definition 2.18. *Let* $\|\cdot\| : M(R, \mu) \to [0, \infty]$ *be a mapping satisfying* $\| |f| \| = \|f\|$ *for all* $f \in M$*. We say that* $\|\cdot\|$ *is a Banach function quasinorm if its restriction to* M_+ *satisfies the axioms (P2), (P3), (P4), (P5) of Banach function norms together with a weaker version of axiom (P1), namely*

(Q1) *it is a quasinorm, in the sense that it satisfies the following three conditions:*

- *(a) it is positively homogeneous, i.e.* $\forall a \in \mathbb{C} \; \forall f \in M_+ : \|af\| = |a| \|f\|$*,*
- *(b) it satisfies* $\|f\| = 0 \Leftrightarrow f = 0$ μ*-a.e.,*
- *(c) there is a constant* $C > 0$*, called the modulus of concavity of* $\|\cdot\|$*, such that it is subadditive up to this constant, i.e.*

$$\forall f, g \in M_+ : \|f + g\| \leq C(\|f\| + \|g\|).$$

Definition 2.19. *Let* $\|\cdot\|_X$ *be a Banach function quasinorm. We then define the corresponding quasi-Banach function space* X *as the set*

$$X = \{f \in M; \|f\|_X < \infty\}.$$

Again, it is easy to see that a Banach function quasinorm restricted to the space it defines is a quasinorm in the sense of Definition 2.8. Analogues of the properties of Banach function spaces that are stated in Theorem 2.15 and Theorem 2.16 are proved in Chapter 3, namely in Theorem 3.6 and Theorem 3.7.

Finally, let us now define one last property that a Banach function quasinorm can have that we will take a special interest in. Note that the class od Banach function quasinorms contains that of Banach function norms so it is not necessary to provide separate definitions.

Definition 2.20. *Let* $\|\cdot\|_X$ *be a Banach function quasinorm. We say that* $\|\cdot\|_X$ *is rearrangement invariant, abbreviated r.i., if* $\|f\|_X = \|g\|_X$ *whenever* $f, g \in M$ *are equimeasurable (in the sense of Definition 2.3).*

Furthermore, if the above condition holds, the corresponding space X *will be called rearrangement invariant too.*

An important property of r.i. Banach function spaces over $([0,\infty), \lambda)$ is that the dilation operator is bounded on those spaces, as stated in the following theorem. For proof see [2, Chapter 3, Proposition 5.11].

Theorem 2.21. *Let* X *be an r.i. Banach function space over* $([0,\infty), \lambda)$ *and consider the dilation operator* D_t *defined on* $M([0,\infty), \lambda)$ *by*

$$D_t f(s) = f(ts).$$

Then $D_t : X \to X$ *is a bounded operator.*

2.4 Associate space

An important concept in the theory of Banach function spaces and their generalisations is that of an associate space. The detailed study of associate spaces of Banach function spaces can be found in [2, Chapter 1, Sections 2, 3 and 4].

We will approach the issue in a slightly more general way. The very definition of an associate space requires no assumptions on the functional defining the original space.

Definition 2.22. *Let* $\|\cdot\|_X : M \to [0,\infty]$ *be some non-negative functional and put*

$$X = \{f \in M; \|f\|_X < \infty\}.$$

Then the functional $\|\cdot\|_{X'}$ *defined for* $f \in M$ *by*

$$\|f\|_{X'} = \sup_{g \in X} \frac{1}{\|g\|_X} \int_R |fg| \, d\mu, \tag{2.1}$$

where we interpret $\frac{0}{0} = 0$ *and* $\frac{a}{0} = \infty$ *for any* $a > 0$*, will be called the associate functional of* $\|\cdot\|_X$ *while the set*

$$X' = \{f \in M; \|f\|_{X'} < \infty\}$$

will be called the associate space of X*.*

As suggested by the notation, we will be interested mainly in the case when $\|\cdot\|_X$ is at least a quasinorm, but we wanted to indicate that such assumption is not necessary for the definition. In fact, it is not even required for the following result, which is the Hölder inequality for associate spaces.

Theorem 2.23. *Let $\|\cdot\|_X : M \to [0, \infty]$ be some non-negative functional and denote by $\|\cdot\|_{X'}$ its associate functional. Then it holds for all $f \in M$ that*

$$\int_R |fg| \, d\mu \leq \|g\|_X \|f\|_{X'}$$

provided that we interpret $0 \cdot \infty = -\infty \cdot \infty = \infty$.

The last result we will present in this generality is the following proposition concerning embeddings. Although the proof is an easy modification of that in [2, Chapter 2, Proposition 2.10] we provide it to show that it truly does not require any assumptions on the original functional.

Proposition 2.24. *Let $\|\cdot\|_X : M \to [0, \infty]$ and $\|\cdot\|_Y : M \to [0, \infty]$ be two non-negative functionals satisfying that there is a constant $C > 0$ such that it holds for all $f \in M$ that*

$$\|f\|_X \leq C\|f\|_Y.$$

Then the associate functionals $\|\cdot\|_{X'}$ and $\|\cdot\|_{Y'}$ satisfy, with the same constant C,

$$\|f\|_{Y'} \leq C\|f\|_{X'}$$

for all $f \in M$.

Proof. Our assumptions guarantee that $Y \subseteq X$ and therefore

$$\begin{aligned} \|f\|_{Y'} &= \sup_{g \in Y} \frac{1}{\|g\|_Y} \int_R |fg| \, d\mu \leq \\ &\leq \sup_{g \in Y} \frac{C}{\|g\|_X} \int_R |fg| \, d\mu \leq \\ &\leq \sup_{g \in X} \frac{C}{\|g\|_X} \int_R |fg| \, d\mu = C\|f\|_{X'}. \end{aligned}$$

□

Let us now turn our attention to the case when $\|\cdot\|_X$ is a Banach function quasinorm. Note that in this case the supremum in (2.1) can be taken only over the unit sphere in X.

The following result, due to Gogatishvili and Soudský in [8], shows that this is more than enough to ensure that the associate functional is a Banach function norm.

Theorem 2.25. *Let $\|\cdot\|_X : M \to [0, \infty]$ be a functional that satisfies the axioms (P4) and (P5) from the definition of Banach function spaces and which also satisfies for all $f \in M$ that $\|f\|_X = \||f|\|_X$. Then the functional $\|\cdot\|_{X'}$ is a Banach function norm. In addition, $\|\cdot\|_X$ is equivalent to a Banach function norm if and only if there is some constant $C > 0$ such that it holds for all $f \in M$ that*

$$C^{-1}\|f\|_{X''} \leq \|f\|_X \leq C\|f\|_{X''}, \tag{2.2}$$

where $\|\cdot\|_{X''}$ denotes the associate functional of $\|\cdot\|_{X'}$.

Additionally, if $\|\cdot\|_X$ is a Banach function norm then (2.2) holds with constant one. This is a classical result of Lorenz and Luxemburg, proof of which can be found for example in [2, Chapter 1, Theorem 2.7].

Theorem 2.26. *Let $\|\cdot\|_X$ be a Banach function norm, then $\|\cdot\|_X = \|\cdot\|_{X''}$ where $\|\cdot\|_{X''}$ is the associate functional of $\|\cdot\|_{X'}$.*

To conclude this section, we observe that, provided the underlying measure space is resonant, the associate functional of an r.i. Banach function quasinorm can be expressed in the terms of non-increasing rearrangement. The proof is the same as in [2, Chapter 2, Proposition 4.2].

Proposition 2.27. *Let $\|\cdot\|_X$ be an r.i. Banach function quasinorm over a resonant measure space. Then its associate functional $\|\cdot\|_{X'}$ satisfies*

$$\|f\|_{X'} = \sup_{\|g\|_X \leq 1} \int_0^\infty f^* g^* \, d\lambda.$$

An obvious consequence of Proposition 2.27 is that an associate space of an r.i. quasi-Banach function space (over a resonant measure space) is also rearrangement invariant.

3 Quasi-Banach function spaces

The aim of this chapter is to show analogues of Theorem 2.15 and Theorem 2.16 for quasi-Banach function spaces. In order to do so we also obtain an analogue of the Riesz–Fisher theorem which is interesting on its own.

The core observation of this section is the following Nekvinda's trick due to Nekvinda in [13]. Although it is in fact quite simple to prove, it is extremely useful as it provides the critical insight needed in order to generalise the standard proofs from the theory of normed spaces.

Lemma 3.1 (Nekvinda's trick). *Let X be a quasinormed space equipped with the quasinorm $\|\cdot\|_X$ and denote by C its modulus of concavity. Let x_n be a sequence of points in X. Then*

$$\left\| \sum_{n=0}^{N} x_n \right\|_X \leq \sum_{n=0}^{N} C^{n+1} \|x_n\|_X$$

for every $N \in \mathbb{N}$.

Proof. The estimate follows by fixing an $N \in \mathbb{N}$ and using the triangle inequality, up to a multiplicative constant, of $\|\cdot\|_X$ $(N+1)$ times to obtain

$$\left\| \sum_{n=0}^{N} x_n \right\|_X \leq C\|x_0\|_X + C \left\| \sum_{n=1}^{N} x_n \right\|_X \leq \cdots \leq \sum_{n=0}^{N} C^{n+1} \|x_n\|_X.$$

□

Firstly, we use this trick to prove a generalised version of the classical Riesz–Fisher theorem. This result is known in slightly less general setting, see for example [12], but, as far as we know, it has not been proven in full generality.

Let us first define a generalised version of the classical Riesz–Fisher property.

Definition 3.2. *Let X be a quasinormed space equipped with the quasinorm $\|\cdot\|_X$ and let $C \in [1, \infty)$. We say that X has the Riesz–Fisher property with constant C if for every sequence x_n of points in X that satisfies*

$$\sum_{n=0}^{\infty} C^{n+1} \|x_n\|_X < \infty$$

there is a point $x \in X$ such that

$$\lim_{N \to \infty} \sum_{n=0}^{N} x_n = x$$

in the quasinormed topology of X.

