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Nonabsolutely convergent integrals

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To my supervisor Honza Malý. Thank You.

Title: Nonabsolutely convergent integrals

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Abstract: In this thesis we develop the theory of nonabsolutely convergent Henstock-Kurzweil type packing integrals in different spaces. In the framework of metric spaces we define the packing integral and the uniformly controlled integral of a function with respect to metric distributions. Applying the theory to the notion of currents we then prove a generalization of the Stokes theorem. In $\mathbb{R}^n$ we introduce the packing $\mathcal{R}$ and $\mathcal{R}^*$ integrals, which are defined as charges – additive functionals on sets of bounded variation. We provide comparison with miscellaneous types of integrals such as $\mathcal{R}$ and $\mathcal{R}^*$ integral in $\mathbb{R}^n$ or MC_α integral in $\mathbb{R}$. On the real line we then study a scale of integrals based on the so called p-oscillation. We show that our indefinite integrals are a.e. approximately differentiable and we give comparison with other nonabsolutely convergent integrals.

Keywords: Nonabsolutely convergent integrals, BV sets, Henstock-Kurzweil integral, Divergence theorem, Analysis in metric measure spaces

Contents

Introduction

The thesis consists of the following articles:

- K. Kuncová and J. Malý. *Non-absolutely convergent integrals in metric spaces.* J. Math. Anal. Appl., 401(2):578–600, 2013. [19]
- K. Kuncová. *BV-packing integral in $\mathbb{R}^n$.* Submitted. [17]
- K. Kuncová and J. Malý. *On a generalization of Henstock-Kurzweil integrals.* Math. Bohem., to appear. [18]

The main topic of the thesis are the nonabsolutely convergent Henstock-Kurzweil type integrals in various spaces with special attention to the Gauss-Green-Stokes type theorems and comparison to miscellaneous definitions of integrals. All these integrals are also connected by the idea of packings - pairwise disjoint finite systems of balls.

The original motivation of the theory of nonabsolutely convergent integrals is to define integral which integrates all derivatives and contains the Lebesgue integral at the same time. The first successful definitions on the real line was done at the beginning of the 20th century by Denjoy [6] and then by Lusin [22] and Perron [28].

Further progress was brought by Kurzweil [20] and Henstock [9] who improved the definition of Riemann integral in the fifties. Not long after that, it was shown that the Henstock-Kurzweil integral is equivalent to the Denjoy-Perron definition. More details on one-dimensional nonabsolutely convergent integration can be found in [31] and [27].

A natural question is how to modify the theory of integrals for the case of the multivariable functions. Both Henstock and Kurzweil provided the multidimensional version of their integrals [10], [11], [21]. The Riemann type of integrals, such as Henstock-Kurzweil integral, are based on interval partitions in $\mathbb{R}$. Hence the main issue is how to generalize the partitions to the multidimensional case. The natural choice – the class of n-dimensional intervals – leads to definition which loses integrability of all derivatives and brings problems in applications.

The problem of integrating all derivatives was solved by Mawhin [26], who established the notion of regularity of a set. Further development of multidimensional nonabsolutely convergent integrals led to improvements at the definition and application to the Gauss-Green theorem, see Jarník, Kurzweil, Schwabik [15] and Jarník and Kurzweil [14], [13].

The idea of regularity was then generalized to the setting of BV sets by Pfeffer [30]. Note that BV sets are the most general ones for which we can consider the Gauss-Green theorem with the boundary integral with respect to a measure.

However, integral defined as a set function is not suitable for the metric measure spaces. In [23], Malý established the integral as a functional on test functions.

In the next sections we describe further development of nonabsolutely convergent integrals. We will introduce so called packing integrals in the metric spaces, $\mathbb{R}^n$ and on the real line and we show that these contain all of the previous improvements. Moreover we provide the comparison within wide classes of integrals and finally we state the Gauss-Green theorem in various versions.

Packing integrals

In 2011, Malý and Bendová introduced monotonically controlled integral (MC integral) in $\mathbb{R}$ and $\mathbb{R}^n$.

Definition A. Let $I = (a, b) \subset \mathbb{R}$ be an interval and $G : (a, b) \to \mathbb{R}$ be a function. We say that $F\colon I \to \mathbb{R}$ is an *indefinite MC-Stieltjes integral* of $f\colon I \to \mathbb{R}$ with respect to G if there is a strictly increasing "control function" $\varphi\colon I \to \mathbb{R}$ such that

$$\lim_{y\to x} \frac{F(y) - F(x) - f(x)(G(y) - G(x))}{\varphi(y) - \varphi(x)} = 0$$

for each $x \in I$.

Definition B. Let $\mathbf{F}$, $\mathbf{G}$ be additive interval functions (defined on the system of all bounded subintervals of $\mathbb{R}^n$) and $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We say that $\mathbf{F}$ is an *indefinite MC-integral of f* with respect to $\mathbf{G}$ if there exists a strictly positive additive function of interval Φ (control function) such that for each $x \in \mathbb{R}^n$ and for each sequence $(Q_k)_k$ of bounded intervals such that $x \in \bigcap_k Q_k$ and $\operatorname{diam} Q_k \to 0$ we have

$$\lim_{k\to\infty} \frac{\mathbf{F}(Q_k) - f(x)\mathbf{G}(Q_k)}{\Phi(Q_k)} = 0.$$

Definition A should be compared with the descriptive definition of Denjoy integrals due to Lusin.

Definition C. Let $I = (a, b)$ be an open interval. A function $F : I \to \mathbb{R}$ is said to be *AC* (*absolutely continuous*) on a set $E \subset I$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite sequence $([a_j, b_j])_{j=1}^m$ of nonoverlapping intervals with endpoints in E we have

$$\sum_{j=1}^m (b_j - a_j) < \delta \implies \sum_{j=1}^m |F(b_j) - F(a_j)| < \varepsilon.$$

We say that F is *ACG* (*generalized absolutely continuous*) on I if F is continuous on I and there exists a sequence $(E_k)_k$ of subsets of I such that $I = \bigcup_k E_k$ and F is *AC* on each E_k.

Given a function $f : I \to \mathbb{R}$, we say that $F : I \to \mathbb{R}$ is an *indefinite Denjoy-Khintchine integral* of f if F is *ACG* on I and f is the approximate derivative of F a.e.

Definition D. Let $I = (a, b)$ be an open interval. A function $F : I \to \mathbb{R}$ is said to be AC^* on a set $E \subset I$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite sequence $([a_j, b_j])_{j=1}^m$ of nonoverlapping intervals with endpoints in E we have

$$\sum_{j=1}^m (b_j - a_j) < \delta \implies \sum_{j=1}^m \operatorname{osc}(F, [a_j, b_j]) < \varepsilon.$$

We say that F is ACG^* on I if F is continuous on I and there exists a sequence $(E_k)_k$ of subsets of I such that $I = \bigcup_k E_k$ and F is AC^* on each E_k.

Given a function $f : I \to \mathbb{R}$, we say that $F : I \to \mathbb{R}$ is an *indefinite Denjoy-Perron integral* of f if F is ACG^* on I and f is the approximate derivative of F a.e.

Notice that the definition of MC integral requires neither the complicated definition of absolute continuity nor the deep notion of Lebesgue measure.

The definition of the MC integral was then developed in various ways: Malý in [23] brought the definition of an integral with respect to a distribution - the UC integral. Further, Malý and Kuncová modified the definition and described the UC integral in metric spaces. Finally, Ball and Preiss [3] enriched the definition by a scaling factor α and introduced MC_α integral on the real line.

Packing integrals in metric spaces

Let us start with the UC integral in metric spaces. The interval functions and interval partitions are not suitable any more. Instead we consider functions on more general sets – functionals on test functions. Especially, $\mathbf{F}$ and $\mathbf{G}$ are (metric) distributions defined on Lipschitz functions and the control function is a Radon measure. The interval partitions is replaced by partition of unity. The last idea was inspired by Jarník and Kurzweil (see PU integral in [14]) and then by Malý, who introduced the distributional approach in [23].

To be more precise, let us give some definitions. The UC integral is defined on a complete separable metric space X equipped with a doubling measure λ.

Definition E (Lipschitz spaces and Lipschitz preduals)**.** We start from the Banach space $\mathcal{D}(X)$ of all bounded Lipschitz functions u on X ("test functions") endowed with the norm

$$\|u\|_{\mathcal{D}(X)} = \max\{\|u\|_{\mathcal{C}},\ |u|_{\mathrm{Lip}}\},$$

where $|u|_{\mathrm{Lip}}$ is the Lipschitz constant of u. Recall that $\mathcal{D}(X)^*$ denotes the dual Banach space to $\mathcal{D}(X)$. Each element of the space $\mathcal{C}_0(X)^*$ is identified with the continuous linear functional

$$u \mapsto \int_X u\, d\nu, \qquad u \in \mathcal{C}_b(X),$$

where ν is the signed measure representing the given functional. Since $\mathcal{D}(X)$ is trivially continuously embedded into $\mathcal{C}_b(X)$, by the dual process, $\mathcal{C}_0(X)^*$ is naturally embedded into $\mathcal{D}(X)^*$ (and, in what follows, identified with the corresponding subspace of $\mathcal{D}(X)^*$). The closure of $C_0(X)^*$ in $\mathcal{D}(X)^*$, denoted by $\mathcal{D}'(X)$, is the space of *convergent (metric) distributions* on X.

Definition F (Localization)**.** The co-Lipschitz spaces can be localized as follows. If $K \subset X$ is compact, let $\mathcal{C}_K(X)$ be the subspace of $\mathcal{C}_0(X)$ consisting of all $\mathcal{C}_0(X)$ function with support in K and $\mathcal{D}_K(X)$ be the subspace of $\mathcal{D}(X)$ consisting of all test functions with support in K. Then $\mathcal{D}'_K(X)$ is defined as the closure of $\mathcal{C}_K(X)^*$ in $\mathcal{D}_K(X)^*$.We define $\mathcal{D}_c(X)$ as the union of all $\mathcal{D}_K(X)$ over all compact $K \subset X$ and $\mathcal{D}'_{\mathrm{loc}}(X)$ as the intersection of $\mathcal{D}'_K(X)$ over all compact $K \subset X$. The elements of $\mathcal{D}'_{\mathrm{loc}}(X)$ will be called *(metric) distributions.*

Definition G. If $x \in X$ and $r > 0$, we denote

$$\mathcal{D}(x,r) = \{\varphi \in \mathcal{D}(X),\ \operatorname{spt}\varphi \in B(x,r),\ |\varphi|_{\mathrm{Lip}} \le 1/r,\ \|\varphi\|_{\mathcal{C}} \le 1\}.$$

If $\mathcal{T} \in \mathcal{D}'(X)$, we write

$$\|\mathcal{T}\|_{x,r} = \sup\Big\{|\mathcal{T}(\varphi)| :\ \varphi \in \mathcal{D}(x,r)\Big\}. \tag{1}$$

Definition H. A pairwise disjoint finite system of balls in X is called a *packing*. Let $\delta : X \to (0, \infty)$ be a function (called a *gauge*). Then a packing $(B_i)_{i=1}^m$, where $B_i = B(x_i, r_i)$, is said to be *δ-fine* if $r_i < \delta(x_i)$ for each $i = 1, \dots, m$.

Definition I. Let $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ and $\mathscr{H} \colon X \to \mathcal{D}'_{\mathrm{loc}}(X)$ be a distribution-valued function. We say that $\mathcal{F}$ is an *indefinite UC integral* of $\mathscr{H}$ if there exist $\tau \in (0, 1]$ and a finite Radon measure μ on X such that

$$\lim_{r \to 0_+} \frac{\|\mathcal{F} - \mathscr{H}(x)\|_{x,\tau r}}{\mu(B(x, r))} = 0$$

holds for each $x \in X$. In particular, we define the *UC* integral of a function $f : X \to \mathbb{R}$ with respect to a distribution $\mathcal{G} \in \mathcal{D}'_{\mathrm{loc}}(X)$ as the *UC* integral of $x \mapsto f(x)\mathcal{G}$.

The scaling factor τ is used to avoid the specific geometry of balls and to allow the bilipschitz change of variables. (The scaling factor will be later replaced by $\alpha = 1/\tau$.)

The definition of the *UC* integral is descriptive – it follows the Newton's definitions. If we define the integral in a Riemann style, as a limit of sums, we obtain the packing integral in metric spaces.

Definition J. We say that a functional $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ is an *indefinite packing integral* of a distribution-valued function $\mathscr{H} : X \to \mathcal{D}'_{\mathrm{loc}}(X)$ if there exists $\tau \in (0, 1]$ such that for every $\varepsilon > 0$ there exists a gauge $\delta \colon X \to (0, \infty)$ such that for every δ-fine packing $(B(x_i, r_i))_{i=1}^m$ we have

$$\sum_{i=1}^{m} \|\mathcal{F} - \mathscr{H}(x_i)\|_{x_i,\tau r_i} < \varepsilon.$$

If $\mathscr{H}$ has the form $\mathscr{H}(x) = f(x)\mathcal{G}$, where $f \colon X \to \mathbb{R}$ is a function and $\mathcal{G}$ is a distribution, (which is mostly the case), we say that $\mathcal{F}$ is an indefinite packing integral of f with respect to $\mathcal{G}$.

Later in Theorems I.25 and I.29 we prove that both integrals are equivalent, which is one of the remarkable results in the thesis.

Theorem K. *Let $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ be a metric distribution and $\mathscr{H} \colon X \to \mathcal{D}'_{\mathrm{loc}}(X)$ be a distribution valued function. Then $\mathcal{F}$ is an indefinite packing integral of $\mathscr{H}$ if and only if $\mathcal{F}$ is an indefinite UC integral of $\mathscr{H}$.*

Another essential result is the uniqueness of the packing integral (the uniqueness of the *UC* integral follows). The proof of Uniqueness Theorem I.20 for such wide class of functions is based on Vitali covering of a set, construction of a partition of unity in Lemma I.18 and applying the "doubling" Lemma I.17.

Theorem L. *Let $\mathscr{H}$ be a distribution-valued function on X. Then there exists at most one indefinite packing integral of $\mathscr{H}$.*

The thesis provides also the definition of packing integral based on currents. This is necessary for the formulation and proof of the formula on integration by parts and the Gauss-Green theorem, which is discussed in its own section.

The BV packing integral in $\mathbb{R}^n$

Pfeffer gave the multidimensional version of the Denjoy-Lusin integral in [30] and [29]. He used regular sets of bounded variations and additive functions on BV sets, so called charges. As a result he introduced the $\mathcal{R}$ integral. For simplicity, we give the Riemann-like version of its definition.

Proposition M. *An additive function $\mathcal{F}$ on $\boldsymbol{\mathcal{BV}}$ is a charge if and only if either of the following conditions is satisfied.*

1. *For given $\varepsilon > 0$ there is a $\theta > 0$ such that for every BV set $B \subset B(1/\varepsilon)$ we have*
$$|\mathcal{F}(B)| < \theta|B| + \varepsilon(\|B\| + 1).$$
2. $\lim \mathcal{F}(A_i) = 0$ *for each sequence $\{A_i\}$ with $A_i \to \emptyset$ in $\boldsymbol{\mathcal{BV}}$.*

Definition N. Let $E \subset \mathbb{R}^n$ be a bounded BV set and let $x \in \mathbb{R}^n$. The *regularity* of the set E is the number

$$r(E) = \begin{cases} \frac{|E|}{d(E)\|E\|} & \text{if } |E| > 0, \\ 0 & \text{if } |E| = 0. \end{cases}$$

The *regularity* of the pair (E, x) is the number

$$r(E, x) = r(E \cup \{x\}) = \begin{cases} \frac{|E|}{d(E\cup\{x\})\|E\|} & \text{if } |E| > 0, \\ 0 & \text{if } |E| = 0. \end{cases}$$

Let $\varepsilon > 0$. We say that the set E and the pair (E, x) are *ε-regular* if $r(E) > \varepsilon$ and $r(E, x) > \varepsilon$, respectively. A system $P = \{(A_1, x_1), \dots, (A_m, x_m)\}$, $A_i \subset \mathbb{R}^n$ and $x_i \in \mathbb{R}^n$, is called ε-regular if $r(A_i, x_i) > \varepsilon$ for $i = 1, \dots, m$.

Definition O. A function $\delta : E \to [0, \infty)$, where $E \subset \mathbb{R}^n$, is called a *gage* if the set $N = \{x; \delta(x) = 0\}$ is of σ-finite $\mathscr{H}^{n-1}$ Hausdorff measure.

We say that a system $P = \{(A_1, x_1), \dots, (A_k, x_k)\}$, $A_i \subset \mathbb{R}^n$ and $x_i \in \mathbb{R}^n$, is *δ-fine* if $d(A_i \cup x_i) < \delta(x_i)$. Let us remark that we do not require $x_i \in A_i$.

Definition P. Let A be a locally BV set and $\mathcal{F}$, $\mathcal{G}$ be charges in A. Let f be a function defined on $\operatorname{cl}_* A$. We say that $\mathcal{F}$ is an *indefinite $\mathcal{R}$ integral* of f in A with respect to $\mathcal{G}$ if for given $\varepsilon > 0$ we can find a gage $\delta : \operatorname{cl}_* A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} \Big|\mathcal{F}(A_i) - f(x_i)\mathcal{G}(A_i)\Big| < \varepsilon$$

for each ε-regular δ-fine BV partition $\{(A_1, x_1), \dots, (A_k, x_k)\}$ with $x_i \in \operatorname{cl}_* A$ for $i = 1, \dots, k$.

The $\mathcal{R}$ integral integrates all derivatives and Pfeffer also provided a version of the Gauss-Green theorem, which will be discussed later. However, the additivity with respect to the domain of integration is missing and also a convergence theorem with respect to the domain cannot be stated. To solve these issues, Pfeffer introduced the generalized $\mathcal{R}$ integral ($\mathcal{GR}$ integral), which is modification of the $\mathcal{R}$ integral closed to the convergence of BV sets.

Another approach was brought by Malý and Pfeffer in [24], who added the isoperimetric condition and introduced the $\mathcal{R}^*$ integral. The $\mathcal{R}^*$ integral extends the $\mathcal{R}$ and also the $\mathcal{GR}$ integral and provides some type of the convergence theorem as well as the Gauss-Green theorem.

Definition Q. Let $\varepsilon > 0$ and $E \subset \mathbb{R}^n$ be a bounded BV set. We say that E is *ε-isoperimetric* if for each $T \in \boldsymbol{\mathcal{BV}}$

$$\min\{P(E \cap T), P(E \setminus T)\} \leq \frac{1}{\varepsilon} P(T, \mathrm{in}\, E).$$

For the definition of the perimeters $P(\cdot)$, $P(\cdot, \mathrm{in}\, E)$ see Definition II.9.

Definition R. Let $\varepsilon > 0$. We say that an ε-regular BV partition $\{(A_1, x_1), \dots, (A_k, x_k)\}$ is *strongly ε-regular* if A_i is ε-isoperimetric and $x_i \in \mathrm{cl}_* A_i$ for $i = 1, \dots, k$.

Definition S. Let $A \subset \mathbb{R}^n$ be a locally BV set. We say that a charge $\mathcal{F}$ in A is an *indefinite $\mathcal{R}^*$ integral* of a function $f : \mathrm{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathcal{G}$ if for given $\varepsilon > 0$ we can find a gage $\delta : \mathrm{cl}_* A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} \Big| \mathcal{F}(A_i) - f(x_i)\mathcal{G}(A_i) \Big| < \varepsilon$$

for each strongly ε-regular δ-fine BV partition $\{(A_1, x_1), \dots, (A_k, x_k)\}$.

Instead of the expression in (1) we can use various seminorms, which idea was presented by Honzík and Malý in [12]. If we choose suitable seminorms and the idea of packing, we obtain the packing $\mathcal{R}$ integral and packing $\mathcal{R}^*$ integral, which generalize the Pfeffer $\mathcal{R}$ integral.

Notation T. *Let $x \in \mathbb{R}^n$, $r, \varepsilon > 0$ and $\mathcal{F}$ be a charge. Then we will use the seminorms*

$$\bar{p}^{\varepsilon}_{x,r}(\mathcal{F}) = \sup\left\{ |\mathcal{F}(E)|; E \subset\subset B(x,r), E \in \boldsymbol{\mathcal{BV}}, (E, x) \textit{ is } \varepsilon\textit{-regular} \right\}.$$

Definition U. Let $A \subset \mathbb{R}^n$ be a locally BV set. We say that a charge $\mathcal{F}$ in A is an *indefinite packing $\mathcal{R}$ integral* of a function $f : \mathrm{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathcal{G}$ if there exists $\tau \in (0, 1]$ such that for every $\varepsilon > 0$ there exists a gage $\delta\colon \mathrm{cl}_* A \to [0, \infty)$ such that for every δ-fine packing $(B(x_i, r_i))_{i=1}^{k}$, $x_i \in \mathrm{cl}_* A$, we have

$$\sum_{i=1}^{k} \bar{p}^{\varepsilon}_{x_i, \tau r_i}(\mathcal{F} - f(x_i)\mathcal{G}) < \varepsilon.$$

Notation V. *Let $x \in \mathbb{R}^n$, $r, \varepsilon > 0$ and $\mathcal{F}$ be a charge. Then we will use the seminorms*

$$\begin{gathered}\bar{q}^{\varepsilon}_{x,r}(\mathcal{F}) = \sup\{|\mathcal{F}(E)|; E \subset\subset B(x,r), E \in \boldsymbol{\mathcal{BV}}, x \in \mathrm{cl}_* E, \\ (E, x) \textit{ is } \varepsilon\textit{-regular and } E \textit{ is } \varepsilon\textit{-isoperimetric}\}.\end{gathered}$$

Definition W. Let $A \subset \mathbb{R}^n$ be a locally BV set. We say that a charge $\mathcal{F}$ in A is an *indefinite packing $\mathcal{R}^*$ integral* of a function $f\colon \mathrm{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathcal{G}$ if there exists $\tau \in (0, 1]$ such that for every $\varepsilon > 0$ there exists a gage

$\delta\colon \operatorname{cl}_* A \to [0,\infty)$ such that for every δ-fine packing $(B(x_i,r_i))_{i=1}^k$, $x_i \in \operatorname{cl}_* A$, we have

$$\sum_{i=1}^{k} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathcal{G}) < \varepsilon.$$

In the case $A = \mathbb{R}^n$ we say that $\mathcal{F}$ is just an *indefinite packing* $\mathcal{R}^*$ *integral* of f with respect to $\mathcal{G}$.

The uniqueness of this integral is highly nontrivial, see Theorem II.40. The proof works with similar ideas as the proof in [19], but uses the partition of intervals instead of partition of unity. Since the packing $\mathcal{R}$ integral is included in the packing $\mathcal{R}^*$ integral, it is unique as well.

Theorem X (Uniqueness of the integral)**.** *Let f be a function and $\mathcal{G}$ be a charge. Then there exists at most one indefinite packing $\mathcal{R}^*$ integral of f with respect to $\mathcal{G}$.*

In Theorems II.71 and II.80 it is shown that the packing $\mathcal{R}$ integral includes the $\mathcal{R}$ integral and the packing $\mathcal{R}^*$ integral includes the $\mathcal{R}^*$ integral. Moreover, the convergence theorem and the Gauss-Green theorem for the packing $\mathcal{R}^*$ integral hold. Details and other comparisons are provided in the next sections.

The HK_α integral

Motivated by the presence of the scaling parameter τ in the definition of packing integrals, Ball and Preiss [3] enriched the original MC integral by a scaling factor α. In contrary to the use of τ, the action of α is one-sided.

Definition Y. Let $\alpha > 0$ be a real number and $f, F, G : \mathbb{R} \to \mathbb{R}$ be functions, let G be continuous. We say that F is an *indefinite MC_α integral* of f with respect to G if there exists a strictly increasing control function $\varphi : \mathbb{R} \to \mathbb{R}$ such that for each $x \in \mathbb{R}$ we have

$$\lim_{h\to 0} \frac{F(x+h) - F(x) - f(x)(G(x+h) - G(x))}{\varphi(x+\alpha h) - \varphi(x)} = 0.$$

The families of all MC_α integrable functions with respect to G and all MC_α integrable functions with respect to identity function are denoted by $MC_\alpha(G)$ and MC_α, respectively.

Especially, if $\alpha = 1$, we say that F is an *indefinite MC integral of f with respect to G*. We write $MC(G) = MC_1(G)$ and $MC = MC_1$.

The new definition brought a whole new scale of integrals to comparison. Especially, the MC integral coincides with the Henstock-Kurzweil integral.

Ball and Preiss was followed by Kuncová and Malý in [18], whose effort led to a constructive definition of (centered) HKS^p_α integral based on the notion of p-oscillation.

Definition Z (Oscillations)**.** Let $[a,b] \subset \mathbb{R}$ be a closed interval and $p \in [1,\infty]$. We define the p-oscillation of a measurable function $F\colon [a,b] \to \mathbb{R}$ as

$$\operatorname{osc}_p(F,[a,b]) = (b-a)^{-1/p} \inf\{\|F - c\|_{L^p([a,b])} \colon c \in \mathbb{R}\}. \tag{2}$$

Here and in the sequel $1/p = 0$ if $p = \infty$.

The ordinary oscillation

$$\operatorname{osc}(F, [a,b]) = \operatorname{osc}_C(F, [a,b]) := \tfrac{1}{2} \sup_{x,y\in[a,b]} |F(y) - F(x)|$$

differs from $\operatorname{osc}_\infty$ in the aspect that it does not neglect Lebesgue null sets. The subscript C refers to the space of continuous function and the somewhat unusual factor $\frac{1}{2}$ is an output of the usage of the supremum norm instead of the L^p-norm in (2). To simplify the presentation, we consider the symbol C as a possible value of p and $1/p$ is 0 for $p = C$. This convention will be used to include the choices of oscillation all at once.

The definition of integrals follows. Let us mention that they differs just in the use of partitions or centered partitions.

Definition AA. Let $\alpha \geq 1$, $p \in [1,\infty] \cup \{C\}$ and $I \subset \mathbb{R}$ be an open interval. A finite family $([a_i, b_i], x_i)_{i=1}^m$, where $[a_i, b_i] \subset I$ are closed intervals and $x_i \in [a_i, b_i]$, is called an *α-partition* in I if the intervals $(\bar{a}_i, \bar{b}_i)$, where

$$\bar{a}_i - x_i = \alpha(a_i - x_i), \quad \bar{b}_i - x_i = \alpha(b_i - x_i),$$

are subsets of I and pairwise disjoint. We say that a partition $([a_i, b_i], x_i)_{i=1}^m$ is *centered* if each x_i is the center of $[a_i, b_i]$. Let $\delta\colon I \to (0,\infty)$ be a *gage* (this means just a strictly positive function). We say that the α-partition is *δ-fine* if $[a_i, b_i] \subset (x_i - \delta(x_i),\, x_i + \delta(x_i))$, $i = 1, \dots, m$.

Definition AB (HKS^p_α integrals)**.** Let $I \subset \mathbb{R}$ be an open interval, $p \in [1,\infty] \cup \{C\}$ and $\alpha \geq 1$. Let F, G, f be measurable functions on I. We say that F is an *indefinite HKS^p_α-integral* (HKS refers to Henstock-Kurzweil-Stieltjes) of f with respect to G if for each $\varepsilon > 0$ there exists a gage $\delta\colon I \to (0,\infty)$ such that for each δ-fine α-partition $([a_i, b_i], x_i)_{i=1}^m$ in I we have

$$\sum_{i=1}^m \operatorname{osc}_p(F - f(x_i)G,\ [a_i, b_i]) < \varepsilon.$$

We denote $HKS_\alpha = HKS^C_\alpha$. We reduce the symbol to HK^p_α or HK_α, respectively, if G is the identity $\mathrm{Id}(x) = x$. We call the integral *centered* if only centered α-partitions are taken into account. We say that F is a *free indefinite HKS^p-integral* of f with respect to G if there exists $\alpha \geq 1$ such that F is an *indefinite HKS^p_α-integral* of f with respect to G, similarly to centered versions.

These integrals cover the theory on the real line. They include the MC_α integrals, packing integral in metric spaces, packing $\mathcal{R}^*$ integral and of course the Henstock-Kurzweil integral. See the next section.

Comparison of integrals

This section brings comparison of classes of integrals. Let us start with the setting of real line.

Ball and Preiss showed the coincidence of the Henstock-Kurzweil (HK) integral and MC_α integral for $\alpha \in [1,2]$. They also showed that there is strict inclusion for different α: $MC_\alpha \subsetneq MC_\beta$ for $2 \leq \alpha < \beta$.

Pfeffer proved that in $\mathbb{R}^1$ the $\mathcal{GR}$ integral coincides with the $\mathcal{R}$ integral. Both are the proper subset of Henstock-Kurzweil integral. Malý and Pfeffer then proved equality of $\mathcal{R}^*$ and Henstock-Kurzweil integral.

Kuncová further showed that that the packing $\mathcal{R}$ integral contains the whole class of MC_α integrals and the inclusion between packing $\mathcal{R}$ and packing $\mathcal{R}^*$ integral.

The results are summarized in the diagram.

$$\begin{array}{ccccccccccccc}
\mathcal{GR} & = & \mathcal{R} & \subsetneq & HK & = & MC & \subsetneq & MC_\beta & \subsetneq & \mathscr{P}\mathcal{R} & \subset & \mathscr{P}\mathcal{R}^* \\
 & & & & \parallel & & \parallel & & & & & & \\
 & & & & \mathcal{R}^* & & MC_\alpha & & & & & &
\end{array}$$

The HK_α^p brought the new wide class of integrals and new relationships between existing definitions. Let us mention the main results.

The class of integrable functions increases with increasing α and shrinks with increasing p. Also the noncentered integrals are included in the centered versions.

The HK_α integral is equivalent to MC_α, HK_1 to Henstock-Kurzweil integral, the centered HK_α to packing $\mathcal{R}$ integral and the centered HK_α^1 to packing integral (denoted α-Lip integral in the following diagram).

The thesis also contains some counterexamples. At first, there is a distinction between classes of HK_α^p integrable functions for different p and also between classes of centered and uncentered HK_α^p integrable functions.

