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Report on the PhD thesis of Erika Maringova

The PhD thesis under the title: "Mathematical analysis of models arising in continuum mechanics with implicitly given rheology and boundary condition" consists of the manuscript and included attachment (*published paper: Maringová, Erika; Zabensky, Josef: On a Navier-Stokes-Fourier-like system capturing transitions between viscous and inviscid fluid regimes and between no-slip and perfect-slip boundary conditions. Nonlinear Anal. Real World Appl. 41 (2018), 152–178.*). The common objective of these two parts is the study of homogenous incompressible fluids in bounded domains. Such flows are driven by balance equations supplemented with constitutive relations which describe particular fluid properties, and to close the system also by initial and boundary conditions. The author imposes two kind of constitutive relations: one determines the properties of the fluid inside the domain, the other captures the relation between the stress and slip velocity (in tangential direction) on the boundary, which together with impermability condition (zero velocity in normal direction) constitute boundary condition. In the simplest case (considered in the thesis in chapter 5) the system of equations consists of:

$$\partial_t v - \mathrm{div}_{\mathrm{x}} S + \nabla_x p = f \text{ in } Q$$

which is conservation of linear momentum satisfied in space-time cylinder Q,

$$\mathrm{div}_{\mathrm{x}} v = 0 \text{ in } Q$$

which is equivalent to conservation of mass of the fluid,

$$-(Sn)_\tau = \alpha s + \beta \partial_t v \text{ on } \Gamma$$

which states the relation between stress, velocity and acceleration of the fluid and is satisfied at the boundary for arbitrary time,

$$v \cdot n = 0 \text{ on } \Gamma$$

which is the usual impermability condition stating that the fluid is not passing through the boundary. In this case the system is supplemented with two explicit constitutive relations: $S = 2\nu Dv$ and $s = v$ (where D stands for the symmetric part of spatial gradient). In this context, in my opinion, the main novelty of the thesis follows from the fact that in the boundary condition there appears a new term $\beta\partial_t v$. Adding this new term however requires defining new functional spaces to construct solutions. These are the spaces of L^p type, both in the domain and on the boundary, defined in section 3.1. The manuscript is built as follows:

After the introduction there is a chapter with several explicit examples in simple geometry, which gives the motivation for the new type of boundary conditions.

The aim of the next chapter is to introduce the function spaces structure to this problem, particularly Gelfand triple and construction of the Galerkin basis.

The following chapter is devoted to the discussion on the properties of r-maximal monotone graphs.

Chapter 5 is a first chapter dealing with PDEs. As it was mentioned above in this section the author investigates the simplest case of linear Stokes equations with linear rheology of boundary conditions. The proof is based on energy estimates for Galerkin approximation.

Chapter 6 introduces the study of the nonlinear Stokes model with 2-maximal monotone graph in strain-stress relation and the same for the boundary conditions. In the proof the author first showed the energy estimates for Galerkin approximations, next with Aubin-Lions Lemma and monotonicity method the proper characterization of the limit of nonlinearities is proven. Finally the author proved uniqueness in this case.

Chapter 7 is an extension of chapter 6 aiming in replacing 2-maximal monotone graph with r-maximal monotone graphs. The idea of the proof is to approximate locally r-maximal monotone graph by 2-maximal monotone graph.

Chapter 8 is dealing with the theory of Navier-Stokes system with this particular new boundary condition. Due to the fact that $S = 2\nu Dv$ and $s = v$ only one nonlinearity is hidden in convective term. Based on this, after constructing approximation with Galerkin basis and proving energy estimates it is enough to use Aubin-Lions Lemma. The last section of this chapter is devoted to proving strong-weak uniqueness result.

Chapter 9 deals not with a real system coming from fluid mechanics, but with some artificial approximation, where the term $\mathrm{div}_{\mathrm{x}}(v \otimes v)$ is replaced by $\mathrm{div}_{\mathrm{x}}[(v \otimes v)\Phi_\delta(|v|^2)]$ for some cut-off function and the implicit rheological relations are given by 2-maximal monotone graphs. It looks at the first glance unnatural however it appears to be useful further. Indeed, in this case a solution can not be taken as a test function in a weak formulation due to inequality $11/5 > 2$, and anyway one is not able to pass with δ to the limit without the use of truncation method (at least L^∞). Therefore this chapter is only preparation for the next chapter (construction of the approximate solutions to general system with implicit rheological relations given by r-maximal monotone graph).

Chapter 10 is proposing the most general result collecting together the difficulties of convective term and general r-maximal monotone graphs both in the stress-strain relation as well as in the boundary condition. In fact this generality is restricted only to $r > 6/5$ what is completely natural (which is minimal regularity requirement for convective term to be well defined). As well as the application of lemma 10.3 (Lipschitz truncation method). The proof of this chapter is neither long nor technical however it uses quite technical lemma 10.3

(see the original paper by Breit & all.)

The separate part of the PhD thesis is a common paper with J. Zabensky, containing the study of Navier-Stokes-Fourier like system in this case $\beta = 0$ and the implicit constitutive relations are 2-maximal monotone graphs which depend on one extra variable e which is an internal energy. To close the system the equation of conservation of total energy is added to the system, as well as inequality for internal energy. Let us note that this is a classical well accepted physical system of equations (particularly inequality for internal energy balance) up to the fact, that in the previous studies only constitutive function were present in place of the implicit constitutive relations. The new difficulty here is the dependence of 2-maximal monotone graph on additional variable e. Authors are also using the method of L^∞ truncations since $9/5 < 2 < 11/5$ (velocity itself is not an appropriate test function in momentum equation).

Conclusion: In my opinion this manuscript is a very nice and complete piece of mathematics and can be definitely considered as a PhD thesis. Erika Maringová has shown to be well accustomed with modern methods and tools used in nonlinear partial differential equations. She was able to develop new approaches to the problems that has not been studied before. The text is written in a very clear way, explaining all the steps of reasoning. The submitted thesis meets all the customary and formal requirements posed to doctoral dissertations and I request for admitting Erika Maringová to the next stages of the doctoral procedure. Moreover I would like to propose a distinction for this thesis.

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