It is obvious that if $C = 1$ then this property coincides with the classical Riesz–Fisher property and also that the greater the constant C the weaker the required condition is. However, this property with any constant is still stronger than the completeness of the space. On the other hand, the completeness of the space is equivalent to having the Ries–Fisher property with constant equal to the modulus of concavity.

Theorem 3.3. *Let X be a quasinormed space equipped with the quasinorm $\|\cdot\|_X$ and denote by C its modulus of concavity. Then*

1. *if X is complete then it has the Riesz–Fisher property with constant C,*
2. *if X has Riesz–Fisher property with an arbitrary constant $C' \in [1, \infty)$ then X is complete.*

Proof. Suppose that X is complete and that x_n is a sequence of points in X that satisfies

$$\sum_{n=0}^{\infty} C^{n+1} \|x_n\|_X < \infty.$$

Then, thanks to Lemma 3.1, it holds for arbitrary natural numbers $N \leq M$ that

$$\left\| \sum_{n=0}^{N} x_n - \sum_{n=0}^{M} x_n \right\|_X \leq \sum_{n=N}^{M} C^{n-N+1} \|x_n\|_X \leq \sum_{n=N}^{\infty} C^{n+1} \|x_n\|_X,$$

which tends to 0 as $N \to \infty$. Hence, the sequence $\sum_{n=0}^{N} x_n$ is Cauchy and therefore by our assumption convergent.

Suppose now that the space X has the Riesz–Fisher property with constant C' and fix some Cauchy sequence x_n. Proceed to find a non-decreasing and unbounded sequence of natural numbers k_n such that it holds for all natural numbers $i, j \geq k_n$ that

$$\|x_i - x_j\|_X \leq (2C')^{-n-2}.$$

Now, let us consider sequence y_n of points in X defined by

$$\begin{aligned} y_0 &= x_{k_0}, \\ y_n &= x_{k_n} - x_{k_{n-1}} \text{ for } n \geq 1. \end{aligned}$$

Then the sequence y_n satisfies

$$\sum_{n=0}^{\infty} C'^{n+1} \|y_n\|_X \leq C' \|x_{k_0}\| + \sum_{n=1}^{\infty} C'^{n+1} \|x_{k_n} - x_{k_{n-1}}\|_X \leq C' \|x_{k_0}\| + \sum_{n=1}^{\infty} 2^{-n-1} < \infty,$$

which means that, by our assumption on X, there is some limit $y \in X$ of the sequence $\sum_{n=0}^{N} y_n$. Since

$$\sum_{n=0}^{N} y_n = x_{k_0} + \sum_{n=1}^{N} x_{k_n} - x_{k_{n-1}} = x_{k_N}$$

we have shown that the sequence x_n has a convergent subsequence with limit y. Because x_n is Cauchy, the standard argument yields that y is also the limit of x_n. □

Note that this means that, for a given quasinormed linear space, the Riesz–Fisher property with constant greater than the modulus of concavity is equivalent to that with constant equal to it.

We now turn our attention to quasi-Banach function spaces as defined in Section 2.3 and show that they have the same basic properties as their normed counterparts. For the proofs of the classical versions of these results see [2, Chapter 1, Section 1].

The first result relates quasi-Banach function spaces with the set of simple functions and M_0.

Theorem 3.4. *Let $\|\cdot\|_X$ be a Banach function quasinorm and let X be the corresponding quasi-Banach function space. Then X is a linear space satisfying*

$$S \subseteq X \hookrightarrow M_0$$

where S denotes the set of all simple functions (supported on a set of finite measure) and M_0 is equipped with the topology of convergence in measure on the sets of finite measure.

The proof is omitted since it does not differ at all from the classical version, i.e. [2, Chapter 1, Theorem 1.4]. This is also the case of the following lemma, the classical version of which can be found in [2, Chapter 1, Lemma 1.5], hence we omit the proof again.

Lemma 3.5. *Let $\|\cdot\|_X$ be a Banach function quasinorm and let X be the corresponding quasi-Banach function space. Consider a sequence f_n of functions in X and $f \in X$. Then the following two assertions hold.*

1. *If $0 \leq f_n \uparrow f$ μ-a.e., then either $f \notin X$ and $\|f_n\|_X \uparrow \infty$ or $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.*
2. *If $f_n \to f$ μ-a.e. and $\liminf_{n\to\infty} \|f_n\|_X < \infty$, then $f \in X$ and*

$$\|f\|_X \leq \liminf_{n\to\infty} \|f_n\|_X.$$

The following result, when combined with Theorem 3.3, establishes the completeness of quasi-Banach function spaces.

Theorem 3.6. *Let $\|\cdot\|_X$ be a Banach function quasinorm and let X be the corresponding quasi-Banach function space. Denote by C the modulus of concavity of $\|\cdot\|_X$. Then X has the Riesz-Fisher property with constant C.*

Proof. Fix some sequence f_n in X such that

$$\sum_{n=0}^{\infty} C^{n+1} \|f_n\|_X < \infty. \tag{3.1}$$

Denote by t and t_N the following pointwise sums:

$$t = \sum_{n=0}^{\infty} |f_n|,$$
$$t_N = \sum_{n=0}^{N} |f_n|.$$

Then $t_N \uparrow t$ and since it holds by Lemma 3.1 that

$$\|t_N\|_X \leq \sum_{n=0}^{N} C^{n+1} \|f_n\|_X \leq \sum_{n=0}^{\infty} C^{n+1} \|f_n\|_X < \infty,$$

we get by part 1 of Lemma 3.5 that $t \in X$. Thanks to Theorem 3.4 the series $\sum_{n=0}^{\infty} |f_n|$ converges almost everywhere and therefore the series $\sum_{n=0}^{\infty} f_n$ does too.

Denote now by f ans s_N the following pointwise sums:

$$f = \sum_{n=0}^{\infty} f_n,$$
$$s_N = \sum_{n=0}^{N} f_n.$$

Then $s_N \to f$ μ-a.e., hence, for any M, we get that $s_N - s_M \to f - s_M$ μ-a.e. as $N \to \infty$. Furthermore, using Lemma 3.1 again, we get that

$$\liminf_{N\to\infty} \|s_N - s_M\|_X \leq \liminf_{N\to\infty} \sum_{n=M+1}^{N} C^{n+1}\|f_n\|_X \leq \sum_{n=M+1}^{\infty} C^{n+1}\|f_n\|_X,$$

which tends to 0 as $M \to \infty$ thanks to (3.1). Therefore, by part 2 of Lemma 3.5, $f - s_M \in X$ (which implies that also $f \in X$) and also $\|f - s_M\|_X \to 0$ as $M \to \infty$. □

We conclude this chapter by proving an important result that tells us that an embedding between two quasi-Banach function spaces is always continuous.

Theorem 3.7. *Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be Banach function quasinorms and let X and Y be the corresponding quasi-Banach function spaces. If $Y \subseteq X$ then also $Y \hookrightarrow X$.*

Proof. Denote by C the modulus of concavity of both $\|\cdot\|_X$ and $\|\cdot\|_Y$.

Suppose that $Y \subseteq X$ but the embedding is not continuous. Then there is a sequence f_n of functions in Y such that

$$\|f_n\|_Y \leq 1,$$
$$\|f_n\|_X \geq n(2C)^{n+1}.$$

We may, without loss of generality, assume that these functions are non-negative. Then, by Lemma 3.1,

$$\sum_{n=0}^{\infty} C^{n+1}\|(2C)^{-n-1}f_n\|_Y \leq \sum_{n=0}^{\infty} 2^{-n-1} < \infty,$$

and therefore $\sum_{n=0}^{\infty}(2C)^{-n-1}f_n \in Y$ as follows from Theorem 3.6. Note that it follows from Theorem 3.4 that $\sum_{n=0}^{N}(2C)^{-n-1}f_n$ converges to $\sum_{n=0}^{\infty}(2C)^{-n-1}f_n$ μ-a.e. as $N \to \infty$. But this means that, by our assumption that f_n are non-negative, it holds for every $k \in \mathbb{N}$ that

$$\sum_{n=0}^{\infty}(2C)^{-n-1}f_n \geq \sum_{n=0}^{k}(2C)^{-n-1}f_n \geq (2C)^{-k-1}f_k \ \mu\text{-a.e.},$$

from which we obtain by the means of the property (P2) of $\|\cdot\|_X$ that

$$\left\|\sum_{n=0}^{\infty}(2C)^{-n-1}f_n\right\|_X \geq \|(2C)^{-k-1}f_k\|_X \geq k$$

for all $k \in \mathbb{N}$, which shows that $\sum_{n=0}^{\infty}(2C)^{-n-1}f_n \notin X$. □

4 Wiener amalgam spaces

In this chapter we investigate the classical Wiener amalgams and show that they preserve neither the properties of Banach function spaces nor the property of being rearrangement invariant. We restrict ourselves to the case when $(R, \mu) = ([0, \infty), \lambda)$ because this allows us to use a definition of Wiener amalgams which can be handled without technical difficulties. Note that this restriction also ensures that our underlying measure space is resonant. Furthermore, for the sake of simplicity, we restrict ourselves to the case when the global component is the classical Lebesgue norm. Even in such a restricted case, it is easy to find pairs of spaces that generate a Wiener amalgam that is neither rearrangement invariant nor a Banach function space.

Let us begin by presenting our definition of Wiener amalgams.

