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RIGORÓZNÍ PRÁCE

Collateralized Debt Obligation: Valuation and Sensitivity Analysis

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Akademický rok: **2008/2009**

Prohlášení

Prohlašuji, že jsem rigorózní práci vypracovala samostatně a použila pouze uvedené prameny a literaturu.

V Praze dne 18.5.2009 Petra Benešová

Poděkování

Na tomto místě bych ráda poděkovala PhDr. Petrovi Teplému, vedoucímu této práce, za konzultace a cenné připomínky. Dále děkuji všem blízkým za podporu a trpělivost.

Abstract

A collateralized debt obligation (CDO) is a highly leverage structured credit product linked to credit events of a pool of underlying debt securities. CDO can be understood as an insurance against a credit risk of the pool where its issuer is a protection buyer and its investor is a protection seller. Whereas a CDO issuance has boomed in recent years, by the end of 2008 two thirds of CDOs were in a formal state of default.

The aim of this thesis is to clear up the course of events which lead to the suspension of the CDO market and to deduce recommendations for its future development. To do so we develop a valuation program in MS Excel VBA based on a One Factor Gaussian Copula model. Using the program we first apply a sensitivity analysis, than we model value of a CDO tranche before the financial crisis stroke and after it to value a loss of investors based on a change in expected cash-flows.

We detect four main deficiencies. First, the market was not properly diversified. Second, the valuation model was often not deeply understood which led to a mispricing of CDO tranches. Third, this resulted in a mispriced base correlation. We also numerically demonstrate the fourth deficiency, i.e. the mark-to-market valuation obligation which can have destructive effects. Recommendations to remove these deficiencies are suggested.

Keywords: collateralized debt obligations, Gaussian Copula, valuation, securitization

JEL: G01, G32, C63

Abstrakt

Zajištěná dluhová obligace (Collateralized debt obligation - CDO) je kreditní strukturovaný produkt navázaný na množinu podkladových dluhových cenných papírů s výrazným pákovým efektem. Zjednodušeně lze říci, že se jedná o formu pojištění proti kreditnímu riziku, ve kterém vydávající CDO představuje pojištěnce a investor pojistitele. Zatímco emisní aktivita CDO v posledních letech výrazně rostla, na konci roku 2008 byly dvě třetiny CDO ve stavu úpadku.

Cílem této práce je objasnit, proč k úpadkům došlo a vyvodit takové důsledky, které umožní další existenci trhu s CDO. Vyvinuli jsme tedy oceňovací program v MS Excel VBA založený na jednofaktorovém modelu gaussovské kopule, s jehož využitím pak aplikujeme citlivostní analýzu a modelujeme cenu tranše CDO před prvními zásahy současné finanční krize a po nich. Získáváme tak ztrátu investorů v důsledku změny očekávaných hotovostních toků.

Přicházíme na čtyři hlavní nedostatky trhu s CDO. Za prvé, diverzifikace podkladových aktiv byla na nízké úrovni. Za druhé, pochopení modelu nebylo úplné, což vedlo ke špatnému ocenění CDO. Za třetí, v důsledku toho byla špatně oceněna i korelace mezi aktivy. Dále pak na základě výpočtu demonstrujeme čtvrtý nedostatek - povinnost oceňovat aktiva podle jejich tržní hodnoty, což může mít ničivé důsledky. Navrhneme také doporučení pro odstranění těchto nedostatků.

Project of the Thesis

Term of master examination: June 2009
Author: Petra Benešová
Leader of the thesis: PhDr. Petr Teplý

Preliminary title: Collateralized Debt Obligation: Valuation and Sensitivity Analysis

Characteristics of the theme:

CDOs belong to a group of asset-backed securities (ABS). By issuing CDO a bank transfers a credit risk of a pool of assets (bonds) off its balance sheet. A CDO investor accepts this risk in exchange for a regular coupon payment.

As quick was a rise of a volume of CDOs issued since the beginning of the century as dramatic was its fall since 2007. Nowadays we see CDOs as one of the main causes for massive mark-to-market losses or even defaults of many financial institutions. The aim of this thesis is to contribute to understanding of this structure and to clarify its valuation principles, sensitivity to different market scenarios and main risks.

Based on the risk profile CDO is divided into tranches, whereas in case of a default of an underlying asset an owner of a lowest (junior) tranche loses his payment as a first one. If a percentage of underlying assets' defaults exceeds predefined bounds, holders of higher tranches lose their coupon payments subsequently. In exchange for a relatively high risk related to lower tranches and a high leverage, investor obtains a higher risk premium over the swap rate.

In a CDO valuation Copula Functions including Factor Models or Monte-Carlo Simulation are used to distinguish a correlation matrix among the underlying assets and to evaluate the premium payment for each tranche.

In this thesis we firstly introduce basic principles of securitization and incentives for CDO issuers and holders. Then we present a model used for a CDO valuation – a One Factor Gaussian Copula Model and apply it to a concrete pool of underlying assets. Consequently, we implement comparative static and sensitivity analysis to parameters (e.g. correlation, hazard rate, width of tranches or number of passed defaults). Finally, we simulate a value of a CDO as before the financial crisis and compare it with a current model value to show a loss of CDO investors based on a change in expected cash-flows.

Basic outline:

1. Introduction
2. Basic principles of CDOs
    a. Definitions
    b. Synthetic CDOs and CDO indices
    c. Main risks and current financial crisis
3. CDO valuation
    a. Fair premium determination
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    c. One Factor Model
4. Methodology
    a. Assumptions about entry parameters

        b. MS Excel VBA valuation process
5. Results of the model
    a. Sensitivity analysis
        i. Hypothesis 1: The higher the asset correlation the lower the risk premium for a junior tranche and the higher the risk premium for a senior tranche
        ii. Hypothesis 2: Base correlation is more stable measure of correlation than implied correlation
        iii. Hypothesis 3: Higher hazard rate increases the premium of all tranches more than proportionally
    b. Mark-to-market loss evaluation
        i. Hypothesis 4: Correlation and hazard rate changed substantially over the past year and a half
        ii. Hypothesis 5: There has been a substantial loss even on the most senior tranche without a necessity to be hit directly by a default
6. Conclusion

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In Prague, September 20, 2008

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1 Introduction

A collateralized debt obligation (CDO) is a credit derivative structured product which enables a transfer of a credit risk of a portfolio of assets from its issuer (protection buyer) to an investor (protection seller)[1]. A CDO can thus be conceived as a form of insurance against a credit event of any of the underlying assets. Moreover it is a structured product. Upon investor's risk profile and risk aversion the investor may choose among a number of CDO tranches. By investing in lower tranches the investor bears the risk of first couple of defaults, the highest tranche on the other hand usually entails the risk of more than 15% of defaults.

A volume of CDOs issued started to boom in 2004 (97 bn. USD) and it multiplied more than four times and reached its top in 2006 (445 bn. USD[2]). In 2006 the CDO market was prospering and the initial issuer purpose of credit risk elimination was replaced by arbitrage, i.e. profit motives. The prices of such contracts were sinking because of a high demand. The highest tranches often earned the highest possible score from independent rating agencies; therefore it was considered a very safe investment.

However, with the first signs of US mortgage crisis, the issuance of CDOs fell dramatically and the premiums the issuers were willing to pay for a credit protection skyrocketed. At the end of 2008 the CDO market was frozen and 67% of the CDOs issued since late 2005 to middle 2007 were in formal state of default[3].

It started by the mortgage crisis, when mortgage backed securities' investors suffered losses. Since many CDOs were linked to these investors their tranches were downgraded. A spiral of losses was then created. Downgrades of CDO tranches created losses of CDO investors. CDO investors holding distinct positions on CDOs were downgraded. And as many CDOs were linked to these investors, further CDO tranches had to be downgraded triggering further losses.

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[1] Fabozzi (2008)
[2] *www.abalert.com*
[3] Thomson Reuters credit news on 22/10/2008

Finally, many institutional investors suffered massive writedowns (e.g. Citigroup, UBS or KBC). Many of them exploited a government bailout to be saved from total bankruptcy (e.g. AIG or Northern Rock), some were acquired by a stronger competitor (e.g. Merrill Lynch, Bank of America or Bear Stearns).

In this thesis our aim is to contribute to the understanding of a CDO with a new light it was spot on it during the current financial crisis. We claim that it is possible only by a deep comprehension of its valuation. CDO is an advanced structured product and the models used for its valuation are very complex, therefore the investors often relied on a rating of a rating agency without a proper understanding of the model. After a clarification of the valuation model and in combination with a course of events since late 2007 we will be able to specify and demonstrate recent weaknesses of the CDO market and provide recommendations for future existence of CDOs.

In Chapter 2 we introduce the main principles of a CDO to reach an understanding of CDOs motivation, functioning, risks and various types. Chapter 3 is devoted to an introduction of a valuation model. Recall that our aim is to clarify the CDO valuation, therefore we will use a relatively simple model which is not that demanding in terms of sophisticated mathematical and probability methods. The One Factor Gaussian Copula Model is a basic model, it thus does not provide as exact results as other models but it is understandable, straightforward and it correctly illustrates the main sensitivities and regularities of a CDO.

Chapter 4 develops the theoretical concept presented in Chapter 3. It shows how to practically implement the valuation by an introduction of simplifying assumptions, description of entry parameters and a brief explanation of the process of CDO valuation using VBA in MS Excel. Finally, in Chapter 5 we test five hypothesizes we proclaimed. We show the results of the valuation, demonstrate the main sensitivities to entry parameters of the model and compare their values before the first credit events with current values. Based on these outcomes we detect main flaws of the CDO market before the crisis and pronounce recommendations that are necessary to ensure more successful future of CDOs.

2 Basic Principles of CDOs

In this chapter our intention is to introduce basic principles of a collateralized debt obligation (CDO). We will make the reader familiar with its properties, functioning, purpose, associated risks and its role in the current financial crisis.

2.1 Definitions

Initial purpose of a CDO was to transfer credit risk linked to a portfolio from an issuer of a CDO to an investor. See below main definitions of a CDO terminology:

- *Protection buyer*: a CDO issuer/originator/sponsor
- *Protection seller*: a CDO investor who absorbs a credit risk of a reference portfolio
- *Reference entity*: a security, which is included into a reference portfolio - a portfolio with a credit risk transfer within a CDO contract
- *Protection payment - premium*: a regular payment carried out by an originator to an investor in exchange for a credit risk removal
- *Credit event*: an event defined in a prospectus of each CDO which defines a default of a reference entity and which triggers a one-off compensation payment from a protection seller to a protection buyer
- *Loss payment*: a compensation payment from a protection seller to a protection buyer in case of a credit event

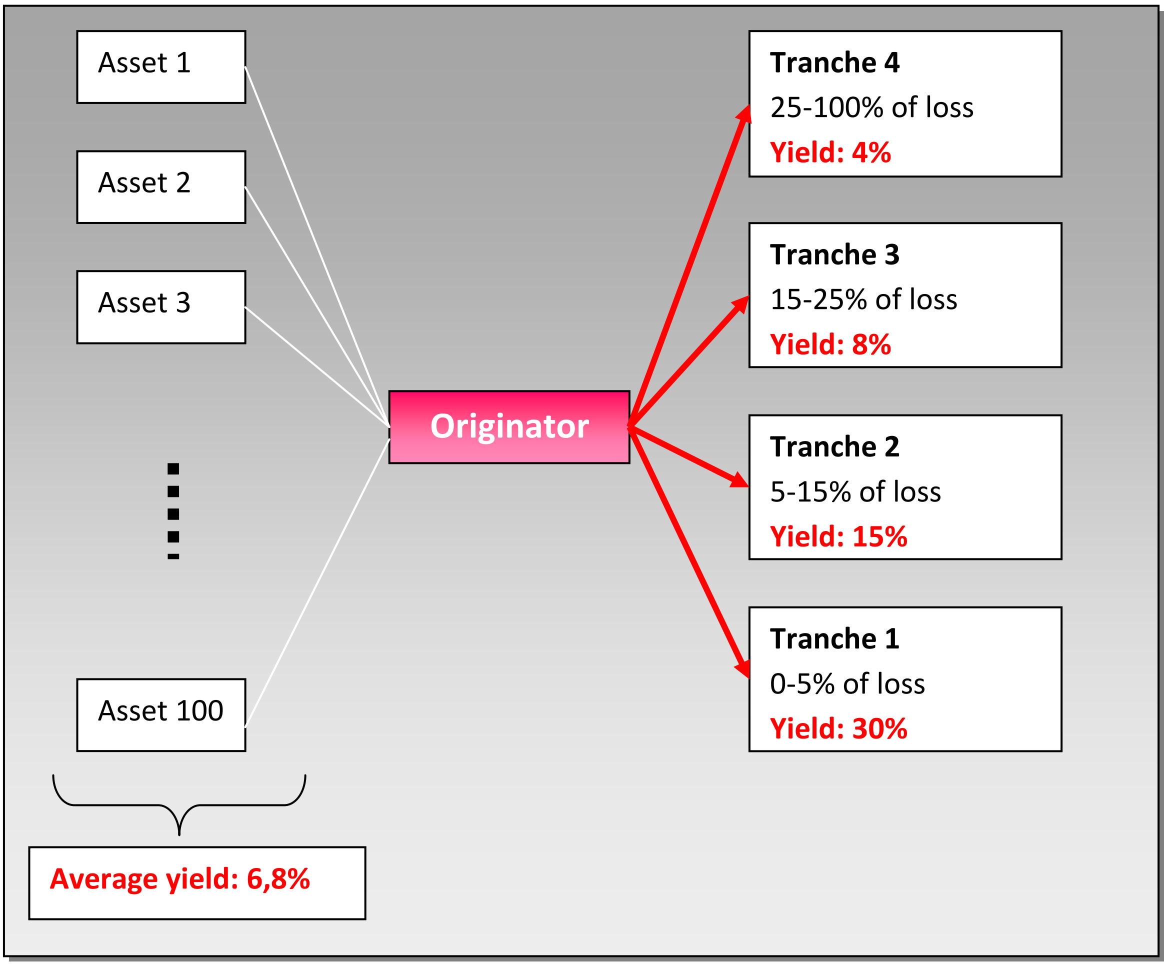
CDO can be though understood as a type of insurance against default. It is a contract between an originator and an investor with specified maturity in which the originator commits to pay the investor regular premium payment until *maturity*. The investor in exchange promises to bear all the credit risk. In case of no default until maturity an originator continues to regularly pay the investor the premium. In case of default the investor compensates the originator the loss the originator suffered.

Moreover CDO represents a product with very diverse risk structure. It offers the investor to choose the amount of credit risk he would absorb based on his risk profile and appetite. This feature lies in *CDO tranching*. The basic principle is sketched in the scheme below.

Imagine an underlying portfolio of 100 securities with the same nominal. CDO in the scheme is divided into four tranches each absorbing a credit risk after all tranches lying below it are hit. The investor into a first tranche is hit first, i.e. immediately when a first default occurs. He compensates the originator for a first five percent of defaults. The second tranche investor doesn't have to make any loss payment to the originator in case of no more than 5 defaults, his tranche is not hit. He compensates the losses of more than five and less than 16 defaults. The same principle holds for higher tranches: they are hit even after all more junior tranches loose 100%.

Each tranche of a CDO is therefore uniquely defined by its *attachment point* and *detachment point*. The attachment point defines the highest percentage of defaults where the tranche is still not hit. The detachment point denotes the lowest percentage of defaults where the whole tranche is hit. E.g. for a (3-7)% tranche 3% is the attachment point and 7% is the detachment point. I.e. if there are 3% of defaults among the underlying assets, the tranche is still not hit. After any further default the tranche suffers a first loss. And similarly, when 7% of underlying assets default the tranche investor just looses 100% of his investment. Any further default doesn't affect him and with lower number of defaults he loses less than 100%.

Figure 1: **Basic structure of a CDO**



Source: author's calculations

The higher the risk the higher the premium received from the originator. Instead of investing in the whole portfolio yielding 6,8% to maturity, the structure allows the investor to choose a specified tranche with much more concrete risk profile and corresponding return.

A CDO integrates a CLO (Collateralized Loan Obligation) where the underlying are wholesale or corporate loans and a CBO (Collateralized Bond Obligation) with underlying bonds. After its issuance, each tranche is usually classified by a rating from an independent rating agency. The lowest tranches are called *junior tranches*; the highest are *senior tranches* and in between are *mezzanine tranches*. Before the recent credit crunch a CDO represented an attractive alternative for a credit investor as it offered higher liquidity and lower bid-offer spreads than a direct bond investment.

Example 1: **CDO Junior Tranche Investor**

- Investor 1 invests 10M USD in a Junior tranche which absorbs 0-5% of loss
- 100 reference entities, weight of each in the reference portfolio is 1%
- CDO maturity is 5 years
- Tranche spread is 1200 bps
- Premium payment: yearly

The premium payment is quoted as a percentage of the investment amount and is usually expressed in basis points. The investor receives each year 1,2M USD from an originator (10M.1200bps).

1) <u>In case of no default</u> the issuer makes five annual payments each amounting 1,2M USD to the investor. The investment yields 12% p.a. to the investor.
2) <u>In case of one default</u>, the tranche is hit. Investor 1 delivers in case of 1-5 defaults, 1 default represents 1/5=20% loss of his initial investment. In other words, one default represents 1% loss on a reference portfolio which amounts 200M USD (10M.1/(5%)). Therefore the investor pays 2M USD to the originator. This also transfers to a 20% loss of a notional in the Junior tranche and investor's further premium reduces to 0,96M USD (1200bps.(10M-2M)).
3) <u>In case of five defaults</u>, the protection buyer pays 10M USD of a loss payment to the protection buyer, which constitutes 100% of his initial investment. The whole reference portfolio loss of 5% though transfers to a 100% loss for a junior tranche investor. There is a 20 times leverage.

For a better illustration of effects of a CDO tranching let us compare the previous example with the following one, where a protection seller invests the same amount into the same reference portfolio CDO, but imagine that the CDO is not tranched. Investor has to make a loss payment in case of any positive number of defaults. Such CDO naturally offers a lower premium.

Example 2: **CDO investor – no tranches**

- Investor 2 invests 10M USD in a CDO which is not tranched
- 100 reference entities, weight of each in the reference portfolio is 1%
- CDO maturity is 5 years
- CDO spread is 250 bps
- Premium payment: yearly

Investor receives 0,25M USD each year.

1)In case of no default the issuer makes five annual payments to investor each amounting 0,25M USD. The investment yields 2,5% p.a. to the investor.

2) In case of one default the investor suffers 1% loss on his portfolio. He makes a loss payment of 0,1M USD (1%.10M). The notional lowers from 10M to 9,9M USD. Since then the investor receives a premium on a reduced notional (9,9M USD). The premium payment is 247 500 USD.

3) In case of five defaults the investor pays 0,5M USD. New notional is 9,5M USD and premium payment reduces to 237 500 USD annually. By a loss payment the investor looses 5% of his initial investment.

Comparison of Examples 1 and 2 is crucial for an illustration of a high leverage associated with a CDO. Default of one reference entity immediately results in a loss of 1% of initial investment for Investor 2 whereas it is 20% for Investor 1 in Example 1. The loss is 20 times higher for Investor 1. After 5 defaults, this investor lost everything whereas Investor 2 lost 5% of the invested amount.

The main economic drivers as e.g. Duffie and Garleanu (2001) note are therefore leverage and diversification.

Leverage

As illustrated above the leverage is a source of high and excessive returns a CDO offers to investors. On the other hand it causes a very high risk for the investor as it multiplies both gains and losses.

Diversification

Is necessary and essential any time leverage is present. It contributes to its decrease. It is indeed not very applicable in cases where an originator really holds some risky, low quality and low liquidity assets and a CDO issuance represents a way how to put these securities off the balance sheet.

2.2 Types of CDOs and Main Features

In this subchapter we will present main CDO division and basic features of each type of CDO. The reason for it is to demonstrate how massively the CDOs developed during their short existence. The structure of today's CDOs is quite sophisticated despite the fact that their main expansion started no more that ten years ago. The division presented here is based on Fabozzi (2008).

2.2.1 Based on Mode for Assets' Acquisition

a) Cash:

- The originator of a cash CDO holds the underlying assets.

b) Synthetic:

- The originator doesn't hold the underlying assets, the CDO is created synthetically.
- It is created such that a CDO originator chooses the underlying assets upon assessment of some parameters (e.g. required risk, return, potential leverage, proper diversification) and buys a credit default swap (CDS) to them on the market[4].
- The result is an unfunded CDO.
- Typically, the CDO is not purely unfunded, the originator usually provides minimum funding required for the most junior tranche to reach a sufficiently high rating or for the most senior tranche to reach a AAA rating. The rest is acquired synthetically.

[4] Credit default swap (CDS) is a credit derivative where credit risk of a single asset is transfered from a CDS seller to a CDS buyer. The principle of regular protection payments until CDS maturity in exchange for a compensation in case of default of the underlying is the same as for CDO. CDS premium is called CDS spread and its magnitude is often used to assess quality of the asset. Though CDS market spread is a useful measure of how market prices the riskiness of any asset.

2.2.2 Based on Underlying Assets

As explained above the CDO pools wholesale or corporate loans and bonds. According to a risk profile and a type of these underlying assets we distinguish:

a) High-yield CDO
b) Investment CDO
c) Emerging market CDO
d) CDO squared
    - A CDO where underlying is composed from other CDOs (called subCDOs) or concrete tranches pooled from the market.
    - Such CDO resecuritizes already securitized assets. The result is even higher leverage and risk.

e) Primary market CDO
    - The loans used as underlying are newly created for the purposes of the CDO.

2.2.3 Based on Change in Underlying

a) Static

- CDO underlying is not managed. It is the same from the inception until maturity of a CDO

b) Managed

- A CDO manager decides about an acquisition of a new asset during the life of a CDO in case of repayment or prepayment of an existing one. He may also decide to sell some asset and substitute it by another one.
- Managed CDOs are less transparent and embody higher administrative costs and management fees.
- There are tests that have to be fulfilled before any change of an underlying is implemented.

2.3 Motives for an Originator

CDOs are related with many internal and external costs such as cost to pay structurers, lawyers and CDO managers (in case of managed CDO), monitoring costs or administrative costs.

