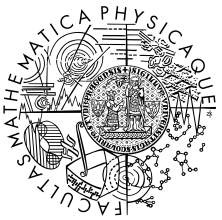

FACULTY
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MASTER THESIS

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Contour methods in the mathematical theory of phase transitions

Department of Mathematical Analysis

Supervisor of the master thesis:	doc. RNDr. Miloš Zahradník, CSc.
Study programme:	Physics
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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague–Holešovice,
date 5 January 2020

Oliver Nagy

Dedication

I would like to express my gratitude to the supervisor of this thesis doc. RNDr. Miloš Zahradník, CSc. for continued guidance and advice not only during writing of this thesis but also during my entire master's and most of my bachelor's studies.

I would also like to thank my family, my partner and my friends for their support.

Title: Contour methods in the mathematical theory of phase transitions

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Abstract: This thesis concerns itself with three topics, namely polymer models, Pirogov–Sinai theory and one-dimensional Dyson models. It contains a short introduction into all three topics. The introduction to Pirogov-Sinai theory will serve as a starting point for a future expanded introductory exposition, since such a material is missing in the contemporary literature. Research result of the first chapter is a detailed combinatorial analysis of cluster expansion of hard-core repulsive polymer model based on 'self-avoiding polymer trees', leading to simplification of the structure of summation in the partition function. In the case of Dyson models we suggest an alternative definition of contours for the one-dimensional Dyson model with the exponent of polynomially-decaying interaction $p \in (1, 2)$ that is usable for study using Pirogov–Sinai methods.

Keywords: Contours, polymers, cluster expansion, Pirogov–Sinai theory, Dyson model;

Název: Konturové metody v matematické teorii fázových přechodů

Autor: Oliver Nagy

Katedra: Katedra matematické analýzy

Vedoucí: doc. RNDr. Miloš Zahradník, CSc., Katedra matematické analýzy

Abstrakt: Práce se zaobírá třemi souvisejícími tématy z matematické statistické fyziky. Jsou to polymerové modely, Pirogov–Sinaiova teorie a Dysonovy modely v dimenzi 1. Práce obsahuje stručný úvod do všech třech uvedených partií. Úvod do Pirogov–Sinaiovy teorie bude využít jako výchozí bod pro budoucí obsáhlejší úvodní text. Takovýto text v soudobé literatuře zatím chybí. Výzkumným přínosem první části práce je detailní kombinatorický rozbor klastrových rozvojů polymerových modelů s tvrdým jádrem založený na pojmu "samovyhýbající se polymerový strom", vedoucí k zjednodušení struktury sumace v partiční funkci. V případě Dysonových modelů navrhujeme alternativní definici kontury pro jednorozměrný Dysonův model s exponentem polynomiálně klesající interakce $p \in (1, 2)$ použitelnou pro zkoumání metodami Pirogov–Sinaiovy teorie.

Klíčová slova: Kontury, polymery, klastrový rozvoj, Pirogov–Sinaiova teorie, Dysonův model;

Contents

Introduction

The following pages should serve both as an introduction to some techniques and an overview of results for a few loosely connected projects in mathematical statistical physics.

The main theses are:

- In the setting of hard-core repulsive polymer model it is possible to simplify the expression of the logarithm of partition function using tree-like structures.
- In Dyson model it is possible to define contours in a way that gives them better energetic separation compared to methods used by other authors.

This work is divided into three parts:

- The first part is concerned with polymer models. First, we introduce this model and provide some examples. We show a way to simplify the structure of partition function for hard-core repulsive polymer model. The main idea is the introduction of a tree-like structure called *umbel*, using which we construct *umbel clusters* and *umbel trees*. Using this technique we can achieve identities of Kotecký–Preiss type and corresponding estimates.
- The second part is an introduction to Pirogov–Sinai theory in its abstract version.
- The third part is concerned with Dyson model on a one-dimensional lattice. The peculiar fact about Dyson model is that it exhibits the phenomenon of phase transition even on a one-dimensional lattice, unlike the Ising model. The reason for this behaviour is the infinite-range slowly-decaying interaction between spins. We provide an overview of results and also an alternative definition of contours that may serve as a starting point for systematic application of Pirogov–Sinai theory to Dyson model.

All of these are active topics of research in mathematical statistical physics, as evidenced by recent publications.

The reader is expected to have some elementary knowledge of graph theory and statistical physics. For graph theory, textbook [Nešetřil and Matoušek, 2009] is a suitable introduction. For statistical physics, there are lots of textbooks an interested reader may choose from, e.g. [Landau and Lifshitz, 1980] for a theoretical physicists' point of view or [Friedli and Velenik, 2017] written from the perspective of the community of mathematical physics.

1 | Polymer models

In this chapter we are developing cluster expansion techniques to work with partition functions of hard-core repulsive polymer model. These partition functions exhibit huge cancellation of terms[1], which can be exploited to significantly simplify the structure of partition function. We define a tree-like structure called *umbel* and reorganize the summation in partition function as a sum over these umbels. We also obtain *identities*, which in our opinion clarify the structure of estimates of Kotecký–Preiss-type.

Cluster expansions are a standard technique of rigorous statistical physics. There are many variations of this idea, for example:

- original approach of [Mayer, 1937];
- tree-graph approach that originated in the article [Penrose, 1963];
- multivariate Taylor expansion approach, see [Dobrushin, 1996];
- combinatorial approach via so-called exponential formula for combinatorial species, see [Faris et al., 2010];
- approach using Möbius inversion formula, as found in [Kotecký and Preiss, 1986];

A comprehensive overview of cluster expansions in a very general setting is given in lecture notes [Procacci, 2005]. Another exposition is given in chapter 4 of [Friedli and Velenik, 2017].

Kotecký–Preiss estimates originated in an article about inductive approach to convergence of cluster expansions [Kotecký and Preiss, 1986]. There were also many others working on this problem, see e.g. [Bovier and Zahradník, 2000; Dobrushin, 1996] and others. The question of convergence of cluster expansion originates in the article [Gruber and Kunz, 1971] where they used at that time traditional, albeit somewhat cumbersome technique of Kirkwood-Salzburg integral equations. In one of the more recent articles [Fernández and Procacci, 2007] authors compare different forms of these estimates and improve upon older results.

[1] i.e. lots of terms like $(1-1)^n$.

1.1 Introductory definitions

Definition 1.1 (Polymer)
Polymer P is an abstract object for which we define its support $\operatorname{supp} P$ *and a pair-wise symmetric relation that determines compatibility with other polymers.*

Definition 1.2 (Hard-core repulsive polymer model)
By polymer model with hard repulsion we mean, as it is standard in mathematical statistical physics, model whose partition function is

$$Z = \sum_{\{P_i \in \mathbb{P}\}} \prod_i w_{P_i}, \tag{1.1}$$

where $\mathbb{P}$ is a collection of all allowed polymers, w_P is a (complex) weight of polymer P and the sum in (1.1) *is taken over all compatible families $\{P_i\}$ of mutually compatible polymers.*

More precisely, let $\mathbb{G}^\bullet(\mathbb{P})$ be an unoriented graph of all incompatible pairs of polymers in $\mathbb{P}$. Since we are dealing with hard-repulsive case, we introduce a symmetric function

$$c \ : \ \mathbb{P} \times \mathbb{P} \to \{0, 1\}; \tag{1.2}$$

$$c(P, P') = \ 1 + i(P, P')$$

$$i(P, P') = \begin{cases} 0, & P \text{ and } P' \text{ are compatible} \\ -1, & P \text{ and } P' \text{ are incompatible.} \end{cases} \tag{1.3}$$

Using this notion of compatibility we can rewrite (1.1) *as*

$$\begin{aligned} Z &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(P_1,\dots,P_N)} \prod_i w_{P_i} \prod_{j<i} c(P_i, P_j) \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{(P_1,\dots,P_N)} \prod_i w_{P_i} \sum_{G \subset \mathbb{G}^\bullet(P_1,\dots,P_n)} (-1)^{|G|}, \end{aligned} \tag{1.4}$$

where the sum is taken over all N-tuples of polymers from $\mathbb{P}$ and all subgraphs $G \in \mathbb{G}^\bullet$. We use the notation $\mathbb{G}^\bullet(P_1, \dots, P_n)$ to denote the graph of all incompatible pairs $\{P_i, P_j\}$ in the ordered collection $(P_1, \dots, P_n)$. We emphasize that polymers $P_1, \dots, P_n$ are in general neither mutually compatible, nor mutually different.

Assumption 1.1 (Self-incompatibility)
We assume that $i(P, P) = -1$ for any $P \in \mathbb{P}$.

Remark. One can consider an equivalent model with

$$i(P, P) = 0 \qquad \forall P; \tag{1.5}$$

and with weight $\widetilde{w}_P$ such that

$$w_P = \exp(\widetilde{w}_P) - 1. \tag{1.6}$$

However, this idea does not seem productive for $|\mathbb{P}| > 2$. At least we are unable to develop a theory of comparable clarity in this equivalent model.

The standard cluster expansion procedure for (1.4) starts with the decomposition of graph G into connected components. As a next step, re-tag the polymers by adding a new tag to any connected component and renumber the polymers inside each component. This makes it possible to lose the tags $1, \ldots, N$ and to re-write (1.4) as

$$Z = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{(\mathscr{C}_1,\ldots,\mathscr{C}_m)} \prod_{j=1}^{m} w_{\mathscr{C}_j}. \tag{1.7}$$

> **Remark.** Each polymer of $\mathscr{C}$ belongs to at least one edge of G and the summation in (1.7) is over all possible collections $(\mathscr{C}_1, \ldots, \mathscr{C}_m)$ and all possible connected subgraphs $G \subset \mathbb{G}^{\bullet}$.

The clusters $\mathscr{C}$ are defined as follows:

Definition 1.3 (Cluster)
Cluster $\mathscr{C}$ is defined as a pair

$$\mathscr{C} = ((P_1, \ldots, P_n), G), \tag{1.8}$$

where $(P_1, \ldots, P_n)$ is some ordered n-tuple of (not necesarilly different) polymers and

$$G \subset \mathbb{G}^{\bullet}(P_1, \ldots, P_n) \tag{1.9}$$

is a connected subgraph of incompatibilities in $(P_1, \ldots, P_n)$.
The weight $w_{\mathscr{C}}$ of a cluster $\mathscr{C}$ is given as

$$w_{\mathscr{C}} = \frac{1}{k!} \prod_{j=1}^{k} w_{P_j} (-1)^{|G|}, \tag{1.10}$$

where $|G|$ denotes the number of edges in G.

Thus (1.7) can be rewritten as (for more details see Section 1.3)

$$Z = \exp\left(\sum_{\mathscr{C}} w_{\mathscr{C}} \right), \tag{1.11}$$

where the sum is taken over all clusters $((P_1, \ldots, P_n), G)$ that can be formed on some ordered set $(P_1, \ldots, P_n)$ of polymers from $\mathbb{P}$.

In this thesis we work on some d-dimensional regular lattice, formally a subset of $\mathbb{Z}^d$, $d \geq 2$. It is convenient to introduce periodic boundary conditions and to systematically work on a torus Λ such that

$$\Lambda = \{0, 1, \ldots, n-1\}^d, \tag{1.12}$$

with the usual structure of an abelian group. Polymers will then be some connected objects on the torus Λ. The family $\mathbb{P} = \mathbb{P}(\Lambda)$ of all polymers will often be determined by some additional requirement $\mathscr{R}$. This additional requirement $\mathscr{R}$ will typically be of the type that only polymers from some specific subset

of Λ are allowed. The collection of allowed polymers w.r.t. $\mathscr{R}$ will be denoted $\mathbb{P}_{\mathscr{R}}$. Write the general form of such a partition function as

$$Z(\mathscr{R}) = Z(\mathbb{P}_{\mathscr{R}}) = \sum_{\{P_i \in \mathbb{P}_{\mathscr{R}}\}} w_{P_i}. \tag{1.13}$$

We can also write clusters in a more detailed way as

$$\mathscr{C} = (\underline{\mathscr{C}}, G), \tag{1.14}$$

where $\underline{\mathscr{C}} = (P_1, \ldots, P_n)$ is an ordered n-tuple of vertices of connected graph G. We will sometimes also use the notation

$$\mathrm{supp}(\mathscr{C}) \equiv \mathrm{supp}(\underline{\mathscr{C}}) = \{P_i\} \tag{1.15}$$

to denote the unordered collection of polymers that are used in $\mathscr{C}$.

1.2 Examples of polymer models

The most well-known examples of polymer models appear in the study of the low- (and high-) temperature Ising model and high-temperature short-range spin systems. Here we provide also some other less known examples. Some of them are classical and well-understood (at least for small enough polymer weights), while others are less explored, with some of them being non-trivial for *any* polymer weight.

1.2.1 Non-ideal lattice gas

This basic example is probably the simplest possible non-trivial polymer model and yet it exhibits several phase transitions of different nature and lots of interesting phenomena. Unfortunately, most of these are out of reach of rigorous study in present-state mathematical physics.

Definition 1.4 (Polymers in non-ideal gas model)
Polymers are points of some lattice Λ[2]. *Two polymers* (P, P') *are incompatible if*

$$\mathrm{dist}(P, P') \leq 1. \tag{1.16}$$

The distance function dist() *is given by our choice of metric on* Λ. *The most common choice is* $\mathbb{Z}^d$ *and the* l_1*-metric*

$$\mathrm{dist}_{l_1}(i, i') = \sum_{j=1}^{d} |i_j - i'_j|. \tag{1.17}$$

[2]Usually $\Lambda = \mathbb{Z}^d$ with a topology of some torus. Other popular choices are hexagonal, quasi-periodic or random lattices. Models of this type can be generalised to $\Lambda = \mathbb{R}^d$.

Definition 1.5 (Partition function of non-ideal gas model)
Partition function of this model is

$$Z = \sum_{M \subset \Lambda} q^{|M|}, \tag{1.18}$$

where the summation is done over all totally disconnected subsets $M \subset \Lambda$, meaning sets where $\forall i, j \in M : \mathrm{dist}(i, j) \geq 2$, or equivalently over all subsets of mutually incompatible polymers. The activity of a polymer $P \in \Lambda$ is given by some parameter $q = e^{-\frac{1}{T}} > 0$, where T plays the role of temperature.

> **Remark.** We choose parameter q to be sufficiently small. For large values of q the behaviour of this model is far from trivial. While for q close to zero this is truly a remarkably simple model of non-ideal gas, for $q \nearrow +\infty$ there are phase transitions from 'gaseous' to 'liquid' and finally to 'solid' phase[3].
>
> While we can only speculate on the behaviour of this model in the 'liquid' regime, on an intuitive level it seems clear how it behaves in the 'solid' phase. For $\Lambda = \mathbb{R}^d$, $d = 2, 3, \ldots$ this is a problem of packing spheres into a given volume with maximal possible density of packing. For $\Lambda = \mathbb{Z}^d$, $d = 2$ and the l_1 metric, the case $q \nearrow +\infty$ seems to correspond to two antiferromagnetic phases, where M is a perturbation of even/odd sublattice of Λ.

1.2.2 Loop models

Two basic examples of this class of models are the classical Ising two-dimensional ferromagnet and the determinant of lattice Laplacian.

Definition 1.6 (Loop)
Let Λ be a torus corresponding to some d-dimensional lattice Λ, e.g. $\Lambda = \mathbb{Z}^d$, $d \geq 2$. A loop[4] *is a closed path in Λ visiting any point at most once. A closed path in Λ is an indexed collection of positions in Λ; $P = (x_0, x_1, \ldots, x_n \equiv x_0)$. Such a path P can be decomposable, i.e. a power of another closed path $Q = (y_0, y_1, \ldots, y_m \equiv y_0)$, $P = Q^k$; meaning that e.g.*

$$P = Q^2 = (\ y_0, y_1, \ldots, y_{m-1}, y_0, y_1, \ldots, y_m \equiv y_0\ ). \tag{1.19}$$

> **Remark.** Simple algebraic argument shows that any closed path P can be written uniquely as $P = Q^k$, with Q indecomposable. Indecomposability means that all the (cyclical) shifts of walk Q are different walks. On the other side, if $P = Q^k$ then there are exactly k rotations of the cycle that give identical path in Λ.

Definition 1.7 (Loop model)
A loop model *is a specific case of a polymer model, where the weight of a loop*

[3] Or probably phases. Also, in contrast to the case of small parameter q, the behaviour of the model substantially depends on the properties of the lattice.

[4] Or cycle – both names are being used in scientific literature.

defined as $P = (x_0, x_1, x_2, \dots, x_{n-1}, x_n \equiv x_0)$ *is given as*

$$w_P = \operatorname{sign}_P \prod_i w_{x_i x_{i+1}}. \tag{1.20}$$

The prescription of the value $\operatorname{sign}(P) = \{\pm 1\}$ *is model-dependent.*

The terms w_{xy} *are given by some matrix* $W = \{w_{xy}, (x, y) \in \Lambda \times \Lambda\}$.

Example. In the case of Laplace operator on a d-dimensional torus Λ related to $\mathbb{Z}^d$, we have

$$W = \{w_{ij}\} \tag{1.21}$$

$$w_{ij} = \begin{cases} \frac{1}{2d} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.22}$$

More generally, consider a matrix

$$(E + W) \tag{1.23}$$

where W is off-diagonal with non-negative elements, such that[5]

$$s_i = \sum_{j \neq i} w_{ji} < 1. \tag{1.24}$$

Compute the determinant

$$\det(E + W) = \sum_{\{C_i\}} \prod_i \operatorname{sign}_{C_i} w_{C_i}, \tag{1.25}$$

where the summation is over all permutations[6] $\{C_i\}$; $\operatorname{sign}_C = \pm 1$ is the usual sign of cycle C defined so that cycles of odd length have positive sign and those of even length have negative; and weight of cycle w_C is given by

$$w_C = \prod_{(j,i) \in C} w_{ji}. \tag{1.26}$$

The product on the right-hand side of the expression above equals to the product of all matrix elements w_{ji} corresponding to edges of a given cycle.

In case of $(E - W)$[7] we get much more tractable expressions. This is one of the simplest examples of exactly solvable models in statistical mechanics. By theorems of linear algebra[8] we have

$$\det(E - W) = \exp\left(\operatorname{Tr}\left(\log(E - W)\right)\right) \tag{1.27}$$

$$= \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr}(W^n)\right) \tag{1.28}$$

$$= \exp\left(-\sum_{n=1}^{\infty} \sum_{P:|P|=n} \frac{1}{n} w_P\right), \tag{1.29}$$

[5] Note that Laplacian is a limit of this more general situation when $s_i \nearrow 1$.

[6] Equivalently over all collections of mutually disjoint cycles; by elementary argument of linear algebra.

[7] in typical examples, like that of a Laplacian operator, the terms of W are non-negative.

