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Measures of non-compactness of Sobolev embeddings

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In Prague, May 9, 2018

Ondřej Bouchala

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Díky!

Title: Measures of non-compactness of Sobolev embeddings

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Abstract: The measure of non-compactness is defined for any continuous mapping $T\colon X \to Y$ between two Banach spaces X and Y as

$$\beta(T) := \inf \left\{ r > 0\colon \begin{array}{l} T(B_X) \text{ can be covered by finitely} \\ \text{many open balls with radius } r \end{array} \right\}.$$

It can easily be shown that $0 \leq \beta(T) \leq \|T\|$ and that $\beta(T) = 0$, if and only if the mapping T is compact.

My supervisor prof. Stanislav Hencl has proved in his paper that the measure of non-compactness of the known embedding $W_0^{k,p}(\Omega) \to L^{p^*}(\Omega)$, where kp is smaller than the dimension, is equal to its norm.

In this thesis we prove that the measure of non-compactness of the embedding between function spaces is under certain general assumptions equal to the norm of that embedding. We apply this theorem to the case of Lorentz spaces to obtain that the measure of non-compactness of the embedding

$$W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)$$

is for suitable p and q equal to its norm.

Keywords: Measure of non-compactness, Sobolev spaces, Lorentz spaces

Contents

Chapter 1

Introduction

In this thesis we study the measure of non-compactness of linear mappings. This notion was firstly introduced by Gohberg, Goldenstein and Markus in 1957 in [1].

1.1 Definition
Let X and Y be Banach spaces and let T be a continuous linear mapping from X into Y. Let us denote the open unit ball in X centered at origin by B_X. We define the **measure of non-compactness** of T as

$$\beta(T) := \inf \left\{ r > 0\colon \begin{array}{l} T(B_X) \text{ can be covered by finitely} \\ \text{many open balls with radius } r \end{array} \right\}.$$

It can be easily shown, that $0 \leq \beta(T) \leq \|T\|$ (see Observation 2.26). Moreover, the mapping T is compact if and only if $\beta(T) = 0$ (see Proposition 2.28).

In particular we are concerned with the measure of non-compactness of the Sobolev embeddings. In the paper [2] it was proven, that the measure of non-compactness of the classical embedding of $W_0^{k,p}(\Omega)$ into $L^{p^*}(\Omega)$ is equal to its norm, i.e.

$$\beta\left(Id\colon\ W_0^{k,p}(\Omega) \to L^{p^*}(\Omega)\right) = \|Id\colon\ W_0^{k,p}(\Omega) \to L^{p^*}(\Omega)\|. \tag{1}$$

Here we simplify the proof of this result, and we generalize the result to Sobolev-Lorentz embedding. For the definitions of Sobolev-Lorentz and Lorentz spaces see Preliminaries.

1.2 Theorem (Non-compactness of embedding into Lorentz spaces)
Let $d \geq 2$, $k \in \mathbb{N}$, $k < d$, $1 \leq p < \frac{d}{k}$, denote $p^* = \frac{dp}{d-kp}$ and let $1 \leq q < \infty$. Let either $p > 1$ or $p = q = 1$. Let Ω be an open subset of $\mathbb{R}^d$ with Lipschitz boundary. Then

$$\beta\left(Id\colon\ W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)\right) = \|Id\colon\ W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)\|.$$

In Chapter 2 we define the function spaces and state or prove the properties needed in the following chapters. Furthermore we define the measure of non-compactness and prove the basic facts about it.

The result (1) from [2] is shown in Chapter 3 with slightly simplified proof.

In Chapter 4 we formulate and prove general statement concerning embeddings and measure of non-compactness, and apply it to prove Theorem 1.2.

And in the last Chapter 5 we use the embedding of $W_0^{1,1}((0,1))$ into the space of continuous functions $\mathcal{C}((0,1))$ to show, that the measure of non-compactness can be smaller then the norm of the embedding.

Chapter 2

Preliminaries

2.1 Sobolev spaces

2.1 Notation

Let X be a Banach space. We denote the open ball with center x and radius r by $B_X(x, r)$, and the open unit ball centered at origin will be denoted by $B_X = B_X(0, 1)$.

Let $x \in X$, $\varepsilon > 0$. By $x + \varepsilon B_X$ we mean the ball $B_X(x, \varepsilon)$.

2.2 Notation

We denote the characteristic function of set E by χ_E, that is

$$\chi_E(x) := \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

2.3 Notation

Let γ be a multi-index, i.e. a finite sequence of non-negative integers. If $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_d)$, then we denote the norm of γ by

$$|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_d.$$

For suitable $f\colon\ \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$ we denote the weak (distributional) derivative by

$$D^\gamma f(x) := \frac{\partial^{|\gamma|} f}{\partial^{\gamma_1} x_1 \partial^{\gamma_2} x_2 \cdots \partial^{\gamma_d} x_d}(x).$$

2.4 Proposition

Let $f\colon\ \mathbb{R}^d \to \mathbb{R}$ have weak derivatives up to the order k and let $|\gamma| \leq k$. Let us denote $f_K(x) := f(Kx)$ for $K \in (0, \infty)$. Then

$$D^\gamma f_K(x) = K^{|\gamma|}(D^\gamma f)(Kx) = K^{|\gamma|}(D^\gamma f)_K(x).$$

Proof:

We know that for $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ the statement holds by the chain rule. In particular it holds for $\frac{1}{K}$, that is

$$D^\gamma \varphi_{\frac{1}{K}}(x) = K^{-|\gamma|}\,(D^\gamma \varphi)\left(\frac{1}{K}x\right) = K^{-|\gamma|}(D^\gamma \varphi)_{\frac{1}{K}}(x).$$

Then we have by the definition of weak derivative and change of variables

$$\begin{aligned}\int_{R^d} D^\gamma f_K(x)\cdot\varphi(x)\,\mathrm{d}x &= (-1)^{|\gamma|}\int_{\mathbb{R}^d} f(Kx)\cdot D^\gamma\varphi(x)\,\mathrm{d}x\\ &= (-1)^{|\gamma|}\int_{\mathbb{R}^d} f(y)\cdot D^\gamma\varphi\left(\frac{1}{K}y\right)\frac{\mathrm{d}y}{K^d}\\ &= (-1)^{|\gamma|}\int_{\mathbb{R}^d} f(y)\cdot K^{|\gamma|}D^\gamma\varphi_{\frac{1}{K}}(y)\frac{\mathrm{d}y}{K^d}\\ &= K^{|\gamma|}\int_{\mathbb{R}^d} D^\gamma f(y)\cdot\varphi\left(\frac{1}{K}y\right)\frac{\mathrm{d}y}{K^d}\\ &= K^{|\gamma|}\int_{\mathbb{R}^d} (D^\gamma f)_K(x)\cdot\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Q.E.D.

2.5 NOTATION (Sobolev space)
Let Ω be an open subset of $\mathbb{R}^d$, $k\in\mathbb{N}$ and $p\in[1,\infty)$. For $f\in W_0^{k,p}(\Omega)$ we consider the norm

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\gamma|\le k}\int_\Omega |D^\gamma g(x)|^p\ \mathrm{d}x\right)^{\frac{1}{p}}.$$

By $W_0^{k,p}(\Omega)$ we denote the set of functions from $W^{k,p}(\Omega)$ with zero traces.

The classical continuous Sobolev embedding into Lebesgue space can be found e.g. in [3, Theorem 2.4.2].

2.6 THEOREM (Sobolev embedding)
Let Ω be an open set, $p\ge 1$ and $kp<n$. Let us denote $p^* := \frac{np}{n-kp}$. Then

$$W_0^{k,p}(\Omega)\hookrightarrow L^{p^*}(\Omega).$$

2.2 Lorentz spaces

2.7 DEFINITION
Let f be a measurable function from measurable set $\Omega\subseteq\mathbb{R}^d$ to $\mathbb{R}$. We define the **distribution function** as

$$f_*(s) := |\{x\in\Omega\colon\ |f(x)|>s\}|,$$

where $s>0$ and $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^d$.

We define the **non-increasing rearrangement** as

$$f^*(t) := \inf\{s>0\colon\ f_*(s)\le t\},\quad t\in(0,\infty)$$

and we define **double-star operator** as

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s),\quad t\in(0,\infty).$$

2.8 Notation

Let $f\colon \Omega \to \mathbb{R}$ be a function. We denote

$$\{f > s\} := \{x \in \Omega\colon\ f(x) > s\},$$

We denote it analogously for other types of (in)equalities $(<, \geq, \leq, =)$.

2.9 Proposition

Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Let $K \in (0, \infty)$ and let us denote $g_K(x) := g(Kx)$. Then

$$(g_K)^*(t) = g^*(K^d t).$$

Proof:

From the definition of the distribution function and change of variables $y = Kx$ it follows, that

$$\begin{aligned}
(g_K)_*(s) &= |\{|g_K| > s\}| \\
&= \int_{\{|g_K(x)|>s\}} 1\,\mathrm{d}x = \int_{\{|g(Kx)|>s\}} 1\,\mathrm{d}x \\
&= \int_{\{|g(y)|>s\}} \frac{1}{K^d}\,\mathrm{d}y \\
&= \frac{g_*(s)}{K^d}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(g_K)^*(t) &= \inf\{s > 0\colon\ (g_K)_*(s) \leq t\} \\
&= \inf\{s > 0\colon\ g_*(s) \leq K^d t\} = g^*(K^d t).
\end{aligned}$$

Q.E.D.