Definition 4.1. *Let $\|\cdot\|_A$ be a norm defined on M and let $\|\cdot\|_{l^p}$, where $p \in [1, \infty]$, be the classical Lebesgue norm defined on $M(\mathbb{N}, m)$. We then define the Wiener norm $\|\cdot\|_{W(A,l^p)}$, for $f \in M$, by*

$$\|f\|_{W(A,l^p)} = \left(\sum_{n=0}^{\infty} \|f\chi_{[n,n+1)}\|_A^p \right)^{1/p} \qquad \text{for } p \in [1, \infty),$$
$$\|f\|_{W(A,l^p)} = \sup_{n\in\mathbb{N}} \|f\chi_{[n,n+1)}\|_A \qquad \text{for } p = \infty,$$

and the corresponding Wiener amalgam space (or just Wiener amalgam) by

$$W(A, l^p) = \{f \in M; \|f\|_{W(A,l^p))} < \infty\}.$$

The following Proposition shows mainly what properties does the Wiener amalgam norm inherit in the case when $\|\cdot\|_A$ is a Banach function norm.

Proposition 4.2. *Let $\|\cdot\|_A$ be a norm defined on M, let $p \in [1, \infty]$ and let $\|\cdot\|_{l^p}$ be the classical Lebesgue norm defined on $M(\mathbb{N}, m)$. Then the Wiener norm $\|\cdot\|_{W(A,l^p)}$ is indeed a norm.*

Moreover, if $\|\cdot\|_A$ is a Banach function norm then the Wiener norm $\|\cdot\|_{W(A,l^p)}$ also satisfies the axioms (P2) and (P3) of Banach function norms together with weaker versions of axioms (P4) and (P5), namely

(P4') *it holds for every bounded $E \subseteq [0, \infty)$ that $\|\chi_E\|_{W(A,l^p)} < \infty$,*

(P5') *it holds for every bounded $E \subseteq [0, \infty)$ that there is a constant $C_E < \infty$ satisfying*

$$\int_E |f| \, d\lambda \le C_E \|f\|_{W(A,l^p)}$$

for every $f \in M$.

Proof. Only the properties (P4') and (P5') in the second assertion require a proof. We will cover only the case $p \in [1, \infty)$ since the remaining case is easier.

Fix bounded $E \subseteq [0, \infty)$. Then there is $n_0 \in \mathbb{N}$ such that $E \cap [n, n+1) = \emptyset$ for every $n \ge n_0$. Thus the assertion (P4') follows from the properties (P2) and (P4) of $\|\cdot\|_A$, since they imply that all the summands in the definition of $\|\cdot\|_{W(A,l^p)}$ are finite and only finitely many of them are greater zero.

Similarly, the property (P5') follows from the property (P5) of $\|\cdot\|_A$, since it allows us to estimate

$$\int_E |f| \, d\lambda = \sum_{n=0}^{\infty} \int_E |f| \chi_{[n,n+1)} \, d\lambda \leq \sum_{n=0}^{n_0} C_E \|f \chi_{[n,n+1)}\|_A \leq$$
$$\leq C_0 C_E \left(\sum_{n=0}^{n_0} \|f \chi_{[n,n+1)}\|_A^p \right)^{\frac{1}{p}} \leq C \|f\|_{W(A,l^p)},$$

where C_E is the constant from the property (P5) of $\|\cdot\|_A$ for the set E and C_0 is the constant from the equivalence of $\|\cdot\|_{l^1}$ and $\|\cdot\|_{l^p}$ norms on $\mathbb{R}^{n_0}$. ☐

The restrictions in (P4') and (P5') are necessary, even in the simplest cases, as shown in the following remark.

Remark 4.3.

1. *Let $1 \leq p < q \leq \infty$. Then the norm $\|\cdot\|_{W(L^q,l^p)}$ does not satisfy (P4).*
2. *Let $1 \leq q < p \leq \infty$. Then the norm $\|\cdot\|_{W(L^q,l^p)}$ does not satisfy (P5).*

Consequently, the norm $\|\cdot\|_{W(L^q,l^p)}$ is a Banach function norm if and only if $1 \leq p = q \leq \infty$, in which case it coincides with the classical Lebesgue norm.

Proof. We will show part 1 for $q < \infty$ since the remaining case is easier. Fix arbitrary $a \in (1, \frac{q}{p})$ and define

$$E = \bigcup_{n \in \mathbb{N}} \left[n, n + \frac{1}{n^a} \right].$$

Then, by our assumptions on a,

$$\lambda(E) = \sum_{n=0}^{\infty} n^{-a} < \infty$$

but

$$\|\chi_E\|_{W(L^q,l^p)}^p = \sum_{n=0}^{\infty} n^{-a\frac{p}{q}} = \infty.$$

As for part 2, we will again show it only for $p < \infty$ since the remaining case is easier. Fix arbitrary $a \in (\frac{q}{p}, 1)$, $b \in (1, \frac{q-a}{q-1})$ (if $q = 1$ then any $b \in (1, \infty)$ will suffice) and define

$$E = \bigcup_{n \in \mathbb{N}} \left[n, n + \frac{1}{n^b} \right],$$
$$f = \sum_{n=0}^{\infty} n^{\frac{b-a}{q}} \chi_{[n,n+n^{-b}]}.$$

Then, by our assumptions on a and b,

$$\lambda(E) = \sum_{n=0}^{\infty} n^{-b} < \infty,$$
$$\|f\|_{W(L^q,l^p)}^p = \sum_{n=0}^{\infty} \left(\int_n^{n+1} |f|^q \, d\lambda \right)^{\frac{p}{q}} = \sum_{n=0}^{\infty} n^{-a\frac{p}{q}} < \infty$$

but

$$\int_E |f| \, d\lambda = \sum_{n=0}^{\infty} n^{\frac{b-a}{q}-b} = \infty.$$

The remaining part is an easy exercise. □

Let us now turn our attention to the property of being rearrangement invariant. We examine the case $p = \infty$ which provides the easiest way to construct counterexamples. Let us first observe that, in this case, the Wiener amalgam norm is, in a way, dominated by $\|\cdot\|_A$.

Remark 4.4. *Let $\|\cdot\|_A$ be an r.i. Banach function norm. Then it holds for all $f \in M$ that*

$$\|f\|_{W(A,l^\infty)} \le \|f^* \chi_{[0,1]}\|_A.$$

Proof. The result follows from a simple calculation, using only the Theorem 2.26 and Proposition 2.27, which reads

$$\begin{aligned}\sup_{n\in\mathbb{N}} \|f\chi_{[n,n+1)}\|_A &= \sup_{n\in\mathbb{N}} \sup_{\|g\|_{A'}\le 1} \int_0^\infty (f\chi_{[n,n+1)})^* g^* \, d\lambda \le \\ &\le \sup_{n\in\mathbb{N}} \sup_{\|g\|_{A'}\le 1} \int_0^\infty f^*\chi_{[0,1]} g^* \, d\lambda = \sup_{n\in\mathbb{N}} \|f^*\chi_{[0,1]}\|_A = \\ &= \|f^*\chi_{[0,1]}\|_A.\end{aligned}$$

□

The following observation provides a characterisation of the Wiener amalgam norm of f^*.

Remark 4.5. *Let $\|\cdot\|_A$ be an r.i. Banach function norm. Then it holds for all $f \in M$ that*

$$\|f^*\|_{W(A,l^\infty)} = \|f^* \chi_{[0,1]}\|_A.$$

Proof. The inequality $\|f^*\|_{W(A,l^\infty)} \ge \|f^*\chi_{[0,1]}\|_A$ is trivial and the inequality $\|f^*\|_{W(A,l^\infty)} \le \|f^*\chi_{[0,1]}\|_A$ follows from Remark 4.4. □

We are now in position to show that, given a simple and non-restrictive assumption on $\|\cdot\|_A$, the Wiener amalgam norm $\|\cdot\|_{W(A,l^\infty)}$ cannot be equivalent to any rearrangement invariant one.

Proposition 4.6. *Let $\|\cdot\|_A$ be an r.i. Banach function norm such that*

$$\lim_{t\to 0_+} \|\chi_{[0,t]}\|_A = 0,$$

then the norm $\|\cdot\|_{W(A,l^\infty)}$ is not equivalent to any rearrangement invariant norm.

Proof. Thanks to Remark 4.5, it suffices to find a sequence of functions $g_k \in M$ such that $\|g_k^*\chi_{[0,1]}\|_A = C > 0$ while

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \|g_k \chi_{[n,n+1)}\|_A = 0.$$

Define g_k for $k \in \mathbb{N}$ by

$$g_k = \sum_{i=0}^{k} \chi_{[i,i+\frac{1}{k+1})},$$

then $g_k^* = \chi_{[0,1]}$ and thus $\|g_k^* \chi_{[0,1]}\|_A = \|\chi_{[0,1]}\|_A > 0$ while, by our assumption,

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \|g_k \chi_{[n,n+1)}\|_A = \lim_{k\to\infty} \|\chi_{[0,\frac{1}{k+1}]}\|_A = 0.$$

□

It follows from Remark 4.3 and Proposition 4.6 that $W(L^q, l^\infty)$ with $q \in [1, \infty)$ is an example of a Wiener amalgam generated by two r.i. Banach function spaces which is neither rearrangement invariant nor a Banach function space. This result is further extended in Corollary 6.4.

5 Wiener–Luxemburg amalgam spaces

In this chapter we introduce the Wiener–Luxemburg amalgam spaces and show that, at the cost of not being necessarily normable, they retain other properties of Banach function spaces as well as rearrangement invariance. We will also show some properties of said spaces, mainly concerning their embeddings, their associate spaces and their relation to the sum and the intersection of their defining spaces.