Theorem AC. *For each $\alpha \geq 1$ and $1 \leq p < q \leq \infty$ there exists a HK_1^p-integrable function which is not centered HK_α^q-integrable.*

Theorem AD. *Let $p \in [1,\infty] \cup \{C\}$. Then for each $\alpha \geq 1$ there exists a centered HK_1-integrable function which is not HK_α^p-integrable.*

We obtain the diagram:

$$\begin{array}{ccccccc}
HK & = & HK_1 & \subsetneq & HK_1^p & \subsetneq & HK_1^1 \\
 & & \rotatebox[origin=c]{-90}{$\subsetneq$} & & \cap & & \cap \\
MC_\alpha & = & HK_\alpha & \subsetneq & HK_\alpha^p & \subsetneq & HK_\alpha^1 \\
 & & \rotatebox[origin=c]{-90}{$\subsetneq$} & & \rotatebox[origin=c]{-90}{$\subsetneq$} & & \rotatebox[origin=c]{-90}{$\subsetneq$} \\
 & & CHK_\alpha & \subsetneq & CHK_\alpha^p & \subsetneq & CHK_\alpha^1 \\
 & & \parallel & & & & \parallel \\
 & & \mathscr{P}\mathcal{R} & & & & \alpha\text{- Lip}
\end{array}$$

There is also another type of comparison – with the Denjoy-Khintchine integral, which was first defined in [7] and [16]. Ball and Preiss proved that there is no inclusion between the Denjoy-Khintchine integral and the MC_α integral for any α.

Analogously, in Theorems III.28 and III.19 we showed that there is no inclusion between (centered or uncentered) HK^p_α integral and Denjoy-Khintchine (Let us note that there is the exception of the case of uncentered HK_α for $\alpha \leq 2$).

The Gauss-Green divergence theorem

Recall that the Denjoy-Perron integral integrates all derivatives on the real line and guarantees validity of the Newton-Leibniz formula

$$\int_a^b F'(x)\ \mathrm{d}x = F(b) - F(a).$$

The generalization to higher dimension leads to the the Gauss-Green divergence theorem

$$\int_A \operatorname{div} \mathbf{u}(x)\ \mathrm{d}x = \int_{\partial_* A} \mathbf{u} \cdot \boldsymbol{\nu}_A \, d\mathscr{H}^{n-1}.$$

Here, $\partial_* A$ is the essential boundary and $\boldsymbol{\nu}_A$ is the measure-theoretic unit exterior normal.

This theorem holds whenever $A \subset \mathbb{R}^n$ is a bounded BV set and $\mathbf{u} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ (see [1]). If we want to allow discontinuous derivatives, the approximation arguments give the formula if $\mathbf{u} \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\operatorname{div} \mathbf{u}(x) \in L^1(\mathbb{R}^n)$. Another idea is to consider the divergence in the sense of distributions. Particularly deep results have been obtained for divergence measure vector fields, see e.g. Chen, Torres and Ziemer [5], Ziemer [35] or Šilhavý [32, 33, 34].

If we go beyond measures, the main task is to find which distributions can be associated with pointwise functions. This leads to nonabsolutely convergent integrals.

Recall that the multidimensional Henstock-Kurzweil integral does not integrate all derivatives (partial derivatives of differentiable functions).

First progress was done by Mawhin in [25] with so called GP integral. Unfortunately, the resulting integrals was defined only for intervals (or not enough general sets) and depended on coordinate system. For other definitions and improvements we refer for example to work of Jarník, Kurzweil and Schwabik [15].

An important improvement was then done by Pfeffer in [29]. The $\mathcal{R}$ integral is defined as a functional on BV sets and its definite version on the left is just the evaluation at the set A, which is the set of finite perimeter.

Pfeffer was followed by Malý in 2011. Theorem AE below is a simplified version of the Gauss-Green theorem, which was proved in [23]. The new approach based on integration with respect to distribution allows to define the boundary integral for some sets of infinite perimeter. Then both integrals in the formula are nonabsolutely convergent. However, the assumptions are quite technical, thus we present a version of the Gauss-Green theorem, where the integral of divergence is absolutely convergent and the boundary integral is the UC integral (Definition I).

Theorem AE. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a countably $(n-1)$-rectifiable boundary. Let $\boldsymbol{f} \in C(\overline{\Omega}) \cap W^{1,1}(\Omega)$ be a vector field. Then*

$$\int_\Omega \operatorname{div} \boldsymbol{f}(x)\ \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{f} \cdot d\boldsymbol{\nu}.$$

In [19] we succeeded to generalize the Gauss-Green theorem to metric spaces. The setting of metric spaces requires different formulation of the theorem and redefinition of the packing integral. The integral is now defined as a current.

Metric currents are linear functionals on differential forms (tuples of Lipschitz functions). This concept was introduced by De Giorgi [8] and developed by Ambrosio and Kirchheim in [2]. Metric currents give a sense to integration of k-forms on metric spaces. Although the classical k-dimensional currents are optimized for k-dimensional integration on manifolds, this integration can be expressed as integration of 1-currents, which gives us a chance to generalize the Gauss-Green theorem to metric spaces.

Definition AF (Metric 1-current)**.** We denote by $\tilde{\mathcal{D}}(X)$ the set of all Lipschitz functions on X. Let $\mathcal{D}^1(X) = \mathcal{D}(X) \times \tilde{D}(X)$. The elements of $\mathcal{D}^1(X)$ are called *test differential* 1*-forms*. The support of a test differential form (φ, ψ) is defined as the support of the product $\varphi\psi$. The family of all compactly supported test differential 1-forms is denoted by $\mathcal{D}^1_c(X)$.

We say that $(\varphi_j, \psi_j) \overset{*}{\to} (\varphi, \psi)$ (converge weak*) in $\mathcal{D}^1(X)$ if the Lipschitz constants of φ_j, ψ_j and $\mathcal{C}$-norms of φ_j form bounded sequences and $\varphi_j \to \varphi$, $\psi_j \to \psi$ pointwise. The $\overset{*}{\to}$ convergence in $\mathcal{D}^1_c(X)$ requires in addition that there is a compact set containing all supports of (φ_j, ψ_j).

We say that a mapping $\mathcal{T} : \mathcal{D}^1_c(X) \to \mathbb{R}$ is a 1*-current* if the following properties are satisfied:

(C-1) $\mathcal{T}$ is bilinear in variables φ, ψ.

(C-2) $(\varphi_j, \psi_j) \overset{*}{\to} (\varphi, \psi)$ in $\mathcal{D}^1_c(X) \implies \mathcal{T}(\varphi_j, \psi_j) \to \mathcal{T}(\varphi, \psi)$.

(C-3) $\varphi\psi = 0 \implies \mathcal{T}(\varphi, \psi) = 0$.

(C-4) $\mathcal{T}(\varphi, 1) = 0$.

The collection of all 1-currents on X is denoted by $(\mathcal{D}^1_c)'(X)$. The family $(\mathcal{D}^1)'(X)$ of all *convergent currents* on X is defined analogously, with the difference that they are defined on $\mathcal{D}^1(X)$ and continuous with respect to the $\mathcal{D}^1(X)$-convergence.

The family of all $\mathbb{R}^m$-valued 1-currents on X is denoted by $(\mathcal{D}^1_c)'(X; \mathbb{R}^m)$; we define them coordinate-wise.

We say that a 1-current $\mathcal{T}$ has a *locally finite mass* if there exists a Radon measure μ on X such that

$$|\mathcal{T}(\varphi, \psi)| \le |\psi|_{\mathrm{Lip}} \int_X |\varphi| \, d\mu, \qquad (\varphi, \psi) \in \mathcal{D}^1_c(X). \tag{3}$$

We also say that $\mathcal{T}$ is *dominated by* μ if (3) is satisfied.

The *boundary* of a 1-current $\mathcal{T}$ is defined as

$$\partial\mathcal{T}(\varphi) = \mathcal{T}(1, \varphi), \qquad \varphi \in \mathcal{D}_c(X),$$

it is a metric distribution. We say that $\mathcal{T}$ is *boundary-free* if $\partial\mathcal{T} = 0$.

We use also the alternative notation $\mathcal{T}(\varphi \, d\psi)$ for $\mathcal{T}(\varphi, \psi)$.

In the following, we define the packing integral as a 1-current and we add the definition of the definite integral.

Definition AG (Integral with respect to a 1-current)**.** Let $f : X \to \mathbb{R}$ be a function and $\mathcal{F}, \mathcal{G} \in (\mathcal{D}_c^1)'(X)$ be 1-currents. We say that $\mathcal{F}$ is an *indefinite integral* of f with respect to $\mathcal{G}$, if $\mathcal{F}(\cdot, \psi)$ is an indefinite packing integral of f with respect to $\mathcal{G}(\cdot, \psi)$ for each Lipschitz function ψ on X. This means

$$\int f \,\mathrm{d}\mathcal{G}(\varphi \, d\psi) = \int f \,\mathrm{d}\mathcal{G}(\cdot, \psi)(\varphi), \qquad \varphi \in \mathcal{D}_c(X). \tag{4}$$

We denote it by $\mathcal{F} = \int f \,\mathrm{d}\mathcal{G}$.

The definite integral $\int_X f \ \mathrm{d}\mathcal{G}$ is defined by

$$\int_X f \ \mathrm{d}\mathcal{G}(d\psi) = \int f \ \mathrm{d}\mathcal{G}(1, \psi)$$

if the indefinite integral $\int f \ \mathrm{d}\mathcal{G}$ can be continuously extended to $\mathcal{D}^1(X)$.

In applications we sometimes integrate $\mathbb{R}^m$ valued functions and/or with respect to $\mathbb{R}^d$-valued currents; this is understood coordinate-wise and the result is an m-dimensional vector if $d = 1$ or a d-dimensional vector if $m = 1$. If both $d > 1$, $m > 1$, the most important situation is $d = m$. If $\boldsymbol{\mathcal{G}} \in (\mathcal{D}_c^1)'(X, \mathbb{R}^m)$ and $\boldsymbol{f} : X \to \mathbb{R}^m$ is a vector field, we define the *inner product integral* $\int \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\mathcal{G}}$ as a 1-current satisfying

$$\int \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\mathcal{G}}(\varphi \, d\psi) = \int \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\mathcal{G}}(\cdot, \psi)(\varphi), \qquad \varphi \in \mathcal{D}_c(X), \ \psi \in \tilde{\mathcal{D}}(X).$$

The Gauss-Green divergence theorem in a metric space X then can be stated as

$$\int_A \mathrm{d}\partial\Big(\int \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\mathcal{T}}\Big) = -\int_X \boldsymbol{f} \cdot \mathrm{d}\partial\Big(\int \chi_A \,\mathrm{d}\boldsymbol{\mathcal{T}}\Big).$$

Here $\mathcal{T}$ is a boundary-free 1-current and A is a set. The packing integral $\mathcal{F}$ of $\boldsymbol{f}$ with respect to $\mathcal{T}$ is denoted by $\mathcal{F} = \displaystyle\int \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\mathcal{T}}$ and is also 1-current. Then $\partial\mathcal{F} = \partial \displaystyle\int \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\mathcal{T}}$ denotes its boundary, which is a metric distribution. Then $\displaystyle\int_A \mathrm{d}\partial\Big(\int \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\mathcal{T}}\Big)$ is the definite integral of 1 with respect to $\partial\mathcal{F}$. The right hand part can be explained analogously.

Let us note the main characteristic of the theorem: the integral of the divergence over a set is replaced by the integral of a characteristic function of a set with respect to divergence.

In the thesis we present very general forms of the formula on integration by parts and of the Gauss-Green formula. The assumptions are very complicated and we refer to Theorems I.53 and I.55 for the precise statements.

Let us concentrate on $\mathbb{R}^n$ again. The Pfeffer's $\mathcal{R}$ integral was followed by the $\mathcal{R}^*$ integral by Malý and Pfeffer. The $\mathcal{R}^*$ integral includes the Pfeffer integral and the Gauss-Green theorem is also provided.

The next progress was done by Kuncová, who introduced the packing $\mathcal{R}^*$ integral [17]. For the proof see Theorem II.62.

Definition AH. Let $A \subset \mathbb{R}^n$ be a locally BV set and let $f : \mathrm{cl}_* A \to \mathbb{R}$ and $\mathbf{u} \in C(\bar{A}, \mathbb{R}^n)$ be functions. Further, let a charge $\mathcal{F}$ be the flux of $\mathbf{u}$ in A. We say that f is a *generalized divergence* of $\mathbf{u}$ in A if $\mathcal{F}$ is an indefinite packing $\mathcal{R}^*$ integral of f in A. The generalized divergence of $\mathbf{u}$ will be denoted by $\operatorname{Div} \mathbf{u}$.

Theorem AI (Gauss-Green divergence theorem)**.** *Let* $A \subset \mathbb{R}^n$ *be a bounded BV set, let* $\mathbf{u} \in C(\bar{A}, \mathbb{R}^n)$*. Let us suppose that there exists a generalized divergence* $\operatorname{Div} \mathbf{u}$ *in* A*. Then*

$$\int_A \operatorname{Div} \mathbf{u}(x)\, \mathrm{d}x = \int_{\partial_* A} \mathbf{u} \cdot \boldsymbol{\nu}_A \, d\mathscr{H}^{n-1},$$

where the integral on the left is the definite packing $\mathcal{R}^*$ *integral.*

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I. Non-absolutely convergent integrals in metric spaces

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ABSTRACT. We develop the theory of Henstock-Kurzweil type integral of functions with respect to metric distributions in the framework of metric spaces. In the setting of metric currents (as originated by E. De Giorgi, L. Ambrosio and B. Kirchheim) we apply the new integral to study a generalization of the Stokes theorem.

I.1 Introduction

The origin of the advanced theory of nonabsolutely convergent integrals is connected with the question to find a concept of integral which would simultaneously contain the Lebesgue integral and integrate all derivatives. First such a definition is due to Denjoy [15] in 1912, shortly followed by Luzin [40] and Perron [45]. In late fifties, a "generalized Riemann integral" has been introduced independently by Kurzweil [33] and Henstock [26], see also [35], [27], [28]. This definition leads to the same family of Denjoy-Perron integrable functions. Its advantage is that it resembles the Riemann definition which is respected as the most transparent and comprehensible definition of integral. For another elementary definition of nonabsolutely convergent integral on the real line see recent contribution [9].

The task to develop a similar theory (of nonabsolutely convergent integral) for functions of several variables brings new significant difficulties. The one-dimensional integration can be understood as an inverse process to differentiation. In the multidimensional case, an attempt to develop parallel thoughts leads to differentiation of set functions. If we want to obtain a wide non-absolutely convergent integral, we need to start from a narrow class of sets. The class of intervals is the most natural choice. (One could also think about the class of all balls, but it has inconvenient partitioning properties.) Thus, two kinds of definitions appear: constructive definitions based on partitions into intervals or descriptive definitions based on differentiation of interval functions.

The Denjoy-Perron integral has been generalized to higher dimension by Bauer [8]. Both Kurzweil and Henstock considered also multidimensional or abstract versions of their integral, [27], [28], [35]. For various other definition of multidimensional integrals, discussion of problems and further bibliography we refer to [10], [14], [29], [31], [32], [38], [37], [44], [47], [48].

For the multidimensional Henstock-Kurzweil integral, the Fubini theorem is available, [34], [35]. However, in contrast with the one-dimensional case, this integral does not integrate all derivatives. Mawhin [42] restricted the class of admissible partitions to obtain an integral which integrates all derivatives. His partitions are *regular*, roughly speaking, the proportions of intervals in the partition are controlled. Such regular integrals integrate all derivatives, however, the

Fubini theorem is lost. This is not a symptom of cumbersomeness of the definition but a general feature. The example in [31] shows that the validity of Fubini's theorem is not compatible with the property of integrating all derivatives.

A disadvantage of the theory of integral based on interval functions is that such integral depends on the choice of the coordinate system and we do not obtain reasonable results on change of variables. In more sophisticated theories, the indefinite integral is a function defined on a more general family of sets. The natural choice for such a system of sets is the family of all BV-sets. This direction has been developed by Pfeffer, see e.g. [46], [48].

The importance of the BV class consist in the feature that it is one of few natural function spaces which distinguishes between "regular" and "irregular" sets: it contains some nontrivial characteristic functions of bounded measurable sets but not all of them.

In our approach, we do not use set functions but functionals. An important step towards this setting was the PU integral (referring to "partition of unity") introduced by Jarník and Kurzweil [29], [30]. Here a smooth partition of unity replaces the partition into intervals. [36] The PU integral fits into the generality of manifolds and has been applied to establish a version of the Stokes theorem. For some purposes, BV partition of unity fit better, see Kurzweil, Mawhin and Pfeffer [36].

In [41], integration with respect to distributions has been introduced. The idea is that the indefinite integral is not a function of a set, but a functional on test functions. Then, the definite integral is obtained as the value of the indefinite integral at the test function 1. Thus, our functional approach falls within the well established theory of distributions. Further, function spaces provides much more possibilities for the choice of test objects than classes of sets. Indeed, characteristic functions of sets are irregular on principle, whereas we use more regular (namely, Lipschitz continuous) functions to test with. The way of integration in [41] was new even for integration with respect to the Lebesgue measure.

Also in our case, the integration is an inverse process to differentiation: we differentiate distributions with respect to distributions similarly as the Radon-Nikodým differentiation (in the geometric setting by Lebesgue and Besicovitch) differentiates measures with respect to measures.

In this paper we develop the theory of nonabsolutely convergent integrals in metric spaces. In Section I.2 we give some preparatory results on Lipschitz spaces and their duals (spaces of metric distributions). In Sections I.3 and I.4 we define new integrals of distribution-valued functions (in particular, the integration of a function with respect to a distribution). The packing integral of Section I.3 is related to Henstock's variational integral [26], whereas the UC integral of Section I.4 is the direct generalization of the integral from [41] to metric spaces. The main results of this section are Theorem I.20, according to which the packing integral is unique, and Theorems I.25 and I.29, which state that the packing integral and the UC integral are equivalent.

One of our main goals is to investigate new versions of the Stokes theorem. In the setting of metric space, we need to work with currents (following Ambrosio and Kirchheim [4]). Integration with respect to metric currents is investigated in Section I.5. In Section I.6, we present a version of the Gauss-Green-Stokes theorem (Theorem I.55) as a particular example of a general result of integration by parts (Theorem I.53).

I.1.1 Notation and conventions

If X is a metric space, $x \in X$ and $r > 0$, then $B(x, r)$ denotes the open ball with center at x and radius r and $\overline{B}(x, r)$ is the corresponding closed ball.

If $\mathcal{E}(X)$ is a space of scalar functions on X, we let $\mathcal{E}(X, \mathbb{R}^m)$ denote the corresponding space of $\mathbb{R}^m$-valued functions (defined coordinatewise).

We use the notation Y^* for the dual space of a normed space Y .

All measures are nonnegative unless labeled as "signed".

If f is a function on X, then $\operatorname{spt} f$ is the smallest closed set F such that $f = 0$ on $X \setminus F$. If μ is a Borel measure on X, then $\operatorname{spt} \mu$ is the smallest closed set F such that $\mu(X \setminus F) = 0$.

If μ is a measure on X and $A \subset X$ is a μ-measurable set, then we define $\mu|_A$ on the same σ-algebra as the measure

$$E \mapsto \mu(E \cap A).$$

If μ is a measure on X, E is a μ-measurable set with $0 < \mu(E) < \infty$ and f is a μ-integrable function on E, we write

$$⨍_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu.$$

We use symbol C for a generic constant which may change at each step of the computation.

I.2 Lipschitz spaces

If we want to define integration of distribution functions on metric spaces, we must keep in mind that Lipschitz smoothness is the ultimate smoothness in the generic metric space setting. Therefore, "distributions" will be functionals not on infinitely smooth functions but just on Lipschitz functions.

It is well known that spaces of Lipschitz functions can be represented as dual spaces. The result goes back to Arens and Eells [6], see [50]. For considerations similar to the material of this section see Bouchitté, Champion and Jimenez [11].

Let (X, ρ) be a metric space. **In this section, our space X will be separable and boundedly compact.** The latter means that all balls are relatively compact. For these spaces, we can present here another approach to duality.

Definition I.1 (Spaces of continuous functions and measures)**.** Let $\mathcal{C}_b(X)$ be the Banach space of all bounded continuous functions on X endowed with the norm

$$\|u\|_{\mathcal{C}} = \sup_{x \in X} |u(x)|$$

and $\mathcal{C}_c(X)$ be the set of all continuous functions on X with compact support. We define the space $\mathcal{C}_0(X)$ as the closure of the set $\mathcal{C}_c(X)$ in $\mathcal{C}_b(X)$.

Linear functionals $\mathcal{F}$ on $\mathcal{C}_c(X)$ satisfying the condition

$$u \geq 0 \implies \mathcal{F}(u) \geq 0$$

are called *Radon integrals.* According to the Riesz representation theorem, there is an one-to-one correspondence between Radon integrals and *Radon measures.* These are defined as nonnegative Borel measures on X such that $\mu(K) < \infty$ for each compact set $K \subset X$. The term "Borel measure" does not mean that we measure only Borel sets, for each fixed measure μ we extend the measure from the Borel σ-algebra to the σ-algebra of *μ-measurable sets* by the completion process.

If possible, we identify Radon integrals with Radon measures.

The dual to the space $\mathcal{C}_0(X)$ is the space $\mathcal{C}_0(X)^*$ of all signed Radon integrals on X. These are integrals with respect to *signed Radon measures.* Signed measures are always real-valued. Hence, there is no inclusion between Radon measures (allowed to attain infinite values) and signed Radon measures.

Definition I.2 (Lipschitz and "co-Lipschitz" spaces)**.** We start from the Banach space $\mathcal{D}(X)$ of all bounded Lipschitz functions u on X ("test functions") endowed with the norm

$$\|u\|_{\mathcal{D}(X)} = \max\{\|u\|_{\mathcal{C}},\ |u|_{\mathrm{Lip}}\},$$

where $|u|_{\mathrm{Lip}}$ is the Lispchitz constant of u. Recall that $\mathcal{D}(X)^*$ denotes the dual Banach space to $\mathcal{D}(X)$. Each element of the space $\mathcal{C}_0(X)^*$ is identified with the continuous linear functional

$$u \mapsto \int_X u\,d\nu, \qquad u \in \mathcal{C}_b(X),$$

where ν is the signed measure representing the given functional. Since $\mathcal{D}(X)$ is trivially continuously embedded into $\mathcal{C}_b(X)$, by the dual process, $\mathcal{C}_0(X)^*$ is naturally embedded into $\mathcal{D}(X)^*$ (and, in what follows, identified with the corresponding subspace of $\mathcal{D}(X)^*$). The closure of $C_0(X)^*$ in $\mathcal{D}(X)^*$, denoted by $\mathcal{D}'(X)$, is the space of *convergent (metric) distributions* on X. The term "convergent" is motivated by the feature that it is possible to determine the definite integral if the indefinite integral is a convergent metric distribution.

Let $u \in \mathcal{D}(X)$. Then the functional

$$\varkappa(u) :\ \mu \mapsto \int_X u\,d\mu, \qquad \mu \in \mathcal{C}_0(X)^*$$

is continuous with respect to the $\mathcal{D}(X)^*$ norm and thus it can be uniquely extended as a continuous linear functional on $\mathcal{D}'(X)$. The mapping

$$\varkappa :\ u \mapsto \varkappa(u) :\ \mathcal{D}(X) \to (\mathcal{D}'(X))^*$$

is called *canonical embedding.*

Definition I.3 (Localization)**.** The co-Lipschitz spaces can be localized as follows. If $K \subset X$ is compact, let $\mathcal{C}_K(X)$ be the subspace of $\mathcal{C}_0(X)$ consisting of all $\mathcal{C}_0(X)$ function with support in K and $\mathcal{D}_K(X)$ be the subspace of $\mathcal{D}(X)$ consisting of all test functions with support in K. Then $\mathcal{D}'_K(X)$ is defined as the closure of $\mathcal{C}_K(X)^*$ in $\mathcal{D}_K(X)^*$.We define $\mathcal{D}_c(X)$ as the union of all $\mathcal{D}_K(X)$ over all compact $K \subset X$ and $\mathcal{D}'_{\mathrm{loc}}(X)$ as the intersection of $\mathcal{D}'_K(X)$ over all compact $K \subset X$. The elements of $\mathcal{D}'_{\mathrm{loc}}(X)$ will be called *(metric) distributions.* In what follows, we study in detail the "global versions" of these spaces.

Remark I.4. It is a standard and useful fact that the set $\mathcal{D}_c(X)$ is dense in $\mathcal{C}_0(X)$. This can be observed, for example, considering the Stone-Weierstrass theorem on the one-point compactification of X. Further, if μ is a Radon measure on X, then $\mathcal{D}_c(X)$ is dense in $L^1(\mu)$. Indeed, it is standard that $\mathcal{C}_c(X)$ is dense in $L^1(\mu)$. If $f \in \mathcal{C}_c(X)$ has to be further approximated, then we set $Y = \{x \in X \colon f(x) \neq 0\}$ and using the above argumentation, we find $g \in \mathcal{D}_c(X)$ with $\operatorname{spt} f \subset Y$ such that g is close to f in the norm of $\mathcal{C}_0(Y)$. Since $\mu(Y) < \infty$, it is clear that g is close to f also in $L^1(\mu)$.

Theorem I.5. *The dual of $\mathcal{D}'(X)$ is (isometrically isomorphic to) $\mathcal{D}(X)$. The dual of $\mathcal{D}'_K(X)$ is (isometrically isomorphic to) $\mathcal{D}_K(X)$.*

Proof. We prove only the first assertion. Obviously the canonical embedding $\varkappa : \mathcal{D}(X) \to (\mathcal{D}'(X))^*$ is an isometric endomorphism, so we need only to show that it is onto. Let $\mathbb{T}$ be a continuous linear functional on $\mathcal{D}'(X)$. We set

$$u(x) = \mathbb{T}(\delta_x), \qquad x \in X,$$

where δ_x is the Dirac measure at x. Then u is a function on X. If $x, y \in X$, then

$$|u(y) - u(x)| = |\mathbb{T}(\delta_y - \delta_x)| \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*} \|\delta_y - \delta_x\|_{\mathcal{D}(X)^*} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}\, \rho(x,y)$$

and

$$|u(x)| = |\mathbb{T}(\delta_x)| \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*} \|\delta_x\|_{\mathcal{D}(X)^*} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}$$

Hence $u \in \mathcal{D}(X)$ and $\|u\|_{\mathcal{D}(X)} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}$.

We want to show that $\mathbb{T} = \varkappa(u)$. Choose $\mu \in \mathcal{C}_0(X)^*$. We claim that there exist signed measures μ_j such that $\mu_j \to \mu$ in $\mathcal{D}(X)^*$ and μ_j are *molecular*, this means, linear combinations of Dirac measures in X. Indeed, since

$$\lim_{k\to\infty} \mu|_{B(x_0,k)} = \mu \qquad \text{in } \mathcal{D}(X)^*$$

and X is boundedly compact, we may assume that μ is supported in a compact set $K \subset X$. Then there exists a sequence (μ_j) of linear combinations of Dirac masses such that $\mu_j \to \mu$ weak* in $C(K)^*$, see e.g. [39, Corollary 2.28]. The embedding

$$I : \; u \mapsto u|_K : \mathcal{D}(X) \to C(K)$$

is compact, hence by the Schauder theorem [17, Theorem 7.7] the dual operator $I^* : C(K)^* \to \mathcal{D}(X)^*$ is compact and the weak* convergence of $\mu_j \in C(K)^*$ to μ is enough to conclude that the convergence is strong in $\mathcal{D}(X)^*$.

For the molecular signed measures μ_j we have

$$\int_X u\, d\mu_j = \mathbb{T}(\mu_j)$$

and passing to the limit we obtain

$$\int_X u\, d\mu = \mathbb{T}(\mu).$$

Then it is easy to conclude that $\mathbb{T} = \varkappa(u)$. $\square$

Remark I.6. We identify $\mathcal{D}(X)$ with $\mathcal{D}'(X)^*$, so that from now, the weak* topology and convergence on $\mathcal{D}(X)$ (and similarly on $\mathcal{D}_K(X)$) are well defined.

Proposition I.7. *Let u, $u_j \in \mathcal{D}(X)$, $j = 1, 2, \dots$. Then the following assertions are equivalent:*

(i) *$u_j \to u$ weak* in $\mathcal{D}(X)$,*

(ii) *u_j is bounded in $\mathcal{D}(X)$ and $u_j \to u$ pointwise,*

(iii) *u_j is bounded in $\mathcal{D}(X)$ and $u_j \to u$ locally uniformly.*

Proof. (i) $\implies$ (ii). By the Banach-Steinhaus theorem [17, Theorem 3.12]), each weak* convergent sequence is bounded. Since each Dirac measure is in $\mathcal{D}'(X)$, the pointwise convergence follows.

(ii) $\implies$ (iii): Boundedness in $\mathcal{D}(X)$ implies equicontinuity and each equicontinuous pointwise convergent sequence converges locally uniformly (this is related to the Arzelà-Ascoli theorem).

(iii) $\implies$ (i): If $u_j \to u$ locally uniformly, then $\varkappa(u_j)(\mu) \to \varkappa(u)(\mu)$ for each measure $\mu \in \mathcal{C}_0(X)^*$. Since $\mathcal{C}_0(X)^*$ is dense $\mathcal{D}'(X)$ and the sequence $(u_j)_j$ is bounded, we conclude that $u_j \to u$ weak* in $\mathcal{D}(X)$. ☐

Proposition I.8. *The following properties of a linear functional $\mathcal{T} : \mathcal{D}(X) \to \mathbb{R}$ are equivalent:*

(i) *$\mathcal{T} \in \mathcal{D}'(X)$,*

(ii) *$\mathcal{T}$ is weak* continuous on $\mathcal{D}(X)$,*

(iii) *$\mathcal{T}$ is sequentially weak* continuous on $\mathcal{D}(X)$.*

Proof. Let S be a countable dense subset of X. Since the linear combinations of Dirac masses concentrated at S form a dense subset of $\mathcal{D}'(X)$, the space $\mathcal{D}'(X)$ is separable. Now, we can use [17, Proposition 3.24, Corollary 4.46]. ☐

Proposition I.9. *$\mathcal{D}_c(X)$ is weak* dense in $\mathcal{D}(X)$.*

Proof. Choose $x_0 \in X$. For $k = 1, 2, \dots$, set

$$\eta_k(x) = \begin{cases} 1, & \rho(x, x_0) \le k - 1, \\ k - \rho(x, x_0), & k - 1 < \rho(x, x_0) \le k, \\ 0, & \rho(x, x_0) > k. \end{cases}$$

Since X is boundedly compact, the functions η_k are compactly supported. If $u \in \mathcal{D}(X)$, using Proposition I.7 we observe that $\eta_k u \to u$ weak* in $\mathcal{D}(X)$. ☐

Remark I.10. If $\mathcal{T}$ is a distribution on $\Omega \subset \mathbb{R}^n$, then it can be extended to a functional from $\mathcal{D}'(\Omega)$ (in our notation) if and only if $\mathcal{T}$ is bounded on $\mathcal{C}_c^\infty(\Omega)$ in the $\mathcal{D}(\Omega)$-norm and

$$\mathcal{T}(\varphi_j) \to 0$$

whenever $\varphi_j \in \mathcal{C}_c^\infty(\Omega)$ and $\varphi_j \to 0$ weak* in $\mathcal{D}(\Omega)$.

