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MASTER THESIS

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Some point–free aspects of connectedness

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In Prague date July 24, 2013 Tomáš Jakl

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Abstrakt: V této práci ukážeme Stoneovu větu o reprezentaci, která je také známa pod názvem Stoneova dualita, v bezbodovém kontextu. Předvedený důkaz je bezvýběrový, a protože se nemusíme starat o jednotlivé body, je mnohem jednodušší než původní důkaz. Ukážeme, že pro každý nekonečný kardinál κ jsou protějšky κ–úplných Booleových algebrer κ–bazicky nesouvislé Stoneovy framy. Také předvedeme přesnou charakterizaci morfismů, které jsou v koresponenci s κ–úplnými Booleovskými homomorfismy. Ikdyž Booleanizace není obecně funktoriální, v části duality extremálně nesouvislých Stoneových framů funktoriální je a dokonce tvoří ekvivalenci kategorií. Na konci práce se zaměříme na De Morganovské (respektive extremálně nesouvislé) framy a ukážeme jejich novou charakterizaci pomocí jejich superhustých sublokálů. Naproti tomu jsou metrizovatelné framy, které nemají žádný netriviální superhustý sublokál, a proto nikdy není jejich netriviální Čech–Stoneova kompaktifikace metrizovatelná.

Klíčová slova: Stoneova dualita, bezbodová topologie, kompaktifikace, De Morganovské framy, konstruktivní matematika

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Abstract: In this thesis we present the Stone representation theorem, generally known as Stone duality in the point–free context. The proof is choice–free and, since we do not have to be concerned with points, it is by far simpler than the original. For each infinite cardinal κ we show that the counterpart of the κ–complete Boolean algebras is constituted by the κ–basically disconnected Stone frames. We also present a precise characterization of the morphisms which correspond to the κ–complete Boolean homomorphisms. Although Booleanization is not functorial in general, in the part of the duality for extremally disconnected Stone frames it is, and constitutes an equivalence of categories. We finish the thesis by focusing on the De Morgan (or extremally disconnected) frames and present a new characterization of these by their *superdense* sublocales. We also show that in contrast with this phenomenon, a metrizable frame has no non-trivial superdense sublocale; in other words, a non-trivial Čech–Stone compactification of a metrizable frame is never metrizable.

Keywords: Stone duality, point–free topology, compactification, De Morgan frames, constructive mathematics

Contents

Chapter I

Introduction

It was a breaktrough both in algebra and in topology when Marshall Stone published in 1936 his paper “The Theory of Representation of Boolean Algebras” [14]. In this article the author presented a structural isomorphism between Boolean algebras and a certain class of topological spaces. Futhermore, the spaces we encounter there, unlike those typically studied before, were not motivated by geometry (subsets of Euclidean spaces and spaces from analysis, and similar) but arised from algebraic structures [9].

Frames (locales), the basic structure of point–free topology are a natural generalization of the concept of a topological space. This generalization is based on the properties of the lattices of open sets of topological spaces (and typically contains all the information one needs). The results in thus extended context often bring new insight into the classical facts. In contrast with the choice dependence of many of them, a large part of the theory of frames can be treated constructively. This phenomenon is aptly expressed by the famous slogan in Banaschewski's [1]:

Choice–free localic argument + suitable choice principle =
classical result in spaces.

(It should be noted, however, that there are numerous results going far beyond the classical context.)

In this thesis we present the Stone representation theorem, generally known as Stone duality in the point–free context. The proof is choice–free, and the Banaschewski's slogan is valid again: using the Boolean Ultrafilter Theorem we then can derive the classical spatial Stone duality. Not surprisingly, the Stone duality (to be more precise, equivalence) is by far simpler than the original. The fact that we do not have to be concerned with points helps a lot.

The point–free approach allows us to analyze easily some particular parts of the Stone duality. For each infinite cardinal κ we show that the counterpart of the κ–complete Boolean algebras is constituted by the κ–basically disconnected Stone frames. We also present a precise characterization of the morphisms which correspond to the κ–complete Boolean homomorphisms.

Booleanization is the construction of a Boolean algebra from a Heyting algebra by taking the set of elements of the form $a = a^{**}$ (for a topological space the Booleanization of its frame of open sets is the system of all regular open subsets; it forms a complete Boolean algebra). In point–free topology it has a useful and somewhat surprising property: it is the smallest dense sublocale of the original frame [8]. This fact has no counterpart in classical topology.

Analogously with the Stone duality for topological spaces, the frames corresponding to the complete Boolean algebras are precisely the extremally disconnected Stone frames. Hence we obtain a proper class of variants of disconnectedness in between the zero-dimensionality and the extremal disconnectedness. Note that although the Booleanization is not functorial in general, in this part of the duality it is, and constitutes an equivalence of the parts in question.

We finish the thesis by focusing on the De Morgan (or extremally disconnected) frames and present a new characterization of these. Let us call a dense sublocale S of a frame L *superdense* if the Čech–Stone compactifications of S and L are isomorphic. Now, a completely regular frame L is De Morgan if and only if each dense sublocale of L is superdense. We also show that in contrast with this phenomenon, a metrizable frame has no non-trivial superdense sublocale; in other words, a non-trivial Čech–Stone compactification of a metrizable frame is never metrizable.

Organization of the thesis

In Chapter II we present the necessary facts from order theory, category theory and topology which we will need in the subsequent chapters. In Chapter III several disconnectedness properties and the Čech–Stone compactification are discussed. In Chapter IV we present the Stone duality in both the point–free and the space setting. Chapter V is devoted to particular parts (fragments) of the Stone duality. In Chapter VI we show that Booleanization is an equivalence of the"complete parts" of the Stone duality. Finally, in Chapter VII we present a new characterization of De Morgan frames and prove the non-metrizability of non-trivial Čech–Stone compactification.

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I would also like to thank my family for their support and my colleagues for creating an inspiring environment.

Chapter II

Preliminaries

In this chapter we present a summary of all relevant and well–known facts to point–free topology which we will need in the following chapters.

We assume the reader has a basic knowledge of set theory, mainly that he knows basic facts about cardinal numbers, cardinalities and Axiom of Choice. We refer to the monograph *Frames and Locales* written by Pultr and Picado [13] as an introduction to point–free topology.

A few comments on notation

For sets X, Y and a mapping $f\colon X \to Y$, denote $f[S] = \{ f(s) \mid s \in S \}$ and $f^{-1}[M] = \{ x \in X \mid f(x) \in M \}$ for any subsets $S \subseteq X$ and $M \subseteq Y$. $[0,1]$ is the closed interval of real numbers greater than or equal to 0 and less than or equal to 1, similarly, $[0,\infty)$ is the set of all non-negative real numbers, and $\mathbb{Q}$ is the set of all rational numbers.

We will often denote a mathematical structure, such as lattice, topological space, frame etc., by the same symbol we use for its underlying set. In expressions, the rightmost unary operation is applied first. For example $\downarrow \bigvee_i a_i^*$ is the same as $\downarrow (\bigvee_i (a_i^*))$.

The propositions which rely on Axiom of Choice or Boolean Ultrafilter Theorem are marked by $(\star)$.

1. Partially ordered set

1.1. Let A be a set and let $\leq$ be a binary relation on A ($(\leq) \subseteq A \times A$). We say that $(A, \leq)$ is a *partially ordered set* if the relation $\leq$ is *partial order*, i.e. satisfies:

(P1) *reflexivity:* $a \leq a$ for all $a \in A$,

(P2) *antisymmetry:* $a \leq b$ and $b \leq a$ implies $a = b$ for all $a, b \in A$,

(P3) *transitivity:* $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in A$.

A partially ordered set $(A, \leq)$ is said to be *with 1* or *with the top* (resp. *with 0* or *with the bottom*) if $1 \in A$ (resp. $0 \in A$) and $1 \geq a$ (resp. $0 \leq a$) for all $a \in A$. A partially ordered set is said to be *bounded* if it is both with 0 and with 1. Denote by $(A, \leq)^{\mathrm{op}}$ the partially ordered set $(A, \leq')$ where $(\leq') = \{ (b, a) \in A \times A \mid a \leq b \}$.

Let S be a subset of A. We say that u is an *upper bound* of S if $u \geq x$ for all $x \in S$. We say that s is a *supremum* of S, and we write $s = \bigvee S$, if s is the least upper bound of S. We say that l is an *lower bound* or an *infimum* if l is an upper bound or the supremum of S in A^{op}.

A partially ordered set $(A, \leq)$ is *join–semilattice* if every two element subset has a supremum. Similarly, *meet–semilattice* has an infimum for every two element subset. A *lattice* is partially ordered set which is both join– and meet–semilattice. We write $a \vee b$ for the supreum (the *meet*) of $\{a, b\}$ and $a \wedge b$ for the infimum (the *join*).

A mapping $f\colon A \to B$ between two partially ordered sets is called *monotone* if $f(x) \leq f(y)$ wherever $x \leq y$. If, moreover, A and B are lattices and $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in A$, we say that f is a *lattice homomorphism.*

1.2. A lattice is called a *complete lattice* if each of its subsets has an infimum and a supremum.

Proposition. *For a lattice L are the following conditions equivalent:*

- *L is a complete lattice;*
- *every subset of L has a supremum;*
- *every subset of L has an infimum.*

For a complete lattice we therefore have two infinitary operations the meet $\bigvee$ and the join $\bigwedge$. Note that every complete lattice is bounded, $0 = \bigvee \emptyset = \bigwedge L$ and $1 = \bigwedge \emptyset = \bigvee L$.

A lattice homomorphism $f\colon A \to B$ between two complete lattices is a *complete lattice homomorphism* if $f(\bigvee X) = \bigvee f[X]$ and $f(\bigwedge X) = \bigwedge f[X]$ for each subset $X \subseteq A$.

1.3. A lattice D is *distributive* if the following equation holds for all $a, b, c \in D$:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Proposition. *For a lattice L, the following conditions are equivalent:*

- *L is a distributive lattice,*
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ *for all* $a, b, c \in L$,
- *For each $a, b, c \in L$ there exists at most one $x \in L$ such that*

$$a \vee x = b$$
$$a \wedge x = c.$$

1.4. Let A and B be two partially ordered sets. Monotone mappings $f\colon A \to B$ and $g\colon B \to A$ are in *Galois connection* or are *Galois adjoint* if

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y) \quad \text{for all } x \in A, y \in B.$$

Or equivalently

$$fg(y) \leq y \quad \text{and} \quad x \leq gf(y) \quad \text{for all } x \in A, y \in B.$$

Then, we say that f is a left Galois adjoint of g and g is a right Galois adjoint of f. For a monotone map f, its left or right Galois adjoint do not need to exists but if it does, it is uniquely determined. We will write f^* for the right adjoint and f_* for the left adjoint.

For two monotone maps in Galois connection f, g we have that

$$fgf = f \quad \text{and} \quad gfg = g.$$

Proposition. *For a monotone map $f\colon A \to B$, if f is a left (resp. right) Galois adjoint then f preserves all existing suprema (resp. infima).*

Moreover, the converse implication also holds if A and B are complete lattices.

1.5. Let L be a lattice with 0. A *pseudocomplement* a^* of an element $a \in L$ is the greatest element x such that

$$x \wedge a = 0.$$

Equivalently, a^* is the pseudocomplement of a if

$$x \wedge a = 0 \quad \text{iff} \quad x \leq a^* \quad \text{for all } x \in L.$$

A lattice is called *pseudocomplemented* if every element has a pseudocomplement. Pseudocomplements, if they exist, satisfy the following properties:

1. $a \leq a^{**}$,
2. $a^* = a^{***}$,
3. $a \leq b$ implies $a^* \geq b^*$,
4. $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

1.6. Let L be a meet semilattice with 0 and with a binary operation $\to$. Then L is said to be a *Heyting algebra* if the following holds

$$a \wedge x \leq b \quad \text{iff} \quad a \leq x \to b \quad \text{for all } a, b, x \in L.$$

Note that each Heyting algebra is pseudocomplemented semilattice with pseudocomplements defined

$$a^* = a \to 0.$$

From the definition we see that the operation $x \to (-)$ is the right Galois adjoint to $(-) \wedge x$. Therefore, each meet semilattice admits at most one Heyting operation.

Proposition. *Let H be pseudocomplemented, in particular a Heyting algebra. Then*

$$(\bigvee_i a_i)^* = \bigwedge_i a_i^*$$

holds whenever $\bigvee_i a_i$ exists in H.

Proposition. *A complete lattice admits a Heyting operation iff the following equation holds*

$$(\bigvee_i a_i) \wedge b = \bigvee_i (a_i \wedge b) \quad \textit{for all } a_i, b.$$

1.7. Let B be a bounded distributive lattice. We say that B is a *Boolean algebra* if for every element $a \in B$ there exists a (*complement*) $a^c \in B$ such that

$$a \vee a^c = 1 \quad \text{and} \quad a \wedge a^c = 0.$$

Since B is a distributive lattice, such a^c is uniquely determined. Each Boolean algebra is a Heyting algebra with the Heyting operation defined as follows

$$a \to b = a^c \vee b.$$

Hence, it is also a pseudocomplemented lattice with pseudocomplements equal to complements.

Let f be a lattice homomorphism between two Boolean algebras. f is said to be a *Boolean homomorphisms* if f preserves 0 and 1. The following equations holds for any Boolean algebra

$$(\bigvee_i a_i)^c = \bigwedge_i a_i^c \quad \text{resp.} \quad (\bigwedge_i a_i)^c = \bigvee_i a_i^c,$$

whenever $\bigvee_i a_i$ resp. $\bigwedge_i a_i$ exists.

1.8. Let L be a bounded distributive lattice. Then a subset $F \subseteq L$ is a *filter* if

(F1) $0 \notin F$ and $1 \in F$,

(F2) $a \in F$ and $a \leq b \in L$ implies $b \in F$, and

(F3) $a, b \in F$ implies $a \wedge b \in F$.

Similarly, an ideal is a subset $I \subseteq L$ such that

(I1) $1 \notin I$ and $0 \in I$,

(I2) $a \in I$ and $a \geq b \in L$ implies $b \in I$, and

(I3) $a, b \in I$ implies $a \vee b \in I$.

A filter $F \subseteq L$ is called *prime filter* if $a \vee b \in F$ implies $a \in F$ or $b \in F$. F is called *completely prime filter* if $\bigvee A \in F$ implies $a \in F$ for some $a \in A$. A *principal filter* is any filter of L of the form $\uparrow a = \{ x \mid x \geq a \}$.

Similarly, an ideal I is called *prime ideal*, *completely prime ideal* or *principal ideal* if I is a prime filter, completely prime filter or principal filter of L^{op}. Principal ideals will be denoted by $\downarrow a = \{ x \mid x \leq a \}$.

Proposition. *Let B be a Boolean algebra and a filter $F \subseteq B$. Then the following are equivalent*

- *F is a maximal filter,*
- *F is a prime filter.*
- *For every $b \in B$, either $b \in F$ or $b^c \in F$.*

From the Proposition we see that the definition of a maximal filter and a prime filter coincide for Boolean algebras, hence we call such filter *ultrafilter*.

1.9. *Axiom of Choice* will be mostly used in its equivalent form – Zorn's Lemma:

(AC) *Let X be a non-empty partially ordered set such that every non-empty chain has an upper bound. Then X has at least one maximal element.*

By *Boolean Ultrafilter Theorem* we mean the following choice principle

(BUT) *Let B be a Boolean algebra and let F be a filter of B. Then there exists a ultrafilter extending F.*

This is equivalent to

(BUT') *Let B be a Boolean algebra and let F be a filter and let I be an ideal of B such that $F \cap I = \emptyset$. Then there exists a prime filter G extending F such that still $G \cap I = \emptyset$.*

Finally, the last choice principle we will need to know is *Axiom of Countable Dependent Choice*

(CDC) *Let R be a binary relation on a set X such that for every $a \in X$ there exists $b \in X$ satisfying aRb. Then there exist a countable sequence $(a_i)_{i=1}^{\infty}$ such that $a_i R a_{i+1}$ for all $i = 1, 2, \ldots$.*

Axiom of Choice is stronger than Boolean Ultrafilter Theorem or Axiom of Countable Dependent Choice; it logically implies both of them.

2. Category theory

2.1. A category $\mathcal{C}$ is a class of objects ($\operatorname{obj}\mathcal{C}$), a class of morphisms ($\operatorname{morph}\mathcal{C}$) and two mappings $\operatorname{dom}, \operatorname{codom}\colon \operatorname{morph}\mathcal{C} \to \operatorname{obj}\mathcal{C}$ (*domain* and *codomain*) satisfying conditions (C1)–(C3) bellow.