We already mentioned that the original purpose for a CDO initiation was a transfer of a credit risk of a portfolio of often low quality and low liquidity assets. Since then besides cash CDOs new types of CDOs such as synthetic CDO, CDO squared, primary market CDO or managed CDOs developed. So what are the main motives for a CDO issuance nowadays?

There are two main purposes to issue a CDO of any type mentioned above:

1. Capital relief and liquidity
2. Arbitrage opportunity[5]

Capital relief or liquidity is a ground for a so-called *balance-sheet CDO* creation. The balance-sheet CDO can be either cash or synthetic. The cash balance-sheet CDO is designed mainly for liquidity purposes. As already discussed, it presents a best way how to transfer credit risk of a portfolio of low quality assets (e.g. low-grade emerging markets' assets) which can't be easily sold for a reasonable price due to low liquidity and high bid-offer spread.

On the other hand, purpose of the synthetic balance-sheet CDO's is the capital relief and risk management purposes of an originator. Securitization reduces a regulatory capital required under Basel II. After a creation of a synthetic balance-sheet CDO the underlying assets are not included into a minimum capital requirement. Only the retained assets are included. E.g. in case of retention of a most junior tranche (0-3%), the assets included in a regulatory capital calculation amount only 3% of the CDO notional.

Nevertheless the main purpose for a CDO creation in recent years has been rather the arbitrage opportunity. Again, *arbitrage CDO* can be both synthetic and cash but the main purpose here is same for both types – to reach a profit. A CDO originator receives either a regular coupon payment from the bond issuer in case of cash CDO or a premium from acquired CDSs. On the costs side he pays investors a regular

[5] Das (2005)

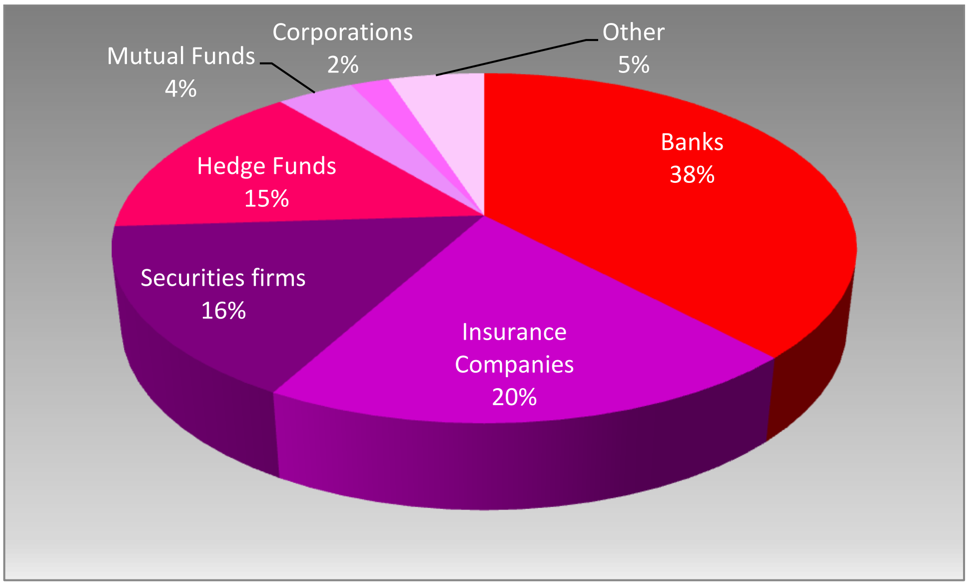
premium payment. The difference between the costs and the revenues represents an arbitrage profit.

A yield of an underlying bond shows an implied default rate as market prices it. On the other hand the CDO pricing unwinds from a rating of a rating agency which assesses the expected default rate based on probability distribution of a downgrade of a bond. A CDO manager's task is to detect those bonds where an implied default rate is higher than an expected default rate and pool them in a CDO.

Duffie and Garleanu (2001) however note this is not a riskfree – arbitrage - profit as on the costs side the counterparty (investor) is different than on the revenues side (issuer, CDS seller). The sponsor risk therefore persists.

Following chart shows percentage representation of protection buyers in 2004 according to notional volume of underlying assets on which the protection was sold. It includes CDOs as well as CDSs. The figure shows that protection was duly bought mainly by banks and insurance companies.

Figure 2: **Protection buyers in 2004**



Source: White (2005)

2.4 Motives for an Investor

Investor can be an initiator of a CDO inception. Globally, CDO offers standardization, liquidity, diversification and solid return. It is a clearly defined structure that guarantees a unified methodology, structure, rating devices, same enhancement levels and easily observable performance. CDO is simple to enter and the investor reaches an access to a whole portfolio of diversified assets in one trade. Moreover upon assessment of his risk aversion he selects the most appropriate tranche.

As the number of underlying usually ranges from 50 to 150 the structure is more transparent that in case of other credit derivatives linked to hundreds and thousands of loans or mortgages. Investor can assess every single underlying bond and evaluate a credit quality of a company.

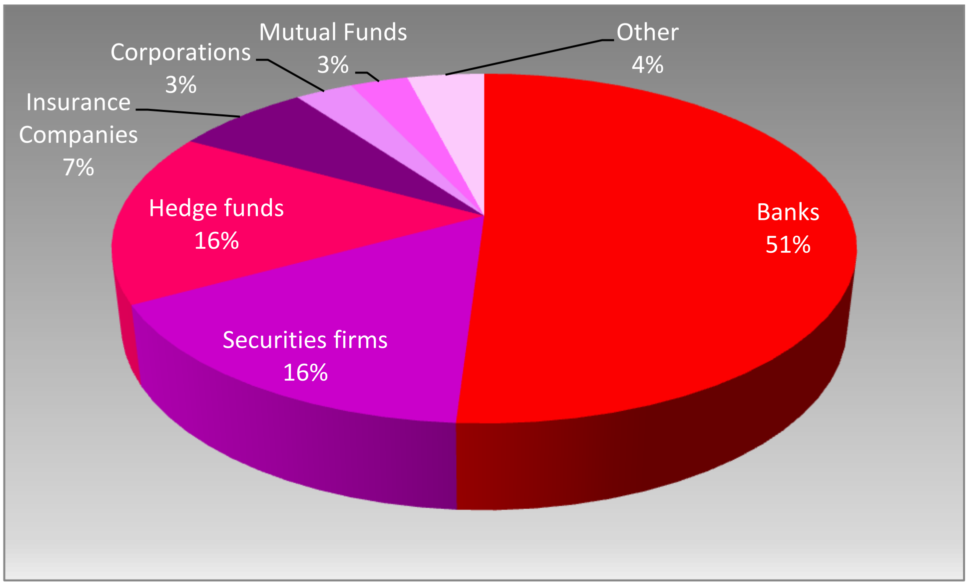
In times of CDOs main expansion, the yield of a tranche was usually higher that the yield of a bond with the same rating. According to UniCredit SpA analyst in Munich in 2006 average yield of a AA European corporate bond has been 9 bps over the money-market rates compared to 50 bps for a AA CDO tranche with European underlying[6]. Therefore CDO was considered very attractive. Recently, CDO spreads have widened sharply due to the credit crisis and their liquidity dropped. The risk associated with a CDO started to be taken more seriously.

CDO represents a structured investment. Its credit quality ranges from a high rated investment (e.g. AAA by S&P) over intermediate rated investment (A or B rated by S&P for mezzanine tranches) to equity investment (unrated). Creation of a AAA rated tranche can be very attractive as there are not many highly rated entities on the market. On the other hand the equity tranche produces a high leverage appreciated by some investors. Motives for AAA tranche holder are distinct from an equity tranche holder. As discussed in the fifth chapter, a sensitivity of a value of a tranche to entry parameters such as pairwise correlation between underlying assets, hazard rate or width of a tranche differ from tranche to tranche and have even opposite character.

[6] *www.bloomberg.com* (news 22.10.2008)

In the next figure we show main investors in CDOs and CDSs in 2004 according to notional volume of assets on which the protection was bought. Banks invested in more than a half of CDSs and CDOs issued. The reason was a high rating of senior tranches which made CDOs generally trusted instruments guaranteeing stable and nearly sure profit. Consequently, banks were not surprisingly first hit in the credit crunch and some of them were forced to default. Also note from this and previous graph that these credit derivatives are nearly exclusively in the interest of financial institutions, corporations represent less than 4% of CDO investors.

Figure 3: **Protection sellers in 2004**



Source: White (2005)

2.5 Synthetic CDO

As outlined above a synthetic CDO is an unfunded CDO where the underlying assets are not factually owned by an originator but they are acquired by selling CDS to chosen assets. The motive for a synthetic CDO originator is thus not a credit risk transfer. For a balance-sheet synthetic CDO the motive is risk management or capital relief. For an arbitrage synthetic CDO the motive is to reach a profit in excess spread.

In the next chapters we will follow only this type of a CDO not only because it markedly outweighs a cash CDO in terms of volume issued but also because the valuation and motives for its issuance are less complex and more straightforward. A cash CDO is initiated by the originator to insure against the default of owned assets.

The underlying is therefore not diversified; it is composed of low grade, illiquid assets, unequally represented in the CDO. The CDO manager is motivated differently, for a valuation of a cash CDO we would have to take into consideration more aspects such as microeconomic environment, competition within the sector or even game theory principles[7].

A creation of a synthetic CDO is depicted in Figure 4 and explained in detail with a use of Example 3.

Example 3: **Synthetic CDO Creation**

Intended parameters of a CDO

- Underlying assets volume - 100M USD
- CDO maturity: 5 years
- Four tranches
- AAA rating for the highest tranche

An originator chooses 100M USD of underlying assets on the market. He sells protection to these assets by a credit default swap to a protection buyer. That's how he synthetically acquires the assets.

The originator works out the probability distribution of default. Minimum internal funding required to reach the desired rating is 2M USD. Moreover the probability of default of more than 30% of an underlying is less than 0,01%. Such certainty is reviewed enough by the originator. Following this reasoning he decides to tranche 30% of a portfolio with 2% enhancement, i.e. first 2% of defaults are accumulated by the originator.

Then he creates a special purpose vehicle (SPV) and buys protection from it using CDS in total value of 28M USD over the first 2M USD. SPV creates four equally large tranches (each 7M USD) and sells a credit linked note (CLN) to each tranche, i.e. it buys credit protection from CLN investors. The proceeds from this operation is used by the SPV to purchase stable high-grade securities (such as government bonds) with low but nearly certain yield.

[7] Fabozzi (2008)

The costs and revenues are therefore:

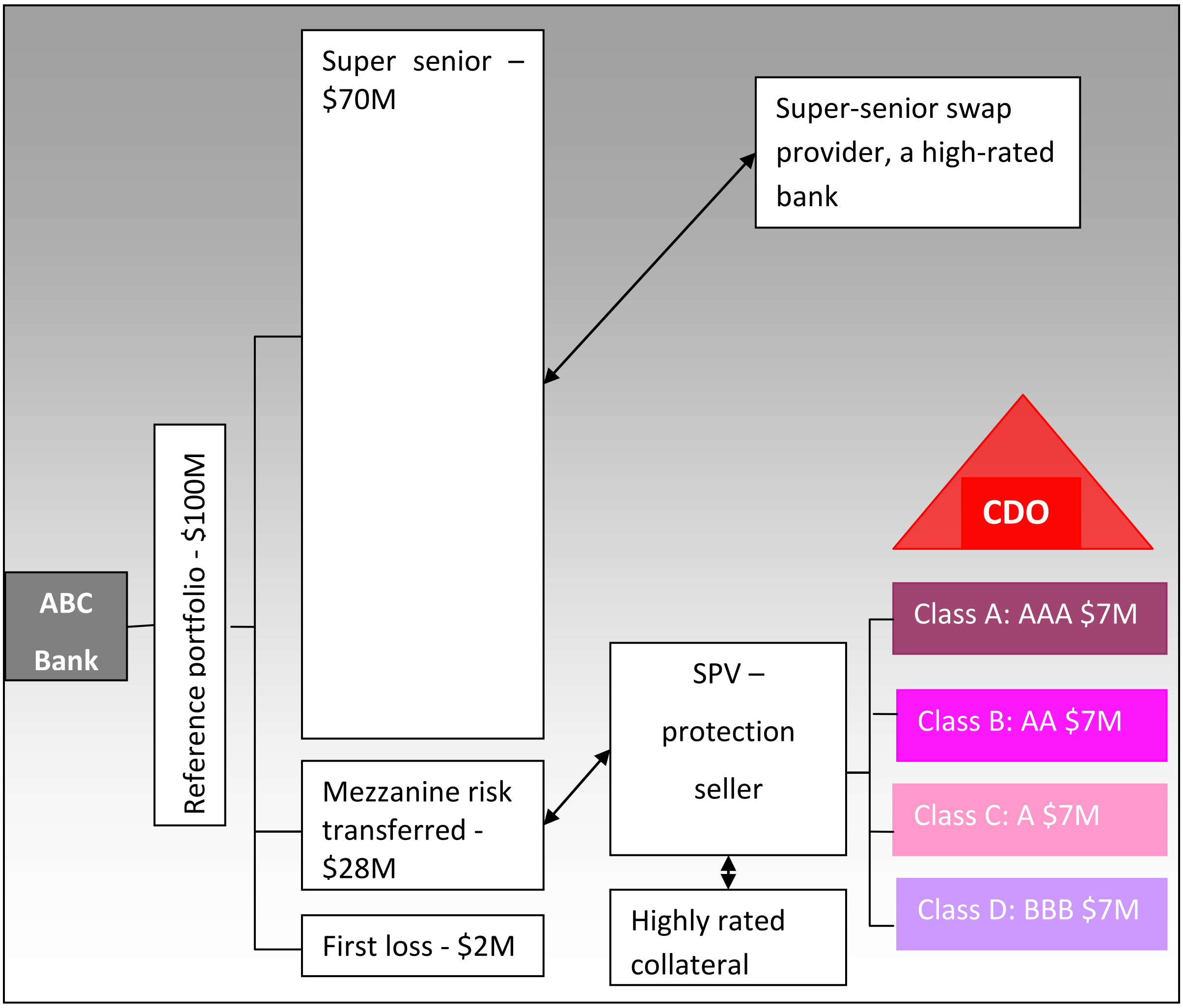
- For the SPV:
    - *Revenues*: premium paid from the originator to 28M USD CDS, coupon from high-grade securities in total volume 28M USD
    - *Costs*: premium paid to CLN investors
- For the originator:
    - *Revenues*: premium from a CDS to securities pooled on the market
    - *Costs*: premium paid to SPV as a protection payment. These costs compensate the SPV for the negative carry (i.e. purchased high-grade securities yield less than what is paid to CLN investors).

Originator now has to absorb first two percent of default and is protected against next 28% of defaults. If this is not sufficient for him, he can sell so called "super-senior" swap to the rest 70M USD notional. By doing so he buys protection to the rest of the notional. This CDS represents a loss for its buyer even after 100% of the highest CDO tranche defaults. And as the highest tranche gained AAA rating, this CDS is called "super-senior". Of course, there exist no official and acknowledged rating score above AAA so the term is not right. Nevertheless, the premium paid on this "super-senior" tranche is very low and doesn't represent any significant costs for the originator.

- In case of no default: the market, originator, SPV and investors continue to pay the abovementioned costs and receive revenues until maturity. At maturity, all swap positions are liquidated; the SPV sells the high-grade securities and pays the investors their notional.
- In case of a default: If the default doesn't exceed 2% of pooled assets volume, it is absorbed by the originator. Neither the SPV nor investors are hit. In case of one more default, the originator as protection seller on one side has to make a loss payment to the CDO tranche holder. The SPV though sells required part of the high-rated collateral and uses the proceeds to pay

the originator. It also decreases the notional for tranche D on which the premium to investors is paid. At maturity the D tranche investor receives lowered principal. After tranche D is fully erased, tranche C is hit and so on.

Figure 4: **Synthetic CDO**



Source: Fabozzi (2008)

Example 3 offers a nice illustration of a low funding needed for a CDO creation. To transfer a credit risk on a portfolio of 100M USD the originator needed only 28M USD of external funding. The synthetic creation of a CDO is more flexible, the assets can be chosen after the risk and default probability assessment. Moreover, the transfer of assets to the SPV is transparent and easier in terms law accordance and accounting as underlying assets are purely synthetic.

2.6 CDO Indices

CDO is an over the counter product. CDO indices were established in times of CDO trades' volume growth to achieve standardization in CDO trading.

There are two main groups of CDS indices:

1. CDX
2. iTraxx

CDX contains North American companies' names. It was established by CDS Index Company and marketed by Markit Group Limited. iTraxx contains European and Asian companies' names and it was introduced by International Index Company. iTraxx is therefore divided into iTraxx Europe and iTraxx Asia. In 2007 Markit Group acquired CDS Index Company and International Index Company, thus it now manages both the CDX and iTraxx[8].

Markit Group Limited is a financial information services company. In was founded in 2001 in order to provide credit default swap quotations' information on a daily basis. Main clients are subjects involved in credit risk trading and assessment - investment banks, hedge funds, asset managers, central banks, insurance companies, auditing firms, regulators or rating agencies. Markit is owned by 15 large banks, 4 hedge funds and its employees.

Both indices' families are composed of various indices representing different underlying assets. They vary according to geographical area, sphere of business and credit risk. An example of an index and all subindices is in the table below.

[8] www.markit.com

Table 1: **iTraxx Europe index family**

	Name	N of assets	
Benchmark	iTraxx Europe	125	Most actively traded names in the CDS six months prior to the index roll
	iTraxx Europe HiVol	30	Highest spread (riskiest) names from iTraxx Europe index
	iTraxx Europe Crossover	50	Sub-investment grade names
Sector	iTraxx Non-Financials	100	Non-financial names
	iTraxx Financials Senior	25	Senior subordination financial names
	iTraxx Financials Sub	25	Junior subordination financial names
	iTraxx TMT	20	Telecommunications, media and technology
	iTraxx Industrials	20	Industrial names
	iTraxx Energy	20	Energy industry names
	iTraxx Consumers	30	Manufacturers of consumer products
	iTraxx Autos	10	Automobile industry names

Source: www.markit.com

There is one main index called iTraxx Europe, which contains 125 names from different sectors with different risk profiles[9]. Then from iTraxx Europe the index manager chooses 30 highest CDS spread names and 50 names rated below investment grade (i.e. below BBB- by S&P) and creates iTraxx Europe HiVol and iTraxx Europe Crossover. Further there are indices representing different sectors of underlying assets.

Moreover iTraxx and CDX main indices are divided into tranches, which are also possible to trade. The tranches are defined by attachment resp. detachment points which are shown in the following table.

[9] For CDX it is CDX North America Investment Grade Index – CDX.NA.IG.

Table 2: **CDS indices tranches**

Tranche	CDX	iTraxx
Equity	0-3%	0-3%
Junior Mezzanine	3-7%	3-6%
Senior Mezzanine	7-10%	6-9%
Senior	10-15%	9-12%
Super Senior	15-30%	12-22%

Source: www.markit.com

Both CDX and iTraxx roll over each 6 months[10]. Prior to the roll date the contributing banks[11] determine the underlying assets for the new index issue. They choose 125 names on which CDS were most traded in past six months. On the day of issue a fixed coupon is determined based on CDS quotations of the underlying assets. The subindices are then identified. Then the index can be actively traded.

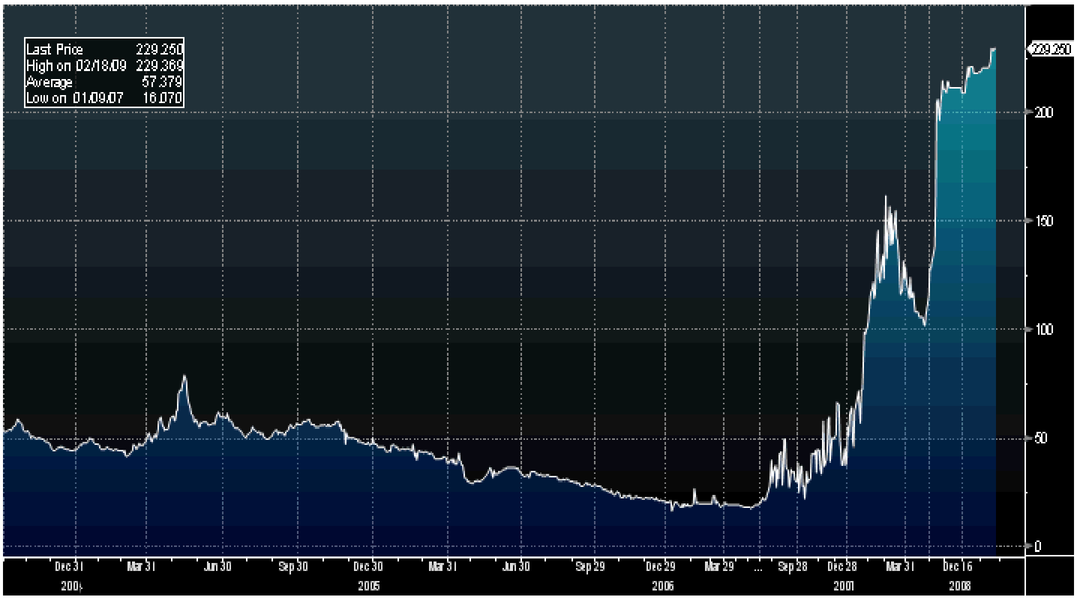
Each issue is denoted by a series number. For example the CDX most recent March 2009 issue is called Series 12. Moreover for each index maturities of 3, 5, 7 and 10 years are issued. The coupon is paid quarterly. Following figures show an evolution of iTraxx and CDX since 2004. These indices are used as a measure of credit risk, financial world affection by a credit crisis is therefore obviously perceptible. Figure 5 shows CDX Investment Grade Index[12] Series 3 market premium development since Series' 3 inception (September 2004). Figure 6 shows iTraxx Europe main index Series 1 since July 2004. Both indices have a maturity of 5 years.

[10] CDX rolls over each March 20 and September 20. iTraxx rolls over each June 20 and September 20.

[11] E.g. for CDX it is Bank of America, BNP Paribas, Barclays Cupital, Citibank, Credit Suisse, Deutsche Bank, Goldman Sachs, HSBC, JP Morgan, Merill Lunch, Morgan Stanley, RBS, UBS and Wells Fargo.