We can consider the example

$$\mathcal{T}(\varphi) = \varphi'(0), \qquad \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{I.1}$$

Then $\mathcal{T}$ is bounded on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in the $\mathcal{D}(\mathbb{R}^n)$-norm and can be extended as a bounded linear functional on $\mathcal{D}(\mathbb{R}^n)$. However, such an extension is not constructive (as it relies on the Hahn-Banach theorem) and is not weak* continuous on $\mathcal{D}(\mathbb{R}^n)$. Hence the functional $\mathcal{T}$ from (I.1) is not a metric distribution.

I.3 Integration in metric spaces

In this section we introduce our concept of integration of distribution-valued functions, prove that the new integral makes sense and investigate some its basic properties.

In the sequel, we assume that X is a complete separable metric space equipped with a doubling measure λ. This means that λ is a Radon measure and there exists a constant c_D such that

$$\lambda(B(x, 2r)) \le c_D \lambda(B(x, r))$$

for each $x \in X$ and $r > 0$. Note that such a space X is always boundedly compact.

Definition I.11. If $x \in X$ and $r > 0$, we denote

$$\mathcal{D}(x, r) = \{\varphi \in \mathcal{D}(X),\ \operatorname{spt}\varphi \in B(x, r),\ |\varphi|_{\mathrm{Lip}} \le 1/r,\ \|\varphi\|_{\mathcal{C}} \le 1\}.$$

If $\mathcal{T} \in \mathcal{D}'(X)$, we write

$$\|\mathcal{T}\|_{x,r} = \sup\Big\{|\mathcal{T}(\varphi)| :\ \varphi \in \mathcal{D}(x, r)\Big\}.$$

Definition I.12. A pairwise disjoint finite system of balls in X is called a *packing*. Let $\delta : X \to (0, \infty)$ be a function (called a *gauge*). Then a packing $(B_i)_{i=1}^m$, where $B_i = B(x_i, r_i)$, is said to be *δ-fine* if $r_i < \delta(x_i)$ for each $i = 1, \dots, m$.

Definition I.13. We say that a functional $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ is an *indefinite packing integral* of a distribution-valued function $\mathscr{H} : X \to \mathcal{D}'_{\mathrm{loc}}(X)$ if there exists $\tau \in (0, 1]$ such that for every $\varepsilon > 0$ there exists a gauge $\delta \colon X \to (0, \infty)$ such that for every δ-fine packing $(B(x_i, r_i))_{i=1}^m$ we have

$$\sum_{i=1}^m \|\mathcal{F} - \mathscr{H}(x_i)\|_{x_i, \tau r_i} < \varepsilon.$$

If $\mathscr{H}$ has the form $\mathscr{H}(x) = f(x)\mathcal{G}$, where $f \colon X \to \mathbb{R}$ is a function and $\mathcal{G}$ is a distribution, (which is mostly the case), we say that $\mathcal{F}$ is an indefinite packing integral of f with respect to $\mathcal{G}$.

We denote the indefinite packing integral of f with respect to $\mathcal{G}$ (which is unique by Theorem I.20 below) by

$$\int f \, \mathrm{d}\mathcal{G}.$$

If $\varphi \in \mathcal{D}_c(X)$ is a test function, then $\int f \, \mathrm{d}\mathcal{G}(\varphi)$ means $\mathcal{F}(\varphi)$, where $\mathcal{F} = \int f \, \mathrm{d}\mathcal{G}$.

Example I.14 (Varifold type structure). Let $E \subset \mathbb{R}^n$, $1 \le k \le n$ and V be a mapping which with each $x \in E$ associates a k-dimensional affine subspace $V(x)$ of $\mathbb{R}^n$ (we may choose $V(x)$ to be a tangent space to E in any level of generality). Let $f : E \to \mathbb{R}$ be a function. Then we may consider $\mathscr{H}$ of type

$$\mathscr{H}(x) = f(x) \int_{V(x)} \varphi \, dH^k,$$

where H^k is the k-dimensional Hausdorff measure. In this example, which may be used to define a kind of surface integration, $\mathscr{H}(x)$ is not of type $f(x)\mathcal{G}$ for a fixed distribution.

Definition I.15 (Definite integral)**.** We say that f is *packing-integrable* with respect to $\mathcal{G}$ if $\int f \, \mathrm{d}\mathcal{G}$ exists and belongs to $\mathcal{D}'(X)$. Then we define the *definite packing integral* of f with respect to $\mathcal{G}$ as

$$\int_X f \, \mathrm{d}\mathcal{G} = \int f \, \mathrm{d}\mathcal{G}(1).$$

We also use the phrases "the integral $\int_X f \, \mathrm{d}\mathcal{G}$ *converges*" in the above situation. By Proposition I.9, the definite integral is determined by the values of $\int f \, \mathrm{d}\mathcal{G}$ on $\mathcal{D}_c(X)$. If $E \subset X$, we define

$$\int_E f \, \mathrm{d}\mathcal{G} = \int_X f \chi_E \, \mathrm{d}\mathcal{G}$$

if this makes sense.

Remark I.16. 1. It is obvious that the family of all pairs $(\mathscr{H}, \mathcal{F})$, where $\mathscr{H}$ is a distribution-valued function and $\mathcal{F}$ is an indefinite packing integral, is a linear space.

2. The purpose of the scaling factor τ is to arrange that the integral is invariant under bilipschitz transformations (see Subsection I.4.3). If we allow $\tau = 1$ only, then the definition still makes sense, but even the invariance with respect to a linear change of variables in $\mathbb{R}^n$ or with respect to the choice of norm in $\mathbb{R}^n$ is unclear and probably false.

3. A packing integral of a Banach space-valued function with respect to a distribution can be defined in a similar way.

4. The definition of the definite integral is only one of many possibilities how to perform the limit process $\varphi \to 1$ from compactly supported test functions to the constant 1. In Definition I.15, we have proposed one possibility how to do it, just to make the theory more complete.

We are convinced that for fine studies of the definite integral, it is better to modify the definition according to particular structures than to insist on universal directives based only on the setting of metric spaces.

We demonstrate the problem in Example I.21 below.

Lemma I.17. *Let $0 < \tau \le 1$. Then there exists a constant c_T (depending only on τ and on the doubling constant of λ) with the following property: for each function $\Phi : \mathbb{R} \to \mathbb{R}$, $x \in X$ and a $R > 0$ there exists $0 < r < R$ such that*

$$\Phi(10r) + \lambda(B(x, 10r)) \le c_T(\Phi(\tau r) + \lambda(B(x, \tau r))).$$

Proof. Let c_D be the doubling constant of λ. We find $k \in \mathbb{N}$ such that $10 \le 2^k\tau$. Then for every $x \in X$ and for every $r > 0$ we have

$$\lambda(B(x, 10r)) \le \lambda(B(x, 2^k\tau r)) \le c_D^k \lambda(B(x, \tau r)). \tag{I.2}$$

Now, we set $c_T := 2c_D^k$, fix $R > 0$ and assume by contradiction that for each $r \in (0, R]$ we have

$$\Phi(10r) + \lambda(B(x, 10r)) > c_T(\Phi(\tau r) + \lambda(B(x, \tau r))).$$

By (I.2),

$$\frac{\Phi(R) + \lambda(B(x, R))}{\lambda(B(x, R))} > \frac{2c_D^k(\Phi(\tau R/10) + \lambda(B(x, \tau R/10)))}{c_D^k \lambda(B(x, \tau R/10))}.$$

Iterating this process we obtain

$$\frac{\Phi(R) + \lambda(B(x, R))}{\lambda(B(x, R))} > 2^n \; \frac{\Phi(\tau^n R/10^n) + \lambda(B(x, \tau^n R/10^n))}{\lambda(B(x, \tau^n R/10^n))} \ge 2^n$$

for each $n \in \mathbb{N}$, which is the contradiction. □

Lemma I.18 (Partition of unity)**.** *Let $K \subset X$ be a compact set and $((B(x_i, R_i))_{i=1}^m$ be a system of balls in X covering K. Then there exists a system $(\omega_i)_{i=1}^m$ of Lipschitz functions such that $\sum_{i=1}^m \omega_i = 1$ on K and the following properties are satisfied for each $i = 1, \dots, m$:*

$$\omega_i \ge 0 \text{ on } X, \tag{I.3}$$

$$\omega_i \in \mathcal{D}(x_i, 2R_i)(X), \tag{I.4}$$

$$|\omega_i|_{\text{Lip}} \le 2/R_i. \tag{I.5}$$

Proof. We may assume that the balls are ordered so that

$$R_1 \ge R_2 \ge \cdots \ge R_m. \tag{I.6}$$

For $i = 1, \dots, m$ we set

$$\kappa_i := \begin{cases} 1, & x \in B(x_i, R_i), \\ 1 - \frac{1}{R_i}(\rho(x, x_i) - R_i), & x \in B(x_i, 2R_i) \setminus B(x_i, R_i), \\ 0, & x \notin B(x_i, 2R_i). \end{cases}$$

Further, write

$$\omega_1 = \xi_1 := \kappa_1,$$
$$\xi_i := \max\{\kappa_1, \dots \kappa_i\}$$

and

$$\omega_i := \xi_i - \xi_{i-1}, \quad i = 2, 3, \dots\ .$$

Then

$$\sum_{i=1}^m \omega_i = 1, \qquad x \in K$$

and the properties (I.3)–(I.5) are obviously satisfied. □

Lemma I.19. *Let $\mathscr{H}$ be a distribution-valued function on X. Assume that for each $x \in X$ there exists $\delta_1(x) > 0$ such that for each $\varphi \in \mathcal{D}(x, \delta_1)(X)$ we have*

$$\varphi \geq 0 \implies \mathscr{H}(x)(\varphi) \geq 0.$$

Let $\mathcal{F}$ be an indefinite packing integral of $\mathscr{H}$ and $\eta \in \mathcal{D}_c(X)$, $\eta \geq 0$. Then $\mathcal{F}(\eta) \geq 0$.

Proof. Step 1. Denote $K = \operatorname{spt} \eta$ and fix a bounded open set $U \supset K$. Since X is boundedly compact, we have $\lambda(U) < \infty$.

Step 2. We choose $\varepsilon > 0$ and construct a covering of the set K. Since $\mathcal{F}$ is a packing integral of $\mathscr{H}$, there exist $\tau \in (0, 1]$ and a gauge $\delta \colon X \to (0, \infty)$ such that for every δ-fine packing $(B(x_i, r_i))_{i=1}^m$ we have

$$\sum_{i=1}^{m} \|\mathcal{F} - \mathscr{H}(x)\|_{x_i, \tau r_i} < \varepsilon.$$

We may assume that $10\delta(x) < \delta_1(x)$ for each $x \in X$. Let us fix $x \in K$. Let c_D be the doubling constant of λ. Since the measure $\varepsilon\lambda$ is doubling with the same doubling constant c_D, applying Lemma I.17 to $\Phi(r) := \|\mathcal{F} - \mathscr{H}(x)\|_{x,r}$ we can find a constant c_T and a radius $r(x) \in (0, \frac{1}{10})$ such that $r(x) < \delta(x)$, $B(x, r(x)) \subset U$ and

$$\begin{aligned} &\|\mathcal{F} - \mathscr{H}(x)\|_{x, 10r(x)} + \varepsilon\lambda(B(x, 10r(x))) \\ &\quad \leq c_T\Big(\|\mathcal{F} - \mathscr{H}(x)\|_{x, \tau r(x)} + \varepsilon\lambda(B(x, \tau r(x)))\Big). \end{aligned} \tag{I.7}$$

Consider the covering $\mathcal{V} = (\bar{B}(x, r(x)))_{x \in K}$ of K. Then we apply a Vitali type Theorem (see e.g. [52, Theorem 1.3.1]) and obtain a disjointed subsystem $\mathcal{V}'$ of $\mathcal{V}$ such that

$$K \subset \bigcup_{\bar{B}(x,r) \in \mathcal{V}'} B(x, 5r).$$

Now, we set

$$\mathcal{W} := \{B(x, 5r) \colon \bar{B}(x, r) \in \mathcal{V}'\}.$$

This is a system of open balls which covers K. Since K is compact, we can find a finite subcovering $\{B(x_1, R_1), \dots, B(x_m, R_m)\}$ of K such that $B(x_i, R_i) \in \mathcal{W}$ for each $i = 1, \dots, m$ (so that $\frac{1}{5}R_i = r_i := r(x_i)$).

Step 3. By Lemma I.18, there exists a partition of unity $(\omega_i)_{i=1}^m$ such that the properties (I.3)–(I.5) are satisfied. Then η can be written as $\eta = \sum_{i=1}^m \omega_i \eta$. We see that $\omega_i \eta \in \mathcal{D}_c(X)$ and $\omega_i \eta = 0$ on $X \setminus B(x_i, 2R_i)$. Since $2R_i = 10r(x_i) \leq 1$, we obtain

$$\begin{aligned} 2R_i |\omega_i \eta|_{\mathrm{Lip}} &\leq 2R_i\Big(\|\eta\|_{\mathcal{C}} |\omega_i|_{\mathrm{Lip}} + \|\omega_i\|_{\mathcal{C}} |\eta|_{\mathrm{Lip}}\Big) \\ &\leq C\|\eta\|_{\mathcal{D}(X)} \end{aligned} \tag{I.8}$$

and

$$\|\omega_i \eta\|_{\mathcal{C}} \leq \|\eta\|_{\mathcal{D}(X)}. \tag{I.9}$$

It follows that $\frac{\omega_i \eta}{C} \in \mathcal{D}(x_i, 2R_i)$. Notice also that $\omega_i \eta \geq 0$ and $2R_i < \delta_1(x_i)$. Thus

$$\begin{aligned} -\mathcal{F}(\omega_i \eta) &= -\mathscr{H}(x_i)(\omega_i \eta) - (\mathcal{F} - \mathscr{H}(x_i))(\omega_i \eta) \\ &\leq 0 + C\|\mathcal{F} - \mathscr{H}(x_i)\|_{x_i, 2R_i}, \qquad i = 1, \dots, m. \end{aligned}$$

Summing over i, by (I.7) we have

$$\begin{aligned} -\mathcal{F}(\eta) &= -\sum_{i=1}^{m} \mathcal{F}(\omega_i \eta) \le C \sum_{i=1}^{m} \|\mathcal{F} - \mathscr{H}(x_i)\|_{x_i, 2R_i} \\ &\le C c_T \sum_{i=1}^{m} \Big(\|\mathcal{F} - \mathscr{H}(x_i)\|_{x_i, \tau r_i} + \varepsilon \lambda(B(x_i, \tau r_i)) \Big) \\ &\le C c_T \varepsilon (1 + \lambda(U)). \end{aligned}$$

Letting $\varepsilon \to 0$ we conclude that $-\mathcal{F}(\eta) \le 0$. ☐

Theorem I.20. *Let $\mathscr{H}$ be a distribution-valued function on X. Then there exists at most one indefinite packing integral of $\mathscr{H}$.*

Proof. If $\mathcal{F}_1$, $\mathcal{F}_2$ are indefinite packing integrals of $\mathscr{H}$, then $\mathcal{F} := \mathcal{F}_1 - \mathcal{F}_2$ is an indefinite packing integral of 0. From Lemma I.19 we infer that $\mathcal{F}(\eta) = 0$ for each nonnegative $\eta \in \mathcal{D}_c(X)$. The conclusion follows by decomposition of a general $\eta \in \mathcal{D}_c(X)$ into the positive and negative parts. ☐

Example I.21. Let $\mathscr{L}$ be the integration with respect to the Lebesgue measure on $X = (0, \infty)$ and $f(x) = \frac{\sin x}{x}$. Then

$$\mathcal{F} : \varphi \mapsto \int_0^\infty \frac{\sin x}{x} \varphi(x)\, dx, \qquad \varphi \in \mathcal{D}_c(X)$$

is the indefinite packing integral of f with respect to $\mathscr{L}$. However, concerning the definite integral according to Definition I.15, f is not packing-integrable. Indeed, $\mathcal{F}$ cannot be continuously extended to $\mathcal{D}(X)$, as the integral diverges if we choose $\varphi(x) = \sin x$. In order to give a reasonable sense to the definite packing integral

$$\int_0^\infty \frac{\sin x}{x}\, dx, \tag{I.10}$$

let us change the distance function on X to the "hyperbolic distance"

$$\tilde{\rho}(x, y) = \left| \log \frac{y}{x} \right|.$$

Denote $\tilde{X} = ((0, \infty), \tilde{\rho})$. The space $\tilde{X}$ is obviously equipped with a doubling measure (for example, the measure with density $1/x$). We set

$$\mathcal{F}(\varphi) = \int_0^\infty \frac{1 - \cos x}{x^2} \Big(\varphi(x) - x\varphi'(x) \Big)\, dx, \qquad \varphi \in \mathcal{D}(\tilde{X}).$$

If $\varphi \in \mathcal{D}(\tilde{X})$, then the Lipschitz condition implies that $|\varphi'(x)| \le \frac{C}{x}$ a.e. and thus

$$\mathcal{F}(\varphi) \le C \|\varphi\|_{\mathcal{D}(\tilde{X})}.$$

Assume that $\varphi_j \to 0$ weak* in $\mathcal{D}(\tilde{X})$ and $u \in \mathcal{C}_c^\infty((0, \infty))$. Write $\psi_j(x) := x\varphi_j'(x)$. Then

$$\int_0^\infty u(x)\, \psi_j(x)\, dx = \int_0^\infty x u(x)\, \varphi_j'(x)\, dx = -\int_0^\infty \Big(u(x) + x u'(x) \Big) \varphi_j(x)\, dx \to 0.$$

Since $(\psi_j)_j$ is bounded in $L^\infty((0,\infty))$ and $\mathcal{C}_c^\infty((0,\infty))$ is dense in $L^1((0,\infty))$, it follows that $\psi_j \to 0$ weak* in $L^\infty((0,\infty))$. Since the function

$$x \mapsto \frac{1-\cos x}{x^2}$$

belongs to $L^1((0,\infty))$, we deduce that $\mathcal{F}(\varphi_j) \to 0$. Hence $\mathcal{F} \in \mathcal{D}'(\tilde{X})$. It is easy to verify that $\mathcal{F}$ is a weak* continuous extension of the functional

$$\varphi \mapsto \int_0^\infty \frac{\sin x}{x}\,\varphi(x)\,dx, \qquad \varphi \in \mathcal{D}_c(\tilde{X})$$

to $\mathcal{D}(\tilde{X})$ and thus we may define the definite packing integral (I.10) as $\mathcal{F}(1)$.

I.4 *UC* integral

In this section we present another approach to the definition of integral. In the Euclidean framework, this integral has been introduced in [41]. We will show that the *UC* integral is equivalent to the packing integral.

Definition I.22. Let $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ and $\mathscr{H}\colon X \to \mathcal{D}'_{\mathrm{loc}}(X)$ be a distribution-valued function. We say that $\mathcal{F}$ is an *indefinite UC integral* of $\mathscr{H}$ if there exist $\tau \in (0,1]$ and a finite Radon measure μ on X such that

$$\lim_{r\to 0_+} \frac{\|\mathcal{F} - \mathscr{H}(x)\|_{x,\tau r}}{\mu(B(x,r))} = 0 \tag{I.11}$$

holds for each $x \in X$. In particular, we define the *UC* integral of a function $f : X \to \mathbb{R}$ with respect to a distribution $\mathcal{G} \in \mathcal{D}'_{\mathrm{loc}}(X)$ as the *UC* integral of $x \mapsto f(x)\mathcal{G}$.

We define the class of *UC* integrable functions and the definite *UC* integral analogously to the case of packing integral.

Remark I.23. Since the role of μ is local and X is boundedly compact, the control measure μ from definition I.22 can be always made finite. Indeed, if $\tilde{\mu}$ is an arbitrary control measure, $x_0 \in X$ and μ_k are defined as $\tilde{\mu}|_{B(x_0,k)}$, then there exist $\alpha_k > 0$ such that

$$\sum_{k=1}^{\infty} \alpha_k \mu_k(X) < \infty.$$

Then the measure

$$\mu = \sum_{k=1}^{\infty} \alpha_k \mu_k$$

controls (I.11) as well.

Remark I.24. Sometimes we rewrite (I.11) as

$$\lim_{r\to 0_+} \frac{\|\mathcal{F} - \mathscr{H}(x)\|_{x,r}}{\mu(B(x,\sigma r))} = 0, \tag{I.12}$$

For th where $\sigma = \frac{1}{\tau}$.

I.4.1 Equivalence

In this section we prove the equivalence of the UC integral to the packing integral.

Theorem I.25. *Let $\mathcal{F}$ be an indefinite UC integral of $\mathscr{H}\colon X\to\mathcal{D}'_{\mathrm{loc}}(X)$. Then $\mathcal{F}$ is also an indefinite packing integral of $\mathscr{H}$.*

Proof. Let τ and μ be as in Definition I.22. By Remark I.23 we may assume that $\mu(X)<\infty$. Choose $\varepsilon>0$. Given $x\in X$, using (I.11) we find $\delta(x)>0$ such that

$$\|\mathcal{F}-\mathscr{H}(x)\|_{x,\tau r}<\varepsilon\mu(B(x,r)),\qquad 0<r<\delta(x). \tag{I.13}$$

Consider a δ-fine packing $(B(x_i,r_i))_{i=1}^m$. Then, by (I.13),

$$\sum_{i=1}^m\|\mathcal{F}-\mathscr{H}(x)\|_{x_i,\tau r_i}<\varepsilon\sum_{i=1}^m\mu(B(x_i,r_i))<\varepsilon\mu(X),$$

which we needed. $\square$

For the converse implication, we need first to prove some lemmas.

Lemma I.26. *Let I be a finite set, $(B(x_i,r_i))_{i\in I}$ be a system of balls and $q\in\mathbb{N}$. Assume that*

$$\sum_{i\in I}\chi_{B(x_i,5r_i)}\le q\quad\text{on } G:=\bigcup_{i\in I}B(x_i,r_i).$$

Then there exists a partition $I=I_1\cup\dots\cup I_q$ of the family of indices such that each the system $(B(x_i,r_i))_{i\in I_j}$ is pairwise disjoint.

Proof. We use induction by q. If $q=1$, then the balls $B(x_i,r_i)$, $i\in I$, are already pairwise disjoint. Assume that the assertion holds for $q-1$. Using the Vitali covering theorem, we select $I_q\subset I$ such that the balls $B(x_i,r_i)$, $i\in I_q$, are pairwise disjoint and

$$\sum_{i\in I_q}\chi_{B(x_i,5r_i)}\ge 1\qquad\text{on } G.$$

Let

$$G_{q-1}=\bigcup_{i\in I\setminus I_q}B(x_i,r_i).$$

Then

$$\sum_{i\in I\setminus I_q}\chi_{B(x_i,5r_i)}\le q-1\quad\text{on } G_{q-1}$$

and we can use the induction hypothesis to partition the set $I\setminus I_q$. $\square$

Definition I.27. Let $\mathcal{R}$ be a function of ball. Given an open set $G\subset X$ and a gauge $\delta:X\to(0,\infty)$, we set

$$V_\delta(\mathcal{R},G)=\sup\Bigl\{\sum_i\mathcal{R}(B(x_i,r_i)):\ \{B(x_i,r_i)\}\text{ is a }\delta\text{-fine packing of }G\Bigr\}.$$

Lemma I.28. *Let $\mathcal{R}$ be a function of ball and $\delta\colon X\to(0,\infty)$ be a gauge. Suppose that*

$$V_\delta(\mathcal{R},X)\le 1.$$

Then there exists a Radon measure μ on X such that $\mu(X)\le 1$ and for each ball $B(x,r)$ we have

$$r<\delta(x)\implies\mathcal{R}(B(x,r))\le\mu(\overline{B}(x,5r)).$$

Proof. We follow a method invented by Csörnyei [12]. Let us say that $(B(x_i, r_i))_{i=1}^m$ is a δ-fine system if $r_i < \delta(x_i)$, $i = 1, \dots, m$ (we do not assume the balls to be disjointed). Set

$$V^*(f) = \sup\Big\{\sum_{i=1}^m \alpha_i \mathcal{R}(B(x_i, r_i)) : (B(x_i, r_i))_{i=1}^m \text{ is a } \delta\text{-fine system}, \ \alpha_i > 0, \\ \sum_{i=1}^m \alpha_i \chi_{B(x_i, 5r_i)} \leq f\Big\}, \qquad f \in C_0(X).$$

Then V^* is a superadditive and positively homogeneous functional. Using rational approximation, we can rewrite V^* in a different way. Indeed, we will consider non-disjoint systems of balls and replace integer coefficients by repeating the use of considered balls. Now, we see that

$$V^*(f) = \sup\Big\{\frac{1}{q}\sum_{i=1}^m \mathcal{R}(B(x_i, r_i)) : (B(x_i, r_i))_{i=1}^m \text{ is a } \delta\text{-fine system}, \ q \in \mathbb{N}, \\ \frac{1}{q}\sum_{i=1}^m \chi_{B(x_i, 5r_i)} \leq f\Big\}, \qquad f \in C_0(X).$$

We claim that

$$V^*(f) \leq \|f\|_{\mathcal{C}_0(X)}, \qquad f \in \mathcal{C}_0(X). \tag{I.14}$$

Given a δ-fine system $(B(x_i, r_i))_{i=1}^m$ and $q \in \mathbb{N}$ such that

$$\sum_{i=1}^m \chi_{B(x_i, 5r_i)} \leq qf,$$

we need to show that

$$\sum_{i=1}^m \mathcal{R}(B(x_i, r_i)) \leq q\|f\|_{\mathcal{C}_0(X)}. \tag{I.15}$$

Without loss of generality we may assume that $\|f\|_{\mathcal{C}_0(X)} \leq 1$. Then

$$\sum_i \chi_{B(x_i, 5r_i)} \leq q$$

and by Lemma I.26 we can divide the index set $I = \{1, \dots, m\}$ into q classes I_j, $j = 1, \dots, q$, such that the balls $B(x_i, r_i)$, $i \in I_j$, are pairwise disjoint for each $j = 1, \dots, q$. Then

$$\sum_{i \in I_j} \mathcal{R}(B(x_i, r_i)) \leq V_\delta(\mathcal{R}, X) \leq 1, \qquad j = 1, \dots, q$$

and summing over $j = 1, \dots, q$ we obtain (I.15). This verifies (I.14).

Now, let us define sets

$$H_1 := \{f \in C_0(X) \colon \ f \geq 0, \ V^*(f) > 1\},$$

$$H_2 := \{f \in C_0(X) \colon \ \|f^+\|_{\mathcal{C}_0(X)} \leq 1\}.$$

Then H_1, H_2 are convex subset of $\mathcal{C}_0(X)$ (we use superadditivity and homogeneity of V^*), H_2 is a neighborhood of the origin and $H_1 \cap H_2 = \emptyset$ (this follows from

(I.14)). By the Hahn-Banach Theorem [17, Corollary 2.13], there exists a bounded linear functional Φ on $\mathcal{C}_0(X)$ such that $\Phi \geq 1$ on H_1 and $\Phi \leq 1$ on H_2. Let $f \in \mathcal{C}_0(X)$, $f \geq 0$. Then $-\frac{1}{\alpha} f \in H_2$ for each $\alpha > 0$ and thus

$$\Phi(f) \geq -\alpha, \qquad \alpha > 0.$$

It follows that Φ is a nonnegative linear functional on $\mathcal{C}_0(X)$. By the Riesz Representation Theorem, there exists a Radon measure μ on X such that

$$\Phi(f) = \int_X f \, d\mu, \qquad f \in C_0(X). \tag{I.16}$$

We have

$$\mu(X) = \sup\{\Phi(f) : \ f \in H_2\} \leq 1.$$

Notice that

$$V^*(f) \leq \Phi(f), \qquad f \in C_0(X), \ f \geq 0. \tag{I.17}$$

This is clear if $V^*(f) = 0$ as Φ is nonnegative. Otherwise we use that $tf \in H_1$ for each $t > 1/V^*(f)$. Now, let us choose $\varepsilon > 0$ and apply (I.16) and (I.17) to a function f with

$$\chi_{B(x,5r)} \leq f \leq \chi_{B(x,5r+\varepsilon)}.$$

Then

$$\mathcal{R}(B(x,r)) \leq V^*(f) \leq \Phi(f) = \int_X f \, d\mu \leq \mu(B(x, 5r+\varepsilon))$$

and letting $\varepsilon \to 0$ we conclude the proof. ☐

Theorem I.29. *Let $\mathcal{F}$ be an indefinite packing integral of $\mathscr{H} \colon X \to \mathcal{D}'_{\mathrm{loc}}(X)$. Then $\mathcal{F}$ is also an indefinite UC integral of $\mathscr{H}$.*

Proof. Let τ be as in Definition I.13. We define

$$\mathcal{R}_k(B(x,r)) := 4^k \|\mathcal{F} - \mathscr{H}(x)\|_{x,\tau r}\,, \qquad k = 1, 2, \dots .$$

Fix $k \in \mathbb{N}$. By Definition I.13, there exists a gauge $\delta_k : X \to (0, \infty)$ such that

$$V_{\delta_k}(\mathcal{R}_k, X) \leq 1.$$

Further, by Lemma I.28 there exists a Radon measure μ_k on X such that $\mu_k(X) \leq 1$ and for each ball $B(x,r)$ we have

$$r < \delta_k(x) \implies \mathcal{R}_k(B(x,r)) \leq \mu_k(\overline{B}(x, 5r)).$$

We define

$$\mu := \sum_{k=1}^{\infty} 2^{-k} \mu_k.$$

Then μ is a Radon measure on X and $\mu(X) \leq 1$. Choose $x \in X$. We want to verify (I.11). Given $k \in \mathbb{N}$, for each $r \in (0, \delta_k(x))$ we have

$$\|\mathcal{F} - \mathscr{H}(x)\|_{x,\tau r} = 4^{-k} \mathcal{R}_k(B(x,r)) \leq 4^{-k} \mu_k(\overline{B}(x, 5r)) \leq 2^{-k} \mu(B(x, 6r)).$$

Since k was arbitrary, we infer that

$$\lim_{r \to 0_+} \frac{\|\mathcal{F} - \mathscr{H}(x)\|_{x,\tau r}}{\mu(B(x, 6r))} = 0.$$

It follows that (I.11) holds with $\tau/6$ replacing τ. ☐

I.4.2 Integration with respect to a measure

In [41] it is proven that the indefinite UC integral with respect to Lebesgue measure on $\mathbb{R}$ includes the indefinite Denjoy-Perron integral. Further, it is shown that the Lebesgue integral of a nonnegative function with respect to a Radon measure ν on $\mathbb{R}^n$ is the same as its UC integral with respect to ν. (Of course, if we admit functions changing a sign, the class of the UC integrable functions is much wider.) In this section we formulate these results in the metric space setting. We omit the proofs as the arguments used in [41] can be repeated almost verbatim.