[8] Those theorems are $\det \exp A = \exp \operatorname{Tr} A$; and $\det AB = \det A \cdot \det B$

where the final sum (1.29) is over all paths with a marked starting point and

$$w_P = \prod_{i=0}^{n-1} w_{x_i x_{i+1}}. \tag{1.30}$$

Notice that each closed path P can be uniquely expressed as

$$P = (Q)^k \tag{1.31}$$

where Q is some indecomposable path. 'Forgetting' the starting point of Q and noticing that there are $m = |Q|$ different ways how to choose the starting point of Q[9] we get the formula

$$\det(E - W) = \exp\left(- \sum_{Q \text{ indecomp.}} \sum_{k=1}^{\infty} \frac{(w_Q)^k}{k} \right) \tag{1.32}$$

$$= \prod_{Q \text{ indecomp.}} (1 - w_Q)\,. \tag{1.33}$$

It would be an achievement of cluster expansion method if we could obtain this result directly via combinatorial considerations without taking refuge in methods of linear algebra. Fortunately this is possible using a modified version of the technique explained here. Namely, if we have a polymer model where compatibility of polymers is defined through the relation

$$\operatorname{supp} P \cap \operatorname{supp} \widetilde{P} = \emptyset \tag{1.34}$$

there is another resummation method that is developed in our parallel article [Nagy and Zahradník]. Also see bachelor's thesis [Nagy, 2017] for some preliminary ideas along these lines.

1.2.3 'Dotted-circle' polymers

The following model appears in the study of perturbations of massless Gaussian fields by potentials of the ϕ^4 kind. It is a *very* difficult model and the discussion here is a very tentative one.

Definition 1.8 (Polymers in dotted-circle model)
Polymers are connected structures

$$\widetilde{P} \equiv (\{P_i\}, S) \tag{1.35}$$

where P_i are 'dotted' walks on a three-dimensional torus Λ ($\Lambda \nearrow \mathbb{Z}^3$) and $S \subset \Lambda$ is the support of $\widetilde{P}$.

Walk P (i.e. closed path on Λ) is called dotted *if some steps $n_1, \ldots, n_k$ of the sequence $(x_1, \ldots, x_n = x_1)$ are points of S.*

[9] Or alternatively that all the rotations of Q are mutually different.

We assume that any point $s \in S$ contains at least one such step of some walk P_i and moreover the entire $\widetilde{P}$ is connected, meaning that a path exists between any two walks P_i, P_j of the system $\widetilde{P}$ using either other walks of $\widetilde{P}$ or jumping between two walks at the pair of their common 'dotted' points.

The weight of $\widetilde{P}$ is given by

$$w_{\widetilde{P}} = \prod_{j \in S} w_j \prod_i w_{P_i} \tag{1.36}$$

where w_{P_i} is the usual weight of a symmetric random walk on Λ. The quantities w_j depend on the number of dotted visits of the system $\widetilde{P}$ in s.

Example (Massless Gaussian model)**.** In the case of perturbations of massless Gaussian model with quadratic form

$$\sum_{|i-j|=1} (x_i - x_j)^2 \tag{1.37}$$

and one-spin perturbing potential U, the quantities w_j are equal to a_m where m is the number of 'dotted' visits of walks $\widetilde{P}$ in s, the numbers a_m being the coefficients of Taylor expansion of

$$\exp\left(-U(x_j)\right) = 1 + \sum_{m=2}^{\infty} \frac{a_m}{m!} \left(-x_j\right)^{2m}. \tag{1.38}$$

We need conditions like

$$a_2 < 0 \tag{1.39}$$

to be able to prove the convergence of cluster expansions of such systems. This is a delicate analytical question and work on this problem is in progress [Zahradník]. A reasonable candidate for U satisfying the properties sketched above is

$$\exp\left(-U(y)\right) = 1 - \varepsilon \int_0^1 e^{-\lambda y^2} \,\mathrm{d}\nu(\lambda) \tag{1.40}$$

where ν is a suitable distribution on $(0,1)$ of the type $\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\phi$ where ϕ is something close to $\lambda e^{-\kappa y^2}$. Then

$$\int_0^1 e^{-\lambda y^2} \,\mathrm{d}\nu(\lambda) = y^4 \int_0^1 \phi(\lambda) e^{-\lambda y^2} \,\mathrm{d}\lambda \tag{1.41}$$

and

$$a_m = \int_0^1 \phi(\lambda)(-\lambda)^{m-2} \,\mathrm{d}\lambda. \tag{1.42}$$

When summing over all possible positions of dotted points between N visits of paths of $\widetilde{P}$ at a given point $j \in S$ one needs something like

$$\int_0^1 \sum_{m=1}^{N} \binom{N}{m} (-\lambda\ )^m \phi(\lambda)\, \mathrm{d}\lambda\ < 0, \tag{1.43}$$

i. e.

$$\int_0^1 (1-\lambda)^m \phi(\lambda)\, \mathrm{d}\lambda < 0 \tag{1.44}$$

for each m, most importantly for $m = 2$.

1.2.4 Reflective random walks

Another non-trivial and highly interesting polymer model appears in the study of *gradient* models close to Gaussian, i.e. models on a three-dimensional torus Λ, such that $\Lambda \nearrow \mathbb{Z}^3$; with Hamiltonian

$$H(x_\Lambda) = \sum_{|i-j|=1} (x_i - x_j)^2 + \sum_{|i-j|=1} U(x_i - x_j), \tag{1.45}$$

where U is some small perturbing potential of the type $U(y) \propto y^4$.

Let P be a closed walk on Λ, $P = \{x_1, x_2, \ldots, x_n = x_1\}$. Imagine horizontal obstructions as seen on Figure 1.1.

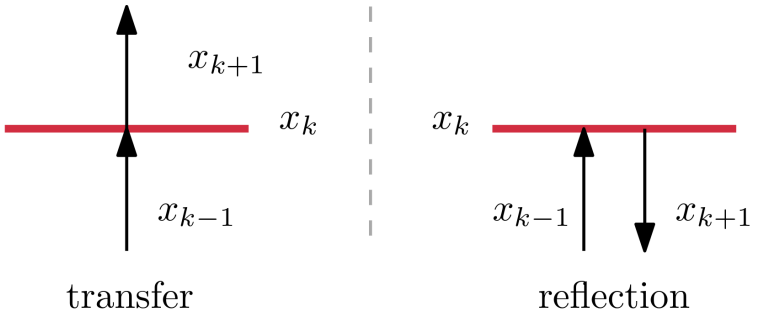


Figure 1.1: Illustration of reflective random walk.

In the case of pass through a barrier some weight w_k is assigned to path P whereas in the second case it is $-w_k$. The full weight of a closed walk P is then given by

$$w_P \prod_{\text{barriers}} (\pm w_k)\,, \tag{1.46}$$

where w_P is the ordinary weight of a random walk P and the additional factors $\pm w_B$ are assigned to P at the places where P meets an obstruction (and either passes through or reflects from it).

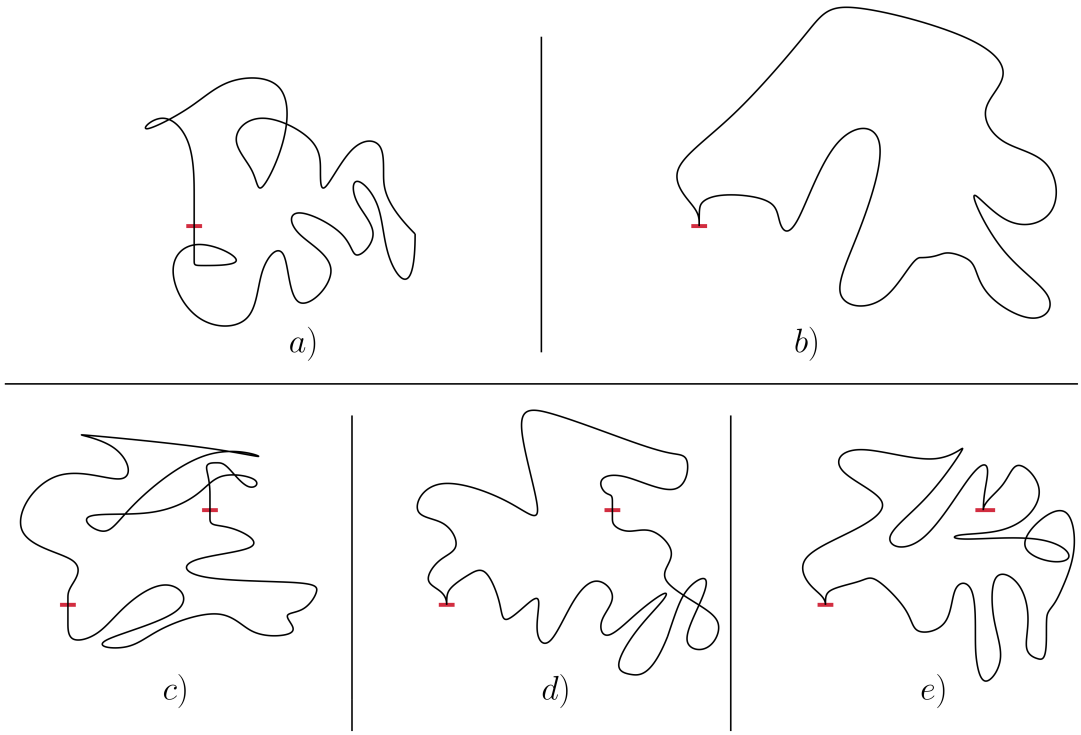


Figure 1.2: Illustration of different reflective random walks. In the case (a) we see a walk that passes through once and so it has weight $+\varepsilon w_P$. In the case (b) we see reflection and therefore the weight is $-\varepsilon w_P$. Cases (c) and (e) have the same weight $+w_p\varepsilon^2$. Only in case of one reflection and one pass-through (case (d)) the weight is $-w_p\varepsilon^2$.

Example. To illustrate the model see Figure 1.2.

Take for example the case (c). The weight of obstruction is some quantity

$$w_B = \begin{cases} +\varepsilon & \text{pass-through} \\ -\varepsilon & \text{reflection} \end{cases} \tag{1.47}$$

and since walk P passes through a barrier twice its final weight is $+w_P\varepsilon^2$.

The study of such polymer models seems to be a deep and technically delicate problem. This is related to precise estimates of probabilities of random walks reaching the barrier at a prescribed height m.

To carry out these estimates, one needs to analyse the Fourier coefficients of functions $\mathbb{R}^3 \to \mathbb{R}$

$$f(x,y,z) = \frac{\sin^2 x}{\sin^2 x + \sin^2 y + \sin^2 z} \tag{1.48}$$

$$g(x,y,z) = \frac{1}{\sin^2 x + \sin^2 y + \sin^2 z}, \tag{1.49}$$

which is a fine problem of mathematical analysis.

1.2.5 Kac polymer models

The model described below appears in the study of low-temperature ferromagnetic phases of Kac-Ising model with Hamiltonian

$$H(x_\Lambda) = \frac{1}{2R^d} \sum_{|i-j|\le R} (x_i - x_j)^2; \qquad x_\Lambda \in \{-1, 1\}. \tag{1.50}$$

If we consider a restricted ensemble of configurations such that some high enough fraction[10] of all spins in any cube

$$\square_i = \{j \in \Lambda, |j - i| \le R\} \tag{1.51}$$

has the value $+1$ then such a model can be shown to be closely related to a specific kind of polymer model — ensemble of constellations.

Definition 1.9 (Polymer in constellation model)
In this case, polymer is a collection $P = (S, G)$ where $S \subset \Lambda$ and G is a connected tree on S. The weight of such a polymer is

$$w_P = \varepsilon^{|S|} \prod_{\{i,j\}\in G} \delta_{ij} \tag{1.52}$$

where ε is a small quantity and δ_{ij} are some quantities satisfying

$$\sum_{j\in\Lambda} \delta_{ij} \le 1 \tag{1.53}$$

$$\delta_{ij} = 0 \qquad \text{if } |i - j| \ge R. \tag{1.54}$$

1.2.6 Dyson polymers

An analogy of the above-mentioned model appears in the study of one-dimensional Dyson-Ising model with long-range part of Hamiltonian being

$$H_L(x_\Lambda) = \sum_{\{i,j\}\subset\Lambda} \frac{1}{|i - j|^p} x_i x_j, \tag{1.55}$$

for $1 < p < 2$.[11] In Chapter 3 we will return to this model in greater detail.

Here we do not consider general graphs and restrict our attention only to a very simplified situation where polymers are intervals P of $\mathbb{Z}$ having weight

$$w_P = C|i - j|^{2-p} \tag{1.56}$$

Compatibility of two intervals (i, j) and $(\widetilde{i}, \widetilde{j})$ is defined by the requirement that the sets $\{i, j\}$ and $\{\widetilde{i}, \widetilde{j}\}$ are *disjoint*[12].

[10] 90 %, for example.
[11] It is possible to work with $p = 2$, but this is a different and *delicate* situation.
[12] But the intervals themselves may overlap.

1.3 Cluster expansion

Proposition 1.1 (Partition function as a sum over graphs)
Let

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{(P_1,\dots,P_n) \\ compatible}} \prod_i w_{P_i} \tag{1.57}$$

where the sum is over all compatible ordered collections of polymers $(P_1, \dots, P_n)$ *be the partition function of studied model.*

Rewrite it as a sum over all collections $(P_1, \dots, P_n)$

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(P_1,\dots,P_n)} \prod_i w_{P_i} \prod_{i<j} \left(1 + c_{P_i P_j}\right) \tag{1.58}$$

where $c_{PP'} \in \{0, -1\}$ *is the compatibility factor. Expand the parentheses and write*

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(P_1,\dots,P_n)} \prod_i w_{P_i} \sum_{G \subset G^\bullet(P_1,\dots,P_n)} (-1)^{|G|} \tag{1.59}$$

with the sum taken over all subgraphs of the graph $G^\bullet(P_1, \dots, P_n)$ *on the set* $\{1, 2, \dots, n\}$ *with edges* $\{i, j\}$ *corresponding to all incompatible pairs* (P_i, P_j).

Until now, polymers were tagged by numbers $1, 2, \dots, n$[13]. Now let us introduce two new enumerations. First, we arbitrarily enumerate using tags $j \in \{1, 2, \dots, k\}$ the connected components of G. Second, we enumerate by integers $\{1, 2, \dots, m_j\}$ the polymers in any connected component $\mathcal{C}_j$, $j = 1, \dots, k$ enumerated in previous step. Now, every polymer is identified by these two integers and we can forget their original enumeration.

We will use the formula

Proposition 1.2
Let the values x_i *be indexed by* $i \in \mathcal{I}$ *for some index set* $\mathcal{I}$. *Then*

$$\exp\left(\sum_{i \in \mathcal{I}} x_i\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(y_1,\dots,y_k)} \prod_{j=1}^{k} y_j, \tag{1.60}$$

where the second sum is over all k*-tuples* $(y_1, \dots, y_k)$ *such that* $y_j = i_j$ *for some* $i_j \in \mathcal{I}$, $(i_1, \dots, i_k) \in \mathcal{I}^k$.

When applying this formula we take $\mathcal{I}$ as the set of all clusters

$$\mathscr{C} = \left((P_1, \dots, P_m), G\right) \tag{1.61}$$

with weights

$$w_{\mathscr{C}} = \frac{1}{m!} (-1)^{|G|} \prod_i w_{P_i} \tag{1.62}$$

where G is a connected graph. Then (1.59) can be written as

[13] Not excluding the case where several P_i are actually copies of the same polymer.

Proposition 1.3 (Cluster expansion)
Using the above-mentioned new enumeration and Proposition 1.2 we can write (1.59) *as*

$$Z = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(\mathscr{C}_1,\dots,\mathscr{C}_k)} \prod_{j=1}^{k} w_{\mathscr{C}_j} \tag{1.63}$$

$$= \exp\left(\sum_{\mathscr{C}} w_{\mathscr{C}}\right). \tag{1.64}$$

Remark. Another way to understand the transformation from (1.59) to (1.64) is to first introduce a new numbering of polymers in any cluster. Then introduce a new auxiliary enumeration of the copies of the same[14] clusters whenever needed[15]. Then we can rewrite

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(P_1,\dots,P_n)} \prod_i w_{P_i} \sum_{G \subset G^\bullet (P_1,\dots,P_n)} (-1)^{|G|} \tag{1.59 revisited}$$

$$= \prod_{\text{all clusters } \mathscr{C}} \left(1 + w_{\mathscr{C}} + \frac{w_{\mathscr{C}}^2}{2!} + \frac{w_{\mathscr{C}}^3}{3!} + \dots\right) \tag{1.65}$$

$$= \exp\left(\sum_{\mathscr{C}} w_{\mathscr{C}}\right). \tag{1.64 revisited}$$

Let us stress that polymers of any cluster $\mathscr{C}$ are enumerated as $(P_1, \dots, P_n)$ and not only different permutations of this enumeration with the same connected graph G are treated as different clusters but also clusters with the same indices of their polymers may appear in several copies in expression (1.63).

Later we will investigate structure of clusters in the specific situation of hard-core repulsive polymer model.

Remark. Observe that expression (1.63) can be interpreted as a partition function of an ideal gas where all the molecules, in this case the clusters, are mutually compatible.

1.3.1 Multilevel expansion

Cluster expansion is the key technical tool in the analysis of the structure of phases and phase transitions of spin models on a lattice. However, it is often not clear from the very beginning what parts of the model can be effectively expanded. The idea of multilevel expansion is thus useful in e.g. Pirogov–Sinai theory.

Multilevel expansion means that we first expand the partition function of some *restricted* ensemble, where the result of expansion will be more straightforward to work with[16], and then try to proceed with the expansion of more complicated objects and their contributions to the partition function.

In the context of polymer models the idea can be formalized in a following way:

[14] 'The same' here means also identical numbering of polymers.
[15] It is not needed to introduce a new tagging for clusters that are different.
[16] e.g. quickly converging series.

Definition 1.10 (Decomposition of set of polymers)
Decompose the set $\mathbb{P}$ *of all considered polymers as*

$$\mathbb{P} = \mathbb{P}_{large} \cup \mathbb{P}_{small}; \qquad \mathbb{P}_{large} \cap \mathbb{P}_{small} = \emptyset; \tag{1.66}$$

where the polymers from $\mathbb{P}_{small}$ *are the 'easy-to-handle' ones and the former are the complicated objects.*

In previous text, we have already described cluster expansion done at once. A more sequential approach is possible.