Throughout this chapter we restrict ourselves to the case when $(R, \mu) = ([0, \infty), \lambda)$. This allows us to make the proofs more elegant and less technical as well as ensures that the underlying measure space is resonant. Note that this comes at no loss of generality, since any r.i. Banach function space over an arbitrary resonant measure space can be represented by some r.i. Banach function space over $([0, \infty), \lambda)$, as follows from the classical Luxemburg representation theorem. For details and the proof of this theorem we refer the reader to [2, Chapter 2, Theorem 4.10]

5.1 Wiener–Luxemburg quasinorms

Definition 5.1. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms. We then define the Wiener–Luxemburg quasinorm $\|\cdot\|_{WL(A,B)}$, for $f \in M$, by*

$$\|f\|_{WL(A,B)} = \|f^* \chi_{[0,1]}\|_A + \|f^* \chi_{(1,\infty)}\|_B \tag{5.1}$$

and the corresponding Wiener–Luxemburg amalgam space $WL(A, B)$ as

$$WL(A, B) = \{f \in M; \|f\|_{WL(A,B)} < \infty\}.$$

Furthermore, we will call the first summand in (5.1) the local component of $\|\cdot\|_{WL(A,B)}$ while the second summand will be called the global component of $\|\cdot\|_{WL(A,B)}$.

For the sake of brevity we will sometimes write just Wiener–Luxemburg amalgams instead of Wiener–Luxemburg amalgam spaces.

Let us at first note that this concept somehow generalises the concept of the r.i. Banach function spaces in the sense that every r.i. Banach function space is, up to equivalence of the defining functionals, a Wiener–Luxemburg amalgam of itself.

Remark 5.2. *Let $\|\cdot\|_A$ be an r.i. Banach function norm. Then*

$$\|f\|_A \leq \|f\|_{WL(A,A)} \leq 2\|f\|_A$$

for every $f \in M$.

Consequently, it makes s good sense to talk about local and global components of arbitrary r.i. Banach function norms.

Consider now only the local component of some r.i. Banach function norm. The following proposition shows that it behaves nicely and also provides an example of Wiener–Luxemburg amalgam space that is an r.i. Banach function space.

Proposition 5.3. *Let $\|\cdot\|_A$ be an r.i. Banach function norm. Then the functional*

$$f \mapsto \|f^* \chi_{[0,1]}\|_A$$

is also an r.i. Banach function norm.

Furthermore, there is a constant $C > 0$ such that it holds for all $f \in M$ that

$$\|f^* \chi_{[0,1]}\|_A \leq \|f\|_{WL(A,L^\infty)} \leq C \|f^* \chi_{[0,1]}\|_A, \tag{5.2}$$

and thus the space $WL(A, L^\infty)$ is an r.i. Banach function space.

Proof. That the functional in question satisfies the axioms (P2), (P3) and (P4) as well as parts (a) and (b) of the axiom (P1) is an easy consequence of the respective properties of $\|\cdot\|_A$ and the properties of non-increasing rearrangement. Furthermore, the rearrangement invariance is obvious.

As for (P5), fix some set $E \subseteq [0, \infty)$ of finite measure. We may, without loss of generality, assume that $\lambda(E) > 1$, because otherwise the proof is similar but simpler. Then, by Hardy–Littlewood inequality (Theorem 2.4), it holds for every $f \in M$ that

$$\begin{aligned} \int_E f \, d\lambda \leq \int_0^{\lambda(E)} f^* \, d\lambda = \int_0^1 f^* \, d\lambda + \int_1^{\lambda(E)} f^* \, d\lambda \leq \\ \leq \int_0^1 f^* \, d\lambda + (\lambda(E) - 1) f^*(1) \leq \lambda(E) \int_0^1 f^* \, d\lambda \leq \lambda(E) C_{[0,1]} \|f^* \chi_{[0,1]}\|_A, \end{aligned}$$

where $C_{[0,1]}$ is the constant from the property (P5) of $\|\cdot\|_A$ for the set $[0, 1]$.

For the triangle inequality (part (c) of axiom (P1)) we employ the associate definition of $\|\cdot\|_A$ (Proposition 2.27) and the fact that $[0, \infty)$ is resonant to get for an arbitrary pair of functions $f, g \in M$ that

$$\begin{aligned} \|(f+g)^* \chi_{[0,1]}\|_A &= \sup_{\|h\|_{A'} \leq 1} \int_0^\infty (f+g)^* \chi_{[0,1]} h^* \, d\lambda = \\ &= \sup_{\|h\|_{A'} \leq 1} \sup_{\tilde{h}^* = h^* \chi_{[0,1]}} \int_0^\infty (f+g) \tilde{h} \, d\lambda \leq \\ &\leq \sup_{\|h\|_{A'} \leq 1} \sup_{\tilde{h}^* = h^* \chi_{[0,1]}} \int_0^\infty f \tilde{h} \, d\lambda + \sup_{\|h\|_{A'} \leq 1} \sup_{\tilde{h}^* = h^* \chi_{[0,1]}} \int_0^\infty g \tilde{h} \, d\lambda = \\ &= \sup_{\|h\|_{A'} \leq 1} \int_0^\infty f^* \chi_{[0,1]} h^* \, d\lambda + \sup_{\|h\|_{A'} \leq 1} \int_0^\infty g^* \chi_{[0,1]} h^* \, d\lambda = \\ &= \|f^* \chi_{[0,1]}\|_A + \|g^* \chi_{[0,1]}\|_A. \end{aligned}$$

Thus we have shown that the functional in question is an r.i. Banach function norm. It remains to show (5.2).

The first inequality in (5.2) is trivial. For the second estimate, it suffices to notice that

$$\|f^* \chi_{(1,\infty)}\|_{L^\infty} = f^*(1) \leq C_{[0,1]} \|f^*\|_A,$$

where $C_{[0,1]}$ is the constant from the property (P5) of $\|\cdot\|_A$ for the set $[0, 1]$. □

While the local component is an r.i. Banach function norm, the global component is much less well behaved. Indeed, it is fairly easy to see that it cannot have the properties (P1) and (P5) (in (P1) only part (a) can possibly hold). Thus it makes no sense to consider it separately.

The following Theorem shows that although Wiener–Luxemburg quasinorm needs not to be a norm, it satisfies all the remaining axioms of r.i. Banach function norms.

Theorem 5.4. *The Wiener–Luxemburg quasinorms, as defined in Definition 5.1, are rearrangement invariant Banach function quasinorms. Consequently, the corresponding Wiener–Luxemburg amalgam spaces are rearrangement invariant quasi-Banach function spaces.*

Proof. The properties (P2), (P3) and (P4) as well as those from parts (a) and (b) of the axiom (Q1) are an easy consequence of the respective properties of $\|\cdot\|_A$ and $\|\cdot\|_B$ and the properties of non-increasing rearrangement. Furthermore, the rearrangement invariance is obvious.

To show (P5), fix some set $E \subseteq [0, \infty)$ of finite measure. We may, without loss of generality, assume that $\lambda(E) > 1$, since otherwise the proof is similar but simpler. Then, by Hardy–Littlewood inequality (Theorem 2.4), it holds for every $f \in M_+$ that

$$\int_E f \, d\lambda \leq \int_0^{\lambda(E)} f^* \, d\lambda = \int_0^1 f^* \, d\lambda + \int_1^{\lambda(E)} f^* \, d\lambda \leq \\ \leq C_{[0,1]} \|f^* \chi_{[0,1]}\|_A + C_{(1,\lambda(E))} \|f^* \chi_{(1,\infty)}\|_B,$$

where $C_{[0,1]}$ is the constant from the property (P5) of $\|\cdot\|_A$ for the set $[0, 1]$ and $C_{(1,\lambda(E))}$ is the constant from the same property of $\|\cdot\|_B$ for the set $(1, \lambda(E))$.

Finally, for the triangle inequality up to a multiplicative constant (part (c) of the axiom (Q1)), consider the dilation operator $D_{\frac{1}{2}}$, defined as in Theorem 2.21, and use at first only the appropriate properties of non-increasing rearrangement and those of $\|\cdot\|_A$ and $\|\cdot\|_B$ to calculate

$$\begin{aligned} \|f + g\|_{WL(A,B)} &= \|(f + g)^* \chi_{[0,1]}\|_A + \|(f + g)^* \chi_{(1,\infty)}\|_B \leq \\ &\leq \|(D_{\frac{1}{2}} f^* + D_{\frac{1}{2}} g^*) \chi_{[0,1]}\|_A + \\ &\qquad + \|(D_{\frac{1}{2}} f^* + D_{\frac{1}{2}} g^*) \chi_{(1,\infty)}\|_B \leq \\ &\leq \|D_{\frac{1}{2}} f^* \chi_{[0,1]}\|_A + \|D_{\frac{1}{2}} g^* \chi_{[0,1]}\|_A + \\ &\qquad + \|D_{\frac{1}{2}} f^* \chi_{(1,\infty)}\|_B + \|D_{\frac{1}{2}} g^* \chi_{(1,\infty)}\|_B, \end{aligned}$$

which shows that it is sufficient to prove that there is some $C \in (0, \infty)$ such that

$$\|D_{\frac{1}{2}} f^* \chi_{[0,1]}\|_A + \|D_{\frac{1}{2}} f^* \chi_{(1,\infty)}\|_B \leq C \|f\|_{WL(A,B)}$$

for all $f \in M_+$. Actually, it suffices to show

$$\|D_{\frac{1}{2}} f^* \chi_{(1,\infty)}\|_B \leq C \|f\|_{WL(A,B)}, \tag{5.3}$$

because $D_{\frac{1}{2}}$ is bounded on A (by Theorem 2.21) and thus

$$\|D_{\frac{1}{2}} f^* \chi_{[0,1]}\|_A = \|D_{\frac{1}{2}} (f^* \chi_{[0,\frac{1}{2}]})\|_A \leq \|D_{\frac{1}{2}}\| \|f^* \chi_{[0,\frac{1}{2}]}\|_A \leq \|D_{\frac{1}{2}}\| \|f^* \chi_{[0,1]}\|_A.$$