If ν is a Radon measure on X, we write $\mathcal{G}_\nu$ for the metric distribution

$$\varphi \mapsto \int_X \varphi \, d\nu, \qquad \varphi \in \mathcal{D}_c(X).$$

We do not identify ν with $\mathcal{G}_\nu$ in this subsection; this help us to distinguish the integrals: integrals $\dots d\nu$ will be Lebesgue integrals, whereas integrals $\dots d\mathcal{G}_\nu$ will be UC integrals.

Theorem I.30. *Let ν be a Radon measure on X and $f \in L^1_{\mathrm{loc}}(\nu)$. Then the indefinite UC integral $\int f \, \mathrm{d}\mathcal{G}_\nu$ exists and*

$$\int f \, \mathrm{d}\mathcal{G}_\nu \, (\varphi) = \int_X f\varphi \, d\nu, \qquad \varphi \in \mathcal{D}_c(X).$$

If, moreover, $f \in L^1(\nu)$, then the UC-integral $\int_X f \, \mathrm{d}\mathcal{G}_\nu$ converges and

$$\int_X f \, \mathrm{d}\mathcal{G}_\nu = \int_X f \, d\nu.$$

Theorem I.31. *Let ν be a Radon measure on X. Let a function $f : X \to \mathbb{R}$ be UC integrable with respect to $\mathcal{G}_\nu$. Then*

(a) *f is ν-measurable,*

(b) *if $\int f \, \mathrm{d}\mathcal{G}_\nu = 0$, then $f = 0$ ν-a.e.,*

(c) *if $f \geq 0$, then*

$$0 \leq \int_X f \, d\nu < \infty.$$

Corollary I.32. Let ν be a Radon measure on X, $f : X \to \mathbb{R}$ be a function. Then f is ν-integrable if and only if the UC integral of f with respect to $\mathcal{G}_\nu$ "converges absolutely", this means, both f and $|f|$ are UC integrable.

I.4.3 Change of variables

In this subsection we describe what we mean by the phrase that "the UC integral is invariant with respect to a bilipschitz change of variables". We assume that X, Y are locally compact separable metric spaces equipped with doubling measures λ_X, λ_Y, respectively.

Definition I.33 (Push forward)**.** Let $\mathcal{G} \in \mathcal{D}'(X)$ and $\Phi : X \to Y$ be a bilipschitz mapping. Then we define the *push forward* $\Phi_\sharp \mathcal{G}$ as

$$\Phi_\sharp \mathcal{G}(\psi) = \mathcal{G}(\psi \circ \Phi), \qquad \psi \in \mathcal{D}(\Phi(X)).$$

Theorem I.34. *Let $\mathcal{G} \in \mathcal{D}'(X)$ and $\Phi : X \to Y$ be a bilipschitz mapping. Let $f : \Phi(X) \to \mathbb{R}$ be a function. Suppose that the UC-indefinite integral $\mathcal{F} := \int f \circ \Phi \, \mathrm{d}\mathcal{G}$ exists. Then there exists a UC-indefinite integral $\int f \, \mathrm{d}\Phi_\sharp \mathcal{G}$ and*

$$\int f \, \mathrm{d}\Phi_\sharp \mathcal{G} = \Phi_\sharp \mathcal{F}.$$

Proof. The proof is the same as in the Euclidean setting, see [41]. □

I.5 Application to currents

We are motivated by "integration of non-smooth differential forms". In the setting of Sobolev and L^p-differential forms on Lipschitz manifolds, related questions have been studied e.g. in [22], [21]. In geometric measure theory, as in [18], the duality between differential forms and currents is used to describe non-smooth integration. In our approach, the non-smooth objects are on the current part of the duality.

The idea of currents in metric spaces goes back to De Giorgi [13] and has been developed in the pioneering paper by Ambrosio and Kirchheim [4]. It is natural to study k-forms on k-dimensional manifold. However, we model a situation of a metric space on which the question of dimension may be meaningless. We will show that 1-forms and 1-currents can be used to describe a general situation including manifolds of arbitrary dimension.

Definition I.35 (1-current)**.** We denote by $\tilde{\mathcal{D}}(X)$ the set of all Lipschitz functions on X. Let $\mathcal{D}^1(X) = \mathcal{D}(X) \times \tilde{D}(X)$. The elements of $\mathcal{D}^1(X)$ are called *test differential* 1*-forms.* The support of a test differential form (φ, ψ) is defined as the support of the product $\varphi\psi$. The family of all compactly supported test differential 1-forms is denoted by $\mathcal{D}^1_c(X)$.

We say that $(\varphi_j, \psi_j) \xrightarrow{*} (\varphi, \psi)$ (converge weak*) in $\mathcal{D}^1(X)$ if the Lipschitz constants of φ_j, ψ_j and $\mathcal{C}$-norms of φ_j form bounded sequences and $\varphi_j \to \varphi$, $\psi_j \to \psi$ pointwise. The $\xrightarrow{*}$ convergence in $\mathcal{D}^1_c(X)$ requires in addition that there is a compact set containing all supports of (φ_j, ψ_j).

We say that a mapping $\mathcal{T} : \mathcal{D}^1_c(X) \to \mathbb{R}$ is a 1-*current* if the following properties are satisfied:

(C-1) $\mathcal{T}$ is bilinear in variables φ, ψ.

(C-2) $(\varphi_j, \psi_j) \xrightarrow{*} (\varphi, \psi)$ in $\mathcal{D}^1_c(X) \implies \mathcal{T}(\varphi_j, \psi_j) \to \mathcal{T}(\varphi, \psi)$.

(C-3) $\varphi\psi = 0 \implies \mathcal{T}(\varphi, \psi) = 0$.

(C-4) $\mathcal{T}(\varphi, 1) = 0$.

The collection of all 1-currents on X is denoted by $(\mathcal{D}^1_c)'(X)$. The family $(\mathcal{D}^1)'(X)$ of all *convergent currents* on X is defined analogously, with the difference that they are defined on $\mathcal{D}^1(X)$ and continuous with respect to the $\mathcal{D}^1(X)$-convergence.

The family of all $\mathbb{R}^m$-valued 1-currents on X is denoted by $(\mathcal{D}^1_c)'(X; \mathbb{R}^m)$; we define them coordinate-wise.

We say that a 1-current $\mathcal{T}$ has a *locally finite mass* if there exists a Radon measure μ on X such that

$$|\mathcal{T}(\varphi,\psi)| \le |\psi|_{\mathrm{Lip}} \int_X |\varphi|\,d\mu, \qquad (\varphi,\psi) \in \mathcal{D}_c^1(X). \tag{I.18}$$

We also say that $\mathcal{T}$ is *dominated by* μ if (I.18) is satisfied.

The *boundary* of a 1-current $\mathcal{T}$ is defined as

$$\partial\mathcal{T}(\varphi) = \mathcal{T}(1,\varphi), \qquad \varphi \in \mathcal{D}_c(X),$$

it is a metric distribution. We say that $\mathcal{T}$ is *boundary-free* if $\partial\mathcal{T} = 0$.

We use also the alternative (and more intuitive) notation $\mathcal{T}(\varphi\,d\psi)$ for $\mathcal{T}(\varphi,\,\psi)$.

Definition I.36 (Integral with respect to a 1-current)**.** Let $f : X \to \mathbb{R}$ be a function and $\mathcal{F},\mathcal{G} \in (\mathcal{D}_c^1)'(X)$ be 1-currents. We say that $\mathcal{F}$ is an *indefinite integral* of f with respect to $\mathcal{G}$, denoted by $\mathcal{F} = \int f\,\mathrm{d}\mathcal{G}$, if $\mathcal{F}(\cdot,\psi)$ is an indefinite packing integral of f with respect to $\mathcal{G}(\cdot,\psi)$ for each Lipschitz function ψ on X; this means

$$\int f\,\mathrm{d}\mathcal{G}(\varphi\,d\psi) = \int f\,\mathrm{d}\mathcal{G}(\cdot,\psi)(\varphi), \qquad \varphi \in \mathcal{D}_c(X). \tag{I.19}$$

(We consider $\mathcal{G}(\cdot,\psi)$ as an functional on $\mathcal{D}_c(X)$ although strictly formally the domain of this functional may be wider for some choices of ψ, similarly with $\mathcal{F}(\cdot,\psi)$.) The indefinite integral is uniquely determined by $\mathcal{G}$ and f: the formula (I.19) determines the values on the space $\mathcal{D}_c(X) \times \tilde{D}(X)$ which is sequentially weak* dense in $\mathcal{D}_c^1(X)$. Then we can use the (C-2) property of $\mathcal{F}$. The definite integral is defined by

$$\int_X f\,\mathrm{d}\mathcal{G}(d\psi) = \int f\,\mathrm{d}\mathcal{G}(1,\psi)$$

if $\int f\,\mathrm{d}\mathcal{G} \in (\mathcal{D}^1)'(X)$.

In applications we sometimes integrate $\mathbb{R}^m$ valued functions and/or with respect to $\mathbb{R}^d$-valued currents; this is understood coordinate-wise and the result is an m-dimensional vector if $d = 1$ or a d-dimensional vector if $m = 1$. If both $d > 1$, $m > 1$, the most important situation is $d = m$. If $\boldsymbol{\mathcal{G}} \in (\mathcal{D}_c^1)'(X,\mathbb{R}^m)$ and $\boldsymbol{f} : X \to \mathbb{R}^m$ is a vector field, we define the *inner product integral* $\int \boldsymbol{f}\cdot\mathrm{d}\boldsymbol{\mathcal{G}}$ as a 1-current satisfying

$$\int \boldsymbol{f}\cdot\mathrm{d}\boldsymbol{\mathcal{G}}(\varphi\,d\psi) = \int \boldsymbol{f}\cdot\mathrm{d}\boldsymbol{\mathcal{G}}(\cdot,\psi)(\varphi), \qquad \varphi \in \mathcal{D}_c(X),\ \psi \in \tilde{\mathcal{D}}(X).$$

Example I.37. Let $\boldsymbol{f} = (f_1, f_2, f_3)$ be a smooth vector field on a smooth 2-dimensional manifold $\mathcal{M} \subset \mathbb{R}^3$. Let Ω be relatively compact and relatively open subset of $\mathcal{M}$ and Γ be the relative boundary of Ω in $\mathcal{M}$. Let φ be a smooth test function on $\mathcal{M}$ (this means, $\varphi \in \mathcal{C}_c^\infty(\mathcal{M})$). For the discussions on the Stokes' theorem, the following integrals are relevant:

$$\int_\Gamma \varphi(f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3) \tag{I.20}$$

and

$$\int_\Omega \varphi(g_1\,dx_2 \wedge dx_3 + g_2\,dx_3 \wedge dx_1 + g_3\,dx_1 \wedge dx_2), \tag{I.21}$$

where $\boldsymbol{g} = (g_1, g_2, g_3) = \mathbf{curl}\,\boldsymbol{f}$. Consider currents

$$\begin{aligned}\mathcal{T}_i(\varphi\, d\psi) &= \int_{\mathcal{M}} \varphi\, d\psi \wedge dx_i, \qquad i = 1,2,3,\\ \mathcal{T}_i^{\Omega}(\varphi\, d\psi) &= \int_{\Omega} \varphi\, d\psi \wedge dx_i, \qquad i = 1,2,3.\end{aligned}$$

Then

$$\mathcal{T}_i^{\Omega} = \int \chi_{\Omega}\, d\mathcal{T}_i, \qquad i = 1,2,3.$$

For (I.20) we have

$$\int_{\Gamma} \varphi(f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3) = \int \sum_{i=1}^{3} f_i\, \mathrm{d}\mathcal{S}_i(\varphi)$$

where

$$\mathcal{S}_i(\varphi) = \int_{\Gamma} \varphi\, dx_i = \int_{\Omega} d\varphi \wedge dx_i = \partial\mathcal{T}_i^{\Omega}(\varphi).$$

The integral (I.21) can be viewed as

$$\int_{\Omega} \varphi(g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2) = \int \chi_{\Omega}\, \mathrm{d}\mathcal{R}(\varphi)$$

where

$$\begin{aligned}\mathcal{R}(\varphi) &= \int_{\mathcal{M}} \varphi(g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2)\\ &= -\int_{\mathcal{M}} (f_1\, d\varphi \wedge dx_1 + f_2\, d\varphi \wedge dx_2 + f_3\, d\varphi \wedge dx_3)\\ &= -\partial\Big(\int \sum_{i=1}^{3} f_i\, \mathrm{d}\mathcal{T}_i\Big)(\varphi).\end{aligned}$$

Example I.38. Similarly, if $\Omega \subset \mathbb{R}^3$ is an open set with a smooth boundary Γ, $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ is a smooth vector field and $\varphi \in \mathcal{C}_c^{\infty}(\Omega)$, then

$$\int_{\Omega} \varphi(x) \operatorname{div} \mathbf{f}(x)\, dx = -\int \chi_{\Omega}\, \mathrm{d}\partial\Big(\int \sum_{i=1}^{3} f_i\, \mathrm{d}\mathcal{T}_i\Big)(\varphi), \tag{I.22}$$

where now

$$\mathcal{T}_i(\varphi, \psi) = \int_{\Omega} \varphi\, \frac{\partial \psi}{\partial x_i}\, dx, \quad i = 1,2,3.$$

For the boundary integral we have

$$\int_{\Gamma} \varphi(f_1\, dx_2 \wedge dx_3 + f_2\, dx_3 \wedge dx_1 + f_3\, dx_1 \wedge dx_2) = \int \sum_{i=1}^{3} f_i\; \mathrm{d}\partial\Big(\int \chi_{\Omega}\, \mathrm{d}\mathcal{T}_i\Big)(\varphi).$$

This example and the preceding one show that we need to investigate (in the non-smooth case) the criteria for validity of formulas of type

$$\int_X \chi_{\Omega}\, \mathrm{d}\partial\Big(\int \sum_{i=1}^{m} f_i\, \mathrm{d}\mathcal{T}_i\Big) + \int_X \sum_{i=1}^{m} f_i\; \mathrm{d}\partial\Big(\int \chi_{\Omega}\, \mathrm{d}\mathcal{T}_i\Big) = 0. \tag{I.23}$$

Remark I.39. If we sum scalar versions of (I.23), namely

$$\int_X \chi_\Omega \partial\Big(\int f_i \,\mathrm{d}\mathcal{T}_i\Big) + \int_X f_i \;\mathrm{d}\partial\Big(\int \chi_\Omega \,\mathrm{d}\mathcal{T}_i\Big) = 0, \quad i = 1, \dots, m,$$

we obtain

$$\sum_{i=1}^m \int_X \chi_\Omega \partial\Big(\int f_i \,\mathrm{d}\mathcal{T}_i\Big) + \sum_{i=1}^m \int_X f_i \;\mathrm{d}\partial\Big(\int \chi_\Omega \,\mathrm{d}\mathcal{T}_i\Big) = 0, \tag{I.24}$$

which is weaker than (I.23). We will demonstrate the difference on situation of Example I.38. Indeed, note that

$$-\int_X \chi_\Omega \partial\Big(\int f_i \,\mathrm{d}\mathcal{T}_i\Big) = \int_\Omega \varphi(x) \frac{\partial f_i}{\partial x_i}\, dx.$$

Due to cancellation effects, the distribution div $\boldsymbol{f}$ can be meaningful even if the summands $\frac{\partial f_i}{\partial x_i}$ do not make sense. Also for the boundary integration, the sum of integrals is not the same as the integral of a sum. For example, eventual orthogonality of $\boldsymbol{f}$ to the element of boundary integration can override singularity of the integrand.

Proposition I.40. *Let $\mathcal{T}$ be a 1-current dominated by a Radon measure μ. Then there exists a unique bilinear functional $\overline{\mathcal{T}} : L^1(\mu) \times \tilde{\mathcal{D}}(X) \to \mathbb{R}$ such that*

$$\overline{\mathcal{T}} = \mathcal{T} \text{ on } \mathcal{D}_c(X) \times \tilde{\mathcal{D}}(X), \tag{I.25}$$

and

$$\overline{\mathcal{T}}(\cdot, \psi) \in L^1(\mu)^*, \qquad \psi \in \tilde{\mathcal{D}}(X). \tag{I.26}$$

Moreover, given $\psi \in \tilde{\mathcal{D}}(X)$, there exists a unique $\omega_\psi \in L^\infty(\mu)$ such that

$$\overline{\mathcal{T}}(f, \psi) = \int_X \omega_\psi f \, d\mu, \qquad f \in L^1(\mu). \tag{I.27}$$

This ω_ψ satisfies

$$\|\omega_\psi\|_{L^\infty(\mu)} \le |\psi|_{\mathrm{Lip}}. \tag{I.28}$$

The mapping

$$\psi \mapsto \omega_\psi : \quad \tilde{\mathcal{D}}(X) \to L^\infty(X)$$

is linear.

Proof. By Remark I.4, $\mathcal{D}_c(X)$ is dense in $L^1(\mu)$. Let $\psi \in \tilde{D}(X)$. By (I.18), the functional

$$\mathcal{T}(\cdot, \psi) \colon \mathcal{D}_c(X) \to \mathbb{R}$$

is linear and uniformly continuous in the $L^1(\mu)$ norm, so that it can be uniquely extended into a functional $\overline{\mathcal{T}}(\cdot, \psi) \in L^1(\mu)^*$. Then $\overline{\mathcal{T}}$ is bilinear and satisfies (I.25) and (I.26). Also, the estimate $\|\overline{\mathcal{T}}(\cdot, \psi)\|_{L^1(\mu)^*} \le |\psi|_{\mathrm{Lip}}$ follows from (I.18). The representation theorem yields a unique $\omega_\psi \in L^\infty(\mu)$ such that (I.27) is satisfied, and

$$\|\omega_\psi\|_{L^\infty(\mu)} = \|\overline{\mathcal{T}}(\cdot, \psi)\|_{L^1(\mu)^*} \le |\psi|_{\mathrm{Lip}}.$$

Since $\overline{\mathcal{T}}$ is bilinear, the mapping

$$\psi \to \overline{\mathcal{T}}(\cdot, \psi) : \ \tilde{\mathcal{D}}(X) \to L^1(\mu)^*$$

is linear, and thus also the mapping

$$\psi \to \omega_\psi : \ \tilde{\mathcal{D}}(X) \to L^\infty(\mu)$$

is linear. $\square$

Proposition I.41. *Let $\mathcal{T}$ be a 1-current dominated by a Radon measure μ and $f \in L^1_{\mathrm{loc}}(\mu)$. Then the functional $\mathcal{F}$ defined on $\mathcal{D}^1_c(X)$ by*

$$\mathcal{F}(\varphi, \psi) = \int_X f \varphi \omega_\psi \, d\mu, \tag{I.29}$$

where ω_ψ is as in Proposition I.40, defines a 1-current. If $f \in L^1(\mu)$, then (I.29) *defines a convergent 1-current (now $\mathcal{F}$ is defined on $\mathcal{D}^1(X)$).*

Proof. Since the mapping $\psi \mapsto \omega_\psi$ is linear, $\mathcal{F}$ is bilinear. By (C-4) for $\mathcal{T}$, $\mathcal{T}(\cdot, 1) = 0$ and thus $\omega_1 = 0$. It follows that (C-4) is satisfied by $\mathcal{F}$. For verification of (C-2) and (C-3), we distinguish two cases.

Case A: $f \in L^1(\mu)$. Fix $(\varphi, \psi) \in \mathcal{D}^1(X)$ with $\varphi\psi = 0$. By Remark I.4 we can find $f_j \in \mathcal{D}_c(X)$ such that $f_j \to f$ in $L^1(\mu)$. Since $\mathcal{T}$ satisfies (C-3), we have

$$\int_X f_j \varphi \omega_\psi \, d\mu = \mathcal{T}(f_j \varphi, \psi) = 0.$$

The passage to limit yields

$$\mathcal{F}(\varphi, \psi) = 0.$$

This verifies (C-3). For (C-2), let $(\varphi_j, \psi_j) \xrightarrow{*} (\varphi, \psi)$ in $\mathcal{D}^1(X)$. We will show that $\mathcal{F}(\varphi_j, \psi_j) \to \mathcal{F}(\varphi, \psi)$. Choose $\varepsilon > 0$. By Remark I.4, we find $h \in \mathcal{D}_c(X)$ such that

$$\int_X |f - h| \, d\mu < \varepsilon.$$

Using (I.28), we estimate

$$\begin{aligned}
|\mathcal{F}(\varphi_j, \psi_j) - \mathcal{F}(\varphi, \psi)| &\le |\mathcal{F}(\varphi_j, \psi_j) - \mathcal{T}(h\varphi_j, \psi_j)| + |\mathcal{T}(h\varphi_j, \psi_j) - \mathcal{T}(h\varphi, \psi)| \\
&\quad + |\mathcal{T}(h\varphi, \psi) - \mathcal{F}(\varphi, \psi)| \\
&= \left| \int_X (f - h) \omega_{\psi_j} \varphi_j \, d\mu \right| + |\mathcal{T}(h\varphi_j, \psi_j) - \mathcal{T}(h\varphi, \psi)| \\
&\quad + \left| \int_X (f - h) \omega_\psi \varphi \, d\mu \right| \\
&\le |\psi_j|_{\mathrm{Lip}} \|\varphi_j\|_{\mathcal{C}} \int_X |f - h| \, d\mu + |\mathcal{T}(h\varphi_j, \psi_j) - \mathcal{T}(h\varphi, \psi)| \\
&\quad + |\psi|_{\mathrm{Lip}} \|\varphi\|_{\mathcal{C}} \int_X |f - h| \, d\mu.
\end{aligned}$$

In view of the convergence $(\varphi_j, \psi_j) \xrightarrow{*} (\varphi, \psi)$ in $\mathcal{D}^1(X)$, we know that $(|\psi_j|_{\mathrm{Lip}})_j$ and $(\|\varphi_j\|_{\mathcal{C}})_j$ are bounded sequences. The middle term tends to 0 as $\mathcal{T}$ is a 1-current. Hence, we obtain

$$\limsup_{j \to \infty} |\mathcal{F}(\varphi_j, \psi_j) - \mathcal{F}(\varphi, \psi)| \le C\varepsilon,$$

which concludes the proof in case that $f \in L^1(\mu)$.

CASE B: $f \in L^1_{\mathrm{loc}}(\mu)$. Consider $(\varphi, \psi) \in \mathcal{D}^1_c(X)$ such that $\varphi\psi = 0$. Then for each relatively compact μ-measurable set $E \subset X$, $f\chi_E \in L^1(\mu)$ and thus by Case A we have

$$\int_E f\varphi\omega_\psi \, d\mu = 0.$$

It follows that $f\varphi\omega_\psi = 0$ μ-a.e. and thus $\mathcal{F}(\varphi, \psi) = 0$. Hence $\mathcal{F}$ satisfies (C-3). Finally, let $(\varphi_j, \psi_j) \xrightarrow{*} (\varphi, \psi)$ in $\mathcal{D}^1_c(X)$. Then there exists a compact set K such that $\operatorname{spt} \varphi_j\psi_j \subset K$ for each j. Find $\eta \in \mathcal{D}_c(X)$ such that $\eta = 1$ on K. Then, $\varphi_j\psi_j(1 - \eta) = 0$ and by the preceding part of the proof,

$$\mathcal{F}(\varphi_j, \psi_j) = \mathcal{F}(\varphi_j\eta, \psi_j),$$

similarly

$$\mathcal{F}(\varphi, \psi) = \mathcal{F}(\varphi\eta, \psi).$$

Now, it remains to observe that $f\eta \in L^1(\mu)$ and thus by Case A we have

$$\int_X \eta f \omega_{\psi_j} \varphi_j \, d\mu \to \int_X \eta f \omega_\psi \varphi \, d\mu$$

which means

$$\mathcal{F}(\varphi_j\eta, \psi_j) \to \mathcal{F}(\varphi\eta, \psi).$$

This shows that $\mathcal{F}$ satisfies (C-2) and concludes the proof. □

Lemma I.42. *Let $\mathcal{T} \in (\mathcal{D}^1_c)'(X)$ be a 1-current dominated by a Radon measure μ and $W \subset X$ be a relatively compact open set. Let $\psi \mapsto \omega_\psi$ be as in Proposition I.40. If $\psi \in \tilde{\mathcal{D}}(X)$, $\psi = 0$ on $X \setminus W$, then $\omega_\psi = 0$ a.e. on $X \setminus W$.*

Proof. Fix $f \in L^1(\mu)$ with the property that $f = 0$ on W and define $\mathcal{F}$ as in Proposition I.41 by

$$\mathcal{F}(\varphi, \psi) = \int_X f\varphi\omega_\psi \, d\mu.$$

Let $\psi \in \mathcal{D}_c(X)$ be such that $\operatorname{spt} \psi \subset W$. Then there exist $f_j \in \mathcal{D}_c(X)$, $j = 1, 2, \dots$, such that $f_j\psi = 0$ and $f_j \to f$ in $L^1(\mu)$. Then

$$\int_X f_j\varphi\omega_\psi \, d\mu = \mathcal{T}(f_j\varphi, \psi) = 0$$

by (C-3). Passing to the limit we obtain

$$\mathcal{F}(\varphi, \psi) = \int_X f\varphi\omega_\psi \, d\mu = 0.$$

This, however, does not work if merely $\{x : \psi(x) \neq 0\} \subset W$. Therefore we need to approximate ψ. We may assume that $\psi \geq 0$. Set

$$\psi_j = (\psi - 2^{-j})^+, \qquad j = 1, 2, \dots.$$

Then for each j we have $\mathcal{F}(1, \psi_j) = 0$ by the preceding part of the proof. Since $(1, \psi_j) \xrightarrow{*} (1, \psi)$ in $\mathcal{D}^1_c(X)$ and $\mathcal{F}$ is a 1-current by Proposition I.41, passing to the limit we obtain

$$\overline{\mathcal{T}}(f, \psi) = \mathcal{F}(1, \psi) = 0,$$

where $\overline{\mathcal{T}}$ is the extension of $\mathcal{T}$ as in Proposition I.40. Now, consider ψ fixed and f variable. Recall that ω_ψ is defined as the L^∞-representation of the functional $\overline{\mathcal{T}}(\cdot,\psi) \in L^1(\mu)^*$. From the preceding part of the proof we see that

$$f \in L^1(\mu),\ f = 0 \text{ on } W \implies \overline{\mathcal{T}}(f,\psi) = 0.$$

Using the properties of the L^∞-representation we deduce that

$$\omega_\psi = 0 \text{ a.e. on } X \setminus W.$$

□

Proposition I.43. *Let $\mathcal{T} \in (\mathcal{D}_c^1)'(X)$ be a 1-current dominated by μ. Suppose that $f \in L^1_{\text{loc}}(\mu)$. Then the functional*

$$\mathcal{F} : (\varphi,\psi) \mapsto \int_X f\varphi\omega_\psi\, d\mu, \qquad (\varphi,\psi) \in \mathcal{D}_c^1(X),$$

where ω_ψ is as in Proposition I.40, is the indefinite UC integral of f with respect to $\mathcal{T}$.

Proof. In Propositions I.40 and I.41 we have verified that $\mathcal{F}$ is a 1-current and that

$$\mathcal{T}(\varphi,\psi) = \overline{\mathcal{T}}(\varphi,\psi) = \int_X \varphi\,\omega_\psi\, d\mu, \qquad \varphi \in \mathcal{D}_c(X),\ \psi \in \tilde{\mathcal{D}}(X).$$

By Theorem I.30, $\mathcal{F}$ is the indefinite *UC* integral of f with respect to $\mathcal{T}$. □

Proposition I.44. *Let $\mathcal{T} \in (\mathcal{D}_c^1)'(X)$ be a 1-current dominated by μ. Suppose that $f \in L^1_{\text{loc}}(\mu)$ and $(\varphi,\psi) \in \mathcal{D}(X) \times \mathcal{D}_c(X)$. Then*

$$\left|\int f\, \mathrm{d}\mathcal{T}(\varphi\, d\psi)\right| \le |\psi|_{\text{Lip}} \int_W |f\varphi|\, d\mu, \tag{I.30}$$

where $W = \{x \in X : \psi(x) \neq 0\}$.

Proof. By Proposition I.40, Proposition I.43, and Lemma I.42, there exists ω_ψ in $L^\infty(\mu)$ such that

$$\|\omega_\psi\|_{L^\infty} \le |\psi|_{\text{Lip}}, \qquad \omega_\psi = 0 \text{ on } X \setminus W$$

and

$$\int f\, \mathrm{d}\mathcal{T}(\varphi\, d\psi) = \int_X f\varphi\omega_\psi\, d\mu.$$

Hence the estimate (I.30) easily follows. □

Remark I.45. For simplicity, the results of Proposition I.40–Proposition I.44 have been formulated in the scalar setting, but it is straightforward that $\mathbb{R}^m$-valued versions of these results hold with the same proofs as well.

I.6 Integration by parts

I.6.1 Pointwise BV functions

In the metric measure space setting, BV functions can be defined as functions, for which a Poincaré-type inequality holds. In this approach, we say that $u \in L^1(X, \lambda)$ is a BV *function* if there exists a constant $\sigma \geq 1$ and a Radon measure ν such that the inequality

$$\int_{B(x,r)} |u - \bar{u}(x,r)| \, d\lambda \leq r\nu(B(x, \sigma r)) \tag{I.31}$$

holds for each ball $B(x,r) \subset X$. Here $\bar{u}(x,r)$ is the integral average of u over $B(x,r)$. Recall that λ is the underlying doubling measure, see Section I.3. The measure ν appearing in (I.31) has the role of an "upper gradient" to u. Self-improving properties of Sobolev-Poincaré inequalities allow us to improve the inequality (I.31) to

$$\left(\fint_{B(x,r)} |u - \bar{u}(x,r)|^p \, d\lambda \right)^{1/p} \leq Cr \, \frac{\nu(B(x,\sigma r))}{\lambda(B(x,\sigma r))} \tag{I.32}$$

with some $p > 1$ and a constant C. For the BV theory in the metric space setting see [43], [1], [2], [7], [25].

We obtain a much more wider class of functions if we let (I.32) hold for small balls only and with the constant C depending on the point.

Definition I.46. We say that $u \in L^1_{\mathrm{loc}}(X, \lambda)$ is a *pointwise* BV *function* (we abbreviate a PBV function) if there exist a constant $\sigma \geq 1$, an exponent $p > 1$ and a Radon measure ν such that the asymptotic behavior

$$\limsup_{r \to 0_+} \frac{\left(\fint_{B(x,r)} |u - \bar{u}(x,r)|^p \, d\lambda \right)^{\frac{1}{p}} \lambda(B(x,\sigma r))}{r \, \nu(B(x,\sigma r))} < \infty \tag{I.33}$$

holds for each $x \in X$.