Notation: For a morphism $f \in \operatorname{morph}\mathcal{C}$, we write $f\colon A \to B$ or $A \xrightarrow{f} B$ whenever we want to express the fact that $A = \operatorname{dom}(f)$ and $B = \operatorname{codom}(f)$.

(C1) Let $f\colon A \to B, g\colon B \to C$ be two morphisms of $\mathcal{C}$. Then there exists their *composition*, the morphism $g \cdot f\colon A \to C$.

(C2) The composition satisfies the *associativity law*: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ whenever the compositions are defined.

(C3) For each object $A \in \operatorname{obj}\mathcal{C}$ there exists an *identity morphism* 1_A satisfying $1_A \cdot f = f$ and $g \cdot 1_A = g$ whenever the compositions are defined.

We say that a category is *small* if $\operatorname{morph}\mathcal{C}$ is a set. A *dual category* $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ is the category with the same class of objects as $\mathcal{C}$ has and with all morphisms of $\mathcal{C}$ reversed, i.e. for each morphism $f\colon A \to B \in \operatorname{morph}\mathcal{C}$, $\hat{f}\colon B \to A$ is a morphism of $\mathcal{C}^{\mathrm{op}}$. The composition $\diamond$ of $\mathcal{C}^{\mathrm{op}}$ is defined as

$$\hat{f} \diamond \hat{g} = \widehat{g \cdot f}.$$

To simplify the notation, we will write fg instead of $f \cdot g$, $A \in \mathcal{C}$ instead of $A \in \operatorname{obj}\mathcal{C}$ and similarly for morph, whenever there is no danger of confusion.

Examples. The most natural example of a category is the category **Set** of all sets, all maps between them and composition defined as map composition.

An important example is the category **Bool** of all Boolean algeras and all Boolean homomorphisms. Also, an arbitrary partially ordered set $(X, \leq)$ is a category with the set X as objects and one morphism between $x, y \in X$ whenever $x \leq y$ (composition holds thanks to transitivity of $\leq$).

Observe that the dual category of $(X, \leq)$ is precisely the category $(X, \leq)^{\mathrm{op}}$.

2.2. A morphism f is a *monomorphism* whenever $fg = fh$ implies $g = h$. Analogously, f is an *epimorphism* if $gf = hf$ implies $g = h$. Finally, f is an *isomorphism* if there exists a f^{-1} such that

$$f \cdot f^{-1} = 1 \quad \text{and} \quad f^{-1} \cdot f = 1.$$

If $fg = 1$, then f is an epimorphism and g is a monomorphism.

2.3. For categories $\mathcal{C}, \mathcal{D}$ the mappings $F\colon \operatorname{obj}\mathcal{C} \to \operatorname{obj}\mathcal{D}$ and $F\colon \operatorname{morph}\mathcal{C} \to \operatorname{morph}\mathcal{D}$ constitute a *(covariant) functor* if

$$F(f)\colon F(A) \to F(B) \quad \text{for any } f\colon A \to B \in \operatorname{morph}\mathcal{C},$$

$$F(1_A) = 1_{F(A)} \quad \text{for any } A \in \operatorname{obj}\mathcal{C}, \quad \text{and} \quad F(gh) = F(g)F(h).$$

Similarly, we say that F is a *contravariant functor* if

$$F(f)\colon F(B) \to F(A) \quad \text{for any } f\colon A \to B \in \operatorname{morph}\mathcal{C},$$

$$F(1_A) = 1_{F(A)} \quad \text{for any } A \in \operatorname{obj}\mathcal{C}, \quad \text{and} \quad F(gh) = F(h)F(g).$$

It is sometimes possible to think of the contravariant functor as of the functor of the form $F\colon \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$. The composition of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ is denoted by $G \circ F\colon \mathcal{C} \to \mathcal{E}$ or simply GF.

Examples. Let $\mathcal{C}$ be a category. The *identity functor* on $\mathcal{C}$ is defined by the mappings

$$\mathrm{Id}_{\mathcal{C}}(A) = A\colon \operatorname{obj}\mathcal{C} \to \operatorname{obj}\mathcal{C}, \quad \text{and} \quad \mathrm{Id}_{\mathcal{C}}(f) = f\colon \operatorname{morph}\mathcal{C} \to \operatorname{morph}\mathcal{C}.$$

Let $\mathcal{D}$ be a non-empty category and let T be an object of $\mathcal{D}$. Then the *constant functor* is defined as follows

$$K(A) = T \quad \text{and} \quad K(f) = 1_T \quad \text{for all } A, f \in \mathcal{C}.$$

2.4. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A collection of morphisms $m = (m_A)_{A \in \operatorname{obj}\mathcal{C}}$ is a *natural transformation* between F and G, $m\colon F \xrightarrow{\bullet} G$, if

$$m_A\colon F(A) \to G(A), \quad \text{for all } A \in \operatorname{obj}\mathcal{C},$$

and the following diagram commutes

$$\begin{array}{ccc} F(A) & \xrightarrow{m_A} & G(A) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{m_B} & G(B) \end{array}$$

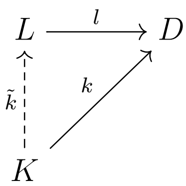
for all $f\colon A \to B$, morphisms of $\mathcal{C}$.

If a natural transformation m is a collection of isomorphism, we say that m is a *natural equivalence*. For two functors F and G, if there exists a natural equivalence $m\colon F \xrightarrow{\bullet} G$, we say that F and G are *naturally equivalent* and write $F \cong G$.

Example. For a functor $F\colon \mathcal{C} \to \mathcal{D}$, the collection of morphisms $(1_{F(A)})_{A \in \mathcal{C}}$ is an *identity natural transformation* $F \xrightarrow{\bullet} F$.

2.5. A *limit* of a *diagram* D in the category $\mathcal{C}$ (that is, of a functor $D\colon \mathcal{D} \to \mathcal{C}$ for a small category $\mathcal{D}$) is a constant functor $L\colon \mathcal{D} \to \mathcal{C}$ and a natural transformation $l\colon L \xrightarrow{\bullet} D$ satisfying universal property.

In other words, l is such that for any constant functor $K\colon \mathcal{D} \to \mathcal{C}$ and a natural transformation $k\colon K \xrightarrow{\bullet} D$ there exists an unique natural transformation $\tilde{k}\colon K \xrightarrow{\bullet} L$ such that the following diagram commutes



(in the category $[\mathcal{D}, \mathcal{C}]$ of all functors from $\mathcal{D}$ to $\mathcal{C}$ and natural transformations as morphisms). From the definition, we see that limits are determined uniquely up to isomorphisms.

A colimit is defined dually to limit. A *colimit* of a diagram is a constant functor $L\colon \mathcal{D} \to \mathcal{C}$ and a natural transformation $l\colon D \xrightarrow{\bullet} L$ such that for any constant functor $K\colon \mathcal{D} \to \mathcal{C}$ and a natural transformation $k\colon D \xrightarrow{\bullet} K$ there exists an unique natural transformation $\tilde{k}\colon L \xrightarrow{\bullet} K$ satisfying $k = \tilde{k}\, l$.

A category is said to be *(co)complete* if there is a (co)limit for every diagram.

Examples. A well–known example of a limit in **Set** is the (cartesian) product of sets $\prod_{i\in I} X_i$ together with projections $(\prod_{i\in I} X_i \to X_j)_{j\in I}$.

Similarly, a natural example of colimit in **Set** is the coproduct, the disjoint union, of sets $\coprod_{i\in I} X_i$ together with injections $(X_j \to \coprod_{i\in I} X_i)_{j\in I}$.

2.6. Functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ are *adjoint* (with F on the left and G on the right) if there exist *units of adjuction*, that is, natural transformations (*the unit and the counit*)

$$\lambda\colon FG \xrightarrow{\bullet} \mathrm{Id}_{\mathcal{D}} \quad \text{and} \quad \rho\colon \mathrm{Id}_{\mathcal{C}} \xrightarrow{\bullet} GF,$$

such that the following compositions of natural transformations

$$F \xrightarrow{F\rho} FGF \xrightarrow{\lambda_F} F \quad \text{and} \quad G \xrightarrow{\rho_G} GFG \xrightarrow{G\lambda} G$$

are equal to identity natural transformations on F and on G, or more precisely: $F(\rho_A) \cdot \lambda_{F(A)} = 1_{F(A)}$ and $\rho_{G(B)} \cdot G(\lambda_B) = 1_{F(B)}$ for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$.

Proposition. *Right adjoinits preserve limits and left adjoinits preserve colimits.*

Two categories $\mathcal{C}, \mathcal{D}$ are said to be *equivalent*, and we write $\mathcal{C} \cong \mathcal{D}$, if there exists adjoint functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ with units of adjunction consisting of natural equivalences.

2.7. A category $\mathcal{C}$ is said to be *subcategory* of a category $\mathcal{D}$ if $\operatorname{obj}\mathcal{C} \subseteq \operatorname{obj}\mathcal{D}$, $\operatorname{morph}\mathcal{C} \subseteq \operatorname{morph}\mathcal{D}$ and the composition of morphisms in $\mathcal{C}$ coincide with that in $\mathcal{D}$. Further, if $\{ f \mid f\colon A \to B \in \mathcal{C} \} = \{ f \mid f\colon A \to B \in \mathcal{D} \}$ for every $A, B \in \operatorname{obj}\mathcal{C}$, we say that $\mathcal{C}$ is a *full subcategory* of $\mathcal{D}$.

A full subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is *reflexive* (or *coreflexive*) if the embedding functor $J\colon \mathcal{C} \xrightarrow{\subseteq} \mathcal{D}$ is a right (or left) adjoint.

Example. The category **Bool** is reflexive subcategory of the category of bounded distributive lattices and lattice homomorphisms.

Proposition. *Each reflexive subcategory of a (co)complete category is (co)complete. Similarly for coreflexive categories.*

3. Topology and point–free topology

3.1. Let X be a set and τ an arbitrary set of subsets of X. Then the pair (X, τ) is a *topological space* if

(T1) $\emptyset, X \in \tau$,

(T2) $\mathcal{M} \subseteq \tau$ implies $\bigcup \mathcal{M} \in \tau$, and

(T3) for $U, V \in \tau$, also $U \cap V \in \tau$.

A subset of X which is an elements of τ is called *open* and the complement of an open set is called *closed*. A subset of X is said to be *clopen* if it is both open and closed. The *closure* of a set is the least closed set containing it.

Let $(X, \tau), (Y, \sigma)$ be topological spaces. A mapping $f\colon X \to Y$ is said to be *continuous (with respect to τ and σ)* if

$$f^{-1}[U] \in \tau, \quad \text{for all } U \in \sigma.$$

Denote by **Top** the category of all topological spaces and continuous mappings. Isomorphisms of **Top** are called *homeomorphisms*.

As we see from the conditions (T1)–(T3), the set τ ordered by inclusion is a complete lattice. Moreover, we have $(\bigcup_i U_i) \cap V = \bigcup_i (U_i \cap V)$ for $U_i, V \in \tau$. Therefore, we can generalize the notion of topological space:

Let L be a complete lattice. We say that L is a *frame* if

$$(\bigvee A) \wedge b = \bigvee_{a \in A} (a \wedge b)$$

for any $A \subseteq L$ and $b \in L$. We see that frames are just complete Heyting algebras and, consequently, pseudocomplemented lattices. The pseudocomplements are given by the formula

$$a^* = \bigvee \{ x \mid x \wedge a = 0 \}.$$

For an open set of a space, taking pseudocomplements in the corresponding frame of open sets is the same as taking the interior of the complement.

Let L and M be frames and $f\colon L \to M$ a monotone map. f is said to be a *frame homomorphism* if f preserves all joins, finite meets, the top and the bottom (1 and 0). Denote by **Frm** the resulting category of all frames and frame homomorphisms.

We have the obvious (contravariant) functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$

$$\Omega(X, \tau) = \tau \quad \text{and} \quad \Omega(f)\colon U \mapsto f^{-1}[U],$$

for $(X, \tau), f \in \mathbf{Top}$.

Since a frame homomorphism f preserves suprema it has a right adjoint f^*. It is called a *localic map*. The category of frames (in this context called *locales*) and localic maps will be called **Loc**. Trivially $\mathbf{Loc} \cong \mathbf{Frm}^{\mathrm{op}}$.

Again, we have a functor $\mathrm{Lc}\colon \mathbf{Top} \to \mathbf{Loc}$ (covariant this time)

$$\mathrm{Lc}(X, \tau) = \tau \quad \text{and} \quad \mathrm{Lc}(f) = \Omega(f)^*.$$

3.2. Separation axioms. For a topological space (X, τ), we say that it is

- T_0: if for every two distinct points $x, y \in X$ there exist an open $U \in \tau$ such that $x \in U \not\ni y$ or $x \notin U \ni y$.
- T_1: if for every two distinct points $x, y \in X$ there exist an open $U \in \tau$ such that $x \in U \not\ni y$.
- T_2: if for every two distinct points $x, y \in X$ there exist disjoint open sets $U, V \in \tau$ separating x and y, i.e. $x \in U \not\ni y$ and $x \notin V \ni y$.
- T_3: if for each point $x \in X$ and each closed $F \subseteq X$ such that $x \notin F$ there exist disjoint open sets separating x and F.
- $T_{3.5}$: if for each point $x \in X$ and each closed $F \subseteq X$ such that $x \notin F$ there exist a continuous function $f\colon X \to [0, 1]$ separating x and F, i.e. $f(x) = 0$ and $f[F] \subseteq \{1\}$.
- T_4: if for every two disjoin closed subsets of X there exist two disjoint open sets separating them, or equivalently, there exists a continuous function separating them.

We see that

$$T_4 \ \&\ T_1 \implies T_{3.5} \ \&\ T_0 \implies T_3 \ \&\ T_0 \implies T_2 \implies T_1 \implies T_0.$$

Topological spaces satisfying T_2, T_3 and T_0, $T_{3.5}$ and T_0, or (just) T_4 are called, in this order, *Hausdorff*, *regular*, *completely regular*, or *normal*. The condition T_3 is equivalent to the condition

$$U = \bigcup \left\{ V \subseteq X \,\middle|\, \overline{V} \subseteq U \right\}, \quad \text{for all open } U \in \tau.$$

Observe that $\overline{V} \subseteq U$ if and only if $V^* \cup U = X$ in the frame $\Omega(X)$. Define,

$$a \prec b \quad \overset{\text{def}}{\equiv} \quad a^* \vee b = 1,$$

A frame L is said to be *regular* if

$$a = \bigvee \{ x \mid x \prec a \},$$

for all $a \in L$. Then, a space X is regular if and only if the frame $\Omega(X)$ is regular. Analogously for complete regularity, define

$$a \prec\!\!\prec b \quad \overset{\text{def}}{\equiv} \quad \begin{array}{l} \text{there exists } a_i \in L, \text{ for all } i \in \mathbb{Q} \cap [0,1], \text{ such that} \\ a_0 = a, \ a_1 = b \text{ and } a_i \prec a_j \text{ whenever } i < j. \end{array}$$

Then, a frame L is *completely regular* if

$$a = \bigvee \{ x \mid x \prec\!\!\prec a \},$$

for all $a \in L$. Again, a space X is completely regular if and only if the frame $\Omega(X)$ is.

Finally, a frame is said to be *normal* if

$$a \vee b = 1 \quad \Longrightarrow \quad \exists c \text{ such that } a \vee c = 1 \text{ and } c^* \vee b = 1.$$

Similarly as in topological spaces we have that *any normal and regular frame is completely regular.*

> (*Proof.* For $x \prec y$, the $x^* \vee y = 1$ holds, from normality there exists a q, such that $x^* \vee q = 1$ and $q^* \vee y = 1$, thus $x \prec q \prec y$. Hence, by (CDC): $\{ x \mid x \prec b \} = \{ x \mid x \prec\!\!\prec b \}$ □)

Observation. *Let* $\lhd \in \{\prec, \prec\!\!\prec\}$. *Then*

1. $a' \leq a \lhd b \leq b$ *implies* $a' \lhd b'$,
2. $a_1 \lhd b_1$ *and* $a_2 \lhd b_2$ *implies* $a_1 \wedge a_2 \lhd b_1 \wedge b_2$ *and* $a_1 \vee a_2 \lhd b_1 \vee b_2$, *and*
3. $a \lhd b$ *implies* $b^* \lhd a^*$.

3.3. For a topological space X, a continuous map $i\colon \{\star\} \to X$ is identified with a point of X. For each such map, we have the associated frame homomorphism $\Omega(i)\colon \Omega X \to \mathbf{2}$, where $\mathbf{2}$ is the frame $\Omega(\{\star\}) = \{0 < 1\}$. Therefore, it is convenient to define a *point* of a frame L as a frame homomorphism $L \to \mathbf{2}$.