To show (5.3), fix some $f \in M_+$ and calculate

$$\begin{aligned}\|D_{\frac{1}{2}} f^* \chi_{(1,\infty)}\|_B &= \|D_{\frac{1}{2}}(f^* \chi_{(\frac{1}{2},\infty)})\|_B \leq \|D_{\frac{1}{2}}\| \|f^* \chi_{(\frac{1}{2},\infty)}\|_B \leq \\ &\leq \|D_{\frac{1}{2}}\|(\|f^* \chi_{(1,\infty)}\|_B + \|f^* \chi_{(\frac{1}{2},1)}\|_B) \leq \\ &\leq \|D_{\frac{1}{2}}\|(\|f^* \chi_{(1,\infty)}\|_B + f^*(\tfrac{1}{2})\|\chi_{(\frac{1}{2},1)}\|_B) \leq \\ &\leq \|D_{\frac{1}{2}}\|(\|f^* \chi_{(1,\infty)}\|_B + 2\|\chi_{(\frac{1}{2},1)}\|_B \int_0^{\frac{1}{2}} f^* \, d\lambda) \leq \\ &\leq \|D_{\frac{1}{2}}\|(\|f^* \chi_{(1,\infty)}\|_B + 2\|\chi_{(\frac{1}{2},1)}\|_B C_{[0,\frac{1}{2}]} \|f^* \chi_{[0,1]}\|_A) \leq \\ &\leq \|D_{\frac{1}{2}}\| \max\{1, 2\|\chi_{(\frac{1}{2},1)}\|_B C_{[0,\frac{1}{2}]}\} \|f\|_{WL(A,B)},\end{aligned}$$

where $C_{[0,\frac{1}{2}]}$ is the constant from property (P5) of $\|\cdot\|_A$ for the set $[0, \frac{1}{2}]$. □

5.2 Associate spaces of Wiener–Luxemburg amalgams

Let us now turn our attention to the associate spaces of Wiener–Luxemburg amalgams. As can be seen in the following theorems, when one consider the Wiener–Luxemburg amalgam of two arbitrary spaces A and B and the Wiener–Luxemburg amalgam of their respective associate spaces A' and B', one obtains easily an analogue of the Hölder inequality for associate spaces (Theorem 2.23), namely that it holds for arbitrary $f, g \in M$ that

$$\int_0^\infty |fg| \, d\lambda \leq \|f\|_{WL(A,B)} \|g\|_{WL(A',B')},$$

but the question whether this estimate is a sharp one is more difficult and requires an additional assumption on the original spaces. We will therefore treat those two question separately and formulate those results in the terms of embeddings between $WL(A', B')$ and $(WL(A, B))'$.

Let us first show the easier embedding which holds for any A and B.

Theorem 5.5. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms and let $\|\cdot\|_{A'}$ and $\|\cdot\|_{B'}$ be their respective associate norms. Then the associate norm $\|\cdot\|_{(WL(A,B))'}$ of $\|\cdot\|_{WL(A,B)}$ satisfies*

$$\|f\|_{(WL(A,B))'} \leq \|f\|_{WL(A',B')}$$

for every $f \in M$.

Consequently, the corresponding associate space satisfies

$$WL(A', B') \hookrightarrow (WL(A, B))'.$$

Proof. Fix some $f \in M$ and arbitrary $g \in M$ satisfying $\|g\|_{WL(A,B)} \leq 1$. Then it follows from the Hölder inequality for associate spaces (Theorem 2.23) and Proposition 2.27 that

$$\begin{aligned}\int_0^\infty f^* g^* \, d\lambda &= \int_0^\infty f^* \chi_{[0,1]} g^* \, d\lambda + \int_0^\infty f^* \chi_{(1,\infty)} g^* \, d\lambda \leq \\ &\leq \|f^* \chi_{[0,1]}\|_{A'} \|g^* \chi_{[0,1]}\|_A + \|f^* \chi_{(1,\infty)}\|_{B'} \|g^* \chi_{(1,\infty)}\|_B \leq \\ &\leq \max\{\|f^* \chi_{[0,1]}\|_{A'}, \|f^* \chi_{(1,\infty)}\|_{B'}\} \cdot \|g\|_{WL(A,B)} \leq \|f\|_{WL(A',B')}.\end{aligned}$$

The result now follows by taking the supremum over the unit ball in $WL(A, B)$. □

For the inverse embedding an additional assumption is needed.

Theorem 5.6. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms and let $\|\cdot\|_{A'}$ and $\|\cdot\|_{B'}$ be their respective associate norms. Suppose that $B \hookrightarrow WL(A,B)$. Then there is a constant $C > 0$ such that the associate norm $\|\cdot\|_{(WL(A,B))'}$ of $\|\cdot\|_{WL(A,B)}$ satisfies*

$$\|f\|_{WL(A',B')} \leq C\|f\|_{(WL(A,B))'}$$

for every $f \in M$.

Consequently, the corresponding associate space satisfies

$$(WL(A,B))' \hookrightarrow WL(A',B').$$

An equivalent condition for $B \hookrightarrow WL(A,B)$ is provided in part 1 of Theorem 5.9.

Proof. Fix some $f \in M$ and $\varepsilon > 0$. Assume at first that $\|f\|_{WL(A',B')} < \infty$. Then $\|f^*\chi_{[0,1]}\|_{A'}$ and $\|f^*\chi_{(1,\infty)}\|_{B'}$ are also finite and we can find g_f^0 in the unit ball of A and g_f^1 in the unit ball of B such that

$$\int_0^\infty f^*\chi_{[0,1]}g_f^0 \, d\lambda \geq \|f^*\chi_{[0,1]}\|_{A'} - \varepsilon, \tag{5.4}$$

$$\int_0^\infty f^*\chi_{(1,\infty)}g_f^1 \, d\lambda \geq \|f^*\chi_{(1,\infty)}\|_{B'} - \varepsilon. \tag{5.5}$$

Note that we may assume that g_f^0 is zero on $(1,\infty)$ and that g_f^1 is zero on $[0,1]$, because neither the expressions (5.4) and (5.5) nor the conditions on the size of g_f^0 and g_f^1 get violated by taking the appropriate restrictions.

Set $g_f = g_f^0 + g_f^1$. Then, by Hardy–Littlewood inequality (Theorem 2.4) and Proposition 2.27, we get that

$$\begin{aligned}\|f\|_{(WL(A,B))'}\|g_f\|_{WL(A,B)} &\geq \int_0^\infty f^*g_f^* \, d\lambda \geq \int_0^\infty f^*(g_f^0 + g_f^1) \, d\lambda = \\ &= \int_0^\infty f^*\chi_{[0,1]}g_f^0 \, d\lambda + \int_0^\infty f^*\chi_{(1,\infty)}g_f^1 \, d\lambda \geq \\ &\geq \|f\|_{WL(A',B')} - 2\varepsilon.\end{aligned}$$

Thus, to obtain the desired estimate, one only has to find some upper bound for $\|g_f\|_{WL(A,B)}$, independent of f and ε and then take supremum over all $\varepsilon > 0$.

Because $\|\cdot\|_{WL(A,B)}$ is a quasinorm (see Theorem 5.4), we immediately get that there is some constant $C_0 > 0$ such that

$$\|g_f\|_{WL(A,B)} \leq C_0(\|g_f^0\|_{WL(A,B)} + \|g_f^1\|_{WL(A,B)}).$$

The estimate for the first term of the right-hand side is easy, because g_f^0, and consequently $(g_f^0)^*$, is zero on $(1,\infty)$ and thus

$$\|g_f^0\|_{WL(A,B)} = \|(g_f^0)^*\chi_{[0,1]}\|_A + \|(g_f^0)^*\chi_{(1,\infty)}\|_B = \|(g_f^0)^*\|_A \leq 1.$$

The estimate for the second term of the right-hand side follows thanks to the assumption that $B \hookrightarrow WL(A,B)$, since we may use it to get

$$\|g_f^1\|_{WL(A,B)} \leq C_1\|g_f^1\|_B \leq C_1,$$

where C_1 is the constant of the embedding $B \hookrightarrow WL(A, B)$. Thus we have shown that the desired estimate holds (with constant $C = C_0(1 + C_1)$) for functions $f \in M$ satisfying $\|f\|_{WL(A',B')} < \infty$.

The case $\|f\|_{WL(A',B')} = \infty$ is very similar. One proceeds as above, with the only difference being that one puts K as the right-hand side of (5.4) or (5.5), as appropriate, and subsequently takes supremum over all $K \in \mathbb{N}$. □

We conclude this section by presenting two corollaries to the preceding theorems. Their proofs are of course trivial, but we believe the results to be interesting enough to be worth stating nonetheless.

Corollary 5.7. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms and let $\|\cdot\|_{A'}$ and $\|\cdot\|_{B'}$ be their respective associate norms. Suppose that $B \hookrightarrow WL(A, B)$. Then*

$$(WL(A, B))' = WL(A', B')$$

up to equivalence of quasinorms. Consequently, $WL(A', B')$ is an r.i. Banach function space.

Corollary 5.8. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms and let $\|\cdot\|_{A'}$ and $\|\cdot\|_{B'}$ be their respective associate norms. Suppose that $B \hookrightarrow WL(A, B)$ and that $\|\cdot\|_{WL(A,B)}$ is equivalent to an r.i. Banach function norm. Then*

$$WL(A, B) = (WL(A', B'))'$$

up to equivalence of quasinorms.

5.3 Embeddings

In this section we examine the embeddings of Wiener–Luxemburg amalgams. First we characterise the embeddings between two Wiener–Luxemburg amalgams, then between a Wiener–Luxemburg amalgam of two spaces and the sum or intersection of these spaces and finally we examine the case when either the local or the global component is either L^1 or L^∞.

The first theorem provides the characterisation of embeddings between two Wiener–Luxemburg amalgams.