The measures serving for (I.33) are called *pointwise upper gradients* to u. The class of all pointwise upper gradients to u is denoted by $\mathrm{PUG}(u)$.

Example I.47. A typical example of a PBV function is a Sobolev function on an Euclidean domain. However, it is easily seen that also each pointwise differentiable function is PBV, with the Lebesgue measure serving as an pointwise upper gradient. Of course, not all pointwise derivatives are Lebesgue integrable; recall that this was one of main motivation for the theory of nonabsolutely convergent integrals.

Example I.48. Another typical example of a BV (and thus PBV) function in the Euclidean setting is a characteristic function of a set of finite perimeter. However, if the (topological) boundary of a set $G \subset \mathbb{R}^n$ is countably $(n-1)$-rectifiable (this means, covered by a countable union of Lipschitz images of $\mathbb{R}^{n-1}$), then χ_G is also a PBV function. In particular, we may consider the case that there exists an exterior normal vector at each boundary point x to G, see [3, Theorem 2.61] for the proof of countable rectifiability in such a situation. Note that we do not require the normal vector to depend continuously on x and thus the perimeter of G may happen to be locally infinite.

Motivated by Examples I.37 and I.38, we are interested in validity of formulas of type (I.23). We write this aim in a more symmetric form

$$\partial\Big(\int\sum_{i=1}^{m}\sum_{j=1}^{n} f_i g_j \,\mathrm{d}\mathcal{T}_{ij}\Big)$$
$$= \int\sum_{j=1}^{n} g_j \,\mathrm{d}\partial\Big(\int\sum_{i=1}^{m} f_i \,\mathrm{d}\mathcal{T}_{ij}\Big) + \int\sum_{i=1}^{m} f_i \;\mathrm{d}\partial\Big(\int\sum_{j=1}^{n} g_i \,\mathrm{d}\mathcal{T}_{ij}\Big),$$

where $\mathcal{T}_{ij}$ are 1-currents and f_i and g_j are functions. Notice that the metric distribution on the left vanishes at $\varphi = 1$, so that the sum of the integrals on the right vanishes as well if they are considered as definite.

Definition I.49 (Concepts of the Lebesgue differentiation theory)**.** By the Lebesgue decomposition theorem, each Radon measure μ on X can be decomposed as $\mu_a + \mu_s$ where μ_a is the *absolutely continuous part* and μ_s is the *singular part*, both with respect to λ. Since λ is doubling, the Lebesgue differentiation theory is valid (see e.g. [5, Theorem 5.2.6]) and λ-a.e. point is a *Lebesgue point* for the Radon-Nikodým derivative $h = \frac{d\mu_a}{d\lambda}$, this means that

$$\lim_{r\to 0+} \fint_{B(x,r)} |h - h(x)|\, d\lambda = 0.$$

Clearly, each Lebesgue point is a *weak Lebesgue point*, this means that

$$\lim_{r\to 0+} \bar{h}(x,r) = h(x), \quad \text{where} \quad \bar{h}(x,r) = \fint_{B(x,r)} h\, d\lambda.$$

Recall also the *maximal function* of the measure μ:

$$M\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}.$$

By [5, Theorem 5.2.4], the maximal function of a Radon measure μ with respect to the doubling measure λ is finite λ-a.e.

Lemma I.50. *Let μ be a Radon measure on X and $N \subset X$ be a μ-null set. Then there exist a Radon measure μ^* on X which is absolutely continuous with respect to μ and a lower semicontinuous function w on X such that $w \geq 1$, $w = \infty$ on N and $d\mu^* = w\, d\mu$.*

Proof. For each $j = 1, 2, \dots$ we find an open set $W_j \subset X$ such that $N \subset W_j$ and $\mu(W_j) < 4^{-j}$. Then the function

$$w = 1 + \sum_j 2^j \chi_{W_j}$$

and μ^* determined by $d\mu^* = w\, d\mu$ obviously have the required properties. $\square$

Lemma I.51. *Let u be a bounded PBV function on X, $\nu \in \mathrm{PUG}(u)$ and $x \in X$. If $M\nu(x) < \infty$, then*

$$\lim_{r\to 0_+} \frac{1}{r} \fint_{B(x,r)} |u - \bar{u}(x,r)|^2\, d\lambda = 0.$$

Proof. Consider $p > 1$ such that

$$\limsup_{r\to 0_+} \frac{\left(\fint_{B(x,r)} |u-\bar u(x,r)|^p\,d\lambda\right)^{\frac1p}\lambda(B(x,\sigma r))}{r\,\nu(B(x,\sigma r))} < \infty, \qquad x\in X.$$

Pick $x \in X$ such that $M\nu(x) < \infty$. Then

$$\limsup_{r\to 0_+} \frac1r\Big(\fint_{B(x,r)} |u-\bar u(x,r)|^p\,d\lambda\Big)^{\frac1p} < \infty.$$

Appealing to the Hölder inequality we may assume that $p < 2$. Since u is bounded,

$$\frac1r\fint_{B(x,r)}|u-\bar u(x,r)|^2 \le \frac{C}{r}\|u\|_\infty^{2-p}\fint_{B(x,r)}|u-\bar u(x,r)|^p \le Cr^{p-1}\to 0 \qquad \text{as } r\to 0_+,$$

where the last constant depends on u. □

Remark I.52. We do not know whether the conclusion of Lemma I.51 continues to hold if we allow $p = 1$ in Definition I.46.

Theorem I.53 (Integration by parts)**.** *Let $\boldsymbol f = (f_1,\dots,f_m)$, $\boldsymbol g = (g_1,\dots,g_n)$, be bounded (vector-valued) PBV functions on X, $\alpha \in \mathrm{PUG}(\boldsymbol f)$, $\beta\in\mathrm{PUG}(\boldsymbol g)$. Suppose that β_s and α_s are mutually singular, that $\boldsymbol f$ has weak Lebesgue points β-a.e. and $\boldsymbol g$ has weak Lebesgue points α-a.e. Let $\mathcal T_{ij}$ be boundary-free* 1*-currents on X dominated by λ, $i = 1,\dots,m$, $j = 1,\dots,n$. Then*

$$\begin{aligned}&\partial\Big(\int\sum_{i=1}^m\sum_{j=1}^n f_ig_j\,\mathrm d\mathcal T_{ij}\Big)\\&\qquad=\int\sum_{j=1}^n g_j\,\mathrm d\partial\Big(\int\sum_{i=1}^m f_i\,\mathrm d\mathcal T_{ij}\Big)+\int\sum_{i=1}^m f_i\,\mathrm d\partial\Big(\int\sum_{j=1}^n g_i\,\mathrm d\mathcal T_{ij}\Big)\end{aligned}\tag{I.34}$$

holds if at least one of the ("exterior") indefinite integrals on the right makes sense.

Proof. We write $\boldsymbol{\mathcal F} = (\mathcal F_1,\dots,\mathcal F_n)$, $\boldsymbol{\mathcal G} = (\mathcal G_1,\dots,\mathcal G_m)$,

$$\begin{aligned}\mathcal F_j &= \int\sum_{i=1}^m f_i\,\mathrm d\mathcal T_{ij}, \quad j=1,\dots,n,\\ \mathcal G_i &= \int\sum_{j=1}^n g_j\,\mathrm d\mathcal T_{ij}, \quad i=1,\dots,m,\\ \mathcal R &= \int\sum_{i=1}^m\sum_{j=1}^n f_ig_j\,\mathrm d\mathcal T_{ij}.\end{aligned}$$

Assume that $\int \boldsymbol g\cdot \mathrm d\partial\boldsymbol{\mathcal F}$ makes sense. We want to show that $\partial\mathcal R - \int\boldsymbol g\cdot\mathrm d\partial\boldsymbol{\mathcal F}$ is an indefinite *UC* integral of $\boldsymbol f$ with respect to $\partial\boldsymbol{\mathcal G}$. We fix $x\in X$ and $r>0$. Then

$$\begin{aligned}&\Big\|\partial\mathcal R-\int\boldsymbol g\cdot\mathrm d\partial\boldsymbol{\mathcal F}-\boldsymbol f(x)\cdot\partial\boldsymbol{\mathcal G}\Big\|_{x,r}\\&\quad\le\|\partial\mathcal R-\boldsymbol g(x)\cdot\partial\boldsymbol{\mathcal F}-\boldsymbol f(x)\cdot\partial\boldsymbol{\mathcal G}\|_{x,r}\\&\qquad+\Big\|\int\boldsymbol g\cdot\mathrm d\partial\boldsymbol{\mathcal F}-\boldsymbol g(x)\cdot\partial\boldsymbol{\mathcal F}\Big\|_{x,r}.\end{aligned}$$

For the second term it is enough to use the assumption of integrability of $\boldsymbol{g}$ with respect to $\partial\boldsymbol{\mathcal{F}}$. We need to estimate the first norm, namely the norm of

$$\begin{aligned}
&\partial\mathcal{R} - \boldsymbol{g}(x)\cdot\partial\boldsymbol{\mathcal{F}} - \boldsymbol{f}(x)\cdot\partial\boldsymbol{\mathcal{G}}\\
&\quad= \partial\int\sum_{i=1}^{m}\sum_{j=1}^{n}(f_i - f_i(x))(g_j - g_j(x))\,\mathrm{d}\mathcal{T}_{ij}\\
&\quad= \partial\int\sum_{i=1}^{m}\sum_{j=1}^{n}(f_i - \bar{f}_i(x,r))(g_j - \bar{g}_j(x,r))\,\mathrm{d}\mathcal{T}_{ij}\\
&\qquad+ \partial\int\sum_{i=1}^{m}\sum_{j=1}^{n}(f_i - \bar{f}_i(x,r))(\bar{g}_j(x,r) - g_j(x))\,\mathrm{d}\mathcal{T}_{ij}\\
&\qquad+ \partial\int\sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{f}_i(x,r)) - f_i(x))(g_j - \bar{g}_j(x,r))\,\mathrm{d}\mathcal{T}_{ij}\\
&\qquad+ \partial\int\sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{f}_i(x,r)) - f_i(x))(\bar{g}_j(x,r) - g_j(x))\,\mathrm{d}\mathcal{T}_{ij}\\
&\quad= \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4,
\end{aligned}$$

where, for $i = 1,\dots,m$, $j = 1,\dots,n$,

$$\bar{f}_i(x,r) = \fint_{B(x,r)} f_i\,d\lambda,\quad \bar{g}_j(x,r) = \fint_{B(x,r)} g_j\,d\lambda.$$

Now, we are going to estimate the expressions $\mathcal{S}_1$–$\mathcal{S}_4$ in terms of the pointwise upper gradients of $\boldsymbol{f}$ and $\boldsymbol{g}$. By the singularity assumption, there exist Borel sets P, Q, R such that $X = P\cup Q\cup R$ and

$$\alpha(Q) = \beta(P) = \alpha_s(R) = \beta_s(R) = \lambda(P\cup Q) = 0.$$

Further, there exist an α-null set S, a β-null set T and a λ-null set $N\subset R$ such that f has weak Lebesgue points everywhere on $X\setminus T$, g has weak Lebesgue points everywhere on $X\setminus S$, $M\alpha<\infty$ and $M\beta<\infty$ on $R\setminus N$. By Lemma I.50, there exist measures α^*, β^*, λ^* and lower semicontinuous functions a, b, w such that $a=\infty$ on $Q\cup S\cup N$, $b=\infty$ on $P\cup T\cup N$, $w=\infty$ on N and

$$d\alpha^* = a\,d\alpha,\quad d\beta^* = b\,d\beta,\quad d\lambda^* = w\,d\lambda.$$

We will use Proposition I.44 to estimate expressions $\mathcal{S}_1$–$\mathcal{S}_4$. Choose $x\in X$, $r>0$ and $\varphi\in\mathcal{D}(x,r)$. Then

$$\begin{aligned}
|\mathcal{S}_1(\varphi)| &= \Big|\partial\int\sum_{i=1}^{m}\sum_{j=1}^{n}(f_i - \bar{f}_i(x,r))(g_j - \bar{g}_j(x,r)\,\mathrm{d}\mathcal{T}_{ij}\,(\varphi)\Big|\\
&= \Big|\int\sum_{i=1}^{m}\sum_{j=1}^{n}(f_i - \bar{f}_i(x,r))(g_j - \bar{g}_j(x,r))\,\mathrm{d}\mathcal{T}_{ij}\,(d\varphi)\Big|\\
&\le C|\varphi|_{\mathrm{Lip}}\int_{B(x,r)}|\boldsymbol{f} - \bar{\boldsymbol{f}}(x,r)||\boldsymbol{g} - \bar{\boldsymbol{g}}(x,r)|\,d\lambda\\
&\le C|\varphi|_{\mathrm{Lip}}\Big(\int_{B(x,r)}|\boldsymbol{f} - \bar{\boldsymbol{f}}(x,r)|^2\,d\lambda\Big)^{\frac12}\Big(\int_{B(x,r)}|\boldsymbol{g} - \bar{\boldsymbol{g}}(x,r)|^2\,d\lambda\Big)^{\frac12}.
\end{aligned}$$

By Lemma I.51

$$\lim_{r\to0_+}\frac{1}{r}\fint_{B(x,r)}|\boldsymbol{f} - \bar{\boldsymbol{f}}(x,r)|^2\,d\lambda = 0$$

on $\{x \in X : M\alpha(x) < \infty\}$, similarly

$$\lim_{r \to 0_+} \frac{1}{r} \fint_{B(x,r)} |\boldsymbol{g} - \bar{\boldsymbol{g}}(x,r)|^2 \, d\lambda = 0$$

on $\{x \in X : M\beta(x) < \infty\}$. This yields a control

$$\lim_{r \to 0_+} \frac{\|\mathcal{S}_1\|_{x,r}}{r\lambda(B(x,r))} = 0$$

if $x \in R \setminus N$. If $x \in N \cup Q$ and p is as in Definition I.46, then (with C depending on $\|g\|_\infty$)

$$\begin{aligned}
&\int_{B(x,r)} |\boldsymbol{f} - \bar{\boldsymbol{f}}(x,r)||\boldsymbol{g} - \bar{\boldsymbol{g}}(x,r)| \, d\lambda \\
&\quad \le C\lambda(B(x,r)) \fint_{B(x,r)} |\boldsymbol{f} - \bar{\boldsymbol{f}}(x,r)| \, d\lambda \\
&\quad \le C\lambda(B(x,r)) \Big(\fint_{B(x,r)} |\boldsymbol{f} - \bar{\boldsymbol{f}}(x,r)|^p \, d\lambda \Big)^{\frac{1}{p}} \\
&\quad \le Cr\alpha(B(x,\sigma r))
\end{aligned}$$

and

$$\lim_{r \to 0_+} \frac{\alpha(B(x,\sigma r))}{\alpha^*(B(x,\sigma r))} = 0.$$

If $x \in P$, then the situation is similar as if $x \in Q$, we interchange the role of f and g and of α and β.

For $\mathcal{S}_2$ we have the estimate

$$\begin{aligned}
|\mathcal{S}_2(\varphi)| &= \Big| \partial \int \sum_{i=1}^m \sum_{j=1}^n (f_i - \bar{f}_i(x,r))(\bar{g}_j(x,r) - g_j(x)) \, \mathrm{d}\mathcal{T}_{ij} \, (\varphi) \Big| \\
&= \Big| \int \sum_{i=1}^m \sum_{j=1}^n (f_i - \bar{f}_i(x,r))(\bar{g}_j(x,r) - g_j(x)) \, \mathrm{d}\mathcal{T}_{ij} \, (d\varphi) \Big| \\
&\le C|\varphi|_{\mathrm{Lip}} |\bar{\boldsymbol{g}}(x,r) - \boldsymbol{g}(x)| \int_{B(x,r)} |\boldsymbol{f} - \bar{\boldsymbol{f}}(x,r)| \, d\lambda \\
&\le C|\varphi|_{\mathrm{Lip}} |\bar{\boldsymbol{g}}(x,r) - \boldsymbol{g}(x)| r\alpha(B(x,\sigma r)).
\end{aligned}$$

Then we can use the fact that

$$|\bar{\boldsymbol{g}}(x,r) - \boldsymbol{g}(x)| \, \frac{\alpha(B(x,\sigma r))}{\alpha^*(B(x,\sigma r))} \to 0.$$

Indeed, either x is a weak Lebesgue point for $\boldsymbol{g}$ or $a(x) = \infty$.

The term $\mathcal{S}_3$ behaves similarly to $\mathcal{S}_2$. Finally, since $\mathcal{T}$ is boundary-free, $\mathcal{S}_4$ vanishes.

□

Corollary I.54. Let $\boldsymbol{f} = (f_1, \dots, f_m)$, $\boldsymbol{g} = (g_1, \dots, g_n)$, be bounded (vector-valued) *PBV* functions on X, $\alpha \in \mathrm{PUG}(\boldsymbol{f})$, $\beta \in \mathrm{PUG}(\boldsymbol{g})$. Suppose that $f_i g_j \in L^1(X, \lambda)$, $i = 1, \dots, m$, $j = 1, \dots, n$, β_s and α_s are mutually singular, that $\boldsymbol{f}$ has weak Lebesgue points β-a.e. and $\boldsymbol{g}$ has weak Lebesgue points α-a.e. Let $\mathcal{T}_{ij}$ be boundary-free 1-currents on X dominated by λ, $i = 1, \dots, m$, $j = 1, \dots, n$. Then

$$\int_X \sum_{j=1}^n g_j \, \mathrm{d}\partial \Big(\int \sum_{i=1}^m f_i \, \mathrm{d}\mathcal{T}_{ij} \Big) = - \int_X \sum_{i=1}^m f_i \, \mathrm{d}\partial \Big(\int \sum_{j=1}^n g_i \, \mathrm{d}\mathcal{T}_{ij} \Big),$$

if at least one of these integrals makes sense.

Proof. By Propositions I.41 and I.43, the integral $\mathcal{S} := \int \sum_{i=1}^{m} \sum_{j=1}^{n} f_i g_j \,\mathrm{d}\mathcal{T}_{ij}$ defines a convergent 1-current. By (C-4), $\partial\mathcal{S}(1) = \mathcal{S}(1,1) = 0$. Hence the result follows from Theorem I.53. □

I.6.2 Gauss-Green-Stokes formula

A prevalent motivation for investigation of nonabsolutely convergent integrals is an effort to find a general setting for formulae of integral calculus like the Gauss-Green (divergence) theorem or Stokes theorem.

Within the framework absolutely convergent integration, the Gauss-Green formula for BV-sets (or, sets of finite perimeter) is the ultimate version. See e.g. [3] [16], [52] for exposition of BV-theory and historical comments.

However, only the non-absolutely convergent integrals can integrate all pointwise derivatives. A version of Stokes formula with nonabsolutely convergent integration over manifolds with boundaries is in [29].

The Pfeffer integral allows to prove Gauss-Green formula for sets of finite perimeter, where the "interior" integral of divergence is non-absolutely convergent [46], [48], [14].

Our aim is to allow also the "boundary integral" be non-absolutely convergent. Various abstract theories allow us to study the Stokes theorem beyond rectifiable sets. For different approaches see [51], [18], [23], [24], [49]. The generalized integrals can be defined by duality or by approximation. Our integrals are not as general, but look more as genuine integrals.

A version of the Gauss-Green theorem for the UC integral in the Euclidean setting is proposed in [41].

Theorem I.55 (Stokes theorem)**.** *Let $G \subset X$ be PBV set (this means, χ_G is a PBV function). Let $\boldsymbol{f}\colon X \to \mathbb{R}^m$ be a bounded PBV function, $\alpha \in \mathrm{PUG}(\boldsymbol{f})$, $\beta \in \mathrm{PUG}(\chi_G)$. Suppose that β_s and α_s are mutually singular, that $\boldsymbol{f}$ has weak Lebesgue points β-a.e. and $\chi_{_G}$ has weak Lebesgue points α-a.e. Let $\boldsymbol{\mathcal{T}} \in \mathcal{D}_1'(X, \mathbb{R}^m)$ be a boundary-free 1-current on X dominated by λ. Assume that $\int_G |\boldsymbol{f}|\, d\lambda$ converges. Then*

$$\int_G \mathrm{d}\partial\Big(\int \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{\mathcal{T}}\Big) = -\int_X \boldsymbol{f} \cdot \mathrm{d}\partial\Big(\int \chi_{_G}\, \mathrm{d}\boldsymbol{\mathcal{T}}\Big)$$

if at least one of these integrals makes sense.

Proof. It is enough to set $g = \chi_G$ in Corollary I.54. □

Remark I.56. As a special, "absolutely convergent" case we obtain the following situation:

Let $\boldsymbol{f} \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ be a bounded vector field and $G \subset \mathbb{R}^n$ be a set of finite perimeter. Then we can take α to be absolutely continuous with respect to the Lebesgue measure (say, with density $|D\boldsymbol{f}|$) and β to be the perimeter measure (which is the total variation of $D\chi_G$). By results of Federer, Fleming and Ziemer, [18], [20], [19], see also [16, 4.8., 5.6.3], $\boldsymbol{f}$ has Lebesgue points $\mathscr{H}^{n-1}$-a.e. and thus β-a.e. On the other hand, by the Lebesgue density theorem, a.e. point of G is a density point for G and a.e. point of $\mathbb{R}^n \setminus G$ is a density point for $\mathbb{R}^n \setminus G$.

Hence, χ_G has Lebesgue points a.e. with respect to the Lebesgue measure and thus α-a.e. This absolutely convergent case is not new, but in view of Examples I.47 and I.48, it is clear that our result covers much more situations.

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II. BV-packing integral in $\mathbb{R}^n$

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abstract. We introduce new integrals (called packing $\mathcal{R}$ and $\mathcal{R}^*$ integrals) which combine advantages of integrals developed by Pfeffer [17], Malý [12], Kuncová and Malý [10] and Malý and Pfeffer [13]. We prove Gauss-Green theorem in generality of the new integrals and provide comparison with the integrals mentioned above and some others (like MC_α by Ball and Preiss [2]).

II.1 Introduction

The Gauss-Green divergence theorem

$$\int_A \operatorname{div} \mathbf{u}(x)\, dx = \int_{\partial_* A} \mathbf{u} \cdot \boldsymbol{\nu}_A \, d\mathscr{H}^{n-1} \tag{II.1}$$

holds whenever $A \subset \mathbb{R}^n$ is a bounded BV set (or, in another terminology, a bounded set of finite perimeter) and $\mathbf{u} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Here, $\partial_* A$ is the essential boundary and $\boldsymbol{\nu}_A$ is the measure-theoretic unit exterior normal. This setting and its history can be found e.g. in [1]. If we want to allow discontinuous derivatives, routine approximation arguments give (II.1) if $\mathbf{u} \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\operatorname{div} \mathbf{u}(x) \in L^1(\mathbb{R}^n)$. Beyond Lebesgue integrability of $\operatorname{div} \mathbf{u}(x)$, a natural idea is to consider the divergence in the sense of distributions. Particularly deep results have been obtained for divergence measure vector fields, see e.g. Chen, Torres and Ziemer [5], Ziemer [22] or Šilhavý [19, 20, 21].

We pursue another direction. If $\mathbf{u}$ is differentiable, the divergence formula still holds even if the divergence is not Lebesgue integrable. This phenomenon indicates that the L^1 setting is not the ultimate generality if we want to consider the divergence as *a pointwise function*. Such a divergence still plays the role of divergence in the sense of distributions, but the task is to what extent non-absolutely integrable pointwise functions can be represented as distributions. The problem exists already in the one-dimensional case where it has been solved by the Denjoy-Perron integral. The multidimensional case has been treated by many authors, among the most important contribution we mention [9, 14, 7]. The most important progress in this direction has been done by Pfeffer [17], who developed a theory which can be used for the divergence theorem on BV sets. In his setting, indefinite integral is a function on BV sets, so that the definite integral on the left of (II.1) is the evaluation of the indefinite integral at A. An interesting extension has been introduced by Pfeffer and Malý in [13]. Their effort leads to the $\mathcal{R}^*$ integral, which is stable under reasonable operations and has a rich family of integrable functions. In particular, the $\mathcal{R}^*$ integral includes Pfeffer's $\mathcal{R}$ integral [17] and the 1-dimensional Henstock-Kurzweil integral.

In a series of papers [10, 12, 8], a new non-absolutely convergent integral with respect to distributions, called *packing integral*, has been introduced. Since main motivation comes from the divergence theorem and related results again, it is natural to ask on comparison of this integral with Pfeffer's approach. In its original setting, the indefinite packing integral is a functional on smooth (or Lipschitz) test functions and its evaluation at BV sets does not make sense. Therefore, the definite integral on the left of (II.1) is the evaluation of the indefinite integral of $\chi_A \operatorname{div} \mathbf{u}$ at a test function which is 1 on a neighborhood of ∂A.

The Pfeffer integral (one of the equivalent versions) is based on Riemann-type sums

$$\sum_{i=1}^{m} \Big|\mathcal{F}(E_i) - f(x_i)\mathcal{L}(E_i)\Big|$$

where $E_i \subset \mathbb{R}^n$ are disjointed BV sets, $x_i \in \mathbb{R}^n$ are tags, $\mathcal{L}$ is Lebesgue measure and $\mathcal{F}$ is the candidate for the indefinite integral. In our setting, we also use sums

$$\sum_{i=1}^{m} q_{x_i,r_i}(\mathcal{F} - f(x_i)\mathcal{L}) \tag{II.2}$$

where $(q_{x,r})_{x,r}$ is a system of suitable seminorms.

Let $A \subset \mathbb{R}^n$ be a bounded BV set. Suppose that $\mathbf{u} \in C(\mathbb{R}^n, \mathbb{R}^n)$ and the indefinite packing integral of a function f is the flux of $\mathbf{u}$, so that $f = \operatorname{Div} \mathbf{u}$ in a general sense. We would be happy to see that

$$\int_A \operatorname{Div} \mathbf{u}(x)\, dx = \int_{\partial_* A} \mathbf{u} \cdot \boldsymbol{\nu}\, d\mathscr{H}^{n-1},$$

where the integral on the left means the integration of $f\chi_A$. (In other words, the characteristic function of A acts as a multiplier for the integration of f.) However, in the setting of [10] it is not clear how to estimate the sums (II.2) (and it is probably impossible without additional hypotheses). It helps if we can omit x_i belonging to a small set, say of σ-finite $\mathscr{H}^{n-1}$ Hausdorff measure, namely, just $\partial_* A$. This change of definition requires the indefinite integral to be a *charge*, a functional on $BV \cap L^\infty$ functions continuous with respect to a convergence specified below. Charges can be represented as functions on BV sets, and by this series of thoughts we recover most ingredients of Pfeffer's setting.

In this paper we present modifications of the packing integral which contains Pfeffer's $\mathcal{R}$ integral and Pfeffer's and Malý's $\mathcal{R}^*$ integral. We apply the new integrals to obtain more general versions of the divergence theorem. In the end we discuss the relationships between particular integrals including the one-dimensional Henstock-Kurzweil-Stieltjes integral and MC_α integral.

II.2 Notation and Preliminaries

Notation II.1. Let E be a subset of $\mathbb{R}^n$. Then $d(E)$ denotes the diameter of E, i.e.

$$d(E) = \sup\{|y - x|; x, y \in E\}.$$

Let $x \in \mathbb{R}^n$ and $r > 0$. Then $B(x, r)$ denotes the open ball

$$B(x, r) = \{y \in \mathbb{R}^n; |y - x| < r\}$$

and $\bar{B}(x, r)$ denotes the closed ball

$$\bar{B}(x, r) = \{y \in \mathbb{R}^n; |y - x| \le r\}.$$

The Lebesgue measure of E is denoted by $|E|$ or $\mathcal{L}(E)$.

Definition II.2. We say, that measurable sets A and B are *equivalent* (or A and B belong to the same *equivalence class*) if $|A \triangle B| = 0$, where $A \triangle B$ denotes the symmetric difference of the sets A and B.

Definition II.3. Let $s \geq 0$. The *s-dimensional outer Hausdorff measure* of a set $E \subset \mathbb{R}^n$ is defined as $\mathscr{H}^s(E) = \lim_{\delta \to 0+} \mathscr{H}^s_\delta(E)$, where

$$\mathscr{H}^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} \alpha_s \left(\frac{\operatorname{diam}(C_i)}{2} \right)^s ; C_i \subset \mathbb{R}^n, E \subset \bigcup_{i=1}^{\infty} C_i, \operatorname{diam}(C_i) < \delta \right\}$$

and $\alpha_s = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)}$.

Proposition II.4. *Let $A \subset \mathbb{R}^n$ be a set and let $\varphi : A \to \mathbb{R}^n$ be a Lipschitz mapping. Then $\mathscr{H}^{n-1}(\varphi(A)) \leq (\operatorname{Lip} \varphi)^{n-1} \mathscr{H}^{n-1}(A)$.*

Proof. For the proof and further details see [6, Section 2.4.1]. ☐

Definition II.5. Let $A \subset \mathbb{R}^n$ be a measurable set and let $x \in \mathbb{R}^n$. Then we define the *lower density* of A at x as

$$\underline{\Theta}(A, x) := \liminf_{r \to 0+} \frac{|A \cap B(x,r)|}{|B(x,r)|}$$

and the *upper density* of A at x as

$$\overline{\Theta}(A, x) := \limsup_{r \to 0+} \frac{|A \cap B(x,r)|}{|B(x,r)|}.$$

The *essential closure* $\operatorname{cl}_* A$, *essential interior* $\operatorname{int}_* A$ and *essential boundary* $\partial_* A$ are then defined as

$$\operatorname{cl}_* A = \{x \in \mathbb{R}^n ; \overline{\Theta}(A, x) > 0\},$$

$$\operatorname{int}_* A = \{x \in \mathbb{R}^n ; \underline{\Theta}(A, x) = 1\}$$

and

$$\partial_* A = \operatorname{cl}_* A \setminus \operatorname{int}_* A.$$

Definition II.6. We say that a measurable set $A \subset \mathbb{R}^n$ is *admissible* if $\operatorname{int}_* A \subset A \subset \operatorname{cl}_* A$.

Remark II.7. Our definition of admissible set differs from that used by Malý and Pfeffer in [13], according to which ∂A is required to be compact.

Remark II.8. Let A, A' be measurable sets such that $|A \triangle A'| = 0$. Then $\operatorname{cl}_* A = \operatorname{cl}_* A'$, $\operatorname{int}_* A = \operatorname{int}_* A'$ and $\partial_* A = \partial_* A'$.

Hence, for every bounded measurable set A we can find an admissible set A' such that $|A \triangle A'| = 0$.