Note that every frame homomorphism $p\colon L \to \mathbf{2}$ determines a completely prime filter (as $p^{-1}[\{1\}]$), and vice versa. The completely prime filters in topological spaces represents the neighbourhoods of points.

Another way of defining points of a frame is by prime elements. An element $p \neq 1$ is *prime* or *meet–irreducible* if $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. Then, the frame homomorphisms $p\colon L \to \mathbf{2}$ are in one-one correspondence with prime elements (as $\bigvee p^{-1}[\{0\}]$). The prime elements of topological space corresponds to the open sets $X \setminus \overline{\{x\}}$.

A frame L is said to be *spatial* or has *enough points* if $L = \Omega(X)$ for some space X.

Proposition. *A frame is spatial iff each of its elements is a meet of prime elements.*

3.4. A *subspace* $(Y \subseteq X, \tau\big|_Y)$ of a topological space (X, τ) (where $\tau\big|_Y = \{\, Y \cap U \mid U \in \tau \,\}$) determines the one-one inclusion (and continuous) mappings $j\colon Y \subseteq X$. The associated localic map $\mathrm{Lc}(j)\colon \mathrm{Lc}(Y) \subseteq \mathrm{Lc}(X)$ is one-one and its adjoint is an onto frame homomorphisms $\Omega(j)\colon \Omega(X) \to \Omega(Y)$.

For a locale L and a subset $S \subseteq L$, S is said to be a *sublocale* of L if the inclusion (one-one) mapping $j\colon S \subseteq L$ is a localic map and, for $s, t \in S$, $s \le t$ whenever $j(s) \le j(t)$. Or equivalently, there exists an onto frame homomorphisms $j_*\colon L \to S$.

One has an another useful characterisation of sublocales. Let L be a frame, a subset $S \subseteq L$ is sublocale if and only if

(S1) S is closed under all meets, and

(S2) $x \to s \in S$ for each $s \in S$ and $x \in L$.

Denote by $\mathfrak{Sl}(L)$ the set of all sublocales of L. Then, $\mathfrak{Sl}(L)$ ordered by inclusion is a co-frame (in other words, $\mathfrak{Sl}(L)^{\mathrm{op}}$ is a frame). The empty sublocale $\mathsf{O} = \{1\}$ is the least sublocale and L is the greatest.

3.5. A sublocale is said to be *open* resp. *closed* if it is of the form

$$\mathfrak{o}(a) = \{\, a \to x \mid x \in L \,\} \quad \text{resp.} \quad \mathfrak{c}(a) = {\uparrow} a,$$

for some a. A sublocale which is both open and closed is called *clopen*. An inspiration for this definition comes naturally from topological spaces. For a topological space (X, τ) and an open subset $U \in \tau$, we have the inclusion mapping

$$j\colon (U, \{\, V \in \tau \mid V \subseteq U \,\}) \to (X, \tau)$$

and an onto frame homomorphisms $\Omega(j)\colon V \mapsto V \cap U$. Hence, we have the formula

$$\mathrm{Lc}(j)\colon V \mapsto U \to V,$$

for the associated localic embedding, the right adjoint to $\Omega(j)$. The sublocales $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are mutually complemented in $\mathfrak{Sl}(L)$ ($\mathfrak{c}(a) \vee \mathfrak{o}(a) = L$ and $\mathfrak{c}(a) \wedge \mathfrak{o}(a) = \mathsf{O}$). Moreover, we have

$$\bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_i a_i), \qquad \mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b),$$
$$\bigwedge_i \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_i a_i), \quad \text{and} \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b).$$

For a sublocale $S \subseteq L$, its *closure*, the least closed sublocale containing S, is given by the formula

$$\overline{S} = {\uparrow}(\bigwedge S).$$

From that, we immediately see that a sublocale $S \subseteq L$ is *dense* (that is, $\overline{S} = L$) iff $0_L \in S$.

The closure of sublocales satisfies the familiar properties: $S \subseteq \overline{S}$, $\overline{\mathsf{O}} = \mathsf{O}$, $\overline{\overline{S}} = \overline{S}$ and $\overline{S} \vee \overline{T} = \overline{S \vee T}$; and also $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$.

Proposition. *Preimage of a closed (resp. open) sublocale under a localic map f is closed (resp. open). Moreover,*

$$f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f_*(a)) \quad \textit{and} \quad f^{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f_*(a)),$$

where f_* is the left Galois adjoint to f.

3.6. Nuclei. Let L be a frame. A *nucleus* ν on L is a monotone map $\nu\colon L \to L$ satisfying the following four properties:

(N1) $a \leq \nu(a)$,

(N2) $a \leq b$ implies $\nu(a) \leq \nu(b)$,

(N3) $\nu\nu(a) = \nu(a)$, and

(N4) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$.

For a frame L and a nucleus $\nu\colon L \to L$, set $S = \nu(L)$. The set S, together with suprema and infima defined

$$\bigsqcup a_i = \nu(\bigvee a_i) \quad \text{and} \quad a \sqcap b = a \wedge b,$$

is a sublocale of L. Indeed, from (N4), we know that $\sqcap$ is the infimum and, from (N2), we know that $\bigsqcup$ is the supremum. Moreover, we have

$$(\bigsqcup a_i) \sqcap b = \nu(\bigvee a_i) \wedge \nu(b) = \nu(\bigvee (a_i \wedge b)) = \bigsqcup (a_i \sqcap b).$$

Hence $\nu(L)$ is a frame, we will show that the monotone map defined as $(\bar{\nu}\colon a \mapsto \nu(a))\colon L \to \nu(L)$ is an onto frame homomorphism. Then, we know that $\nu(L)$ is a sublocale of L. It is enough to show that

$$\nu(\bigvee a_i) = \nu(\bigvee \nu(a_i)) \quad (= \bigsqcup \nu(a_i)).$$

We have $\bigvee a_i \leq \bigvee \nu(a_i)$ by (N1) and $\nu(\bigvee a_i) \leq \nu(\bigvee \nu(a_i))$ by (N2). On the other hand $\nu(a_i) \leq \nu(\bigvee a_i)$ by (N2). Hence $\bigvee \nu(a_i) \leq \nu(\bigvee a_i)$ and $\nu(\bigvee \nu(a_i)) \leq \nu\nu(\bigvee a_i) = \nu(\bigvee a_i)$ by (N2) and (N3).

Thus, nuclei deteremine sublocales. One has more, there is an one-one correspondence between nuclei and sublocales.

Chapter III

Connectedness and compactification

Compactness, connectedness and variants of disconnectedness are standard properties of topological spaces. In this chapter, we will prove some well–known facts from Set theoretical topology in the context of point–free topology.

1. Connectedness and variants of disconnectedness

1.1. Definition. We say that a frame L is *disconnected* if L contains non-trivial elements a and b (that is, different from the top and the bottom of L) such that $a \vee b = 1$ and $a \wedge b = 0$. If a frame is not disconnected, we say that it is *connected*.

1.2. Observation. *For a frame L, the following conditions are equivalent:*

1. *L is disconnected.*
2. *There exists a non-trivial (that is, other than O and L) sublocale that is both open and closed.*
3. *There exists an one-one frame homomorphism $f\colon B_2 \to L$, where B_2 is the Boolean algebra on four elements.*

1.3. Lemma. *Let L be a frame. The following are equivalent:*

1. *Closure of each open sublocale of L is open.*
2. *For all $a \in L$: $\overline{\mathfrak{o}(a)} = \mathfrak{o}(a^{**})$.*
3. *For all $a \in L$: $a^{**} \vee a^{*} = 1$.*

Proof. The implication from 2 to 1 is trivial. For the implication from 3 to 2, first observe that

$$\mathfrak{o}(a^{**}) \vee \mathfrak{o}(a^{*}) = \mathfrak{o}(a^{**} \vee a^{*}) = \mathfrak{o}(1) = L, \text{ and}$$
$$\mathfrak{o}(a^{**}) \wedge \mathfrak{o}(a^{*}) = \mathfrak{o}(a^{**} \wedge a^{*}) = \mathfrak{o}(0) = \mathsf{O}.$$

However, $\mathfrak{o}(a^*)$ is complemented with $\mathfrak{c}(a^*)$. In other words

$$\begin{aligned}\mathfrak{c}(a^*) \vee \mathfrak{o}(a^*) &= L, \text{ and}\\ \mathfrak{c}(a^*) \wedge \mathfrak{o}(a^*) &= \mathsf{O}.\end{aligned}$$

By distributivity of the frame of all all sublocales of L, $\mathfrak{o}(a^{**}) = \mathfrak{c}(a^*) = \overline{\mathfrak{o}(a)}$.

Finally, for the implication from 1 to 3: $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*) = \mathfrak{o}(b)$ for some $b \in L$, hence

$$\begin{aligned}\mathfrak{o}(a^* \vee b) &= \mathfrak{o}(a^*) \vee \mathfrak{o}(b) = L, \text{ and}\\ \mathfrak{o}(a^* \wedge b) &= \mathfrak{o}(a^*) \wedge \mathfrak{o}(b) = \mathsf{O}.\end{aligned}$$

Thus a^* is complemented with b and $b = a^{**}$ from the uniqueness of complements. □

1.4. Definition. A frame satisfying conditions 1, 2 and 3 from the previous Lemma is called *extremally disconnected* or *De Morgan*.

A frame L is said to be *zero–dimensional* if every element is a join of complemented elements.

1.5. Observation. *1. Any zero–dimensional frame is completely regular.*

2. Any regular and extremally disconnected frame is zero–dimensional.

Proof. 1. For every complemented a, we have $a \prec\!\!\prec a$. Therefore by zero–dimensionality,

$$e = \bigvee \{ c \mid c \leq e, c \text{ is complemented} \} \leq \bigvee \{ x \mid x \prec\!\!\prec e \} \leq e$$

for each e.

2. Note that $x \prec y$ implies $x^{**} \prec y$. Since $x \leq x^{**}$ and every element of the form $x = x^{**}$ is complemented in extremally disconnected frame, we obtain a frame is zero–dimensional.

□

1.6. Note that in the classical terminology, a topological space is *connected*, *disconnected*, *zero–dimensional* or *extremally disconnected* if and only if the frame of its open sets is connected, disconnected, zero–dimensional or extremally disconnected.

2. Compactness and compactification

2.1. Definition. A frame L is *compact* if for every *cover*, that is a subset $C \subseteq L$ such that $\bigvee C = 1$, there exists a finite subcover (a finite $F \subseteq C$ such that $\bigvee F = 1$).

2.2. Proposition. *If L is a compact regular frame then L is completely regular.*

Proof. It suffices to check that L is normal. Then by the Proposition in II.3.2, L is also completely regular. Let $a, b \in L$ and let $a \vee b = 1$. Then from regularity, we know that

$$a = \bigvee \{ x \mid x \prec a \} \quad \text{and} \quad b = \bigvee \{ y \mid y \prec b \}.$$

Set $A = \{ x \mid x \prec a \}$ and $B = \{ y \mid y \prec b \}$. Since $\bigvee A \vee \bigvee B = 1$, by compactness, there exists a finite $F \subseteq A \cup B$, such that $\bigvee F = 1$. From II.3.2, $f_a = \bigvee (F \cap A) \prec a$ and $f_b = \bigvee (F \cap B) \prec b$. Further,

$$(f_a \wedge f_b^*) \vee b = (f_a \vee f_b) \wedge (f_b^* \vee b) = (\bigvee F) \wedge 1 = 1.$$

By the same argument, $(f_b \wedge f_a^*) \vee a = 1$. Further, $(f_b \wedge f_a^*) \wedge (f_a \wedge f_b^*) = 0$, therefore L is normal. □

2.3. The frame of ideals. Let L be a join–semilattice with 0. Denote by $\mathfrak{J}L$ the set of all ideals of L. We will show that $\mathfrak{J}L$ ordered by inclusion is a frame. Set intersections of ideals are again an ideal and hence (by II.1.2) $\mathfrak{J}L$ is a complete lattice. For the suprema we have the explicit formula.

Let $I_i \in \mathfrak{J}L$, for $i \in J$, and set

$$\bigvee_{i \in J} I_i = \{ \bigvee F \mid F \text{ is a finite subset of } \bigcup_{i \in J} I_i \}. \qquad \text{(Idl-}\bigvee\text{)}$$

The set defined this way is again an ideal and it is the supremum of $\{ I_i \mid i \in J \}$. Also, for any ideals J and I_i of L, for $i \in J$, the following equality holds

$$(\bigvee_i I_i) \cap J = \bigvee_i (I_i \cap J).$$

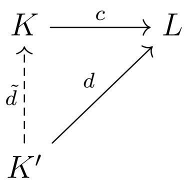
The $\supseteq$ inclusion is trivial, for the other inclusion take $x \in J$ such that $x = \bigvee F$ for some finite $F \subseteq \bigcup_i I_i$. Then $F \subseteq J$ (as J is an ideal) and also $F \subseteq \bigcup_i (I_i \cap J)$. We get, $x \in \bigvee_i (I_i \cap J)$. Thus, $\mathfrak{J}L$ is a frame.

Moreover, $\mathfrak{J}L$ is a compact frame: Let $I_i \in \mathfrak{J}L$, for $i \in J$, such that $\bigvee_i I_i = L = 1_{\mathfrak{J}L}$. Then there exists a finite $F \subseteq \bigcup_i I_i$ such that $\bigvee F = 1$. Set $i(f) \in J$ such that $f \in I_{i(f)}$ for all $f \in F$. Then also $\bigvee_{f \in F} I_{i(f)} = 1_{\mathfrak{J}L}$. Therefore $\mathfrak{J}L$ is a compact frame.

Conclusion. *The set $\mathfrak{J}L$ ordered by inclusion is a compact frame.*

2.4. Definition. We say a frame homomorphism $f\colon L \to M$ is *dense* if $f(a) = 0$ implies $a = 0$.

We say a compact frame K together with a frame homomorphism $c\colon K \to L$ is the *(Čech–Stone) compactification* of a frame L if c is dense and for every dense frame homomorphism $d\colon K' \to L$, with K' compact regular, there exists an unique frame homomorphism $\tilde{d}\colon K \to K$ such that the following diagram commutes



2.5. Observation. *If the Čech–Stone compactification exists, it is determined uniquely, up to isomorphism.*

The following construction is due to Banaschewski and Mulvey [2].

2.6. Regular ideals and a frame of regular ideals. Let L be a completely regular frame. We say an ideal I is *regular* if for any $a \in I$ there exists a $b \in I$ such that $a \prec\!\!\prec b$. Denote by $\mathfrak{R}L$ the set of all regular ideals of L.

For two regular ideals $I_1, I_2 \in \mathfrak{R}L$, their set intersection is again a regular ideal. Indeed, take any $a \in I_1 \cap I_2$, then $a \prec\!\!\prec b_i$ for some $b_i \in I_i$ and $a \prec\!\!\prec b_1 \wedge b_2 \in I_1 \cap I_2$ from II.3.2.

For any regular ideals J and I_i of L, for $i \in J$, from previous we know that $\bigvee_{i\in J} I_i$ is an ideal. We will show that it is a regular ideal. For $a \in \bigvee_{i\in J} I_i$, by (Idl-$\bigvee$), there exists a finite $F \subseteq \bigcup_{i\in J} I_i$ such that $a = \bigvee F$. For every $f \in F$ there exists an $e_f \in I_j$, for some $j \in J$, such that $f \prec\!\!\prec e_f$. We see that $\bigvee_{f\in F} e_f \in \bigvee_{i\in J} I_i$ and, from II.3.2, $\bigvee F \prec\!\!\prec \bigvee_{f\in F} e_f$.

Hence, the set $\mathfrak{R}L$ is a subframe of $\mathfrak{J}L$. Consequently, $\mathfrak{R}L$ is compact.

2.7. For an $a \in L$, set

$$\sigma_L(a) = \{\, x \mid x \prec\!\!\prec a \,\}\,.$$

This set is a regular ideal. We will omit the subscript if the frame L is obvious.

The following property of σ will be expedient: $\sigma(a) \prec \sigma(b)$ for any $a \prec\!\!\prec b$.