Theorem 5.9. *Let $\|\cdot\|_A$, $\|\cdot\|_B$ and $\|\cdot\|_C$ be r.i. Banach function norms. Then the following assertions are true:*

1. *The embedding $WL(A, C) \hookrightarrow WL(B, C)$ is true if and only if the local component of $\|\cdot\|_A$ is stronger that that of $\|\cdot\|_B$, in the sense that for every $f \in M$ the implication*

$$\|f^* \chi_{[0,1]}\|_A < \infty \Rightarrow \|f^* \chi_{[0,1]}\|_B < \infty$$

   holds.

2. *The embedding $WL(A, B) \hookrightarrow WL(A, C)$ is true if and only if the global component of $\|\cdot\|_B$ is stronger that that of $\|\cdot\|_C$, in the sense that for every $f \in M$ such that $f^*(1) < \infty$ the implication*

$$\|f^* \chi_{(1,\infty)}\|_B < \infty \Rightarrow \|f^* \chi_{(1,\infty)}\|_C < \infty$$

   holds.

Proof. The sufficiency follows in both cases directly from Theorem 3.7 and Definition 5.1, only in the second case one has to realise that all $f \in WL(A,B)$ satisfy $f^*(1) < \infty$.

The necessity in the case 1 can be shown as follows. Fix some $f_0 \in M$ such that $\|f_0^*\chi_{[0,1]}\|_A < \infty$ but $\|f_0^*\chi_{[0,1]}\|_B = \infty$. Then $f = f_0^*\chi_{[0,1]}$ belongs to $WL(A,C)$, since

$$\|f^*\chi_{[0,1]}\|_A = \|f_0^*\chi_{[0,1]}\|_A < \infty,$$
$$\|f^*\chi_{(1,\infty)}\|_C = \|0\|_C = 0,$$

but not to $WL(B,C)$, since

$$\|f^*\chi_{[0,1]}\|_B = \|f_0^*\chi_{[0,1]}\|_B = \infty.$$

As for the case 2, fix some $f_0 \in M$ such that $f^*(1) < \infty$ and $\|f_0^*\chi_{(1,\infty)}\|_B < \infty$ while $\|f_0^*\chi_{(1,\infty)}\|_C = \infty$. Then $f = f_0^*(1)\chi_{[0,1]} + f_0^*\chi_{(1,\infty)}$ belongs to $WL(A,B)$, since

$$\|f^*\chi_{[0,1]}\|_A = \|f_0^*(1)\chi_{[0,1]}\|_A = f_0^*(1)\|\chi_{[0,1]}\|_A < \infty,$$
$$\|f^*\chi_{(1,\infty)}\|_B = \|f_0^*\chi_{(1,\infty)}\|_B < \infty,$$

but not to $WL(A,C)$, since

$$\|f^*\chi_{(1,\infty)}\|_C = \|f_0^*\chi_{(1,\infty)}\|_C = \infty.$$

□

To provide an example we turn to the classical Lebesgue spaces. It is well known that Lebesgue spaces over $[0,\infty)$ are not ordered, but it is easy to show that their local and global component are, as is formalised in the following remark.

Remark 5.10. *Let $p, q \in [1,\infty]$ Then it holds that*

1. *the local component of $\|\cdot\|_{L^p}$ is stronger than that of $\|\cdot\|_{L^q}$ if and only if $p \geq q$,*
2. *the global component of $\|\cdot\|_{L^p}$ is stronger than that of $\|\cdot\|_{L^q}$ if and only if $p \leq q$.*

We now put $WL(A,B)$ in relation with the sum and intersection of A and B. We first show that $WL(A,B)$ is always sandwiched between them.

Theorem 5.11. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms. Then*

$$A \cap B \hookrightarrow WL(A,B) \hookrightarrow A + B.$$

Proof. Fix some $f \in M$. Then

$$\|f\|_{WL(A,B)} = \|f^*\chi_{[0,1]}\|_A + \|f^*\chi_{(1,\infty)}\|_B \leq \|f\|_A + \|f\|_B \leq 2\|f\|_{A\cap B}$$

which establishes the first embedding.

As for the second embedding, note that we may consider f to be non-negative, since it is an easy exercise to show that, provided that $\|\cdot\|_A$ and $\|\cdot\|_B$ are Banach function norms, it holds that $\|f\|_{A+B} = \| \, |f| \, \|_{A+B}$ for every $f \in M$.

Consider now functions g and h defined by

$$g = \max\{f - f^*(1), 0\},$$
$$h = \min\{f, f^*(1)\}.$$

Then $f = g + h$ and thus

$$\|f\|_{A+B} \leq \|g\|_A + \|h\|_B = \|g^*\|_A + \|h^*\|_B$$

thanks to rearrangement invariance of both $\|\cdot\|_A$ and $\|\cdot\|_B$. Furthermore, thanks to f beeing non-negative, it is an exercise to verify that

$$g^* = (f^* - f^*(1))\chi_{[0,1]},$$
$$h^* = f^*(1)\chi_{[0,1]} + f^*\chi_{(1,\infty)},$$

and therefore

$$\begin{aligned}\|f\|_{A+B} &\leq \|f^*\chi_{[0,1]}\|_A + \|f^*(1)\chi_{[0,1]}\|_B + \|f^*\chi_{(1,\infty)}\|_B \leq \\ &\leq \|f\|_{WL(A,B)} + \|\chi_{[0,1]}\|_B \int_0^1 f^* \, d\lambda \leq \\ &\leq (1 + C_{[0,1]}\|\chi_{[0,1]}\|_B)\|f\|_{WL(A,B)},\end{aligned}$$

where $C_{[0,1]}$ is the constant from the property (P5) of $\|\cdot\|_A$ for the set $[0,1]$. This establishes the second embedding. ☐

Moreover, as stated in the following result, in the case when we have proper relations between the respective components of A and B we can describe their sum and intersection in terms of Wiener–Luxemburg amalgams, at least in the set theoretical sense.

Corollary 5.12. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms. Suppose that the local component of $\|\cdot\|_A$ is stronger than of $\|\cdot\|_B$ while the global component of $\|\cdot\|_B$ is stronger than that of $\|\cdot\|_A$. Then*

$$A \cap B = WL(A, B)$$

up to equivalence of quasinorms while

$$A + B = WL(B, A)$$

as a set.

Proof. Thanks to Proposition 2.17, Theorem 3.7 and Theorem 5.11 it suffices to prove that $WL(A, B) \subseteq A \cap B$ and $A + B \subseteq WL(B, A)$. But this is provided by Theorem 5.9, which, thanks to our assumptions, yield

$$\begin{aligned} WL(A, B) &\hookrightarrow A, \\ WL(A, B) &\hookrightarrow B, \\ A &\hookrightarrow WL(B, A), \\ B &\hookrightarrow WL(B, A), \end{aligned}$$

which, when combined with the fact that $WL(B, A)$ is a linear set, is sufficient for the inclusions. ☐

The reason for the second equality holding only in the set theoretical sense is of course the fact that $A+B$ is not necessarily a Banach function space. In the case when $A+B$ is a Banach function space we obtain from Theorem 3.7 the equivalence of quasinorms and consequently also a normability of the appropriate Wiener–Luxemburg amalgam.

Corollary 5.13. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms. Suppose that the local component of $\|\cdot\|_B$ is stronger than of $\|\cdot\|_A$ while the global component of $\|\cdot\|_A$ is stronger than that of $\|\cdot\|_B$. Suppose also that $A+B$ is a Banach function space. Then the Wiener–Luxemburg amalgam space $WL(A,B)$ is an r.i. Banach function space.*

Proof. Our assumptions together with Theorem 3.7 and Corollary 5.12 imply that A + B = WL(A,B) up to equivalence of norms, which is sufficient since we assume that $A+B$ is a Banach function space. □

In the next theorem, we show that the classical Lebesgue space L^1 has the weakest local component, as well as the strongest global component, among all r.i. Banach function spaces, while L^∞ has, in the same context, the strongest local component as well as the weakest global component.

Theorem 5.14. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms. Then*

1. $WL(L^\infty, B) \hookrightarrow WL(A,B)$,
2. $WL(A, L^1) \hookrightarrow WL(A,B)$,
3. $WL(A,B) \hookrightarrow WL(L^1, B)$,
4. $WL(A,B) \hookrightarrow WL(A, L^\infty)$.

Proof. Fix $f \in M$. The first embedding follows from the estimate

$$\|f^*\chi_{[0,1]}\|_A \leq \|f^*\chi_{[0,1]}\|_{L^\infty}\|\chi_{[0,1]}\|_A$$

and part 1 of Theorem 5.9.

The third embedding also uses part 1 of Theorem 5.9 but this time paired with the estimate

$$\|f^*\chi_{[0,1]}\|_{L^1} = \int_0^1 f^*\chi_{[0,1]}\,d\lambda \leq C_{[0,1]}\|f^*\chi_{[0,1]}\|_A,$$

where $C_{[0,1]}$ is the constant from property (P5) of $\|\cdot\|_A$.

The fourth embedding is a trivial consequence of Proposition 5.3, specifically of (5.2).

The second embedding is most involved. First step is to show that

$$WL(L^\infty, L^1) \hookrightarrow C \tag{5.6}$$

for any r.i. Banach function space C. Fix such a space and denote by C' its associate space. Then we know from part 4, which has already been proven, that $C' \hookrightarrow WL(C', L^\infty)$. Thus it follows from the Theorem 5.5 and Proposition 2.24 that

$$WL(C, L^1) \hookrightarrow (WL(C', L^\infty))' \hookrightarrow C'' = C.$$

To obtain (5.6), it remains only to combine this with part 1, which has also already been proved.