Proposition 1.4 (Multilevel cluster expansion)
First, fix the choice $\{\Gamma_i\}$ *of* large *polymers from* $\mathbb{P}_{large}$ *and expand the partition function*

$$Z_{\{\Gamma_i\}}(M) = \left(\prod_i w_{\Gamma_i}\right) Z(M \backslash \{\Gamma_i\}) \tag{1.67}$$

$$= \left(\prod_i w_{\Gamma_i}\right) \exp\left(\sum_\varphi w_\varphi\right) \tag{1.68}$$

$$= \left(\prod_i w_{\Gamma_i}\right) \prod_{\mathscr{C}} \left(1 + w_\varphi + \frac{w_\varphi^2}{2!} + \dots\right) \tag{1.69}$$

where M *is a given volume*[17] *and the sum in final expression is taken over all clusters of polymers from* $\mathbb{P}_{small}$ *satisfying the condition that they 'belong to volume* M*' and are compatible with any* Γ_i*.*

Remark. The expression

$$\sum_{\{\Gamma_i\}} \prod_i w_{\Gamma_i} \prod_{\mathscr{C}} \left(1 + w_{\mathscr{C}} + \frac{w_{\mathscr{C}}^2}{2!} + \dots\right) \tag{1.70}$$

is the partition function of an artificial polymer model containing polymers of two types:

1. polymers Γ_i from $\mathbb{P}_{\text{large}}$
2. 'pseudo-polymers' that are clusters $\mathscr{C}$ of polymers from $\mathbb{P}_{\text{small}}$.

The 'pseudo-polymers' are all mutually compatible[18] and the compatibility of $\Gamma \in \mathbb{P}_{\text{large}}$ and a cluster $\mathscr{C}$ formed from polymers $\{P_j\}$ from $\mathbb{P}_{\text{small}}$ is naturally defined so that Γ is compatible with $\mathscr{C}$ if Γ is compatible with *any* P_j.

Now we can further expand this new artificial polymer model. Following simple observation is useful:

Proposition 1.5 (Equivalence of two approaches to cluster expansion)
The result of 'one-step' and 'multilevel' expansion is the same.

[17] More correctly, it is a subset of $\mathbb{P}_{\text{small}}$
[18] Also with themselves, see Assumption 1.1.

Proof. Consider a decomposition into connected components of any cluster, where Γ_i are the 'large' and Γ_j the 'small' ones, into components arising after removing all the incompatibility bonds between large polymers Γ_i. Also replace the sum $\sum_H (-1)^{|H|}$ over all graphs H connecting a ordered cluster $\widetilde{\mathscr{C}}$ of small polymers to a large polymer Γ by a single term (-1) representing the incompatibility bond between Γ and the cluster $\widetilde{\mathscr{C}}$. The result follows. ■

Remark. More general formulation of this proposition can be obtained in a more abstract setting where the graph $\mathbb{G}^\bullet$ of all incompatibilities between the polymers decomposes as

$$\mathbb{G}^\bullet = G_1 \cup G_2 \tag{1.71}$$

and the expansion in the first step is performed only over $\mathscr{C}_1$, i.e. the incompatible pairs $(\Gamma, \Gamma') \in G_2$ are *not* introduced but later they are represented by terms

$$(1 + c_{\Gamma\ \Gamma'}) \tag{1.72}$$

in the sum

$$\sum_{(\Gamma_1,\dots,\Gamma_n)} \frac{1}{n!} \prod_i w_{\Gamma_i}. \tag{1.73}$$

1.4 Tree-like structures in hard-core repulsive polymer models

In this section we describe our new results[19].

1.4.1 Rooted clusters

Our aim is to investigate the sum

$$w_{\underline{\mathscr{C}}} = \sum_G w_{\mathscr{C}} \tag{1.74}$$

and to show that in the hard-core repulsive polymer model there are significant cancellations to be exploited in order to simplify the structure of partition function of this model.

Let us start with a trivial observation.

Proposition 1.6
Given some incompatibility graph G, notice that

$$w_{\underline{\mathscr{C}},G} = -w_{\underline{\mathscr{C}},\widetilde{G}}, \tag{1.75}$$

if $\widetilde{G} = G \cup \{P_i, P_j\}$ is a connected graph with one more edge than G.

[19] Hopefully original. We are not aware that this or similar construction would appear elsewhere in the literature.

We will show that by careful examination of cancellations arising from this flipping of sign, we can rewrite (1.74) as a sum over tree-like structures on $\mathscr{C}$. In order to achieve this, we will introduce some additional definitions.

Definition 1.11 (Rooted cluster)
Cluster $\mathscr{C} = (\underline{\mathscr{C}}, G)$, $\underline{\mathscr{C}} = (P_1, \dots, P_n)$ with selected polymer $R \in \underline{\mathscr{C}}$ is called rooted cluster *with root R, denoted as*

$$\mathscr{C} = (R, m, (P_1, \dots, P_{n-m}), G), \ m \in \mathbb{N}; \tag{1.76}$$

where m is the multiplicity of root polymer. If no multiplicity is stated, then $m = 1$ by convention and we will call $\mathscr{C}$ a simply rooted *cluster.*

The following elementary result — so-called circular lemma — will play a crucial role in our transformation of cluster expansion sums. First, let us introduce some further notation.

Definition 1.12 (Miscellaneous notation)
On a set $N = \{1, 2, \dots n\}$ denote by $G^{\text{full}}(N)$ the complete graph consisting of all the edges $\{i, j\}$; $i, j \in \mathbb{N}$, $i \neq j$. We will study the sum

$$S = \sum_{G^{\text{conn}} \subset G^{\text{full}}(N)} (-1)^{|G|} \tag{1.77}$$

over all connected graphs on N, i.e. on all connected subgraphs of $G^{\text{full}}(N)$. Here $|G|$ denotes the number of edges in G^{conn}. We will transform S to a sum over cycles on N, i.e. strings of general structure

$$C = (1, \pi(2), \pi(3), \dots, \pi(n)), \tag{1.78}$$

where π is a permutation of $\{2, \dots, n\}$. These circles will be depicted as

$$C = 1 \circledast \pi(2) \circledast \dots \circledast \pi(n). \tag{1.79}$$

Newly defined symbols $\circledast$ will be called bonds, to distinguish them from edges of a graph.

Now let us introduce the circular lemma:

Lemma 1.1 (Circular lemma)

$$\sum_{G^{\text{conn}} \subset G^{\text{full}}(N)} (-1)^{|G|} = (-1)^{n-1} (n-1)! \tag{1.80}$$

> ***Proof***. By induction. There are several other, more elegant proofs. The most standard one is perhaps using the following result on coefficients of power series:

Lemma 1.2
Let $g(x) = \sum_{m=1}^{\infty} \frac{a_m}{m!} x^m$. *Then*

$$\exp\left(g(x)\right) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \tag{1.81}$$

where

$$b_n = \sum_{\substack{\{M_i\}: \bigcup M_i = N \\ N=\{1,2,\ldots,n\}}} \prod_i a_{|M_i|} \tag{1.82}$$

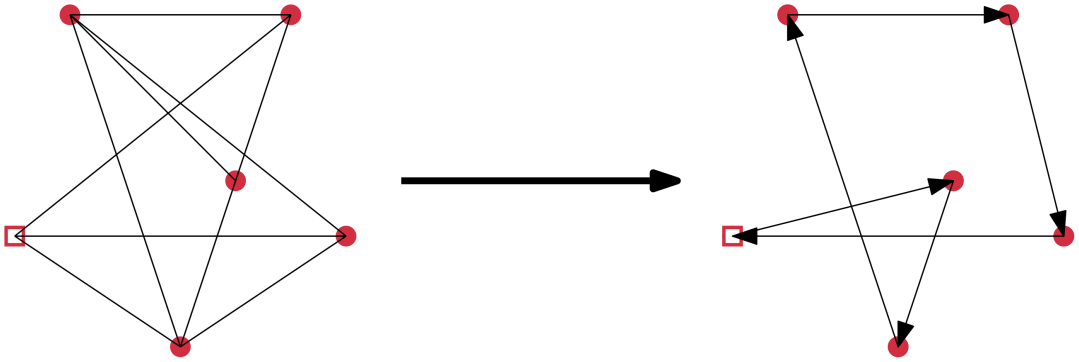
with the sum being taken over all decompositions $N = \bigcup M_i$.

Compare the two examples

$$b_n = (1-1)^{\binom{n}{2}} = \sum_{\text{all graphs on } N=\{1,2,\ldots,n\}} (-1)^{|G|} \tag{1.83}$$

$$b_n = \sum_{\text{all permutations on } N} \operatorname{sgn} \pi \tag{1.84}$$

for $n \geq 2$. The corresponding quantities a_n represent the left- and right-hand sides of the equality in Lemma 1.2. ■



sum over connected graphs sum over cycles

Figure 1.3: Illustration of transformation from connected to circular structure.

1.4.2 Umbels

Let $\mathbb{P}$ be a collection of polymers, with some relation $(P \; i \; Q)$ of mutual incompatibility of polymers from $\mathbb{P}$. Let us define umbels as:

Definition 1.13 (Umbel)
Umbel U *is a pair*

$$U = \left(R, \{H_R^Q; Q \in \mathbb{P}\}\right), \tag{1.85}$$

where $R \in \mathbb{P}$ is the root of umbel U and H_R^Q are the 'handles' of U, indexed by polymers Q incompatible with R. For all $Q \in \mathbb{P}$, $(Q\ i\ R)$ there will be at most one non-trivial handle H_R^Q defined as a string

$$H_R^Q = R \circledast Q \circledast \ldots \circledast Q, \tag{1.86}$$

where $Q \circledast \ldots \circledast Q$ is a string of copies of the same polymer Q connected by bonds $\circledast$.

Let us once again remind the reader about self-incompatibility of polymers in considered models, see Assumption 1.1.

Now we will define umbel-clusters.

Definition 1.14 (Endpoints of an umbel)
Denote by $E(U)$ the collection of all polymer positions (and polymers Q attached to them) forming the handles of cluster U. Elements of $E(U)$ will be called endpoints.

> **Remark.** Note that polymers in each handle are copies of the very same polymer, but they have different positions.

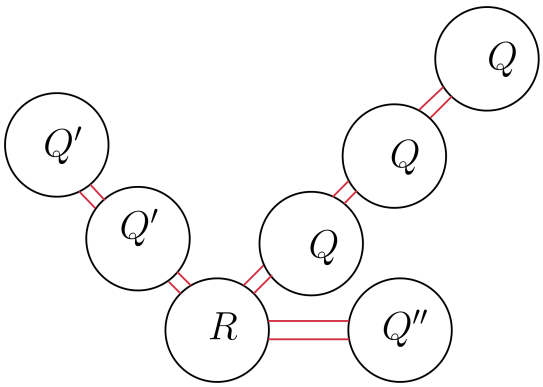


Figure 1.4: Illustration of an umbel with root R and 6 endpoints.

Definition 1.15 (Umbel-cluster)
Umbel-cluster $U_{\mathscr{C}}$ is defined as

$$U_{\mathscr{C}} = \{U, \{\mathscr{C}_P\}_{P \in \mathbb{P}(U)}\}, \tag{1.87}$$

where U is an umbel and $\mathscr{C}_P$ is a cluster simply rooted in P.

We will require that the umbel-clusters we use have the property of self-repulsion, defined as follows:

Definition 1.16 (Self-repulsive umbel-cluster)
Assume there exists a total order $\prec$ of polymers $Q \in \mathbb{P}, Q\ i\ R$, root R fixed. We say that umbel-cluster $U_{\mathscr{C}}$ is self-repulsive *if for each cluster $\mathscr{C}_P$ polymer $P \in E(U)$ is the first polymer w.r.t. $\prec$ from the set of all polymers of $\mathscr{C}_P$ incompatible with R.*

Remark. *Self-repulsive* means that the polymers that are forming the clusters $\mathscr{C}_P$ not only avoid R (and P — except for the root of $\mathscr{C}_P$) but also all polymers of $\mathbb{P}$ incompatible with R and preceding P in the relation $\prec$.

The choice of $\prec$ brings some arbitrariness into the way we analyse the structure of clusters contributing to the cluster expansion. However, there are models where the particular choice of $\prec$ is of (very) minor importance. For example we can choose $\prec'$ to be a *partial* ordering given by the distance of the centre of polymer $Q, Q\ i\ P$; to the centre of polymer P and then extend it to a total ordering $\prec$.

For models defined in continuum note that the event that two polymers Q, Q' incompatible with P have the same distance to P happens almost never.

The key step in our transformation procedure is the following lemma:

Lemma 1.3 (Main lemma)
Let $R \in \mathbb{P}$ and let $\prec$ be a total ordering on the set $\{P \in \mathbb{P}, (P\ i\ R)\}$. Then the sum

$$\sum_{\mathscr{C}_R \in \mathbb{C}_R(\mathbb{P})} w_{\mathscr{C}_R}, \tag{1.88}$$

taken over all clusters formed from polymers in family $\mathbb{P}$ with R being a simple root, can be expressed as

$$\sum_{\mathscr{C}_R \in \mathbb{C}_R(\mathbb{P})} w_{\mathscr{C}_R} = \sum_{U_R \in \mathbb{U}_R(\mathbb{P})} w_{U_R}, \tag{1.89}$$

where the sum on the r.h.s. is taken over all umbel-clusters

$$U_R = (U, \{\mathscr{C}_E, E \in E(U)\}) \tag{1.90}$$

satisfying the relation $\prec$.

Remark. The fact that umbel-clusters satisfy relation $\prec$ essentially means that E is the least element (w.r.t. $\prec$) chosen from $\mathscr{C}_E$ that is incompatible with R.

Proof. *(of Lemma 1.3).* Let

$$\mathscr{C}_R = (R, P_1, \ldots, P_n, G) \tag{1.91}$$

be a cluster appearing on the l.h.s. of (1.89). Decompose the interaction graph G of $\mathscr{C}_R$ in the following way:

- First, temporarily erase all the edges of the type (R, P_i) from G. Denote by $G^\star$ the remainder of G.
- The system $(P_1, \ldots, P_n, G^\star)$ naturally decomposes into connected components. Take any such component, denote it by $\mathscr{C}'$, and look for the last (w.r.t. $\prec$) element of its support which is incompatible with R. Denote this polymer as P'.

- Then erase all the edges (P', P') in $\mathscr{C}'$ (if there are any). Then the remainder of $\mathscr{C}'$ further decomposes into smaller components — denote them as $\mathscr{C}''$. Notice that each $\mathscr{C}''$ must contain P'. Also observe that each $\mathscr{C}''$ can be either simply or multiply rooted in P'. The latter case will be called self-subtracting. Later we will show that contributions of all clusters containing such a multiply rooted $\mathscr{C}''$ have zero sum.
- Now let us focus on polymers $P^\star$ such that

$$P^\star \in \mathscr{C}'; \qquad P^\star \; i \; R; \qquad P^\star \prec P'. \tag{1.92}$$

  We claim that situations where these polymers appear are of no importance. More precisely, these are the 'self-subtracting' situations whose total contribution to the sum over clusters is zero. By fixing P', the highest polymer incompatible with R, the bonds of between R and $P^\star$ become 'dummy' and a term $(1-1)^k$, k being the number of $P^\star$-type polymers, can be extracted in front of the sum over these situations.
- There can be several components $\mathscr{C}''$ containing the polymer P'. Write them as $(\mathscr{C}''_1, \ldots, \mathscr{C}''_m)$ and restore the edges (P', P') previously removed from $\mathscr{C}''$. Do this for all components $\mathscr{C}''$ that are simply rooted in P'.

The incompatibility graph on the ordered collection of polymers $(R, P^{1\prime}, P^{2\prime}, \ldots, P^{m\prime})$ is complete[20]. Below (see Figure 1.5) are shown two equivalent ways to depict this.

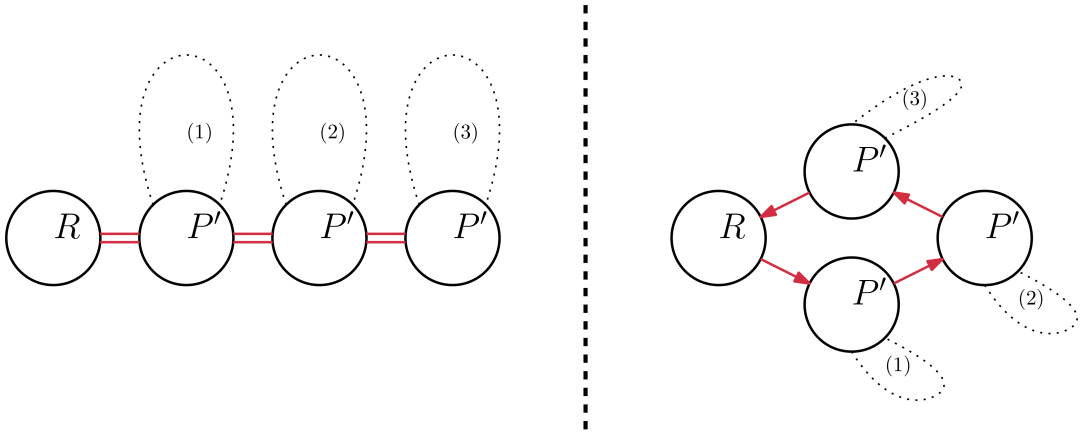


Figure 1.5: Linear and circular rearrangement of a simply rooted cluster rooted in R.

We replace the sum over (non-self-subtracting) clusters by sum over umbel-clusters $U_R = (U_R, \mathscr{C}''_E)$ where $\mathscr{C}''_E$ are the above-mentioned components, indexed by the end points of umbels of the type (see Figure 1.5, left part) where P' are the roots of components $\mathscr{C}'$ mentioned in the first step.

[20] These are copies of the same polymer, and all polymer are incompatible with themselves, viz Assumption 1.1.

In the self-subtracting case, notice that these clusters have 'dummy' bonds of the type (P', P') inside the component U'' described above. As mentioned above, we can bracket the sum over such $w_{\mathscr{C}}$ in a way that we first fix the edges in the component $\mathscr{C}''$ containing more than one polymer P' and then the summation over 'dummy' variables with $i(P', P') = -1$ gives us the term $(1-1)^k$, where k is the number of copies of P' in such a cluster U''. ■

Corollary 1.1
By iterating the expression (1.89) *from Lemma 1.3 we arrive to a structure called* umbel tree*, see definition below.*

Definition 1.17 (Self-avoiding umbel tree)
Self-avoiding umbel tree is a tree recursively defined by the following requirements

1. *It is simply rooted in some polymer P.*
2. *It has a form*

$$\mathcal{U} = (R, U, \{\mathcal{U}_P, P \in E(U)\}), \tag{1.93}$$

where R is the root of an umbel U and $\mathcal{U}_P$ are umbel trees attached to the endpoints of U. The umbel trees $\mathcal{U}_P$ satisfy the property of self-avoidance, meaning that among all the polymers of $\mathcal{U}_P$ incompatible with R the polymer P is the first one w.r.t. ordering $\prec$.

Theorem 1.1 (Main theorem)
The cluster expansion sum (1.88) *can be rewritten as*

$$\sum_{\mathscr{C}_R \in \mathbb{C}_R(\mathbb{P})} w_{\mathscr{C}_R} = \sum_{\mathcal{U}_R \in \mathbb{U}_R} w_{\mathcal{U}_R}, \tag{1.94}$$

where $\mathbb{U}_R$ is the family of all umbel trees simply rooted in R.