> (*Proof.* First, interpolate $a \prec\!\!\prec x \prec\!\!\prec y \prec\!\!\prec b$. By II.3.2, $x^* \prec\!\!\prec a^*$ and $x^* \in \sigma(a^*) \subseteq \sigma(a)^*$. Since $y \in \sigma(b)$ and $x^* \vee y = 1$, we have $\sigma(a)^* \vee \sigma(b) = 1_{\mathfrak{R}L}$. $\square$)

2.8. Proposition. *$\mathfrak{R}L$ is completely regular.*

Proof. We will show that $\mathfrak{R}L$ is regular, then it is also completely regular from Proposition 2.2 and from the fact that $\mathfrak{R}L$ is compact. Given any $I \in \mathfrak{R}L$,

$$I = \bigcup \{\, \sigma(a) \mid a \in I \,\} = \bigvee \{\, \sigma(a) \mid a \in I \,\} = \bigvee \{\, \sigma(a) \mid a \prec\!\!\prec b \in I \,\}\,.$$

We know that $a \prec\!\!\prec b$ implies $\sigma(a) \prec \sigma(b)$ and we also know that $b \in I$ implies $\sigma(b) \subseteq I$, hence

$$I \subseteq \bigvee \{\, \sigma(a) \mid \sigma(a) \prec \sigma(b) \subseteq I \,\} \subseteq \bigvee \{\, \sigma(a) \mid \sigma(a) \prec I \,\} \subseteq \bigvee \{\, K \mid K \prec I \,\} \subseteq I. \quad \square$$

2.9. The functor $\mathcal{R}$. Now, we are ready to define a functor from the category of completely regular frames to the category of compact regular frames. Denote by

$$\mathcal{R}\colon \mathbf{CRegFrm} \to \mathbf{RegKFrm}$$

the following two mappings. On objects:

$$\mathcal{R}(A) = \mathfrak{R}A,$$

for every completely regular frame A, and on morphisms:

$$(\mathcal{R}f)(I) = {\downarrow} f[I],$$

for every morphism frame homomorphism $f\colon L \to M$ and $I \in \mathfrak{R}L$.

The set $f[I]$ is obviously closed under finite meets, hence ${\downarrow} f[I]$ is an ideal. From the fact that $a \prec\!\!\prec b$ implies $f(a) \prec\!\!\prec f(b)$, we see that ${\downarrow} f[I]$ is a regular ideal. Further, we observe that

- $\mathcal{R}f$ preserves joins:

$$\begin{aligned}(\mathcal{R}f)(\bigvee_i I_i) &= {\downarrow} f[\{\bigvee F \mid F \text{ a finite subset of } \bigcup_i I_i\}] \\ &= {\downarrow}\{\bigvee F \mid F \text{ a finite subset of } \bigcup_i f[I_i]\} \\ &= \{\bigvee F \mid F \text{ a finite subset of } \bigcup_i {\downarrow} f[I_i]\} \\ &= \bigvee_i (\mathcal{R}f)(I_i);\ \text{and that}\end{aligned}$$

- $\mathcal{R}f$ preserves finite meets:

$$(\mathcal{R}f)(I_1 \cap I_2) = {\downarrow} f[I_1 \cap I_2] \subseteq {\downarrow} f[I_1] \cap {\downarrow} f[I_2] = (\mathcal{R}f)(I_1) \cap (\mathcal{R}f)(I_2).$$

  For $x \in {\downarrow} f[I_1] \cap {\downarrow} f[I_2]$, there exists $y_i \in I_i$, for $i = 1, 2$, such that $f(y_1) = f(y_2) = x$. Since f is a frame homomorphism, f preserves finite meets, hence $f(y_1 \wedge y_2) = f(y_1) \wedge f(y_2) = x$. We know that $y_1 \wedge y_2 \in I_1 \cap I_2$, therefore $x \in {\downarrow} f[I_1 \cap I_2]$.

We obtain that $\mathcal{R}f$ is a frame homomorphism. As a result, we have the following

Proposition. *$\mathcal{R}$ is a functor.*

2.10. For a completely regular frame L, define $\gamma_L\colon \mathcal{R}L \to L$ as follows:

$$\gamma_L\colon I \mapsto \bigvee I.$$

We see that

$$\gamma_L \sigma(a) = a \quad \text{and} \quad \sigma\gamma_L(I) \subseteq I. \tag{III.1}$$

Since both γ_L and σ are monotone maps, we see that they form a Galois ajdunction, with γ_L to the left and σ to the right. Hence, γ_L preserves joins. It also preserves finite meets: $\bigvee I_1 \wedge \bigvee I_2 = \bigvee \{ a_1 \wedge a_2 \mid a_i \in I_i \} \leq \bigvee \{ a \mid a \in I_1 \cap I_2 \} = \bigvee (I_1 \cap I_2) \leq \bigvee I_1 \wedge \bigvee I_2$.

Consequently, γ_L is a frame homomorphism and σ is a localic map. Observe that γ_L is also dense: $\bigvee I = 0$ implies $I = \{0\}$.

2.11. The following diagram commutes

$$\begin{array}{ccc} \mathcal{R}L & \xrightarrow{\ \gamma_L\ } & L \\ \Big\downarrow{\scriptstyle \mathcal{R}f} & & \Big\downarrow{\scriptstyle f} \\ \mathcal{R}M & \xrightarrow{\ \gamma_M\ } & M \end{array}$$

for any frame homomorphism $f\colon L \to M$ between completely regular frames (since $(\gamma_M \cdot \mathcal{R}f)(I) = \bigvee(\downarrow f[I]) = \bigvee f[I] = f[\bigvee I] = f\gamma_M(I)$). Thus, $\gamma\colon \mathcal{R} \xrightarrow{\bullet} \mathrm{Id}$ is a natural transformation

2.12. Lemma. *If L is a compact regular frame, then γ_L is an isomorphism and $L \cong \mathcal{R}L$.*

Proof. Let I be a regular ideal of L. We already know that $\sigma\gamma_L(I) \supseteq I$. For $x \in \sigma\gamma_L(I)$, we have $x \prec\!\!\prec \bigvee I$, hence $x^* \vee \bigvee I = 1$. By compactness of L, there exists a finite $F \subseteq I$ such that $x^* \vee \bigvee F = 1$. We see that $\bigvee F \in I$ and $x \prec\!\!\prec \bigvee F$, hence x belongs to I and $\sigma\gamma_L(I) \subseteq I$.

Regular ideals of L are precisely the ideals of the form $\sigma(a)$. Thus, γ_L is an one-one frame homomorphism and consequently γ_L is an isomorphism. □

2.13. From the previous we obtain

Theorem. *The functor $\mathcal{R}$ and the natural transformation γ provides a coreflection of the category of completely regular frames onto the category of compact regular frames.*

In other words, for a completely regular frame L, the mapping $\gamma_L\colon \mathcal{R}L \to L$ is the compactification of L.

2.14. Note that a topological space is *compact* if and only if the frame of its open sets is compact.

2.15. As a consequence of Hofmann–Lawson's Duality [7] (which depends on the Axiom of Choice), we obtain

Theorem. *The category of compact regular frames is equivalent to the dual of the category of compact regular spaces.*

In particular each regular compact frame is spatial. $(\star)$

3. Properties of compactification with respect to disconnectedness

3.1. Proposition. *A completely regular frame is disconnected iff its Čech–Stone compactification is disconnected.*

Proof. For a completely regular frame L, if L is disconnected, then by 1.2 there exists a one-one frame homomorphism $f\colon B_2 \to L$. From 2.13, we know that there exists an extension, a frame homomorphism $\tilde{f}\colon B_2 \to \mathcal{R}L$, such that $\gamma_L \tilde{f} = f$. Since f is one-one, $\tilde{f}$ is also one-one.

For converse, suppose $\mathcal{R}L$ is disconnected. There exists a non-trivial clopen sublocale S of $\mathcal{R}L$, and then $S \cap L$ is a non-trivial clopen sublocale of L. □

For the clarity of proofs, when there is no danger of confusion, we will write just $\overline{S}$ instead of $\overline{S}^{\mathcal{R}L}$ for the closure of a sublocale $S \subseteq \mathcal{R}L$, whereas we will write $\overline{S}^L$ for the closure of S in L.

3.2. Lemma. *Let S be a dense sublocale of L and let $U \subseteq L$ be an open sublocale. Then $\overline{U \cap S}^L = \overline{U}^L$.*

Proof. Let $U = \mathfrak{o}(a)$, for some a. Then $\bigwedge(\mathfrak{o}(a) \cap S) = \bigwedge \{ a \to s \mid s \in S \} = a \to \bigwedge S$. We have $\bigwedge S = 0$, since S is dense in L (and includes 0_L). Hence $\overline{U \cap S}^L = {\uparrow}(\bigwedge(U \cap S)) = {\uparrow}(a \to 0) = \mathfrak{c}(a^*) = \overline{U}^L$. □

3.3. Lemma. *Let L be a completely regular frame and let M be a clopen sublocale of L. Then the closure of M in $\mathcal{R}L$ is also clopen.*

Proof. Let N be the complement sublocale of M in L. L is dense in $\mathcal{R}L$, hence we have

$$\overline{M} \vee \overline{N} = \overline{M \vee N} = \overline{\overline{M \vee N}^L} = \overline{L} = \mathcal{R}L.$$

Since M and N are complemented in L, there exists an onto localic map $f\colon L \to B_2$ such that

$$f[M] \subseteq {\uparrow}a \quad \text{and} \quad f[N] \subseteq {\uparrow}b,$$

where a and b are the two complemented elements of B_2 different from 0 and 1. From Theorem 2.13, we know that there exists a localic map $\tilde{f}\colon \mathcal{R}L \to B_2$ such that $\tilde{f}\Big|_M = f$. Since localic maps preserve meets, we know that $f[\overline{M}] \subseteq \overline{f[M]}$, hence

$$\tilde{f}[\overline{M}] \subseteq \overline{\tilde{f}[M]} = \overline{f[M]} \subseteq {\uparrow}a \quad \text{and} \quad \tilde{f}[\overline{N}] \subseteq {\uparrow}b.$$

By the Proposition in II.3.5,

$$\overline{M} \subseteq f^{-1}[{\uparrow}a] = {\uparrow}f_*(a) \quad \text{and} \quad \overline{N} \subseteq f^{-1}[{\uparrow}b] = {\uparrow}f_*(b),$$

and therefore, for $m = \bigwedge M$ and $n = \bigwedge N$,

$$m \geq f_*(a) \quad \text{and} \quad n \geq f_*(b).$$

Finally, from $m \vee n \geq f_*(a) \vee f_*(b) = f_*(a \vee b) = 1$ we have

$$\overline{M} \wedge \overline{N} = \mathfrak{c}(m) \wedge \mathfrak{c}(n) = \mathfrak{c}(m \vee n) = \mathfrak{c}(1) = \mathsf{O}.$$

Hence, $\overline{M}$ and $\overline{N}$ are mutually complemented sublocales of $\mathcal{R}L$. □

3.4. Proposition. *Let L be an extremally disconnected frame. Then $\mathcal{R}L$ is also extremally disconnected.*

Proof. Let U be an open sublocale of $\mathcal{R}L$. Take $V = U \cap L$ an open sublocale of L. We know that $\overline{V}^L = \overline{U \cap L}^L = \overline{U \cap L} \cap L = \overline{U} \cap L$ (the last equality follows from Lemma 3.2). From extremal disconnectedness, $\overline{V}^L$ is clopen in L and, from Lemma 3.3, $\overline{\overline{V}^L}$ is clopen in $\mathcal{R}L$. Then

$$\overline{U} = \overline{U \cap L} \subseteq \overline{\overline{V}^L} = \overline{\overline{U} \cap L} \subseteq \overline{\overline{U}} = \overline{U}.$$

□

3.5. Remark. Analogously to set topology, it is not always the case that Čech–Stone compactification of zero–dimensional frames is again zero–dimensional. Frames in which this is true are called strongly zero–dimensional [10].

Chapter IV

Stone duality

In 1936 Marshall Stone published a paper called "The Theory of Representations of Boolean Algebras" [14], describing a duality of two categories, the category of Boolean algebras and Boolean homomorphisms and the category of compact Hausdorff zero–dimensional topological spaces and continuous maps. At that time he could not use the language of category theory, of course.

The word "duality" means that one category is equivalent to the dual of the other category. The duality at that time was a big surprise because it has shown similarities between a objects with nice algebraic structure, the Boolean algebras, with, another mathematical objects witch was thought to have no algebraic structure at all, topological spaces.

In this chapter, we will first discuss the Stone duality in the point–free context. Since the adjunction between topological spaces and frames is contravariant, the construction will be covariant and we speak of the Stone correspondence. In the second part we will obtain the standard Stone duality for topological spaces.

1. Stone correspondence for frames

1.1. Definition. We say a frame is *Stone frame* if it is compact and zero–dimensional.

By **StoneFrm** denote the category of Stone frames and frame homomorphisms.

1.2. Definition. Let B be a Boolean algebra. Define $\mathfrak{J}_\omega B$ to be the set of all ideals of B. For a Boolean homomorphism $f\colon A \to B$, define $\mathfrak{J}_\omega f\colon \mathfrak{J}_\omega A \to \mathfrak{J}_\omega B$ as

$$(\mathfrak{J}_\omega f)(I) = {\downarrow} f[I].$$

Note. We do not assume B complete, hence we depart from the notation used in III.2.3.

1.3. Lemma. *Let B be a Boolean algebra and $I \in \mathfrak{J}_\omega B$. Then I is complemented iff $I = {\downarrow} b$ for some $b \in B$.*

Proof. Let I be a complemented ideal. Since $I \vee I^c = 1_{\mathfrak{J}_\omega B}$ there exists $a \in I$ and $b \in I^c$ such that $a \vee b = 1$. From $I \wedge I^c = 0_{\mathfrak{J}_\omega B}$ we have $a \wedge b = 0$ and $I \wedge {\downarrow} b = 0_{\mathfrak{J}_\omega B}$. From the uniques of complements we get $I^c = {\downarrow}b$, indeed $I \vee {\downarrow}b = 1_{\mathfrak{J}_\omega B}$ and $I \wedge {\downarrow}b = 0_{\mathfrak{J}_\omega B}$. Using the same argument we get $I = {\downarrow}a$.

The converse implication is trivial, since ${\downarrow}a \vee {\downarrow}a^c = 1_{\mathfrak{J}_\omega B}$ and ${\downarrow}a \wedge {\downarrow}a^c = 0_{\mathfrak{J}_\omega B}$. □

1.4. Proposition. $\mathfrak{J}_\omega\colon$ **Bool** $\to$ **StoneFrm** *is a functor.*

Proof. $\mathfrak{J}_\omega B$ is a compact frame by III.2.3. Lemma 1.3 implies that ${\downarrow}a$ is complemented and ${\downarrow}a \prec {\downarrow}a$ for all $a \in B$. Thus for any ideal $I \in \mathfrak{J}_\omega B$, we obtain

$$I = \bigvee \{\, {\downarrow}a \mid a \in I \,\} = \bigvee \{\, {\downarrow}a \mid {\downarrow}a \prec I \,\} \subseteq \bigvee \{\, J \mid J \prec I \,\} \subseteq I.$$

Hence, $\mathfrak{J}_\omega B$ is zero–dimensional. If f is a Boolean homomorphism, then $\mathfrak{J}_\omega f$ is a frame homomorphism (for the same reason as the $\mathcal{R}f$ is in III.2.9). □

1.5. Definition. Let L be a Stone frame. Define $\mathbb{B}_\omega L$ to be the set of all complemented elements of L and for a frame homomorphism between Stone frames $f\colon L \to M$ define

$$\mathbb{B}_\omega f = f\big|_{\mathbb{B}_\omega L}\colon \mathbb{B}_\omega L \to \mathbb{B}_\omega M.$$

From the fact that a homomorphic image of a complemented element is a complemented element and since join or meet of two complemented elements is again complemented (for complemented elements a and b, $a^c \vee b^c$ is the complement of $a \wedge b$ and $a^c \wedge b^c$ is the complement of $a \vee b$), one can see that $\mathbb{B}_\omega L$ is a Boolean algebra and $\mathbb{B}_\omega f$ is well–defined Boolean homomorphism.

1.6. Observation. $\mathbb{B}_\omega\colon$ **StoneFrm** $\to$ **Bool** *is a functor.*

1.7. For a Boolean algebra B, define $i_B\colon B \to \mathbb{B}_\omega\mathfrak{J}_\omega(B)$ as follows

$$i_B\colon b \mapsto {\downarrow}b.$$

The definition is sound by Lemma 1.3 and i_B is a Boolean homomorphism: indeed ${\downarrow}a \vee {\downarrow}b = {\downarrow}(a \vee b)$, ${\downarrow}a \wedge {\downarrow}b = {\downarrow}(a \wedge b)$, and ${\downarrow}1 = B$ respectively ${\downarrow}0 = \{0\}$ is top respectively bottom of $\mathbb{B}_\omega\mathfrak{J}_\omega(B)$.