Consider now function $f \in M$ such that $f^*(1) < \infty$ which also satisfies $\|f^* \chi_{(1,\infty)}\|_{L^1} < \infty$. Since it holds for any $t \in (0, \infty)$ that

$$(f^* \chi_{(1,\infty)})^*(t) = (f^* \chi_{(1,\infty)})(t+1),$$

it is obvious that $f^* \chi_{(1,\infty)}$ belongs to $WL(L^\infty, L^1)$ and consequently, by (5.6) applied to the space B, to B. This shows that the global component of $\|\cdot\|_{L^1}$ is stronger than that of $\|\cdot\|_B$ and therefore the desired embedding follows from the part 2 of Theorem 5.9. ☐

As a corollary, we obtain the following well-known classical result, for which we thus provide an alternative proof.

Corollary 5.15. *Let A be an r.i. Banach function space. Then*

$$L^1 \cap L^\infty \hookrightarrow A \hookrightarrow L^1 + L^\infty.$$

Proof. This corollary is a direct consequence of Remark 5.10, Corollary 5.12, Theorem 5.14, Theorem 3.7 and the fact that $L^1 + L^\infty$ is an r.i. Banach function space. ☐

Another application of Theorem 5.14 is that it provides a framework for a separate examination of the local and global components of an r.i. Banach function norms. This is employed in the proof of the following proposition.

Proposition 5.16. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms and denote by $\|\cdot\|_{A'}$ and $\|\cdot\|_{B'}$ the respective associate norms. Suppose that the local component of $\|\cdot\|_A$ is stronger than that of $\|\cdot\|_B$. Then the local component of $\|\cdot\|_{B'}$ is stronger than that of $\|\cdot\|_{A'}$.*

Similarly, if the global component of $\|\cdot\|_A$ is stronger than that of $\|\cdot\|_B$, then the global component of $\|\cdot\|_{B'}$ is stronger than that of $\|\cdot\|_{A'}$

Proof. By our assumption and part 1 of Theorem 5.9 we get that

$$WL(A, L^\infty) \hookrightarrow WL(B, L^\infty).$$

Consequently, it follows from part 1 of Theorem 5.14, Corollary 5.7 and Proposition 2.24 that

$$WL(B', L^1) = (WL(B, L^\infty))' \hookrightarrow (WL(A, L^\infty))' = WL(A', L^1),$$

that is, the local component of $\|\cdot\|_{B'}$ is stronger than that of $\|\cdot\|_{A'}$.

The second claim can be proved in similar manner, only using $WL(L^1, A)$ and $WL(L^1, B)$ instead of $WL(A, L^\infty)$ and $WL(B, L^\infty)$. ☐

As a direct consequence of Proposition 5.16 and Corollary 5.7 we obtain the following result concerning normability.

Corollary 5.17. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be r.i. Banach function norms and suppose that the local component of $\|\cdot\|_A$ is stronger than that of $\|\cdot\|_B$. Then $WL(A, B)$ is an r.i. Banach function space.*

5.4 Normability of Wiener–Luxemburg amalgams

In the following result we collect, for the reader's convenience, all the results concerning normability.

Theorem 5.18. *The Wiener–Luxemburg amalgam space $WL(A, B)$ is an r.i. Banach function space provided at least one of the following conditions holds:*

1. $B = L^\infty$.
2. *The local component of $\|\cdot\|_A$ is stronger than that of $\|\cdot\|_B$.*
3. *The local component of $\|\cdot\|_B$ is stronger than that of $\|\cdot\|_A$ while the global component of $\|\cdot\|_A$ is stronger than that of $\|\cdot\|_B$ and $A + B$ is a Banach function space.*

6 Examples of applications

In this chapter we apply the general theory developed in Chapter 5 to some concrete examples spaces to show how it can easily provide some interesting results.

6.1 Wiener–Luxemburg amalgams of Lebesgue spaces

We first turn our attention to the most natural and well known r.i. Banach function spaces, namely the Lebesgue spaces. In this context we will also compare Wiener–Luxemburg amalgams with the more classical Wiener ones.

Let us first recall Remark 5.10 which, thanks to Theorem 5.9, provides a characterisation of the embeddings among Wiener–Luxemburg amalgams of Lebesgue spaces. This leads to the following result.

Theorem 6.1. *Let $1 \leq p \leq q \leq \infty$. Then*

1.
$$WL(L^q, L^p) = L^p \cap L^q$$
up to equivalence of norms,

2.
$$WL(L^p, L^q) = L^p + L^q$$
up to equivalence of norms.

Consequently, the Wiener–Luxemburg amalgam space of any two Lebesgue spaces is an r.i. Banach function space.

Proof. The first assertion as well as the set theoretical part of the second assertion follows immediately from Corollary 5.12 and Remark 5.10, while the equivalence of norms in the second assertion follows from Theorem 3.7 and the classical result that $L^p + L^q$ is always a Banach function space. □

Note that the fact that $L^p + L^q$ is always a Banach function space follows from the fact that either at least one of $L^{p'}$, $L^{q'}$, where $p' = \frac{p}{p-1}$ if $p < \infty$ and $p' = 1$ otherwise, has an absolutely continuous norm or $L^p + L^q = L^1$ (see [2, Chapter 3, Exercise 5]), and thus our proof does not depend on any deep results.

Important consequence of Theorem 6.1 (and also of Remark 5.10 and Theorem 5.9) is the following classical interpolation theorem.

Theorem 6.2. *Let $1 \leq p \leq s \leq q \leq \infty$. Then*

$$L^p \cap L^q \hookrightarrow L^s \hookrightarrow L^p + L^q.$$

Those results are of course well known, Theorem 6.2 can be proved using classical Hölder inequality while Theorem 6.1 follows from the Holmstedt formula (see [10]), but we believe that our proofs provide more insight.

Let us now return to the Wiener amalgams as studied in Chapter 4. We already know that $W(L^q, l^p)$ is in many cases not rearrangement invariant and almost never a Banach function space. A natural approach to fixing this issue is to consider, instead of $\|\cdot\|_{W(L^q,l^p)}$, a functional

$$f \mapsto \|f^*\|_{W(L^q,l^p)}.$$

As the next theorem shows, this approach leads to the Wiener–Luxemburg amalgam norm.

Theorem 6.3. *Let $p, q \in [1, \infty]$. Then there is a constant $C > 0$ such that*

$$C^{-1}\|f^*\|_{W(L^q,l^p)} \leq \|f\|_{WL(L^q,L^p)} \leq C\|f^*\|_{W(L^q,l^p)},$$

for every $f \in M$.

This result follows from Theorem 3.7 once one shows that the functional

$$f \mapsto \|f^*\|_{W(L^q,l^p)}$$

is a Banach function quasinorm and that the set

$$\left\{f \in M; \|f^*\|_{W(L^q,l^p)} < \infty\right\}$$

coincides with $WL(L^q, L^p)$, both of which is easy to do. The details are left as an exercise to the reader.

Thanks to Proposition 4.6 we also get the following result.

Corollary 6.4. *Let $p, q \in [1, \infty]$. Then the Wiener amalgam norm $\|\cdot\|_{W(L^q,l^p)}$ is equivalent to a rearrangement invariant norm if and only if $p = q$ in which case it is equivalent to the classical Lebesgue norm $\|\cdot\|_p$.*

6.2 Wiener–Luxemburg amalgams of Orlicz spaces

The second class of spaces we present here as an example of possible application of our theory is the class of Orlicz spaces. The general theory of these spaces has been covered in depth for example in [15, Chapter 4], which will be our reference for this section.

Let us first provide a (very) brief introduction of said spaces. For details and further information we refer the reader to [15, Chapter 4, Sections 4.1-4.6]. Note that we still assume that $(R, \mu) = ([0, \infty), \lambda)$.

The first step is to define a Young function.

Definition 6.5. *We say, that function $\Phi : [0, \infty) \to [0, \infty)$ is a Young function if there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$\Phi(t) = \int_0^t \varphi \, d\lambda$$

for all $t \in [0, \infty)$ and the function φ satisfies the following conditions:

1. *$\varphi(0) = 0$,*
2. *$\varphi(s) > 0$ for $s > 0$,*
3. *φ is right-continuous on $[0, \infty)$,*

4. φ *is non-decreasing on* $[0,\infty)$,

5. $\lim_{s\to\infty}\varphi(s)=\infty$.

The next necessary definition is that of a complementary function.

Definition 6.6. *Let* Φ *be a Young function generated by* φ*, that is*

$$\Phi(t)=\int_0^t \varphi\,d\lambda.$$

Define a function ψ *for* $t\in[0,\infty)$ *by*

$$\psi(t)=\sup_{\varphi(s)\le t} s.$$

Then the function $\Psi:[0,\infty)\to[0,\infty)$*, defined for any* $t\in[0,\infty)$ *by*

$$\Psi(t)=\int_0^t \phi\,d\lambda,$$

will be called the complementary function to Φ.

Thus equipped we can now define the Orlicz norm corresponding to a Young function Φ.

Definition 6.7. *Let* Φ *be a Young function and* Ψ *its complementary function. We then define the Orlicz norm* $\|\cdot\|_\Phi$ *for* $f\in M_0$ *by*

$$\|f\|_\Phi=\sup_g\int_0^\infty |fg|\,d\lambda,$$

where the supremum is taken over all $g\in M_0$ *such that*

$$\int_0^\infty \Psi(|f|)\,d\lambda\le 1.$$

Furthermore, we define the corresponding Orlicz space L^Φ *as the set*

$$L^\Phi=\{f\in M_0;\|f\|_\Phi<\infty\}.$$

The Orlicz spaces, as defined above, are rearrangement invariant Banach function spaces (see [2, Chapter 4, Theorem 8.9]).

Having provided the basic definitions, we may now apply our general theory on these spaces. We first prove the following theorem concerning embeddings.