Proof. By induction over the size of considered objects. The summation over clusters will be replaced by summation over umbel-clusters, then by summation over umbel-umbel-clusters etc. ■

To generalise Theorem (1.1) to sums over multiply rooted clusters we need another notion:

Definition 1.18 (Circular umbel)
A circular umbel is a circularly ordered collection of umbels $\{U_R^i, i \in C\} \equiv U^\circ$ where $\circ$ is a cycle $(1, \dots, m)$ and all umbels U_R^i have the same root R. The integer m is called multiplicity of circular umbel U°.

Definition 1.19 (Circular umbel tree)
Using the concept of circular umbel we define circular umbel tree $\mathcal{T}^\circ$ so that the structure of branches of $\mathcal{T}^\circ$ is the same as for simply rooted umbel trees but the root of such an object is a cycle, see Figure 1.6.

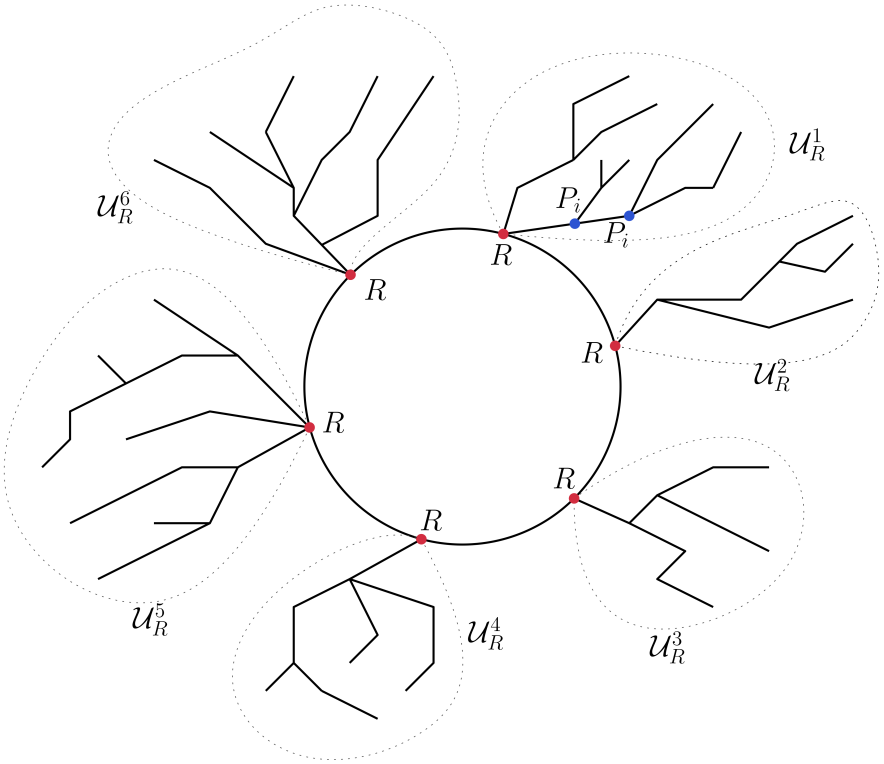


Figure 1.6: Illustration of a circular umbel tree. Only root polymers R and one pair of polymers P_i in the P_i-th handle of the umbel tree $\mathcal{U}_R^1$ is shown.

Now it is easy to extend the statement of Theorem 1.1 also to sums over clusters multiply rooted in some R.

Theorem 1.2
Denote by $\mathbb{C}_R^{full}(\mathbb{P})$ the collection of all clusters of polymers from $\mathbb{P}$ that contain at least one polymer R.

Then

$$\sum_{\mathscr{C}\in\mathbb{C}_R^{full}(\mathbb{P})} w_{\mathscr{C}} = \sum_{U_R^\circ\in\mathbb{U}_R^\circ(\mathbb{P})} w_{U_R^\circ} \tag{1.95}$$

where the sum on the r.h.s. is over all circular umbel trees.

Proof. Similar to the proof of Lemma 1.3 — the difference is only in the first step: remove all the bonds of a cluster $\mathscr{C}$ containing the polymer R, in particular remove all the bonds of the type (R, R). If some of the arising components $\mathscr{C}_i$ of $\mathscr{C}$ is multiply rooted then there are 'dummy' variables i_{RR} not affecting the definition of $\{\mathscr{C}_i\}$. By suitable bracketing of the sum there will appear a term $(1-1)^k$, where k is the multiplicity of R in the above-mentioned $\mathscr{C}_i$.
So after removing all the bonds containing R from $\mathscr{C}$ we may and will assume that all the components $\mathscr{C}_i$ are *simply* rooted in R. Using Lemma 1.1 we obtain the result. ■

Remark. In what follows the word 'umbel' will be used to denote a circular umbel and 'umbel tree' will mean a circular umbel tree. If

we need to precisely differentiate between different root multiplicities, we will use adjectives 'simply/multiply rooted'.

1.4.3 Kotecký–Preiss-type estimates

Definition 1.20
Let

$$v_P = \sum_{\substack{\mathcal{U}\ni P \\ \text{simply rooted}}} w_{\mathcal{U}} \tag{1.96}$$

$$V_P = \sum_{\mathcal{U}\ni P} w_{\mathcal{U}}, \tag{1.97}$$

where the first sum is over umbel trees simply rooted in P and the second sum is over all umbel trees rooted in P. Notation $\mathcal{U}\ni P$ means that $\mathcal{U}$ is rooted in P.

Remark. The relationship between these two quantities is

$$V_P = v_P - \frac{v_P^2}{2} + \frac{v_P^3}{3} - (\ldots) = \log(1+v_P). \tag{1.98}$$

by construction.

Introduce the following quantities subjected to conditions

Definition 1.21
Define

$$v_P^{(R)} = \sum_{\substack{\mathcal{U}\ni P \\ \text{simply rooted}}}^{(R)} w_{\mathcal{U}} \tag{1.99}$$

where the condition (R) means that if

$$P' \in \operatorname{supp}\mathcal{U}; \qquad P \ i \ R \tag{1.100}$$

then $P \prec P'$.
Analogously define

$$V_P^{(R)} = \sum_{\mathcal{U}\ni P}^{(R)} w_{\mathcal{U}}. \tag{1.101}$$

Theorem 1.3

$$\begin{aligned} v_P &= w_P \prod_{Q \ i \ P} \left(1 - v_Q^{(P)} + (v_Q^{(P)})^2 - (v_Q^{(P)})^3 + (\ldots)\right) \\ &= w_P \prod_{Q \ i \ P} \frac{1}{1+v_Q^{(P)}} \end{aligned} \tag{1.102}$$

where the superscript $^{(P)}$ means that the trees $\mathcal{T}_Q$ used in the definition of $v_Q^{(P)}$ satisfy also the condition that if $\widetilde{Q} \in \operatorname{supp}\mathcal{T}_Q$ is incompatible with P then we have $Q \prec \widetilde{Q}$.

Proof. Use the formula for the sum of geometric series. ■

Let us also define the following auxiliary quantities, which enable us to write identities and inequalities resembling the usual form of Kotecký–Preiss inequalities.

Definition 1.22
Put

$$A_P = \sum_{\mathscr{C} \ni P} w_{\mathscr{C}} = \sum_{Q \ i \ P} V_Q^{(P)} \tag{1.103}$$

and a slightly less transparently defined

$$a_P = \sum_{Q \ i \ P} v_Q^{(P)} = \sum_{\mathscr{C} \ni P}^{\star} w_{\mathscr{C}}, \tag{1.104}$$

where the condition $\star$ means a restriction to only those $\mathscr{C}$ satisfying the property that if Q is the smallest polymer (w.r.t. $\prec$) of $\mathscr{C}$ incompatible with P then $\mathscr{C}$ contains only one such polymer Q (i.e. is simply rooted in Q).

Using these, eq. (1.102) can be rewritten as

$$\begin{aligned} v_P &= w_P \prod_{Q \ i \ P} \frac{1}{1 + v_Q^{(P)}} \\ &= w_P \exp\left(- \sum_{Q \ i \ P} \log(1 + v_Q^{(P)}) \right) \\ &= w_P \exp\left(- \sum_{Q \ i \ P} V_Q^{(P)} \right). \end{aligned} \tag{1.105}$$

Now the quantities above are summed over all $(P \ i \ M)$, where M is a fixed polymer. Writing the above-mentioned auxiliary quantities as

$$a_M = \sum_{P \ i \ M} v_P^{(M)} \tag{1.106}$$

$$A_M = \sum_{P \ i \ M} V_P^{(M)} \tag{1.107}$$

we can rewrite eq. (1.105) into the form

$$v_P = w_P \exp\left(-A_P^{(P)} \right) \tag{1.108}$$

where $A_P^{(P)}$ is defined as

$$A_P^{(P)} = \sum_{Q \ i \ P} V_Q^{(P)} \tag{1.109}$$

$$V_Q^{(P)} = \sum_{\mathcal{U} \ni Q} w_{\mathcal{U}} = \sum_{\substack{\mathcal{U} \ni Q \\ \text{simply rooted}}} \log(1 + w_{\mathcal{U}}) \tag{1.110}$$

where the sum in $V_Q^{(P)}$ runs over those umbel trees $\mathcal{U}$ rooted in Q that can be attached to P such that the whole graph $(P \cup \mathcal{U})$ is a correctly defined umbel tree simply rooted in P.

Finally, we have

Corollary 1.2

$$\sum_{P\ i\ M} v_P = \sum_{P\ i\ M} w_P \exp(-A_P^{(P)}) \tag{1.111}$$

$$a_M = \sum_{P\ i\ M} w_P \exp(-A_P^{(P)}) \tag{1.112}$$

Corollary 1.3

$$A_M = \sum_{P\ i\ M} \log\left(1 + w_P \exp(-A_P^{(P)})\right) \tag{1.113}$$

Remark. Writing the last equation approximately as

$$A_M \approx \sum_{P\ i\ M} w_P \exp(-A_P^{(P)}) \tag{1.114}$$

we obtain an equation in form similar to the usual inequality of Kotecký and Preiss.

Remark. For *negative* weights w_P, all the cluster expansion terms (when rewritten in terms of umbel trees) share the same *negative* sign. This makes the discussion of convergence very transparent.

However, we are usually more interested in polymer models with *positive* values of w_P and then the cluster expansion sum over umbel trees has a form

$$v_P = \sum_{n=1}^{\infty} (-1)^{(n-1)} \alpha_n, \tag{1.115}$$

where $\alpha_n = \sum w_{\mathcal{U}}$, with the sum now being taken over all umbel trees $\mathcal{U}$ having n vertices.

The numerical study of the quantities α_n is an interesting problem; one expects approximate relations $\alpha_n \approx C^n$ for some model-dependent constants.

Let us also provide relations analogous to (1.102) and (1.112) for the sums of absolute values

$$\widetilde{v}_P = \sum_{\mathcal{U} \text{ simply rooted in } P} |w_{\mathcal{U}}| \tag{1.116}$$

$$\widetilde{V}_P = \sum_{\mathcal{U} \text{ rooted in } P} |w_{\mathcal{U}}| \tag{1.117}$$

and analogously for $\widetilde{v}_P^{(R)}$ and $\widetilde{V}_P^{(R)}$.

Define also

$$\widetilde{a}_P = \sum_{\mathcal{U} \text{ simply rooted in some } Q\ i\ P}^{\star} |w_{\mathcal{U}}| \tag{1.118}$$

$$\widetilde{A}_P = \sum_{\mathcal{U} \text{ rooted in some } Q\ i\ P}^{\star} |w_{\mathcal{U}}|, \tag{1.119}$$

where the sums marked by $\star$ are only over those $\mathcal{U}$ that satisfy the condition from Definition 1.22.

Then we can write

$$\widetilde{v}_P = |w_P| \prod_{Q\ i\ P} \frac{1}{1 - v_Q^{(P)}} \tag{1.120}$$

and from the relation (analogous to eq. (1.98))

$$\widetilde{V}_P = \widetilde{v}_p + \frac{\widetilde{v}_P^2}{2} + \ldots = -\log(1 - \widetilde{v}_P) \tag{1.121}$$

we get

$$\widetilde{v}_P = |w_P| \exp\left(- \sum_{Q\ i\ P} \log(1 - \widetilde{v}_Q^{(P)}) \right) \tag{1.122}$$

$$= |w_P| \exp\left(\sum_{Q\ i\ P} \widetilde{V}_Q^{(P)} \right). \tag{1.123}$$

We get the relations

$$\begin{aligned} \widetilde{a}_M &= \sum_{P\ i\ M} |w_P| \exp(\widetilde{A}_P^{(P)}) \\ \widetilde{A}_M &= \sum_{P\ i\ M} -\log\left(1 - |w_P| \exp(\widetilde{A}_P^{(P)})\right) \end{aligned} \tag{1.124}$$

Now we could be look at (1.124) as a chain of integral equations for the quantities $\widetilde{A}_M$. We want to show the existence of a solution for such a chain of equations. This can be done using the bounds of the type

$$\sum_{P\ i\ M} |w_P| \exp(B_P) < B_M. \tag{1.125}$$

The equation (1.124) can be written in a more general way as

$$\widetilde{A}_M^{\mathscr{R}} = - \sum_{P\ i\ M} \log(1 - |w_P| \exp(\widetilde{A}_P^{(\mathscr{R}\&P)})) \tag{1.126}$$

where $\mathscr{R}$ is some additional condition for umbel trees $\mathcal{U}$ entering the sums in definitions above. Typically if $\mathscr{R}$ is not empty it is a condition that our umbel trees are in fact branches of some bigger umbel tree whose 'root component' was already explored.

Proposition 1.7
Let B_M be a quantity such that

$$B_M > \sum_{P\ i\ M} -\log\left(1 - |w_P| \exp(B_P)\right) \tag{1.127}$$

Then eq. (1.126) *has a solution such that*

$$|\widetilde{A}_M^{(\mathscr{R})}| \leq B_M \tag{1.128}$$

for any restriction $\mathscr{R}$.

In practice we start the iterative procedure by putting

$$\widetilde{A}_M^{(\mathscr{R})}[0] = \sum_{P \ i \ M} |w_P| \tag{1.129}$$

Insert this into (1.126) to compute

$$\widetilde{A}_M^{(\mathscr{R})}[1] = \sum_{P \ i \ M} -\log(1 - |w_P| \exp(\widetilde{A}_{P[0]}^{\mathscr{R}\&P})) \tag{1.130}$$

and iterate this process until done. By condition (1.127), the sequence of $\{\widetilde{A}\}_k$ monotonously converges to some limiting value satisfying (1.126).

1.4.4 Umbel-cluster view of non-ideal gas model

Let us return to the *non-ideal gas* model on some lattice Λ. This is our main example and in future we will be numerically testing our umbel-tree based resummation method on it. Possible choices of Λ are e.g.

- torus $\{0, 1, \ldots, n-1\}^d$
- other lattices with periodic boundary conditions, for example triangular or hexagonal lattices.
- random lattices formed by Poisson process on a torus $\Lambda = [0,1]^d$[21] etc.

It is straightforward to define this model also in continuum with appropriate changes in formulations.

Let us remind the reader that polymers of this non-ideal gas model are *points* of Λ and their *in*compatibility is given by requirement of the type

$$P \ i \ \widetilde{P} \Leftrightarrow \mathrm{dist}(P, \widetilde{P}) \leq 1. \tag{1.131}$$

Function $\mathrm{dist}(\cdot,\cdot)$ is given by some metric, which can be chosen as Euclidean, l_1, l_∞ or other. In the low-temperature regime the precise behaviour of the model depends also on the structure of Λ and the chosen metric. However, here we study the high-temperature (small polymer weights) regime where the behaviour is roughly the same for all kinds of lattices Λ.

Umbel trees of a non-ideal gas model are rooted[22] self-avoiding trees $\mathcal{T}$ with some additional properties described below. For the sake of simplicity of notation, we will consider only the trees $\mathcal{T}$ simply rooted in the origin $\mathcal{O} \in \Lambda$, where Λ is a chosen lattice.

Any umbel tree $\mathcal{T}$ rooted in $\mathcal{O}$ can be described as a collection of its branches

$$(\mathcal{O} \equiv x_0, x_1, \ldots, x_k) \tag{1.132}$$

[21] Quotient group $\mathbb{R}^d/\mathbb{Z}^d$.
[22] Either simply or multiply.

where $|x_i - x_{i+1}| \leq 1, \forall i \in [0, k-1]$. A branch is therefore a non-backtracking walk from the root to some chosen leaf[23].

Self-avoidance is most conveniently imagined in the case when Λ is continuum and distance is given by e.g. Euclidean metric. In this case self-avoidance means that for any i all the polymers $x_{i+1}, x_{i+2}, \ldots, x_k$ have their distance from x_{i-1} not smaller than $|x_i - x_{i-1}|$. The case $x_i = x_{i+1}$ is possible but it happens almost never and so we may ignore it. For illustration see Figure 1.7.

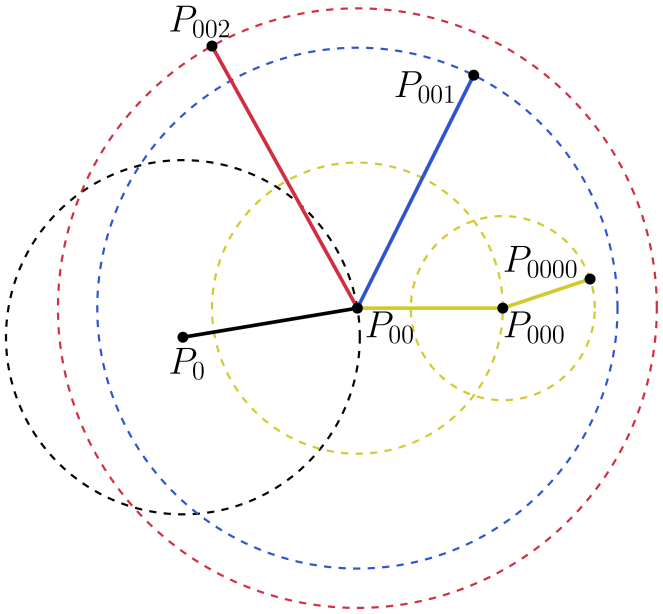


Figure 1.7: Illustration of non-ideal gas model in continuum. Labels of polymers describe the history of branching. Circles are regions into which corresponding branch may never return.

For discrete lattices Λ the ordering of polymers incompatible with given x_{i-1} based on their distance to x_{i-1} is *not* a total ordering so we extend it by introducing some auxiliary local ordering $\prec$. For any point of $\mathbb{Z}^2$, let us define an ordering of its neighbours as seen on Figure 1.8.

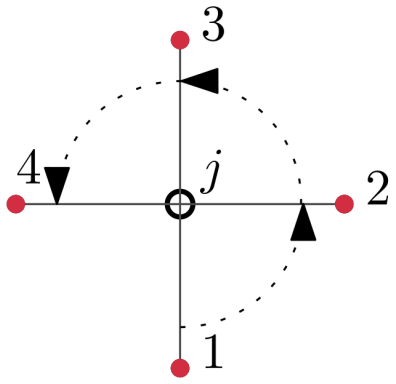


Figure 1.8: Example of local ordering in $\mathbb{Z}^2$.