From Lemma 1.3, we also see that i_B is an isomorphism and the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\;i_A\;} & \mathbb{B}_\omega\mathfrak{J}_\omega(A) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{B}_\omega\mathfrak{J}_\omega(f)} \\ B & \xrightarrow{\;i_B\;} & \mathbb{B}_\omega\mathfrak{J}_\omega(B) \end{array}$$

for any Boolean homomorphism $f\colon A \to B$ (for any $a \in A$, $\mathbb{B}_\omega\mathfrak{J}_\omega(f)\, i_A(a) = \mathbb{B}_\omega\mathfrak{J}_\omega(f)({\downarrow}a) = {\downarrow}f[{\downarrow}a] = {\downarrow}f(a) = i_B\, f(a)$). From previous we have

Proposition. *The collection $i = (i_A)_A$ of Boolean homomorphisms forms a natural equivalence between $\mathbb{B}_\omega\mathfrak{J}_\omega$ and the identity functor on* **Bool**.

1.8. Similarly, for a Stone frame L, we have a mapping $v_L\colon \mathfrak{J}_\omega\mathbb{B}_\omega(L) \to L$ defined as

$$v_L\colon I \mapsto \bigvee I,$$

and a mapping in the opposite direction $\iota\colon L \to \mathfrak{J}_\omega\mathbb{B}_\omega(L)$:

$$\iota\colon e \mapsto {\downarrow} e \cap \mathbb{B}_\omega L.$$

We can see that both v_L and ι are monotone maps, $v_L\iota = \mathrm{id}_L$ (by zero–dimensionality of L) and $\mathrm{id}_{\mathfrak{J}_\omega\mathbb{B}_\omega(L)} \subseteq \iota v_L$. Therefore v_L is the left Galois adjoint to ι and hence v_L preserves all suprema. Since $\bigvee I_1 \wedge \bigvee I_2 = \bigvee \{ a_1 \wedge a_2 \mid a_i \in I_i \} \leq \bigvee \{ a \mid a \in I_1 \cap I_2 \} = \bigvee (I_1 \cap I_2) \leq \bigvee I_1 \wedge \bigvee I_2$, v_L also preserves finite infima which makes v_L a frame homomorphism.

Finally, $\mathrm{id}_{\mathfrak{J}_\omega\mathbb{B}_\omega(L)} = \iota v_L$: take any $x \in \iota v_L(I)$. From the definitions we immediately see that $x \leq \bigvee I$ and x is complemented in L. By the fact that

$$1 = x \vee x^c \leq \bigvee I \vee x^c$$

and by compactness of L there is a finite $F \subseteq I$ such that $\bigvee F \vee x^c = 1$. Since $x = 1 \wedge x = (\bigvee F \vee x^c) \wedge x = (\bigvee F \wedge x) \vee (x^c \wedge x) = \bigvee F \wedge x$ we get that $x \leq \bigvee F$ and therefore $x \in I$.

From previous observations, we know that v_L is an isomorphism of L and $\mathfrak{J}_\omega\mathbb{B}_\omega(L)$. Also by direct computation, for any homomorphism of Stone frames $f\colon L \to M$, we see that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{J}_\omega\mathbb{B}_\omega(L) & \xrightarrow{v_L} & L \\ {\scriptstyle \mathfrak{J}_\omega\mathbb{B}_\omega(f)}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathfrak{J}_\omega\mathbb{B}_\omega(M) & \xrightarrow{v_M} & M \end{array}$$

(for any $I \in \mathfrak{J}_\omega\mathbb{B}_\omega(L)$, $(v_M \cdot \mathfrak{J}_\omega\mathbb{B}_\omega(f))(I) = \bigvee({\downarrow} f[I]) = \bigvee f[I] = f(\bigvee I) = f\, v_L(I))$. Again, as a consequence of previous paragraphs we obtain

Proposition. *The collection $v = (v_L)_L$ of frame homomorphisms is a natural equivalence between $\mathfrak{J}_\omega\mathbb{B}_\omega$ and the identity functor on* **StoneFrm**.

1.9. Using previous facts we obtain the main result of this section.

Theorem. *Functors $\mathbb{B}_\omega$ and $\mathfrak{J}_\omega$ constitute an equivalence of categories* **StoneFrm** *and* **Bool**.

2. Stone duality for spaces

The classical spatial version of Stone duality follows from the Stone correspondence between Stone frames and Boolean algebras from previous section. We have the duality between the category of compact zero–dimensional frames and the category compact zero–dimensional spaces (restricted Hofmann–Lawson's Duality, Theorem III.2.15).

However the Hofmann–Lawson's Duality depends on the Axiom of Choice. In this section we will prove the Stone duality by assuming only the Boolean Ultrafilter Theorem.

2.1. Similarly to frames:

Definition. We say a topological space is *Stone space* if it is compact, Hausdorff and zero–dimensional.

By **StoneSp** denote the category of Stone spaces and continuous mappings.

2.2. Let B be a Boolean algebra. Define

$$X_B = \{\, F \subseteq B \mid F \text{ is an ultrafilter of } B \,\} \qquad \text{and} \qquad \tau_B = \{\, W_I \mid I \in \mathfrak{J}_\omega B \,\},$$

where W_I is the set $\{\, F \in X_B \mid F \cap I \neq \emptyset \,\}$. The pair (X_B, τ_B) is a topological space:

(T1) τ_B contains $\emptyset = W_{\downarrow 0}$ and $X_B = W_{\downarrow 1}$.

(T2) $\bigcup_i W_{J_i} = W_{\bigvee_i J_i}$: $\subseteq$ holds trivially since $\bigcup J_i \subseteq \bigvee J_i$. On the other hand, for an ultrafilter F such that $F \cap \bigvee_i J_i \neq \emptyset$, there is an $e = \bigvee E$ where E is a finite subset of $\bigcup J_i$ and $e \in F$. Since F is prime, there is an $e' \in E$ such that $e' \in F$ and therefore $J_j \ni e'$ and $F \cap J_j \neq \emptyset$ for some j.

(T3) $W_I \cap W_J = W_{I \wedge J}$: $F \in W_I \cap W_J$ iff $F \cap I \neq \emptyset$ and $F \cap J \neq \emptyset$ iff $F \cap I \cap J \neq \emptyset$ iff $F \in W_{I \cap J}$.

We will denote the topological space (X_B, τ_B) by $\mathcal{S}B$.

2.3. Lemma. *Let B be a Boolean algebra. Then $\mathcal{S}B$ is a Stone space.*

Proof. Compactness follows directly from compactness of $\mathfrak{J}_\omega B$. Note that the complemented elements are of the form $W_{\downarrow a}$ for some $a \in B$: $W_{\downarrow a} \cup W_{\downarrow a^c} = W_{\downarrow (a \vee a^c)} = X_B$ and $W_{\downarrow a} \cap W_{\downarrow a^c} = W_{\downarrow (a \wedge a^c)} = \emptyset$. Since every open set is an union of sets of the form $W_{\downarrow a}$, the $\mathcal{S}B$ is zero–dimensional.

To show that $\mathcal{S}B$ is also Hausdorff, take any ultrafilters $E \neq F$. Without any loss of generality there is some $e \in E \setminus F$. We have $e \vee e^c = 1$ and since F is an ultrafilter we have $e^c \in F$. We also have $e \wedge e^c = 0$ and so $W_{\downarrow e}$ and $W_{\downarrow e^c}$ separates E and F. □

2.4. Lemma. *Let $f \colon A \to B$ be a Boolean homomorphism and let F be an ultrafilter of B. Then $f^{-1}[F]$ is an ultrafilter of A.*

Proof. Set $E = f^{-1}[F]$. First, we will show that E is a filter. Trivially, $1 \in E$ and $0 \notin E$. For $x, y \in E$, there are $x', y' \in F$ such that $x' = f(x)$ and $y' = f(y)$. Hence $x' \wedge y' = f(x) \wedge f(y) = f(x \wedge y)$ and $x \wedge y \in E$. For the upwards closeness, take any $x \in E$ and $y \geq x$. $y \in E$ as $f(y) \geq f(x) \in F$.

E is also an ultrafilter. For $a, b \in A$ such that $a \vee b \in E$, $f(a) \vee f(b) = f(a \vee b) \in F$ and so $f(a)$ or $f(b)$ is in F and therefore a or b is in E. □

2.5. Let $f\colon A \to B$ be a Boolean homomorphism. Denote by $\mathcal{S}f\colon \mathcal{S}B \to \mathcal{S}A$ the map defined as

$$\mathcal{S}f\colon F \mapsto f^{-1}[F].$$

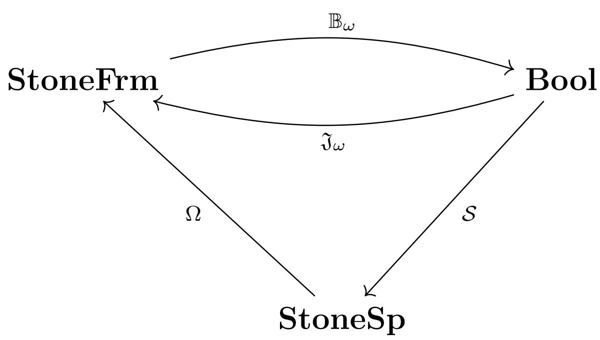
From the previous Lemma, we see that the definition is sound. We will show that $\mathcal{S}f$ is also continuous. Take any $W_I \in \tau_B$. We have

$$\begin{aligned}(\mathcal{S}f)^{-1}[W_I] &= \left\{ (\mathcal{S}f)^{-1}(F) \;\middle|\; F \cap I \neq \emptyset \right\} \\ &= \{ E \mid \exists F \subseteq B \text{ ultrafilter}, (\mathcal{S}f)(E) = F, \text{ and } F \cap I \neq \emptyset \} \\ &= \left\{ E \;\middle|\; f^{-1}(E) \cap I \neq \emptyset \right\} = \{ E \mid E \cap f[I] \neq \emptyset \} \\ &= \{ E \mid E \cap {\downarrow} f[I] \neq \emptyset \} = W_{(\mathfrak{J}_\omega h)(I)}.\end{aligned}$$

2.6. Theorem. $\mathcal{S}\colon$ **Bool** $\to$ **StoneSp** *is a functor.*

Proof. Follows immediately from 2.3 and 2.5. □

2.7. Our situation is as follows



2.8. Proposition. *The collection of morphisms* $\pi_B\colon \mathfrak{J}_\omega B \to \Omega\mathcal{S}(B)$ *defined for* $B \in$ **Bool** *by* $I \mapsto W_I$*, constitutes a natural equivalence* $\mathfrak{J}_\omega \cong \Omega \circ \mathcal{S}$. $(\star)$

Proof. From the definition of $\mathcal{S}B$, we can see that π_B is an onto frame homomorphism.

Take any $I \neq J$, $I, J \in \mathfrak{J}_\omega B$. Without loss of generality take $e \in I \setminus J$. Now the filter ${\uparrow}e$ is disjoint with the ideal J and by Boolean Ultrafilter Theorem there exists an ultrafilter $U \supseteq {\uparrow}e$ disjoint with J. The intersection $U \cap I$ is not empty, it contains e, and therefore $W_I \neq W_J$.

The naturalness of π follows from the commutativity of the following diagram

$$\begin{array}{ccc} \mathfrak{J}_\omega A & \xrightarrow{\pi_A} & \Omega\mathcal{S}(A) \\ {\scriptstyle \mathfrak{J}_\omega f}\Big\downarrow & & \Big\downarrow{\scriptstyle \Omega\mathcal{S}(f)} \\ \mathfrak{J}_\omega B & \xrightarrow{\pi_B} & \Omega\mathcal{S}(B) \end{array}$$

for any Boolean homomorphism $f\colon A \to B$. Indeed, by 2.5, we have

$$\Omega\mathcal{S}(f)(W_I) = (\mathcal{S}f)^{-1}[W_I] = W_{(\mathfrak{J}_\omega h)(I)},$$

for any $I \in \mathfrak{J}_\omega B$. □

2.9. Conclusion. $\mathbb{B}_\omega \circ \Omega \circ \mathcal{S} \cong \mathrm{Id}_{\mathbf{Bool}}$. $(\star)$

Proof. By the previous Proposition and the Stone correspondence for Stone frames and Boolean algebras (Theorem 1.9), we have $\mathbb{B}_\omega \circ \Omega \circ \mathcal{S} \cong \mathbb{B}_\omega \circ \mathfrak{J}_\omega \cong \mathrm{Id}_{\mathbf{Bool}}$. □

2.10. Proposition. *The collection of morphisms* $\rho_X\colon X \to \mathcal{S}\mathbb{B}_\omega\Omega(X)$ *defined for* $X \in$ **StoneSp** *by* $x \mapsto F_x = \{\, U \text{ clopen} \mid x \in U \,\}$, *constitutes a natural equivalence* $\mathrm{Id}_{\mathbf{StoneSp}} \cong \mathcal{S} \circ \mathbb{B}_\omega \circ \Omega$.

Proof. We will show that ρ_X is a homeomorphism.

- ρ_X is one–one: For two points $x_1 \neq x_2$ of X, from Hausdorff property there are U_1, U_2 such that $U_1 \cap U_2 = \emptyset$ and $x_i \in U_i$. From zero–dimensionality of X there are two clopen subsets $M_1 \subseteq U_1, M_2 \subseteq U_2$, and $x_i \in M_i$. Hence $F_{x_1} \neq F_{x_2}$.

- ρ_X is onto: Take any F ultrafilter. We will prove $F \subseteq F_x$ for some $x \in X$. Suppose it is not the case, then $\bigcap F = \emptyset$. Further, $X = X \setminus \bigcap F = \bigcup_{U \in F}(X \setminus U)$. Therefore $\mathcal{C} = \{\, X \setminus U \mid U \in F \,\}$ is a cover and from compactness there exist a finite subcover $\mathcal{C}' \subseteq \mathcal{C}$. Then, $X = \bigcup_{X\setminus U \in \mathcal{C}'}(X \setminus U) = X \setminus \bigcap_{(X\setminus U)\in\mathcal{C}'} U$. This is a contradiction, since $\bigcap_{(X\setminus U)\in\mathcal{C}'}$ is empty set.
  And so $F \subseteq F_x$; but F is an ultrafilter, hence maximal, and hence $F = F_x$.

- ρ_X and ρ_X^{-1} are continuous: From previous (•), we know that each ultrafilter of X is of the form F_x, for some x. Hence

$$\begin{aligned} \rho_X^{-1}[W_I] &= \left\{\, \rho_X^{-1}(F_x) = x \;\middle|\; F_x \cap I \neq \emptyset \,\right\} = \{\, x \mid x \in M \in I \,\} = \bigcup I \in \tau_X, \text{ and} \\ \rho_X[W] &= \{\, F_x \mid x \in M \subseteq W, M \text{ is clopen} \,\} \\ &= \{\, F \mid F \cap ({\downarrow}W \cap \mathbb{B}_\omega\Omega(X)) \neq \emptyset \,\} = W_{{\downarrow}W \cap \mathbb{B}_\omega\Omega(X)} \in \tau_{\mathcal{S}\mathbb{B}_\omega\Omega(X)}. \end{aligned}$$

Finally, $\rho = (\rho_X)_X$ is a natural equivalence. The following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{\rho_X} & \mathcal{S}\mathbb{B}_\omega\Omega(X) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{S}\mathbb{B}_\omega\Omega(f)} \\
Y & \xrightarrow{\rho_Y} & \mathcal{S}\mathbb{B}_\omega\Omega(Y)
\end{array}
$$

for any continuous map $f\colon X \to Y$. Indeed,

$$
\begin{aligned}
\mathcal{S}\mathbb{B}_\omega\Omega(f)(F_x) &= (\mathbb{B}_\omega\Omega(f))^{-1}[F_x] = \Omega(f)^{-1}[F_x] \\
&= \left\{ \Omega(f)^{-1}(M) \;\middle|\; M \text{ clopen}, x \in M \right\} \\
&= \{ N \mid x \in M, M \text{ clopen}, f[M] = N \} \\
&= \{ N \mid f(x) \in N \} = F_{f(x)} = \rho_Y(f(x)).
\end{aligned}
$$

□

2.11. From the previous, we have

Theorem. *Functors $\Omega \circ \mathbb{B}_\omega$ and $\mathcal{S}$ are mutually inverse. Thus, categories* **Bool** *and* **StoneSp**^{op} *are equivalent.* (⋆)

2.12. Similarly to the previous section, we obtained an equivalence of categories. However, the adjoint functors of this equivalence are contravariant, thus we obtained a duality of two categories instead of a correspondence.

Note that we can make the correspondence, from the previous section, into duality by the duality between the category of frames and the category of locales.

3. Notes on constructivity

One can check that the whole correspondence between Stone frames and Boolean algebras has been proved constructively. Therefore, the necessity of a choice principle needed in classical Stone duality between Stone spaces and Boolean algebras is necessary only to show that spaces constructed from Boolean algebras have enough points.

Analogously, the compactification of completely regular frames described in the previous chapter was constructive; its counterpart for topological spaces is equivalent to Boolean Ultrafilter Theorem [2].