[12] It is the main index. The notation is following: CDX.NA.IG S3 5Y - CDX North America Investment Grade Index, Series 3, maturity 5 years.

Figure 5: **CDX IG 5Y index Series 3, September 2004 – February 2009**



Source: Bloomberg

Figure 6: **iTraxx Europe 5Y index Series 1, June 2004 – February 2009**

Source: Bloomberg

The advantage of the index over a single CDS is diversification. The advantage over a CDO is transparency, standardization and liquidity that guarantees lower bid-offer spread. Unlike CDO it gives a possibility to invest only in a selected geographical area or business sector. By investing in a tranche of an index CDO investors can proxy hedges against CDO tranches they hold. Index trades are also used to take positions and express outlook on a credit market. Indices are is also used as a benchmark for credit risk assessment.

The credit index trade can be either funded or unfunded. The funded trade is comparable to a purchase of a bond by a protection seller. For CDS index it is carried out using a CLN. The way of a funded contract implementation is similar to the one explained in Example 3. In case of a default a new version of the current Series of the index is introduced. The defaulted underlying is removed from the basket and a notional is appropriately decreased. Also, there is a cash settlement in case of a default and at maturity.

Nevertheless, the index trades are mostly unfunded. The following example shows how such trade is settled at inception, in case of a default and at maturity[13].

Example 4: **Unfunded investment in iTraxx Europe Series 10 on 22/9/2008**

Suppose an insurance company ABC which wants to invest 10M EUR in 5-year iTraxx Europe Series 10 (maturing in December 2013) in order to proxy hedge against its senior CDO credit portfolio.

On the roll-over date (i.e. 20/9/2008) the coupon was fixed at 120 bps. Now, two days later the index is trading at 120,56 bps. on the market. The difference between the market spread and the fixed one is compensated to the ABC by an up-front payment. The expected present value of 0,56 bps. until maturity is computed and compensated to the ABC at T+3 (25/9/2008). In this computation we must take into account the probability distribution of default so it is not that straightforward. Bloomberg summed it to 2446 EUR.

As the next coupon payment occurs in 3 months less two days and ABC was not holding the index in these two days the accrued interest of 667 EUR (10M.0,012/360.2) is deducted. Therefore on 25/9/2008 the ABC receives 1779 EUR.

[13] The example was taken from iTraxx Series 10 presentation on www.markit.com

Then ABC receives 120 bps per annum in quarterly payments from the market maker. In case of no default, it continues to receive it until maturity.

Now, let's assume one default after 3 years. As each assets' weight in a portfolio is 0,8%, ABC pays to the market maker 80 000 EUR. The market maker in exchange delivers the defaulted asset in 80 000 EUR nominal to ABC. The notional amount on which the premium is paid now reduces to 9 920 000 EUR and ABC receives an annual premium of 120 bps on this reduced notional until maturity.

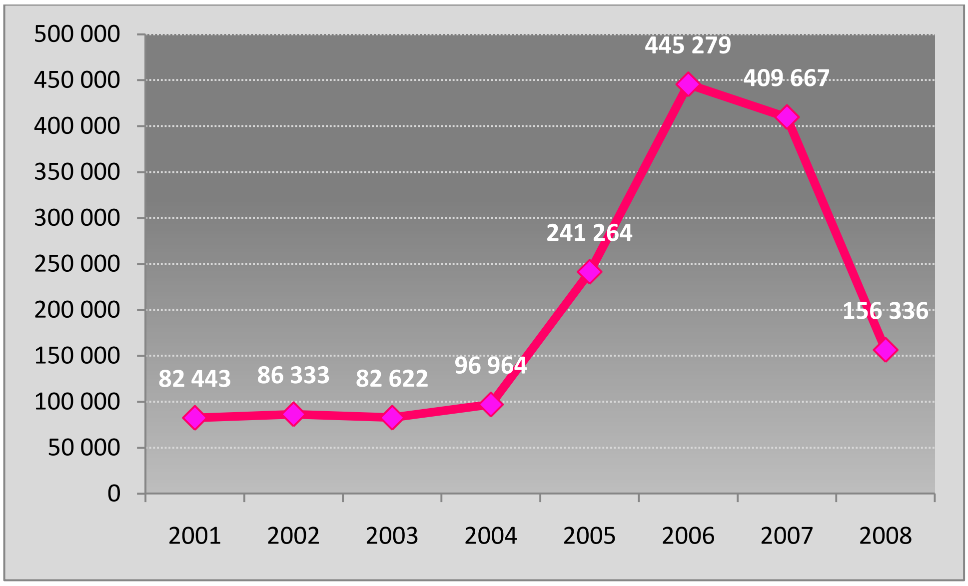
There is thus a physical settlement in an unfunded contract in case of a default.

2.7 CDO Issuance

CDOs were first presented in 1980s and their issuance registered an outstanding growth since 2001 when majority of CDO received a rating as the rating agencies became more familiar with rating CDOs[14]. In 2002 CDOs in total volume of 86 bn. USD were issued. In 2003 the volume slightly decreased to 83 bn. USD. Since then a sharp growth followed. The peak was reached in 2006 when a volume of 445 bn. USD was issued (i.e. more than five times the 2003 issuance). With a credit crisis which was noticed around mid 2007 in USA by mortgage loans' defaults and which spread around the world, the CDO issuance fell as dramatically as was its previous growth. In 2008 only 156 bn. USD was issued, which shows an annual fall by 62%.

[14] Fabozzi (2008)

Figure 7: **CDO issuance in millions USD**

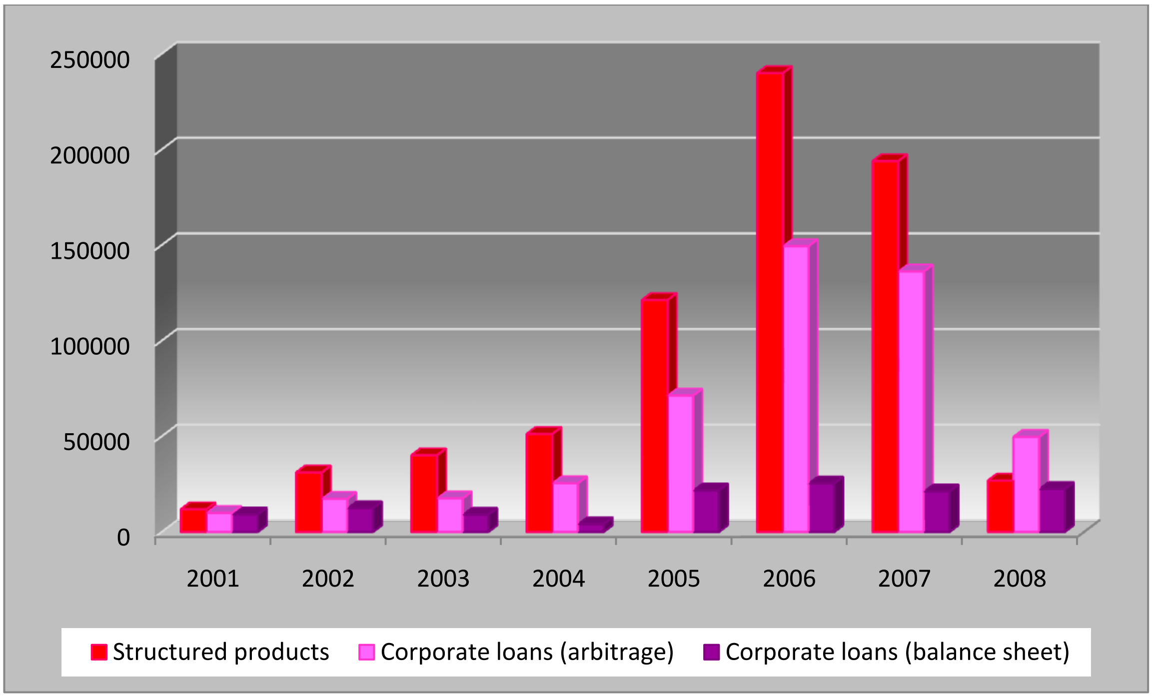


Source: www.abalert.com

It is also interesting to note that the data in Figure 7 show only a funded part of CDOs. For synthetic CDOs there is much less need of funding and the pool on which the credit risk is transferred is much higher (remember Example 3 where the funding was nearly four times lower that the pool of assets).

The structure of CDOs changed a lot during past 6 years. In 2006 most of them was issued due to arbitrage motives. There was a strong demand for a high leverage on the growing markets to outperform a return on equities' market. The leverage was assured by CDOs with other CDOs as an underlying. For a CDO squared the returns as well as risks are multiplied. As markets grew the risk aversion reached low levels and investor's appetite to absorb risk was higher.

Figure 8: **CDO issuance according to primary collateral type (in mio. USD)**

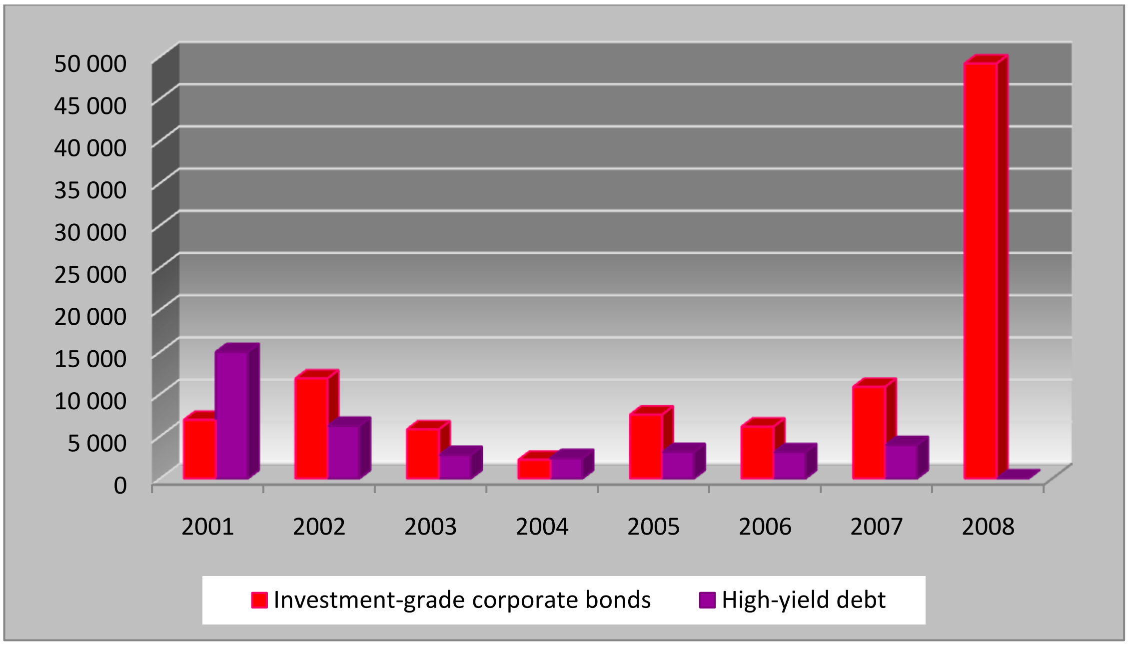


Source: www.abalert.com

Figure 8 shows the underlying assets of CDOs issued. The development is obvious. The main source of growth was structured products' CDO which comprises above all synthetic CDOs based on credit default swaps or CDOs squared. The issuance of these CDOs grew incredibly from 12 bn. USD in 2001 to 240 bn. USD in 2006 and fell back to 26 bn. in 2008 which is less than the 2002 level. The increase between 2001 and 2006 corresponds to 82% growth per annum.

Similar development was recorded by CDOs with corporate loans as underlying assets that were created for arbitrage purposes. The volume of balance-sheet CDOs backed by corporate loans has been more or less stable in recent years. As mentioned in section 2.3 the purpose for a creation of a balance-sheet CDO is either capital relief or liquidity. These motives are subject to issuer's needs. They are not purely profit oriented and therefore they are not so volatile over time or sensitive to credit market changes.

Figure 9: **CDO issuance according to primary collateral type (in mio. USD)**



Source: *www.abalert.com*

Figure 9 shows also a development of CDO issuance based on underlying assets. Here we see the allocation between investment-grade corporate bonds (with S&P rating BBB and above) and high-yield debt (rating below BBB). The graph perfectly illustrates the steep increase in risk aversion between 2007 and 2008. In 2008 the volume of high-yield debt based CDOs was 0.

2.8 Main Risks and the Crisis

CDO represents many risks for the investor. Besides the interest rate risk, cross-currency risk, ramp-up risk or reinvestment risk we would like to highlight the following two:

1. Correlation risk
2. Counterparty risk

By a correlation we mean the correlation between the defaults of underlying assets. The higher the correlation the more fragile is the whole structure. The importance of a correlation differs for different tranches' investors. For a senior tranche holder, the correlation is very important. One default doesn't bother him much, but what he cares about is the probability that there will be more defaults, if one underlying asset defaults. On the other hand an equity tranche holder is less interested in correlation.

Suppose he is hit in case of 1 to 3 defaults. Thereby it is of no importance for him if there will be in total 3 defaults until maturity or if these defaults trigger a series of other 10 defaults. In both cases he loses 100% of his investment.

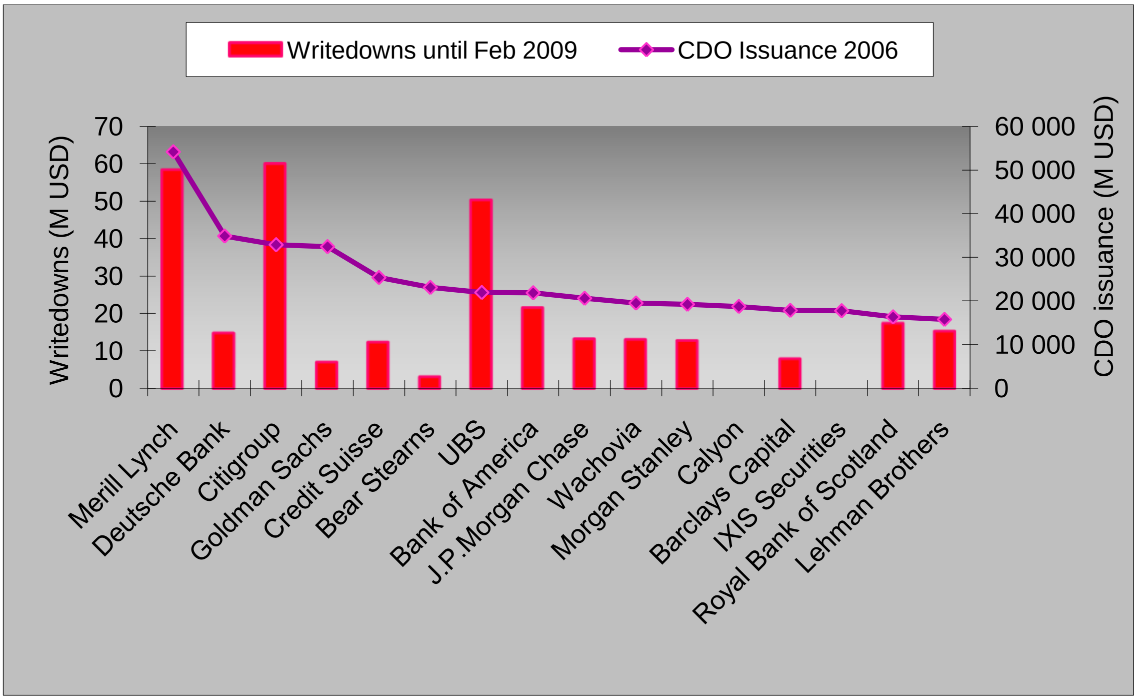
What's more - the correlation changes over time and it is dependent on macroeconomic conditions. In times of a recession the correlation between assets tends to increase, whereas it is low in times of growth[15]. This is a very important feature which is essential to understand when investing in a CDO. Correlation will be discussed in much more details in the following chapters.

The second risk we would like to discuss is a counterparty risk. A CDO investor is subject to a counterparty risk of both a CDO issuer and all underlying assets' issuers. Usually, each CDO tranche is classified by a rating. This rating however can be reviewed and changed by a rating agency at any time. And that's what each investor has to bear in mind. In previous years the rating agencies' assessment of risk was taken too reliably by all its users and they acted as if it was once given and irreversible.

A default of an underlying can cause a downgrade of all tranches of a CDO. Consequently, not only a junior tranche investor is hit by a default but also a senior tranche investor suffers a loss – a mark-to-market loss - as the spread of his tranche soars. Exactly this happened in the crisis and that's why CDOs are often labelled main cause of the unthought-of losses resulting in numerous defaults or financial difficulties of often large and stable companies and financial institutions. The threat of a downgrade of an asset and all its consequences based on numerical evidence will be discussed in Chapter 5.

[15] Kakodkar, A. et al. (2003)

Figure 10: **Top CDO Issuers and their writedowns (in millions USD)**



Ranking	Issuer	Volume
1	Merill Lynch	54 153,5
2	Deutsche Bank	34 866,1
3	Citigroup	32 871,1
4	Goldman Sachs	32 440,3
5	Credit Suisse	25 364,7
6	Bear Stearns	23 091,9
7	UBS	21 937,7
8	Bank of America	21 858,8
9	J.P.Morgan Chase	20 608,4
10	Wachovia	19 519,7
11	Morgan Stanley	19 225,4
12	Calyon	18 744,7
13	Barclays Capital	17 837,6
14	IXIS Securities	17 759,5
15	Royal Bank of Scotland	16 348,5
16	Lehman Brothers	15 776,7

Source: author, based on www.abalert.com

Figure 10 illustrates the above explained risks and their consequences. On the right x-axis we have issuers that issued more than 10 bn. USD nominal of CDOs in the most successful year – 2006. Merrill Lynch was evidently the first with 54 bn., followed by Deutsche Bank with 35 bn. and Citigroup with 33 bn. USD issuance. On the 16^{th} place we see the defaulted Lehman Brothers with 15 bn. USD. The values are shown in the table below the graph. The red slopes then show the writedowns of the institutions since mid 2007. Their value is on the left x-axis on the left. Citigroup with nearly 60M USD of writedowns is on a first place, Merill Lynch is the second and UBS the third.

These writedowns have their roots in a high volume of subprime mortgages offered recklessly to households with a low credibility in the United States. The mortgages were securitized into a mortgage backed securities (MBS) and sold to institutional investors. This way the credit risk of the mortgages was spread to the whole financial sector. After some mortgage defaults the institutions involved in this process were hit and some of them defaulted (e.g. Lehman Brothers or US mortgage agencies - Fannie Mae and Freddie Mac).

These defaults triggered first settlements of CDO contracts, downgrades of MBS holders, consecutive downgrades of CDO holders and massive writedowns of many counterparties included in MBS or CDO business. This also resulted in CDS spreads widening and further mark-to-market losses. Some CDOs had to be terminated before maturity creating even higher losses. Because of high interdependence within a financial sector and its strong link to all business spheres a series of problems of underlying companies led to a serious financial crisis the world is now experiencing.

In the following chapter we will explain the mathematical background for a CDO valuation to develop a CDO pricing program. Then we will observe CDO sensitivities to entry parameters to be finally able to better illustrate the features of a CDO and the risks that have been omitted and to highlight implications essential for any future of CDOs and similar products.

3 CDO Valuation – One Factor Gaussian Copula Model

In this chapter we will introduce One Factor Gaussian Copula model. It is a basic model used for CDO tranches' pricing. Its outcomes are though not as precise as by other recently introduced models[16]. However it is not a purpose of this thesis to present an exact and complex model as it requires sophisticated mathematical methods and deep understanding of advanced mathematics. The purpose here is to introduce a comparatively simpler model which accurately describes basic functioning of a CDO and its sensitivities to parameters. Nevertheless the model is still not trivial. The steps for a pricing are described in the following subchapters.

Basically, the main idea behind the valuation the same for all models is to set the spread of a tranche i.e. the premium such that the present value of premium payments equals to the present value of the loss payments so that total present value of the contract is zero. Both the loss payment and the premium payment depend on a number of defaults in the future and their timing, which further determines a time distribution of loss. As none of this is known, loss is a random variable and we have to determine its expected value.

Firstly, we should determine the probability of default of an obligor by time *t*. Then, having this distribution for each obligor and combining it with a correlation structure between the obligors, we identify the joint distribution function. Factor model enables us to solve the problem with a correlation structure between obligors. And the copula function approach introduces a tractable way how to cope with multidimensional distribution functions. After obtaining the joint distribution it is straightforward to deduce the probability of number of defaults in each time period and the loss distribution.

3.1 Fair Premium Determination

To value a CDO a present value of loss payments and a present value of premium payments has to be determined and set a premium has to be set such that they equal.

[16] Instead of using factor models CDOs can be priced using Monte Carlo Simulation – e.g. Duffie and Garleanu (2001). More exact outcomes can be given either by introducing more than one factor or modification of the factor distribution and copula form (Hull and White (2004), Wang, Rachev and Fabozzi (2006)). In addition these models use more advanced approach to entrance parameters form (Krekel (2008)).

For both these legs appointment we need to estimate the expected loss by each time period.

Each tranche of a CDO is defined by an attachment and detachment point - K_A and K_D. Attachment point denotes a percentage of defaults on an underlying portfolio where the tranche holder starts to incur a loss and a detachment point shows a percentage of defaults where he lost 100% of his investment and is not affected by any further default.

Denote also a cumulative loss on a tranche until time t $M(t)$ and cumulative loss on a whole portfolio until time t $L(t)$. Then

$$M(t) = \begin{cases} 0 & \text{if } L(t) \leq K_A \\ L(t) - K_A & \text{if } K_A \leq L(t) \leq K_D \\ K_D - K_A & \text{if } L(t) \geq K_D \end{cases} \qquad (1)$$

In order to determine $M(t)$ we therefore need to determine $L(t)$.

Suppose we have n obligors – $i = 1, \dots, n$. Denote recovery rate of i-th asset R_i and volume of i-th asset in an underlying portfolio A_i. Recovery rate shows a rate which the asset holder obtains from a defaulted asset (e.g. proceeds from selling the defaulter firm's assets in liquidation). The loss on an obligor in case of default would then be $L_i = (1 - R_i) \cdot A_i$. L_i though shows a loss given default (LGD) of i-th asset.