2.10 Proposition

Let $g\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Let $K \in (0, \infty)$ and let us denote $g_K(x) := g(Kx)$. Then

$$(g_K)^{**}(t) = g^{**}\left(K^d t\right).$$

Proof:

From Proposition 2.9 it follows, that

$$\begin{aligned}
(g_K)^{**}(t) &:= \frac{1}{t}\int_0^t (g_K)^*(s)\,\mathrm{d}s \\
&= \frac{1}{t}\int_0^t g^*(K^d s)\,\mathrm{d}s \\
&= \frac{1}{K^d}\frac{1}{t}\int_0^{K^d t} g^*(s)\,\mathrm{d}s =: g^{**}(K^d t).
\end{aligned}$$

Q.E.D.

2.11 Corollary

Let $g: \mathbb{R}^d \to \mathbb{R}$ have weak derivatives up to the order k, $|\gamma| \leq k$ and let $K > 0$. Let us denote $g_K(x) := g(K \cdot x)$. Then by Proposition 2.4 and Proposition 2.14 i) we have

$$\begin{aligned}(D^\gamma(g_K))^*(t) &= \left(K^{|\gamma|} \cdot (D^\gamma g)_K\right)^*(t) &= K^{|\gamma|} \cdot (D^\gamma g)^*(K^d t) \quad \text{and} \\ (D^\gamma(g_K))^{**}(t) &= \left(K^{|\gamma|} \cdot (D^\gamma g)_K\right)^{**}(t) &= K^{|\gamma|} \cdot (D^\gamma g)^{**}(K^d t).\end{aligned}$$

2.12 Lemma

Let f and g be two functions from $\Omega \subseteq \mathbb{R}^d$ to $\mathbb{R}$ with disjoint supports and let $s > 0$. Then

$$(f+g)_*(s) = f_*(s) + g_*(s).$$

Proof:

Clearly

$$\begin{aligned}(f+g)_*(s) &= |\{|f+g| > s\}| \\ &= |\{|f| > s\} \cup \{|g| > s\}| \\ &= |\{|f| > s\}| + |\{|g| > s\}| \\ &= f_*(s) + g_*(s).\end{aligned}$$

Q.E.D.

2.13 Definition

Let Ω be an open subset of $\mathbb{R}^d$ and let $m, q \in [1, \infty]$. We define **Lorentz space** $L^{m,q}(\Omega)$ as

$$L^{m,q}(\Omega) := \{f\colon\ \Omega \to \mathbb{R},\ \text{such that } \|f\|_{L^{m,q}(\Omega)} < \infty\},$$

where

$$\|f\|_{L^{m,q}(\Omega)} := \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{m}} \cdot f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{when } q < \infty, \\ \sup\limits_{t>0} \left(t^{\frac{1}{m}} \cdot f^*(t)\right) & \text{when } q = \infty. \end{cases}$$

Furthermore we define

$$\|f\|_{L^{(m,q)}(\Omega)} := \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{m}} \cdot f^{**}(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{when } q < \infty, \\ \sup\limits_{t>0} \left(t^{\frac{1}{m}} \cdot f^{**}(t)\right) & \text{when } q = \infty. \end{cases}$$

2.14 Proposition

i) For $c \geq 0$: $(c \cdot f)^* = c \cdot f^*$.

ii) For $p \in [1, \infty]$ it holds that $\|\bullet\|_{L^{p,p}(\Omega)} = \|\bullet\|_{L^p(\Omega)}$.

iii) For $1 \le q \le m$ the functional $\|\bullet\|_{L^{m,q}(\Omega)}$ is a norm and $L^{m,q}(\Omega)$ is a Banach space.

iv) For $1 < m < \infty$ the functional $\|\bullet\|_{L^{(m,q)}(\Omega)}$ is a norm equivalent to $\|\bullet\|_{L^{m,q}(\Omega)}$.

Proof:
Proof of these statements can be found for example in [4].

2.15 Proposition (Inclusions)
Let $1 \le m, q, M, Q \le \infty$, $\Omega \subseteq \mathbb{R}^d$ measurable.

- If $q < Q$, then $\|f\|_{L^{m,Q}(\Omega)} \le C \cdot \|f\|_{L^{m,q}(\Omega)}$,
- if $m < M$ and $|\Omega| < \infty$, then $\|f\|_{L^{m,q}(\Omega)} \le C \cdot \|f\|_{L^{M,Q}(\Omega)}$,

where $C > 0$ is a constant depending on d and $|\Omega|$.

Proof:
See [4, Theorem 3.8].

2.16 Proposition (Lorentz norm via distribution)
Let $m, q \in [1, \infty)$. Then

$$\|f\|^q_{L^{m,q}(\Omega)} = m \int_0^\infty s^{q-1} [f_*(s)]^{\frac{q}{m}} \, \mathrm{d}s.$$

Proof:
See [4, Proposition 3.6].

2.17 Definition
Let $\Omega \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$ and $m, q \in [1, \infty]$. We define **Sobolev-Lorentz space** as

$$W^k L^{m,q}(\Omega) := \left\{ f\colon\ \Omega \to \mathbb{R},\ \text{such that } \|f\|_{W^k L^{m,q}(\Omega)} < \infty \right\},$$

where

$$\|f\|_{W^k L^{m,q}(\Omega)} := \begin{cases} \left(\sum\limits_{|\gamma| \le k} \|D^\gamma f\|^q_{L^{m,q}(\Omega)} \right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \max\limits_{|\gamma| \le k} \|D^\gamma f\|_{L^{m,\infty}(\Omega)} & \text{if } q = \infty. \end{cases}$$

We define $W^k_0 L^{m,q}(\Omega)$ as

$$W^k_0 L^{m,q}(\Omega) := \{ f\colon\ \Omega \to \mathbb{R}\colon\ \widetilde{f} \in W^k L^{m,q}(\mathbb{R}^d) \},$$

where

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \Omega \text{ and} \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

As in the case of Lorentz spaces we can define $\|\bullet\|_{W^k L^{(m,q)}(\Omega)}$ with the same formula where we use $\|\bullet\|_{L^{(m,q)}(\Omega)}$ instead of $\|\bullet\|_{L^{m,q}(\Omega)}$.

The key element in the proof of the main Theorem 1.2 is the following proposition. The proof of the proposition is from [4, Lemma 3.10].

2.18 Proposition

Let $\Omega \subseteq \mathbb{R}^d$, $1 \le q \le m$ and let f_1 and f_2 be two functions from $L^{m,q}(\Omega)$ with disjoint support. Then

$$\|f_1\|^m_{L^{m,q}(\Omega)} + \|f_2\|^m_{L^{m,q}(\Omega)} \le \|f_1 + f_2\|^m_{L^{m,q}(\Omega)}.$$

Proof:

If $q = m$, then $L^{m,q} = L^m$ and the inequality holds because for f_1 and f_2 with disjoint supports we have

$$\|f_1 + f_2\|^m_{L^m(\Omega)} = \int_\Omega |f_1 + f_2|^m = \int_\Omega |f_1|^m + \int_\Omega |f_2|^m = \|f_1\|^m_{L^m(\Omega)} + \|f_2\|^m_{L^m(\Omega)}.$$

So we may assume that $q < m$. From Lemma 2.12 we know that

$$(f_1)_* + (f_2)_* = (f_1 + f_2)_*.$$

Hölder's inequality for measure $s^{q-1}\,\mathrm{d}s$ yields

$$\begin{aligned}
&\left(\int_0^\infty s^{q-1} (f_j)_*^{\frac{q}{m}}(s)\,\mathrm{d}s\right)^{\frac{m}{q}} \\
&\quad= \left(\int_0^\infty s^{q-1} \left((f_j)_*^{\frac{q}{m}}(s)(f_1+f_2)_*^{\frac{q(q-m)}{m^2}}(s)\right)\left((f_1+f_2)_*^{\frac{q(m-q)}{m^2}}\right)\mathrm{d}s\right)^{\frac{m}{q}} \\
&\quad\le \left(\int_0^\infty s^{q-1}(f_j)_*(s)(f_1+f_2)_*^{\frac{q}{m}-1}(s)\,\mathrm{d}s\right)\left(\int_0^\infty s^{q-1}(f_1+f_2)_*^{\frac{q}{m}}(s)\,\mathrm{d}s\right)^{\frac{m}{q}-1}
\end{aligned}$$

for $j = 1, 2$. We apply Proposition 2.16 and sum over j to get with the help of $q < m$ that

$$\begin{aligned}
m^{-\frac{m}{q}} \sum_{j=1}^2 \|f_j\|^m_{L^{m,q}(\Omega)} &= \sum_{j=1}^2 \left(\int_0^\infty s^{q-1}(f_j)_*^{\frac{q}{m}}(s)\,\mathrm{d}s\right)^{\frac{m}{q}} \\
&\le \left(\int_0^\infty s^{q-1}(f_1+f_2)_*^{\frac{q}{m}}(s)\,\mathrm{d}s\right)^{\frac{m}{q}-1} \sum_{j=1}^2 \left(\int_0^\infty s^{q-1}(f_j)_*(s)(f_1+f_2)_*^{\frac{q}{m}-1}(s)\,\mathrm{d}s\right) \\
&= \left(\int_0^\infty s^{q-1}(f_1+f_2)_*^{\frac{q}{m}}(s)\,\mathrm{d}s\right)^{\frac{m}{q}} \\
&= m^{-\frac{m}{q}} \|f_1 + f_2\|^m_{L^{m,q}(\Omega)}.
\end{aligned}$$

Q.E.D.

For the case $q > m$ analogous statement holds as well, but with different power. For that we need the following inequality.