To be more precise, in the proof of the compactification we used the fact that compact regular frames are completely regular (Proposition III.2.2). We construct an interpolative relation $\prec\!\!\prec$ using the Axiom of Countable Dependent Choice (CDC), but by Banaschewski and Pultr [6], we can avoid using CDC by working with strongly regular frames instead of completely regular. The whole construction can then be made constructive, even in the sense of topos theory.

Strongly regular frames are frames in which, for every element x, the following equation holds

$$x = \bigvee \{\, y \mid y \; (\prec)_o \; x \,\}\,,$$

where $(\prec)_o$ is the largest interpolative relation contained in $\prec$. Such a relation can be constructed as the union of all interpolative relations contained in $\prec$. Under CDC, complete regularity is precisely the same as strong regularity [6].

Chapter V

Parts of duality

In this chapter, we analyse some parts of the Stone correspondence for Stone frames and Boolean algebras. Namely, we will show that the category of κ–complete Boolean algebras is in the correspondence with the category of κ–basically disconnected Stone frames.

We will also show how the restriction of the class of Boolean homomorphisms affects the morphisms on the side of Stone frames.

1. κ–complete Boolean algebras

For the rest of this section, let κ be a fixed infinite regular cardinal.

1.1. Definition. A lattice is *κ–complete*, if any subset of cardinality less than κ has a supremum and an infimum. A homomorphism is said to be *κ–complete* if it preserves all suprema and infima of subsets of cardinality less than κ.

Let L be a Stone frame and let s be an element of L. We say that s is *κ–generated* if $s = \bigvee S$ for some $S \subseteq \mathbb{B}_\omega L$ of cardinality less than κ.

1.2. The following Lemma will be often expedient for computations.

Lemma. *Let B be a κ–complete Boolean algebra and let I be a κ–generated element of $\mathfrak{J}_\omega B$. Then $I^* = {\downarrow}(\bigvee I)^c$ in $\mathfrak{J}_\omega B$. In particular, for $a \in B$, $({\downarrow}a)^* = {\downarrow}a^c$.*

Proof. I is of the form $\bigvee \{ {\downarrow}s \mid s \in S \}$ for some $S \subseteq B$ of cardinality less than κ. Since $b \wedge s = 0$ iff $b \leq s^c$, we have

$$\begin{aligned} I^* &= (\bigvee \{ {\downarrow}s \mid s \in S \})^* = \bigvee \{ {\downarrow}b \mid b \in B,\ b \wedge s = 0,\ \text{for all } s \in S \} \\ &= \bigvee \{ {\downarrow}b \mid b \in B,\ b \leq \bigwedge_{s \in S} s^c \} = \bigvee \{ {\downarrow}b \mid b \in B,\ b \leq (\bigvee S)^c \} = {\downarrow}(\bigvee S)^c = {\downarrow}(\bigvee I)^c. \end{aligned}$$

□

1.3. Definition. Let L be a Stone frame. We say that L is *κ–basically disconnected* if $m^{**} \vee m^* = 1$ for all κ–generated elements $m \in L$.

1.4. Lemma. *If B is a κ–complete Boolean algebra then $\mathfrak{J}B$ is a κ–basically disconnected Stone frame.*

Proof. By Lemma IV.1.3, we know that complemented ideals are precisely the principal ideals of B. For a subset M of B of cardinality less than κ set $I = \bigvee\{\,{\downarrow}a \mid a \in M\,\}$. We will show that $I^{**} \vee I^* = 1_{\mathfrak{J}B}$.

Set $m = \bigvee M$ and $J = {\downarrow}m$. Observe that $I^{**} = J$: Trivially from Lemma 1.2 we have $I^{**} \subseteq J^{**} = J$. The other inclusion, the $J \subseteq I^{**}$, follows from

$$I^* = \bigcup\{\,{\downarrow}a \mid {\downarrow}a \wedge I = 0_{\mathfrak{J}B}\,\} = \bigcup\{\,{\downarrow}a \mid a \wedge m = 0\,\} = ({\downarrow}m)^* = J^*.$$

Consequently, $I^{**} \vee I^* = J \vee J^* = {\downarrow}m \vee {\downarrow}m^c = 1_{\mathfrak{J}B}$. □

1.5. The following Lemma will be often very useful. We will use it without further reference.

Lemma. *Let B be a Boolean algebra such that each of its subset of cardinality at most κ has a supremum. Then B is κ–complete.*

Consequently, any Boolean homomorphism preserving all κ–meets (or κ–joins) is κ–complete.

Proof. Let S be an arbitrary subset of B such that $|S| < \kappa$. Set $M = (\bigvee\{\,b^c \mid b \in S\,\})^c$, we will show that M is the infimum of S.

Take an $a \in S$. We have $a \wedge M = a^{cc} \wedge (\bigvee\{\,b^c \mid b \in S\,\})^c = (a^c \vee \bigvee\{\,b^c \mid b \in S\,\})^c = M$. Hence $a \geq M$.

Now suppose $m \leq a$ for all $a \in S$. Then $m \wedge M = (m^c \vee \bigvee\{\,b^c \mid b \in S\,\})^c = (m^c)^c = m$. Hence $m \leq M$. □

1.6. Lemma. *If L is a κ–basically disconnected Stone frame then $\mathbb{B}_\omega L$ is a κ–complete Boolean algebra. The joins in $\mathbb{B}_\omega L$ are defined by the following formula*

$$\bigsqcup M = (\bigvee M)^{**}.$$

Proof. For M a subset of $\mathbb{B}_\omega L$ of cardinality less than κ, set $m = \bigvee M$. Since L is κ–basically disconnected, we have $m^{**} \vee m^* = 1$ and therefore $m^{**} \in \mathbb{B}_\omega L$. So m^{**} is an upper bound for M in $\mathbb{B}_\omega L$.

Now, let n be an arbitrary upper bound for M in $\mathbb{B}_\omega L$. Then n is also an upper bound in L, but $m \leq n$ since m is the supremum of M in L. This gives us the desired relation $m^{**} \leq n^{**} = n$, hence m^{**} is the supremum of M in $\mathbb{B}_\omega L$. □

1.7. From Lemma 1.5 and Lemma 1.6, we conclude that κ–complete Boolean algebras are in (Stone) correspondence with κ–basically disconnected Stone frames. The same holds for topological spaces, a topological space is κ–basically disconnectedness iff any union of cardinality less than κ clopen sets has open closure. Hence, we have a duality between κ–complete Boolean algebras and κ–basically disconnected Stone spaces [12].

Now, we will focus on morphisms.

1.8. Observation. *Let $f\colon A \to B$ be a κ–complete Boolean homomorphism and let I be a κ–generated ideal of A. Then*

$$(\mathfrak{I}_\omega f)(I^*) = (\mathfrak{I}_\omega f)(I)^*.$$

Proof. It is straightforward. We have

$$\begin{aligned}
(\mathfrak{I}_\omega f)(I^*) &= (\mathfrak{I}_\omega f)(\downarrow(\bigvee I)^c) && \text{(Lemma 1.2)}\\
&= \downarrow f[\downarrow(\bigvee I)^c] = \downarrow f((\bigvee I)^c)\\
&= \downarrow f(\bigvee I)^c && \text{(f is a Boolean homomorphism)}\\
&= \downarrow(\bigvee f[I])^c && \text{(κ–completeness of f)}\\
&= \downarrow(\bigvee \downarrow f[I])^c = \downarrow(\bigvee(\mathfrak{I}_\omega f)(I))^c\\
&= (\mathfrak{I}_\omega f)(I)^*.
\end{aligned}$$

The use of κ–completeness of f in the fourth step is valid because I is κ–generated. In particular, $I = \bigvee\{\downarrow s \mid s \in S\}$ for some S a subset of A of the cardinality less then κ. □

1.9. Definition. Let $f\colon L \to M$ be a homomorphism between Stone frames. We say that f is a *κ–basically complete* if $f(a^*) = f(a)^*$ holds for all κ–generated elements $a \in L$.

In other words, the last observation states that the functor $\mathfrak{I}_\omega$ sends any κ–complete Boolean homomorphism to a κ–basically complete frame homomorphism. As we will see in the following Lemma, morphisms of the image of κ–complete part of Boolean algebras in Stone correspondence are characterised precisely this way.

1.10. Lemma. *Let $f\colon L \to M$ be a κ–basically complete frame homomorphism. Then $\mathbb{B}_\omega f\colon \mathbb{B}_\omega L \to \mathbb{B}_\omega M$ is a κ–complete Boolean homomorphism.*

Proof. Let A be an arbitrary subset of $\mathbb{B}_\omega L$ such that $|A| < \kappa$. We have

$$\begin{aligned}
(\mathbb{B}_\omega f)(\bigsqcup A) &= f((\bigvee A)^{**})\\
&= f((\bigvee A)^*)^* && \text{($(\bigvee A)^*$ is complemented)}\\
&= f(\bigvee A)^{**} && \text{($\bigvee A$ is κ–generated)}\\
&= (\bigvee f[A])^{**} && \text{(f is a frame homomorphism)}\\
&= \bigsqcup f[A] = \bigsqcup(\mathbb{B}_\omega f)[A].
\end{aligned}$$

Therefore $\mathbb{B}_\omega f$ is κ–complete. □

1.11. From Lemmas 1.4 and 1.6, we see that the restriction of Stone correspondence to subcategories of κ–complete Boolean algebras on one side and κ–basically disconnected

Stone frames on the other side (without any restriction on morphisms) is still a duality of categories.

If we set κ–**ComplBool** to be the category of κ–complete Boolean algebras and κ–complete Boolean homomorphisms and set κ–**BDStoneFrm** to be the category of κ–basically disconnected Stone frames and κ–basically complete frame homomorphisms, then it is sound (by the previous two Lemmas) to define two functors

$$\mathfrak{J}_\kappa \colon \kappa\text{–}\mathbf{ComplBool} \to \kappa\text{–}\mathbf{BDStoneFrm}, \text{ and}$$
$$\mathbb{B}_\kappa \colon \kappa\text{–}\mathbf{BDStoneFrm} \to \kappa\text{–}\mathbf{ComplBool},$$

as the restriction of $\mathfrak{J}_\omega$ and $\mathbb{B}_\omega$ to the corresponding subcategories. Note that the notation is consistent with the previously defined $\mathfrak{J}_\omega$ and $\mathbb{B}_\omega$. We get the following

Theorem. *The functors $\mathfrak{J}_\kappa$ and $\mathbb{B}_\kappa$ constitute an equivalence between categories κ–**ComplBool** and κ–**BDStoneFrm**.*

Proof. The only thing we need to show is that the morphisms of natural equivalences for identity functors and functors $\mathbb{B}_\omega\mathfrak{J}_\omega$ and $\mathfrak{J}_\omega\mathbb{B}_\omega$ are morphisms of our categories.

For the first part, we will show that $i_B \colon B \to \mathbb{B}_\omega\mathfrak{J}_\omega(B)$ is a κ–complete Boolean homomorphisms for any κ–complete Boolean algebra B. Let B' be an arbitrary subset of B of cardinality less than κ. Then by straightforward computation we get

$$\bigsqcup_{b\in B'} i_B(b) = \bigsqcup_{b\in B'} {\downarrow} b = (\bigvee_{b\in B'} {\downarrow} b)^{**} = {\downarrow}(\bigvee_{b\in B'} b) = i_B(\bigvee_{b\in B'} b),$$

where the third equality follows from Lemma 1.2.

For the second part, we need to show that $v_L \colon \mathfrak{J}_\omega\mathbb{B}_\omega(L) \to L$ is κ–basically complete for any κ–basically disconnected Stone frame L. We will show that v_L is λ–basically complete for any regular cardinal λ. Take any $I \in \mathfrak{J}_\omega\mathbb{B}_\omega(L)$, we have

$$\begin{aligned} v_L(I^*) &= v_L(\bigvee \{ {\downarrow} s \mid {\downarrow} s \wedge I = 0 \}) = \bigvee \{ v_L({\downarrow} s) \mid {\downarrow} s \wedge I = 0 \} \\ &= \bigvee \{ s \mid s \wedge v_L(I) = 0 \} = v_L(I)^*. \end{aligned}$$

□

2. Complete Boolean algebras

By Theorem 1.11 we know that there is an equivalence between the category of κ–complete Boolean algebras and κ–complete Boolean homomorphisms and the category of κ–basically disconnected Stone frames and κ–basically complete frame homomorphisms. Since, there is no limitation or upper bound for the cardinal κ in Theorem 1.11, let us have a look at the part of the correspondence where κ is arbitrary large.

2.1. It is interesting to see, how the Stone frame part of the correspondence looks like. Take any object L and an element $a \in L$. From zero–dimensionality, we know that a is λ–generated for some regular cardinal λ, but L is λ–basically disconnected so that

$$a^{**} \vee a^{*} = 1$$

holds. Since a has been chosen arbitrarily we see that L extremally disconnected. Similarly, any morphism of this part of the duality f with domain L is λ–basically complete, hence the following holds

$$f(a^{*}) = f(a)^{*}. \tag{B.C.}$$

We will call morphisms satisfying (B.C.) for all elements a *basically complete* frame homomorphisms.

We will denote the resulting category of extremally disconnected Stone frames and basically complete frame homomorphisms by **ExtrDStoneFrm**.

2.2. On the side of Boolean algebras, we have that Boolean algebras with arbitrary large joins and Boolean homomorphisms preserving such joins. Thus the observed category is the category of complete Boolean algebras and complete Boolean homomorphisms. We will denote it by **ComplBool**.

2.3. Recall that, the definitions of $i_B \colon B \to \mathbb{B}_\omega \mathfrak{J}_\omega(B)$ and $v_L \colon \mathfrak{J}_\omega \mathbb{B}_\omega(L) \to L$ are

$$i_B \colon b \mapsto {\downarrow} b \quad \text{and} \quad v_L \colon I \mapsto \bigvee I.$$

For a complete Boolean algebra B, take any $\{ a_i \mid i \in I \}$ subset of B. Then

$$\bigsqcup_{i\in I} i_B(a_i) = \bigsqcup_{i\in I} {\downarrow} a_i = (\bigvee_{i\in I} {\downarrow} a_i)^{**} = {\downarrow}(\bigvee_{i\in I} a_i)^{cc} = {\downarrow}(\bigvee_{i\in I} a_i) = i_B(\bigvee_{i\in I} a_i).$$

Consequently, i_B is a complete Boolean homomorphism, that is a morphisms of **ComplBool**.

For any extremally disconnected Stone frame L, from the proof of Theorem 1.11, we know that v_L is λ–basically complete frame homomorphism for all regular cardinals λ, therefore v_L is also basically complete and is a morphisms of **ExtrDStoneFrm**.

Therefore, $(v_L)_{L \in \mathbf{ExtrDStoneFrm}}$ *and* $(i_B)_{B \in \mathbf{ComplBool}}$ *are collection of morphisms of categories* **ExtrDStoneFrm** *and* **ComplBool**.

2.4. As a conclusion, we can define functors

$$\mathfrak{J}_\infty \colon \mathbf{ComplBool} \to \mathbf{ExtrDStoneFrm}, \text{ and}$$
$$\mathbb{B}_\infty \colon \mathbf{ExtrDStoneFrm} \to \mathbf{ComplBool},$$

as the restriction of $\mathfrak{J}_\omega$ and $\mathbb{B}_\omega$ to the corresponding categories and obtain the following

Theorem. *The categories* **ExtrDStoneFrm** *and* **ComplBool** *are equivalent.*

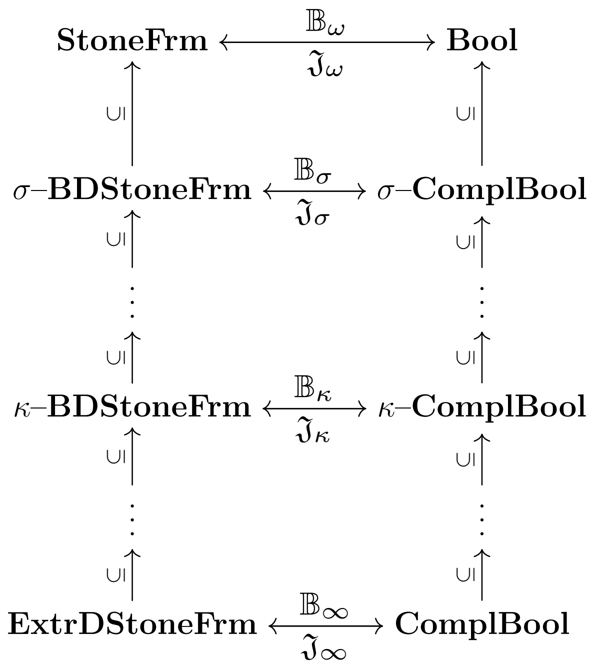


Figure V.1: Diagram of categories in Stone correspondence.