Denote a time-until-default of i-th asset τ_i. The asset defaulted by time t if $\tau_i \leq t$ and it survives if $\tau_i > t$. Let $N(t)$ be a cumulative default loss process which switches from zero to one if the default time of an asset is lower than time t, i.e. if an asset defaults.

$$N(t) = \sum_{i=1}^{n} 1_{\{\tau_i \leq t\}} \qquad (2)$$

$N(t)$ counts the number of defaults by given time. Suppose the same recovery for all obligors $R_i = R$ and the same volume of each asset in a portfolio $A_i = A$. Then also $L_i = L$ and loss on a portfolio $L(t)$ is

$$L(t) = N(t) \cdot L \qquad (3)$$

Expected percentage cumulative loss on a tranche $EL_{(K_A,K_D)}(t_k)$ until time t_k where $k = 1, \dots, T$ is now

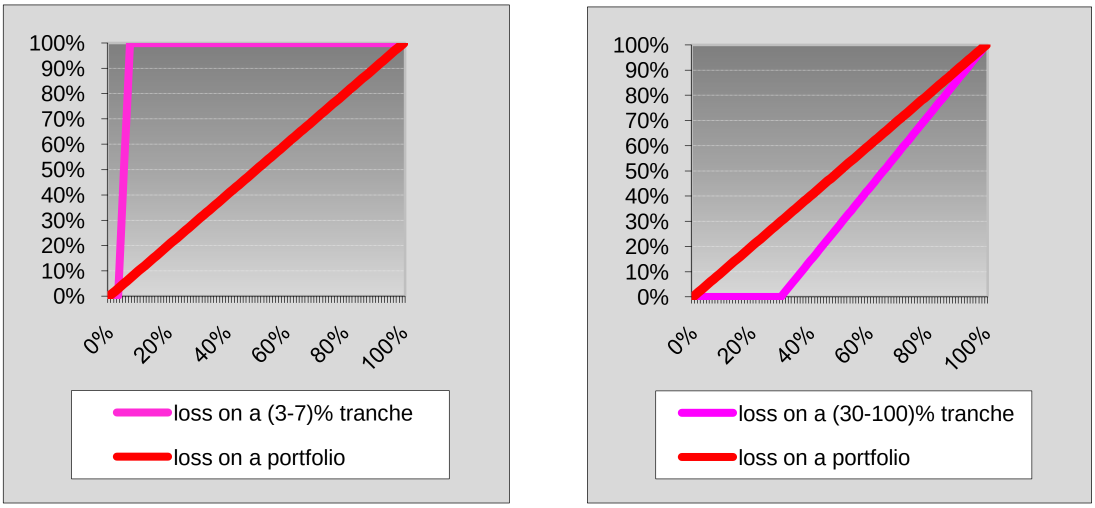
$$EL_{(K_A,K_D)}(t_k) = \frac{1}{K_D - K_A} \sum_{j=1}^{n} \left[min\left(\frac{L_j(t_k)}{A \cdot n}; K_D \right) - K_A \right]^{+} \cdot P(N(t_k) = j), \qquad (4)$$

where $L_j(t_k)$ denotes a cumulative loss on a whole portfolio in equation (3) with $N(t_k) = j$. Simply said, to reach $EL_{(K_A,K_D)}(t_k)$ we count loss on a tranche for all cases of $j = 0,1, \dots, n$ defaults $\left[min\left(\frac{L_j(t_k)}{A \cdot n}; K_D \right) - K_A \right]^{+}$ and sum them weighted by their probability $P(N(t_k) = j)$[17]. The expression in brackets after sum can thus be explained in the following way. If the rate of cumulative loss on a whole portfolio $\frac{L_j(t_k)}{A \cdot n}$ is lower than attachment point K_A, the tranche is not hit; the expression would be negative, so it is set to zero. If $\frac{L_j(t_k)}{A \cdot n}$ is between attachment and detachment point the tranche is partially hit and if it is above or equal the detachment point K_D the tranche is fully hit and its owner looses 100% of his investment.

Equation (4) evidences the leverage – one of the main features of a CDO explained in the previous chapter. The tranche's owner's loss is determined only by few percentage points of losses. E.g. a loss of a holder of a tranche with $K_A = 3\%$, $K_D = 7\%$ and 100 obligors is entirely determined by an occurrence of four defaults following first three. During these four defaults transforming in a loss of 4% on a portfolio the owner looses everything instead of just 4%. His loss after 7 defaults is 14 times higher than the loss on a portfolio (100/7). His tranche's loss depending on a loss on a portfolio is depicted in the following figure[18]. We can compare it with a holder of a more senior tranche. His loss is lower or equal to a loss on a portfolio but it is again more rapid once first default hurts him and other defaults follow.

[17] Note that we denote i an underlying asset and j number of defaults: $i = 1, \dots, n$ whereas $j = 0,1, \dots, n$.
[18] In the scheme we omit discrete character of number of defaults and consider it continuous, i.e. number of underlying assets approaching ∞.

Figure 11: **Loss on a tranche versus loss on a portfolio**



With a use of $EL_{(K_A,K_D)}(t_k)$ we can now determine a present value of premium and loss payment. We will treat time as a continuous variable. Suppose there are T payment moments left to a maturity of a CDO. The payments take place in time t_k with $k = 1, \dots, T$, where $0 = t_0 < t_1 < \cdots < t_T$. Then the present value of a premium payment leg in percentage would be:

$$PV_{Premium} = \sum_{k=1}^{T} B(t_0, t_k)(t_k - t_{k-1})V\left[1 - EL_{(K_A,K_D)}(t_k)\right] \qquad (5)$$

where V is the premium or spread percentage and $B(t_0, t_k)$ is the discount factor: $B(t_0, t_k) = exp\left(-\int_0^k f(0,s)ds\right)$, where $f(0,s)$ is the spot forward interest rate. $(t_k - t_{k-1})$ is the time spread between each two moments of payment; usually the spread is paid quarterly. The duration between the payment moments can also differ with a day convention we choose.

Note that with an event of default the nominal of the defaulted asset is deduced from the nominal of a portfolio. The premium V is always paid on an outstanding volume of a portfolio (here expressed in percentage) by time t_k $\left[1 - EL_{(K_A,K_D)}(t_k)\right]$. Check for example that in case of a percentage loss on a portfolio exceeding K_D no premium will be paid at all.

Present value of the default leg in percentage is deduced similarly:

$$PV_{Loss} = \sum_{k=1}^{T} B(t_0, t_k)\left[EL_{(K_A,K_D)}(t_k) - EL_{(K_A,K_D)}(t_{k-1})\right] \qquad (6)$$

where $EL_{(K_A,K_D)}(t_0) = 0$. Each period the CDO tranche's buyer has to compensate the seller the loss which occurred since the last period. The present value of loss payments is therefore a sum of the discounted differences between expected cumulative losses in consequential periods.

The premium payment V should be initially set so that $PV_{Premium} = PV_{Loss}$:

$$\sum_{k=1}^{T} B(t_0, t_k)(t_k - t_{k-1})V\left[1 - EL_{(K_A,K_D)}(t_k)\right]$$
$$= \sum_{k=1}^{T} B(t_0, t_k)\left[EL_{(K_A,K_D)}(t_k) - EL_{(K_A,K_D)}(t_{k-1})\right]$$

The fair premium is therefore:

$$V^* = \frac{\sum_{k=1}^{T} B(t_0, t_k)\left[EL_{(K_A,K_D)}(t_k) - EL_{(K_A,K_D)}(t_{k-1})\right]}{\sum_{k=1}^{T} B(t_0, t_k)(t_k - t_{k-1})\left[1 - EL_{(K_A,K_D)}(t_k)\right]} \qquad (7)$$

Our task is now to determine $EL_{(K_A,K_D)}(t_k)$ to which we need the derivation of $P(N(t_k) = j)$ for all *j* and t_k.

To determine the probability of *0,…,n* defaults by time *1,…,T* we first need to derive the marginal probability distribution of a time-to-default of each obligor τ_i. We will do so in section 3.2. Then, in sectio 3.3, considering the dependence structure between obligors we can derive a joint distribution function conditional on a value of a factor in One Factor Model. And finally, in section 3.4 we will determine the conditional probability of *j* defaults by time t_k on a factor value and by integrating it over all factor values we will finally determine $P(N(t_k) = j)$.

3.2 Determination of Marginal Distribution Function of τ_i

Recall τ_i is the time-to-default or the default time of an i-th obligor in a portfolio of a CDO. τ_i was introduced by Li (2000). The definition of a default or credit event differs and it should be specified in a termsheet of each CDO contract.[19] Luscher (2005) uses a general definition that the default occurs if a stochastic variable falls below a critical threshold (e.g. when the assets of a firm fall below the value of firms liabilities).

In our case the stochastic variable is τ_i and we use the threshold time *t*. *t* is considered a continuous variable expressed in years and $t \geq 0$. The obligor defaults by threshold time *t* if $\tau_i \leq t$.

The distribution function of τ_i is called the default distribution function $F_i(t)$. Suppose that $F_i(t)$ is continuously differentiable. We also define the survival distribution function of the obligor $S_i(t)$:

$$F_i(t) = P(\tau_i \leq t)$$
$$S_i(t) = 1 - F_i(t) = P(\tau_i > t) \qquad (8)$$

We have $F_i(0) = 0$ and $S_i(0) = 1$. The probability density function for τ_i is derived as:

$$f_i(t) = F_i{'}(t) = -S_i{'}(t) = \lim_{\Delta \to 0} \frac{P(t \leq \tau_i \leq t + \Delta t)}{\Delta t} \qquad (9)$$

where $F_i{'}(t) = \frac{dF_i(t)}{dt}$. Besides the default and survival distribution function there are other measures how to express the probability distribution of τ_i. One of them which is very useful is a hazard rate function $h_i(t)$ defined e.g. in Li (2000). It returns a

[19] Definition of default from www.investorwords.com: failure to make required debt payments on a timely basis or to comply with other conditions of an obligation or agreement.

value of the density function of τ_i at an exact age t conditional on the survival until this age:[20]

$$h_i(t) = \frac{f_i(t)}{1 - F_i(t)} \qquad (10)$$

Using equations (8) and (9) result in: $h_i(t) = -\frac{S_i'(t)}{S_i(t)}$ and by taking an integral of this equation we reach that:

$$S_i(t) = exp\left(-\int_0^t h_i(s)ds\right) \quad and \quad F_i(t) = 1 - exp\left(-\int_0^t h_i(s)ds\right) \ (11)$$

From equation (10) it follows that $h_i(t) = \frac{f_i(t)}{S_i(t)}$. Therefore from equation (11) we get

$$f_i(t) = h_i(t)exp\left(-\int_0^t h_i(s)ds\right) \qquad (12)$$

3.3 Determination of Conditional Joint Distribution Function of τ_1, τ_2, ... , τ_n

In this subchapter we make following steps to finally get a conditional joint default distribution function of default times:

1. Introduction of Copula functions
2. Introduction One Factor Model
3. Conditional default probability derivation

Copula function approach provides an understandable framework to reach the joint multidimensional distribution function using univariate marginal distribution

[20] The hazard rate function of the i-th asset is derived as:

$$P(t \leq \tau_i \leq t + \Delta t | \tau_i > t) = \frac{F_i(t + \Delta t) - F_i(t)}{1 - F_i(t)} \sim \frac{f_i(t)}{1 - F_i(t)} \cdot \Delta t = h_i(t)$$

functions and some correlation structure between the obligors. In the previous subchapter we derived the marginal default distribution function (equation (11)). Now, One Factor Model will help simplify the correlation matrix problem and it will enable us to deduce the correlation structure between underlying assets.

Under the One Factor Copula approach the joint distribution function is first derived conditionally on the factor M value, i.e. $P(\tau_i \leq t|M = m)$ is deduced. In section 3.4 the probability of j defaults by time t_k conditionally on a factor value $P(N(t_k) = jM=m$ will be determined. Finally, a use of integral over all factor values m enables us to reach the required unconditional probability $P(N(t_k) = j)$.

3.3.1 Copula Functions

In this subchapter we introduce basic properties of copula functions so that we can apply them to the distribution of a vector of default times $(\tau_1, \tau_2, \dots, \tau_n)$.[21]

First, it is convenient to refer following statement accompanied by the proof which clarifies some relations between variables we will use in the consequent text.

Statement 1: Suppose random variable X with distribution function F. Then

i) F(X) has a uniform distribution R(0,1)

ii) If $U \sim R(0,1)$ then random variable $F^{-1}(U)$ has a distribution function F.

Proof:

i) $P(F(X) \leq x) = P\big(X \leq F^{-1}(x)\big) = F(F^{-1}(x)) = x$

ii) $P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$

A copula is a multivariate distribution function with standard uniform margins.

[21] The basic definitions and theorem are taken from D-fine (2004)

Definition 1: **Copula of U**. A function C: $[0,1]^n \rightarrow [0,1]$ is a copula if:

(a) There are random variables $U_1, U_2, \dots, U_n$ taking values in $[0,1]$ such that C is their distribution function

(b) C has uniform marginal distributions, i.e. for all $i \in \{1, \dots, n\}$, $u_i \in [0,1]$

$$C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$$

The word "copula" was first used by Sklar in 1959. Sklar's theorem says that for any set of random variables their multivariate distribution function can be decomposed to marginal distribution functions and an appropriate copula.

Theorem 1: **Sklar's theorem.** Let $X_1, \dots, X_n$ be random variables with marginal distribution functions $F_1, \dots, F_n$ and joint distribution function *F*. Then there exists an n-dimensional copula C such that for all $(x_1, \dots, x_n) \in R^n$

$$F\,(x_1, \dots, x_n) = C\big(F_1(x_1), \dots, F_n(x_n)\big) \qquad (15)$$

i.e. C is the distribution function of $\big(F_1(X_1), \dots, F_n(X_n)\big)$. If $F_1, \dots, F_n$ are continuous, then C is unique.

And conversely, if C is a copula and $F_1, \dots, F_n$ are univariate distribution functions then a function F satisfying:

$$C(u_1, \dots, u_n) = F\big(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)\big) \qquad (16)$$

is a multivariate distribution function with margins $F_1, \dots, F_n$.

Our modelling problem therefore divides into two parts. First part is represented by the marginal distribution functions and the second is represented by a specification of copula which considers the dependency structure between the obligors. Having both of these parts we derive the joint distribution function.

In our model we use a Gaussian copula function.

Definition 2: **The Gaussian Copula**. Let $X_1, \dots, X_n$ be normally distributed random variables with means $\mu_1, \dots, \mu_n$, standard deviations $\sigma_1, \dots, \sigma_n$ and a correlation matrix Σ. Then the distribution function $C_G(u_1, \dots, u_n)$ of the random variables

$$U_i = \Phi\left(\frac{X_i - \mu_i}{\sigma_i}\right), \quad i \in \{1, \dots, n\} \qquad (17)$$

(where $\Phi(\cdot)$ denotes the univariate standard normal distribution function) is a copula and it is called the Gaussian copula with the correlation matrix Σ.

That means for a Gaussian copula C_G it holds

$$C_G(u_1, \dots, u_n) = \Phi_\Sigma\big(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)\big) \qquad (18)$$

where Φ_Σ denotes a multivariate normal distribution function with correlation matrix Σ and standard normal marginals.

In our model we will assume that the default times τ_is, are connected to each other by a Gaussian copula function. Gaussian copula is quite popular and often used because of the fact that it is very easy to draw random samples from it and because the dependence structure of a multivariate normal distribution is easily understandable.

Our task is therefore to determine the joint distribution function of the default times:

$$P(\tau_1 \le t_1, \tau_2 \le t_2, \dots, \tau_n \le t_n) = F(t_1, t_2, \dots, t_n) \qquad (19)$$

where *F* denotes n-dimensional joint distribution function.

3.3.2 One Factor Gaussian Copula Model

In previous section we showed that all we need to know to deduce the joint distribution function of the default times is their marginal distribution function and correlation structure. Factor model helps us solve the correlation matrix problem. As our random variable is the default time, our task is to calculate the correlation structure between the default times. Instead of a simulation of a correlation matrix among large number of obligors using Monte Carlo method the Factor Model assumes that there are common factors for all default times that reflect their correlation. Let us now present the model.

The One Factor Gaussian Copula model assumes that random variables default times $\tau_1, \tau_2, \dots, \tau_n$ are linked to random variables $X_1, X_2, \dots, X_n$. It holds that:

$$X_i = \rho_i M + \sqrt{1-\rho_i^2} \cdot \varepsilon_i, \qquad i = 1, \dots, n \qquad (20)$$

where *M* is standard normally distributed random variable representing common *factor*

$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are standard normally distributed random variables

M, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are independent

ρ_i is a constant, called a *loading factor* and $|\rho_i| < 1$. It is also often assumed that ρ_i is nonnegative, i.e. the correlation between the default times is not negative

Equation (20) shows that X_i consists of two parts. The first one is represented by a common market factor M which links all X_i. The second part is individual for each X_i, it is an idiosyncratic part represented by random variable ε_i.

Here comes the reason why normal distribution assumption of M and ε_is is so convenient for this model: it is stable under convolution and therefore X_i is also standard normally distributed with distribution function $P(X_i \leq x) = \Phi(x)$.

Moreover:

$$cov(X_i, X_j) = \rho_i \rho_j \quad if\ i \neq j \qquad and$$
$$cov(X_i, X_j) = 1 \quad if\ i = j \qquad (21)$$

for all $i, j = 1, \dots, n$.

Regarding the link between X_i and τ_i for fixed i suppose $\Phi(x)$ is the distribution function of X_i and $F_i(t)$ is the distribution function of τ_i. If F_i is increasing, then there exist bilaterally unique correspondence between $t \in D_{F_i}$ and $x \in D_{\Phi}$[22] such that:

$$F_i(t) = P(\tau_i \leq t) = P(X_i \leq x) = \Phi(x) \qquad (22)$$

i.e.

$$t = F_i^{-1}(\Phi(x)) \quad resp. \quad x = \Phi^{-1}(\mathrm{F_i(t)}) \qquad (23)$$

X_is are mapped to τ_i using a percentile-to-percentile transformation. That means that a ten percentile point in a probability distribution of X_i transforms to a ten percentile point on a probability distribution of τ_i.

From the property of τ_i in equation (22) and the fact that X_i has a standard normal distribution it follows that:

1. The dependency structure of the default times is given by a Gaussian Copula, which follows from Definition 2 (Gaussian Copula).
2. The task of modelling $\frac{1}{2}(n-1)^2$ elements of a correlation matrix of the default times simplifies to modelling just *n* elements $\rho_1, \rho_2, \dots, \rho_n$.
3. Conditional on a realization of a common market factor *M*, the random variables $X_1, X_2, \dots, X_n$ are independent. Also, (from equation (22)) $\tau_1, \tau_2, \dots, \tau_n$ are conditionally independent given *M*.

[22] *D* is a standard notation of the definition scope

Implication 1 says that we can write the joint distribution of default times denoted by F from equation (19) using the copula terminology as:

$$F(t_1, t_2, \dots, t_n) = C_G\big(F_1(t_1), F_2(t_2), \dots, F_n(t_n)\big) \qquad (24)$$

From Theorem 1 and Definition 2 we also get:

$$C_G(x_1, x_2, \dots, x_n) = \Phi_\Sigma\big(\Phi^{-1}(x_1), \Phi^{-1}(x_2), \dots, \Phi^{-1}(x_n)\big) \qquad (25)$$

where Φ_Σ is n-dimensional normal distribution function with standard normal marginals given correlation matrix Σ between $X_i s$. Using equation (25) and combining it with equation (20) we can solve equation (24).

The reasoning for the third implication is that if we condition $X_1, X_2, \dots, X_n$ on the common factor, the X_i only differ by the idiosyncratic factor which is assumed independently distributed for all *n* and independent of *M*. Therefore $X_1, X_2, \dots, X_n$ are conditionally independent given *M*.

3.3.3 Conditional Default Probability of i-th Asset

Now we can compute the conditional default probability of the i-th obligor by time t given *M*. If the value of a common factor is known: $M = m$, then we compute the conditional probability that i-th obligor defaults by time t as:

$$P(\tau_i \leq t_i | M = m) = P(X_i \leq x_i | M = m) = P\left(\rho_i M + \sqrt{1-\rho_i^2} \cdot \varepsilon_i \leq x_i | M = m\right)$$

$$= P\left(\frac{x_i - \rho_i M}{\sqrt{1-\rho_i^2}} \geq \varepsilon_i | M = m\right)$$

$$= P\left(\varepsilon_i \leq \frac{\Phi^{-1}(F_i(t_i) - \rho_i M)}{\sqrt{1-\rho_i^2}} | M = m\right)$$

This calculation results from the properties deduced in equations (20) and (22).

Since ε_i is standard normally distributed, we finally get the joint distribution function of default times $\tau_1, \tau_2, \dots, \tau_n$ conditionally on M :

$$P(\tau_i \leq t_i | M = m) = \Phi\left(\frac{\Phi^{-1}(F_i(t_i) - \rho_i \cdot m)}{\sqrt{1-\rho_i^2}}\right) \qquad (26)$$

$$F(t_1, t_2, \dots, t_n | M = m) = \prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(F_i(t_i) - \rho_i \cdot m)}{\sqrt{1-\rho_i^2}}\right) \qquad (27)$$

Equation (11): $F_i(t_i) = 1 - exp\left(-\int_0^{t_i} h_i(s)ds\right)$ shows that $F_i(t_i)$ can be computed with a knowledge of $h_i(t)$. $h_i(t)$ can be derived from market quotes, which will be shown in the following chapter. Knowing ρ_i we can now directly compute the probability of a default of an asset by time *t* given factor value *m* considering its interdependence with other assets in the pool.

3.4 Distribution of Number of Defaults N(t)

Up to now we have been deriving the probability of a default of a specific asset within the pool. Firstly, we derived the default probability of an individual asset in section 3.2, then, we considered and included the dependence structure between assets in the pool in section 3.3.

Nevertheless, in case of a CDO contract an investor wants to derive the loss distribution of the whole pool of underlying assets; the knowledge of individual asset default distribution is not satisfactory. As in most cases the weight of each asset in the basket is the same and we assume it here too, the investor is rather interested in how many assets are expected to default in year 1, 2, up to the maturity of a CDO than concretely which assets are concerned. Therefore wee look for the distribution of the number of defaults by time t $N(t)$ rather than for the distribution of the default times λ_i.

