Referee's Comments on the Doctoral Thesis

Mathematical analysis of models arising in continuum mechanics with implicitly given rheology and boundary conditions

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This excellent and thoughtfully structured doctoral thesis is concerned with the mathematical analysis of an important class of models of non-Newtonian fluids.

The mathematical analysis of nonlinear partial differential equations that arise in models of non-Newtonian fluids is an active and important field of current research. Non-Newtonian fluids are ubiquitous in one's daily life, ranging from cosmetic and household products, such as creams, gels, paints, pastes, and foodstuff, and they are also of important use in industry. Most biological fluids are also of non-Newtonian nature, and their modelling is therefore of significant importance in the biological and life sciences.

The thesis of Erika Maringová addresses an important class of mathematical models: fluids with implicit constitutive relations, for both the bulk flow and the boundary conditions.

The thesis is structured as follows. The first part of the thesis is devoted to the study of the dynamic slip model. This part consists of ten chapters, which are, in turn, organized into four parts. Chapters 1 and 2 introduce the problem under consideration, in conjunction with a collection of auxiliary results. They also explain the Candidate's motivation for undertaking the study, including several explicit, numerically computed results. Specifically, Chapter 1 formulates the problem, gives an overview of implicit constitutive theory in the context of the general class of problems under consideration, surveys Navier–Stokes–Fourier-like systems, describes the dynamic slip model, and in addition to introducing the relevant notation it states a number of auxiliary lemmas (such as the Aubin–Lions lemma concerning compact embedding, Chacon's biting lemma, Caratheodory's local existence theorem, Egorov's theorem, Gronwall's lemma, the Sobolev embedding theorem, an interpolation inequality for L^p spaces, an integration-by-parts lemma for Sobolev–Bochner functions, Korn's inequality, trace theorems, Vitali's theorem, and Young's inequality), which are used throughout the thesis. Chapter 2 formulates two explicit examples, which in simple geometries demonstrate the dynamic slip phenomenon. The first example concerns a model of a steady shear experiment, where a Dirichlet boundary condition is assumed on one part of the boundary and a dynamic slip condition is assumed on the rest of the boundary. The Candidate compares the dynamic slip condition for the parameter ratio α/β less than and greater than one (but finite) with the case where $\alpha > 0$ and $\beta = 0$, which corresponds to the static Navier-slip boundary condition. The example clearly shows that the use of the dynamic slip condition is much more natural than the use of a standard static slip condition. The second example concerns a time-periodic flow with the same boundary conditions, and it is compared with the same problem with a Dirichlet boundary condition throughout the boundary. The effects of each parameter α and β are simulated separately, the reference Dirichlet solution being independent of these parameters.

Chapters 3 and 4 provide the theoretical background for the subsequent study. The results are formulated for the case of an arbitrary number d of spatial dimensions, and the Candidate rigorously defines the mathematical tools suggested in Chapter 2. In particular, Chapter 3 introduces the relevant function spaces which are needed in order to rigorously state the weak formulation of the problem. The Candidate also constructs a suitable Galerkin basis for the space V appearing in the weak formulation, and proves its existence and the continuity of a projection operator onto this space. Chapter 4 introduces the notion of a maximal monotone r-graph, including the definition of approximating 2-graphs, with statements and proofs of some key properties. Specifically, for a maximal monotone r-graph, for which one does not *a priori* assume the existence of a measurable selection, the Candidate constructs a sequence of 2-graphs with a measurable selection, and shows that this sequence approaches the original r-graph as the associated approximation parameter tends to 0_+.

Chapters 5, 6 and 7 study a Stokes-like problem; the chapters are distinguished by the use of different constitutive relations. In particular, in Chapter 5, the Candidate proves the existence of a weak solution to a class of problems with simple linear constitutive relations. Particular attention is payed here to explaining in detail the crucial properties of the space V. In Chapter 6, the Candidate notes that a linear relation is a very special case of a maximal monotone 2-graph with a selection, and generalizes the results from Chapter 5 to this, more general, setting. Finally, she combines the results from Chapter 4 concerning the convergence of the approximating graphs with the results of Chapter 6, to prove the existence of a weak solution in the general setting with constitutive relations described by maximal monotone r-graphs without a measurable selection.

In Chapters 8, 9 and 10, the Candidate extends the results of Chapters 5, 6, and 7, to the case of a Navier–Stokes-like problem, again with different constitutive relations. In Chapter 8, she uses linear constitutive relations, in Chapter 9, she generalizes her previous result for maximal monotone 2-graphs with selection, however, now with a cut-off function in the convective term, thus extending the results from Chapter 6 to the case of a problem with a convective term; and finally, in Chapter 10, she proves the existence of a weak solution in the general setting of a maximal monotone r-graph without selection. This is achieved in two steps: the first step is based on the use of r-graphs, and in the second step she eliminates the cut-off function.

The second part of the thesis contains the results obtained by the Candidate in collaboration with Josef Žabenský. This work was published under the title

- *On a Navier–Stokes–Fourier-like system capturing transitions between viscous and inviscid fluid regimes and between no-slip and perfect-slip boundary conditions*, Nonlinear Analysis: Real World Applications 41 (2018) 152–178.

The paper studies a generalization of the Navier–Stokes–Fourier system for an incompressible fluid where the deviatoric part of the Cauchy stress tensor is related to the symmetric part of the velocity gradient via a maximal monotone 2-graph that is continuously parametrized by the temperature. As such, the fluid considered may exhibit transitions between three of the following regimes: it can flow as a Bingham fluid for a specific value of the temperature, it can behave as the Navier–Stokes fluid for another value of the temperature or; for other values of temperature, it can respond as an Euler fluid until a certain activation initiates the response of a Navier–Stokes fluid. The paper also considers a generalized threshold slip on the boundary that may also go through various regimes continuously as a function of the temperature. All material coefficients, including the kinematic viscosity, the friction, and the activation coefficients, are assumed to be temperature-dependent. The paper establishes the large-data and long-time existence of weak solutions, by applying an L^∞-truncation technique to approximate the velocity field.

The Candidate has two other excellent publications in high-quality journals:

- L. Beck, M. Bulíček and E. Maringová: Globally Lipschitz minimizers for variational problems with linear growth, ESAIM: COCV, 24, Number 4, 1395–1413, (2018).
- M. Bulíček, E. Maringová, B. Stroffolini, and A. Verde: A boundary regularity result for minimizers of variational integrals with nonstandard growth, Nonlinear Analysis, 177, Part A, 153.168, (2018).

Conclusions

The exposition in the thesis is clear and scholarly throughout. The work reported in the thesis represents a significant original scientific contribution to the mathematical analysis of the existence of large-data global-in-time weak solutions to models of incompressible implicitly constituted non-Newtonian fluids, and clearly demonstrates the Candidate's ability to undertake high-quality original and creative scientific work.