(Note that the subcategories indicated as $\subseteq$ are not full.)

The whole situation is depicted in Figure V.1 where κ is a regular cardinals greater or equal to ω_1, σ–**ComplBool** denotes the category ω_1–**ComplBool** and σ–**BDStoneFrm** denotes the category ω_1–**BDStoneFrm**.

2.5. Here we present an alternative point of view on what we already know from the discussion above.

Proposition. *Let B be a complete Boolean algebra. Then the frame $\mathfrak{J}_\omega B$ is an extremally disconnected Stone frame.*

Proof. In a Boolean algebra the relations $\prec$ and $\leq$ coincide. Moreover, each complete Boolean algebra is a (completely regular extremally disconnected) Boolean frame. Thus $\mathfrak{J}_\omega B$ equals $\mathcal{R}B$.

From Lemma III.3.4, we know that compactification preserves extremal disconnectedness and therefore $\mathfrak{J}_\omega B$ is also extremally disconnected. We also know that $\mathfrak{J}_\omega B$ is a Stone frame by Lemma IV.1.4. $\square$

The proof of the previous Proposition unveils an important fact about the complete part of Stone correspondence. It shows that this part of the correspondence is an equivalence of two subcategories of the category of frames (on Boolean side, without any restriction to frame homomorphisms) and that the correspondence is provided by a purely topological construction by compactification.

Chapter VI

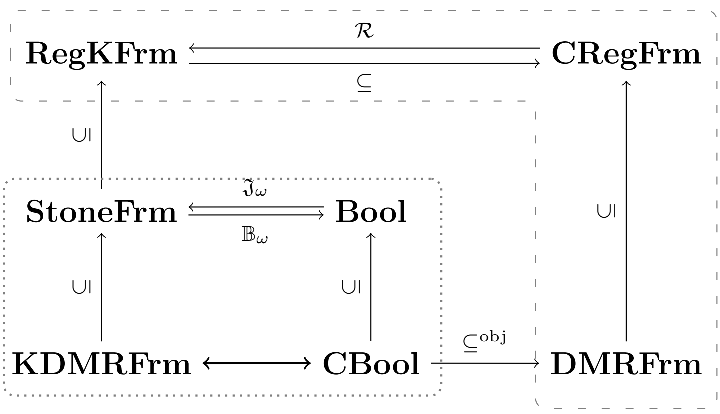
Construction of $\mathfrak{R}$

1. Summarization

Recall Theorem III.2.13. The compactification of a complete regular frames is obtained as the frame of all regular ideals. The construction of a Stone frame in the Stone correspondence (Theorem IV.1.9) is given in similar fashion. The functor $\mathfrak{J}_\omega$ maps a Boolean algebra to the frame of all ideals of the Boolean algebra. We can equivalently say that any Stone frame is obtained from a Boolean algebra by taking the frame of all regular ideals since in any Boolean algebra the relations $\leq$, $\prec$ and $\prec\!\!\prec$ coincide.

In classical topology, the compactification and the Stone representation of Boolean algebras are the same construction as well. The compactification of a completely regular topological space is homeomophic to the space of all ultrafilters of that space. Similarly for the Stone duality, each Stone space is constructed as the space of all ultrafilters of some Boolean algebra.

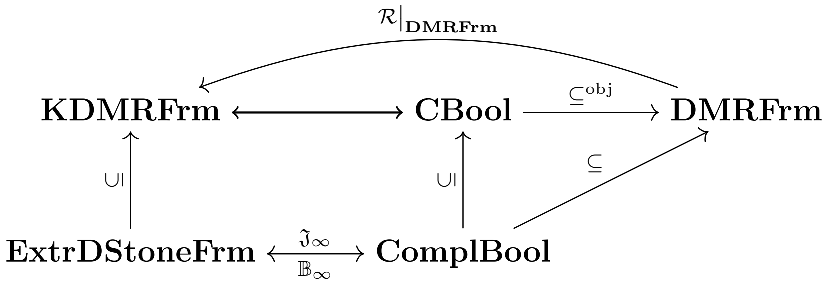
The whole (point-free) situation is depicted in the diagram below.



where the symbol $\subseteq^{\text{obj}}$ denotes an inclusion of objects of one category to another. (New categories in the diagram: **CBool** denotes the category of complete Boolean algebras and *all* Boolean homomorphisms. The category **KDMRFrm** is the category of all extremally disconnected Stone spaces, or in other words all compact De Morgan regular frames, and *all* frame homomorphisms. **RegKFrm** denotes the category of compact regular frames, **CRegFrm** the category of completely regular frames.)

In the diagram, Stone duality is drawn in the area surrounded by the dotted rectangle and the compactification is drawn in the area surrounded by the dashed curve. As we know from the discussion above, the construction on objects is exactly the same for both marked parts of the diagram. However, the Stone correspondence is an equivalence of categories, whereas the compactification is just a coreflection.

Taking the frame of all regular ideals of a bounded pseudocomplemented lattice is a general construction of compact frames. A natural question arises: Is it possible to extend the category **ComplBool** to a wider subcategory of **DMRFrm** and again obtain the equivalence of categories carried by $\mathfrak{R}$? From the following diagram we can see why it is not possible.



Objects of **CBool** are in correspondence with objects of **KDMRFrm**. On the other hand, $\mathfrak{R}$ provides a coreflection of the category **DMRFrm** onto **KDMRFrm**. Since **ComplBool** is a subcategory of **CBool**, we cannot hope to expand the correspondence by extending the category **ComplBool** into a wider subcategory of **DMRFrm**.

2. Booleanization

In this section we will prove that Booleanization is functorial in the part of the Stone correspondence for extremally disconnected Stone frames, and that it constitutes an equivalence of categories.

2.1. Definition. Let H be a Heyting algebra. By *Booleanization* of H we mean the set

$$\mathfrak{B}H = \{\, a^{**} \mid a \in H \,\}\,.$$

2.2. Proposition. *If H be a Heyting algebra then $\mathfrak{B}H$, with joins and meets defined*

$$a \sqcup b = (a^* \wedge b^*)^* \quad \textit{and} \quad a \sqcap b = a \wedge b,$$

is a Boolean algebra.

Proof. First, we will show that the operations $\sqcap$ and $\sqcup$ are really the meet and the join on $\mathfrak{B}H$. Let $a, b, c \in \mathfrak{B}H$:

- From II.1.5, $a \sqcap b = a^{**} \wedge b^{**} = (a \wedge b)^{**} \in \mathfrak{B}H$.
- Whenever $a, b \leq c$ then $a^*, b^* \geq c^*$ and so $a^* \wedge b^* \geq c^*$, hence $a \sqcup b = (a^* \wedge b^*)^* \leq c^{**} = c$. Trivially $a \sqcup b \in \mathfrak{B}H$.

For $a \in \mathfrak{B}H$: $a^* = a^{***} \in \mathfrak{B}H$ too, $a \sqcup a^* = (a^* \wedge a^{**})^* = 0^* = 1$ and also $a \sqcap a^* = a \wedge a^* = 0$. Hence, each element is complemented and $\mathfrak{B}H$ is a Boolean algebra. ☐

2.3. Let L be a frame. The mapping defined as $a \mapsto a^{**}$ is a nucleus:

(N1) $a \leq a^{**}$: it is a standard inequality (from II.1.5);

(N2) $a \leq b$ implies $a^{**} \leq b^{**}$: since taking pseudocomplements is antitone;

(N3) $a^{**\,**} = a^{**}$: follows from $a^* = a^{***}$; and

(N4) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$: it is again a standard equality (from II.1.5).

Therefore, by II.3.6, $\mathfrak{B}L$ is a sublocale of L and consequently a complete Boolean algebra. As a direct consequence of II.3.6, the mapping

$$\beta_L \colon L \to \mathfrak{B}L, \quad a \mapsto a^{**},$$

is a frame homomorphism.

2.4. Note. $\mathfrak{B}L$ is the smallest dense sublocale of L; joins are given by the formula $(a \vee b)^{**}$ (since $a \mapsto a^{**}$ is a nucleus).

2.5. Definition. For a frame homomorphism $f \colon L \to M$ set $\mathfrak{B}f \colon \mathfrak{B}L \to \mathfrak{B}M$ to be the mapping

$$\mathfrak{B}f \colon a \mapsto f(a)^{**}.$$

2.6. The following Proposition is taken from [4].

Proposition. *Let $f \colon L \to M$ be a frame homomorphism. Then $\mathfrak{B}f$ is a frame homomorphism such that the following diagram commutes*

$$\begin{array}{ccc} L & \xrightarrow{\beta_L} & \mathfrak{B}L \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathfrak{B}f} \\ M & \xrightarrow{\beta_M} & \mathfrak{B}M \end{array}$$

if and only if $f(a^{**}) \leq f(a)^{**}$ for all $a \in L$.

Proof. First observe that

$$f(a^{**})^{**} = f(a)^{**} \iff f(a^{**}) \leq f(a)^{**}, \quad \text{for all } a \in L. \tag{W.O.}$$

$\Leftarrow$ is straightforward and $\Rightarrow$ follows from $f(a^{**}) \leq f(a^{**})^{**}$. Commutativity of the diagram is precisely the equality $f(a^{**})^{**} = f(a)^{**}$.

The last thing we need to show is that $f(a^{**}) \leq f(a)^{**}$ implies that $\mathfrak{B}f$ is a frame homomorphism. It is straightforward to see that $\mathfrak{B}f$ preserves 0, 1 and $\sqcap$. For $\sqcup$–preserving take $a, b \in \mathfrak{B}L$ and compute

$$(\mathfrak{B}f)(a) \sqcup (\mathfrak{B}f)(b) = (f(a)^{**} \vee f(b)^{**})^{**} \geq f(a^{**} \vee b^{**})^{**} = f(a \vee b)^{**} = (\mathfrak{B}f)(a \sqcup b),$$

where the middle and the last equality hold by (W.O.). The opposite inequality follows from $f(a)^{**} \vee f(b)^{**} \leq f(a \vee b)^{**}$. Thus $\mathfrak{B}f$ is a Boolean homomorphism.

We will show that $\mathfrak{B}f$ preserves arbitrary joins. Again using (W.O.), we get

$$(\mathfrak{B}f)(\bigsqcup A) = f((\bigvee A)^{**})^{**} = f(\bigvee A)^{**} = (\bigvee f[A])^{**}.$$

Now take any $a \in A$, from $f(a) \leq \bigvee f[A]$ we have $f(a)^{**} \leq (\bigvee f[A])^{**}$. Since a was chosen arbitrary, we also have $\bigvee_{a \in A} f(a)^{**} \leq (\bigvee f[A])^{**}$ and therefore $(\bigvee_{a \in A} f(a)^{**})^{**} \leq (\bigvee f[A])^{**}$. The opposite inequality is trivial, hence $\bigsqcup (\mathfrak{B}f)[A] = (\bigvee f[A])^{**}$.

Thus we obtain

$$\bigsqcup (\mathfrak{B}f)[A] = (\bigvee f[A])^{**} = (\mathfrak{B}f)(\bigsqcup A),$$

which is what we wanted. $\square$

2.7. Proposition. *Let $\mathcal{C}$ be a subcategory of the category of frames and frame homomorphisms satisfying (W.O.). Then*

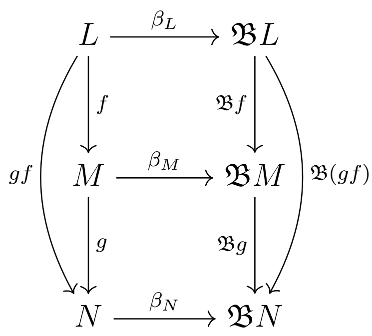
$$\mathfrak{B} \colon \mathcal{C} \to \mathbf{ComplBool},$$

defined on objects as in 2.1 and on morphisms as in 2.5, is a functor.

Proof. From 2.3 we know that $\mathfrak{B}L$ is a complete Boolean algebra. As a direct implication of Proposition 2.6 we get that for any morphism $f \colon L \to M$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc} L & \xrightarrow{\beta_L} & \mathfrak{B}L \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathfrak{B}f} \\ M & \xrightarrow{\beta_M} & \mathfrak{B}M \end{array}$$

$\mathfrak{B}f$ is a frame homomorphisms, but it is also a complete Boolean homomorphisms, as we know from V.1.5. Since the following diagram also commutes for any morphisms f, g we see that $\mathfrak{B}$ respects morphisms composition.



Finally, for any identity frame homomorphism i_L, $\mathfrak{B}(i_L)$ is the identity on $\mathfrak{B}L$. Consequently, $\mathfrak{B}$ is a functor. □

2.8. Lemma. *Let $f\colon L \to M$ be a frame homomorphism and $a \in L$ such that $a^{**} \vee a^* = 1$. Then*

$$f(a^{**})^* = f(a^*).$$

In particular, for $a = a^{**}$ we have

$$f(a)^* = f(a^*).$$

Proof. From the assumptions we see that the following holds

$$f(a^{**}) \vee f(a^*) = f(a^{**} \vee a^*) = 1,$$
$$f(a^{**}) \wedge f(a^*) = f(a^{**} \wedge a^*) = 0.$$

Hence, $f(a^{**})$ is complemented and $f(a^*)$ is its complement. From distributivity of M we know that complements are unique; thus $f(a^*) = f(a^{**})^*$. □

2.9. For any frame homomorphism between two extremally disconnected Stone frames $f\colon L \to M$. By Lemma 2.8, $f(a^{**})^* = f(a^*)$ for any $a \in L$. Note that we have

$$f(a^{**})^* = f(a^*) \text{ and } f(a^{**}) \le f(a)^{**} \iff f(a^*) = f(a)^*, \quad \text{for all } a \in L. \qquad \text{(N.O.)}$$

(Indeed, the $f(a^*) \le f(a)^*$ is always true and $f(a^*) = f(a^{**})^* \ge f(a)^{***} = f(a)^*$, the opposite direction is straightforward).

Hence, for a category of extremally disconnected frames, in order to satisfy the conditions of Proposition 2.7, we need to restrict morphisms only to those frame homomorphisms satisfying (N.O.).

Observe that the category **ExtrDStoneFrm** satisfies the conditions of Proposition 2.7. For a frame homomorphism, to be basically complete is precisely the same condition as to satisfy (N.O.).

Conclusion. $\mathfrak{B}$: **ExtrDStoneFrm** → **ComplBool** *is a functor.*

In literature the frame homomorphisms satisfying (W.O.) are called *weakly open* and frame homomorphisms satisfying (N.O.) are called *nearly open* [3].

2.10. Observation. *$\mathfrak{B}L = \mathbb{B}_\infty L$ for any extremally disconnected Stone frame L.*

Proof. $\mathfrak{B}L \supseteq \mathbb{B}_\infty L$ holds always. Let $x \in \mathfrak{B}L$. Then $x = x^{**}$ and from extremal disconnectedness also $x^{**} \vee x^* = 1$, hence $x \in \mathbb{B}_\infty L$. □

2.11. Until the end of this section, we will use the following Lemma frequently and without further reference.

Lemma. *Let L be a frame. Then:*

1. *For all $J \in \mathfrak{J}L$: $J^* = {\downarrow}(\bigvee J)^*$.*
2. *For all $a \in L$: $({\downarrow}a)^* = {\downarrow}a^*$ in $\mathfrak{J}L$.*
3. *If L is Boolean, then for all $J \in \mathfrak{J}L$: $J^{**} = {\downarrow}\bigvee J$.*

Proof. We will prove just the first statement, the others follow directly from it. Let J be any ideal on L. Observe that $a \wedge \bigvee J = 0$ iff ${\downarrow}a \wedge J = 0_{\mathfrak{B}L}$. Since

$$\begin{aligned} J^* &= \bigvee \{\, L \mid L \wedge J = 0_{\mathfrak{B}L} \,\} = \bigcup \{\, {\downarrow}a \mid {\downarrow}a \wedge J = 0_{\mathfrak{B}L} \,\} \text{ and} \\ (\bigvee J)^* &= \bigvee \{\, a \mid a \wedge \bigvee J = 0 \,\}, \end{aligned}$$

we see that ${\downarrow}a \subseteq J^*$ iff $a \in J^*$ iff $a \leq (\bigvee J)^*$; hence $J^* = {\downarrow}(\bigvee J)^*$. □

2.12. Proposition. *The functor $\mathfrak{B} \circ \mathfrak{J}_\infty$ is naturally equivalent to the identity functor on* **ComplBool***.*

Proof. Let B be a Boolean frame. We see that $J \in \mathfrak{B}\mathfrak{J}_\infty(B)$ iff $J = J^{**} = {\downarrow}\bigvee J$.