2.19 Lemma

Let $a, b \ge 0$ and let $1 \le p < \infty$. Then

$$(a + b)^p \ge a^p + b^p.$$

Proof:
If $a = b = 0$, then there is nothing to prove. So let $a+b > 0$. The function x^p is convex, therefore

$$a^p = \left(\frac{b}{a+b}\cdot 0 + \frac{a}{a+b}\cdot(a+b)\right)^p \le \frac{b}{a+b}\cdot 0^p + \frac{a}{a+b}\cdot(a+b)^p \quad \text{and}$$
$$b^p = \left(\frac{a}{a+b}\cdot 0 + \frac{b}{a+b}\cdot(a+b)\right)^p \le \frac{a}{a+b}\cdot 0^p + \frac{b}{a+b}\cdot(a+b)^p.$$

Summing these two inequalities gives us the statement.

Q.E.D.

2.20 Proposition

Let $\Omega \subseteq \mathbb{R}^d$, $1 \le m < q < \infty$ and let f and g be two functions from $L^{m,q}(\Omega)$ with disjoint supports. Then

$$\|f\|^q_{L^{m,q}(\Omega)} + \|g\|^q_{L^{m,q}(\Omega)} \le \|f+g\|^q_{L^{m,q}(\Omega)}.$$

Proof:
Thanks to Lemma 2.12, Proposition 2.16 and Lemma 2.19 for $p := \frac{q}{m} > 1$ we have

$$\begin{aligned}
\|f+g\|^q_{L^{m,q}(\Omega)} &= m\int_0^\infty s^{q-1}(f+g)_*^{\frac{q}{m}}(s)\,\mathrm{d}s \\
&= m\int_0^\infty s^{q-1}(f_*+g_*)^{\frac{q}{m}}(s)\,\mathrm{d}s \\
&\ge m\int_0^\infty s^{q-1}(f_*)^{\frac{q}{m}}(s)\,\mathrm{d}s + m\int_0^\infty s^{q-1}(g_*)^{\frac{q}{m}}(s)\,\mathrm{d}s \\
&= \|f\|^q_{L^{m,q}(\Omega)} + \|g\|^q_{L^{m,q}(\Omega)}.
\end{aligned}$$

Q.E.D.

2.21 Definition

Let X be a Banach space of functions from $\Omega \subseteq \mathbb{R}^d$ to $\mathbb{R}$ and let $1 \le m < \infty$. We say that X is **disjointedly m-superadditive**, if there is a constant $M > 0$ such that for any finite sequence of functions $\{f_i\}_{i=1}^k \subseteq X$ with disjoint supports it holds, that

$$\sum_{i=1}^k \|f_i\|_X^m \le M\left\|\sum_{i=1}^k f_i\right\|_X^m.$$

Furthermore we say that X is **monotone** if restricting decreases norm, that is if $E \subseteq \Omega$ and $f \in X$, then

$$\|f\cdot\chi_E\|_X \le \|f\|_X.$$

2.22 Remark

The Lebesgue spaces L^m and Lorentz spaces $L^{m,q}$ are for $1 \le m < \infty$ clearly monotone.

If $q \leq m$, then Proposition 2.18 implies that $L^{m,q}$ (and therefore L^m) is disjointedly m-superadditive with $M = 1$.

For $q > m$ we must be a bit more careful, because the functional $\|\bullet\|_{L^{m,q}(\Omega)}$ is not a norm. But for $m > 1$ it is equivalent to the norm $\|\bullet\|_{L^{(m,q)}(\Omega)}$, and thanks to Proposition 2.20 we know that for $q < \infty$

$$\sum_{i=1}^{k} \|f_i\|_{L^{(m,q)}}^q \leq \sum_{i=1}^{k} v^q \|f_i\|_{L^{m,q}}^q \leq v^q \left\| \sum_{i=1}^{k} f_i \right\|_{L^{m,q}}^q \leq v^q V \left\| \sum_{i=1}^{k} f_i \right\|_{L^{m,q}}^q ,$$

where v and V are the constants from the equivalence of the functionals. Therefore for $\infty > q > m > 1$ the space $L^{m,q}$ equipped with the norm $\|\bullet\|_{L^{(m,q)}}$ is disjointedly q-superadditive.

The embeddings between Sobolev-Lorentz spaces and Lorentz spaces we study in the next chapters are ensured by [5, Theorem 6.9].

2.23 Theorem (Sobolev-Lorentz embedding)
Let $\Omega \subseteq \mathbb{R}^d$ be an open set with Lipschitz boundary, $d \geq 2$, $k \in \mathbb{N}$, $k < d$. Let $1 < p < \frac{d}{k}$ and $1 \leq q \leq \infty$. Denote $p^* := \frac{dp}{d-kp}$. Then there is a continuous embedding

$$W_0^k L^{p,q}(\Omega) \hookrightarrow L^{p^*,q}(\Omega).$$

In particular if we choose $q = p$ we can use Proposition 2.15 to get the continuous embeddings

$$W_0^{k,p}(\Omega) \hookrightarrow L^{p^*,p}(\Omega) \hookrightarrow L^{p^*}(\Omega).$$

2.24 Remarks

- The embedding holds even for $p = q = 1$. See again [5, Theorem 6.9].
- But if $p = 1$ and $q > 1$, then the functional $\|\bullet\|_{L^{p,q}(\Omega)}$ is not equivalent to any norm, so the situation is more complicated and will not be dealt with here.
- Let $q > p > 1$. Because we do not care about the constant of the embedding, and because the functionals $\|\bullet\|_{L^{p,q}(\Omega)}$ and $\|\bullet\|_{L^{(p,q)}(\Omega)}$ are equivalent, we may consider either of them in the definition of Lorentz or Sobolev-Lorentz space in the embedding.

2.3 Measure of non-compactness

It is more convenient to define the measure of non-compactness using the entropy numbers. The following definition is clearly equivalent to Definition 1.1.

Here we only need the definition of entropy numbers and definition of measure of non-compactness, for further properties and applications see for example [6] and references given there.

2.25 Definition

Let X and Y be two Banach spaces. Let $T\colon\ X \to Y$ be a bounded linear mapping. We define **entropy numbers** for $k \in \mathbb{N}$ as

$$e_k(T) := \inf \left\{ \varepsilon > 0\colon \text{ there exist } c_j \in Y, \text{ such that } T(B_X) \subseteq \bigcup_{j=1}^{2^{k-1}} (c_j + \varepsilon B_Y) \right\},$$

and we define the **measure of non-compactness** as

$$\beta(T) := \lim_{k\to\infty} e_k(T).$$

2.26 Observation

We can easily show that $0 \leq e_k(T) \leq \|T\|$. Furthermore, the numbers $e_k(T)$ are clearly non-increasing as $k \to \infty$, so $\beta(T)$ exists and $0 \leq \beta(T) \leq \|T\|$.

Proof:

Let us fix $k \in \mathbb{N}$. If $e_k(T) > \|T\|$, then there would be $\varepsilon > \|T\|$ such that $T(B_X)$ is not contained in 2^{k-1} balls with radius ε. But from the definition of the norm of T we know that $T(B_X) \subseteq \varepsilon B_Y$. The rest is obvious.

Q.E.D.

2.27 Theorem

Let K be a subset of a metric space. Then K is compact if and only if K is complete and totally bounded.

Proof:

See [7, Theorems 4.3.27-4.3.29].

2.28 Proposition

The mapping T between Banach spaces is compact if and only if $\beta(T) = 0$.

Proof:

"$\Rightarrow$":

Let $\varepsilon > 0$. The set $\overline{T(B_X)}$ is compact, and therefore it is totally bounded thanks to Theorem 2.27. Therefore there exists a finite ε-net for $T(B_X)$, that is there exist $k \in \mathbb{N}$ and at most 2^{k-1} points c_j in Y such that

$$T(B_X) \subseteq \bigcup_{j=1}^{2^{k-1}} (c_j + \varepsilon B_Y).$$

Therefore $e_k(T) \leq \varepsilon$ and so $\beta(T) \leq \varepsilon$. We conclude by sending ε to 0.

“$\Leftarrow$”:
Because Y is complete we know that $\overline{T(B_X)}$ is complete and thanks to Theorem 2.27 it suffices to show that $\overline{T(B_X)}$ is totally bounded. For that it is enough to show that for fixed $\varepsilon > 0$ there is a ε-net for $T(B_X)$. We know that $\beta(T) = 0$, so there is $k \in \mathbb{N}$ such that $e_k(T) < \frac{1}{2}\varepsilon$. The definition of $e_k(T)$ clearly ensures the existence of ε-net.

Q.E.D.

Chapter 3

Embeddings into Lebesgue spaces

In this chapter we show the result concerning the measure of non-compactness of the embedding of Sobolev space into Lebesgue space from the paper [2] with slightly less technical proof. These results are corollaries of the theorems proven in Chapter 4.

3.1 Notation

In $\mathbb{R}^n$ we will also use the l^p norm, that is

$$|x|_p := \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

and we denote the open ball in this norm by

$$B_p(x, r) := \{y \in \mathbb{R}^n\colon\ |x - y|_p < r\}.$$

3.2 Lemma

Let $\{b^i\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. Let $1 < p < \infty$. Let $x \in \mathbb{R}^n$ and $t > 0$ be such that every b^i is in $B_p(x, t)$. Then

$$t^p > \left(1 - \frac{1}{1 + (n-1)^{\frac{1}{p-1}}}\right)^p + \frac{n-1}{\left(1 + (n-1)^{\frac{1}{p-1}}\right)^p}.$$

Proof:

It is clearly enough to show, that

$$\left(1 - \frac{1}{1 + (n-1)^{\frac{1}{p-1}}}\right)^p + \frac{n-1}{\left(1 + (n-1)^{\frac{1}{p-1}}\right)^p} = \inf_{x\in\mathbb{R}^n} \max_{1\le i\le n} |b^i - x|_p^p. \qquad (2)$$

Obviously

$$\inf_{x\in\mathbb{R}^n} \max_{1\le i\le n} |b^i - x|_p^p = \inf_{x\in[-5,5]^n} \max_{1\le i\le n} |b^i - x|_p^p = \min_{x\in[-5,5]^n} \max_{1\le i\le n} |b^i - x|_p^p.$$

Because $1 < p < \infty$, the function $|b^i - x|_p^p$ is strictly convex in x for every $i \in \{1, 2, \ldots, n\}$, thus $M(x) := \max_{1\le i\le n} |b^i - x|_p^p$ is strictly convex as well.