[23] Let $\mathcal{S}$ be a rooted tree. Any vertex x_k of $\mathcal{S}$ with degree 1 that is not a root $\mathcal{O}$ is called a leaf.

In this particular case, condition of self-avoidance of a branch $(x_0, \ldots, x_k)$ would be that for any $j < l < k$ not only $x_l \neq x_j$ but also that x_l must not have the value preceding x_j among the collection of all the nearest neighbours of x_{j-1}.

For the case of $\mathbb{Z}^2$ an alternate way to describe the branches of $\mathcal{T}$ is

$$(d_1, d_2, \ldots, d_k) \tag{1.133}$$

where d_i are differences $(x_{i+1} - x_i)$ between positions $(x_0, x_1, \ldots, x_k)$. We may imagine d_i being assembled from an alphabet of six symbols (up, down, right, left, stay, empty).

The vertices of tree $\mathcal{T}$ can be uniquely characterized by segments $(x_0, \ldots, x_l)$, $l < k$; of the full branch $(x_0, \ldots, x_k)$ that holds the leaf x_k. It is important not only to specify the leaves and branches, but also all of the vertices of $\mathcal{T}$ that are *not* 'fully occupied' in the sense that one could stick another leaf in them in a self-avoiding way — thus obtaining a larger self-avoiding tree.

Denote by $\mathcal{A}(\mathcal{T})$ the collection of all *active* positions of $\mathcal{T}$ formed by nearest neighbour pairs (x_k, x) where x is a active vertex of $\mathcal{T}$, i.e. the endpoint of corresponding branch $(x_0, \ldots, x_k)$.

Consider the collection of all *active* vertices of $\mathcal{T}$, i.e. the collection of all subsegments $v = (x_0, \ldots, x_l)$ of some branch $(x_0, \ldots, x_k)$ of $\mathcal{T}$ such that a new leaf can be added for which $|y - x_l| = 1$ and $(x_0, \ldots, x_l, y)$ is a new self-avoiding branch of a new tree extending $\mathcal{T}$. Let us call the pair (v, y) an *active position* of $\mathcal{T}$ and denote by $\mathcal{A}(\mathcal{T})$ the collection of all active positions of $\mathcal{T}$ and by $a(\mathcal{T}) = |\mathcal{A}(\mathcal{T})|$ its cardinality. Analogously denote by $\mathcal{E}(\mathcal{T})$ the collection of all leaves and by $e(\mathcal{T})$ its cardinality. We are interested in the behaviour of quantities $a(\mathcal{T}), e(\mathcal{T})$ and particularly in their ratio $\frac{a(\mathcal{T})}{e(\mathcal{T})}$.

The statistical ensemble we want to study is the collection of all self-avoiding trees $\mathcal{T}$ rooted in $\mathcal{O}$ with the weight $w_{\mathcal{T}} = q^{|\mathcal{V}(\mathcal{T})|}$. The partition function of this ensemble is closely related to the free energy h of the non-ideal gas model via relation

$$h = \log(1 + f) \tag{1.134}$$

where

$$f = \sum_{N=1}^{\infty} (-q)^N S(N) \tag{1.135}$$

where $S(N)$ is the number of self-avoiding trees having exactly N vertices.

Dynamics of self-avoiding trees

Let us provide a suggestion for an algorithm to generate self-avoiding trees from an ensemble of all trees $\mathbb{T}$. To fix notation, let $\mathcal{T}$ be a tree. Sets $\mathcal{V}(\mathcal{T}), \mathcal{L}(\mathcal{T})$ and $\mathcal{A}(\mathcal{T})$ are the sets of vertices, leaves and active positions of $\mathcal{T}$.

Definition 1.23 (Weight of a tree)
Let $\mathcal{T} \in \mathbb{T}$. Define weight of $\mathcal{T}$ as

$$w_{\mathcal{T}} = q^{|\mathcal{V}(\mathcal{T})|}. \tag{1.136}$$

Parameter q can be interpreted as temperature.

Proposition 1.8 (Dynamics of self-avoiding trees)

1. *In all $a \in \mathcal{A}(\mathcal{T}), l \in \mathcal{L}(\mathcal{T})$ install 'Poisson clocks', i.e. let there be a Poisson process happening in any of these positions or vertices. Let the speed of 'clocks' in active vertices be q-times that of those in leaves.*

2. *When $\mathcal{A}$-clock ticks, add a vertex to some its neighbours determined by the given clock. On the other hand when $\mathcal{L}$-clock ticks, remove the corresponding leaf.*

This dynamic is of the 'detailed balance' kind with equilibrium distribution given by

$$P_q(\mathcal{T}) = \frac{1}{Z_q} q^{|\mathcal{V}(\mathcal{T})|} \tag{1.137}$$

where

$$Z_q = \sum_{\mathcal{T}} q^{|\mathcal{V}(\mathcal{T})|} \tag{1.138}$$

with the sum being taken over all $\mathcal{T}$ rooted in $\mathcal{O}$.

The mean values

$$a_q = \langle a(\mathcal{T}) \rangle_{P_q} \tag{1.139}$$
$$e_q = \langle e(\mathcal{T}) \rangle_{P_q} \tag{1.140}$$

in this ensemble satisfy the relation

$$q \cdot a_q = e_q. \tag{1.141}$$

Having this dynamic, look for the radius of convergence R of the cluster expansion series

$$\sum_{\mathcal{T}} q^{|\mathcal{V}(\mathcal{T})|}. \tag{1.142}$$

Conjecture 1
For $q \nearrow R$ the quantities a_q and e_q go to infinity and the structure of a typical tree $\mathcal{T}$ in the ensemble given by P_q attains some universality properties. This will be a subject of future numerical research.

2 | Introduction to Pirogov–Sinai theory

2.1 Introduction

Pirogov–Sinai theory first appeared in 1975 in the article [Pirogov and Sinai, 1975]. It was inspired (at least partially) by the previous work [Minlos and Sinai, 1967] on the Ising model. It was later simplified in the article [Zahradník, 1984]. The Pirogov–Sinai theory is based on an idea of a *contour functional* $F(\Gamma)$ and it is typically applied to models without any symmetries of its Hamiltonian. Generally it is applicable to a wide class of Ising-like lattice models, usually with short-range interactions, in their low-temperature regime. Considered models must have Hamiltonians that have a reasonable structure of ground states[24] in the unperturbed case, or a slightly perturbed case, and also there must be a significant energy barrier between these ground states. The idea of energy barriers between local 'almost-ground' states is formalized by the definition of contours and in the Peierls condition for them.

Another feature of the Pirogov–Sinai theory is the possibility to rigorously estimate the 'surface tension' terms[25] in the expressions of $\log \mathcal{Z}^q$ for so-called diluted partition functions $\mathcal{Z}^q$.

The cluster expansion for polymer models is the usual technical tool-of-choice used to establish control of 'surface tension'. There were attempts to use other methods like Dobrushin uniqueness condition[26] but in our opinion the method of cluster expansions is far more powerful and flexible for these purposes.

An application of Pirogov–Sinai theory consists of two steps:

1. The first step is the preparation of the studied model. We need to find a way to determine its *local ground states* (i.e. ground states of either the original unperturbed Hamiltonian or a Hamiltonian close to it). Suitable definition of contours is the next step and this can be a rather straightforward task as in the Ising model or quite a refined construction as in some other models. This is the truly creative part of application of Pirogov–Sinai theory.

2. Then comes the representation of configurations as *compatible collections*

[24] e.g. constant configurations

[25] more precisely, two-sided estimates for the stable configurations and bounds from above for the general case

[26] e.g. [Lebowitz et al., 1999]

of contours, together with the re-expression of the Hamiltonian in terms of energy densities e_q of the local ground states and energy barriers $E(\Gamma)$.

The first step is model dependent and there is no general way of doing it, but after finding the local ground states and after proper definition of contours and their energies the application of Pirogov–Sinai theory is more or less universal for all situations. We will concentrate here on this second step, on the essentials of the Pirogov–Sinai machinery.

Introductions to Pirogov–Sinai theory are quite rare in the literature. One could take a look at the book [Sinai, 1982] or lecture notes [Zahradník, 1998].

In recent articles one might find Pirogov–Sinai theory applied to Dyson models, see [Bissacot et al., 2018], or to problems on the interface of computer science, combinatorics and statistical physics, see [Helmuth et al., 2019].

There are further important applications that are beyond the scope of this thesis. Interested reader can take a look e.g. at the following articles on continuous-spin models [Dobrushin and Zahradník, 1986; Zahradník, 2000].

We will present a rather general approach having in mind the two basic examples, namely the standard low-temperature ferromagnetic Ising model and 'thick' contours of the original Pirogov–Sinai approach.

2.2 Basic objects of Pirogov–Sinai theory

In the following section suppose that we have a model (with finite-range interactions) whose states are described by $S^{\mathbb{Z}^d}$ with $d \geq 1$, where S is the set of admissible configurations of a single spin. As mentioned above, the application of Pirogov–Sinai theory starts with identification of structure of the ground states of our model. In the 'abstract Pirogov–Sinai' we assume that we have already obtained a fixed finite set of colours $Q = \{q_i\}$. This set of colours is often chosen as a subset of S and is used to denote the local ground states of our model. We assume for simplicity that these ground states are constant configurations of spins[27].

When talking about local ground states, we mean ground states in some not too large volume. More precisely

Definition 2.1 (Local ground state)
Local ground state is a constant configuration that is an infinite-volume minimiser of Hamiltonian H with a suitable small *perturbation.*

> **Remark.** As stated above, we assume that we have found *all* the local ground states of our considered model and that configurations different from these local ground states have significantly more energy. This is later formalised by the definition of contours and the Peierls condition, see subsection 2.2.4

[27] It is possible to generalise Pirogov–Sinai theory to encompass also models with periodic or quasi-periodic ground states.

Should we miss a local ground state and operate with an incomplete set of them we will usually see that we are unable to establish the validity of Peierls condition.

A local ground state is often called 'colour'.

2.2.1 Contours and their compatible systems

Everywhere in this chapter we will be working with l_1 metric on some torus Λ. For convenience we will define the following term

Definition 2.2 (Infinity on a finite torus)
Choose a point or a simply connected subset $\Lambda_\infty(\in / \subset)\Lambda$. *This point will be called infinity on a finite torus.*

Lemma 2.1 (Topological lemma)
If S *is a connected subset of* Λ *and* $\Lambda\backslash S$ *is decomposed into simply connected components. The component containing* Λ_∞ *is called* $\mathrm{ext}(S)$ *and the union of remaining components is called* $\mathrm{int}(S)$.

Definition 2.3 (Contour)
Contour is an object for which we define its support $\operatorname{supp}\Gamma$, *which is a connected set in lattice* Λ[28]. *Recalling Lemma 2.1, we define interior* $\mathrm{int}(\Gamma)$ *and exterior* $\mathrm{ext}(\Gamma)$. *Part of the definition is also a colour prescription for every component of* $\operatorname{int}\Gamma$ *and for* $\operatorname{ext}\Gamma$.

> **Remark.** Sometimes it is useful to consider Λ_∞ as a special kind of contour and to assign it some colour q.

In what follows, we will work with *compatible* systems of contours. We stress that this notion will *not* be defined merely by testing the pairs of contours from this system.

Now we will need some new definitions:

Definition 2.4 (Communicating points/contours)
Say that two points $s, t \in \Lambda$ *communicate in the contour system* $\{\Gamma_i\}$ *if there is a connected path from* s *to* t *avoiding the supports* $\operatorname{supp}\Gamma_i$ *of all the contours of the system.*

Say that two contours Γ, Γ' *from a system of contours* $\{\Gamma\}^q$ *communicate if there is a connected path* $t_1, t_2, \ldots, t_N$ *such that* $t_1 \in \operatorname{supp}\Gamma$, $t_N \in \operatorname{supp}\Gamma'$ *and such that the points* t_2 *and* t_{N-1} *communicate (e.g. through the above mentioned path) in the sense described above.*

Say that a point $t \in \Lambda$ *communicates with* Γ *if there is a connected path* $t, t_2, \ldots, t_N$ *such that* $t_N \in \operatorname{supp}\Gamma$ *and points* t *and* t_{N-1} *communicate in the sense mentioned above.*

[28] Or its dual $\Lambda^\star$. In that case we would need some modification of our definitions.

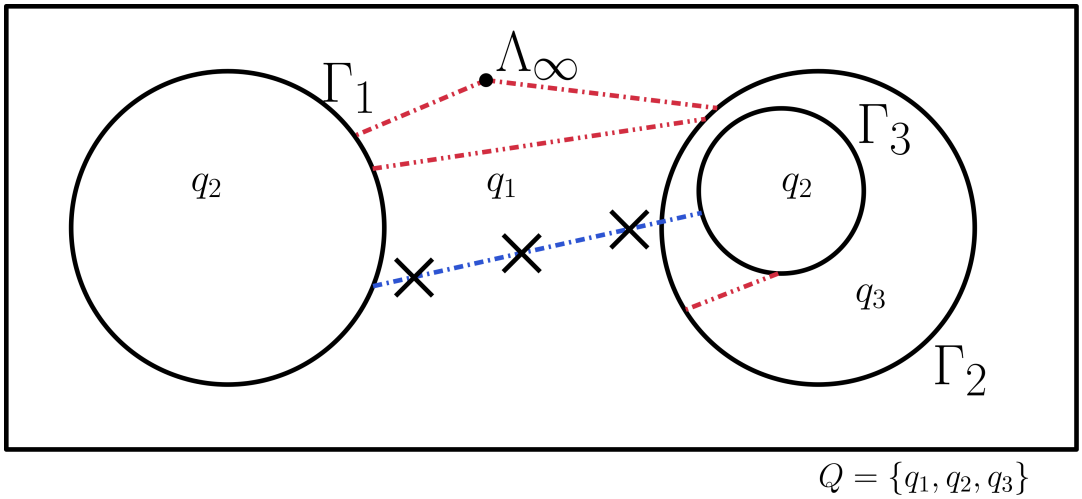


Figure 2.1: Communicating contours are connected by red lines. The two contours that do not communicate are connected by a blue line.

Definition 2.5 (Compatible system of contours)
Contour system is compatible if the colours imposed on its components by contours of the system match. Matching of colours means that if two contours communicate then colour prescriptions at the beginning and the end of the connected path agree.

Remark. It is clear that to any point t such that

$$t \in \Lambda \setminus \bigcup_i \operatorname{supp} \Gamma_i \tag{2.1}$$

we can now assign in a natural way a unique colour q, which will be the same also for all other points of Λ communicating with t. We write

$$t \in \Lambda_q \tag{2.2}$$

in such a case.

Definition 2.6 (External contour of a system)
Let $\{\Gamma_i\}$ be a compatible system of contours. Contours $\{\Gamma_j\}$ are external contours of a compatible system $\{\Gamma_i\}$ if any contour from $\{\Gamma_j\}$ communicates with Λ_∞.

Definition 2.7 (Contours lying inside one another)
We say that contour Γ_j lies inside of Γ_i if Γ_j communicates with Γ_i via path P, $P = (t_1, \ldots, t_N)$, where $\forall t \in \{t_2, \ldots, t_{N-1}\}$ it holds that $t \in \operatorname{int}(\Gamma_i)$ and the path P does not cross the support of any third contour Γ_k.

The compatible system of contours can be also viewed as a forest whose roots are external contours, all having the same external colour q. The ordered nearest-neighbour pairs of vertices on the branches of trees of such a forest traversed from the root to a chosen leaf correspond to pairs $(\widetilde{\Gamma}, \Gamma)$ where the second contour lies inside of the first one with non-conflicting choice of colours.

Generally, compatibility of contours in the Pirogov–Sinai theory is not merely a pair relation specified by mutual compatibility of supports, like in the case

of polymer models. In Pirogov–Sinai theory we have possibly different colours (local ground states) attached to the interior and exterior boundaries of contours and their compatibility is another important aspect[29].

Let us provide some examples of contours and their compatible systems for specific models:

1. **The standard Ising model in low-temperature regime.**
   Consider dimension $d = 2$ for simplicity. This is a well-known case, with contours being closed curves in the dual lattice $\Lambda^\star$ separating regions of '+' spin from the '−' ones. Notice that there is still some freedom in defining the notion of connectedness (see Figure 2.2).

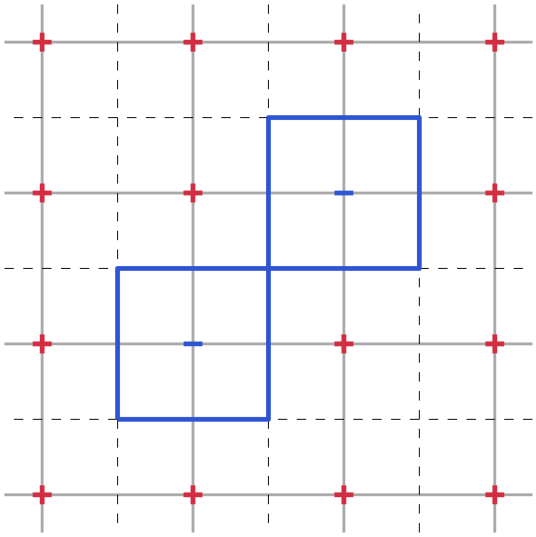


Figure 2.2: Are these two contours, or just a single one?

   These fine distinctions are important when searching for the exact solution (see the seminal work of Onsager on two-dimensional free Ising ferromagnet [Onsager, 1944]).

2. **Standard Pirogov–Sinai abstract model**
   This starts with a choice of local ground states $\{q\}$. The crucial notion is that of a q-correct point of a configuration $x = \{x_i, i \in \Lambda\}$ defined as a point where all the spins in the r-neighbourhood[30] of i share the same value q. The *in*correct points (which are q-correct for no choice of q) are split into connected components. These will be supports of the contours denoted by $\operatorname{supp}\Gamma$. Contours Γ are defined as restrictions of the considered configuration x to such components $\operatorname{supp}\Gamma$. Now, any configuration on Λ is interpreted as a compatible collection of contours. Compatibility of two contours $(\Gamma, \widetilde{\Gamma})$ simply means that

$$\operatorname{dist}(\operatorname{supp}\Gamma, \operatorname{supp}\widetilde{\Gamma}) > 1 \tag{2.3}$$

[29] This aspect is *totally* missing in the theory of polymer models where the colour is only one — the colour filling the void between polymers.

[30] r is the range of interactions of the original spin model

along with the implicitly defined requirement on the choice of colours of $\operatorname{ext}\Gamma$ and $\operatorname{int}\Gamma$, so that the contour system is compatible in the 'matching colours' sense and therefore also in the sense of unique colour prescription for paths connecting communicating contours.

2.2.2 Expression of the Hamiltonian using e_q and E

For a system with Hamiltonian

$$H(x_\Lambda) = \sum_{\substack{A\subset\Lambda \\ \operatorname{diam} A\leq r}} \Phi_A(x_A). \tag{2.4}$$

it is straightforward to define the following quantities

Definition 2.8 (Density of energy)

$$e_q(i) = \sum_{A\ni i} \frac{\Phi_A(x_A^q)}{|A|} \tag{2.5}$$

> **Remark.** In the translation-invariant case these quantities do *not* depend on the choice of i.