Our derivation will consist of two steps:

1. Conditional distribution of number of defaults *N(t)*
2. Unconditional distribution of number of defaults *N(t)*

First, using the conditional probability of default derived in previous section in equation (26) we will derive the probability of *j* defaults by time t conditional on factor value: $P(N(t) = j|M = m)$. Second, using the iterated expectations theorem and integrating $P(N(t) = j|M = m)$ over all M we will derive the unconditional probability of *j* defaults by time *t* $P(N(t) = j)$.

3.4.1 Conditional Distribution of N(t)

In this part our intention is to derive $P(N(t) = j|M = m)$. Recall equation (3) that $N(t) = \sum_{i=1}^{n} 1_{\{\tau_i \leq t\}}$, i.e. $N(t)$ counts the number of defaults by time t and $j = 0, \dots, n$. In section 3.1. we proceeded as if it was known. Now, we accept it is a random variable distribution of which we are looking for.

Recall that we assumed recovery rates the same for all obligors $R_i = R$, the weight of each asset in the pool is the same $A_i = A$. Moreover in this section we add two other assumptions: we assume a homogenous portfolio in the sense that all obligors have the same threshold $t_i = t$ and that asset correlation is the same between all obligors $\rho_i = \rho$. These assumptions of a large homogenous portfolio also return that $P(\tau_i \leq$ *tiM=m=Pτ≤tM=m.* Finally, as mentioned at the end of section 3.3.2. as an implication number 3, conditionally on factor M the default times of obligors are mutually independent.

As a consequence we have *n* assets; as time passes we observe for each obligor if it defaults or if it survives. As there are only two outcomes possible for each asset, random variable $N(t)$ follows a binomial distribution conditionally on the fact that the market factor equals m. The parameter of the distribution is n and the probability is $P(\tau \leq t|M = m)$:

$$N(t)_{|M=m} \approx Bin\big(n, P(\tau \leq t|M = m)\big) \qquad (28)$$

As both parameters are now known it directly follows the binomial distribution that:

$$P(N(t) = j|M = m) = \binom{n}{j} \cdot P^j(\tau \leq t|M = m) \cdot \big(1 - P(\tau \leq t|M = m)\big)^{n-j} \quad (29)$$

3.4.2 Unconditional Distribution: N(t) and Joint Distribution of Default Times

The single factor M has a standard normal distribution. The aim is to reach the unconditional distribution function of $N(t)$. To that purpose we will use the iterated expectations theorem discussed for example by D-fine (2004).[23]

But first, let us look back shortly. As the theorem shows how to get unconditional distributions from the conditional ones we will just to crown our theory developed in section 3.3. By the end of this section we reached the conditional distribution of default times $F(t_1, \dots, t_n|M = m)$. Since then, we abandoned the distribution of default times as and started to devote to the distribution of $N(t)$. Now, with a help of the theorem, we have an opportunity to deduce the unconditional joint distribution function of $\tau_1, \dots, \tau_n$. It says that:

$$\begin{aligned} F(t_1, t_2, \dots, t_n) &= P(\tau_1 \leq t_1, \tau_2 \leq t_2, \dots, \tau_n \leq t_n) \\ &= E[P(\tau_1 \leq t_1, \tau_2 \leq t_2, \dots, \tau_n \leq t_n|M = m)] \qquad (30) \end{aligned}$$

The basic principle to reach the unconditional distribution function is to derive the expected value of conditional default probabilities. That is to integrate the

[23] The iterated expectations theorem says that with a probability space (Ω, F, P) and $X, Y: \Omega \to R$ random variables, it holds that $E[X] = E\big[E[X|Y]\big]$. By further argumentation it follows that $P(A) = E[P(A|Y)]$ for any subset $A \subset \Omega$.

conditional default probability with respect to the distribution of *M*. So the joint distribution function of default times is:

$$
\begin{aligned}
F(t_1, \dots, t_n) &= E\left[\prod_{i=1}^{n} P(\tau_i \leq t_i | M = m)\right] \\
&= \int_{-\infty}^{\infty} \phi(m) \prod_{i=1}^{n} P(\tau_i \leq t_i | M = m) dm \\
&= \int_{-\infty}^{\infty} \phi(m) \prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(F_i(t_i) - \rho_i \cdot m)}{\sqrt{1 - \rho_i^2}}\right) dm \qquad (31)
\end{aligned}
$$

where $\phi(m)$ is a Gaussian density function of M and $F_i(t_i) = 1 - exp\left(-\int_0^{t_i} h_i(s)ds\right)$. From (31) the joint distribution function of default times can now be solved.

Turning back to the distribution of number of defaults, using the same principle it follows that:

$$
P(N(t) = j) = \int_{-\infty}^{\infty} P(N(t) = j | M = m) \cdot \phi(m) dm \qquad (32)
$$

where $\phi(m)$ is the standard normal density function of *M*. Substituting (26) and (29) in (32) we get:

$$
\begin{aligned}
P(N(t) = j) = \int_{-\infty}^{\infty} \binom{n}{j} \cdot \left(\Phi\left(\frac{\Phi^{-1}(F(t) - \rho \cdot m)}{\sqrt{1 - \rho^2}}\right)\right)^{j} \\
\cdot \left(1 - \Phi\left(\frac{\Phi^{-1}(F(t) - \rho \cdot m)}{\sqrt{1 - \rho^2}}\right)\right)^{n-j} \cdot \phi(m) dm \qquad (33)
\end{aligned}
$$

The probability distribution function of k defaults ($k = 0, \dots, n$) or less by time t is therefore:

$$
\begin{aligned}
B(k) &= P(N(t) \leq k) \\
&= \sum_{j=0}^{k} \int_{-\infty}^{\infty} \binom{n}{j} \cdot \left(\Phi\left(\frac{\Phi^{-1}(F(t) - \rho \cdot m)}{\sqrt{1-\rho^2}} \right) \right)^{j} \\
&\quad \cdot \left(1 - \Phi\left(\frac{\Phi^{-1}(F(t) - \rho \cdot m)}{\sqrt{1-\rho^2}} \right) \right)^{n-j} \cdot \phi(m)dm \qquad (34)
\end{aligned}
$$

3.5 Summary

We will briefly summarize what we deduced in Chapter 3. The process of a CDO tranche valuation is not straightforward, thus now we draw up the basic line of our further computation to make the process clear.

Basic principle is same as with other financial products' valuation. Our task is to set the premium payment *V* for each tranche such that the present value of the tranche is zero. Remind equation (7):

$$
V^* = \frac{\sum_{k=1}^{T} B(t_0, t_k) \left[EL_{(K_A, K_D)}(t_k) - EL_{(K_A, K_D)}(t_{k-1}) \right]}{\sum_{k=1}^{T} B(t_0, t_k)(t_k - t_{k-1}) \left[1 - EL_{(K_A, K_D)}(t_k) \right]}
$$

To determine optimal premium V^* we need the expected loss function $EL_{(K_A, K_D)}(t_k)$ which is given by equation (4):

$$
EL_{(K_A, K_D)}(t_k) = \frac{1}{K_D - K_A} \sum_{j=1}^{n} \left[min\left(\frac{L_j(t_k)}{A \cdot n}; K_D \right) - K_A \right]^{+} \cdot P(N(t_k) = j)
$$

where we need to derive the probability of exactly j defaults by time t_k $P(N(t_k) = j)$. The derivation is not direct. In the first step we condition the probability on one factor M in equation (29):

$$P(N(t) = j|M = m) = \binom{n}{j} \cdot P^j(\tau \leq t|M = m) \cdot \big(1 - P(\tau \leq t|M = m)\big)^{n-j}$$

where $P(\tau \leq t|M = m)$ is derived from equation (26) using One Factor Gaussian Copula approach and considering a homogenous portfolio with $\tau_i = \tau$ for all $i = 1,2, \ldots, n$:

$$P(\tau_i \leq t_i|M = m) = \Phi\left(\frac{\Phi^{-1}(\mathrm{F_i}(\mathrm{t_i}) - \rho_\mathrm{i} \cdot \mathrm{m})}{\sqrt{1 - \rho_i^2}}\right)$$

In the second step using integral over all *M* we derive the unconditional probability in equation (33):

$$P(N(t) = j) = \int_{-\infty}^{\infty} \binom{n}{j} \cdot \left(\Phi\left(\frac{\Phi^{-1}(F(t) - \rho \cdot m)}{\sqrt{1 - \rho^2}}\right)\right)^j \cdot \left(1 - \Phi\left(\frac{\Phi^{-1}(F(t) - \rho \cdot m)}{\sqrt{1 - \rho^2}}\right)\right)^{n-j} \cdot \phi(m)dm$$

Given the mathematical background, in the following chapter we will present the valuation program we designed in VBA. Then parameters of the model will be chosen appropriately given recent advances in CDO pricing. And finally, we will value CDX index and its tranches, implement comparative statics and assess sensitivity on each parameter in context with current financial crisis.

4 Implementation of CDO Valuation

In this chapter we will show how to implement the valuation of a CDO contract following the theoretical concept we introduced in the previous chapter.

4.1 Introduction of the Index Priced

As discussed in section 2.7 and 2.8 the volume of CDO trades fell dramatically in 2008 after years of distinctive growth and the liquidity of the market disappeared. Therefore we will evaluate a CDS index as it is the only bearer of satisfactory level of persisting liquidity.

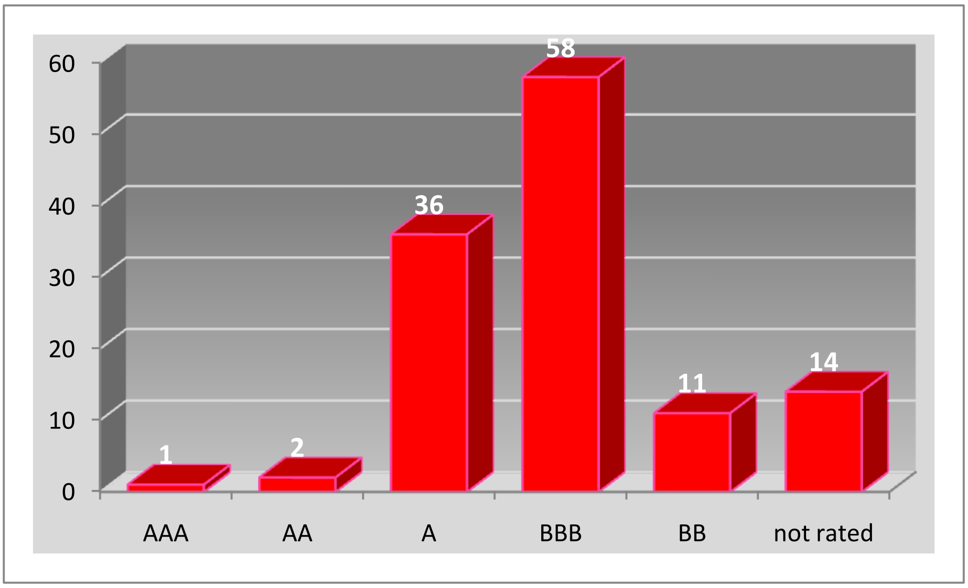
A principle of index trades is explained in section 2.6. We choose the main CDX index and use its tranches' market quotes from Thomson Reuters. Specifically, for the sake of low liquidity we chose the most traded series – 9 and maturity – 5 years. This index originally counted 125 underlying bonds issued by North American companies. The effective date of the index was on 21/9/2007. Since then there have been 3 defaults of underlying companies:

1. Washington Mutual Inc.
2. Federal Home Loan Mortgage Corporation
3. Federal National Mortgage Association

So that the index could continually exist, it is reversioned after each event of default. This means that the defaulted asset is displaced from the index, number of underlying companies is decreased whereas the percentage definition of attachment and detachment points of each tranche remains the same and a version number in the index name is changed. Since second two defaults were settled at once, there have been two reversions of the index. Our intension is therefore to price CDX North American Investment Grade index, Series 9, Version 3 with 5 years maturity – denoted by the Index in further text. The valuation date is 28/2/2009.

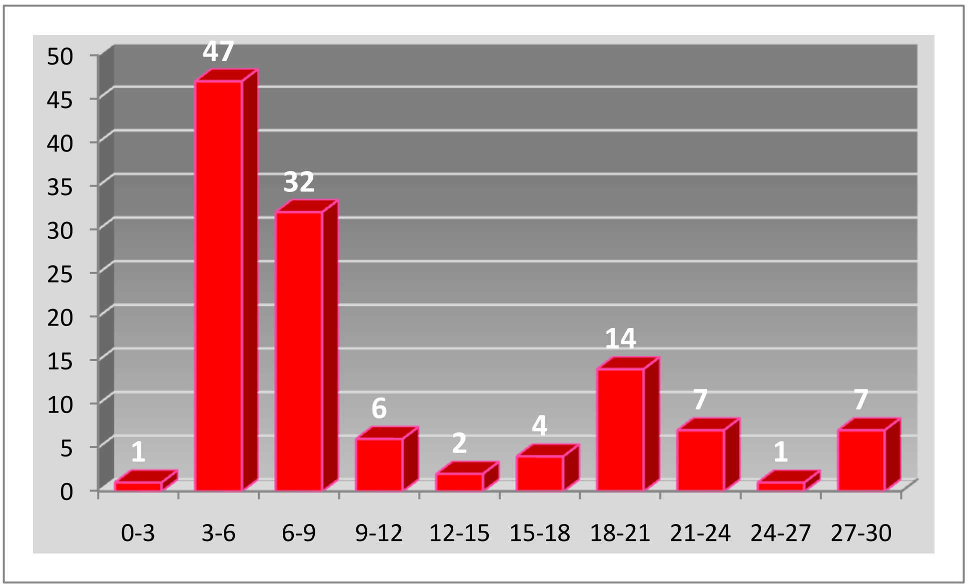
For an illustration of main features of the underlying bonds of the Index their maturities and ratings are showed in the following two figures:

Figure 12: **Rating and number of Index bonds in each category (on 28/2/2009)**



Source: www.markit.com

Figure 13: **Number of Index bonds in each maturity bucket (on 28/2/2009)**



Source: www.markit.com

To implement the valuation it is necessary to adopt some assumptions about the entry parameters such as distribution of τ_i, pairwise correlations, hazard rates, recovery rates and discount rates and their mutual relation. We will do so in section 4.2. Then we draw up the steps of the valuation using VBA in MS Excel in section 4.3.

All market data used are taken from Bloomberg.

4.2 Assumptions and Inputs

4.2.1 Distribution of τ_i

Up to now no specific assumption about the distribution of time-to-default of i-th asset τ_i was needed. Firstly, we will assume the same distribution function for all τ_is:

$$F_i(t) = F(t) \text{ for all } i = 1, \dots, n$$

Next, let us recall equation (10) with the definition of hazard rate function:

$$h(t) = \frac{f(t)}{1 - F(t)}$$

and equation (12) with derived probability density of a default time:

$$f(t) = h(t)exp\left(-\int_0^t h(s)ds\right)$$

Common assumption is that the hazard rate function is constant at some level λ (Li (2000), O'Kane and Turnbull (2003)). Hazard rate function of an asset can be then derived from its CDS market quotes. Based on this assumption the density of the default time simplifies to an exponential one:

$$f(t) = \lambda \cdot exp(-\lambda t) \qquad (35)$$

The reasoning for the exponential distribution of default times is explicit. From the inception of the CDO contract (i.e. at $t = 0$) we can mark on a timeline realizations of random variables – defaults of underlying assets. The hazard rate parameter λ shows a rate at which the realization of random variable τ arrives in time. However the terminology of a Poisson process cannot be used without consequence. The reason is that under this process the default times would have to be independent which is not the case here. As showed in section 3.3.2 the default times are independent only if we condition on factor M.

To be more specific, the distribution function and survival function of the default time is:

$$F(t) = 1 - exp(-\lambda t) \quad \text{and} \quad S(t) = exp(-\lambda t) \qquad (36)$$

with mean $E(\tau) = 1/\lambda$ and variance $var\tau = 1/\lambda^2$.

A derivation of the appropriate value of hazard rate will be discussed in section 4.2.3.

4.2.2 Correlation

The correlation parameter ρ^2 is another input in the model which needs to be specified. It is defined by equation (20) as a loading factor in a One Factor Model. As assumed in section 3.4.1. it is supposed to be the same for all obligors.

The correlation between default times was introduced by Li (2000) and its properties are discussed e.g. in Das (2005) or Luscher (2005). There are two approaches to correlation determination:

1. Implied correlation
2. Base correlation

In both approaches the correlation is determined endogenously by the model. Implied correlation is defined as a correlation for which the net present value of a tranche equals zero. Therefore, given the market quote of a CDO tranche premium we will determine the correlation parameter.

The base correlation approach is a little more complex. Suppose a CDX index with following tranching: (0-3)%, (3-7)%, (7-10)%, (10-15)% and (15-30)%. Now imagine a non-existing series of tranches (0-7)%, (0-10)%, (0-15)% and (0-30)%. Such tranches includes first 2 to 5 tranches of a CDX index. The cash-flows on the non-existing tranche (0-7)% combine two parts: the part of (0-3)% defaults is paid the same as first tranche of the index and the part of (3-7)% is paid the same as the second tranche in the CDX index. Given market quotes of these first two tranches the base correlation is a correlation input under which the price of the non-existing tranche would be zero. The practise is such that the correlation parameter is set at some level, then present values of tranches are summed and the task is to iterate such correlation under which the sum of these present values is zero.

Implied and base correlation should be the same for all tranches and subsets of tranches under perfect model and perfect markets. The real market resulting correlations (both implied and base) using our model will be discussed in the next chapter.

It is also notable that in our model the pairwise correlation is constant, whereas as shown in Das (2005) it differs over time and it depends on macroeconomic conditions of the markets. Generally, it is observed that in good times the correlation between assets tend to be lower than in recession.

4.2.3 Hazard Rate

Hazard rate function was supposed to be the same for all obligors and constant at a level λ – hazard rate. These assumptions are often used (Li (2000), Wang et al. (2006) or O'Kane and Turnbull (2003)) and will simplify our calculation in such a way that it enables us to use exponential distribution of default times.

Hazard rate of an asset is deduced from the market quotes of credit default swap (CDS). In most cases CDSs are quoted only for five years' maturity as the liquidity on other maturities is low. That's why the assumption of constant hazard rate is reasonable.

Let's now present a main idea behind the determination. CDO consists of n underlying assets. Now let's assume one single asset from these. It has a CDS spread quoted on the market. Basically, this premium is determined using the similar but simpler approach as in case of a CDO as there is only one underlying asset. Nevertheless, the probability of default of the asset is used to derive the expected loss on the contract. Then the fair premium is determined so that the present value of the premium payments equals the present value of loss payment. The default time of the asset has an exponential distribution with rate parameter λ. Then, by definition, implied λ is such value of λ that sets the present value of CDS contract to zero given market quote. After some derivations we arrive at simple approximation of λ:

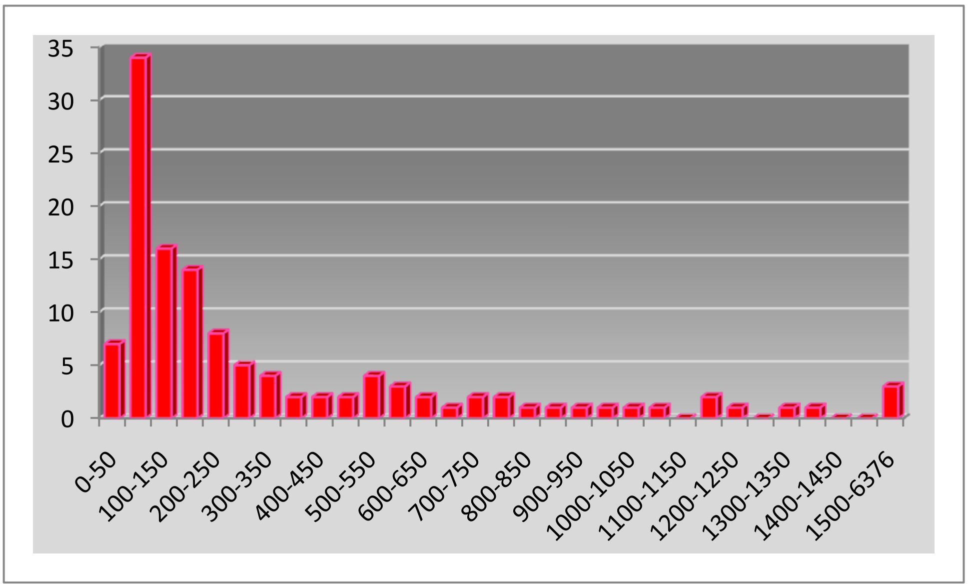
$$\lambda \cong \frac{CDS}{1-R} \qquad (37)$$

where *CDS* denotes quoted *CDS* spread and *R* is the recovery rate of the asset. For details of this method see Li (1998). It is assumed that the term structure of interest rate is flat and independent of hazard rate λ.

Intuitive explanation of the equation above is such that λ – as a parameter of an exponential distribution - represents a risk that the asset will default and CDS spread quote should embody not only this risk but it should consider that there is some recovery which lowers the part of investment which is exposed to the risk.

In the following graph there is a distribution of CDS spreads in the Index we price. We assign to each interval of 5 years CDS quoted spread the number of assets with corresponding spread.

Figure 14: **Number of assets from the Index with CDS spread in given interval (28/2/09)**



Source: Thomson Reuters

In our model we assume a homogenous portfolio of underlying assets. Therefore we use the average level of hazard rate based on individual CDS spreads and recovery rates of underlying assets.