Therefore it has unique minimizer in $[-5,5]^n$, which we denote by $c = (c_1,\dots,c_n)$. The uniqueness implies $c_1 = c_2 = \dots = c_n$, because if we permute the coordinates in c we do not change the value of $M(c)$, so if we have two different coordinates in c, than their transposition gives us different minimizer, which is a contradiction with the uniqueness.

Therefore $c = (s,s,\dots,s)$ for some $s \in \mathbb{R}$. Clearly if $s < 0$ then $M(c) > M((0,\dots,0))$, and if $1 < s$ then $M(c) > M((1,\dots,1))$, so $0 \le s \le 1$. And thus we know that s minimizes

$$f(s) := \max_{1\le i\le n} |b^i - (s,s,\dots,s)|_p^p = (1-s)^p + (n-1)s^p, \quad s \in [0,1].$$

This function is smooth and $f'(s) = p(n-1)s^{p-1} - p(1-s)^{p-1}$. From that we can deduce that the function f is decreasing on $\left(0, \frac{1}{1+(n-1)^{\frac{1}{p-1}}}\right)$ and increasing on $\left(\frac{1}{1+(n-1)^{\frac{1}{p-1}}}, 1\right)$. Therefore the minimum of f is at $s = \frac{1}{1+(n-1)^{\frac{1}{p-1}}}$ and that gives us (2).

Q.E.D.

3.3 Lemma

Let $1 \le p < \infty$ and let $f \in L^p(\Omega)$, $\|f\|_{L^p} \neq 0$. Then there exists a function $g \in L^{p'}$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) such that

$$\begin{aligned} \|g\|_{L^{p'}} &= 1, \\ \operatorname{supp}(g) &= \operatorname{supp}(f) \quad \text{and} \\ \int_\Omega gf &= \|f\|_{L^p}. \end{aligned}$$

Proof:

If $p = 1$ take $g = \operatorname{sgn} f$. If $p > 1$ take

$$g(x) := \operatorname{sgn}(f(x)) \frac{|f(x)|^{p-1}}{\|f\|_{L^p}^{p-1}}.$$

In both cases g clearly satisfies the given conditions.

Q.E.D.

3.4 Lemma

Let $p \in [1,\infty)$, and let $f_1,\dots,f_k \in L^p(\Omega)$ have pairwise disjoint supports. Then there exists a linear projection $P\colon L^p(\Omega) \to \operatorname{span}\{f_1,f_2,\dots,f_k\}$ with norm 1.

Proof:

Without loss of generality we may assume that each $\|f_i\|_{L^p} > 0$. We use Lemma 3.3 to get functions $g_i \in L^{p'}$, such that

$$\begin{aligned} \|g_i\|_{L^{p'}} &= 1, \\ \operatorname{supp}(g_i) &= \operatorname{supp}(f_i) \quad \text{and} \\ \int_\Omega g_i f_i &= \|f_i\|_{L^p} \text{ for all } i \in \{1,\dots,k\}. \end{aligned}$$

Now, for every $f \in L^p(\Omega)$ we set

$$P(f) := \sum_{i=1}^{k} \left(\int_\Omega f g_i \right) \frac{f_i}{\|f_i\|_{L^p}}.$$

Clearly P is a linear projection of L^p onto $\operatorname{span}\{f_1, \dots, f_n\}$. Since the supports of f_i (and therefore g_i) are pairwise disjoint, we can use Hölder's inequality to obtain that

$$\begin{aligned}
\|P(f)\|_{L^p}^p &= \sum_{i=1}^{k} \left| \int_\Omega f g_i \right|^p \frac{\int_\Omega |f_i|^p}{\|f_i\|_{L^p}^p} = \sum_{i=1}^{k} \left| \int_\Omega f g_i \right|^p \\
&\leq \sum_{i=1}^{k} \|f \chi_{\operatorname{supp} g_i}\|_{L^p}^p \|g_i\|_{L^{p'}}^p \\
&= \sum_{i=1}^{k} \int_{\operatorname{supp} g_i} |f|^p \leq \|f\|_{L^p(\Omega)}^p,
\end{aligned}$$

and therefore the norm of P is at most one. Now $P(f_i) = f_i$ implies that the norm is equal to one.

Q.E.D.

3.5 Lemma

Let $1 \leq p < \infty$, $\alpha > 0$. Let X be a Banach space and let $T\colon X \to L^p(\Omega)$ be a continuous linear map. Assume that for every $\varepsilon > 0$ there exist a sequence of points $\{x_i\}_{i=1}^\infty \subseteq X$ and sequence of functions $\{g_i\}_{i=1}^\infty \subseteq L^p(\Omega)$, such that the supports of g_i are pairwise disjoint and that

$$\begin{aligned}
\|x_i\|_X &< 1, \\
\|T(x_i) - g_i\|_{L^p(\Omega)} &< \varepsilon \quad \text{and} \\
\|T(x_i)\|_{L^p(\Omega)} &\geq \alpha - \varepsilon.
\end{aligned} \tag{3}$$

Then $\beta(T) \geq \alpha$.

Proof:

Firstly we prove the statement for $p = 1$. Basically we will show (for $A = \alpha - 3\varepsilon$), that the distance between $T(x_i)$ and $T(x_j)$ for $i, j \in \mathbb{N}$, $i \neq j$ is greater or equal than $2A$, and therefore they can not fit in finitely many balls of diameter smaller than A in the definition of entropy numbers.

So let us fix $\varepsilon \in \left(0, \frac{\alpha}{3}\right)$, find x_i and g_i as in (3) and denote $f_i := T(x_i)$, $A := \alpha - 3\varepsilon$. From (3) and triangle inequality it follows trivially for every $i \in \mathbb{N}$ that

$$\|g_i\|_{L^1} \geq \|T(x_i)\|_{L^1} - \|T(x_i) - g_i\|_{L^1} \geq \alpha - 2\varepsilon = A + \varepsilon.$$

Then for $i \neq j$ we have

$$\|f_i - f_j\|_{L^1} \geq \|g_i - g_j\|_{L^1} - \|f_i - g_i\|_{L^1} - \|f_j - g_j\|_{L^1} \geq \|g_i - g_j\|_{L^1} - 2\varepsilon,$$

and because the supports of g_i and g_j are disjoint, we have

$$\begin{aligned}\|f_i - f_j\|_{L^1} &\geq \|g_i - g_j\|_{L^1} - 2\varepsilon \\ &= \|g_i\|_{L^1} + \|g_j\|_{L^1} - 2\varepsilon \\ &\geq 2(A+\varepsilon) - 2\varepsilon \\ &= 2A.\end{aligned} \tag{4}$$

Now we claim that $\beta(T) \geq A$. Assume for contradiction that there is $k \in \mathbb{N}$ such that $e_k(T) < A$. From the definition of e_k we have $\{c_j\}_{j=1}^{2^{k-1}}$ in $L^1(\Omega)$ such that

$$\{f_i\}_{i=1}^{\infty} \subseteq T(B_X) \subseteq \bigcup_{j=1}^{2^{k-1}} (c_j + AB_{L^1}).$$

For every ball $c_j + AB_{L^1}$ there is at most one $i \in \mathbb{N}$ such that the function $f_i \in c_j + AB_{L^1}$, because $f_{i_1}, f_{i_2} \in c_j + AB_{L^1}$ implies together with (4) that

$$2A \leq \|f_{i_1} - f_{i_2}\|_{L^1} \leq \|f_{i_1} - c_j\|_{L^1} + \|c_j - f_{i_2}\|_{L^1} < A + A = 2A.$$

We cannot put infinitely many functions separately in finitely many balls and thus we have a contradiction.

Finally we conclude by sending $\varepsilon \to 0$ in $\beta(T) \geq A = \alpha - 3\varepsilon$ to get $\beta(T) \geq \alpha$.

Now we consider the case $p \in (1, \infty)$. Fix $\varepsilon \in \left(0, \frac{\alpha}{3}\right)$ and find x_i, g_i as in (3). Suppose (for contradiction), that $\beta(T) < \alpha - 3\varepsilon$. We will project into finitely many dimensions (using Lemma 3.4) and arrive at contradiction with Lemma 3.2.

So, firstly denote

$$t(p,n) := \sqrt[p]{\left(1 - \frac{1}{1 + (n-1)^{\frac{1}{p-1}}}\right)^p + \frac{n-1}{\left(1 + (n-1)^{\frac{1}{p-1}}\right)^p}}.$$

Clearly $\lim_{n\to\infty} t(p,n) = 1$, so there exists $n \in \mathbb{N}$ such that

$$\beta(T) < (\alpha - 2\varepsilon) \cdot t(p,n) - \varepsilon. \tag{5}$$

For p and this n fixed denote $t := t(p,n)$, $A := \alpha - 2\varepsilon$, $f_i := T(x_i)$ and $E_i := \operatorname{supp} g_i$. From the definition of measure of non-compactness and (5) we obtain that there exists $k \in \mathbb{N}$ such that $e_k(T) < At - \varepsilon$. Therefore for some $c_j \in L^p(\Omega)$ we have

$$\{f_i\}_{i=1}^{\infty} \subseteq T(B_X) \subseteq \bigcup_{j=1}^{2^{k-1}} (c_j + (At - \varepsilon)B_{L^p}). \tag{6}$$

Then we claim that for every such ball $(c_j + (At - \varepsilon)B_{L^p})$, $j \in \{1, \ldots, 2^{k-1}\}$, there are at most $n-1$ functions f_i, such that $f_i \in (c_j + (At - \varepsilon)B_{L^p})$.