Definition 2.9 (Energy of a single contour)

$$\widetilde{E}(\Gamma) = \sum_{A\ \cap\ \operatorname{supp}\Gamma\neq\emptyset} \frac{\Phi_A(x_A^\Gamma)}{|A|}\,|A\cap\operatorname{supp}\Gamma|\,, \tag{2.6}$$

where x^Γ is the configuration which has exactly one contour Γ.

Using these we can rewrite the Hamiltonian as

$$H(x_\Gamma) = \sum_q e_q|\widetilde{\Lambda}^q| + \sum_i \widetilde{E}(\Gamma^i), \tag{2.7}$$

where $\widetilde{\Lambda}^q$ is the collection of q-correct points of x_Λ.
Next, we rewrite (2.7) by putting

$$\Lambda = \bigcup_q \Lambda^q. \tag{2.8}$$

This can be attained, for example, by adding all the points $i\in\operatorname{supp}\Gamma_q$ to $\widetilde{\Lambda}^q$. To be precise, we can formulate the following definition.

Definition 2.10 (q-coloured part of lattice Λ_q)
Define q-coloured part of the lattice Λ as

$$\Lambda_q = \left\{ \bigcup_{q\text{-correct}} i : i\in\Lambda \right\} \cup \left\{ \bigcup_{\substack{i \\ \operatorname{color}(\Gamma_i)=q}} \operatorname{supp}\Gamma_i^q \right\} \tag{2.9}$$

Finally, we define

Definition 2.11 (Energy of a contour)

$$E(\Gamma_q) = \widetilde{E(\Gamma_q)} - e_q|\operatorname{supp}\Gamma_q|. \tag{2.10}$$

Now we can finally re-express the Hamiltonian in a following way:

Proposition 2.1

$$H(x_\Lambda) = \sum_q e_q|\Lambda^q| + \sum_i E(\Gamma^i). \tag{2.11}$$

2.2.3 Diluted configuration in a volume $M = \operatorname{int}\Gamma$

We define diluted configuration in the following way:

Definition 2.12 (Diluted configuration in $\operatorname{int}_q\Gamma$)
If $M = \operatorname{int}_q\Gamma$ and $\{\Gamma_i\}$ is a compatible system of contours then this system will be called a diluted configuration if the whole system $\{\{\Gamma_i\}\ \&\ \Gamma\}$ is compatible.

Definition 2.13 (q-diluted configuration in M)
Let M be a simply connected set, $M \subset \Lambda\backslash\Lambda_\infty$. Say that a compatible collection of contours $\{\Gamma_i\}$ that has external colour q is diluted in M (alternatively, q-diluted) if

$$\operatorname{dist}\left(\operatorname{supp}\Gamma_i, M^C\right) \geq 2 \tag{2.12}$$

for all i.

> **Remark.** In this chapter we will mostly use the term 'diluted configuration' in the sense of Definition 2.12. The second version (Definition 2.13) generalises the first one to more general cases of M.

2.2.4 Peierls condition

Peierls condition is one of the basic assumptions of Pirogov–Sinai theory. In its most basic form it is stated as

Assumption 2.1 (Peierls condition)

$$E(\Gamma) \geq \tau\,|\operatorname{supp}\Gamma|; \qquad \tau \gg 1. \tag{2.13}$$

There may be also other definitions more suitable for situations where the definition of the interior of contour Γ is problematic, e.g. stratified Ising models with rigid interfaces. But these applications are far beyond our current scope.

Remark. In our notation temperature is not explicit, since we include the factor $\beta = \frac{1}{k_B T}$ into the Hamiltonian. Therefore temperature is just one of the parameters of the Hamiltonian that will be explicitly mentioned only later when describing the properties of phase diagram in detail. However, notice the large constant τ. This τ will often be of form $\tau = \frac{\text{const.}}{T}$ where T is a (small) temperature of the original spin model.

The optimal choice of τ is a delicate question that is connected to the convergence of cluster expansion in the underlying polymer model. Should we choose τ too small, we might run into trouble with the cluster expansion. On the other hand, if we start with τ too big, we can have problems with applicability of the machinery. Here we will just assume that τ is reasonably large and we will make no effort to approach the optimal value.

2.2.5 Contour functional

The contour functional $F(\Gamma)$ is without any doubt the most important concept in Pirogov–Sinai theory. To understand how it appeared, let us return to the original problem that had later led to formulation of Pirogov–Sinai theory:

Problem
Assume that we have a Hamiltonian H without any symmetries[31] *and that we have managed to find a complete set of its ground states*[32].

Now the question is: Suppose that the number of ground states is $|Q| = n+1$ and we can perturb H by an additional small Hamiltonian $\widetilde{H}$ depending in a regular way on some n-dimensional parameter λ[33]*. Is it possible for each temperature to fix the parameter λ such that the coexistence of a* ***maximal possible number*** *$n+1$ of phases can be assured?*

It was this question where the ingenious idea of a contour functional $F(\Gamma)$ was first developed and Pirogov and Sinai formulated a chain of integral equations for $F(\Gamma)$ for this case. These integral equations contained also the volume terms and they were solved using the following trick:

- Simplify this complicated chain of equations 'not caring for the volume terms'[34], namely assuming that all h_q in the expression that provides the separation of volume and surface terms of a diluted partition function *are the same.*
- Then compute h_q from the contour functional $F(\Gamma)$ just obtained.

[31] The symmetric case is technically much simpler and was understood immediately after the rigorous description of low temperature Ising ferromagnet was made.
[32] Assumed for brevity to be constant configurations, see Definition 2.1.
[33] This is the so-called degeneracy-removing condition in the original paper [Pirogov and Sinai, 1975]. Also observe we do not include temperature in the list of parameters λ here.
[34] These are actually the most important terms if the quantities h_q are not the same.

– If h_q are not the same then the previous computation was nonsensical and the computed quantities $F(\Gamma)$ have no reasonable interpretation.
– However, if we manage to adjust the parameter λ such that the computed quantities h_q are the same we get the solution for the problem mentioned above and we have found the value of λ for which the coexistence of a maximal number of phases is assured.

Next step was to generalise this idea to the remaining parts of the phase diagram (where the number of phases is not the maximal possible) and this was solved by Pirogov and Sinai by introducing a concept of a contour model with parameter. A simpler approach to the problem was found in [Zahradník, 1984] and we will present here its modernized version.

Abstract Pirogov–Sinai contour functional

Let Γ be a contour with an external colour q and with an interior

$$\operatorname{int}\Gamma = \bigcup_{q'} \operatorname{int}_{q'}\Gamma \tag{2.14}$$

The idea behind the construction of F is that we would like to have the same colour q also inside of the contour Γ, which is the case for polymer models[35]. However, now the colour inside Γ could be different, which means that the partition function inside $\operatorname{int}\Gamma$ is *not* the same as it would be if the contour Γ was erased and the external colour q would spread to the inside of $\operatorname{int}\Gamma$. What to do with this problem? Just compare available partition functions inside Γ with what we would like to have instead of them. We are trying to construct a polymer model where the partition function of external contours would be the *same* as in the original 'physical' model. We do not care how this auxiliary model behaves *inside* of external contours. Thus we define

Definition 2.14 (Contour functional)

$$F(\Gamma) = E(\Gamma) - \log Z^q(\operatorname{int}\Gamma) + \sum_{q'} \log Z^{q'}\left(\operatorname{int}_{q'}\Gamma\right). \tag{2.15}$$

Remark.

- Normally $\operatorname{int}\Gamma$ has only one component. Denote its colour (the internal colour of Γ_q) as q'. Then

$$F(\Gamma) = E(\Gamma) - \log Z^q(\operatorname{int}\Gamma) + \log Z^{q'}(\operatorname{int}\Gamma). \tag{2.16}$$

- If Γ is of such a small size that no interior contour can be pushed into it, the relation above simplifies to

$$F(\Gamma) = E(\Gamma) - (e_{q'} - e_q)|\operatorname{int}\Gamma| \tag{2.17}$$

[35] Where there are no colours or equivalently there is only one colour filling the space around supports of polymers.

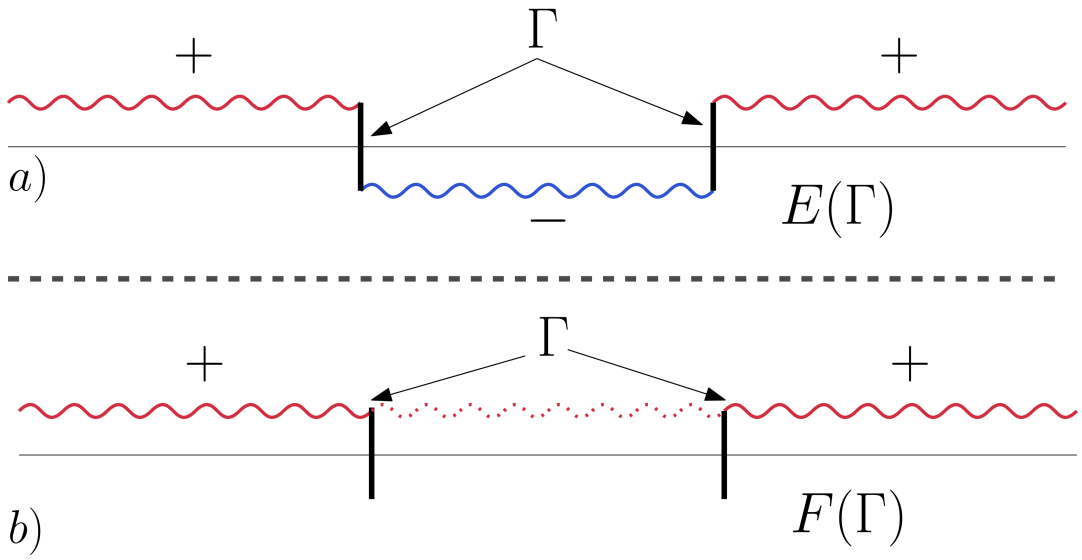


Figure 2.3: Idea behind the contour functional. In the case a) we have two colours and we want to rewrite this model into a polymer model (a model with a single colour) with the same partition function. This can be achieved using newly defined weights $F(\Gamma)$.

and this relation[36] already shows the difficulties we might run into: the value of the r.h.s. of (2.17) can be negative for very large Γ and $(e_{q'} - e_q)$.

It is a simple arithmetic (by induction over volume) to prove that the partition function for the system of external contours diluted in volume M is the *same* for the original 'physical' system with the Hamiltonian (2.11) as well as for the 'artificial' polymer model whose polymers are q-contours Γ^q and polymer weights are given by

$$w_{\Gamma^q} = \exp\left(-F(\Gamma^q)\right). \tag{2.18}$$

For more details see subsection 2.2.6 and Proposition 2.2

Remark. Here we assume that the pair-wise compatibility of external contours in the original physical system (2.11) and the pair-wise compatibility of contours in the polymer model are both defined in the same way, usually by requiring that

$$\operatorname{dist}\left(\operatorname{supp}\Gamma, \operatorname{supp}\widetilde{\Gamma}\right) \geq 2. \tag{2.19}$$

[36] Approximate for large contours, for sufficiently small ones it is exact

2.2.6 Construction of an auxiliary polymer model

In this section we will use the term 'diluted configuration' in the following situations:

Definition 2.15 (Diluted configuration)

- *If* $M = \operatorname{int}_q \widetilde{\Gamma}$ *for some* $\widetilde{\Gamma}$ *then diluteness of* $\{\Gamma_i\}$ *in* $\widetilde{\Gamma}$ *will denote just the compatibility of* $\{\Gamma_i\}$ *and* $\widetilde{\Gamma}$ *(in the traditional Pirogov–Sinai sense). This is the requirement that*

$$\operatorname{dist}(\operatorname{supp}\Gamma_i, \operatorname{supp}\widetilde{\Gamma}) \geq 2 \qquad \forall i \tag{2.20}$$

  or equivalently

$$\operatorname{dist}(\operatorname{supp}\Gamma_i, M^C) \geq 2 \qquad \forall i. \tag{2.21}$$

- *If* $M \subset \Lambda \backslash \Lambda_\infty$ *then the diluteness of* $\{\Gamma_i\}$ *will have the meaning that* $\forall\, i$ Γ_i *does not touch* M^C*. By 'not touching' we once again mean the relation expressed in* (2.19).

Consider the polymer model on M with polymers being contours Γ^q diluted in M and the compatibility relation defined as follows:

- for mutually external contours Γ^q and $\widetilde{\Gamma^q}$ (i.e. such that $\operatorname{int}\Gamma^q \cap \operatorname{int}\widetilde{\Gamma^q} = \emptyset$) we use the same notion of compatibility as in the original abstract Pirogov–Sinai model mentioned above.
- If Γ^q is diluted in a volume $\operatorname{int}\widetilde{\Gamma^q}$ then we treat polymers Γ^q, $\widetilde{\Gamma^q}$ as compatible ones.

We study an abstract Pirogov–Sinai model with partition function

$$Z^q(M) = \sum_{\substack{\{\Gamma_i\} \text{ diluted in } M \\ \text{with external colour } q}} \exp(-H(\{\Gamma_i\})), \tag{2.22}$$

where

$$H_M(\{\Gamma_i\}) = \sum_{q'} e_{q'} |M_{q'}| + \sum_i E(\Gamma_i). \tag{2.23}$$

Furthermore, we assume the validity of Peierls condition, see Definition (2.13). Using the definition of contour functional (2.15)

$$F(\Gamma^q) = E(\Gamma^q) - \log Z^q(\operatorname{int}\Gamma^q) + \sum_{q' \in Q} Z^{q'}(\operatorname{int}_{q'}\Gamma_q) \tag{2.24}$$

we can rewrite the partition function Z^q in a following way:

Proposition 2.2

$$Z^q(M) = \exp(-e_q|M|)\mathcal{Z}^q_M, \tag{2.25}$$

where $\mathcal{Z}^q_M$ is the polymer partition function in volume M of the polymer model with polymers being contours Γ^q diluted in M and having weights

$$w_{\Gamma^q} = \exp(-F(\Gamma^q)). \tag{2.26}$$

We will organise the technical tools of the Pirogov–Sinai theory in an inductive way w.r.t. 'complexity' of considered models, meaning that only contours whose size is not larger than n will be allowed in the n^{th} step. For an outline of some details see the next subsection.

2.2.7 The n-controlled Pirogov–Sinai model

Definition 2.16 (n-controlled Pirogov–Sinai model)
This is an abstract Pirogov–Sinai model with an additional requirement that only the contours Γ such that

$$\operatorname{diam}\Gamma \leq n; \qquad \operatorname{diam}\Gamma \equiv \operatorname{diam}(\operatorname{int}\Gamma) \tag{2.27}$$

are allowed.

Next, we define notions of stability and metastability — more precisely, n-stability and n-metastability. This will be done by induction over n. Below we construct, also by induction, the free energy of n-metastable model of colour q, denoted as

$$h^{(n)}_q = e_q - s^n_q, \tag{2.28}$$

where

$$s^n_q = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \mathcal{Z}^q_\Lambda \tag{2.29}$$

is the free energy of the polymer model with weights

$$w_{\Gamma^q} = \exp\left(-F(\Gamma^q)\right) \tag{2.30}$$

where allowed contours must satisfy

$$\operatorname{diam}\,\Gamma^q \leq n \tag{2.31}$$

and moreover must be *stable*.

The notion of stability is crucial, but there is a lot of freedom in its definition. The most technically convenient seems to be the following one:

Definition 2.17 (Stable contour)
Say that a contour Γ^q *of size*

$$\operatorname{diam} \Gamma^q \geq n+1 \tag{2.32}$$

is stable if

$$a_q |\operatorname{int} \Gamma^q| \geq \widetilde{\tau} |\operatorname{supp} \Gamma^q|, \tag{2.33}$$

where $a_q \equiv a_q^{(n)}$ *is the temporary*[37] *measure of instability of* q*, given as*

$$a_q^{(n)} = h_q^{(n)} - h^{(n)} \tag{2.34}$$

and

$$h^{(n)} = \min_{q' \in Q} \{h_{q'}^{(n)}\} \tag{2.35}$$

is the free energy of the 'level n*' Pirogov–Sinai model, i.e. such that only contours satisfying the relation*

$$\operatorname{diam} \Gamma \leq n \tag{2.36}$$

are allowed.

Corollary 2.1
Unstable contours satisfy the relation

$$a_q^{(n)} \operatorname{diam} \Gamma^q \geq \widetilde{\tau} \tag{2.37}$$

Proof. This follows immediately from the isoperimetric inequality

$$|\operatorname{int} \Gamma| \leq |\operatorname{supp} \Gamma| \cdot \operatorname{diam} \Gamma \tag{2.38}$$

■

Corollary 2.2
Stable contours satisfy a variant of Peierls condition

$$F(\Gamma^q) > \widetilde{\widetilde{\tau}}, \tag{2.39}$$

where $\widetilde{\widetilde{\tau}} = \tau - \widetilde{\tau} - \varepsilon$.

Proof. This is an immediate consequence of the following bounds (see below)

$$\log Z^{q'}(\operatorname{int}_{q'} \Gamma) \geq h_{q'} |\operatorname{int}_{q'} \Gamma| - \varepsilon |\operatorname{supp} \Gamma| \tag{2.40}$$

$$\log Z^{q}(\operatorname{int}_{q} \Gamma) \leq h |\operatorname{int} \Gamma| + \varepsilon |\operatorname{supp} \Gamma| \tag{2.41}$$

and the stability of Γ. ■

[37] it will change slightly with growing n. See more detailed discussion below.

Remark. It may happen that some contours Γ^q classified at the level n as stable will become unstable at higher induction levels $m > n$ and vice versa. These marginal effects will cause no problem. There will always be some freedom in in the definition of stable contours Γ^q for unstable q. This freedom disappears in small enough volumes Λ, and for stable q (minimising the value $h^q = h$) it disappears in *any* volume. In such a case all the external contours of any q-diluted configuration will be stable and our 'metastable' model will be equal to the 'full physical' model of q-diluted configuration in a volume Λ.

The quantities

$$h_q = h_q^{(n)} \tag{2.42}$$

$$h = \min\{h'_q, q' \in Q\} \tag{2.43}$$

$$a_q = h_q - h \tag{2.44}$$

are dependent on our choice of n (and τ), but the difference between various $h_q^{(n)}$ will be very small, of the order

$$|h_q^{(n)} - h_q^{(m)}| \leq \exp\left(-\widetilde{\tau} n\right); \qquad m > n. \tag{2.45}$$

How does this affect our decision on (in)stability of contours Γ of size n? The quantities[38]

$$a_q^{(n)} |\operatorname{int}\Gamma|; \qquad a_q^{(m)} |\operatorname{int}\Gamma| \tag{2.46}$$

in the instability test (see Corollary 2.1) differ by a value of order

$$n^\alpha \exp(-\widetilde{\tau} n) \tag{2.47}$$

for $m > n$, where d is the dimension of Λ. This value is negligible compared to $E(\Gamma)$.

