Report on the doctoral thesis by Rastislav Oľhava

“Inequalities for discrete and continuous supremum operators”

The thesis of Rastislav Oľhava is devoted to the investigation of certain specific operators of Hardy type of both continuous and discrete form. The results of the thesis are subject to three papers.

The main common feature of all the operators studied is the appearance of the operation of taking the supremum over an interval having a variable bound. Such operators have been studied mainly during the last two decades and have proved to be of interest and to have many important applications for example in the theory of interpolation or in the theory of function spaces. To name just one such application, let us note that it has been shown in the paper of Cianchi, Kerman, Opic and Pick (Studia Math., 2000) that a supremum-type operator materializes the Calderón operator for the fractional maximal function and hence can be very useful for obtaining sharp boundedness results for this type of operators.

The second main ingredient of the thesis is the discretization of inequalities involving integral operators acting on functions defined on an interval, turning them into inequalities involving sums. This technique has proved to be particularly useful mainly thanks to the discovery of the so-called blocking technique developed by Grosse-Erdmann (Springer 1998) and its field of applications was substantially extended by its combination with the anti-discretization methods established by Gogatishvili and Pick (Publ. Mat. 2003).

The third main ingredient is the study of iterated operators. Again, it constitutes an important and rapidly developing topic. Iteration of operators has been recently used for example in order to deal with problems dealing with higher-order derivatives, typically in all sorts of Sobolev-type inequalities. Iteration of operators involving suprema has not been so widely studied yet, and it is likely that the results contained in the thesis will be useful for the mathematical community.

The text of the thesis can be divided into three parts. In the first part the author studies continuous and discrete operators involving suprema. The boundedness results obtained are immediately applied to a characterization of interpolation spaces between two endpoint spaces of Marcinkiewicz type. Such spaces are known as certain cornerstones of classes of rearrangement-invariant function spaces that enjoy the common fundamental function and are very important.

In the second part the author studies the action of operators involving suprema on classical Lorentz spaces. These are intensively studied function spaces defined through the nonincreasing rearrangement and they constitute a wide extension of Lebesgue, Lorentz and Lorentz-Zygmund spaces. The most notorious ones are traditionally called “of Lambda type” and “of Gamma type”, the terminology going back to Lorentz (1951) and Sawyer (1990), and the authors studies both of them in detail. In the case when the integrability parameter of the domain space is majorized by that of the target space, the results obtained are complete. An interesting technique of inserting auxiliary spaces is used here, reminding one somewhat of earlier classical work by Sinnamon and by Neugebauer. In the case when the parameters are reversed, only a partial result is obtained. In this case the task seems to be quite difficult and currently known techniques do not work. Even so, the author is able to obtain at least some conditions of discrete nature.

The third part of the thesis contains the author’s results on a three-weight inequality for an iterated discrete operator. The main idea contained here is the discovery that in the case when the integrability
parameter is less than one (and hence, as is well known, a corresponding integral condition is impossible for any nontrivial weights), there is still a chance to obtain an equivalent condition in the continuous world, but one has to use certain scaling property and a striking equivalence between certain three types of inequalities which fails if the parameter of integrability is bigger than 1.

The thesis is very well and carefully written. It is not just a collection of papers; instead, the author apparently invested a lot of effort in writing up a consistent and self-contained text which is sufficiently reader friendly. The thesis contains several new interesting results. Although the nature of the research is quite abstract and technical, it is a widely well-known fact that all kinds of Hardy-type inequalities have a wide field of applications.

It is my opinion that the author has undoubtedly proved his skills for performing a sensible mathematical research and obtaining interesting new results. The thesis certainly fulfils the requirements for doctoral dissertations.

I recommend that the title Ph.D. is awarded to Rastislav Of'hava.