Indeed, suppose for contradiction that there is $j \in \mathbb{N}$ and distinct numbers $i_1, \ldots, i_n$ such that $f_{i_1}, \ldots, f_{i_n} \in c_j + (At - \varepsilon)B_{L^p}$. Let P denote a norm one projection of $L^p(\Omega)$ onto the linear span of functions $g_{i_1}, g_{i_2}, \ldots g_{i_n}$ given by Lemma 3.4. Let q_s denote the coordinates of c_j in the projection, that is $P(c_j) = \sum_{s=1}^{n} q_s g_{i_s}$. Using (3) we obtain for every $r \in \{1, 2, \ldots, n\}$ that

$$
\begin{aligned}
At-\varepsilon &> \|c_j - f_{i_r}\|_{L^p} \\
&\geq \|c_j - g_{i_r}\|_{L^p} - \|g_{i_r} - f_{i_r}\| \\
&\geq \|P(c_j - g_{i_r})\|_{L^p} - \varepsilon \\
&= \left\| \sum_{s=1}^{n} q_s g_{i_s} - g_{i_r} \right\|_{L^p} - \varepsilon \\
&\geq \sqrt[p]{\left(\sum_{s=1, s\neq r}^{n} \int_{E_{i_s}} |q_s g_{i_s}|^p \right) + \int_{E_{i_r}} |1-q_r|^p |g_{i_r}|^p} - \varepsilon.
\end{aligned}
$$

Now thanks to (3) we have $\|g_i\|_{L^p} \geq \alpha - 2\varepsilon$, therefore

$$
At - \varepsilon > (\alpha - 2\varepsilon) \sqrt[p]{\left(\sum_{s=1, s\neq r}^{n} |q_s|^p \right) + |1-q_r|^p} - \varepsilon = A|b^r - q|_p - \varepsilon,
$$

where $\{b^1, \dots, b^n\}$ is the canonical basis of $\mathbb{R}^n$ and $q = (q_1, \dots, q_n)$. From this it follows that for every $r \in \{1, \dots, n\}$ the vector b^r is in $B_p(q, t)$, which contradicts Lemma 3.2.

So we proved that inside every ball in (6) there are at most $n-1$ functions f_i. But that contradicts the fact that there are infinitely many functions f_i and finitely many balls.

Q.E.D.

3.6 Corollary

Let X be a Banach space, $1 \leq p < \infty$, $\alpha > 0$ and let $T\colon\ X \to L^p(\Omega)$ be a continuous map. Suppose that there exists a sequence of points $\{x_i\}_{i=1}^{\infty} \subseteq X$ such that the supports of $T(x_i)$ are pairwise disjoint and that

$$
\begin{aligned}
\|x_i\|_X &< 1, \\
\|T(x_i)\|_{L^p(\Omega)} &\geq \alpha.
\end{aligned}
$$

Then $\beta(T) \geq \alpha$.

Proof:

Obviously just take $g_i := T(x_i)$.

Q.E.D.

3.7 Theorem (Non-compactness of embedding into Lebesgue spaces)

Let $\Omega \subseteq \mathbb{R}^d$ be an open set with Lipschitz boundary. Let $k \in \mathbb{N}$, $1 \leq p < \infty$, $kp < d$ and denote $p^* = \frac{dp}{d-kp}$. Denote by I the embedding of $W_0^{k,p}(\Omega)$ into $L^{p^*}(\Omega)$. Then

$$
\beta(I) = \|I\|.
$$

Proof:

For given $r > 0$ denote by a_r the norm of the embedding of $W_0^{k,p}(B(x,r))$ into $L^{p^*}(B(x,r))$ (it clearly does not depend on $x \in \mathbb{R}^d$). If $r > s > 0$, then trivially from definition $a_r \geq a_s \geq 0$, so we can define $a = \lim_{r\to 0^+} a_r$ (the limit exists).

We claim, that $\beta(I) \geq a$. To prove that we find sequence of pairwise disjoint balls $B_i(x_i, r_i) \subseteq \Omega$. Fix $\delta > 0$. For every $i \in \mathbb{N}$ there is a function $g_i \in W_0^{k,p}(B_i)$, such that

$$\|g_i\|_{W^{k,p}(\Omega)} < 1 \quad \text{and}$$
$$\|g_i\|_{L^{p^*}(\Omega)} > a_r - \delta \geq a - \delta.$$

Corollary 3.6 applied to $T = I$, $x_i = g_i$ and $\alpha = a - \delta$ gives us $\beta(I) \geq a - \delta$. We conclude by sending δ to 0.

Previous inequality and Observation 2.26 give us, that

$$a \leq \beta(I) \leq \|I\|.$$

So it suffices to prove that $a = \|I\|$.

If we assume, that for all $r > s > 0$ we have $a_r = a_s$ (and therefore for all $r > 0$: $a_r = a$), then we have $\|\operatorname{Id}\colon W_0^{k,p}(\mathbb{R}^d) \to L^{p^*}(\mathbb{R}^d)\| = a$. And from that and the inequalities $a = \|\operatorname{Id}\colon W_0^{k,p}(\mathbb{R}^d) \to L^{p^*}(\mathbb{R}^d)\| \geq \|I\| \geq \beta(I) \geq a$ we obtain, that

$$\|\operatorname{Id}\colon W_0^{k,p}(\mathbb{R}^d) \to L^{p^*}(\mathbb{R}^d)\| = \|I\| = \beta(I) = a.$$

It remains to prove that for every $r > s > 0$ we have $a_r = a_s$. We fix such r, s and $\varepsilon > 0$. Then we find $g \in W_0^{k,p}(B(0,r))$ such that

$$\|g\|_{W^{k,p}(B(0,r))} = 1 \quad \text{and}$$
$$\|g\|_{L^{p^*}(B(0,r))} > a_r - \varepsilon.$$

Now consider the function $h\colon B(0,s) \to \mathbb{R}$ given by $h(x) = cg\left(\frac{r}{s}x\right)$, where c is a positive constant such that $\|h\|_{W^{k,p}(B(0,s))} = 1$. Then the change of variables $y = \frac{r}{s}x$ and Proposition 2.4 give us

$$\begin{aligned} 1 = \|h\|^p_{W^{k,p}(B(0,s))} &= \int_{B(0,s)} c^p \sum_{|\gamma|\leq k} \left| \left(\frac{r}{s}\right)^{|\gamma|} (D^\gamma g)\left(\frac{r}{s}x\right)\right|^p \,\mathrm{d}x \\ &= \frac{c^p}{\left(\frac{r}{s}\right)^{d-kp}} \int_{B(0,r)} \sum_{|\gamma|\leq k} \left(\frac{r}{s}\right)^{|\gamma|p-kp} |D^\gamma g(y)|^p \,\mathrm{d}y. \end{aligned} \tag{7}$$

Because $|\gamma|p - kp \leq 0$, $\frac{r}{s} > 1$ and $\|g\|_{W^{k,p}(B(0,r))} = 1$ we can continue with

$$1 \leq \frac{c^p}{\left(\frac{r}{s}\right)^{d-kp}} \int_{B(0,r)} \sum_{|\gamma|\leq k} |D^\gamma g(y)|^p \,\mathrm{d}y = \frac{c^p}{\left(\frac{r}{s}\right)^{d-kp}}.$$

This inequality and again change of variables $y = \frac{r}{s}x$ give us

$$\begin{aligned} \|h\|^{p^*}_{L^{p^*}(B(0,s))} &= \int_{B(0,s)} c^{p^*} \left| g\left(\frac{r}{s}x\right)\right|^{p^*} \,\mathrm{d}x \\ &= \frac{c^{p^*}}{\left(\frac{r}{s}\right)^d} \int_{B(0,r)} |g(y)|^{p^*} \,\mathrm{d}y \\ &> \frac{c^{p^*}}{\left(\frac{r}{s}\right)^d} (a_r - \varepsilon)^{p^*} \\ &= \left(\frac{c^p}{\left(\frac{r}{s}\right)^{d-kp}} \right)^{\frac{d}{d-kp}} (a_r - \varepsilon)^{p^*} \geq (a_r - \varepsilon)^{p^*}. \end{aligned}$$

Therefore the function h proves that $a_s \geq a_r - \varepsilon$. Sending $\varepsilon \to 0$ gives us $a_s \geq a_r$, and since $r > s$ we know that $a_s \leq a_r$.

Q.E.D.

Chapter 4

Embeddings into Lorentz spaces

In this chapter we generalize the results from Chapter 3. We formulate it in quite general setting using the property of disjoint monotonicity from Definition 2.21. Then we apply this general theorem to the embeddings into Lorentz spaces.

4.1 General theorem

4.1 Lemma

Let $1 \leq m < \infty$, $\alpha > 0$. Let X and Y be Banach spaces and let Y be disjointedly m-superadditive and monotone function space. Let $T\colon X \to Y$ be a continuous linear map. Assume that there exists a sequence of points $\{x_i\}_{i=1}^{\infty} \subseteq X$, such that the supports of $T(x_i)$ are pairwise disjoint and that

$$\begin{aligned} \|x_i\|_X < 1 \quad \text{and} \\ \|T(x_i)\|_Y \geq \alpha. \end{aligned} \tag{8}$$

Then $\beta(T) \geq \alpha$.