II.3 BV sets and charges

In this section we will present some basic facts about spaces of sets of bounded variation (BV sets) and about charges which will be essential in further definitions. For details see [17], [16] and [4].

Definition II.9. Let $U \subset \mathbb{R}^n$ be an open set. For a measurable set $E \subset \mathbb{R}^n$ we define the *perimeter of E in U* as

$$P(E, U) = \sup\left\{ \int_{U \cap E} \operatorname{div} \varphi \colon \varphi \in C^1_c(U),\ \|\varphi\|_\infty \leq 1 \right\}.$$

If $P(E, U) < \infty$, then the distributional gradient $D\chi_E$ of χ_E in U is a vector-valued Radon measure and $P(E, U)$ is exactly its total variation. By the De Giorgi–Federer theorem, we can compute $P(E, U)$ as

$$P(E, U) = \mathscr{H}^{n-1}(\partial_* E \cap U).$$

The particular choice $U = \mathbb{R}^n$ gives the *perimeter* of E

$$P(E) = \|E\| = \mathscr{H}^{n-1}(\partial_* E).$$

If $A \subset \mathbb{R}^n$ is just measurable, we define also the *relative perimeter of E in A* as

$$P(E, \operatorname{in} A) = \mathscr{H}^{n-1}(\partial_* E \cap \operatorname{int}_* A).$$

There is a distinction between $P(E, \operatorname{in} U)$ and $P(E, U)$ if U is open, see Example II.10 below.

We say that a measurable set E is a *locally BV set*, if $P(E, A) < \infty$ for each bounded open set A. A measurable set E is called a *BV set*, if $|E| + \|E\| < \infty$.

The family of all BV sets and all locally BV sets is denoted by $\mathcal{BV}$ and $\mathcal{BV}_{\mathrm{loc}}$, respectively. The family of all bounded BV sets is denoted by $\boldsymbol{\mathcal{BV}}$.

Example II.10. Let $E = B(0,1)$ and $A = B(0,2) \setminus \{x \in \mathbb{R}^2 : |x| = 1\}$ be subsets of $\mathbb{R}^2$. Then $P(E, A) = \mathscr{H}^{n-1}(\emptyset) = 0$, whereas $P(E, \operatorname{in} A) = \mathscr{H}^{n-1}\Big(\partial\left(B\left(0,1\right)\right)\Big) = 2\pi$.

Remark II.11. If $n = 1$, each BV set E is equivalent to a set $\bigcup_{i=1}^k (a_i, b_i)$, where $a_1 < b_1 < \cdots < a_k < b_k$ are real numbers. In this case, $\|E\| = 2k$.

Definition II.12. Let A be a locally BV set. Then we define the *critical boundary* of A as

$$\partial_c A = \left\{ x \in \mathbb{R}^n ; \limsup_{r \to 0+} \frac{P(A, B(x,r))}{r^{n-1}} > 0 \right\}.$$

The *critical interior* $\operatorname{int}_c A$ and *critical exterior* $\operatorname{ext}_c A$ are then defined as

$$\operatorname{int}_c A = \operatorname{int}_* A \setminus \partial_c A, \qquad \operatorname{ext}_c A = \operatorname{ext}_* A \setminus \partial_c A.$$

In the following, we will define the regularity of a BV set. This concept has been first introduced by Kurzweil, Mawhin and Pfeffer in [11]. In this article, we use the modification established by Pfeffer in [16].

Definition II.13. Let $E \subset \mathbb{R}^n$ be a bounded BV set and let $x \in \mathbb{R}^n$. The *regularity* of the set E is the number

$$r(E) = \begin{cases} \frac{|E|}{d(E)\|E\|} & \text{if } |E| > 0, \\ 0 & \text{if } |E| = 0. \end{cases}$$

The *regularity* of the pair (E, x) is the number

$$r(E, x) = r(E \cup \{x\}) = \begin{cases} \frac{|E|}{d(E\cup\{x\})\|E\|} & \text{if } |E| > 0, \\ 0 & \text{if } |E| = 0. \end{cases}$$

Let $\varepsilon > 0$. We say that the set E and the pair (E, x) are *ε-regular* if $r(E) > \varepsilon$ and $r(E, x) > \varepsilon$, respectively. A system $P = \{(A_1, x_1), \dots, (A_m, x_m)\}$, $A_i \subset \mathbb{R}^n$ and $x_i \in \mathbb{R}^n$, is called ε-regular if $r(A_i, x_i) > \varepsilon$ for $i = 1, \dots, m$.

Let us note that every ε-regular BV set is bounded.

Remark II.14. For every bounded BV set E we have the estimate $r(E) \leq 1/(2n)$. Especially, the regularity of a ball is equal to $1/(2n)$ (see [17, Chapter 2.3]).

Definition II.15. A *dyadic cube* is an interval

$$\prod_{i=1}^{n} \left[\frac{k_i}{2^m}, \frac{k_i + 1}{2^m} \right],$$

where $m, k_1, \dots, k_n$ are integers. A dyadic cube C' is called the *mother* of a dyadic cube C if C' is the smallest (with respect to inclusion) dyadic cube properly containing C.

A finite (possibly empty) union of nondegenerate compact intervals in $\mathbb{R}^n$ is called a *figure.* A *dyadic figure* is a figure that is a union of finitely many dyadic cubes.

Definition II.16. Let B be a bounded BV set. We say that a sequence $\{B_i\} \subset \boldsymbol{\mathcal{BV}}$ *converges* to B in $\boldsymbol{\mathcal{BV}}$ if

1. $\bigcup_{i=1}^{\infty} B_i$ is a bounded set,
2. $\lim_{i\to\infty} |B_i \triangle B| = 0$ and $\sup_i \|B_i\| < \infty$.

Lemma II.17. *Let A be a bounded BV set. Then there exists a sequence $\{A_i\}$ of dyadic figures which converges to A in $\boldsymbol{\mathcal{BV}}$.*

Proof. See [17, Proposition 1.10.3]. □

Definition II.18. We say that a function $\mathcal{F} : \boldsymbol{\mathcal{BV}} \to \mathbb{R}$ is a *charge* if $\mathcal{F}$ satisfies the following conditions:

1. $\mathcal{F}(A \cup B) = \mathcal{F}(A) + \mathcal{F}(B)$ for each disjoint bounded BV sets A and B.
2. Given $\varepsilon > 0$ there exists an $\eta > 0$ such that $|\mathcal{F}(C)| < \varepsilon$ for each BV set $C \subset B(0, 1/\varepsilon)$ with $\|C\| < 1/\varepsilon$ and $|C| < \eta$.

Remark II.19. Let E be a bounded BV set and $\mathcal{F}$ be a charge. Since $\mathcal{F}$ is additive and vanishes on bounded negligible sets, $\mathcal{F}(E)$ depends only on the equivalence class of the set E.

Notation II.20. Let E be a locally BV set and $\mathcal{F}$ be a charge. Then $\mathcal{F}\lfloor_E$ denotes the charge $\mathcal{F}\lfloor_E(A) := \mathcal{F}(A \cap E)$, $A \in \boldsymbol{\mathcal{BV}}$.

Definition II.21. Let E be a locally BV set and let $\mathcal{F}$ be a charge. We say that $\mathcal{F}$ is a *charge in* E if $\mathcal{F} = \mathcal{F}\lfloor_E$.

Proposition II.22. *An additive function $\mathcal{F}$ on $\boldsymbol{\mathcal{BV}}$ is a charge if and only if either of the following conditions is satisfied.*

1. *For given ε there is a $\theta > 0$ such that for every BV set $B \subset B(1/\varepsilon)$ we have*
$$|\mathcal{F}(B)| < \theta|B| + \varepsilon(\|B\| + 1).$$
2. $\lim \mathcal{F}(A_i) = 0$ *for each sequence $\{A_i\}$ with $A_i \to \emptyset$ in $\boldsymbol{\mathcal{BV}}$.*

Proof. See [17, Proposition 2.2.6, Proposition 2.1.2]. □

Definition II.23. Let A be a locally BV set. We say that an additive function $\mathcal{F} : \boldsymbol{\mathcal{BV}} \to \mathbb{R}$ is a *flux in* A of a vector field $\mathbf{u} \in C(\bar{A}, \mathbb{R}^n)$, if for each $E \in \boldsymbol{\mathcal{BV}}$ we have
$$\mathcal{F}(E) = \int_{\partial_*(E\cap A)} \mathbf{u} \cdot \boldsymbol{\nu}_{E\cap A} \, \mathrm{d}\mathscr{H}^{n-1},$$
where $\boldsymbol{\nu}_{E\cap A}$ denotes the unit exterior normal of $E \cap A$.

In the case $A = \mathbb{R}^n$ we say that $\mathcal{F}$ is just a *flux* of $\mathbf{u}$.

Examples II.24.

1. Let $n = 1$. Since every bounded set $E \subset \mathbb{R}$ is equivalent to a finite disjoint union of compact intervals $\bigcup_{i=1}^{k}[a_i, b_i]$, each additive function $\mathcal{F}$ on $\boldsymbol{\mathcal{BV}}$ can be written as
$$\mathcal{F}(E) = \sum_{i=1}^{k} \left(u(b_i) - u(a_i)\right),$$
where $u : \mathbb{R} \to \mathbb{R}$. The additive function $\mathcal{F}$ is a charge if and only if u is continuous (see [17, Remark 2.1.5]). In other words, $\mathcal{F}$ can be represented as the distributional derivative of a continuous function u.

2. Let $\mathcal{F}$ be a flux in A of a continuous vector field $\mathbf{u} \in C(\bar{A}, \mathbb{R}^n)$, where A is a locally BV set. Then $\mathcal{F}$ is a charge (see [17, Example 2.1.4]). On the other hand, a charge needs not to be of this form. For an example see [17, Example 2.1.10].

3. Let $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a function. Then the function $\mathcal{F} : \boldsymbol{\mathcal{BV}} \to \mathbb{R}$ defined as
$$\mathcal{F}(A) = \int_A f \, \mathrm{d}\boldsymbol{\mathcal{L}}$$
is a charge. (See [17, Example 2.1.3].)

4. Let A be a measurable set with $\mathscr{H}^{n-1}(A) > 0$. Then the function $\mathcal{F} : \boldsymbol{\mathcal{BV}} \to \mathbb{R}$ defined as $\mathcal{F}(E) = \mathscr{H}^{n-1}(E \cap A)$ is not a charge.

   Without loss of generality we may assume A to be bounded. At first let us suppose that $\mathscr{H}^{n-1}(A) < \infty$. Then there is a constant c such that for every $k \in \mathbb{N}$ we can find a sequence of balls $\{B_i\}$ with $A \subset \bigcup_{i=1}^{\infty} B_i$, $\operatorname{diam} B_i < 1/k$

and $\sum_{i=1}^{\infty} \|B_i\| < c$. Then for $E_k := \bigcup_i B_i$ we have $E_k \subset \bigcup_{x\in A} B(x,1)$, $\|E_k\| < c$ and $|E_k| < \frac{c}{k}$.

It follows that $E_k \to \emptyset$ in $\boldsymbol{\mathcal{BV}}$, whereas $\mathcal{F}(E_k) = \mathscr{H}^{n-1}(E_k \cap A) = \mathscr{H}^{n-1}(A) > 0$. By Proposition II.22 $\mathcal{F}$ cannot be a charge.

It is easy to check that $\mathcal{F}$ is not a charge if $\mathscr{H}^{n-1}(A) = \infty$.

II.4 Packing $\mathcal{R}$ integral

In this section we set up concept of the packing $\mathcal{R}$ integral, which will be further developed in the next section.

Definition II.25. A pairwise disjoint finite system of balls $(B(x_i,r_i))_{i=1}^k$ in $\mathbb{R}^n$ is called a *packing*.

A function $\delta : E \to [0,\infty)$, where $E \subset \mathbb{R}^n$, is called a *gage* if the set $N = \{x; \delta(x) = 0\}$ is of σ-finite $\mathscr{H}^{n-1}$ Hausdorff measure.

We say that a system $P = \{(A_1,x_1),\dots,(A_k,x_k)\}$, $A_i \subset \mathbb{R}^n$ and $x_i \in \mathbb{R}^n$, is *δ-fine* if $d(A_i \cup x_i) < \delta(x_i)$. Let us remark that we do not require $x_i \in A_i$.

Especially, a packing $(B(x_i,r_i))_{i=1}^k$ is δ-fine if and only if $2r_i < \delta(x_i)$ for $i = 1,\dots,k$.

Notation II.26. Let $x \in \mathbb{R}^n$, $r, \varepsilon > 0$ and $\mathcal{F}$ be a charge. Then we will use the seminorms

$$\bar{p}^{\varepsilon}_{x,r}(\mathcal{F}) = \sup\left\{|\mathcal{F}(E)|; E \subset\subset B(x,r), E \in \boldsymbol{\mathcal{BV}}, (E,x) \text{ is } \varepsilon\text{-regular}\right\}.$$

Definition II.27. Let $A \subset \mathbb{R}^n$ be a locally BV set. We say that a charge $\mathcal{F}$ in A is an *indefinite packing $\mathcal{R}$ integral* of a function $f : \operatorname{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathscr{G}$ if there exists $\tau \in (0,1]$ such that for every $\varepsilon > 0$ there exists a gage $\delta\colon \operatorname{cl}_* A \to [0,\infty)$ such that for every δ-fine packing $(B(x_i,r_i))_{i=1}^k$, $x_i \in \operatorname{cl}_* A$, we have

$$\sum_{i=1}^{k} \bar{p}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) < \varepsilon.$$

Remark II.28. In the previous definition, as well as in forthcoming Definitions II.33, II.51, II.57, II.64, II.66 and II.78 it is possible to consider a function f defined only on $\operatorname{cl}_* A \setminus T$, where T is of σ-finite $\mathscr{H}^{n-1}$ Hausdorff measure. The integral is well defined since we can consider gages δ with $\delta = 0$ on T. For the same reason, the indefinite packing $\mathcal{R}$ integral with respect to any charge $\mathscr{G}$ does not depend on values of f on a set of σ-finite $\mathscr{H}^{n-1}$ Hausdorff measure.

Remark II.29. The uniqueness of the indefinite packing integral of f in A will be discussed later.

Remark II.30. The indefinite packing $\mathcal{R}$ integral is linear with respect to a function f.

II.5 Packing $\mathcal{R}^*$ integral

Let us continue with so called packing $\mathcal{R}^*$ integral. We will prove its uniqueness, basic properties and finally we will formulate and prove the Gauss-Green theorem.

Its definition relies on the concept of an ε-isoperimetric set, which was introduced by Malý and Pfeffer in [13]. We will be inspired by their work also further in this section.

Definition II.31. Let $\varepsilon > 0$ and $E \subset \mathbb{R}^n$ be a bounded BV set. We say that E is *ε-isoperimetric* if for each $T \in \boldsymbol{\mathcal{BV}}$

$$\min\{P(E \cap T), P(E \setminus T)\} \leq \frac{1}{\varepsilon} P(T, \operatorname{in} E).$$

Since $P(T, \operatorname{in} E) = P(E \cap T, \operatorname{in} E)$, it is enough to consider only $T \subset E$. (See [13, Lemma 2.1].)

Notation II.32. Let $x \in \mathbb{R}^n$, $r, \varepsilon > 0$ and $\mathcal{F}$ be a charge. Then we will use the seminorms

$$\bar{q}^{\varepsilon}_{x,r}(\mathcal{F}) = \sup\{|\mathcal{F}(E)|; E \subset\subset B(x,r), E \in \boldsymbol{\mathcal{BV}}, x \in \operatorname{cl}_* E,$$
$$(E, x) \text{ is } \varepsilon\text{-regular and } E \text{ is } \varepsilon\text{-isoperimetric}\}.$$

Definition II.33. Let $A \subset \mathbb{R}^n$ be a locally BV set. We say that a charge $\mathcal{F}$ in A is an *indefinite packing $\mathcal{R}^*$ integral* of a function $f\colon \operatorname{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathscr{G}$ if there exists $\tau \in (0,1]$ such that for every $\varepsilon > 0$ there exists a gage $\delta\colon \operatorname{cl}_* A \to [0, \infty)$ such that for every δ-fine packing $(B(x_i, r_i))_{i=1}^k$, $x_i \in \operatorname{cl}_* A$, we have

$$\sum_{i=1}^k \bar{q}^{\varepsilon}_{x_i, \tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) < \varepsilon.$$

In the case $A = \mathbb{R}^n$ we say that $\mathcal{F}$ is just an *indefinite packing $\mathcal{R}^*$ integral* of f with respect to $\mathscr{G}$.

The family of all functions packing $\mathcal{R}^*$ integrable with respect to a charge $\mathscr{G}$ is denoted by $\mathscr{P}\mathcal{R}^*(\mathscr{G})$.

Lemma II.34. *Let $\tau \in (0,1]$ and $\varepsilon > 0$. Then there exists a constant c_T (depending only on τ and n) with the following property: for each function $\Phi : \mathbb{R} \to (0, \infty)$, $x \in \mathbb{R}^n$ and $R > 0$ there exists $0 < r < R$ such that*

$$\Phi(10r) + \varepsilon|B(x, 10r)| \leq c_T\Big(\Phi(\tau r) + \varepsilon|B(x, \tau r)|\Big).$$

Proof. See [10, Lemma 3.7]. ☐

Lemma II.35. *Let $0 < \varepsilon \leq 1/(2n)$ and $Q = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$ be an ε-regular interval. Then*

$$\max\{a_1, \dots, a_n\} \leq \frac{1}{\varepsilon} \min\{a_1, \dots, a_n\}.$$

Proof. For simplicity, let us suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$. Since Q is ε-regular, we can estimate

$$a_n(a_2 \cdots a_n) \leq d(Q)\|Q\| < \frac{1}{\varepsilon}|Q| = \frac{1}{\varepsilon} a_1 a_2 \cdots a_n.$$

Dividing by $a_2 \cdots a_n$ we obtain $a_n < \frac{1}{\varepsilon} a_1$, which establishes the formula. ☐

Lemma II.36. *Let $\varepsilon > 0$, Q be an ε-regular interval and $T \in \boldsymbol{\mathcal{BV}}$, $T \subset Q$ satisfying $|T| \le |Q|/2$. Then there exists a constant $\gamma = \gamma(\varepsilon, n)$ such that*

$$\mathscr{H}^{n-1}(\partial Q \cap \partial_* T) \le \gamma \mathscr{H}^{n-1}(\operatorname{int} Q \cap \partial_* T). \tag{II.3}$$

Proof. At first let Q be a cube. By [18, Lemma 6.7.2] there exists a constant $\eta = \eta(n)$ such that

$$\mathscr{H}^{n-1}(\partial Q \cap \partial_* T) \le \eta \mathscr{H}^{n-1}(\operatorname{int} Q \cap \partial_* T). \tag{II.4}$$

Further, let Q be an ε-regular interval. We can suppose $Q = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$, $a_1 \le a_2 \le \cdots \le a_n$. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping represented by the diagonal matrix

$$\begin{pmatrix} a_n/a_1 & 0 & \cdots & 0 \\ 0 & a_n/a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then $L(Q)$ is a cube and $|L(T)| \le |L(Q)|/2$. Moreover, $\operatorname{int} L(Q) \cap \partial_* L(T) = L(\operatorname{int} Q \cap \partial_* T)$. Further, we can estimate the Lipschitz constant of L as $\operatorname{Lip}(L) = \max_i\{a_i/a_n\} \le \frac{1}{\varepsilon}$, which follows from Lemma II.35. Since L^{-1} can be represented by the matrix

$$\begin{pmatrix} a_1/a_n & 0 & \cdots & 0 \\ 0 & a_2/a_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

we have $\operatorname{Lip}(L^{-1}) = 1$.

Applying Lemma II.4, inequality (II.4) and properties of L we obtain

$$\begin{aligned} \mathscr{H}^{n-1}(\partial Q \cap \partial_* T) &= \mathscr{H}^{n-1}(L^{-1}(L(\partial Q \cap \partial_* T))) \le \mathscr{H}^{n-1}(\partial L(Q) \cap \partial_* L(T)) \\ &\le \eta \mathscr{H}^{n-1}(\operatorname{int} L(Q) \cap \partial_* L(T)) = \eta \mathscr{H}^{n-1}(L(\operatorname{int} Q \cap \partial_* T)) \\ &\le \frac{\eta}{\varepsilon^{n-1}} \mathscr{H}^{n-1}(\operatorname{int} Q \cap \partial_* T). \end{aligned}$$

Hence (II.3) holds with $\gamma(\varepsilon, n) := \frac{\eta}{\varepsilon^{n-1}}$. □

Lemma II.37. *For every $n \in \mathbb{N}$ there exists an increasing function $\beta : (0, \infty) \to \mathbb{R}$ such that every ε-regular interval $Q \subset \mathbb{R}^n$ is $\beta(\varepsilon)$-isoperimetric.*

Proof. We set $\beta(\varepsilon) = 1/(1 + \gamma(\varepsilon, n))$, the constant $\gamma(\varepsilon, n)$ being as in Lemma II.36. Now let us fix an ε-regular interval Q and a set $T \in \boldsymbol{\mathcal{BV}}$, $T \subset Q$. We need to show that

$$\min\{P(Q \cap T), P(Q \setminus T)\} \le \frac{1}{\beta(\varepsilon)} P(T, \operatorname{in} Q),$$

Let us assume $|T| \le |Q|/2$. Since Q is an interval, we have $\operatorname{int} Q = \operatorname{int}_* Q$. Then by Lemma II.36 there exists a $\gamma = \gamma(\varepsilon, n)$ such that

$$\begin{aligned} P(T) &\le \mathscr{H}^{n-1}(\operatorname{int} Q \cap \partial_* T) + \mathscr{H}^{n-1}(\partial Q \cap \partial_* T) \le (1 + \gamma) \mathscr{H}^{n-1}(\operatorname{int} Q \cap \partial_* T) \\ &\le (1 + \gamma) P(T, \operatorname{in} Q) = \frac{1}{\beta(\varepsilon)} P(T, \operatorname{in} Q). \end{aligned}$$

In the case $|T| > |Q|/2$ we have $|Q \setminus T| < |Q|/2$ and then we obtain

$$P(Q \setminus T) = P(Q \cap (Q \setminus T)) \leq (1+\gamma) P(Q \setminus T, \mathrm{in}\, Q) = \frac{1}{\beta(\varepsilon)} P(T, \mathrm{in}\, Q).$$

□

Lemma II.38. *Let $r > 0$, $x \in \mathbb{R}^n$ and $Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be an interval such that $Q \subset B(x, 2r)$ and $\frac{r}{2\sqrt{n}} \leq \min_l\{|b_l - a_l|\}$. Then (Q, x) is ρ-regular, where $\rho = \rho(n) = \frac{1}{n^{\frac{n+1}{2}} 2^{3n-2}}$.*

Proof. Let us denote $s := \min_l\{|b_l - a_l|\}$ and $w := \max_l\{|b_l - a_l|\}$. Since $\frac{r}{2\sqrt{n}} \leq s$, $w \leq 4r$ and $\operatorname{diam}(Q \cup \{x\}) \leq 4r$, we can estimate the regularity of Q as

$$\begin{aligned} r(Q,x) &= \frac{|Q|}{\operatorname{diam}(Q \cup \{x\}) \cdot \|Q\|} \\ &\geq \frac{s^{n-1} w}{4r \cdot 2n w^{n-1}} \\ &\geq \frac{\left(\frac{r}{2\sqrt{n}}\right)^{n-1}}{8rn\,(4r)^{n-2}} \\ &= \frac{1}{n^{\frac{n+1}{2}} 2^{3n-2}} = \rho(n). \end{aligned}$$

□

Lemma II.39. *Let $\mathcal{F}$ be a charge and $B(x,r) \subset \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, be a ball. Further, let $Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ be an interval such that $Q \subset B(x, 2r)$ and $\frac{r}{2\sqrt{n}} \leq \min_l\{|b_l - a_l|\}$. Then*

$$|\mathcal{F}(Q)| \leq 2^m \bar{q}^{\varepsilon}_{x,2r}(\mathcal{F}), \tag{II.5}$$

where $m = \#\{l; x_l \notin [a_l, b_l]\}$, $\varepsilon = \min\{\beta(\rho), \rho\}$ and β and ρ are as in Lemma II.37 and Lemma II.38.

Proof. The proof proceeds by induction on m. First, for $m = 0$ we have $x \in Q$. Since $Q \subset B(x, 2r)$ and $\frac{r}{2\sqrt{n}} \leq \min_l\{|b_l - a_l|\}$, Q is $\rho(n)$-regular. Furthermore, by Lemma II.37 we obtain Q is also $\beta(\rho)$-isoperimetric. Then we can estimate

$$|\mathcal{F}(Q)| \leq \bar{q}^{\varepsilon}_{x,2r}(\mathcal{F}).$$

Now let us fix $m \geq 1$ and suppose that (II.5) holds for $m-1$. Without loss of generality we can assume that $x_l \notin [a_l, b_l]$ for $l = 1, \ldots, m$.

Our next purpose is to define an auxiliary interval

$$\widetilde{Q} = [a_1, b_1] \times \cdots \times [a_{m-1}, b_{m-1}] \times [\tilde{a}_m, \tilde{b}_m] \times [a_{m+1}, b_{m+1}] \times \cdots \times [a_n, b_n],$$

where $[\tilde{a}_m, \tilde{b}_m]$ is defined as follows:

In the case $x_m < a_m$ let us set $\tilde{a}_m = x_m - (b_m - x_m)$ and $\tilde{b}_m = b_m$. If $x_m > b_m$, let us set $\tilde{a}_m = a_m$, $\tilde{b}_m = x_m + (x_m - a_m)$.

We see that $Q \subset \widetilde{Q} \subset B(x, 2r)$ and $x \in \widetilde{Q}$. For simplicity, let us assume $x_m < a_m$. Then

$$Q = \widetilde{Q} \setminus \widetilde{Q}',$$

where

$$\widetilde{Q}' = [a_1, b_1] \times \cdots \times [a_{m-1}, b_{m-1}] \times [\tilde{a}_m, a_m] \times [a_{m+1}, b_{m+1}] \times \cdots \times [a_n, b_n].$$

In the following, we need to estimate the regularity of subintervals $\widetilde{Q}$ and $\widetilde{Q'}$. Since $\min_l\{|b_l - a_l|\} \geq \frac{r}{2\sqrt{n}}$ and $\widetilde{Q} \subset B(x, 2r)$, by Lemma II.38 we obtain $\widetilde{Q}$ is $\rho(n)$-regular. Analogously we obtain the regularity of $\widetilde{Q}'$.

By Lemma II.37 we have $\widetilde{Q}$ and $\widetilde{Q}'$ are $\beta(\rho)$-isoperimetric. Using the additivity of $\mathcal{F}$ and the inductive assumption we obtain

$$|\mathcal{F}(Q)| \leq |\mathcal{F}(\widetilde{Q})| + |\mathcal{F}(\widetilde{Q}')| \leq 2 \cdot 2^{m-1} \bar{q}^{\varepsilon}_{x,2r}(\mathcal{F}) = 2^m \bar{q}^{\varepsilon}_{x,2r}(\mathcal{F}),$$

which completes the proof.

□

Theorem II.40 (Uniqueness of the integral). *Let f be a function and $\mathscr{G}$ be a charge. Then there exists at most one indefinite packing $\mathcal{R}^*$ integral of f with respect to $\mathscr{G}$.*

Proof. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be indefinite packing $\mathcal{R}^*$ integrals of f with respect to $\mathscr{G}$. Then $\mathcal{F}_1 - \mathcal{F}_2$ is the integral of 0 with respect to $\mathscr{G}$. So it is sufficient to show that if $\mathcal{F}$ is an indefinite packing $\mathcal{R}^*$ integral of $f \equiv 0$, then $\mathcal{F} \equiv 0$.

By Lemma II.17 it is enough to prove that $\mathcal{F}(K) = 0$ for each dyadic cube K. Let τ be as in Definition II.33. Now, let us fix a dyadic cube K of side a_0 and choose $\varepsilon > 0$ such that $\varepsilon < \min\{\beta(\rho), \rho\}$, where β and ρ are as in Lemma II.37 and Lemma II.38. Finally, denote $K_0 := \bigcup_{x \in K} B(x, 1)$.

STEP 1.

Since $\mathcal{F}$ is an indefinite packing $\mathcal{R}^*$ integral of $f \equiv 0$, there exists a gage $\delta : \mathbb{R}^n \to [0, \infty)$ such that for every δ-fine packing $(B(x_i, r_i))_{i=1}^h$we have

$$\sum_{i=1}^{h} \bar{q}^{\varepsilon}_{x_i, \tau r_i}(\mathcal{F}) < \varepsilon. \tag{II.6}$$

STEP 2.

In this step we construct the covering of the set $K \setminus N$, where $N = \{x; \delta(x) = 0\}$.

By Lemma II.34, applied to $\Phi(r) := \bar{q}^{\varepsilon}_{x,r}(\mathcal{F})$, we can find a constant c_T such that for every x there exists $r(x) < \delta(x)$, $10r(x) < 1$, with the following properties:

$$20r(x) < a_0 \tag{II.7}$$

and

$$\bar{q}^{\varepsilon}_{x,10r(x)}(\mathcal{F}) + \varepsilon|B(x, 10r(x))| \leq c_T(\bar{q}^{\varepsilon}_{x,\tau r(x)}(\mathcal{F}) + \varepsilon|B(x, \tau r(x))|). \tag{II.8}$$

Now, let us consider the covering $\mathcal{C} = \{\bar{B}(x, r(x)); x \in K \setminus N\}$. By the Vitali theorem we can construct a pairwise disjoint subsystem $\mathcal{C}' \subset \mathcal{C}$, such that $\bigcup_{B(x,R) \in \mathcal{C}''} B(x, R) \supset K \setminus N$, where $\mathcal{C}'' = \{B(x, 5r); \bar{B}(x, r) \in \mathcal{C}'\}$.

STEP 3.

Now we will cover the set N.