4.2.4 Recovery Rate

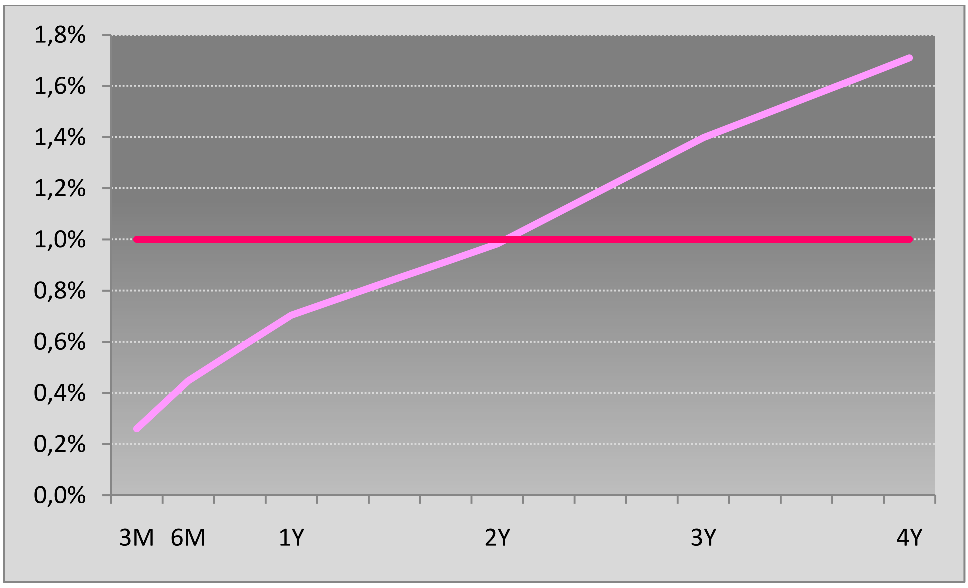
Recovery rate is a part of any loss exposure that is likely to be recovered (Das (2005)). It depends on industry, type of an instrument, its seniority, rating (given by expected loss), jurisdiction and its enhancement. Recovery rates are provided by rating agencies.

In a computation of individual hazard rates we used recovery rates assumed by Société Générale. To determine the expected loss the average recovery rate of 39% was used.

4.2.5 Discount Rates

As a discount rate we use US zero bond yield curve. Its term structure on 28/2/2009 is depicted in the following graph. As it was assumed that the discount rate is constant over time, we set it at level $B(t_0, t_k) = r = 1\%$ for all $k = 1, \dots, 16$.

Figure 15: **US zero bond yield curve on 27/2/2009**



Source: Bloomberg

The premium is paid quarterly, on each 20/3, 20/6, 20/9, 20/12 until maturity. In case the premium day is a weekend or holiday, the premium is paid next business day. For the Index there are 16 payments left until maturity. Their dates are given in the following table together with the distance from the valuation date expressed in years and corresponding discount factor:

Table 3: **Discount factors on 28/2/2009**

Payment date	Year fraction - t	Discount factor - exp(-rt)
20.3.2009	0,06	0,999
22.6.2009	0,32	0,997
21.9.2009	0,56	0,994
21.12.2009	0,81	0,992
22.3.2010	1,06	0,989
21.6.2010	1,31	0,987
20.9.2010	1,56	0,985
20.12.2010	1,81	0,982
21.3.2011	2,06	0,980
20.6.2011	2,31	0,977
20.9.2011	2,56	0,975
20.12.2011	2,81	0,972
20.3.2012	3,06	0,970
20.6.2012	3,31	0,967
20.9.2012	3,56	0,965
20.12.2012	3,81	0,963

Source: author's calculations based on Bloomberg

4.2.6 Market Quotes of the Index – Upfront Payment vs. Running Spread

There is one more complication we need to overcome. Nowadays, first three Index tranches' quotes consist from two parts:

1. Upfront payment
2. Running spread 500 bps

Suppose for example that the quote of the most junior tranche of the Index is (81%UF + 500bps). Then the protection seller receives upfront 81% of the invested nominal which is settled as a spot transaction, then he continues to receive 500 bps per annum (running spread) in form of quarterly payments quarter until the maturity of a contract.

Since our model returns the premium in form of a pure running spread, we need to discount part of it to the valuation date and leave only 500 bps of running spread. Therefore we will derive the present value of a risky basis point (PV01). It is clear that PV01 will vary for each tranche. The present value of one basis point premium received each year up to maturity would be lower for a holder of an equity tranche compared to a holder of a senior tranche. The latter faces a huge risk that his tranche will we affected by a default and the notional on which the premium point is paid will decrease significantly. Therefore he discounts premium payments by a lower discount factor. On the other hand the discount factor for a most senior tranche holder will converge to the market discount factor based on the zero curve.

Recall equation (5) from the previous chapter which shows the present value of a premium payment for a CDO tranche buyer:

$$PV_{Premium} = \sum_{k=1}^{T} B(t_0, t_k)(t_k - t_{k-1})V\left[1 - EL_{(K_A,K_D)}(t_k)\right]$$

where $B(t_0, t_k)$ is the risk-free discount rate from t_k to t_0, V is the premium in percentage and $EL_{(K_A,K_D)}(t_k)$ is the expected percentage cumulative loss on a tranche by time t_k. Present value of one risky basis point received on a tranche is then:

$$PV01_{(K_A,K_D)} = 0{,}0001 \cdot \sum_{k=1}^{T} B(t_0, t_k)(t_k - t_{k-1})\left[1 - EL_{(K_A,K_D)}(t_k)\right] \qquad (38)$$

Suppose for example an equity tranche contract maturing in 4 years. Present value of 1 bp received each year is 1,25 bps. As the fair running spread is 69%, its present value is thus 69%.1,25=86,25%. The investor would therefore be indifferent between these two scenarios:

1. Receive 86,25% in form of an upfront payment
2. Receiving 69% annually for four years

From this example a substantial risk of an equity tranche investment perceived by the market is evident.

However, investor still receives 500 pbs running. Denoting V^* the fair spread the upfront payment *UF* quoted on the market will thus be:

$$UF = (V^* - 500bps) \cdot PV01_{(K_A,K_D)}$$

4.3 VBA Valuation Process

The valuation is implemented in MS Excel VBA. A file with the valuation program is attached to this thesis. This section lists a sequence of steps needed for the valuation. There are three main parts:

1. Conditional probability of default derivation
2. Unconditional probability of default derivation
3. Expected loss determination and fair spread value

The outcome of first two steps is a matrix with *(n x T)* elements giving us conditional resp. unconditional probability of a number of defaults by given time. Unconditional probabilities will enable us to determine the expected loss for each time period, that will serve to an evaluation of the present value of both loss and premium payment. The program will than find a value of a fair premium equalizing both values.

4.3.1 Conditional Default Probability Derivation

Conditional probability of default is derived on MS Excel list called Conditional Probabilities. The probability of exactly *j* defaults ($j = 0,1,\dots,n$) given market situation *m* by given time *t* (values of *t* are in Table 3) is given in equation (29):

$$P(N(t) = j|M = m) = \binom{n}{j} \cdot P^j(\tau \leq t|M = m) \cdot \big(1 - P(\tau \leq t|M = m)\big)^{n-j}$$

where the probability of default by time t given m is:

$$P(\tau \leq t|M = m) = \Phi\left(\frac{\Phi^{-1}(\mathrm{F(t)} - \rho \cdot \mathrm{m})}{\sqrt{1-\rho^2}}\right)$$

(equation (26))

Entry parameters we need are:

- Number of assets – $n = 122$
- Pairwise standard deviation - ρ
- Distribution function of default times τ - $F(t) = 1 - exp(-\lambda t)$
- Hazard rate - λ
- Market situation – m. Recall that common market factor M is a random variable with standard normal distribution. As M is defined on $\mathbb{R}$, in this stage of computation m can be chosen arbitrarily from real numbers.

4.3.2 Unconditional Default Probability Derivation

Unconditional probability of default is derived on the attached MS Excel list called Unconditional Probabilities. The theory requires integration of conditional probabilities of default for given j and t over M:

$$P(N(t) = j) = \int_{-\infty}^{\infty} P(N(t) = j|M = m) \cdot \phi(m)dm$$

(equation (29)). In our computation the number of realizations of the market factor M is considered finite and the conditional probabilities are approximated using the specification of M. The user of the program will define:

1. *MaxM*: maximum value of M considered in calculation, $MaxM \in \mathbb{R}^+$
2. *MinM*: minimum value of M considered in calculation, $MinM \in \mathbb{R}^-$
3. Number of realizations of M denoted as c

Then, the difference between *MaxM* and *MinM* will be divided into *c-1* intervals of equal size with endpoints:

$$MinM = m_1, m_2, \ldots, MaxM = m_c \qquad (39)$$

The unconditional probability of default is then computed as:

$$P(N(t)=j)=\sum_{k=2}^{c-1}\left[\frac{\Phi(m_{k+1})-\Phi(m_{k-1})}{2}\cdot P(N(t)=j|M=m_k)\right] \qquad (40)$$

Where $\Phi(\cdot)$ is a standard normal distribution function. Unconditional probability is then a sum of probabilities conditioned on *M* weighted by the probability of each realization of *M*.

The expression of the sum of all weights in equation (40) should be zero. The verification follows:

$$\begin{aligned}\sum_{k=2}^{c-1}\frac{\Phi(m_{k+1})-\Phi(m_{k-1})}{2}& \\ &= \frac{1}{2}\big(\Phi(m_3)-\Phi(m_1)+\Phi(m_4)-\Phi(m_2)+\Phi(m_5)-\Phi(m_3)\\ &+\Phi(m_6)-\Phi(m_4)+\Phi(m_7)-\Phi(m_5)+\cdots+\Phi(m_{c-3})-\Phi(m_{c-5})\\ &+\Phi(m_{c-2})-\Phi(m_{c-4})+\Phi(m_{c-1})-\Phi(m_{c-3})+\Phi(m_c)\\ &-\Phi(m_{c-1})\big)=\frac{1}{2}\big(-\Phi(m_1)-\Phi(m_2)+\Phi(m_{c-1})+\Phi(m_c)\big)\approx 1.\end{aligned} \qquad (41)$$

If we choose sufficiently low m_1 the first two elements of the sum in the last row are nearly zero, similarly, by choosing sufficiently high m_c second two elements approach 1. Then the sum of weights is very close to 1.[24]

4.3.3 Expected Loss Determination and Fair Spread Value

Knowing $P\big(N_j(t_k)=j\big)$ the computation of expected percentage cumulative loss and fair premium is straightforward:

$$EL_{(K_A,K_D)}(t_k)=\frac{1}{K_D-K_A}\sum_{j=1}^{n}\left[min\left(\frac{N_j(t_k)}{A\cdot n};K_D\right)-K_A\right]^{+}\cdot P(N(t_k)=j)$$

[24] For example selecting $m_1=-10$ and $m_c=10$ the deviation from 1 is noted on 15th decimal place.

$$V^* = \frac{\sum_{k=1}^{T} B(t_0, t_k)\left[EL_{(K_A,K_D)}(t_k) - EL_{(K_A,K_D)}(t_{k-1})\right]}{\sum_{k=1}^{T} B(t_0, t_k)(t_k - t_{k-1})\left[1 - EL_{(K_A,K_D)}(t_k)\right]}$$

(equations (4) resp. (7))

5 Results of the Model

The object of this chapter is to illustrate a behavior of a CDO using the model we presented in Chapter 3 based on assumptions taken in Chapter 4. We run the calculation introduced above to show a market risk of a CDO and to model the mark-to-market loss of a holder of senior tranches. I.e. we demonstrate what were the consequences of massive and reckless investment in AAA rated tranches and why the CDO market nearly ceased to exist. Thus it is not our intention to find the best fit to market quotes. The intention in Chapter 3 was rather to introduce relatively simple but functional model which is used to show main sensitivities and risks of a CDO.

The chapter is divided into three main parts. In the first one we determine a relation of each tranche's premium to correlation, hazard rate and event of default. In the second part we show these implications on recent data. As we have data of an index instead of a CDO we estimate how a real CDO would have behaved in two recent years, which were affected by the financial crisis. The last part of the chapter is dedicated to an overall assessment of the CDO market, its weaknesses, its role in the crisis and contribution to the crisis. The main flaws of the market are spotted and their correction is proposed.

Moreover, throughout the two first parts – section 5.1 and 5.2 we test five hypothesizes about the entry parameters. Each hypothesis is presented at the beginning of the subsection which is devoted to its test.

5.1 Sensitivities

5.1.1 Evaluation of Correlation

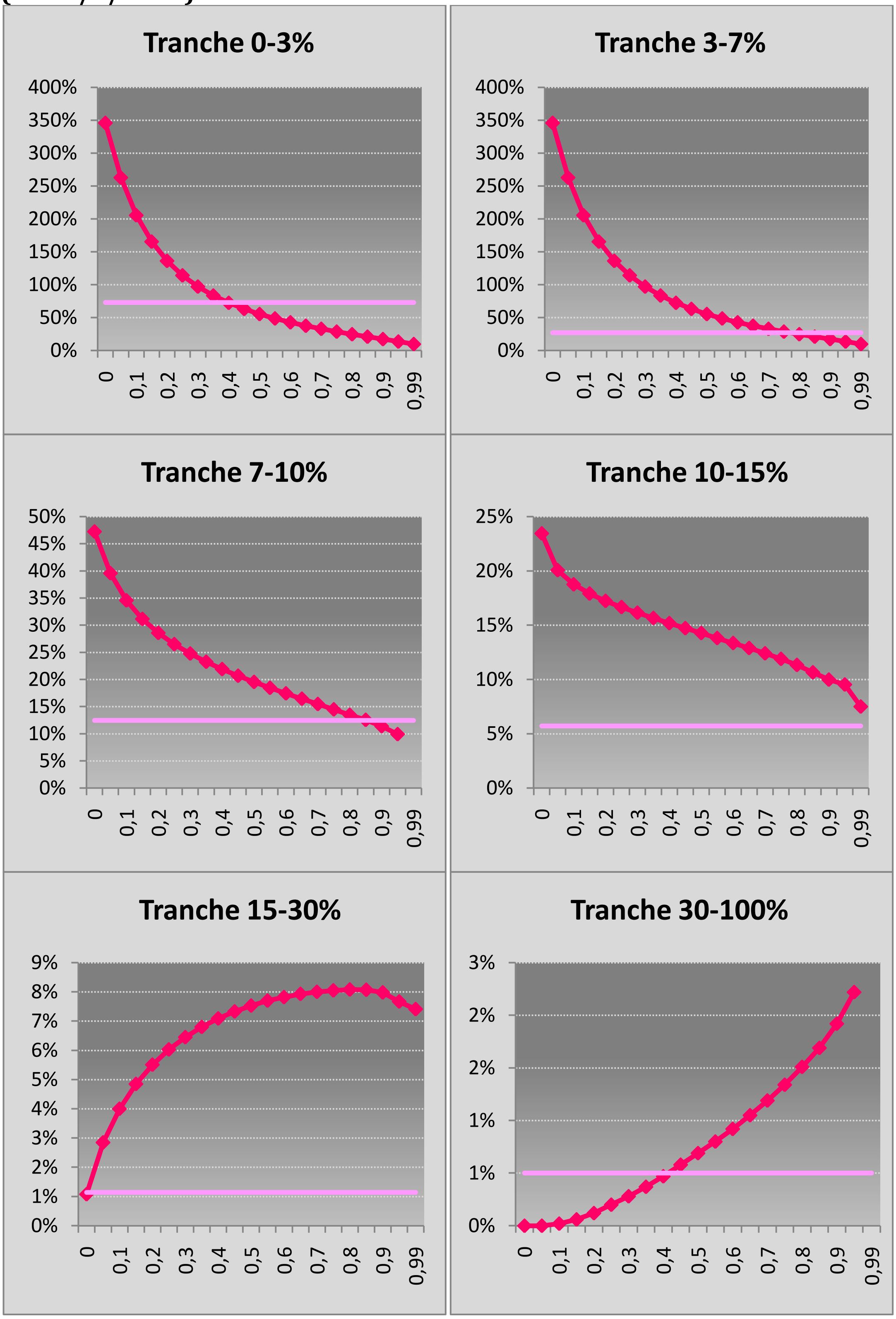
5.1.1.1 Correlation and Tranche Premium

Hypothesis 1: The higher the asset correlation the lower the risk premium for a junior tranche and the higher the risk premium for a senior tranche

Recall that correlation parameter shows the correlation between each pair of underlying assets. Figure 16 shows what the role of correlation in each tranche is.

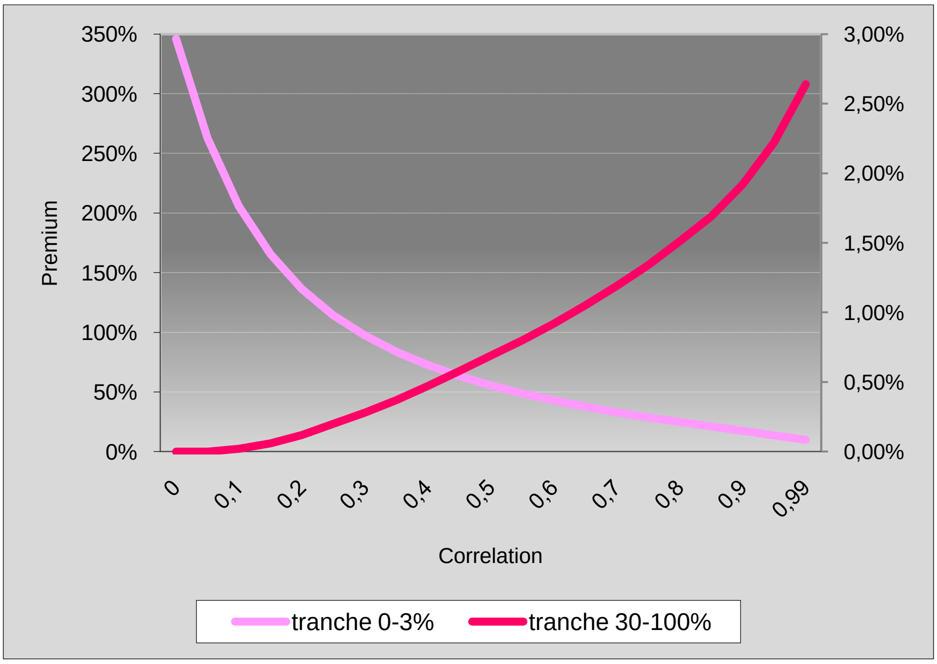
We fixed the hazard rate at 0,07 level and calculated the premium of a tranche for a varying level of correlation on vertical axis.

Figure 16: **Tranche premium for a given level of correlation (on 28/2/2009)**



For the most junior tranche the premium is a decreasing function of correlation whereas for the most senior tranche the premium increases with correlation. The mezzanine tranches are less sensitive to correlation. Moreover the relation between correlation and premium doesn't have to be monotonic – see results for tranche 15-30%. It depends on the definition of attachment and detachment points of the mezzanine tranches. In the Figure 17 the sensitivity of 0-3% tranche and 30-100% tranche is compared.

Figure 17: **Sensitivity of equity and senior tranche to correlation**



Higher correlation has a lower value for someone who buys protection on the equity tranche. He is willing to pay less to the protection buyer. The opposite holds for the senior tranches. Higher correlation has a higher value. Plesner (2008) explains this fact by using a comparison to the mine field and two vehicles: a tank and a motorcycle that are supposed to pass through the field. The motorcycle is destroyed in case of hitting one mine; the tank destroys after hitting 10 mines. Suppose two cases how mines could be located. Firstly, mines are highly correlated, i.e. on the large field there is one center by which all mines are located, elsewhere there are no mines. Secondly, mines are not correlated, i.e. they are equally distributed over the whole field. A survival for the motorcycle is much more probable in the first case. Unless it runs into the center it will survive. Then, when it recovers the center it doesn't care how many mines are in there. In the second case it is nearly sure that it will be destroyed. The motorcycle stands for the equity tranche. It is also hit even in case of one default. The holder of this tranche is concerned about the first 3 defaults, then, if the correlation is high and other defaults follow, he is no more interested as he already lost everything. The opposite holds for the tank. Suppose that in the second case there are usually no more than 10 mines to be hit when crossing the field. Therefore the tank will nearly surely survive. But in the first case, when it reaches the center it will be destroyed. The tank stands for the senior tranche. If the correlation is high, it is valued more since it is more likely that it will be hit than in case of low correlation.

Hypothesis 1 can thus be admitted.

5.1.1.2 Values of Implied Correlation

To be able to value a CDO we need to determine the level of correlation. First, we will use the implied correlation derivation from market quotes as explained is section 4.2.2. See the following table for the results of the valuation based on different values of correlation. In the first row there is a market quote of CDX tranches on 28/2/2009. For the three lowest tranches the values displayed are already recalculated to the upfront payment quotes. Market quote of the tranche is also incorporated in Figure 16 by a light pink line. Its intersection point with the red line determines the implied correlation.