Proof:

Denote $f_i := T(x_i)$. From the continuity we know that

$$\|f_i\|_Y = \|T(x_i)\|_Y \leq \|T\| \cdot \|x_i\|_X \leq \|T\|. \tag{9}$$

Suppose (for contradiction) that $\beta(T) < \alpha$. We can find $\varepsilon > 0$ such that $\beta(T) < \alpha - \varepsilon$. Let us fix $n \in \mathbb{N}$ big enough, such that $\big(\|T\| + \alpha\big)^m < \frac{n}{M} \cdot \varepsilon^m$, where M is the constant from disjoint m-superadditivity.

From the definition of measure of non-compactness we obtain that there exists $k \in \mathbb{N}$ such that $e_k(T) < \alpha - \varepsilon$. Therefore for some functions $c_j \in Y$ we have

$$\{f_i\}_{i=1}^{\infty} \subseteq T(B_X) \subseteq \bigcup_{j=1}^{2^{k-1}} \left(c_j + (\alpha - \varepsilon)B_Y\right). \tag{10}$$

We claim that for every such ball $(c_j + (\alpha - \varepsilon)B_Y)$, $j \in \{1, \ldots, 2^{k-1}\}$, there are at most $n - 1$ functions f_i, such that $f_i \in (c_j + (\alpha - \varepsilon)B_Y)$.

Indeed, suppose for contradiction that there are n distinct numbers $i_1, \dots, i_n$ and in fact any ball with center C and radius $(\alpha - \varepsilon)$ such that

$$f_{i_1}, \dots, f_{i_n} \in C + (\alpha - \varepsilon) B_Y. \tag{11}$$

Let S_r denote the support of f_{i_r}, $S := \bigcup_{1 \le r \le n} S_r$. Put $\widetilde{C} = C \cdot \chi_S$ and note that clearly $\|f_i - \widetilde{C}\|_Y \le \|f_i - C\|_Y$ because of the monotonicity of Y. Therefore without loss of generality we may assume, that C is supported in S.

We observe that S_r are disjoint and therefore we can write C as sum of functions $C_r := C \cdot \chi_{S_r}$ which have disjoint supports, i.e. $C = \sum_{1 \le r \le n} C_r$.

The monotonicity of Y and (11) give us

$$\|f_{i_r} - C_r\|_Y \le \|f_{i_r} - C\|_Y \le (\alpha - \varepsilon).$$

Using this and (8) we estimate for each $1 \le r \le n$

$$\|C_r\|_Y \ge \|f_{i_r}\|_Y - \|f_{i_r} - C_r\|_Y \ge \alpha - (\alpha - \varepsilon) = \varepsilon.$$

Thanks to the disjoint m-superadditivity of Y we obtain the estimate

$$\|C\|_Y^m = \left\| \sum_{r=1}^{n} C_r \right\|_Y^m \ge \frac{1}{M} \sum_{r=1}^{n} \|C_r\|_Y^m \ge \frac{1}{M} n \varepsilon^m > \big(\|T\| + \alpha\big)^m.$$

Using this, (11) and (9) we get

$$\alpha - \varepsilon \ge \|C - f_{i_1}\|_Y \ge \|C\|_Y - \|f_{i_1}\|_Y \ge \big(\|T\| + \alpha\big) - \|T\| = \alpha,$$

which is a contradiction.

We proved that inside every ball in (10) there are at most $n - 1$ functions f_i. But that contradicts the fact that there are infinitely many functions f_i and finitely many balls.

Q.E.D.

4.2 Remark

Corollary 3.6 used in the proof of the main theorem in Chapter 3 is a special case of Lemma 4.1, because the space L^p is clearly monotone and p-superadditive (Remark 2.22).

4.3 Notation

Let $X(\mathbb{R}^d)$ be a space of functions from $\mathbb{R}^d$ to $\mathbb{R}$ and let Ω be an open subset of $\mathbb{R}^d$. We denote

$$X_0(\Omega) := \{f \in X_0(\mathbb{R}^d) \colon \ f(x) = 0 \text{ for } x \in \mathbb{R}^d \setminus \Omega\}.$$

We furthermore denote $\|f\|_{X_0(\Omega)} := \|f\|_{X_0(\mathbb{R}^d)}$.

4.4 Remark

We have $L^p(\Omega) = L_0^p(\Omega)$ and $L^{p,q}(\Omega) = L_0^{p,q}(\Omega)$ in the case of Lebesgue resp. Lorentz space.

4.5 Theorem (Non-compactness of embedding)
Let Ω be an open subset of $\mathbb{R}^d$ and let $X_0(\Omega)$ and $Y_0(\Omega)$ be two Banach spaces of functions from Ω to $\mathbb{R}$. Let $a \in (0, \infty)$ and assume the following conditions:

(i) The space $X_0(\Omega)$ is continuously embedded into $Y_0(\Omega)$ and

$$\|Id\colon\ X_0(\Omega) \to Y_0(\Omega)\| = a. \tag{12}$$

(ii) The space $X_0(B)$ is continuously embedded into $Y_0(B)$ for any open ball $B \subseteq \Omega$ and

$$\|Id\colon\ X_0(B) \to Y_0(B)\| = a. \tag{13}$$

(iii) The space $Y_0(\Omega)$ is monotone and disjointedly m-superadditive.

Denote by I the embedding of $X_0(\Omega)$ into $Y_0(\Omega)$. (The condition (i) states that it is continuous and $\|I\| = a$.) Then

$$\beta(I) = \|I\|.$$

Proof:
We claim that $\beta(I) \geq a$. To prove that we find sequence of pairwise disjoint balls $B_i(x_i, r_i) \subseteq \Omega$. Fix $\delta > 0$. For every $i \in \mathbb{N}$ there is a function $g_i \in X(B_i)$, such that

$$\begin{aligned} &\|g_i\|_{X_0} < 1 \quad \text{and} \\ &\|g_i\|_{Y_0} > a - \delta. \end{aligned}$$

The space $Y_0(\Omega)$ is monotone and disjointedly m-superadditive, so we can use Lemma 4.1 applied to $T = I$, $x_i = g_i$ and $\alpha = a - \delta$ to get $\beta(I) \geq a - \delta$. We conclude by sending δ to 0.

Observation 2.26 furthermore gives us that

$$a \leq \beta(I) \leq \|I\| = a.$$

Q.E.D.

4.6 Remark
Let $B \subseteq \Omega$ and let $X_0(\Omega)$ and $Y_0(\Omega)$ be two Banach spaces of functions. Then

$$\|Id\colon\ X_0(B) \to Y_0(B)\| \leq \|Id\colon\ X_0(\Omega) \to Y_0(\Omega)\|.$$

Proof:
Clearly $X_0(B) \subseteq X_0(\Omega)$, therefore

$$\begin{aligned} \|Id\colon\ X_0(B) \to Y_0(B)\| &:= \sup\{\|f\|_{Y_0(B)}\colon\ f \in X_0(B),\ \|f\|_{X_0(B)} \leq 1\} \\ &= \sup\{\|f\|_{Y_0(\Omega)}\colon\ f \in X_0(B),\ \|f\|_{X_0(\Omega)} \leq 1\} \\ &\leq \sup\{\|f\|_{Y_0(\Omega)}\colon\ f \in X_0(\Omega),\ \|f\|_{X_0(\Omega)} \leq 1\} \\ &=: \|Id\colon\ X_0(\Omega) \to Y_0(\Omega)\|. \end{aligned}$$

Q.E.D.

4.2 Applications

Since the measure of non-compactness depends on the norm, we need to be careful about the definition of Lorentz space $L^{m,q}$. In the case $q \le m$ we consider the norm $\|\cdot\|_{L^{m,q}}$, but in the case $q > m$ we need to use the norm $\|\cdot\|_{L^{(m,q)}}$.

1.2 Theorem (Non-compactness of embedding into Lorentz spaces)
Let $d \ge 2$, $k \in \mathbb{N}$, $k < d$, $1 \le p < \frac{d}{k}$, denote $p^* = \frac{dp}{d-kp}$ and let $1 \le q < \infty$. Let either $p > 1$ or $p = q = 1$. Let Ω be an open subset of $\mathbb{R}^d$ with Lipschitz boundary. Then

$$\beta\left(Id\colon\ W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)\right) = \|Id\colon\ W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)\|.$$

Proof:
Let us denote the embedding of $W_0^k L^{p,q}(\Omega)$ into $L^{p^*,q}(\Omega)$ by I. Firstly we observe, that the definitions of Sobolev, Sobolev-Lorentz, Lebesgue and Lorentz spaces $W_0^{k,p}, W_0^k L^{p,q}, L^p$ and $L^{p,q}$ agree with the Notation 4.3.

Let us denote

$$a_r := \|Id\colon\ W_0^k L^{p,q}\left(B(c,r)\right) \to L^{p^*,q}(B(c,r))\|$$

for $B(c,r) \subseteq \mathbb{R}^d$. Clearly a_r does not depend on $c \in \mathbb{R}^d$. Thanks to Remark 4.6 we have $a_r \ge a_s$ for $r > s > 0$. We claim that $a_r = a_s$ and we denote this value by a (that is for example $a := a_1$).

Assume that we already know that $a_r = a$ for every $r > 0$. We want to use Theorem 4.5. The validity of the embeddings in (i) and (ii) from Theorem 4.5 follows from Theorem 2.23 and Remark 2.24.