Since N is of σ-finite $\mathscr{H}^{n-1}$ measure, we can write out $N = \bigcup_{s=1}^{\infty} N^s$, where $\mathscr{H}^{n-1}(N^s) = c_s < \infty$ for every $s = 1, 2, \ldots$ Let us fix $s \in \mathbb{N}$ and $\varepsilon_s \in (0, \varepsilon)$ such that

$$\varepsilon_s(c_1 c_c 2^{n-1}(c_s + \varepsilon) + 1) < 2^{-s}\varepsilon, \tag{II.9}$$

where $c_1 = 2^n n^{(3-n)/2}$ and $c_c = \alpha_n 2^{2n} n^{n/2}$. By Lemma II.22, with ε_s we can associate θ_s such that for every BV set $E \subset B(1/\varepsilon)$ we have

$$|\mathcal{F}(E)| < \theta_s|E| + \varepsilon_s(\|E\| + 1). \tag{II.10}$$

Furthermore, there exist $\zeta_s < 1/2$ and a system of balls $\mathcal{N}^s = \{B(x_i^s, R_i^s)\}$ covering N^s such that $R_i^s \le \zeta_s$,

$$4R_i^s < a_0, \tag{II.11}$$

$$c_2 \zeta_s \theta_s (c_s + \varepsilon) < 2^{-s} \varepsilon \alpha_{n-1}$$

and

$$\sum_{B(x_i^s, R_i^s) \in \mathcal{N}^s} \alpha_{n-1} \left(\frac{\operatorname{diam} B(x_i^s, R_i^s)}{2} \right)^{n-1} \le (\alpha_{n-1} + 1) \sum_{B(x_i^s, R_i^s) \in \mathcal{N}^s} (R_i^s)^{n-1} < c_s + \varepsilon. \tag{II.12}$$

Note that

$$c_2 \theta_s \sum_{B(x_i^s, R_i^s) \in \mathcal{N}^s} (R_i^s)^n \le c_2 \zeta_s \theta_s \sum_{B(x_i^s, R_i^s) \in \mathcal{N}^s} (R_i^s)^{n-1} < 2^{-s}\varepsilon, \tag{II.13}$$

where $c_2 = \alpha_n c_c 2^n$.

Let us denote $\mathcal{N} := \bigcup_s \mathcal{N}^s$.

Now, let us consider the covering $\mathcal{V} := \mathcal{C}'' \cup \mathcal{N}$. Since $\mathcal{V}$ covers the compact set K, we can choose a finite system of balls $B(x_i, R_i) \in \mathcal{V}$, $i = 1, \ldots, k$, covering K. Without loss of generality we can assume that $B(x_1, R_1), \ldots, B(x_h, R_h) \in \mathcal{C}''$ and $B(x_{h+1}, R_{h+1}), \ldots, B(x_k, R_k) \in \mathcal{N}$.

STEP 4.

In this step we construct a partition of the cube K in the sense that we look for a finite system of nonoverlapping cubes whose union is K.

Recall that Q' denotes the mother cube of a cube Q. Let $\mathcal{K}$ denote the family of all dyadic subcubes of K. For fixed $i \in \{1, \ldots, k\}$ set

$$\tilde{\mathscr{Q}}_i = \{Q \in \mathcal{K}; Q \cap B(x_i, R_i) \ne \emptyset,\ Q \subset B(x_i, 2R_i) \text{ and } Q' \not\subset B(x_i, 2R_i)\}.$$

We show that the union $\tilde{\mathscr{Q}} = \bigcup_{i=1}^k \tilde{\mathscr{Q}}_i$ is all of K. Choose $y \in K$. Consider a sequence P_l of dyadic cubes such that $P_0 = K$, $P_{l-1} = P_l'$ for $l = 1, 2, \ldots$ and $\{y\} = \bigcap_{l=0}^{\infty} P_l$. There exists $i \in \{1, \ldots, k\}$ such that $y \in B(x_i, R_i)$. Since $\operatorname{diam} P_l \searrow 0$, there exists l such that $P_l \subset B(x_i, 2R_i)$. We find the smallest l such that $P_l \subset B(x_i, 2R_i)$. By (II.7) and (II.11), $l \ge 1$. We easily verify that $y \in P_l \in \tilde{\mathscr{Q}}$.

Next we show that the system $\tilde{\mathscr{Q}}$ is finite. Let us fix $Q \in \tilde{\mathscr{Q}}_i$ and denote the side length of Q by a. The length of the diagonal can be expressed as $\sqrt{n}a$. Since Q' intersects both $B(x_i, R_i)$ and $B(x_i, 2R_i)^c$, we obtain

$$R_i/2 < \sqrt{n}a. \tag{II.14}$$

Hence the side length of all cubes in $\tilde{\mathscr{Q}}_i$ is bounded from below. Therefore, the systems $\tilde{\mathscr{Q}}_i$ and hence the system $\tilde{\mathscr{Q}}$ are finite.

Now we can define the system of cubes

$$\mathscr{Q} = \tilde{\mathscr{Q}} \setminus \{Q \in \tilde{\mathscr{Q}};\ \exists P \in \tilde{\mathscr{Q}} \text{ such that } P \supsetneq Q\}.$$

Since two dyadic cubes are either in inclusion or nonoverlapping, $\mathscr{Q}$ is a finite partition of K; we enumerate it as $\mathscr{Q} = \{Q_j, j = 1, \dots, m\}$. Finally, let us define the systems

$$\mathscr{Q}_i = \Big\{Q \in \mathscr{Q} \cap \tilde{\mathscr{Q}}_i \colon Q \notin \bigcup_{l<i} \tilde{\mathscr{Q}}_l\Big\}.$$

Let us fix $Q_j \in \mathscr{Q}_i$ and denote it side length by a_j. Recall that the length of the diagonal can be expressed as $\sqrt{n}a_j$ and since Q_j is included in $B(x_i, 2R_i)$, we have $\sqrt{n}a_j < 4R_i$. Hence we can estimate the perimeter of Q_j:

$$\|Q_j\| = 2na_j^{n-1} \leq 2n\left(\frac{4R_i}{\sqrt{n}}\right)^{n-1} = 2^n n^{(3-n)/2} 2^{n-1} R_i^{n-1} = c_1 2^{n-1} R_i^{n-1}. \tag{II.15}$$

Let us estimate the number of the cubes $Q_j \in \mathscr{Q}_i$. Applying (II.14), we obtain

$$\alpha_n(2R_i)^n = |B(x_i, 2R_i)| \geq \left|\bigcup_{Q_j \in \mathscr{Q}_i} Q_j\right| \geq \#\mathscr{Q}_i \left(\frac{R_i}{2\sqrt{n}}\right)^n.$$

Hence $\#\mathscr{Q}_j \leq c_c$. (Let us remind that $c_c = \alpha_n 2^{2n} n^{n/2}$.)

STEP 5.

In this step we estimate $\mathcal{F}(K)$. By the additivity of $\mathcal{F}$ we obtain

$$\mathcal{F}(K) = \mathcal{F}\left(\bigcup_{i=1}^{k} \bigcup_{Q_j \in \mathscr{Q}_i} Q_j\right) = \mathcal{F}\left(\bigcup_{i=1}^{h} \bigcup_{Q_j \in \mathscr{Q}_i} Q_j\right) + \mathcal{F}\left(\bigcup_{i=h+1}^{k} \bigcup_{Q_j \in \mathscr{Q}_i} Q_j\right).$$

Firstly let us suppose that $i \in \{1, \dots, h\}$. Then $B(x_i, R_i) \in \mathcal{C}''$. Let us fix a pair $(Q_j, B(x_i, R_i))$. Since $q_j \geq \frac{R_i}{2\sqrt{n}}$, we can apply Lemma II.39 and obtain

$$|\mathcal{F}(Q_j)| \leq 2^n \bar{q}^{\varepsilon}_{x_i, 2R_i}(\mathcal{F}). \tag{II.16}$$

Using the fact that $\#Q_j \leq c_c$, the system $\{B(x_1, r_1), \dots, B(x_h, r_h)\}$ is a δ-fine packing and applying (II.16), (II.8) and (II.6) we can estimate

$$\begin{aligned}\left|\mathcal{F}\left(\bigcup_{i=1}^{h} \bigcup_{Q_j \in \mathscr{Q}_i} Q_j\right)\right| &= \sum_{i=1}^{h} \sum_{Q_j \in \mathscr{Q}_i} |\mathcal{F}(Q_j)| \\ &\leq \sum_{i=1}^{h} c_c 2^n \bar{q}^{\varepsilon}_{x_i, 2R_i}(\mathcal{F}) \\ &\leq 2^n c_c \sum_{i=1}^{h} c_T(\bar{q}^{\varepsilon}_{x_i, \tau r_i}(\mathcal{F}) + \varepsilon|B(x_i, \tau r_i)|) \\ &< c_c 2^n (c_T\varepsilon + c_T\varepsilon|K_0|) = c_c 2^n c_T \varepsilon(1 + |K_0|).\end{aligned}$$

Secondly, let us fix $s \in \mathbb{N}$ and set $\mathcal{A}^s := \{i \in \{h+1, \dots, k\}; B(x_i, R_i) \in \mathcal{N}^s\}$. Then, applying the fact that $\#\mathscr{Q}_j \le c_c$ and inequalities (II.10), (II.12), (II.9), (II.15) and (II.13) we obtain

$$\begin{aligned}\left|\mathcal{F}\left(\bigcup_{i\in\mathcal{A}^s}\bigcup_{Q_j\in\mathscr{Q}_i} Q_j\right)\right| &\le \theta_s\left|\bigcup_{i\in\mathcal{A}^s}\bigcup_{Q_j\in\mathscr{Q}_i} Q_j\right| + \varepsilon_s\left(\left\|\bigcup_{i\in\mathcal{A}^s}\bigcup_{Q_j\in\mathscr{Q}_i} Q_j\right\| + 1\right)\\ &< \varepsilon_s + \theta_s\alpha_n c_c 2^n \sum_{i\in\mathcal{A}^s} R_i^n + \varepsilon_s \sum_{i\in\mathcal{A}^s}\sum_{Q_j\in\mathscr{Q}_i}\|Q_j\|\\ &< \varepsilon_s + \theta_s c_2 \sum_{i\in\mathcal{A}^s} R_i^n + \varepsilon_s c_c c_1 2^{n-1}\sum_{i\in\mathcal{A}^s} R_i^{n-1}\\ &\le \varepsilon_s + \varepsilon 2^{-s} + \varepsilon_s c_c c_1 2^{n-1}(c_s + \varepsilon) < 2\cdot 2^{-s}\varepsilon.\end{aligned}$$

Since the union $\bigcup_{s=1}^{\infty}\bigcup_{i\in\mathcal{A}^s}\bigcup_{Q_j\in\mathscr{Q}_i} Q_j$ has only finite number of nonempty elements, we can use the additivity of $\mathcal{F}$ and we obtain

$$|\mathcal{F}(K)| < c_c 2^n c_T \varepsilon(1 + |K_0|) + \varepsilon\sum_{s=1}^{\infty} 2^{-s+1} = \varepsilon(c_T c_c 2^n (1 + |K_0|) + 2),$$

which completes the proof.

□

Remark II.41. The indefinite packing $\mathcal{R}^*$ integral of a function f with respect to a charge $\mathscr{G}$ depends linearly on f.

In the preceding, we were concerned with an indefinite packing $\mathcal{R}^*$ integral of a function $f : \mathbb{R}^n \to \mathbb{R}$. Now we will concentrate on a packing $\mathcal{R}^*$ integral in A, where A is a locally BV set.

Theorem II.42. *Let $A \subset \mathbb{R}^n$ be a locally BV set and let a charge $\mathcal{F}$ be an indefinite packing $\mathcal{R}$ integral of a function $f : \operatorname{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathscr{G}$. Then $\mathcal{F}$ is also an indefinite packing $\mathcal{R}^*$ integral of f in A with respect to $\mathscr{G}$.*

Proof. The proof follows from the fact that $\bar{q}^{\varepsilon}_{x,r} \le \bar{p}^{\varepsilon}_{x,r}$. □

Notation II.43. Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}$ be a function. Then $\bar{f}_A$ denotes the zero extension of f:

$$\bar{f}_A = \begin{cases} f(x) & \text{if } x \in A,\\ 0 & \text{if } x \notin A.\end{cases}$$

The two following lemmas with proofs can be found in [13, Lemma 2.5 and 3.7].

Lemma II.44. *Let $\mathcal{F}$ be a charge. Then for $\varepsilon > 0$ there is an absolutely continuous Radon measure μ in $\mathbb{R}^n$ such that for each BV set $E \subset B(0, 1/\varepsilon)$,*

$$|\mathcal{F}(E)| \le \mu(E) + \varepsilon P(E).$$

Lemma II.45. *Let $A \in \mathcal{BV}_{\mathrm{loc}}$ and $\varepsilon > 0$. For each $x \in \operatorname{ext}_c A$, there is $\delta > 0$ such that every strongly ε-regular set E with $x \in \operatorname{cl}_* E$ and $d(E) < \delta$ satisfies*

$$P(E \cap A) \le P(E \setminus A).$$

The proof of the next theorem follows the lines of the proof in [13, Lemma 3.8].

Theorem II.46. *Let $\mathcal{F}$ be a charge and A be an admissible locally BV set. For given $\tau \in (0,1]$ and $\varepsilon > 0$ there is a gage $\delta : \mathbb{R}^n \to [0,\infty)$ such that*

$$\sum_{x_i \in A} \bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A^c}) < \varepsilon \quad \textit{and} \quad \sum_{x_i \notin A} \bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A}) < \varepsilon$$

for each δ-fine packing $(B(x_i, r_i))_{i=1}^k$.

Proof. At first let us suppose that A is bounded. Let us fix $\varepsilon > 0$ such that $\bar{A} \subset B := B(0, 1/\varepsilon')$, where

$$\varepsilon' = \frac{\varepsilon^2}{P(A)}. \tag{II.17}$$

By Lemma II.44, there is an absolutely continuous Radon measure μ in $\mathbb{R}^n$ such that

$$|F(E)| \le \mu(E) + \varepsilon' P(E)$$

for each $E \in BV$, $E \subset B$. Then there exists a compact K such that $K \subset B \setminus A$ and

$$\mu((B \setminus A) \setminus K) < \frac{1}{2}\varepsilon. \tag{II.18}$$

Applying Lemma II.45 to A^c, for each $x \in B \cap \operatorname{ext}_c A^c = B \cap \operatorname{int}_c A$ we can find $\delta_x > 0$ such that $B(x, \delta_x) \subset B$, and

$$P(E \setminus A) \le P(E \cap A) \tag{II.19}$$

for each strongly ε-regular set E with $x \in \operatorname{cl}_* E$ and $d(E) < \delta_x$.

Making δ_x smaller, we may assume that $K \cap B(x, \delta_x) = \emptyset$ for $x \in \operatorname{int}_c A$. Since A and is an admissible set, it follows that also A^c is admissible and hence $\operatorname{int}_c A^c \subset A^c$ and $A^c \cap \operatorname{ext}_c A^c = \emptyset$. Let us set $N := \partial_c A^c = \partial_c A$. Then N is of σ-finite Hausdorff measure $\mathscr{H}^{n-1}$, which follows from the criterion for finite perimeter [6, p. 222]. Now we can define a gage $\tilde{\delta}$ on $\mathbb{R}^n$ in the following way:

$$\tilde{\delta}(x) = \begin{cases} 0 & \text{if } x \in N, \\ 1 & \text{if } x \in \operatorname{ext}_c A, \\ \delta_x & \text{if } x \in \operatorname{int}_c A. \end{cases}$$

Let us fix a $\tilde{\delta}$-fine packing $(B(x_i, r_i))_{i=1}^k$ and sets E_i, where $E_i \subset\subset B(x_i, \tau r_i)$, $E_i \in BV$, (E_i, x_i) is ε-regular and E_i is ε-isoperimetric for each $i = 1, \dots, k$. By the ε-regularity of E_i, inequality (II.19) and definition of $\tilde{\delta}$, we obtain

1. $x_i \notin N$ for $i = 1, \dots, k$;
2. $E_i \setminus A \subset (B \cap A^c) \setminus K$ when $x_i \in \operatorname{int}_c A$;
3. $P(E_i \setminus A) \le (1/\varepsilon) P(A^c, \operatorname{in} E_i)$ when $x_i \in \operatorname{int}_c A$.

Hence, using the inequality (II.18) and the fact that packing is pairwise disjoint, we can estimate

$$\begin{aligned}\sum_{x_i\in A}|\mathcal{F}(E_i\setminus A)| &= \sum_{x_i\in \operatorname{int}_c A}|\mathcal{F}(E_i\setminus A)|\\ &\le \sum_{x_i\in \operatorname{int}_c A}\mu(E_i\setminus A)+\frac{\varepsilon^2}{2P(A)}\,P(E_i\setminus A)\\ &\le \mu(B\cap A^c\setminus K)+\frac{\varepsilon}{2P(A)}\sum_{x_i\in \operatorname{int}_c A}P(A^c,\operatorname{in}E_i)\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}.\end{aligned}$$

Passing to the supremum we obtain

$$\sum_{x_i\in A}\bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A^c})<\varepsilon,$$

which we needed.

We now turn to the case A is unbounded. Let us consider a sequence of balls $\{B_m\}$ which forms a locally finite covering of $\mathbb{R}^n$. Choose $\varepsilon>0$. Let us fix $m\in\mathbb{N}$ and set $A_m=A\cap B_m$. Then A_m is a bounded admissible locally BV set and we can use the previous step to find $\varepsilon_m\le 2^{-m}\varepsilon$ and a gage $\delta_m:\mathbb{R}^n\to[0,\infty)$ such that

$$\sum_{x_i\in A_m}\bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A_m^c})<\varepsilon_m$$

for every δ_m-fine packing $((B(x_i,r_i))_{i=1}^k$.

Further, let us set

$$\tilde{\delta}(x):=\min\{\delta_m(x)\colon x\in B_m\}.$$

It is easily seen that $\tilde{\delta}$ is a gage. Let us fix a $\tilde{\delta}$-fine packing $((B(x_i,r_i))_{i=1}^k$. Then

$$\begin{aligned}\sum_{x_i\in A}\bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A^c}) &\le \sum_{m=1}^{\infty}\sum_{x_i\in A_m}\bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A_m^c})\\ &<\sum_{m=1}^{\infty}2^{-m}\varepsilon=\varepsilon,\end{aligned}$$

which establishes the formula.

Finally, we proceed similarly to find $\tilde{\delta}^c$ which yields the second inequality and set

$$\delta=\min\{\tilde{\delta},\tilde{\delta}^c\},$$

which gives both inequalities at the same time. □

In the proof of the next theorem we are inspired by [13, Proposition 3.9].

Theorem II.47. *Let $\mathscr{G}$, $\mathcal{F}$ be charges, $f\in\mathscr{PR}^*(\mathscr{G})$ and let $\mathcal{F}$ be an indefinite packing $\mathcal{R}^*$ integral of f with respect to $\mathscr{G}$. If A is an admissible locally BV set, then $\chi_A f\in\mathscr{PR}^*(\mathscr{G})$ and $\mathcal{F}\lfloor_A$ is an indefinite packing $\mathcal{R}^*$ integral of $\chi_A f$ with respect to $\mathscr{G}$.*

Proof. Let us fix $\tau \in (0,1]$ as in Definition II.33 and $\varepsilon > 0$. By the definition of packing $\mathcal{R}^*$ integral and Theorem II.46 there exists a gage $\delta : \mathbb{R}^n \to [0,\infty)$ such that for every δ-fine packing $(B(x_i,r_i))_{i=1}^k$ we have

$$\sum_{i=1}^{k} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) < \varepsilon,$$

$$\sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A^c}) < \varepsilon \quad \text{and} \quad \sum_{x_i \notin A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A}) < \varepsilon.$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{k} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_A - f(x_i)\chi_A(x_i)\mathscr{G}) \\
&\quad = \sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_A - f(x_i)\mathscr{G}) + \sum_{x_i \notin A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_A) \\
&\quad < \sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_A - \mathcal{F}) + \sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) + \sum_{x_i \notin A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_A) \\
&\quad = \sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A^c}) + \sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) + \sum_{x_i \notin A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_A) \\
&\quad < 3\varepsilon,
\end{aligned}$$

which completes the proof. ☐

Theorem II.48. *Let A be an admissible locally BV set. Then the charge $\mathcal{F}$ in A is an indefinite packing $\mathcal{R}^*$ integral of a function $f : \operatorname{cl}_* A \to \mathbb{R}$ with respect to a charge $\mathscr{G}$ in A if and only if $\mathcal{F}$ is an indefinite packing $\mathcal{R}^*$ integral of $\overline{f}_A$ with respect to $\mathscr{G}$ in $\mathbb{R}^n$.*

Proof. Let us suppose that $\mathcal{F}$ in A is the indefinite packing $\mathcal{R}^*$ integral of f with respect to $\mathscr{G}$ in A. Let us fix $\varepsilon > 0$. Now let $\tau \in (0,1]$ and a gage δ_1 on $\operatorname{cl}_* A$ be as in Definition II.33 and let δ_2 on $\mathbb{R}^n$ be as in Theorem II.46. Then let us fix a δ-fine packing $(B(x_i,r_i))_{i=1}^k$ and set

$$\delta = \begin{cases} \min\{\delta_1(x), \delta_2(x)\} & \text{if } x \in \operatorname{cl}_* A, \\ \delta_2(x) & \text{if } x \in \mathbb{R}^n \setminus \operatorname{cl}_* A. \end{cases}$$

At first, let us consider the sum over $x_i \in A$. Since $\mathcal{F}$ in A is the indefinite packing $\mathcal{R}^*$ integral of f in A, we have

$$\sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - \overline{f}_A(x_i)\mathscr{G}) = \sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) < \varepsilon.$$

Further, for the case $x_i \notin A$, we have by Theorem II.46 the estimate

$$\sum_{x_i \notin A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - \overline{f}_A(x_i)\mathscr{G}) = \sum_{x_i \notin A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}) < \varepsilon.$$

Therefore we obtain

$$\begin{aligned}
\sum_{i=1}^{k} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - \overline{f}_A(x_i)\mathscr{G}) &= \sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - \overline{f}_A(x_i)\mathscr{G}) + \sum_{x_i \notin A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - \overline{f}_A(x_i)\mathscr{G}) \\
&< 2\varepsilon.
\end{aligned}$$

Hence $\mathcal{F}$ is the indefinite packing $\mathcal{R}^*$ integral of $\overline{f}_A$ with respect to $\mathscr{G}$.

Conversely, let $\mathcal{F}$ be the indefinite packing $\mathcal{R}^*$ integral of $\overline{f}_A$ with respect to $\mathscr{G}$. By Theorem II.47 it follows that $\mathcal{F}\lfloor_A$ is the indefinite packing $\mathcal{R}^*$ integral of $\overline{f}_A$ with respect to $\mathscr{G}$ in $\mathbb{R}^n$. In other words, for fixed $\varepsilon > 0$ there exists a gage $\delta : \mathbb{R}^n \to [0, \infty)$ such that $\delta = 0$ on $\mathrm{cl}_* A \setminus A$ and for every δ-fine packing $(B(x_i, r_i))_{i=1}^k$ we have

$$\sum_{i=1}^{k} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_A - \overline{f}_A(x_i)\mathscr{G}) < \varepsilon.$$

By the uniqueness of packing $\mathcal{R}^*$ integral we have $\mathcal{F}\lfloor_A = \mathcal{F}$ and hence

$$\sum_{x_i \in \mathrm{cl}_* A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) = \sum_{x_i \in A} \overline{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - \overline{f}_A(x_i)\mathscr{G}) < \varepsilon,$$

which we needed.

□

Corollary II.49. Let A be an admissible locally BV set and let a charge $\mathcal{F}$ be an indefinite packing $\mathcal{R}^*$ integral of a function $f : \mathrm{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathscr{G}$. Then, by Theorem II.48, $\mathcal{F}$ is the indefinite packing $\mathcal{R}^*$ integral of $\overline{f}_A$ with respect to $\mathscr{G}$, which is unique by Theorem II.40. Therefore the indefinite packing $\mathcal{R}^*$ integral in A is unique as well.

Further, let A be an admissible locally BV set and let a charge $\mathcal{F}$ be an indefinite packing $\mathcal{R}$ integral of a function $f : \mathrm{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathscr{G}$. Then $\mathcal{F}$ is also the packing $\mathcal{R}^*$ integral of in A with respect to $\mathscr{G}$ by Theorem II.42. Hence the uniqueness holds also for the indefinite packing $\mathcal{R}$ integral in A.

Remark II.50. Since the function f is defined on $\mathrm{cl}_* A$, the requirement that A be admissible might seem to be unnecessary. This is really the case with $\mathscr{G} = \boldsymbol{\mathcal{L}}$, because sets of measure zero (such as $A \triangle \mathrm{cl}_* A$) does not play a role in integration with respect to Lebesgue measure. On the other hand, Lebesgue null sets cannot be neglected in general. For example, the classical Cantor set cannot be neglected for integration with respect to the Cantor measure in $\mathbb{R}$, which is a charge by Example II.24(1).

Definition II.51. Let $A \in \boldsymbol{\mathcal{BV}}$ be an admissible set, $f : \mathrm{cl}_* A \to \mathbb{R}$ be a function and $\mathcal{F}$, $\mathscr{G}$ be charges. We say that the number $\mathcal{F}(A)$ is a *definite packing* $\mathcal{R}^*$ *integral* of f over A with respect to $\mathscr{G}$ if $\mathcal{F}$ is an indefinite packing $\mathcal{R}^*$ integral of $\overline{f}_A$ with respect to $\mathscr{G}$.

More generally: if $A \subset \mathbb{R}^n$ is a bounded measurable set and $\mathcal{F}$ is the indefinite packing $\mathcal{R}^*$ integral of $\overline{f}_A$ with respect to $\mathscr{G}$, then the definite packing $\mathcal{R}^*$ integral of f over A with respect to $\mathscr{G}$ is the number $\mathcal{F}(A')$, where $A' \in \boldsymbol{\mathcal{BV}}$, $A' \supset A$ is a bounded admissible set.

The family of all functions packing $\mathcal{R}^*$ integrable with respect to $\mathscr{G}$ over A is denoted by $\mathscr{P}\mathcal{R}^*(A, \mathscr{G})$.

Remark II.52. The integral does not depend on the choice of A'. Indeed, let A' and A'' be bounded admissible BV sets. Since $\overline{f}_A \cdot \chi_{A'} = \overline{f}_A \cdot \chi_{A''}$, by Theorem II.47 and by the uniqueness of the packing $\mathcal{R}^*$ integral we obtain $\mathcal{F}\lfloor_{A'} = \mathcal{F}\lfloor_{A''}$. Then $\mathcal{F}(A') = \mathcal{F}\lfloor_{A'}(A' \cup A'') = \mathcal{F}\lfloor_{A''}(A' \cup A'') = \mathcal{F}(A'')$.

Remark II.53. Let $A \in \boldsymbol{\mathcal{BV}}$ be an admissible set, $\mathscr{G}$ be a charge and $f \in \mathscr{P}\mathcal{R}^*(A,\mathscr{G})$. Let $\mathcal{F}$ be the indefinite packing $\mathcal{R}^*$ integral of f in A with respect to $\mathscr{G}$. Then the definite $\mathscr{P}\mathcal{R}^*$ integral of f over A with respect to $\mathscr{G}$ is just $\mathcal{F}(A)$. This fact follows from Theorem II.48.

Remark II.54. If f is a merely an indefinite packing $\mathcal{R}^*$ integrable function, it does not make sense to define the definite integral over unbounded sets in general. If we want to set up the definite integral over an unbounded set, we must suppose some additional limiting behaviour of the indefinite integral at infinity. There are several nonequivalent ways how to do it and we do not pursue this direction.

Remark II.55. Let $A \subset \mathbb{R}^n$ be a bounded measurable set and let $f : A \to \mathbb{R}$ be a Lebesgue integrable function. Then $\overline{f}_A$ is also a Lebesgue integrable function.

Then there exists an indefinite packing $\mathcal{R}^*$ integral of $\overline{f}_A$ with respect to Lebesgue measure. Hence the definite packing $\mathcal{R}^*$ integral of f over A is well defined.

In the following theorem, we will focus on the convergence of a sequence of sets. The importance of this property will be demonstrated in Section II.7. The proof uses ideas from Pfeffer and Malý in [13, Theorem 3.20].

Theorem II.56. *Let A be a bounded admissible BV set, $\mathscr{G}$ and $\mathcal{F}$ be charges and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of bounded admissible BV sets such that $A_j \subset A$ for $j = 1, 2, \dots$ and $A_j \to A$ in $\boldsymbol{\mathcal{BV}}$. Further, let $f\chi_{A_j} \in \mathscr{P}\mathcal{R}^*$ and $\mathcal{F}\lfloor_{A_j}$ be an indefinite packing $\mathcal{R}^*$ integral of $f\chi_{A_j}$ with respect to $\mathscr{G}$ with constants τ_j as in Definition II.33. Let $\inf_j \tau_j > 0$. Then there exists an indefinite packing $\mathcal{R}^*$ integral of $f\chi_A$ with respect to $\mathscr{G}$ and is equal to $\mathcal{F}\lfloor_A$.*

Proof. Let us fix $\tau = \inf_j \tau_j$ and let us denote $N := A \setminus \bigcup_{j=1}^{\infty} A_j$. Then N is of σ-finite $\mathscr{H}^{n-1}$ measure (see [17, Cor. 6.2.7]). Let us choose $\varepsilon > 0$. Since $\bar{q}^{\varepsilon}_{x,\tau r} \leq \bar{q}^{\varepsilon}_{x,\tau' r}$ for $\tau \leq \tau'$, we can by the definition of packing $\mathcal{R}^*$ integral and by Theorem II.46 for $j \in \mathbb{N}$ find a gage δ_j such that for each δ_j-fine packing $(B(x_i, r_i))_{i=1}^{k}$ we obtain

$$\sum_{x_i \in A_j} \bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A_j} - f(x_i)\mathscr{G}) < \varepsilon 2^{-j} \tag{II.20}$$

and

$$\sum_{x_i \in A_j} \bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A_j^c}) < \varepsilon 2^{-j}. \tag{II.21}$$

Further, for $x \in \bigcup_{j=1}^{\infty} A_j$ let us set $j_x := \min\{j \in \mathbb{N}; x \in A_j\}$. Now we can define a gage

$$\delta(x) = \begin{cases} \delta_{j_x}(x) & \text{if } x \in \bigcup_{j=1}^{\infty} A_j, \\ 0 & \text{if } x \in N. \end{cases}$$

By Theorem II.48 it is enough to show that $\mathcal{F}\lfloor_A$ is the indefinite packing $\mathcal{R}^*$ integral of f in A with respect to $\mathscr{G}$. Let us choose δ-fine packing $(B(x_i, r_i))_{i=1}^{k}$, $x_i \in A$, and denote $j_i := j_{x_i}$. Using the additivity of $\mathcal{F}$ and estimates (II.20) and (II.21) we can for fixed $p \in \mathbb{N}$ estimate

$$\begin{aligned} \sum_{x_i : j_i = p} \bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_A - f(x_i)\mathscr{G}) &\leq \sum_{x_i : j_i = p} \bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A_j} - f(x_i)\mathscr{G}) + \bar{q}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}\lfloor_{A \setminus A_j}) \\ &< \varepsilon 2^{-p+1}. \end{aligned}$$

Summing over p we obtain

$$\sum_{p=1}^{\infty} \sum_{x_i : j_i = p} \bar{q}^{\varepsilon}_{x_i, \tau r_i}(\mathcal{F}\lfloor_A - f(x_i)\mathscr{G}) < \sum_{p=1}^{\infty} \varepsilon 2^{-p+1} = 2\varepsilon,$$

which completes the proof.