Theorem 2.1 (Bounds for diluted partition functions)
Diluted partition functions satisfy for any q the following bounds

1.

$$Z^q(M) \leq \exp(-h|M| + \varepsilon|\partial M|) \tag{2.48}$$

2.

$$Z^q(M) \geq \exp(-h_q|M| - \varepsilon|\partial M|) \tag{2.49}$$

Proof. The second bound is a trivial consequence of the cluster expansion of the metastable model with the partition function $\mathcal{Z}^q_{met}(M)$. Proof of the first bound is more interesting and it is based on the idea of fixing all unstable external contours of a system $\{\Gamma_i\}$ and then inductively using the first bound for their interiors. In the exterior $\operatorname{ext}\{\Gamma_i\}$ we have a metastable model that we fully control by cluster expansion of the corresponding contour model. The core of the proof is formulated in the bubble lemma, see subsection 2.2.8, Lemma 2.2. ■

[38] actually the 'dangerous volume effects' from the definition of $F(\Gamma)$

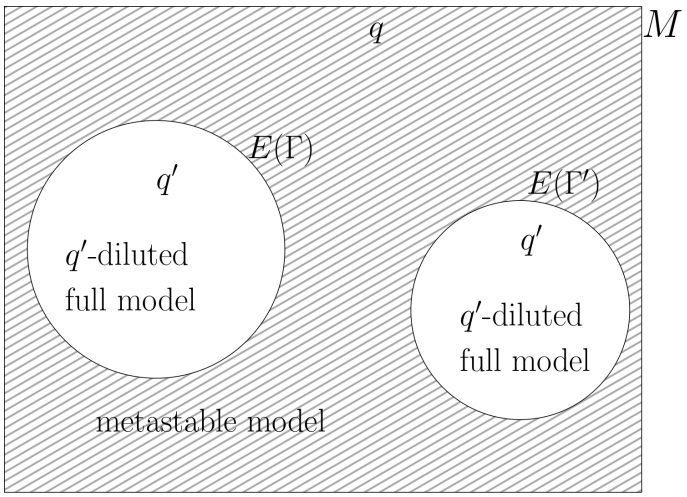


Figure 2.4:

2.2.8 Bubble lemma

Configuration in this section is a collection of mutually external unstable contours $\{\Gamma_i\}^{ext}$ with the same external colour such that

$$a_q|\operatorname{int}\Gamma| \geq \tau|\operatorname{supp}\Gamma| \tag{2.50}$$

for each $\Gamma \in \{\Gamma_i\}^{\text{ext}}$. Notice that these contours are quite large, with diameter $\frac{\tau}{a_q}$. The Hamiltonian is

$$H(\{\Gamma_i\}) = a_q|\operatorname{ext}\{\Gamma_i\}| + \sum_i E(\Gamma_i) + \Delta, \tag{2.51}$$

where Δ is a small term of a complicated nature (dependent also on $\{\Gamma_i\}$) such that

$$|\Delta| \leq \varepsilon\left(|\partial M| + \sum_i \operatorname{supp}\Gamma_i\right). \tag{2.52}$$

We will estimate from above the partition function

$$Z^q_{\text{bubble}}(M) = \sum_{\{\Gamma_i\}^{\text{ext}}} \exp\left(-H(\{\Gamma_i\})\right). \tag{2.53}$$

Lemma 2.2 (Bubble lemma)
For any (stable or unstable) $q \in Q$ we have

$$Z^q_{bubble}(M) \leq \exp\left(\varepsilon|\partial M|\right) \tag{2.54}$$

Remark. In fact, more detailed estimates of Z_{bubble} would be useful in a more detailed analysis of instability in Pirogov–Sinai models. We will not try to investigate these things here. They would be of course crucial in an attempt to describe the behaviour of a typical configuration in volume M subject to an unstable boundary condition q.

Also, what size and shape of M is necessary for one or more big 'bubbles' to occur? We omit all these interesting questions here, as we only aim to determine stable values of q and to analyse the coexistence of various stable q.

Intuitively, one expects stronger bounds (compared to (2.54)) of the type

$$Z^q_{\text{bubble}}(M) \leq \exp\left(-E^q_{\min}(M) + \varepsilon|\partial M|\right) \tag{2.55}$$

where the quantity $E^q_{\min}(M)$ is the minimiser of (2.51) with the term Δ discarded. Clearly, for small volumes

$$E^q_{\min}(M) = a_q|M|, \tag{2.56}$$

whereas for very large (and 'nicely' shaped) volumes M we expect something similar to

$$E^q_{\min} = a_q|\text{ext}\,\Gamma| + E(\Gamma) \tag{2.57}$$

where Γ is the best possible contour such that the system jumps from q to the stable regime. We will not prove these statements here.

Sketch of a proof *(of Lemma 2.2).*
The proof is based around the following idea:

1. If the system is connected (in some loose sense) then showing that the contribution of such configurations to $Z^q_{\text{bubble}}(M)$ satisfies (2.54) is straightforward, from their cluster expansion.
2. If the distances between connected components of $\{\Gamma_i\}$ are of the value $\frac{\tau}{a_q}$ or even larger then in such a case the volume term

$$a_q\,|\text{ext}\,\{\Gamma_i\}| \tag{2.58}$$

also becomes quite large and we may represent it as a sum of energies of some artificial disjoint system of 'square' contours of the size $\frac{\tau}{a_q}$ with density of energy being τ, not a_q. But after doing this we are in the situation of the case 1. See also Figure 2.5

■

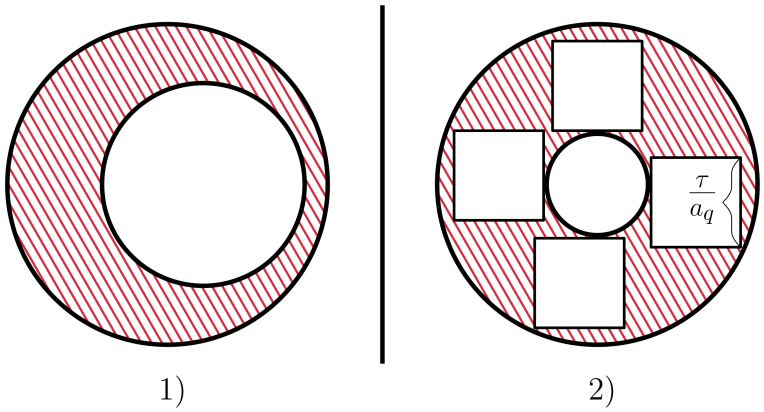


Figure 2.5: Illustration for the proof of Lemma 2.2.

2.2.9 Phase diagram

Let H from (2.4) depend on some tuple of parameters $\vec{\lambda}$[39]. In the previous section we constructed for a given Hamiltonian a collection of metastable free energies

$$\{h_q; q \in Q\} \tag{2.59}$$
$$h = \min h_q, \tag{2.60}$$

and the result of our labour is the map

$$\vec{\lambda} \to \{h_q(\vec{\lambda}); q \in Q\}. \tag{2.61}$$

Denote by $n = |Q|$ the cardinality of the set of all local ground states Q and let $\lambda \in \mathbb{R}^{n-1}$. For concreteness choose $|Q| = 3$ and $\lambda \in \mathbb{R}^2$. We can take related map

$$\vec{\lambda} \to \{q \in Q : h_q(\lambda) = h\} \tag{2.62}$$

and plot this map (see Figure 2.6). This way we obtain the *phase diagram* of a given system.

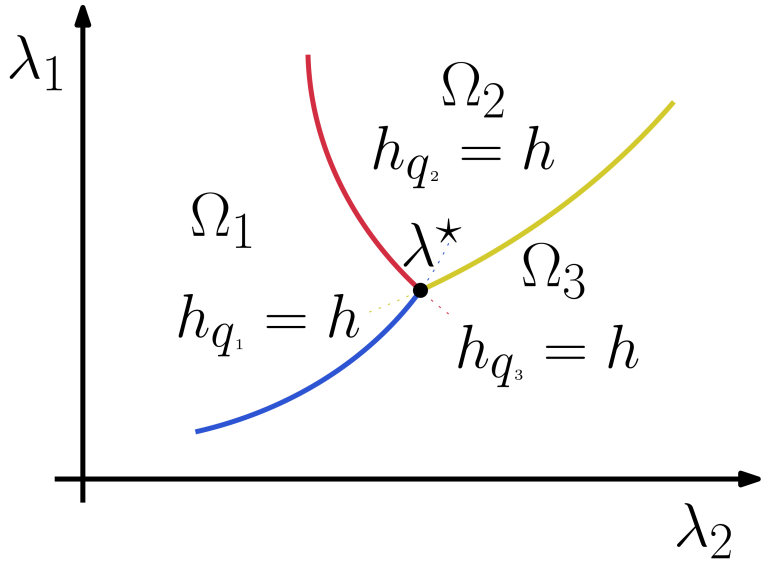


Figure 2.6: Phase diagram for $|Q| = 3$ and $\lambda \in \mathbb{R}^2$.

Notice the region Ω_1 where

$$h_{q_1} = h \tag{2.63}$$
$$h_{q_2} > h \tag{2.64}$$
$$h_{q_3} > h \tag{2.65}$$

and analogously for Ω_1, Ω_2. Also take note of the curves

$$C_{ij} = \{x \in \mathbb{R}^2 : (x \to q) = h_i = h_j\}, \qquad i < j, \qquad i \in \{1, 2, 3\}. \tag{2.66}$$

[39] Therefore also the quantities e_q and $E(\Gamma)$ depend on $\vec{\lambda}$.

and the triple point $\lambda^\star$ where $h_1(\lambda^\star) = h_2(\lambda^\star) = h_3(\lambda^\star) = h$.

For a 'nice' behaviour of the curves C_{ij} we need some assumptions on the properties of the maps

$$\vec{\lambda} \to e_q(\vec{\lambda}) \tag{2.67}$$

$$\vec{\lambda} \to [E(\Gamma)]\,(\vec{\lambda}). \tag{2.68}$$

These assumptions are called 'degeneracy-removing conditions'[40] in the paper [Pirogov and Sinai, 1975].

In the region Ω_i we guarantee the existence of a 'q_i-coloured' phase in the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$. On curves C_{ij} we guarantee coexistence of two phases — one of them 'q_i-coloured', the other one 'q_j-coloured' — also in the thermodynamic limit. Finally in the triple point $\lambda^\star$ we guarantee that in the thermodynamic limit there coexist all three phases.

We are omitting the discussion of non-existence of other translation invariant phases on $\mathbb{Z}^d$ (other than the ones specified in the phase diagram sketched above). Another question, and in this case a definitely non-trivial one, that we fully omit is the discussion of finite size effects under various boundary conditions.

[40] e.g. rank $(n-1)$ of the first differential of $e_q(\vec{\lambda})$

3 | Dyson models on a one-dimensional lattice

3.1 Historical developments

From the beginning of the second half of the twentieth century a lively topic in statistical physics is the phenomenon of phase transition. Following the paper [Onsager, 1944] on the existence of a phase transition in two-dimensional free Ising model, there have been many attempts to generalise and extend this result. One avenue of research in this direction is the study of phase transitions in discrete lattice spin models with long-range interaction.

We start our exposition with the paper [Kac and Thompson, 1969]. Already in 1969 they have conjectured existence of a phase transition for long-range one-dimensional discrete lattice system, whose constants of interaction are non-negative, interaction is *summable* and the first moment of magnetisation is *infinite*.[41] To be more precise, let us introduce some notation

Definition 3.1 (Moments of magnetisation)
The n-th moment of magnetisation, $n \in \mathbb{N}_0$, is defined as

$$M_n = \sum_{i=1}^{\infty} i^n J(i). \tag{3.1}$$

If the zeroth moment is finite we say that the interaction is summable.

Definition 3.2 (Long-range discrete lattice model)
Let Λ be a lattice on $\mathbb{Z}^1$. Let $x_i = \pm 1$ be spins *that exist on a coordinate $i \in \Lambda$. Define constants of interaction*

$$J(i) \in (0, +\infty), \forall i \in \mathbb{N} \tag{3.2}$$

and long-range Hamiltonian

$$H_L = -\sum_{i<j} J(i-j) x_i x_j. \tag{3.3}$$

[41] If the zeroth moment is infinite, there is a infinite energy gap between ground state and other states and therefore there is no phase transition. If only finite number of summands is non-zero, then by the result in [Rushbrooke and Ursell, 1948] the system is without a long-range order at *any* temperature. Also if the first moment is finite no long-range order can be expected since $2M_1$ is the energy one pays for one 'change of phases' on the infinite line and these changes are to be expected with finite density for any finite temperature.

A model on this kind of lattice with the Hamiltonian specified above will be called long-range discrete lattice model.

> **Remark.** A remarkable choice of $J(i)$ that guarantees the assumptions on moments mentioned above is
>
> $$J(i) = Ci^{-p}; \qquad p \in (1, 2]\,,\ C > 0 \tag{3.4}$$

This special choice of constants of interaction gives us the definition of the Dyson model.

Definition 3.3 (Dyson model)
Long-range discrete lattice model with constants of interaction given by (3.4) *is called the Dyson model.*

Also in 1969, Dyson in his attempt to prove the conjecture of Kac and Thompson provided the proof of existence of phase transition in the Dyson model. The exact statement he proved is the following one:

Theorem 3.1 (Phase transition in long-range discrete lattice model)
In the long-range discrete lattice model with non-negative monotonously decreasing constants of interaction there is a phase transition at finite temperature given that M_0 is finite and the modified inverse moment

$$M'_{-3} = \sum_{i=1}^{\infty} \frac{\log\log(i+4)}{i^3 J(i)} \tag{3.5}$$

is finite.

> ***Proof.*** See [Dyson, 1969]. ■

From this it easily follows that

Corollary 3.1 (Phase transition in the Dyson model)
Dyson model has a phase transition for finite temperature for $p \in (1, 2)$.

> ***Proof.*** Since constants of interaction (3.4) with $p \in (1, 2)$ satisfy the requirement of the theorem above, this corollary follows. ■
>
> **Remark.** This leaves the case of $p = 2$ undecided. We will see that this is a rather different situation compared to other choices of p.

Further work on this problem was then done by non-rigorous methods in theoretical physics. First, in the article [Anderson and Yuval, 1969] it was demostrated by asymptotic estimates that the Dyson model with $p = 2$ undergoes a phase transition for finite temperature that is close to $\beta_T J = 0.63$. Later in [Thouless, 1969] it was shown that this model undergoes a peculiar kind of phase transition, nowadays called the Thouless effect. It can be summarized that Thouless effect occurs when magnetisation is discontinuous and energy is continuous as functions of temperature.

This achievement of theoretical physics led Dyson to improve upon his first attempt, resulting in the following theorem

Theorem 3.2 (Phase transition for modified Dyson model with $p = 2$)
For Dyson-like model with Hamiltonian

$$H = -J \sum_{i<j} \sum \frac{\log\log(j-i+3)}{(j-i)^2} x_i x_j \tag{3.6}$$

there is a phase transition for finite temperature.

Proof. See [Dyson, 1971] ■

This paper also contains a rigorous proof of Thouless effect in the model described in Theorem 3.2. These result were once again obtained by comparison to hierarchical models. The resolution of the question of existence of (Thouless-type) phase transition for unmodified Dyson model with $p = 2$ has been provided more than a decade later, in [Fröhlich and Spencer, 1982]. The result was that even this unmodified version of Dyson model exhibits the Thouless effect. Arguments of Fröhlich and Spencer were later formalized in the article [Cassandro et al., 2005]. The result of this paper had two main requirements

1. $p \in \left(3 - \frac{\log(3)}{\log(2)} \approx 1.42, 2\right)$
2. $J(1) \gg 1$.

Both of them are proof-generated, but the second one has an intuitive explanation — it makes the Dyson model behave more like the well-known Ising model. Later in [Littin and Picco, 2017] there has been some attempt to provide interpretation also for the first restriction.

The approach of Fröhlich, Spencer, Cassandro, Ferrari, Merola, Presutti... was based on contour methods. From that moment on contour methods became an effective tool in the study of Dyson model. Recently it has been shown that first the assumption of closeness to Ising and later also the assumption on restriction of values of p can both be dropped, see [Bissacot et al., 2018].

3.2 Contours in Dyson model

In the following section we will be using the following definition:

Definition 3.4 (Configuration positive/negative almost everywhere (a.e.))
Configuration

$$x = \{x_i\}_{i\in\mathbb{Z}}; \qquad x_i \in \{-1, +1\} \tag{3.7}$$

is called positive a.e. if there exist $S \subset \mathbb{Z}$ finite such that

$$x_i = \begin{cases} -1 & i \in S \\ +1 & i \in S^C. \end{cases} \tag{3.8}$$

Analogously define configuration negative a.e.

Remark. In literature this is also called a configuration with a chosen boundary condition. It we do not want to make distinction between positive and negative, we use the expression *constant* a.e.

3.2.1 Introductory estimates of energy of an interval

Proposition 3.1
Dyson energy of a configuration constant a.e. is given by

$$H(x) = \frac{1}{4} \sum_{\{i,j\}\in\mathbb{Z}} \frac{(x_i - x_j)^2}{|i-j|^p} = \frac{1}{4} \sum_{\substack{\{i,j\}\in\mathbb{Z} \\ x_i \neq x_j}} \frac{(x_i - x_j)^2}{|i-j|^p}. \tag{3.9}$$

Specifically for configuration positive a.e. it is

$$H(x) = \sum_{i\in S} \sum_{j\in S^C} \frac{1}{|i-j|^p}. \tag{3.10}$$

Having this expression for energy we proceed to 'localise' the energy. This localisation depends significantly on the value of p. For values of p close to 2 the phase separation is weaker[42] but the discussion gets considerably simpler for small enough temperatures.