Since functions with compact support are dense in $W_0^k L^{p,q}(\Omega)$ it can be easily shown using Remark 4.6 that

$$\|Id\colon\ W_0^k L^{p,q}(\mathbb{R}^d) \to L^{p^*,q}(\mathbb{R}^d)\| = \lim_{r\to\infty} a_r = a.$$

By Remark 4.6 we now have

$$\begin{aligned} a &= \|Id\colon\ W_0^k L^{p,q}(\mathbb{R}^d) \to L^{p^*,q}(\mathbb{R}^d)\| \\ &\ge \|Id\colon\ W_0^k L^{p,q}(\Omega) \to L^{p^*,q}(\Omega)\| \\ &\ge \|Id\colon\ W_0^k L^{p,q}(B) \to L^{p^*,q}(B)\| = a \end{aligned}$$

for any open ball $B \subseteq \Omega$, which shows (12) and (13) of Theorem 4.5.

Finally the condition (iii) follows from the fact that $L^{p^*,q}(\Omega)$ is monotone and disjointedly m-superadditive thanks to Remark 2.22, where $m = p^*$ for $p^* \ge q$ and $m = q$ for $p^* < q$.

It remains to prove, that for $r > s > 0$ we have $a_r \le a_s$, that is

$$\|Id\colon W_0^k L^{p,q}(B(0,r)) \to L^{p^*,q}(B(0,r))\| \le \|Id\colon W_0^k L^{p,q}(B(0,s)) \to L^{p^*,q}(B(0,s))\|. \tag{14}$$

Because of different norms in Lorentz spaces we need to split the proof into three parts depending on the value of q with respect to p and p^*, where $p < p^*$.

Part 1: $\mathbf{q \leq p < p^*}$

In this case on $W_0^k L^{p,q}$ resp. $L^{p^*,q}$ we have the norm $\|\cdot\|_{W^k L^{p,q}}$ resp. $\|\cdot\|_{L^{p^*,q}}$. Let $r > s > 0$ and fix $\varepsilon > 0$. Then we find $g \in W_0^k L^{p,q}(B(0,r))$ such that

$$\begin{aligned} \|g\|_{W^k L^{p,q}(B(0,r))} &= 1 \quad \text{and} \\ \|g\|_{L^{p^*,q}(B(0,r))} &> a_r - \varepsilon \end{aligned}$$

and let us denote

$$h\colon\ B(0,s) \to \mathbb{R}, \quad h(x) = cg\left(\frac{r}{s}x\right),$$

where c is a positive constant such that $\|h\|_{W^k L^{p,q}(B(0,s))} = 1$. From Corollary 2.11 it follows, that

$$(D^\gamma h)^*(t) = c \cdot \left(\frac{r}{s}\right)^{|\gamma|} \cdot (D^\gamma g)^* \left(\left(\frac{r}{s}\right)^d t\right).$$

This and the change of variables $T = \left(\frac{r}{s}\right)^d t$ give us

$$\begin{aligned} 1 &= \|h\|^q_{W^k L^{p,q}(B(0,s))} \\ &= \sum_{|\gamma|\leq k} \int_0^\infty \left(t^{\frac{1}{p}} \left(D^\gamma h\right)^* (t)\right)^q \frac{\mathrm{d}t}{t} \\ &= c^q \sum_{|\gamma|\leq k} \int_0^\infty t^{\frac{q}{p}-1} \left[\left(\frac{r}{s}\right)^{|\gamma|} \cdot (D^\gamma g)^* \left(\left(\frac{r}{s}\right)^d t\right)\right]^q \mathrm{d}t \\ &= c^q \sum_{|\gamma|\leq k} \int_0^\infty \left(\left(\frac{s}{r}\right)^d T\right)^{\frac{q}{p}-1} \left(\frac{r}{s}\right)^{|\gamma|q} \left[(D^\gamma g)^* (T)\right]^q \left(\frac{s}{r}\right)^d \mathrm{d}t \\ &= \frac{c^q}{\left(\frac{r}{s}\right)^{q\left(\frac{d}{p}-k\right)}} \sum_{|\gamma|\leq k} \int_0^\infty \left(\frac{r}{s}\right)^{|\gamma|q-kq} \left(T^{\frac{1}{p}} (D^\gamma g)^*(T)\right)^q \frac{\mathrm{d}T}{T} \end{aligned}$$

Because $|\gamma|q - kq \leq 0$, $\frac{r}{s} > 1$ and $\|g\|^q_{W_0^k L^{p,q}(B(0,r))} = 1$ we can continue with

$$1 \leq \frac{c^q}{\left(\frac{r}{s}\right)^{q\left(\frac{d}{p}-k\right)}} \sum_{|\gamma|\leq k} \int_0^\infty \left(T^{\frac{1}{p}} (D^\gamma g)^*(T)\right)^q \frac{\mathrm{d}T}{T} = \frac{c^q}{\left(\frac{r}{s}\right)^{q\left(\frac{d}{p}-k\right)}}. \tag{15}$$

From Proposition 2.9 it follows, that

$$h^*(t) = c \cdot g^* \left(\left(\frac{r}{s}\right)^d t\right).$$

This combined with inequality (15) and change of variables $T = \left(\frac{r}{s}\right)^d t$ give us

$$
\begin{aligned}
\|h\|^q_{L^{p^*,q}} &= \int_0^\infty \left(t^{\frac{1}{p^*}} \cdot h^*(t)\right)^q \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty t^{\frac{q}{p^*}-1} \cdot \left(c \cdot g^*\left(\left(\frac{r}{s}\right)^d t\right)\right)^q \mathrm{d}t \\
&= \int_0^\infty c^q \left(\left(\frac{s}{r}\right)^d T\right)^{\frac{q}{p^*}-1} (g^*(T))^q \left(\frac{s}{r}\right)^d \mathrm{d}t \\
&= \frac{c^q}{\left(\frac{r}{s}\right)^{d\left(\frac{q}{p^*}-1\right)+d}} \|g\|^q_{L^{p^*,q}} \\
&\geq \left(\frac{c^q}{\left(\frac{r}{s}\right)^{q\left(\frac{d}{p}-k\right)}}\right)(a_r - \varepsilon)^q \geq (a_r - \varepsilon)^q.
\end{aligned} \tag{16}
$$

Therefore the function h proves that $a_s \geq a_r - \varepsilon$. Sending $\varepsilon \to 0$ gives us (14).

Part 2: $\mathbf{p < q \leq p^*}$

In this case we have the same norm on $L^{p^*,q}$, but on $W^k_0 L^{p,q}$ we have the norm $\|\bullet\|_{W^k L^{(p,q)}}$. The proof is the same as in the first case, just everywhere we wrote $\|\bullet\|_{W^k L^{p,q}}$ we now write $\|\bullet\|_{W^k L^{(p,q)}}$ and up to the equation (15) we use the double-star operator ** instead of the rearrangement *. Note that by Corollary 2.11 the double star operator ** scales in the same way as the rearrangement *.

Part 3: $\mathbf{p < p^* < q}$

In this case on $W^k_0 L^{p,q}$ resp. $L^{p^*,q}$ we have the norm $\|\bullet\|_{W^k L^{(p,q)}}$ resp. $\|\bullet\|_{L^{(p^*,q)}}$. The proof is again the same as in the second case, now we replace $\|\bullet\|_{L^{p^*,q}}$ with $\|\bullet\|_{L^{(p^*,q)}}$ and we use ** instead of * everywhere.

Q.E.D.

4.7 Remarks

- We can use Theorem 4.5 to easily prove Theorem 3.7.
- If we take $q = p$ or $q = p^*$ we get that the measure of non-compactness of the embedding $W^{k,p}_0(\Omega) \hookrightarrow L^{p^*,p}$ or $W^k_0 L^{p,p^*} \hookrightarrow L^{p^*}$ is equal to their respective norm.

4.8 Remark

Let $1 \leq q \leq Q < \infty$. Theorem 1.2 holds even for the embedding of $W^k_0 L^{p,q}(\Omega)$ into $L^{p^*,Q}(\Omega)$, i.e. it's measure of non-compactness is equal to it's norm.

Proof:
The validity of the embedding follows from Theorem 2.23 and Proposition 2.15, because

$$W_0^k L^{p,q}(\Omega) \hookrightarrow L^{p^*,q}(\Omega) \hookrightarrow L^{p^*,Q}(\Omega).$$

The rest of the proof is analogous to the proof of Theorem 1.2, we only need to raise the inequality (15) to the power of $\frac{Q}{q}$ and to replace q by Q in inequalities (16).

Q.E.D.

Chapter 5

Embedding into the space of continuous functions

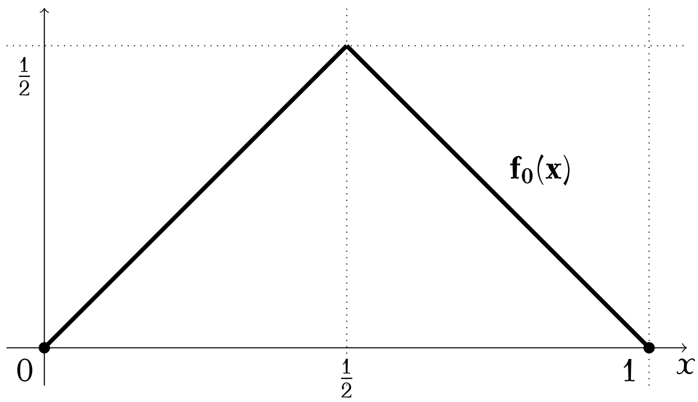
In this chapter we show that the measure of non-compactness of an embedding can be smaller than its norm. For that we consider the Sobolev space $W_0^{1,1}((0,1))$ equipped with the norm $\|u\|_{1,1} := \int_0^1 |u'(x)|\,\mathrm{d}x$ (where u' is the weak derivative), and the space of continuous functions $\mathcal{C}((0,1))$ equipped with the supremum norm $\|u\|_\infty = \sup_{x\in(0,1)} |f(x)|$.