Denote by $\tilde{i}_B \colon B \to \mathfrak{B}\mathfrak{J}_\infty(B)$ the mapping $a \mapsto {\downarrow}a$. From the previous Observation we know that the definitions of $\tilde{i}_B$ and i_B coincide, therefore by IV.1.7, $\tilde{i}_B$ is a Boolean isomorphism. From V.2.3, we see that $\tilde{i}_B$ is a morphisms of **ComplBool**.

Now, for any complete Boolean algebras A and B and for any complete Boolean homomorphism $f \colon A \to B$, the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\tilde{i}_A} & \mathfrak{B}\mathfrak{J}_\infty(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{B}\mathfrak{J}(f)} \\ B & \xrightarrow{\tilde{i}_B} & \mathfrak{B}\mathfrak{J}_\infty(B) \end{array}$$

Indeed, we have

$$\mathfrak{B}\mathfrak{J}_\infty(f)({\downarrow}a) = ({\downarrow}f[{\downarrow}a])^{**} = {\downarrow}f(a)^{**} = {\downarrow}f(a).$$

Hence, the collection $\tilde{i} = (\tilde{i}_B)_B$ of complete Boolean homomorphisms forms a natural equivalence between $\mathfrak{B}\mathfrak{J}_\infty$ and the identity functor on **ComplBool**. □

2.13. Proposition. *The functor $\mathfrak{J}_\infty \circ \mathfrak{B}$ is naturally equivalent to the identity functor on* **ExtrDStoneFrm**.

Proof. Let L be an extremally disconnected Stone frame. Similarly to the general case, define $\tilde{v}_L \colon \mathfrak{J}_\infty\mathfrak{B}(L) \to L$ as

$$\tilde{v}_L \colon I \mapsto \bigvee I.$$

From Observation 2.10, we know the definitions of $\tilde{v}_L$ and v_L are the same, and $\tilde{v}_L$ is a frame isomorphism by IV.1.8. From V.2.3, $\tilde{v}_L$ is a morphism of **ExtrDStoneFrm**.

Further, let L and M be extremally disconnected Stone frames and let $f \colon L \to M$ be a basically complete frame homomorphism. The following diagram commutes

$$\begin{array}{ccc} \mathfrak{J}_\infty\mathfrak{B}(L) & \xrightarrow{\tilde{v}_L} & L \\ {\scriptstyle \mathfrak{J}_\infty\mathfrak{B}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{J}_\infty\mathfrak{B}(M) & \xrightarrow{\tilde{v}_M} & M \end{array}$$

To show that, let $I \in \mathfrak{J}_\infty\mathfrak{B}(L)$. First observe, for $x \in I$, $x = x^{**}$ and since L is extremally disconnected, $x \vee x^* = 1$. Therefore, $f(x^*) = f(x)^*$ by Lemma 2.8. Now, we can compute

$$(\mathfrak{J}_\infty\mathfrak{B})(f)(I) = {\downarrow}\{\, f(x)^{**} \mid x \in I \,\} = {\downarrow}\{\, f(x) \mid x \in I \,\} = {\downarrow} f[I].$$

And from that, we see

$$\tilde{v}_M((\mathfrak{J}_\infty\mathfrak{B})(f)(I)) = \bigvee {\downarrow} f[I] = \bigvee f[I] = f(\bigvee I) = f(\tilde{v}_L(I)).$$

The conclusion follows from commutativity of the diagram above. The collection $\tilde{v} = (\tilde{v}_L)_L$ of basically complete frame homomorphisms is a natural equivalence between $\mathfrak{J}_\infty\mathfrak{B}$ and the identity functor on **ExtrDStoneFrm**. □

2.14. From 2.9—2.13, we obtain the main result of this section.

Theorem. *The functors $\mathfrak{B}$ and $\mathfrak{J}_\infty$ provide an equivalence between the categories* **ExtrDStoneFrm** *and* **ComplBool**.

Chapter VII

More about De Morgan frames

In this chapter we present a new characterisation of De Morgan (that is, extremally disconnectedness) completely regular frames. In the second part we extend the classical spacial result, that the compactification of a metrizable space is not metrizable, to context of point–free topology.

1. Density and superdensity

1.1. Observation. *1. If gf is dense, then f is also dense.*

2. If g is dense and f is dense, then fg is dense.

1.2. Lemma. *Let S be a sublocale of L. S is dense in L iff $\mathfrak{B}S = \mathfrak{B}L$.*

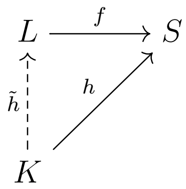
Proof. $\Rightarrow$: Suppose S is dense sublocale, then pseudocomplements in S are also pseudocomplements in L (from the fact that $0_L \in S$ and $a^{*S} = a \to 0_L = a^{*L}$). From the definition of sublocale, we know $\mathfrak{B}L = \{ x \to 0_L \mid x \in L \}$ is a subset of S. Hence, if $x = x^{**} \in \mathfrak{B}L$ then $x \in S$ and also $x \in \mathfrak{B}S$. The other inequality, the $\mathfrak{B}S \subseteq \mathfrak{B}L$, is trivial.

$\Leftarrow$: Follows from the fact that $a \le a^{**}$, for any $a \in L$, and that $0_L = 0_L^{**} \in \mathfrak{B}L = \mathfrak{B}S$. □

1.3. Proposition. *If S is dense in L and L is De Morgan, then S is De Morgan.*

Proof. Let $x \in S$. From the proof of the previous Lemma, we know that pseudocomplements in S coincide with those in L. Since each sublocale is determined by some nucleus, from II.3.6, we know that supremum in S is greater or equal to supremum in L. Thus, $x^{**} \vee_S x^* \ge x^{**} \vee_L x^* = 1$. □

1.4. Definition. Let $f\colon L \to S$ be a dense onto frame homomorphism. We say that S is *superdense* in L or that f is *superdense* if, for any compact regular K and any frame homomorphism $h\colon K \to S$, there exists a frame homomorphism $\tilde{h}\colon K \to L$ such that the following diagram commutes



1.5. Lemma. *Let f be an onto frame homomorphism between completely regular frames. Then $\mathcal{R}f$ is also onto.*

Proof. It immediately follows from the fact that $a \prec\!\!\prec b$ implies $f(a) \prec\!\!\prec f(b)$. $\square$

1.6. Proposition. *Let L be a completely regular frame and let S be a sublocale of $\mathcal{R}L$ such that $L \subseteq S \subseteq \mathcal{R}L$. Then $\mathcal{R}S \cong \mathcal{R}L$.*

Proof. We will translate the proposition to the language of frames and frame homomorphisms. Let $i\colon S \to L$ and $j\colon \mathcal{R}L \to S$ be onto frame homomorphisms such that ij equals to γ_L, the compactification of L. We will prove that $\mathcal{R}j$ is an isomorphism.

From the fact that $\gamma\colon \mathcal{R} \xrightarrow{\bullet} \mathrm{Id}$ is a coreflection, we get that $\mathcal{R}(\gamma_L)$ is an isomorphism. Therefore, $\mathcal{R}j$ is one-one since $\mathcal{R}(\gamma_L) = \mathcal{R}i \cdot \mathcal{R}j$ and monomorphisms in **Frm** are one-one. From the previous Lemma, we know that $\mathcal{R}j$ is onto.

A frame homomorphism which is both onto and one-one is an isomorphism. $\square$

1.7. Observation. *1. Let f and g be superdense. Then fg is also superdense.*

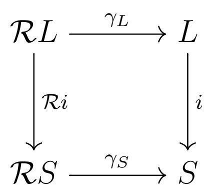
2. If fg is superdense, then f is superdense.

3. Let L be a completely regular and S its sublocale. S is superdense in L iff S is superdense in $\mathcal{R}L$.

4. Let S be a dense sublocale of a completely regular locale L. S is superdense in L iff the Čech–Stone compactifications of L and S are isomorphic.

Proof. 1 and 2 are straightforward. For 3, $\Rightarrow$ follows from 1 and $\Leftarrow$ follows from 2.

For 4, $\Leftarrow$ is again straightforward and for $\Rightarrow$: Let $i\colon L \to S$ be an onto frame homomorphism. From naturalness of γ, we see that the diagram



commutes. From superdensity, there exists a frame homomorphism $\tilde{\gamma}_S\colon \mathcal{R}S \to L$ such that $\gamma_S = i\tilde{\gamma}_S$. Further, there exists an unique frame homomorphism $\tilde{\tilde{\gamma}}_S\colon \mathcal{R}S \to \mathcal{R}L$ such that $\tilde{\gamma}_S = \gamma_L\tilde{\tilde{\gamma}}_S$. Thus, we obtain that

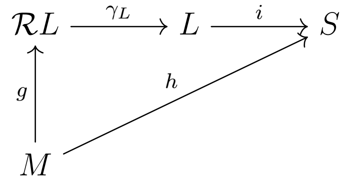
$$i\,\gamma_L = \gamma_S\,\mathcal{R}i = i\,\tilde{\gamma}_S\,\mathcal{R}i = i\,\gamma_L\,\tilde{\tilde{\gamma}}_S\,\mathcal{R}i.$$

From the uniques of compactification, $1_{\mathcal{R}L} = \tilde{\tilde{\gamma}}_S \mathcal{R}i$, hence $\mathcal{R}i$ is one-one. From Lemma 1.5, $\mathcal{R}i$ is also onto. Hence $\mathcal{R}i$ is an isomorphism. □

1.8. Theorem. *Let L be a completely regular frame. Then L is De Morgan iff each dense sublocale $S \subseteq L$ is superdense.*

Proof. $\Rightarrow$: Given a dense sublocale S of L and an onto frame homomorphism $i\colon L \to S$. Let M be a compact regular frame and let $h\colon M \to S$ be a frame homomorphism. We know that S is dense in $\mathcal{R}L$ from Observation 1.1, and $\mathfrak{B}S = \mathfrak{B}\mathcal{R}(L)$ from Lemma 1.2. By Theorem VI.2.14 for the complete part of Stone correspondence, we know that $\mathcal{R}L \cong \mathcal{R}\mathfrak{B}\mathcal{R}(L)$.

Hence, $\mathfrak{B}S = \mathfrak{B}\mathcal{R}(L) \subseteq S \subseteq \mathcal{R}L \cong \mathcal{R}\mathfrak{B}\mathcal{R}(L)$. By Proposition 1.6 we have that $\mathcal{R}S \cong \mathcal{R}L$ and S is superdense in $\mathcal{R}L$. Therefore there exists a frame homomorphism $g\colon M \to \mathcal{R}L$ such that the following diagram commutes



Obviously, $\gamma_L g$ is the desired homomorphism to L.

$\Leftarrow$: We know that $\mathfrak{B}L$ is extremally disconnected. Hence, its compactification, the $\gamma_{\mathfrak{B}L}\colon \mathcal{R}\mathfrak{B}(L) \to \mathfrak{B}L$, is also extremally disconnected by Proposition III.3.4. From the assumption, $\beta_L\colon L \to \mathfrak{B}L$ is superdense, therefore there exists a frame homomorphism $f\colon \mathcal{R}\mathfrak{B}(L) \to L$ such that $\beta_L f = \gamma_{\mathfrak{B}L}$.

From Observation 1.1 we know that f is dense and from Proposition 1.3 we know that L is De Morgan. □

2. Non–metrizability of compactification

In contrast with a De Morgan frame, a metric frame has no non-trivial superdense sublocale. That is a non-trivial Čech–Stone compactification of a metrizable frame is never metrizable.

2.1. Lemma. *Let L be a regular frame. Then, for each prime $p \in L$, $\mathfrak{c}(p) = \uparrow p = \{p, 1\}$.*

Proof. Given $q > p$. From regularity we know that there exists an $a \in L$ such that

$$a \prec q \quad \text{and} \quad a \not\leq p.$$

From meet–irreducibility of p and from $a \wedge a^* = 0$, we have $a^* \leq p$. Hence, $q = p \vee q \geq a^* \vee q = 1$. □

2.2. Proposition. *Let $\mathcal{R}L$ be a spatial frame such that $\mathcal{R}L \ncong L$. Then $\mathcal{R}L \cong \mathcal{R}(\mathfrak{o}(a))$ for some prime $a \in \mathcal{R}L$.*

Proof. Suppose there is no prime $a \in \mathcal{R}L$ such that $L \subseteq \mathfrak{o}(a)$, then $L \cap \mathfrak{c}(a) \neq \mathsf{O} = \{1\}$ for all prime a. By the previous Lemma we know that $\mathfrak{c}(a) = \{a, 1\}$, hence L contains all prime elements of $\mathcal{R}L$.

However, $\mathcal{R}L$ is spatial and each of its element is a meet of prime elements. Then $L \cong \mathcal{R}L$ since every sublocale is closed under meets, a contradiction. Hence, $L \subseteq \mathfrak{o}(a) \subseteq \mathcal{R}L$ for some prime a and from Proposition 1.6 we have $\mathcal{R}L \cong \mathcal{R}(\mathfrak{o}(a))$. □

2.3. Definition. A topological space X is said to be *metrizable* if there exists a *distance function* $\rho\colon X \times X \to [0, \infty)$ such that, for all $x, y, z \in X$,

(M1) $\rho(x, y) = 0$ if and only if $x = y$,

(M2) $\rho(x, y) = \rho(y, x)$,

(M3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$;

and the set $\{\, B_\rho(x, q) \mid x \in X, q \in \mathbb{Q} \,\}$ generates the topology on X, where $B_\rho(x, q) = \{\, y \in X \mid \rho(x, y) < q \,\}$.

2.4. Theorem. *Let $K = \Omega(X)$ be a spatial compact regular frame with X metrizable. Then K has no non-trivial superdense sublocale.*

Proof. Suppose $S \subsetneq K$ is a sublocale such that $\mathcal{R}S \cong K$. By Proposition 2.2, $\mathcal{R}(\mathfrak{o}(a)) \cong K$ for some prime a and $\mathfrak{o}(a) = X \setminus \{x\}$ for some $x \in X$.

We have,

$$B_\rho(x, q) \cap (X \setminus \{x\}) \neq \emptyset, \quad \text{for all } q \in \mathbb{Q}, \tag{VII.1}$$

where ρ is the distance function on X. Otherwise, $X \setminus \{x\}$ would be a closed and compact subspace of X and equal to $\mathcal{R}(X \setminus \{x\})$.

By (VII.1) it is sound to choose a sequence $(a_n)_{n=1}^{\infty}$ (by (CDC)) such that

$$\begin{aligned} a_1 &\in X \setminus \{x\}, \text{ and} \\ a_{n+1} &\in B_\rho\left(x, \frac{\rho(x, a_n)}{2}\right) \cap (X \setminus \{x\}). \end{aligned}$$

For $i = 1, 2, \ldots$, set $U_i = B_\rho(x, \rho(x, a_i)/2) \cap (X \setminus \{x\})$. Then $\overline{U}_{i+1} \subseteq U_i$ and $U_i \setminus \overline{U}_{i+1} \neq \emptyset$, for all i. For a subspace $X_i = \overline{U}_{2i} \setminus U_{2i+3}$, set

$$A_i = \overline{U}_{2i} \setminus U_{2i+1} \quad \text{and} \quad B_i = \overline{U}_{2i+2} \setminus U_{2i+3}.$$

A_i and B_i are disjoint closed subsets of X_i, therefore, from normality, there exists a continuous function f_i such that

$$f_i[A_i] = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad f_i[B_i] = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}.$$

Set $X_0 = (X \setminus \{x\}) \setminus U_2$. Observe that $X \setminus \{x\} = \bigcup_{i=0}^{\infty} X_i$, $X_i \cap X_j = \emptyset$ whenever $0 < i < i+1 < j$ and that $X_i \cap X_{i+1} = A_i = B_{i+1}$ for $i > 0$. Finally, set

$$E = \bigcup_{i>0,\text{ odd}} A_i \cup \bigcup_{i>0,\text{ even}} B_i \quad \text{and} \quad F = \bigcup_{i>0,\text{ even}} A_i \cup \bigcup_{i>0,\text{ odd}} B_i.$$

We see that E and F are disjoint closed subsets of $X \setminus \{x\}$.

Define a constant function f_0 on X_0: $f_0(x) = 1$ for all $x \in X_0$, then $f = \bigcup_{i=0}^{\infty} f_i$ is a function from $X \setminus \{x\}$ to $[0,1]$ which separates E and F. Since $[0,1]$ is compact, there exists an extension $\tilde{f} \colon X \to [0,1]$. However, the value of $\tilde{f}(x)$ should be equal to 0 ($B_\rho(x,q)$ intersects E for every positive $q \in \mathbb{Q}$) and it should also be equal to 1 ($B_\rho(x,q)$ intersects E for every positive $q \in \mathbb{Q}$). We obtained a contradiction, therefore, such $S \subsetneq K$ cannot exist. $\square$

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