Table 4: **Results of valuation on 28/2/2009 with hazard rate 0,07**

	tranche 0-3%	tranche 3-7%	tranche 7-10%	tranche 10-15%	tranche 15-30%	tranche 30-100%
market quote on 28/2	**81,72%**	**53,85%**	**22,97%**	**7,72%**	**1,14%**	**0,50%**
correlation						
0	98,02%	92,86%	86,68%	23,45%	**1,08%**	0,00%
0,05	97,46%	90,37%	75,91%	20,07%	2,85%	0,00%
0,1	96,59%	85,87%	67,40%	18,76%	4,00%	2,00%
0,15	95,21%	80,89%	61,18%	17,91%	4,85%	0,06%
0,2	93,28%	76,01%	56,34%	17,24%	5,51%	0,12%
0,25	90,88%	71,39%	52,34%	16,67%	6,03%	0,20%
0,3	88,08%	67,03%	44,91%	16,15%	6,45%	0,28%
0,35	84,95%	62,98%	45,88%	15,66%	6,80%	0,37%
0,40	**81,54%**	59,12%	43,12%	15,19%	7,09%	**0,47%**
0,45	77,87%	55,44%	40,56%	14,73%	7,33%	0,58%
0,5	73,98%	**51,89%**	38,15%	14,28%	7,53%	0,69%
0,55	69,87%	48,45%	35,84%	13,82%	7,70%	0,80%
0,6	65,53%	45,07%	33,59%	13,36%	7,82%	0,92%
0,65	60,97%	41,72%	31,36%	12,90%	7,93%	1,05%
0,7	56,15%	38,36%	29,13%	12,41%	8,00%	1,19%
0,75	51,02%	34,94%	26,79%	11,90%	8,05%	1,34%
0,8	45,52%	31,40%	24,34%	11,34%	8,08%	1,51%
0,85	39,46%	27,51%	**22,10%**	10,66%	8,07%	1,69%
0,9	32,68%	22,44%	18,82%	9,99%	7,98%	1,92%
0,95	24,52%	17,62%	15,12%	9,54%	7,67%	2,22%
0,99	15,30%	12,67%	8,71%	**7,51%**	7,41%	2,64%
Implied correlation	**0,40**	**0,50**	**0,85**	**0,99**	**0,00**	**0,40**

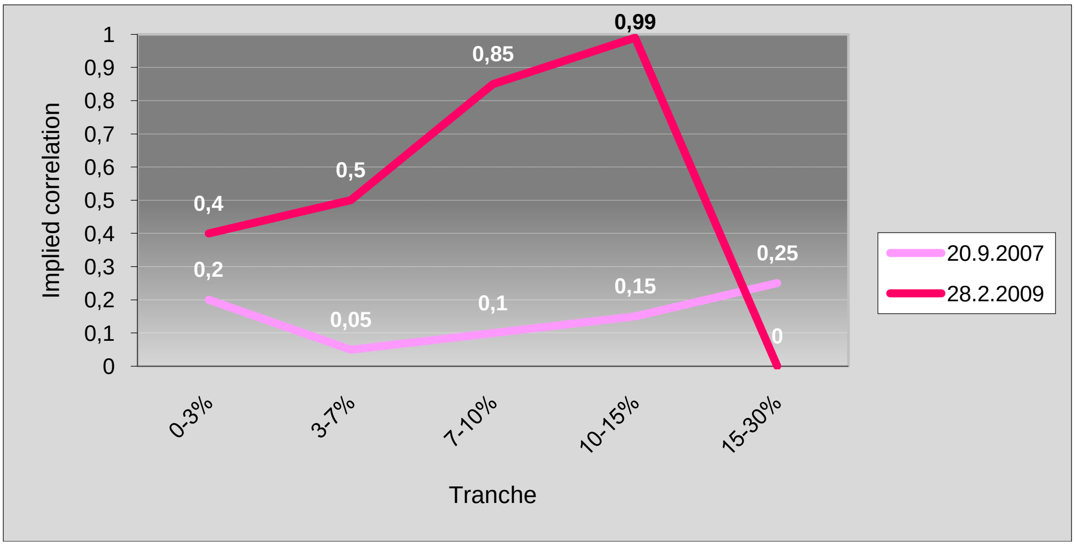
The correlation differs substantially among tranches. The reason why it is one for the 10-15% tranche and 0 for the 15-30% one is that the correlation sensitivity of these tranches is low and the premium doesn't differ much with correlation change. Then, the consequence of either inexactness of the model or a skew of the quote due to market sentiment is that the valuation model has a difficulty to fit some reasonable correlation value.

Comparing the results of the model with market quotes we perceive that for correlation 0,4 Gaussian copula prices fairly the junior and most senior tranches but overvalues the risk of the mezzanine tranches – 7-30%.

The difference between implied correlations is a usual outcome of CDO valuation models (e.g. in Hull and White (2004) or Amato and Gyntelberg (2005)). It is called correlation smile and it points to both imperfection of the model and a fact, that market quotes comprise other factors that are not included in the model.

In our case the variation is significant. It can be due to the financial crisis and false market valuation. A comparison with the past puts up. Figure 18 supplies a comparison with implied correlation we evaluated on the issue date of the index.

Figure 18: **Implied correlation of a tranche on 28/2/2009 compared to 20/9/2007**



Market conditions changed substantially since 20/9/2007. On that date the CDS traded markedly lower. Due to the financial crisis which caused several defaults of trusted corporations or institutions the hazard rate multiplied seven times since then. As we see from the graph above based on Gaussian copula implied correlation was more stable through tranches. Therefore the reason for current huge variations of implied correlation among tranches can be also perceived in distressed markets and inappropriate valuation of tranches.

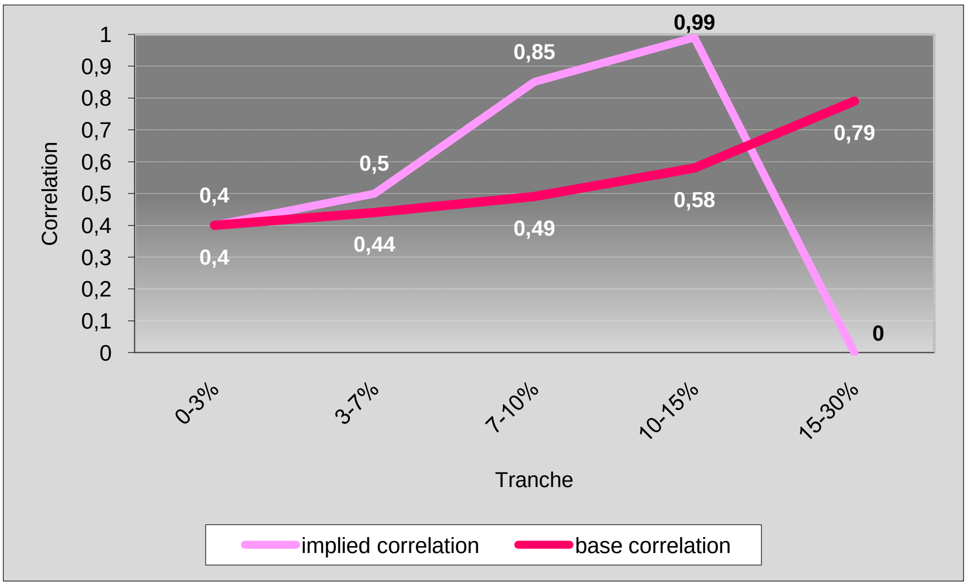
5.1.1.3 Values of Base Correlation

Hypothesis 2: Base correlation is more stable measure of correlation than implied correlation

Hypothesis 3: Correlation and hazard rate changed substantially over the past year and a half

The principle of base correlation derivation was explained in section 4.2.2. Our results of base correlation compared to implied correlation are displayed in the following graph.

Figure 19: **Base and implied correlation for tranches on 28/2/2009**



Base correlation records less variance than the implied one. Suppose an investment in all tranches of a CDO totaling 1 M USD. The distribution of the notional among tranches is given by their attachment and detachment points. E.g. we invest 30 000 USD in equity tranche, 40 000 USD in 3-7% tranche and so on.

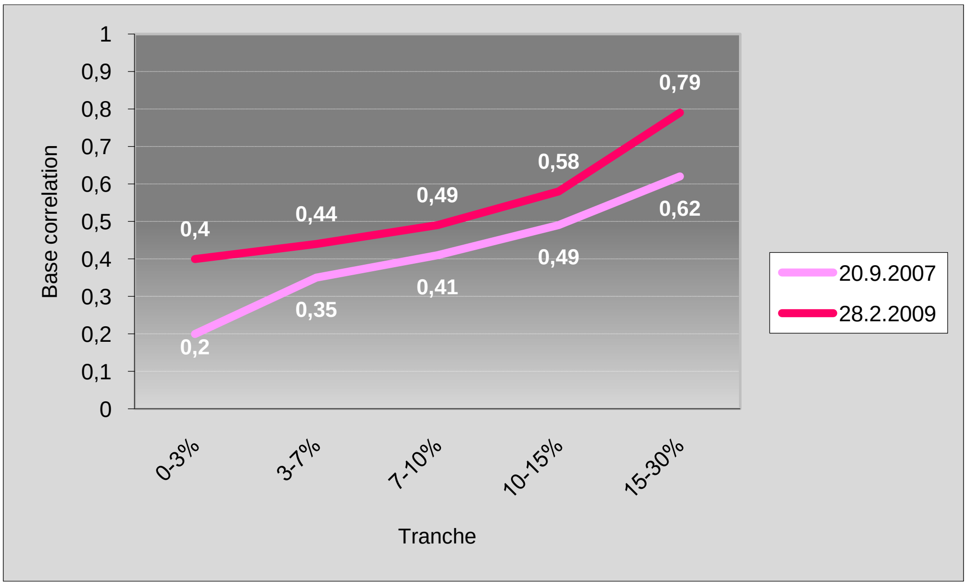
We first start with the equity tranche where the base correlation equals the implied correlation. Then we need to evaluate the base correlation for the two lowest tranches. Taking correlation of 0,4 the present value of the 0-3% tranche is 0 and present value of 3-7% tranche is negative. Both these tranches' premiums are decreasing in correlation. Thus, running the calculation with a higher value of correlation given market quotes increases the present value of both tranches[25]. With correlation of 0,44 and we arrive at present value of 0-3% tranche of approximately +3200 USD and present value of 3-7% tranche of -3200 USD. Hence 0,44 is the base correlation for the two lowest tranches.

The concept of base correlation is thus not that intuitive as it seems to be. To illustrate it, we show how the correlation is appointed for the 15-30% tranche. From Figure 16 we see that all first four tranches are decreasing in correlation; consequently the base correlation has to increase while adding more senior tranches up to the detachment point 15%. Then, after arriving at correlation of 0,56, we add the 15-30% tranche which has a premium increasing in correlation (for reasonable levels of correlation – see Figure 16). Now it depends on its implied correlation which determines its present value. As we see in Figure 16 the implied correlation is approaching 0 thus for a correlation 0,56 the present value of the tranche is deeply negative. Decreasing correlation leads to an increase of a present value of 15-30% tranche from negative values but also to a decrease of all more junior tranches' present values from positive values. On the other hand, increasing correlation would decrease present value of 15-30% tranche to more negative values but it also increases the present value of other tranches. Now, it depends on the sensitivity of the tranches to correlation and on the width of the tranches which effect would be stronger and lead to a 0 sum of present values. Finally, we conclude that it would be the case of an increase in correlation. The base correlation is therefore monotonically increasing in correlation and it is more stable than the implied correlation. Hypothesis 2 was approved.

In case of a usual CDO, the most senior tranche, i.e. 30-100% is not sold to protection sellers and it is retained by the issuer. Therefore it is not included into our calculations. Next graph shows the evolution of base correlation during the financial crisis. We compare the base correlation at issue date the index to the one derived on 28/2/2009.

[25] Positive or negative present value is meant from the protection seller – i.e. CDO investor point of view.

Figure 20: **Base correlation on 20/9/2007 and on 28/2/2009**



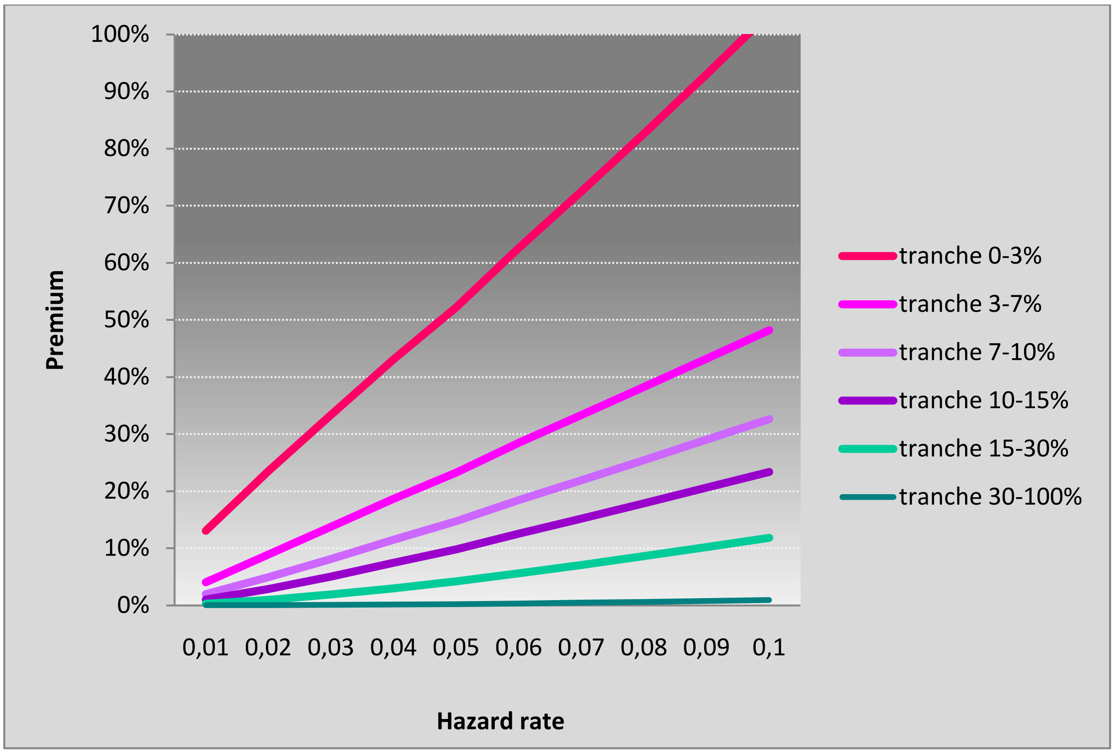
It is evident that the correlation changed in past year and a half. Part of Hypothesis 3 concerning the correlation is satisfied. The second part regarding the hazard rate will be examined in section 5.2.2. The maximum increase can be observed by the equity tranche where the correlation increased by 100%, the minimum increase was by the mezzanine tranche 7-10%. In the model we assumed constant correlation. So, when evaluating a CDO it is essential to take into consideration that the correlation in bad times usually increases and consequently the mark-to-market loss of a tranche can be even more distinctive.

5.1.2 Hazard Rate and Tranche Premium

Hypothesis 4: Higher hazard rate increases the premium of all tranches more than proportionally

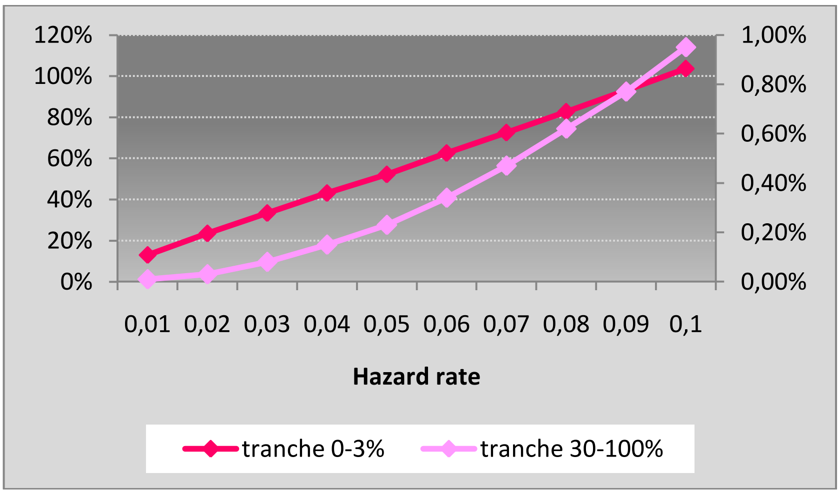
Hazard rate for an asset is calculated from the CDS quote and recovery rate. Higher CDS adverts to a higher credit risk of an asset. The premium of any CDO tranche based on a pool of assets also increases. Higher recovery rate adverts to lower loss given default and therefore the premium of a CDO tranche would be lower. This relation is depicted in the figure below. Correlation is now fixed at 0,4.

Figure 21: **Tranche's premium with respect to a hazard rate (on 28/2/2009)**



From the nature of a hazard rate as a parameter of an exponential distribution of the default times the relation is not linear as it might appear from the graph. The detailed dependence of the premium to a hazard rate is illustrated in Figure 22 where the equity tranche and 30-100% tranche are compared.

Figure 22: **0-3% tranche and 30-100% tranche hazard rate dependence**



The higher the seniority of the tranche the more convex the relation between the premium and hazard rate is. The higher the hazard rate the higher the compensation in form of tranche's premium has to be to offset increased credit risk. Also, the mark-to-market loss on a senior tranche in case of an increase of hazard rate has to be expected higher for higher starting level of hazard rate. Hypothesis 4 is therefore also approved.

5.1.3 Event of Default and Tranche Premium

Based on the data of the CDX index we will now evaluate the change of a premium of each tranche after 1, 2 and 3 defaults as if it was a real CDO. The assumptions are following:

1. Correlation 0,2
2. Hazard rate 0,01
3. Total notional invested in all tranches: 1 M USD (the distribution among tranches is given by their attachment and detachment points)

First, we evaluate the CDO supposing there is no default. Therefore the parameters are fixed and set on their levels before the financial crisis. As a valuation date we choose 28/2/2009.

Then, we make the same valuation assuming that there was 1, 2 and 3 defaults at the valuation date. I.e. the parameters are reset in the following way:

1. New number of assets: Instead of 125, we set 124, 123 resp. 122 for 1, 2, resp. 3 defaults.
2. Lower volume of investment amount: In case of a default, there is a physical or cash settlement and the notional decreases by the nominal of one bond in the portfolio.
3. New attachment and detachment point: Both these points' absolute value decreases by the amount of defaulted notional. E.g. given one default the detachment point of the equity tranche changes from 30 000 USD to 22 000 USD.

The outcome of this valuation is displayed in the table below.

Table 5: **Change of a premium and loss on a tranche after default (on 28/2/2009)**

<table>
<tr><td></td><td></td><td colspan="2">tranche 0-3%</td><td>tranche 3-7%</td><td>tranche 7-10%</td><td>tranche 10-15%</td><td>tranche 15-30%</td></tr>
<tr><td>NO DEFAULT</td><td>Premium</td><td colspan="2">19,38%</td><td>3,83%</td><td>1,12%</td><td>0,35%</td><td>0,03%</td></tr>
<tr><td></td><td></td><td></td><td>Total loss</td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">1 DEFAULT</td><td>Premium</td><td>24,09%</td><td></td><td>5,20%</td><td>1,42%</td><td>0,43%</td><td>0,04%</td></tr>
<tr><td>Loss</td><td>-11,15%</td><td>-24,18%</td><td>-4,71%</td><td>-1,09%</td><td>-0,30%</td><td>-0,04%</td></tr>
<tr><td rowspan="2">2 DEFAULTS</td><td>Premium</td><td>32,17%</td><td></td><td>7,25%</td><td>1,81%</td><td>0,54%</td><td>0,05%</td></tr>
<tr><td>Loss</td><td>-26,34%</td><td>-44,29%</td><td>-11,29%</td><td>-2,53%</td><td>-0,69%</td><td>-0,08%</td></tr>
<tr><td rowspan="2">3 DEFAULTS</td><td>Premium</td><td>48,37%</td><td></td><td>10,66%</td><td>2,36%</td><td>0,67%</td><td>0,06%</td></tr>
<tr><td>Loss</td><td>-46,42%</td><td>-57,28%</td><td>-21,03%</td><td>-4,49%</td><td>-1,20%</td><td>-0,12%</td></tr>
</table>

In the first violet row we see the resulting premiums in case of no default. In the next rows the number of defaults increases. Moreover, for each tranche we displayed firstly, the new market premium and secondly, the loss on a tranche incurred by the investor who bought the tranche just before the defaults came about.

To be more specific about the nature of the loss, it is a loss representing the difference between the expected discounted cash-flows based on the premium agreed and the premium after the default. Therefore it is not exactly the mark-to-market loss as both premiums represent fair (model) prices not market prices.

It is notable, that the loss is not expressed per annum. It is an immediate loss after the defaults. Its per annum size depends on the timing of defaults.

Remark that for the equity tranche there is one more column showing total loss on the tranche. That is the loss has two parts. The first one is the mark-to-market loss displayed in the first column and the second is the loss from settlement of the default bringing obligatory delivery of the defaulted bond to the investor. Total loss therefore depends on the recovery rate of the defaulted bond. E.g. in case of one default and 0,4 recovery rate the investor looses 2453 USD mark-to-market and $8000 \cdot (1 - RR) = 8000 \cdot 0{,}6 = 4800\ USD$ due to the delivery obligation. Total loss is therefore 7253 USD, giving us 24,18% loss on his 30 000 USD investment.

Supposing 40% recovery rate the loss can't exceed 60% of the investment. See that in case of 3 defaults the equity tranche investor looses 57,28% of his investment.

Note also, that the premium of the most senior tranche doubles with 3 defaults. Due to its low quote before the defaults we don't arrive at very high mark-to-market loss. All premiums are at least twice as high after three defaults compared to the no default state (except 10-15% tranche where the premium nearly doubles).

Recall that this calculation is based on fixed correlation and hazard rate value and therefore they represent just a theoretical – ceteris paribus - evaluation of a sensitivity of premiums to events of default. In the next subchapter we will ease these assumptions and base our calculation of mark-to-market losses on real time data.

5.2 CDO and Credit Crunch: Mark-to-Market Loss Valuation

In this subchapter we will present the data describing the situation of a CDO market in past year and a half. In its first part, tranche's market quotes and values of alternative versions of CDXs are showed to illustrate a course of events in fields of credit. In the second part we calculate the losses experienced by CDO investors transforming a CDX index into a CDO.