5.1 Proposition

The norm of the embedding of $W_0^{1,1}((0,1))$ into $\mathcal{C}((0,1))$ is equal to $\frac{1}{2}$.

Proof:

Let $f_0(x) := \frac{1}{2} - \left|x - \frac{1}{2}\right|$.



Then

$$\|f_0\|_{1,1} = \int_0^1 \left|\left(\frac{1}{2} - \left|x - \frac{1}{2}\right|\right)'\right| \mathrm{d}x = \int_0^{\frac{1}{2}} 1 + \int_{\frac{1}{2}}^1 1 = 1,$$

and

$$\|f_0\|_\infty = 1/2.$$

Therefore $\|Id\colon\ W_0^{1,1}((0,1)) \to \mathcal{C}((0,1))\| \geq \frac{1}{2}$.

To show the opposite inequality we consider arbitrary function f from the unit ball in $W_0^{1,1}((0,1))$. From [8, Theorem 8.2 and Theorem 8.12] we know, that f has an absolutely continuous representative (we can without the loss of generality assume that it is f) such that $f(0)=f(1)=0$. Let us fix any point A where the maximum of $|f|$ is attained, i.e.

$$f(A)=\max_{x\in[0,1]}|f(x)|=\|f\|_\infty.$$

Without loss of generality we may assume that $f(A)\geq 0$, otherwise we can consider $-f$. Then

$$\begin{aligned}1&\geq\|f\|_{1,1}\\&=\int_0^1|f'(x)|\,\mathrm{d}x\\&=\int_0^A|f'(x)|\,\mathrm{d}x+\int_A^1|f'(x)|\,\mathrm{d}x\\&\geq\int_0^A f'(x)\,\mathrm{d}x+\int_A^1-f'(x)\,\mathrm{d}x\\&=f(A)-f(0)-f(1)+f(A)=2f(A)=2\|f\|_\infty.\end{aligned}$$

Therefore $\|Id\colon\ W_0^{1,1}((0,1))\to\mathcal{C}((0,1))\|\leq\frac{1}{2}$.

Q.E.D.

5.2 Remark

It is well known that the embedding of $W_0^{1,1}((0,1))$ into $\mathcal{C}((0,1))$ is not compact, therefore

$$\beta\left(Id\colon\ W_0^{1,1}((0,1))\to\mathcal{C}((0,1))\right)>0.$$

5.3 Proposition

The measure of non-compactness of the embedding of $W_0^{1,1}((0,1))$ into $\mathcal{C}((0,1))$ is less or equal than $\frac{1}{3}$.

In particular, the measure of non-compactness of this embedding is smaller than its norm (which is equal to $\frac{1}{2}$).

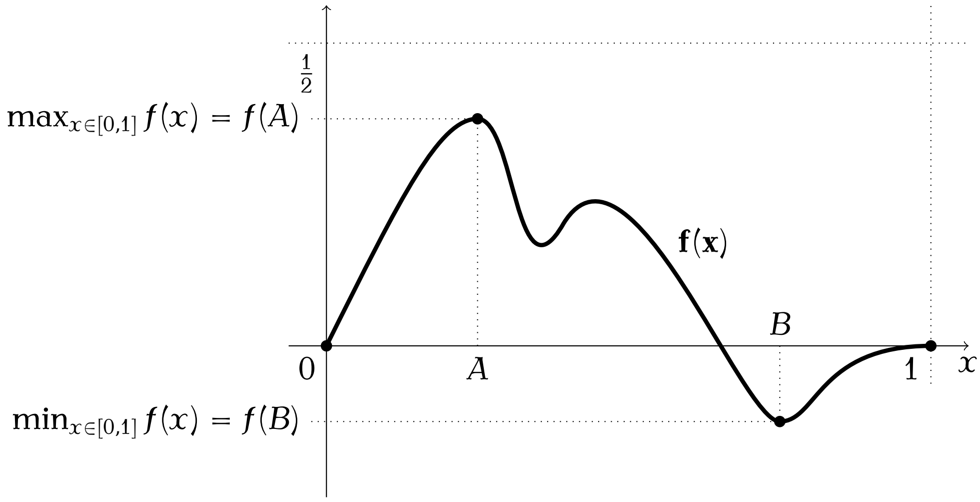
Proof:

It is enough to show that in $\mathcal{C}((0,1))$ there are finitely many balls with radius $\frac{1}{3}$ that cover $B_{W_0^{1,1}((0,1))}$. Clearly we can consider closed balls. We claim that it is enough to consider the balls

$$\begin{aligned}B_{-\frac{1}{6}}&:=B_{\mathcal{C}((0,1))}\left(-\frac{1}{6},\frac{1}{3}\right),\\B_0&:=B_{\mathcal{C}((0,1))}\left(0,\frac{1}{3}\right)\text{ and}\\B_{\frac{1}{6}}&:=B_{\mathcal{C}((0,1))}\left(\frac{1}{6},\frac{1}{3}\right),\end{aligned}$$

where $-\frac{1}{6}$, 0 and $\frac{1}{6}$ are meant as constant functions on $(0,1)$.

Consider arbitrary function f such that $\|f\|_{1,1} \leq 1$. We want to show that it is in one of these three balls. Thanks to [8, Theorem 8.2 and Theorem 8.12] we can without loss of generality assume that f is absolutely continuous and $f(0) = f(1) = 0$. Let us fix A as any point where the maximum of f is attained and B as any point where the minimum is attained.



We know from Proposition 5.1 that $\min_{x\in[0,1]} f(x) = f(B) \in \left[-\frac{1}{2}, 0\right]$ and $\max_{x\in[0,1]} f(x) = f(A) \in \left[0, \frac{1}{2}\right]$. Therefore $f(A) \geq f(B)$ and

$$\frac{f(A)+f(B)}{2} \in \left[-\frac{1}{4}, \frac{1}{4}\right]. \tag{17}$$

Without loss of generality we assume that

$$\frac{f(A)+f(B)}{2} \geq 0, \tag{18}$$

otherwise we may use $-f$ and (if needed) ball $B_{-\frac{1}{6}}$ instead of $B_{\frac{1}{6}}$. Furthermore we can assume that $A \leq B$, otherwise we can use $f(-x)$.

We use the fact that f is absolutely continuous to obtain

$$\begin{aligned}
1 &\geq \|f\|_{1,1} \\
&= \int_0^1 |f'(x)| \,\mathrm{d}x \\
&= \int_0^A |f'(x)| \,\mathrm{d}x + \int_A^B |f'(x)| \,\mathrm{d}x + \int_B^1 |f'(x)| \,\mathrm{d}x \\
&\geq \int_0^A f'(x) \,\mathrm{d}x + \int_A^B -f'(x) \,\mathrm{d}x + \int_B^1 f'(x) \,\mathrm{d}x \\
&= f(A) - f(0) - f(B) + f(A) + f(1) - f(B) = 2(f(A) - f(B)),
\end{aligned}$$

therefore

$$\left|\frac{f(A)-f(B)}{2}\right| \leq \frac{1}{4}. \tag{19}$$

To determine into which of the balls $B_{-\frac{1}{6}}$, B_0 or $B_{\frac{1}{6}}$ the function f belongs we distinguish two cases. Recall (17) and (18).

$$\textbf{Case 1:}\ \frac{\mathbf{f(A)+f(B)}}{\mathbf{2}} \in \left[\mathbf{0}, \frac{\mathbf{1}}{\mathbf{12}}\right] \tag{20}$$

Then we claim that $f \in B_0$. Thanks to estimates (19) and (20) we have

$$\begin{aligned} |f(A)| &\leq \left| f(A) - \frac{f(A)-f(B)}{2} \right| + \left| \frac{f(A)-f(B)}{2} \right| \\ &\leq \left| \frac{f(A)+f(B)}{2} \right| + \left| \frac{f(A)-f(B)}{2} \right| \\ &\leq \frac{1}{12} + \frac{1}{4} = \frac{1}{3} \end{aligned}$$

and symmetrically we can show that $|f(B)| \leq \frac{1}{3}$. Therefore

$$\|f - 0\|_\infty = \max\{|f(A)|, |f(B)|\} \leq \frac{1}{3}.$$

$$\textbf{Case 2:}\ \frac{\mathbf{f(A)+f(B)}}{\mathbf{2}} \in \left(\frac{\mathbf{1}}{\mathbf{12}}, \frac{\mathbf{1}}{\mathbf{4}}\right] \tag{21}$$

Then we claim that $f \in B_{\frac{1}{6}}$. We know that $f(A) \in \left[0, \frac{1}{2}\right]$, which implies $\left|f(A) - \frac{1}{6}\right| \leq \frac{1}{3}$. Furthermore the estimates (19) and (21) yield

$$\begin{aligned} \left| f(B) - \frac{1}{6} \right| &\leq \left| f(B) - \frac{f(A)+f(B)}{2} \right| + \left| \frac{f(A)+f(B)}{2} - \frac{1}{6} \right| \\ &\leq \left| \frac{f(B)-f(A)}{2} \right| + \frac{1}{12} \\ &\leq \frac{1}{4} + \frac{1}{12} = \frac{1}{3}. \end{aligned}$$

Therefore

$$\left\| f - \frac{1}{6} \right\|_\infty = \max\left\{ \left| f(A) - \frac{1}{6} \right|, \left| f(B) - \frac{1}{6} \right| \right\} \leq \frac{1}{3}.$$

Q.E.D.

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