Theorem 6.8. *Let* Φ_1 *and* Φ_2 *be two Young functions and let* $\|\cdot\|_{\Phi_1}$ *and* $\|\cdot\|_{\Phi_2}$ *be the corresponding Orlicz norms. Then*

1. *if there are some constants* $c,T\in(0,\infty)$ *such that*

$$\Phi_2(t)\le\Phi_1(ct)$$

for all $t\in[T,\infty)$ *then the local component of* $\|\cdot\|_{\Phi_1}$ *is stronger than that of* $\|\cdot\|_{\Phi_2}$, *2. if there are some constants $c, T \in (0, \infty)$ such that*

$$\Phi_2(t) \leq \Phi_1(ct)$$

for all $t \in [0, T]$ then the global component of $\|\cdot\|_{\Phi_1}$ is stronger than that of $\|\cdot\|_{\Phi_2}$.

The proof is quite straightforward and the general idea is the same as in the proof of characterisation of the embeddings between two Orlicz spaces, see for example [15, Theorem 4.17.1].

Proof. Suppose that the first condition hold and that $\|f^*\chi_{[0,1]}\|_{\Phi_1} < \infty$. Then there exist some $\gamma \in (0, \infty)$ such that

$$\int_0^\infty \Phi_1(\gamma f^* \chi_{[0,1]})\, d\lambda < \infty.$$

This is a characterisation of Orlicz spaces which can be found for example in [15, Lemma 4.7.2]. Denote now

$$I = \left\{ t \in [0, \infty); f^*\chi_{[0,1]} < \frac{c}{\gamma} T \right\}.$$

Then we immediately get that

$$\Phi_2\left(\frac{\gamma}{c} f^* \chi_{[0,1]}\right) \leq \Phi_1\left(\gamma f^* \chi_{[0,1]}\right)$$

for all $t \in [0, \infty) \setminus I$. Hence

$$\begin{aligned} \int_0^\infty \Phi_2\left(\frac{\gamma}{c} f^* \chi_{[0,1]}\right) d\lambda &= \int_I \Phi_2\left(\frac{\gamma}{c} f^* \chi_{[0,1]}\right) d\lambda + \int_{[0,\infty)\setminus I} \Phi_2\left(\frac{\gamma}{c} f^* \chi_{[0,1]}\right) d\lambda \leq \\ &\leq \lambda(I \cap [0,1]) \Phi_2(T) + \int_{[0,\infty)\setminus I} \Phi_1\left(\gamma f^* \chi_{[0,1]}\right) d\lambda \leq \\ &\leq \Phi_2(T) + \int_0^\infty \Phi_1(\gamma f^* \chi_{[0,1]})\, d\lambda < \infty. \end{aligned}$$

This shows that $\|f^*\chi_{[0,1]}\|_{\Phi_2} < \infty$ (see [15, Remark 4.7.1]).

Suppose now that the second condition holds and that we have some f such that both $f^*(1) < \infty$ and $\|f^*\chi_{(1,\infty)}\|_{\Phi_1} < \infty$. Then again there is some $\gamma \in (0, \infty)$ such that

$$\int_0^\infty \Phi_1(\gamma f^* \chi_{(1,\infty)})\, d\lambda < \infty.$$

Note, that this means that $\lambda(\{t \in [0, \infty); f^*\chi_{(1,\infty)} > a\}) < \infty$ for any $a \in (0, \infty)$ (because Φ_1 is increasing, see [15, Lemma 4.2.2]). Denote now

$$I = \left\{ t \in [0, \infty); f^*\chi_{(1,\infty)} > \frac{c}{\gamma} T \right\}.$$

Then $\lambda(I) < \infty$ by the observation above and of course

$$\Phi_2\left(\frac{\gamma}{c} f^* \chi_{(1,\infty)}\right) \leq \Phi_1\left(\gamma f^* \chi_{(1,\infty)}\right)$$

for all $t \in [0,\infty) \setminus I$. We thus obtain

$$\int_0^\infty \Phi_2\left(\frac{\gamma}{c} f^* \chi_{(1,\infty)}\right) d\lambda = \int_I \Phi_2\left(\frac{\gamma}{c} f^* \chi_{(1,\infty)}\right) d\lambda + \int_{[0,\infty)\setminus I} \Phi_2\left(\frac{\gamma}{c} f^* \chi_{(1,\infty)}\right) d\lambda \leq$$

$$\leq \lambda(I)\Phi_2\left(\frac{\gamma}{c} f^*(1)\right) + \int_{[0,\infty)\setminus I} \Phi_1\left(\gamma f^* \chi_{(1,\infty)}\right) d\lambda \leq$$

$$\leq \lambda(I)\Phi_2\left(\frac{\gamma}{c} f^*(1)\right) + \int_0^\infty \Phi_1(\gamma f^* \chi_{(1,\infty)})\, d\lambda < \infty.$$

Hence it holds that $\|f^* \chi_{(1,\infty)}\|_{\Phi_2} < \infty$. □

This results allows us to obtain the following analogue to Theorem 6.2.

Theorem 6.9. *Let Φ_1, Φ_2 and Φ_3 be three Young functions and let L^{Φ_1}, L^{Φ_2} and L^{Φ_3} be the corresponding Orlicz spaces. Then*

1. *if there are some constants $c_1, c_2, T_1, T_2 \in (0,\infty)$ such that*
$$\Phi_3(t) \leq \Phi_1(c_1 t)$$
for all $t \in [T_1,\infty)$ and
$$\Phi_3(t) \leq \Phi_2(c_2 t)$$
for all $t \in [0,T_2]$ then
$$L^{\Phi_1} \cap L^{\Phi_2} \hookrightarrow L^{\Phi_3},$$

2. *if there are some constants $c_1, c_2, T_1, T_2 \in (0,\infty)$ such that*
$$\Phi_1(t) \leq \Phi_3(c_1 t)$$
for all $t \in [T_1,\infty)$ and
$$\Phi_2(t) \leq \Phi_3(c_2 t)$$
for all $t \in [0,T_2]$ then
$$L^{\Phi_3} \hookrightarrow L^{\Phi_1} + L^{\Phi_2}.$$

Proof. Part 1 holds because

$$L^{\Phi_1} \cap L^{\Phi_2} \hookrightarrow WL(L^{\Phi_1}, L^{\Phi_2}) \hookrightarrow WL(L^{\Phi_3}, L^{\Phi_3}) = L^{\Phi_3},$$

where the first embedding follows from Theorem 5.11 and the second embedding is provided, under our assumptions, by Theorem 5.9 and Theorem 6.8.

Similarly, part 2 is true because

$$L^{\Phi_3} = WL(L^{\Phi_3}, L^{\Phi_3}) \hookrightarrow WL(L^{\Phi_1}, L^{\Phi_2}) \hookrightarrow L^{\Phi_1} + L^{\Phi_2}$$

which follows from the same theorems. □

Bibliography

[1] A. Alberico, A. Cianchi, L. Pick, and L. Slavíková. Sharp Sobolev type embeddings on the entire Euclidean space. *Communications on Pure Applied Analysis*, 17(5):2011–2037, 2018.

[2] C. Bennett and R. Sharpley. *Interpolation of Operators*. Number 129 in Pure and Applied mathematics. Academic Press, 1988.

[3] Y. Benyamini and J. Lindenstrauss. *Geometric nonlinear functional analysis*, volume 1. American Mathematical Society, 2000.

[4] H. G. Feichtinger. Banach convolution algebras of Wiener type. In E. B. Sz.-Nagy and J. Szabados., editors, *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, 1983.

[5] H. G. Feichtinger. Generalized amalgams, with applications to Fourier transform. *Canadian Journal of Mathematics*, 42(3):395–409, 1990.

[6] H. G. Feichtinger. Wiener amalgams over Euclidean spaces and some of their applications. In K. Jarosz, editor, *Function Spaces, Proc Conf, Edwardsville/IL (USA) 1990*, volume 136 of *Lect. Notes Pure Appl. Math.*, pages 123–137. Marcel Dekker, 1992.

[7] J. J. Fournier and J. Stewart. Amalgams of L^p and l^q. *Bulletin of the American Mathematical Society*, 13(1):1–21, 1985.

[8] A. Gogatishvili and F. Soudský. Normability of Lorentz spaces — an alternative approach. *Czechoslovak Mathematical Journal*, 64(3):581–597, 2014.

[9] F. Holland. Harmonic analysis on amalgams of L^P and l^q. *Journal of the London Mathematical Society*, s2-10(3):295–305, 1975.

[10] T. Holmstedt. Interpolation of quasi-normed spaces. *Mathematica Scandinavica*, 26:177, June 1970.

[11] N. Kalton. Chapter 25 - quasi-Banach spaces. In W. Johnson and J. Lindenstrauss, editors, *Handbook of the Geometry of Banach Spaces*, volume 2, pages 1099 – 1130. Elsevier Science B.V., 2003.

[12] L. Malý. Minimal weak upper gradients in Newtonian spaces based on quasi-Banach function lattices. *Annales Academiae Scientiarum Fennicae Mathematica*, 38, 10 2012.

[13] A. Nekvinda. Untitled and unpublished manuscript. 2017.

[14] B. Opic and L. Pick. On generalized Lorentz–Zygmund spaces. *Mathematical Inequalities and Applications*, 2(3):391–467, 1999.

[15] L. Pick, A. Kufner, O. John, and S. Fučík. *Function Spaces, Vol. 1.* Number 14 in De Gruyter Series in Nonlinear Analysis and Applications. Walter de Gruyter, 2 edition, 2013.

[16] J. Vybíral. Optimal Sobolev embeddings on R^n. *Publicacions Matemàtiques*, pages 17–44, 2007.

[17] N. Wiener. On the representation of functions by trigonometrical integrals. *Mathematische Zeitschrift*, 24(1):575–616, Dec 1926.

[18] N. Wiener. Tauberian theorems. *Annals of mathematics*, pages 1–100, 1932.

[19] N. Wiener. The Fourier integral and certain of its applications, 1933.