5.2.1 CDX Historical Prices

5.2.1.1 CDX Versions' Historical Prices

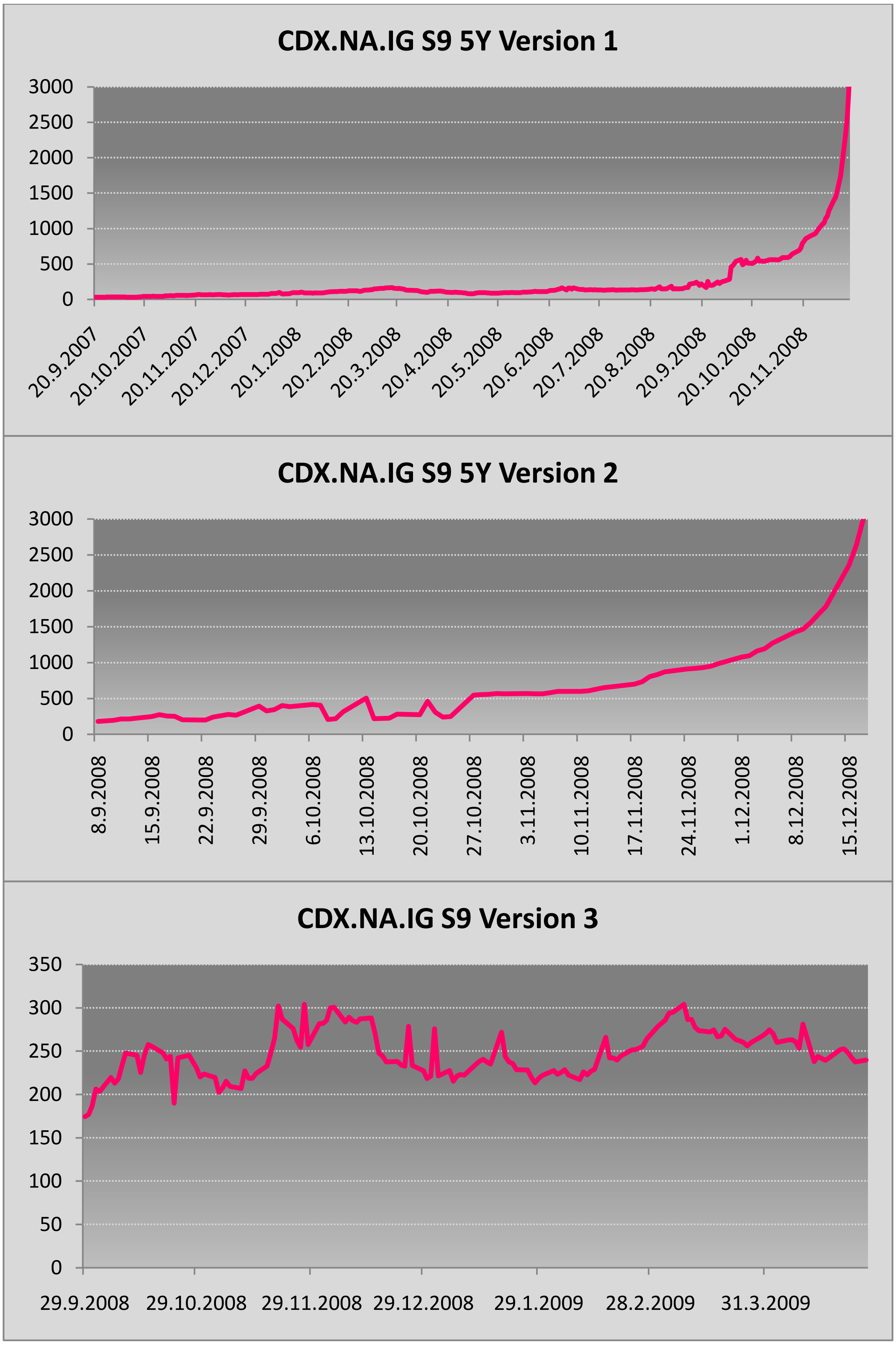
Firstly, to present the overall development on the credit market, see market quotes of the whole CDX.NA.IG Series 9 index with 5 years maturity. Recall that this subindex has the highest liquidity. From Section 4.1. we know that there were three events of default in the index. The information on the defaults are provided in Table 6:

Table 6: **Events of default in CDX.NA.IG S9 underlying**

Defaulted entity	Date of settlement	Final Price
Freddie Mac	6.10.2008	57,00%
Fannie Mae	6.10.2008	94,00%
Washington Mutual	23.10.2008	95,51%

If there is an event of default in an index, a casual settlement follows - the same as in case of a CDO. But subsequently, the version of an index ceases to exist and a new version with lower number of underlying assets is introduced. As the Federal National Mortgage Association (Fannie Mae) and the Federal Home Loan Mortgage Corporation (Freddie Mac), both US government sponsored entersprises, announced their defaults at once, there were only two reversionings of the index.

Figure 23: **Value of alternating versions of CDX.NA.IG Series 9**



Source: Bloomberg

It is evident that reactions of an index to an announcement of a default were enormous. E.g. the first version premium was decreasing since its issue date. It started trading for 30 bps., on 5/9/2008 it was already 148 bps. On 8/9/2008 the Federal Housing Finance Agency announced it had put Fannie Mae and Freddie Mac under its conservatorship, which is considered as a credit event. Washington Mutual nationalization was announced two weeks later. Then, after the auction and settlement of the defaults the market quote skyrocketed to 3000 bps. within two months.

Consequently, the second version wasn't valid long and it was replaced by the third version which is now traded on expressively higher levels that the first version after its issuance – around 250 bps.

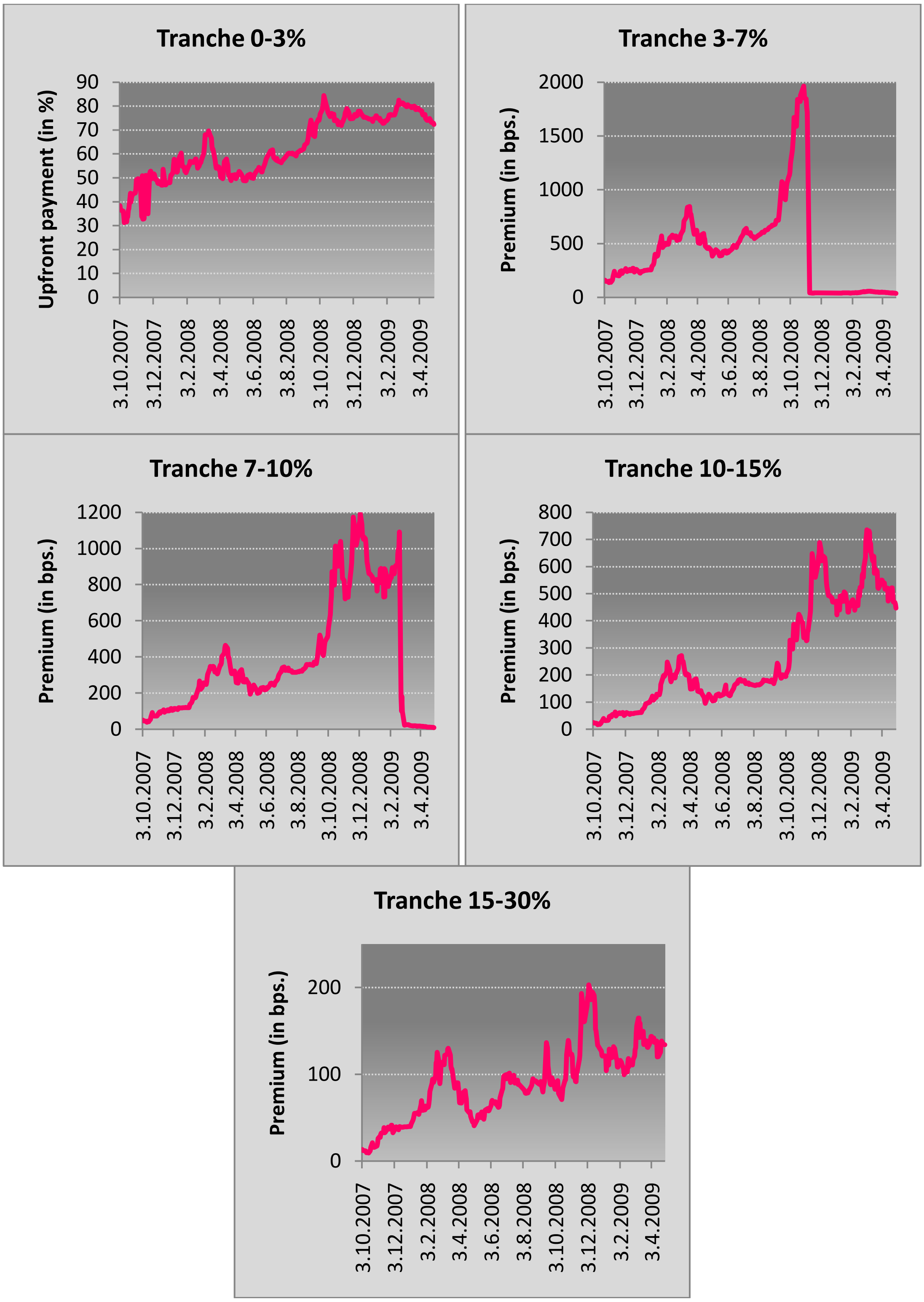
5.2.1.2 CDX Tranche's Historical Prices

In the following figure there are historical quotes of the index tranches. It is for the reader to become acquainted with the market development during the crisis. The most senior tranche is not included as the volumes traded are low. The reason is that CDOs usually do not offer the highest tranche so there are no motives to hedge a CDO position using this tranche of the Index.

On the issue date, only the equity tranche was quoted in form of an upfront payment + running spread of 500 bps. However, during the existence of the index two more senior tranches' quotes had to be transformed to this form as the quote in form of pure running spread was too high to be quoted in basis points. That explains a fierce drop in market quotes by these tranches.

The most serious increases can be observed by mezzanine tranches: if we take the highest and the lowest running spread during the observed period, it grew 42 times by tranche 10-15% and 31 times by tranche 7-10%. The reasoning for the enormous growth is explained by an immense growth of apprehension on the market. It started by the mortgage crisis in the US, consequences of which spread quickly to various business sectors. The link which enabled the expansion of the crisis was securitization. Through securitization the credit risk was transfered from banks to other banks, financial institutions and corporations. Then, a chain reaction of financial troubles and defaults followed.

Figure 24: **CDX tranches' market quotes since 20/9/2007**



Source: Bloomberg

5.2.2 *Transformation of a CDX to a CDO: Loss Evaluation*

Hypothesis 5: There has been a substantial loss even on the most senior tranche without a necessity to be hit directly by a default

An expressive increase of a credit risk is obvious from the previous graph. The premiums of the tranches multiplied many times in the crisis even through the fact that after each default the index was reversioned. In this part, we will transform the CDX to a CDO, i.e. we will abolish the feature of new version following each default.

A loss after a default can be separated to three parts. Loss as a consequence of:

1. increased perception of credit risk
2. new definition of tranche attachment and detachment point
3. settlement of the defaulted asset.

The first part is illustrated by Figure 24. It points to a pure increase of fear on the market. Numerically it is expressed by an increase of hazard rate and correlation between assets.

The second part of the loss was evaluated in section 5.1.3. and its values are listed in Table 5. Numerically it is expressed by a decrease of underlying assets with a fixed volume and decrease of attachment and detachment points' absolute values by the notional of the defaulted assets, both with fixed hazard rate and correlation. The third part of loss bears only one tranche depending on sequence of the default.

In our calculation we suppose a CDO tranche buyer who enters the CDO contract on 20/9/2007 and he holds it until 28/2/2009. It means that his CDO suffers three defaults during its life. Our task is to evaluate his loss on 28/2/2009 based on the difference between the premium he agreed and current fair premium based on expected cash-flows. First, we evaluate the CDO at the issue date. Then, we evaluate it on the valuation date with new parameters:

Table 7: **Change in parameters of the model since 20/9/2007 to 28/2/2009**

	20.9.2007	28.2.2009
Correlation	0,32	0,44
Hazard rate	0,01	0,07
Number of assets	125	122
Notional invested	10 M USD	9,76 M USD
AP and DP		-2,4 percentage points

The hazard rate was deduced from the CDS spreads of the underlying assets based on 0,39 recovery rate. It increased seven times since autumn 2008. Therefore Hypothesis 3 is fully confirmed. The correlation is set as an average base correlation for three lowest tranches (see Figure 20). Results of the valuation are in the next table.

Table 8: **Mark-to-Market loss on a CDO tranche on 28/2/09 with 10 M USD initial investment**

		TRANCHE				
		0-3%	3-7%	7-10%	10-15%	15-30%
20.9.2007	Premium	**14,69%**	**4,21%**	**1,89%**	**0,88%**	**0,19%**
28.2.2009	Premium	**121,10%**	**46,94%**	**26,52%**	**17,76%**	**8,37%**
	% M-t-M Loss	-82,12%	-71,28%	-57,39%	-46,13%	-26,59%
	M-t-M Loss	**1 642 400**[26]	**7 128 000**	**5 739 000**	**4 613 000**	**2 659 000**

To be exact, it is important to recall, that what we call the mark-to-market loss is in fact the loss based on changed values of expected cash-flows (i.e. the loss based on mark-to-market change of entry parameters, then the tranches are still valued by the model). The real mark-to-market loss would have to be derived from the market value of an instrument based on principle of fair value accounting. There are no available market data to a particular CDO but we can deduce (e.g. by looking at first two parts in Figure 23) that this loss would be much higher. Actually, it is probable that such CDO contract would have to be terminated before our valuation date. The reason is that we don't fully include the liquidity risk and market sentiment, notwithstanding that it is partly included in the hazard rate.

[26] 82,2% of loss refers to the rest of the notional of the equity tranche. Since there have already been three defaults, the ivestor lost 80% of the notional (he was partly compensated by the recovered part of the defaulted bond). On the rest – i.e. 20% of the notional - he faces the mark-to-market loss of 82,2%.

The outcomes of our model using the expected cash-flows are alarming. The premium on the most senior tranche increased 44 times since the issue date. The loss on this tranche is 26,50% of the notional. What is even more serious, such tranches usually retain the highest possible rating score from the rating agencies. Hypothesis 5 can now be also approved.

Even though only the equity tranche investors were factually hit by the defaults, all tranches were hit indirectly - in form of mark-to-market losses regardless of their rating. It is still very unlikable that there will be more than 18 defaults so that the 15-30% tranche was hit, but its mark-to-market loss is high. Therefore, even if the investor decides to hold the tranche to maturity, as a financial institution it will have to report a significant loss in its accounting.

5.3 What Went Wrong?

In the next table major writedowns as a result of an obligation of financial institutions to mark assets to market values in their accounting are displayed:

Table 9: **Top writedowns on 19/2/2009 (in Billions USD)**

		USD Billion
1	Citigroup	59,9
2	Merrill Lynch	58,3
3	UBS	50,3
4	AIG	47,3
5	Bank of America	21,6
6	Royal Bank of Scotland	17,5
7	Freddie Mac	17,0
8	Lehman Brothers	15,3
9	Deutsche Bank	14,8
10	J.P. Morgan	13,3

Source: www.abalert.com

The volume of writedowns is immense. What is appealing that absolute majority of these and countless other writedowns result from just seven credit events in total:

1. Washington Mutual Inc.: US largest savings and loan association
2. Lehman Brothers Holdings Inc.: global financial services firm
3. Federal National Mortgage Association (Fannie Mae): US mortgage company

4. Federal Home Loan Mortgage Corporation (Freddie Mac): US mortgage company
5. Glitnir: Icelandic bank
6. Kaupthing: Icelandic bank
7. Landsbanki: Icelandic bank

First four institutions defaulted or were overtaken by state or other institution due to the subprime crisis. To put it briefly, Washington Mutual offered mortgages and credit cards to less creditworthy borrowers and it was hit by their inability to pay back followed by a bank run. Lehman Brothers were holding large positions in mortgage backed securities with lower rating and suffered sharp losses by the settlement of the defaults of borrowers. And Fannie Mae and Freddie Mac were damaged by purchasing and securitization of mortgages. All three major Icelandic banks defaulted and had to be nationalized due to their holding of worldwide asset backed securities and strong link to US and European financial markets.

After the credit events numerous downgrades by rating agencies followed. They downgraded the tranches of CDOs where any of the 7 companies had an underlying asset[27]. Then the companies that held positions in these CDOs had to be downgraded too[28]. Often, these companies were also included in CDOs and therefore further downgrades and mark-to-market losses followed.

Thus, the mortgage crisis was no doubt the trigger of the following complex credit crunch. But what went wrong that the losses were so high and CDO market collapsed after a couple of defaults?

Standard&Poors estimates that in October 2008 there were 3000 CDO contracts[29]. 75% of the synthetic CDOs sold swaps on Lehman Brothers. 376 contracts included Kaupthing, Glitnir or Landsbanki, 1500 contracts included Washington Mutual and 1200 contracts included both Fannie Mae and Freddie Mac. In Europe, 75% of all

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[27] E.g. Standard&Poor's downgraded 791 tranches of CDOs during one week in December 2008.
[28] E.g. AIG, MBIA or Ambac were downgraded due to CDS hedging their CDO positions losses.
[29] Tomson Reuters news on 22/10/2008

CDO deals contained at least one of the 7 defaulted companies. It is obvious that <u>we can't talk about real diversification</u> in case of CDOs.

Not diversified CDOs' portfolios, a high cohesion of international financial markets together with the spiral of mark-to-market losses and downgrades had to have these disasterous consequences. But why didn't the market realize the possibility of such losses? Why didn't the investors price these risks and let the market prices of CDO decrease to low levels? In the following scheme we suggest the main weaknesses of the CDO market, their effects and lessons that should be learnt.

Figure 25: **Main flaws of the CDO market**

What went wrong?				
	1.	2.	3.	4.
Mistakes	Insuficient analysis of underlying assets	Misunderstanding of the valuation model	Mispriced correlation	Use of mark-to-market valuation principle
Explanation	- low diversification of underlying assets - volume of CDOs issued on one bond exceeded the issued volume of the bond - few bonds included in a majority of CDOs	- model based on expected cashflows - mark-to-market loss not considered - reliance on the rating score without understanding the model	base and implied correlation derived from the market quotes which were set on lower than fair levels	obligation to mark assets to market quotes even if held to maturity
Effects	chain reaction of massive downgrades and losses after just one default	- huge mark-to-market losses triggered dy the downgrades - lower premiums required from investors: mispricing	biased CDO pricing	market quotes' overreaction and freeze of the CDO market resulted in multiplied writedowns
Lessons	- deeper analysis done by investors - better diversification - use of more complex methods by rating agencies	stress-testing: changing a hazard rate, correlation and number of defaults for a better appraisal of risks	correlation can be only priced fairly if market values fairly CDO tranches	introduction of a possibility to value assets by the expected cash-flows in some cases

First, CDO investors didn't do any deeper analysis of the underlying assets. It should have been alarming that in many cases the issued volume of a bond was much lower than the total volume included in CDO contracts. Rating agencies should have also comprehended the low diversification and the threat of CDO market breakdown after even a few defaults due to advanced complexity of the market.

Second, the valuation model understanding was often not complete. Even the basic model we introduced in Chapter 3 was not comprehended by the investors. They relied on rating agency score and weren't suspicious why a bond with a AAA rating by S&P yields less than a CDO tranche with the same rating. The valuation model is a probability model which derives a price of a CDO based on probability of default. The extreme case of multiple credit events is taken into account - it is priced in. Its probability is low but not zero, such that it can happen – and it happened.

Moreover, the model is based on future expected cashflows. It shows the value for investors who hold it to maturity and don't have to mark it to market value. If they buy a senior tranche, even after three defaults the chance of being hit is still very low and their cashflow will be unchanged. Thus the basic idea of the model is correct. What the model doesn't take into account are the mark-to-market losses.

This should have been understood by the majority of investors that have to show the mark-to-market value of their assets. Stress tests on what will happen in case of a change in entry parameters – hazard rate and correlation – in combination with credit events should have been run. The same way we did it the previous section. Still, the resulting losses based on model quotes can only be considered as the lower limit of losses, because the market quotes tend to overreact in bad times. Then, these investors should have priced in the results. This complex analysis would lead to a better assessment of a risk and higher premiums required from a CDO seller.

Third, the correlation was obviously mispriced in the model. As explained in Chapter 4, either the implied correlation or the base correlation is derived from the market quotes. In the previous paragraph we argued that the tranches were mispriced and therefore also the correlation value was not correct. Only after the market proper valuation of CDOs the actual value of correlation can be derived.

And last but not least, <u>the mark-to-market valuation principle according to the US law should be reconcidered</u>. After the defaults the CDO market froze and the quotes of tranches skyrocketed. All financial institutions still had to value their assets according to these market quotes, in spite of their intention to hold it until maturity. This duty induced multiple losses. On 3/10/2008 the Emergency Economic Stabilization Act (often referred to as a bailout of the US financial system) was pronounced in the US. Primarily, it set apart 700 Bn. USD to purchase distressed assets and increase the capital in banks. But it also allows in some cases suspending the mark-to-market accounting. Instead, the value of a distressed asset can be derived from the expected value of cash-flows, i.e. it can be valued according to the model. As showed in section 5.2.2, such valuation will still show huge losses after the default and change of entry parameters but it would not be as high as after using the distressed market quotes. Still, valuation rules should be defined clearly and deliberately so that there were no perturbations.

6 Conclusion

A CDO market has undoubtedly experienced a serious shock since late 2007. In this thesis we research what were the main flaws of the market that enabled extensive writedowns from CDOs.

To be able to detect these flaws we present a valuation One Factor model based on a Gaussian Copula and develop a simple valuation program in MS Excel VBA which is attached to this thesis. It this program we run useful simulations to test five hypothesizes proclaimed which result in our sensitivities' analysis. After the output data evaluation we present four main deficiencies of the market and pronounce our recommendation to their elimination.

Specifically, as no data of a particular CDO are available to us, the base of our modeling is an index – CDX, the quotes of which we appropriately transform to CDO quotes. Then we run our valuation with varying entry parameters to show the sensitivities of all tranches. Finally, we compare a model value of a tranche before the CDO market was stroked by the crisis and after it to value a loss of CDO investors based on changed expected cash-flows. It is reasonable to say that this loss constitutes a lower bound of real mark-to-market losses incurred by investors.

First deficiency we perceive is insufficient analysis of underlying assets either by investors or by rating agencies. The fact that seven financial institutions that defaulted since September to December 2008 were included in 75% of all European synthetic CDOs should have been alarming. A result of such a poor diversification was a chain reaction of losses and downgrades of institutions and CDO tranches after a couple of defaults. A deeper analysis of a proper diversification and quality of underlying assets should be implemented.

Second deficiency is that the valuation model was often not understood well. Since the structure and the valuation of a CDO is quite sophisticated investors relied on a high rating of senior CDO tranches without understanding the main risks. The model is based just on expected cash-flows. The possibility of mark-to-market losses of the tranches should have been included in the pricing. Results of a stress-testing of the tranches would increase the expected premium payments.

This leads us to the third deficiency, i.e. the fact that also correlation had to be mispriced. Both implied and base correlation derives from the market quotes which were lowered by improper market optimism. Only after a deep understanding of the model the correlation can be priced correctly.

And finally, as we numerically demonstrated the mark-to-market valuation obligation for financial institutions should be reviewed and it should be possible to back out of it in cases of a frozen market when risk premiums explode. Then the expected cash-flows valuation should be considered, especially if the instrument is held to maturity. Otherwise a next set of writedowns and downgrades may be triggered.

If the previous recommendations will be adhered, the CDO market has a chance to be regenerated. Securitization and credit market is needed, but the trades have to be done rationally and deliberately. And that was not the case of past couple of years. A future CDO market would then be more conscious, driven by smarter motives and definitely less extensive.

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