So, choose $p \lesssim 2$ and let I be an interval on $\mathbb{Z}$, $|I| = l$. Denote by x_I a configuration such that x is positive a.e. and $S = I$. Then we can decompose the energy in a following way:

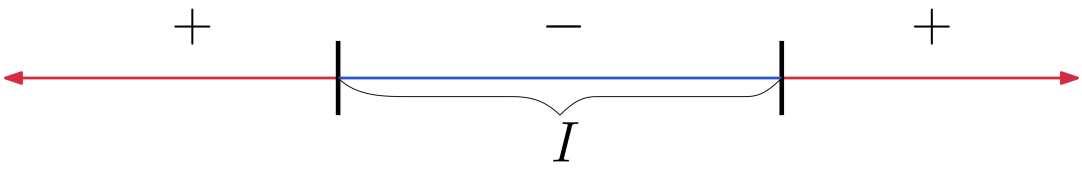


Figure 3.1: Illustration of a configuration positive a.e. whose negative spins are all on interval I.

$$H(x_I) = \hat{E}_I + \breve{E}_I \tag{3.11}$$

where

$$\hat{E}_I = \sum_{i\in I} \sum_{\substack{j\in I^C \\ |i-j|<l}} \frac{1}{|i-j|^p} \tag{3.12}$$

$$\breve{E}_I = \sum_{i\in I} \sum_{\substack{j\in I^C \\ |i-j|\geq l}} \frac{1}{|i-j|^p} \tag{3.13}$$

The $\hat{E}_I$ represents short-range contributions to the energy, while $\breve{E}_I$ accounts for long-range ones. Taking l, $l = |I|$ large enough, we can approximate these

[42] remember that for $p = 2$ we actually see a hybrid kind of transition — Thouless effect. Furthermore there is a critical region near the point of phase transition, see [Imbrie and Newman, 1988].

terms as

$$\hat{E}_I = 2\sum_{n=1}^{l-1} \frac{n}{n^p} \approx 2\int_1^l \frac{1}{x^{p-1}}\,\mathrm{d}x \approx \frac{2l^{2-p}}{2-p} \tag{3.14}$$

$$\breve{E}_I = 2l\sum_{n=l}^{\infty} \frac{1}{n^p} \approx 2l\int_l^{\infty} \frac{1}{x^p}\,\mathrm{d}x \approx \frac{2l^{2-p}}{p-1}. \tag{3.15}$$

Observe that for $p \lesssim 2$

$$\frac{\hat{E}_I}{\breve{E}_I} \approx \frac{p-1}{2-p}. \tag{3.16}$$

For $p \gtrsim 1$ the situation and its description is more delicate because the dominant part of energy comes from $\breve{E}_I$, not $\hat{E}_I$. This begs the following question: at what distance 'sits' the dominant part of energy from $\breve{E}_I$?

Choose an integer L, the 'diameter multiplication coefficient' that will be used in the following construction. Decompose $\breve{E}_I$ into two parts

$$\breve{E}_I = E_{\hat{I}} + E_{\hat{I}}^{\infty} \tag{3.17}$$

where $\hat{I}$ is a 'protective zone' of interval I, see Figure 3.2;

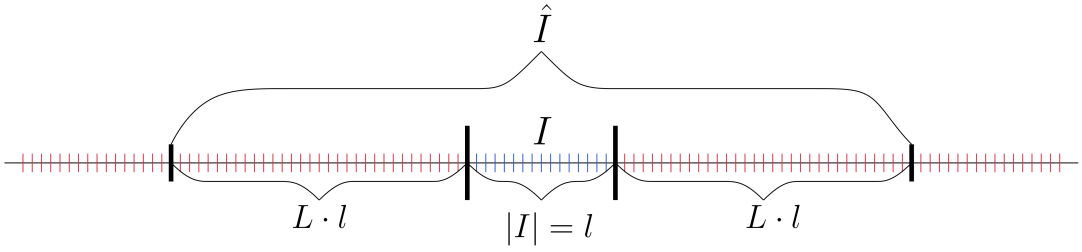


Figure 3.2: Illustration of a protective zone $\hat{I}$ around interval I.

and the two terms are defined as

$$E_{\hat{I}} = \sum_{i\in I}\sum_{j\in\hat{I}} \frac{1}{|i-j|^p} \tag{3.18}$$

$$E_{\hat{I}}^{\infty} = \sum_{i\in I}\sum_{j\in\hat{I}^C} \frac{1}{|i-j|^p}. \tag{3.19}$$

Going forward with rough approximations, we obtain

$$E_{\hat{I}} = 2l\sum_{n=l}^{l(L-1)} \frac{1}{n^p} + 2\sum_{n=l(N-1)}^{Nl} \frac{n-(N-1)l}{n^p} \tag{3.20}$$

$$> 2l\int_l^{Ll} \frac{\mathrm{d}x}{x^p} \quad \approx \frac{2l}{p-1}\left(\left(\frac{1}{l}\right)^{p-1} - \left(\frac{1}{lL}\right)^{p-1}\right) \tag{3.21}$$

$$\approx \frac{2l^{2-p}}{p-1}\left(1-\left(\frac{1}{L}\right)^{p-1}\right). \tag{3.22}$$

$$E_{\hat{I}}^{\infty} > 2l\sum_{n=(L-1)l}^{\infty} \quad \approx \frac{2l^{2-p}}{p-1}\left(\frac{1}{L}\right)^{p-1}. \tag{3.23}$$

For the sake of concreteness, for $p = \frac{3}{2}$ the value of $E_{\hat{I}}$ is roughly twice as big as $E^{\infty}_{\hat{I}}$ if we take $L = 9$[43].

Taking the above-mentioned p-dependent dichotomy into account, we propose *two different* approaches to the definition of contours in Dyson model. First is the classical definition dating back to [Fröhlich and Spencer, 1982] and refined by [Cassandro et al., 2005] and their followers, using a geometric construction we would prefer to avoid. The second definition is more intricate and we believe that it is a starting point for further systematic application of the Pirogov–Sinai theory to Dyson model, which is our research program for near future.

3.2.2 The case of $p \lesssim 2$

As mentioned a few times already, this is the construction of [Fröhlich and Spencer, 1982], [Cassandro et al., 2005]... It is usually given as a geometric construction done in $\mathbb{R}^2$ into which $\mathbb{Z}$ is naturally embedded as a set of equidistant points on a line with help of a dual lattice $\Lambda^\star$ which is used as a set of border points for bases of constructed isosceles triangles. For details see [Cassandro et al., 2005].

In order to avoid all of this we propose an alternative but equivalent construction. Take a configuration that is positive a.e. and decompose it into disjoint intervals I_i such that

$$S = \left\{ \bigcup_i I_i \right\}_{i=1}^{n \in \mathbb{N}} \tag{3.24}$$

where (3.25)

$$x_i = \begin{cases} +C & i \in I_i \\ -C & i \in \{\min(I_i) - 1; \max(I_i) + 1\} \end{cases} \tag{3.26}$$

$$C \in \{-1; +1\}\,. \tag{3.27}$$

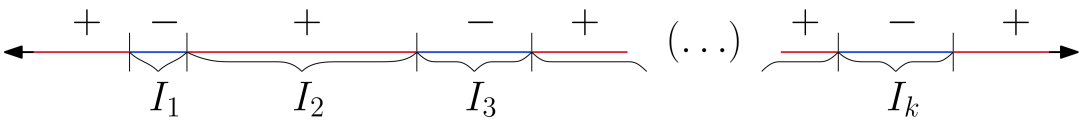


Figure 3.3: Illustration of decomposition into intervals I_k.

Now use the following algorithm

Algorithm 3.1

1. *Take the shortest interval I_S from the collection $\{I_i\}$ and call it a contour.*
2. *Replace all the spins in I_S with the opposite value.*
3. *Iterate until configuration is positive everywhere.*

[43] this has proven to be a convenient choice for $p = \frac{3}{2}$

Taking all the contours identified in step 1 of this algorithm we get a *compatible* system of contours.

> **Remark.** There is some ambiguity in this construction is the smallest interval in any given step is not unique. There are many ways to resolve this, e.g. take the left-most interval, more precisely such a interval I that $I \ni i_M = \min_{x_i \in \bigcup I_S} (x_i)$.

Proposition 3.2
The approach of Algorithm 3.1 and the one in [Cassandro et al., 2005] produce equivalent systems of compatible contours, up to possible ambiguities in the case mentioned in the remark above.

> ***Proof.*** This is an exercise in trigonometry. Since we do not wish to copy the details from [Cassandro et al., 2005] we will not provide the complete construction. ■

This construction originally did work only in the parameter space of p close to 2. By later effort of [Bissacot et al., 2010] it was made to work for any $p \in (1, 2]$. Unfortunately, it is still too coarse to be useful for the application of Pirogov–Sinai theory, especially should one hope to treat *asymmetric*[44] perturbations of the original Hamiltonian. For this purpose we require a definition that provides contours that are energetically better separated.

3.2.3 Case of $p \in (1, 2)$

As we have seen in chapter 2, Pirogov–Sinai theory has two main keywords: *contour* and *local ground state*. Any configuration in a model where the use of Pirogov–Sinai technique is admissible should be thought of as a *compatible collection* of contours with spins outside of contours being in some local ground state. In the simplest cases (and also here) these local ground states are configurations of constant spin. There is also a crucial technical requirement on energies of contours and their interaction; namely that self-energy of any contour $E(\Gamma)$ should be sufficiently large[45] and their interaction should be small compared to $E(\Gamma)$. This is the problematic part with the construction in section 3.2.2. When $\check{E}_I$ is comparable to (or even larger than) $\hat{E}_I$ then in order to use Pirogov–Sinai theory we need a different approach.

The approach we present here will make use of so-called grey segments. For brevity and concreteness set $p = \frac{3}{2}$

Definition 3.5 (Grey segments)
Let x be a configuration positive a.e. Segment $I \subset \mathbb{Z}$ is called grey, where $+1$ is 'white' and -1 'black', if

$$\overline{x}_I \in \left[\frac{-1}{3}; \frac{1}{3}\right] \tag{3.28}$$

$$\overline{x}_I = \frac{1}{|I|} \sum_{i \in I} x_i. \tag{3.29}$$

[44] i.e. spin-flip symmetry breaking.
[45] remember the Peierls condition

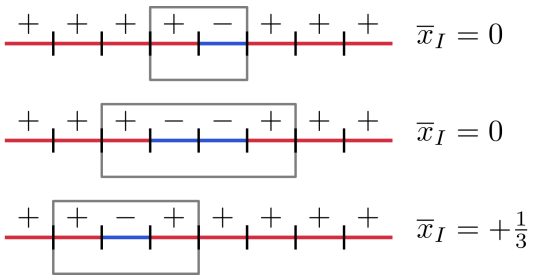


Figure 3.4: Illustration of grey segments.

The choice of constant $\frac{1}{3}$ will be related to the following auxiliary construction

Definition 3.6 (Reference system of squares in $\mathbb{Z}$)
Define system of squares $\mathcal{K}$ as

$$\mathcal{K} = \bigcup_{k=1}^{\infty} \mathcal{K}_k \tag{3.30}$$

$$\mathcal{K}_k = \bigcup_{m\in\mathbb{Z}} \square_m^k \tag{3.31}$$

$$\square_m^k = \left\{a_i + m2^{k+1}; a_i \in \left[0, 2^{k-1}\right]\right\} \tag{3.32}$$

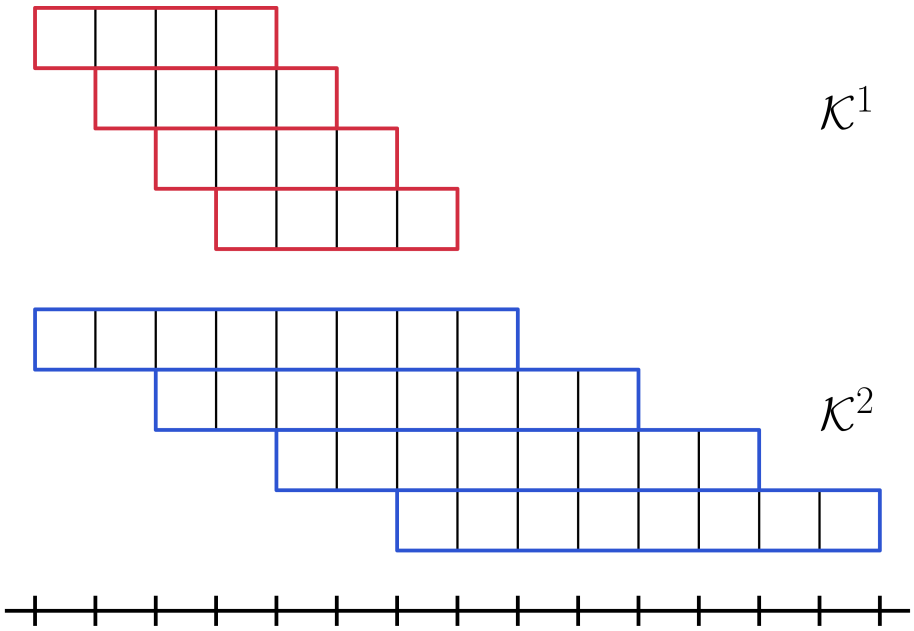


Figure 3.5: Illustration of system of squares $\mathcal{K}$.

Remark. The usefulness of this construction is that it allows us to embed any interval $I \in \mathbb{Z}$ into some element of $\mathcal{K}$ of comparable size.

The quantity

$$H_I^{\bullet}(x) = \sum_{\substack{\{i,j\}\cap I\neq\emptyset \\ |i-j|\geq\frac{l}{2}}} \frac{1}{|i-j|^{\frac{3}{2}}} x_i x_j \tag{3.33}$$

should express the part of full energy $H()$ related to interactions of interval I with entities in range comparable to or larger than $|I|$. It can be shown that for a suitable choice of constant C (e.g. $C = \frac{1}{4}$) we have

$$H(x) > C \sum_{\substack{I \in \mathcal{K} \\ I \text{ grey}}} H_I^\bullet(x). \tag{3.34}$$

Furthermore for any grey interval I(not necesarilly $I \in \mathcal{K}$) we have

$$H_I^\bullet(x) > C'\sqrt{|I|} \tag{3.35}$$

$$> C'|I|^{2-p} \text{ more generally} \tag{3.36}$$

Finally we will arrive to the conclusion that the dominant part of $H_I^\bullet(x)$ sits within $\hat{I}$. This is important from the point of view of Pirogov–Sinai theory. In the symmetric case[46] this is not necessary and contours from section 3.2.2 would be enough.

Using these estimates, let us formulate the definition of contours

Definition 3.7 (Removable segment)
Say that segment of configuration $x_I \subset x$, $I \subset \mathbb{Z}$ is removable from x if the following holds:

1. *Values of x_i on ∂I are the same.*
2. *For any grey segment $J \subset I$ its protection zone $\hat{J}$ is a subset of I.*

Remark. It is easy to see that all the spins on the boundary of a removable segment I have the same value. Therefore we can distinguish removable contours of a given external value.

Definition 3.8 (Interior contour)
Interval $I \subset \mathbb{Z}$ is an interior contour if it is a minimal removable segment from configuration x, meaning that there exists no removable segment J such that $J \subset I$.

Definition 3.9 (Contour)
Let I be an interval and x be a configuration that is positive on I^C. Denote as contour a sub-configuration x_I if it is the only interior contour of x.

It is evident that any configuration can be decomposed into a (compatible) system of contours by induction 'from the inside'.

For technical reasons we require that grey segments that give rise to given contour are connected in an appropriate sense.

Definition 3.10 (Commensurate segments)
Let $\mathcal{K}$ be a system of squares and $I, J \in \mathcal{K}$. If

$$\operatorname{diam} I \leq \operatorname{diam} J \tag{3.37}$$

$$\operatorname{dist}(I, J) \leq L \cdot \operatorname{diam} I, \tag{3.38}$$

where L is the coefficient from the definition of protective zone, then I and J are commensurate.

[46] no external fields and full spin-flip symmetry.

Theorem 3.3
Grey squares from a system of squares $\mathcal{K}$ that are contained in contour are vertices of a connected *graph, where edges are given by the relation of commensurability.*

> ***Idea of a proof.*** Denote by G the system of grey segments of a sub-configuration y that is a contour. Let $G = G_1 \cup G_2$, where G_1 and G_2 are disconnected w.r.t. relation of commensurability. Assume moreover that G_1 is connected. The idea is to show that either G_1 and G_2 are mutually external (see Figure 3.6, case a) or G_2 lies deep inside the interior of G_2.
> The idea in the first case is to show that the sub-configuration packed into the union of protective zones of its grey segments is removable as a whole from the full configuration x. ■

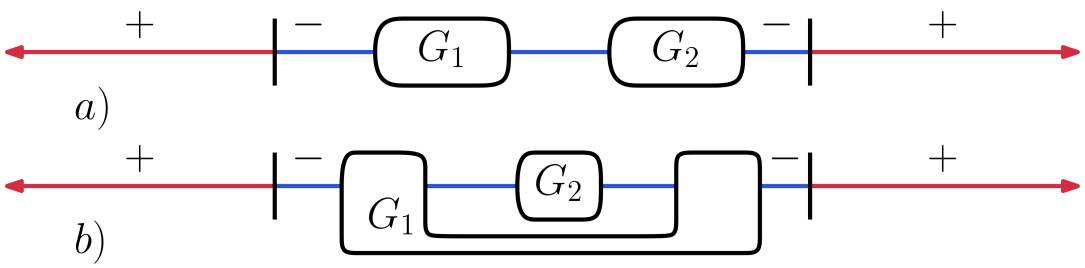


Figure 3.6: Illustration of possible situations in the proof of Theorem 3.3.

Conclusion and future plans

In these pages have been summarised the current results of author's work done under supervision of and in collaboration with his supervisor. Author is well-aware that some formulations could be more elaborate but often at a price of reduced clarity or heavy attention to minute details. Since we do not plan on abandoning these ideas after the submission of this thesis, we will return to these if need be.

To summarize:

- In chapter 1 has been provided an introduction into polymer models, with special focus on hard-core repulsive polymer model and a new resummation method for partition functions of hard-core repulsive model. The most important part is the introduction of umbel trees.
- Chapter 2 serves as a short introduction to some ideas in Pirogov–Sinai theory in its abstract version.
- One-dimensional Dyson models were the topic of chapter 3. After a short introduction of the model and its history an alternative definition of contour is provided.

Plans for the future are:

- Numerically explore the structure of self-avoiding trees and Kotecký–Preiss type equations from chapter 1, especially in the context of non-ideal gas polymer model.
- Further elaborate and extend chapter 2 to become a self-contained introductory exposition into the Pirogov–Sinai theory.
- Using the definition of contours provided in chapter 3 study the one-dimensional Dyson model using the methods of Pirogov–Sinai theory. There is still considerable amount of work ahead.

List of some used notation

- $\mathbb{P}$ – set of polymers in polymer models.
- $\mathbb{G}^\bullet$ – graph of incompatibilities in a polymer model.
- $\mathbb{G}^\bullet(P_1, \ldots, P_n)$ – graph on incompatibilities on polymers $(P_1, \ldots, P_n)$.
- Z – partition function of a polymer model.
- $\mathscr{R}$ – restriction in specification of a structure
- $\mathrm{supp}(P)$ – support of P
- $\mathscr{C}$ – cluster
- $U, U_{\mathscr{C}}, \mathcal{U}$ – umbel, umbel-cluster, umbel tree
- X° – circular structure X
- Λ– lattice, more specifically torus on $\mathbb{Z}^d$.
- Λ_∞– subset of Λ used as an infinity on Λ.
- $\{\Gamma_i\}$ – compatible set of contours.
- Γ – single contour
- $\mathrm{int}\, M$ – interior of M
- $\mathrm{int}_q\, M$ - q-coloured interior of M
- $\mathrm{ext}\, M$ – exterior of M
- $\mathcal{Z}^q\,(M)$ - partition function in volume M with exterior colour q
- S – set of possible spin configurations of a considered model

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