$\square$

Definition II.57. Let $A \subset \mathbb{R}^n$ be a locally BV set and let $f : \operatorname{cl}_* A \to \mathbb{R}$ and $\mathbf{u} \in C(\bar{A}, \mathbb{R}^n)$ be functions. Further, let a charge $\mathcal{F}$ be the flux of $\mathbf{u}$ in A. We say that f is a *generalized divergence* of $\mathbf{u}$ in A if $\mathcal{F}$ is an indefinite packing $\mathcal{R}^*$ integral of f in A. The generalized divergence of $\mathbf{u}$ will be denoted by $\operatorname{Div} \mathbf{u}$.

The following three definitions was mentioned by Pfeffer in Chapters 2.3 and 2.5 of [17].

Definition II.58. Let $A \subset \mathbb{R}^m$ be a measurable set and let $x \in A \cap \operatorname{int}_* A$. A map $\mathbf{u} : A \to \mathbb{R}^n$ is called *differentiable at x relative to A* if there is a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ such that for given $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$|\mathbf{u}(y) - \mathbf{u}(x) - L(y - x)| < \varepsilon |y - x|$$

for each $y \in A \cap B(x, \delta)$. The linear map L is called the *differential of* $\mathbf{u}$ *at x relative to A* and is denoted by $D_A \mathbf{u}(x)$.

Let $x \in \operatorname{int}_* A$ and $\mathbf{u} : \operatorname{cl}_* A \to \mathbb{R}^m$ be a vector field. Let $\mathbf{u}$ be differentiable at x relative to $\operatorname{cl}_* A$. The *divergence of* $\mathbf{u}$ *at x relative to* $\operatorname{cl}_* A$ is the number $\operatorname{div}_* \mathbf{u}(x) := \operatorname{tr} D_{\operatorname{cl}_* A} \mathbf{u}(x)$, where $\operatorname{tr} D_{\operatorname{cl}_* A} \mathbf{u}(x)$ denotes the trace of the matrix representation of the linear transformation $D_{\operatorname{cl}_* A} \mathbf{u}(x) : \mathbb{R}^m \to \mathbb{R}^m$.

By $\operatorname{div} \mathbf{u}$ we will denote the pointwise divergence defined on interior points of A at which $\mathbf{u}$ is differentiable. Especially, $\operatorname{div}_* \mathbf{u} = \operatorname{div} \mathbf{u}$ whenever $\operatorname{div} \mathbf{u}$ is defined.

Definition II.59. Let $\mathcal{F}$ be a charge and let $x \in \mathbb{R}^n$. Then for $\eta \geq 0$ we define

$$\underline{D}_\eta \mathcal{F}(x) := \sup_{\delta > 0} \inf_E \frac{\mathcal{F}(E)}{|E|} \quad \text{and} \quad \overline{D}_\eta \mathcal{F}(x) := \inf_{\delta > 0} \sup_E \frac{\mathcal{F}(E)}{|E|},$$

where $E \in BV$ such that $d(E \cup \{x\}) < \delta$ and $r(E, x) > \eta$.

The *lower* and *upper derivative* of $\mathcal{F}$ at x are defined as

$$\underline{D}\mathcal{F}(x) := \inf_{\eta > 0} \underline{D}_\eta \mathcal{F}(x) \quad \text{and} \quad \overline{D}\mathcal{F}(x) := \sup_{\eta > 0} \overline{D}_\eta \mathcal{F}(x).$$

We say that $\mathcal{F}$ is *derivable* at x, if

$$\underline{D}\mathcal{F}(x) = \overline{D}\mathcal{F}(x) \neq \pm\infty.$$

The *derivative* of $\mathcal{F}$ at x is then defined as $D\mathcal{F}(x) := \underline{D}\mathcal{F}(x) = \overline{D}\mathcal{F}(x)$.

Definition II.60. Let E be a locally BV set, $\mathbf{u} : \operatorname{cl}_* E \to \mathbb{R}^n$ be a bounded Borel measurable vector field and $\mathcal{F}$ be the flux of $\mathbf{u}$. If $\mathcal{F}$ is derivable at $x \in \operatorname{int}_c E$, we call the number $\mathfrak{div}\, \mathbf{u}(x) := D\mathcal{F}(x)$ the *mean divergence* of $\mathbf{u}$ at x.

Applying the inclusion between $\mathcal{R}$ integral and packing $\mathcal{R}^*$ integral we can state sufficient conditions for existence of generalized divergence. For further details see [17, Example 2.3.2, Remark 2.5.9, Theorem 5.1.12, Proposition 2.5.7 and Corollary 5.1.13].

Proposition II.61. *Let A be a locally BV set and $\mathbf{u} \in C(\bar{A}, \mathbb{R}^n)$.*

1. *If $A = \mathbb{R}^n$ and $\mathbf{u}$ is differentiable in $\mathbb{R}^n$, then $\operatorname{div} \mathbf{u}$ is a generalized divergence of $\mathbf{u}$.*
2. *If $\mathbf{u}$ is differentiable relatively to $\operatorname{cl}_* A$ on $\operatorname{int}_c A$, then $\operatorname{div}_* \mathbf{u}$ is a generalized divergence of $\mathbf{u}$.*
3. *If $\mathbf{u}$ is differentiable relatively to $\operatorname{cl}_* A$ on $\operatorname{int}_c A$, then $\mathfrak{div}\, \mathbf{u}$ is a generalized divergence of $\mathbf{u}$.*
4. *If $\mathbf{u}$ is Lipschitz on $\operatorname{cl}_* A \setminus T$, where T is of σ-finite Hausdorff measure $\mathscr{H}^{n-1}$, then $\operatorname{div}_* \mathbf{u}$ is a generalized divergence of $\mathbf{u}$.*

Theorem II.62 (Gauss-Green divergence theorem)**.** *Let $A \subset \mathbb{R}^n$ be a bounded BV set, let $\mathbf{u} \in C(\bar{A}, \mathbb{R}^n)$. Let us suppose that there exists a generalized divergence $\operatorname{Div} \mathbf{u}$ in A. Then*

$$\int_A \operatorname{Div} \mathbf{u}(x)\, dx = \int_{\partial_* A} \mathbf{u} \cdot \boldsymbol{\nu}_A \, d\mathscr{H}^{n-1},$$

where the integral on the left is the definite packing $\mathcal{R}^*$ integral.

Proof. Since $|A \triangle \operatorname{cl}_* A| = 0$, it is enough to show that $\int_{\operatorname{cl}_* A} \operatorname{Div} \mathbf{u}(x)\, dx = \int_{\partial_* A} \mathbf{u} \cdot \boldsymbol{\nu}_A \, d\mathscr{H}^{n-1}$ (see Remark II.50). Let $\mathcal{F}$ denote the indefinite packing $\mathcal{R}^*$ integral of $\operatorname{Div} \mathbf{u}$ in A. Since $\mathcal{F}$ is the flux of $\mathbf{u}$ in A, we have $\mathcal{F}(A) = \int_{\partial_* A} \mathbf{u} \cdot \boldsymbol{\nu}_A \, d\mathscr{H}^{n-1}$. By Theorem II.48 we obtain $\int_{\operatorname{cl}_* A} \operatorname{Div} \mathbf{u}(x)\, dx = \mathcal{F}(A)$, which completes the proof. ☐

II.6 $\mathcal{R}$ integral

In this section we will introduce Pfeffer's $\mathcal{R}$ integral described in [17]. For easier comparison of integrals we use the characterization of $\mathcal{R}$ integral [17, Proposition 5.5.6] rather than original definition.

Definition II.63. A *BV partition* is a system of couples $\{(A_1, x_1), \ldots, (A_k, x_k)\}$ of pairwise disjoint bounded BV sets A_i and points $x_i \in \mathbb{R}^n$ for $i = 1, \ldots, k$. It is not required $x_i \in A_i$.

Definition II.64. Let A be a locally BV set and $\mathcal{F}$, $\mathscr{G}$ be charges in A. Let f be a function defined on $\operatorname{cl}_* A$. We say that $\mathcal{F}$ is an *intrinsic indefinite $\mathcal{R}$ integral* of f in A with respect to $\mathscr{G}$ if for given $\varepsilon > 0$ we can find a gage $\delta : \operatorname{cl}_* A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} \Big| \mathcal{F}(A_i) - f(x_i) \mathscr{G}(A_i) \Big| < \varepsilon$$

for each ε-regular δ-fine BV partition $\{(A_1, x_1), \ldots, (A_k, x_k)\}$ with $\bigcup_{i=1}^{k} A_i \subset A$ and $x_i \in \operatorname{cl}_* A$ for $i = 1, \ldots, k$.

The family of all $\mathcal{R}$ integrable functions in A with respect to $\mathscr{G}$ is denoted by $\mathcal{R}(A, \mathscr{G})$. The family of all $\mathcal{R}$ integrable functions in A with respect to Lebesgue measure is denoted just by $\mathcal{R}(A)$.

Remark II.65. The intrinsic indefinite $\mathcal{R}$ integral is well defined, unique and linear. For the proof and other properties see [17, p. 211-213].

Definition II.66. Let A be a locally BV set and $\mathcal{F}$, $\mathscr{G}$ be charges in A. Let f be a function defined on $\operatorname{cl}_* A$. We say that $\mathcal{F}$ is an *indefinite* $\mathcal{R}$ *integral* of f in A with respect to $\mathscr{G}$ if for given $\varepsilon > 0$ we can find a gage $\delta : \operatorname{cl}_* A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} \Big| \mathcal{F}(A_i) - f(x_i)\mathscr{G}(A_i) \Big| < \varepsilon$$

for each ε-regular δ-fine BV partition $\{(A_1, x_1), \ldots, (A_k, x_k)\}$ with $x_i \in \operatorname{cl}_* A$ for $i = 1, \ldots, k$. (We do not require that $A_i \subset A$.)

The family of all $\mathcal{R}$ integrable functions in A with respect to $\mathscr{G}$ is denoted by $\mathcal{IR}(A, \mathscr{G})$.

Remark II.67. Let us remark that our terminology slightly differs from that used in [17]. Namely, what we call "intrinsic indefinite $\mathcal{R}$ integral in A" is termed simply "indefinite $\mathcal{R}$ integral" in [17]. Furthermore, in [17] it is distinguished between the $\mathcal{R}$ integral (with respect to Lebesgue measure) and $\mathscr{S}$ integral (Stieltjes version; with respect to an arbitrary charge).

Lemma II.68. *Let $\varepsilon > 0$ and $A \subset \mathbb{R}^n$ be an ε-regular bounded BV set. Then $[\operatorname{diam}(A)]^n \leq \frac{1}{\varepsilon^n} c|A|$, where $c = c(n)$ is a constant depending only on n.*

Proof. Since A is ε-regular, we have $\operatorname{diam}(A)P(A) \leq \frac{1}{\varepsilon}|A|$. Further, by the isoperimetric inequality (see [17, Theorem 1.8.7]) we have $|A|^{\frac{n-1}{n}} \leq p(n)P(A)$, where $p(n)$ is a constant depending on n. Thus

$$\operatorname{diam}(A)P(A) \leq \frac{1}{\varepsilon}|A| \leq \frac{1}{\varepsilon} p(n)^{\frac{n}{n-1}} P(A)^{\frac{n}{n-1}}.$$

Hence

$$\begin{aligned}
\operatorname{diam}(A)^{n-1} P(A)^{n-1} &\leq \frac{1}{\varepsilon^{n-1}} p(n)^n P(A)^n, \\
\operatorname{diam}(A)^{n-1} &\leq \frac{1}{\varepsilon^{n-1}} p(n)^n P(A), \\
\operatorname{diam}(A)^{n} &\leq \frac{1}{\varepsilon^{n-1}} p(n)^n P(A) \operatorname{diam}(A) \\
&\leq \frac{1}{\varepsilon^{n}} c|A|,
\end{aligned}$$

where $c = p(n)^n$. ☐

Theorem II.69. *Let A be a locally BV set, $\mathcal{F}$, $\mathscr{G}$ be charges in A. Let f be a function defined on* $\operatorname{cl}_* A$. *If $\mathcal{F}$ is an intrinsic indefinite $\mathcal{R}$ integral of f in A with respect to $\mathscr{G}$, then $\mathcal{F}$ is also an indefinite $\mathcal{R}$ integral of f in A with respect to $\mathscr{G}$.*

Proof. Let us choose $\varepsilon \in (0, 1/(c\alpha_n))$ and set $\varepsilon' = \varepsilon(1 - c\alpha_n\varepsilon)/(1 + c)$, where $c = c(n)$ is as in Lemma II.68. Let us find a gage $\delta_1 : \operatorname{cl}_* A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} \Big| \mathcal{F}(A_i) - f(x_i)\mathscr{G}(A_i) \Big| < \varepsilon'$$

for each ε'-regular δ_1-fine BV partition $\{(A_1, x_1), \dots, (A_k, x_k)\}$ with $\bigcup_{i=1}^k A_i \subset A$ and $x_i \in \operatorname{cl}_* A$ for $i = 1, \dots, k$.

Further, for each $x \in \operatorname{int}_c A$ let us find $R = R(x) > 0$ such that for every $r < R$ we have

$$P(A, B(x, r)) \leq \varepsilon^{n-1} r^{n-1}. \tag{II.22}$$

Since $\operatorname{int}_c A \subset \operatorname{int}_* A$, for every $x \in \operatorname{int}_c A$ we can find $R' = R'(x) > 0$ such that

$$|B \setminus A| < \varepsilon^{n+1} |B| \tag{II.23}$$

for every $B = B(x, r)$, $r < R'$.

Now let us define

$$\delta(x) = \begin{cases} 0 & \text{if } x \in \operatorname{cl}_* A \setminus \operatorname{int}_c A, \\ \min\{\delta_1(x), R(x), R'(x)\} & \text{if } x \in \operatorname{int}_c A. \end{cases}$$

Since the set $\operatorname{cl}_* A \setminus \operatorname{int}_c A$ is of σ-finite $\mathscr{H}^{n-1}$ Hausdorff measure (see [17, Theorem 1.8.2] and [18, Proposition 7.3.1]), δ defines a gage.

Let us fix an ε-regular δ-fine BV partition $\{(A_1, x_1), \dots, (A_k, x_k)\}$ such that $x_i \in \operatorname{int}_c A$ for $i = 1, \dots, k$. We need to show that

$$\sum_{i=1}^{k} \Big| \mathcal{F}(A_i) - f(x_i) \mathscr{G}(A_i) \Big| < \varepsilon.$$

Let us set $A_i' = A_i \cap A$, $i = 1, \dots, k$. Then $\{(A_1', x_1), \dots, (A_k', x_k)\}$ is obviously a δ-fine BV partition. We need to show that the system $\{(A_1', x_1), \dots, (A_k', x_k)\}$ is ε'-regular.

Let us fix $i \in \{1, \dots, k\}$. Since (A_i, x_i) is ε-regular, we have

$$\operatorname{diam}(A_i \cup \{x_i\}) P(A_i) \leq \frac{1}{\varepsilon} |A_i|.$$

Further, let us find a minimal ball $B = B(x_i, r)$ with the property that $A_i \subset \bar{B}$. Then $r < \delta(x)$.

By Lemma II.68 there exists a constant c such that

$$|B| \leq \alpha_n \left(\operatorname{diam}(A_i \cup \{x_i\}) \right)^n \leq \frac{c \alpha_n}{\varepsilon^n} |A_i|.$$

Then applying (II.23) we can estimate

$$\begin{aligned} |A_i| &\leq |A_i \cap A| + |A_i \setminus A| \\ &\leq |A_i \cap A| + |B \setminus A| \\ &\leq |A_i \cap A| + \varepsilon^{n+1} |B| \\ &\leq |A_i \cap A| + c \alpha_n \varepsilon |A_i|. \end{aligned}$$

Hence

$$(1 - c \alpha_n \varepsilon) |A_i| \leq |A_i \cap A|. \tag{II.24}$$

Applying (II.22), (II.24) and Lemma II.68 then gives

$$\begin{aligned}\operatorname{diam}(A_i \cap A \cup \{x_i\})P(A_i \cap A) &\leq \operatorname{diam}(A_i \cup \{x_i\}) \left[P(A_i) + P(A, B)\right] \\ &\leq \frac{1}{\varepsilon}|A_i| + \operatorname{diam}(A_i \cup \{x_i\})\varepsilon^{n-1}r^{n-1} \\ &\leq \frac{1}{\varepsilon}|A_i| + \varepsilon^{n-1}[\operatorname{diam}(A_i \cup \{x_i\})]^n \\ &\leq \left(\frac{1}{\varepsilon} + \frac{c}{\varepsilon}\right)|A_i| \\ &\leq \frac{1+c}{\varepsilon(1 - c\alpha_n\varepsilon)}|A_i \cap A| \\ &= \frac{1}{\varepsilon'}|A_i \cap A|.\end{aligned}$$

Thus the system $\{(A'_1, x_1), \ldots, (A'_k, x_k)\}$ is δ-fine ε'-regular BV partition. Since $\mathcal{F}$ and $\mathscr{G}$ are charges in A, we have $\mathcal{F}(A_i) = \mathcal{F}(A'_i)$ and $\mathscr{G}(A_i) = \mathscr{G}(A'_i)$ for $i = 1, \ldots, k$. Further, since $\mathcal{F}$ is the intrinsic indefinite $\mathcal{R}$ integral of f with respect to $\mathscr{G}$, we can estimate

$$\sum_{i=1}^{k} \left|\mathcal{F}(A_i) - f(x_i)\mathscr{G}(A_i)\right| = \sum_{i=1}^{k} \left|\mathcal{F}(A'_i) - f(x_i)\mathscr{G}(A'_i)\right| < \varepsilon' < \varepsilon,$$

which completes the proof.

□

Corollary II.70. Let A be a locally BV set, $\mathcal{F}$, $\mathscr{G}$ be charges in A. Let f be a function defined on $\operatorname{cl}_* A$. Then $\mathcal{F}$ is an intrinsic indefinite $\mathcal{R}$ integral of f in A with respect to $\mathscr{G}$ if and only if $\mathcal{F}$ is an indefinite $\mathcal{R}$ integral of f in A with respect to $\mathscr{G}$.

Theorem II.71. *Let A be an admissible locally BV set, $\mathcal{F}$, $\mathscr{G}$ be charges in A. Let f be a function defined on $\operatorname{cl}_* A$. Let $\mathcal{F}$ be an (intrinsic) indefinite $\mathcal{R}$ integral of f in A with respect to $\mathscr{G}$. Then $\mathcal{F}$ is also an indefinite packing $\mathcal{R}$ integral of f in A with respect to $\mathscr{G}$.*

Proof. Let us set $\tau := 1$. Now let us choose $\varepsilon > 0$ and find a gage δ as in Definition II.66. Let us fix a δ-fine packing $(B(x_i, r_i))_{i=1}^k$, $x_i \in \operatorname{cl}_* A$. We need to show that

$$\sum_{i=1}^{k} \bar{p}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) < \varepsilon,$$

where $\bar{p}^{\varepsilon}_{x,r}(\mathcal{F}) = \sup\left\{|\mathcal{F}(E)|; E \subset\subset B(x, r), E \in \boldsymbol{\mathcal{BV}}, (E, x) \text{ is } \varepsilon\text{-regular}\right\}.$

Now, let us fix test sets E_i such that $E_i \subset\subset B(x_i, r_i)$, E_i are BV sets and (E_i, x_i) are ε-regular for $i = 1, \ldots, k$. Obviously, the system $\{(E_i, x_i)\}$ is ε-regular δ-fine BV partition and hence by Definition II.66 and Theorem II.69 we have

$$\sum_{i=1}^{k} |\mathcal{F}(E_i) - f(x_i)\mathscr{G}(E_i)| < \varepsilon.$$

Passing to the supremum we obtain

$$\sum_{i=1}^{k} \bar{p}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) \leq \varepsilon.$$

□

II.7 $\mathcal{GR}$ integral

It can happen that a function which is $\mathcal{R}$ integrable in sets A_1 and A_2 is not $\mathcal{R}$ integrable in their union. Also, $\mathcal{R}$ integrability is not closed with respect to BV convergence of sets. To correct this deficiency, Pfeffer [17] extended the definition of the $\mathcal{R}$ integral. Fortunately, the construction based on the closure with respect to BV convergence of sets solves automatically the problem of additivity. The result of this construction is called $\mathcal{GR}$ integral (the generalized Riemann integral). Using our Theorem II.76 we show that also this $\mathcal{GR}$ integral is contained in our packing $\mathcal{R}^*$ integral.

Notation II.72. Let f be a function whose domain contains a locally BV set E and let $\mathcal{F}$ be a charge.

Then we denote by $\boldsymbol{R}(f, \mathcal{F}, E)$ the family of all bounded BV sets $A \subset E$ such that $f\chi_A$ belongs to $\mathcal{R}(A)$ and the charge $\mathcal{F}\lfloor_A$ is the indefinite $\mathcal{R}$ integral of $f\chi_A$.

Further, let us denote $\overline{\boldsymbol{R}}(f, \mathcal{F}, E)$ the minimal system of bounded BV sets containing $\boldsymbol{R}(f, \mathcal{F}, E)$ and closed with respect to convergence in $\boldsymbol{\mathcal{BV}}$.

Definition II.73. Let f be a function defined on a locally BV set E. We say that a charge $\mathcal{F}$ is an *indefinite $\mathcal{GR}$ integral* of f in E if $\overline{\boldsymbol{R}}(f, \mathcal{F}, E) = \boldsymbol{\mathcal{BV}}(E)$, where $\boldsymbol{\mathcal{BV}}(E) = \{A \in \boldsymbol{\mathcal{BV}}; A \subset E\}$. The family of all $\mathcal{GR}$ integrable functions in E is denoted by $\mathcal{GR}(E)$.

Remark II.74. The indefinite $\mathcal{GR}$ integral is well defined, unique and linear. For further details see [17, Sec. 6.3].

The next theorem with proof can be found in [17, Proposition 6.3.12].

Theorem II.75. *Let E be a locally BV set. Then*

1. *If $n = 1$, then $\mathcal{R}(E) = \mathcal{GR}(E)$.*
2. *If $n \geq 2$ and $\operatorname{int} E \neq \emptyset$, then $\mathcal{R}(E) \subsetneq \mathcal{GR}(E)$.*

Theorem II.76. *Let E be a bounded admissible BV set. Then $\mathcal{GR}(E) \subset \mathscr{PR}^*(E)$.*

Proof. The proof follows from Theorems II.42, II.71 and II.56. □

II.8 $\mathcal{R}^*$ integral

The $\mathcal{R}^*$ integral was introduced by Malý and Pfeffer in [13]. It is an alternative approach to overcome drawbacks of the $\mathcal{R}$ integral. Moreover, in $\mathbb{R}^1$ this integral coincides with the Henstock-Kurzweil integral.

Definition II.77. Let $\varepsilon > 0$. We say that an ε-regular BV partition $\{(A_1, x_1), \ldots, (A_k, x_k)\}$ is *strongly ε-regular* if A_i is ε-isoperimetric and $x_i \in \operatorname{cl}_* A_i$ for $i = 1, \ldots, k$.

Definition II.78. Let $A \subset \mathbb{R}^n$ be a locally BV set. We say that a charge $\mathcal{F}$ in A is an *indefinite $\mathcal{R}^*$ integral* of a function $f : \operatorname{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathscr{G}$ if for given $\varepsilon > 0$ we can find a gage $\delta : \operatorname{cl}_* A \to [0, \infty)$ so that

$$\sum_{i=1}^{k} \left| \mathcal{F}(A_i) - f(x_i)\mathscr{G}(A_i) \right| < \varepsilon$$

for each strongly ε-regular δ-fine BV partition $\{(A_1, x_1), \dots, (A_k, x_k)\}$.

The family of all $\mathcal{R}^*$ integrable functions in A is denoted by $\mathcal{R}^*(A, \mathscr{G})$. The family of all $\mathcal{R}^*$ integrable functions in A with respect to Lebesgue measure is denoted just by $\mathcal{R}^*(A)$.

Remark II.79. It is easily seen that for an admissible BV set E we have $\mathcal{R}(E) \subset \mathcal{R}^*(E)$.

Theorem II.80. *Let $A \subset \mathbb{R}^n$ be a locally BV set. Let a charge $\mathcal{F}$ be an indefinite $\mathcal{R}^*$ integral of a function $f : \mathrm{cl}_* A \to \mathbb{R}$ in A with respect to a charge $\mathscr{G}$. Then $\mathcal{F}$ is also an indefinite packing $\mathcal{R}^*$ integral of f in A with respect to $\mathscr{G}$.*

Proof. Let us set $\tau := 1$. Then let us choose $\varepsilon > 0$ and find a gage δ as in Definition II.78.

Let us fix a δ-fine packing $(B(x_i, r_i))_{i=1}^k$, $x_i \in \mathrm{cl}_* A$. We need to show that

$$\sum_{i=1}^{k} \overline{q}^{\varepsilon}_{x_i, \tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) < \varepsilon,$$

where

$$\overline{q}^{\varepsilon}_{x,r}(\mathcal{F}) = \sup\{|\mathcal{F}(E)|; E \subset\subset B(x,r), E \in \boldsymbol{\mathcal{BV}}, x \in \mathrm{cl}_* E, (E,x) \text{ is } \varepsilon\text{-regular and } \varepsilon\text{-isoperimetric}\}.$$

Now let us fix test sets E_i, $E_i \subset\subset B(x_i, r_i)$, $x_i \in \mathrm{cl}_* E_i$, E_i is BV and (E, x) is ε-regular and ε-isoperimetric for $i = 1, \dots, k$.

Obviously, the system $\{(E_i, x_i)\}$ is strongly ε-regular δ-fine BV partition and hence by Definition II.78 we obtain

$$\sum_{i=1}^{k} |\mathcal{F}(E_i) - f(x_i)\mathscr{G}(E_i)| < \varepsilon.$$

Passing to the supremum we obtain

$$\sum_{i=1}^{k} \overline{q}^{\varepsilon}_{x_i, \tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) \leq \varepsilon.$$

□

Remark II.81. Let E be an admissible locally BV set. Then $\mathcal{GR}(E) \subsetneq \mathcal{R}^*(E)$. The inclusion follows from [13, Corollary 3.18] and [13, Theorem 3.20]. An example of function which is $\mathcal{R}^*$ integrable but not $\mathcal{GR}$ integrable can be found in [15, Example 6.9] and [15, Proposition 10.8].

II.9 Henstock-Kurzweil-Stieltjes integral

In the next two sections we will investigate packing $\mathcal{R}$ and packing $\mathcal{R}^*$ integral on the real line. For this purpose let us note that a charge $\mathcal{F}$ in $\mathbb{R}^1$ can be identified with an "ordinary" function F through the relation $\mathcal{F}((u, v)) = F(v) - F(u)$. Since for those integral $\mathcal{F}$ is supposed to be a charge, in the view of Example II.24, F is continuous.

Definition II.82. Let $[a,b] \subset \mathbb{R}^1$ be a compact interval. A finite collection $([a_i,b_i],\xi_i)_{i=1}^k$ of tagged intervals is called a *subpartition* of $[a,b]$ if intervals $[a_i,b_i]$ are nonoverlapping and $\xi_i \in [a_i,b_i]$ for every $i = 1,\dots,k$.

A function $\delta : [a,b] \to (0,\infty)$ is called a *positive gage*. We say that a subpartition is δ-fine if $|b_i - a_i| < \delta(\xi_i)$.

Definition II.83. Let $f, G, F : [a,b] \to \mathbb{R}$ be functions. We say that F is the *strong Henstock-Kurzweil-Stieltjes integral* of f with respect to G if for every $\varepsilon > 0$, there exists a positive gage $\delta : [a,b] \to (0,\infty)$, so that for every δ-fine subpartition $([a_i,b_i],\xi_i)_{i=1}^k$ we have

$$\sum_{i=1}^{k} |F(b_i) - F(a_i) - f(\xi_i)(G(b_i) - G(a_i))| < \varepsilon.$$

In the case G is the identity function we say that F is just the *strong Henstock-Kurzweil integral* of f.

The families of all strongly Henstock-Kurzweil-Stieltjes integrable functions on $[a,b]$ with respect to G and all strongly Henstock-Kurzweil integrable functions on $[a,b]$ are denoted by $HKS([a,b],G)$ and $HK([a,b])$, respectively.

Definition II.84. Let $f, G, F : \mathbb{R} \to \mathbb{R}$ be functions. We say that F is the *indefinite Henstock-Kurzweil-Stieltjes integral* of f with respect to G if F is the strong Henstock-Kurzweil-Stieltjes integral of f with respect to G on every compact interval $[a,b] \subset \mathbb{R}$.

In the case G is the identity function we say that F is just the *indefinite Henstock-Kurzweil integral* of f.

The families of all Henstock-Kurzweil-Stieltjes integrable functions with respect to G and all Henstock-Kurzweil integrable functions are denoted by $HKS(\mathbb{R},G)$ and $HK(\mathbb{R})$, respectively.

The proof of the following proposition can be found in [13, Proposition 3.6].

Proposition II.85. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is $\mathcal{R}^*$ integrable with respect to Lebesgue measure if and only if f is strongly Henstock-Kurzweil integrable on every compact interval $[a,b] \subset \mathbb{R}$.*

Applying Theorem II.75, Remark II.81 and Proposition II.85 we obtain the following theorem.

Theorem II.86. $\mathcal{R}(\mathbb{R}) \subsetneq HK(\mathbb{R})$.

II.10 MC and MC_α integrals

In this section we will introduce MC and MC_α integrals. The monotonically controlled Stieltjes (MC) integral was defined by Bendová and Malý in [3]. The theory of the MC_α integral with respect to Lebesgue measure was further developed by Ball and Preiss in [2]. Their ideas will be used in the proofs of Propositions II.89 and II.90.

Definition II.87. Let $\alpha > 0$ be a real number and $f, F, G : \mathbb{R} \to \mathbb{R}$ be functions, let G be continuous. We say that F is an *indefinite* MC_α *integral* of f with respect to G if there exists a strictly increasing control function $\varphi : \mathbb{R} \to \mathbb{R}$ such that for each $x \in \mathbb{R}$ we have

$$\lim_{h\to 0} \frac{F(x+h) - F(x) - f(x)(G(x+h) - G(x))}{\varphi(x+\alpha h) - \varphi(x)} = 0.$$

The families of all MC_α integrable functions with respect to G and all MC_α integrable functions with respect to identity function are denoted by $MC_\alpha(G)$ and MC_α, respectively.

Especially, if $\alpha = 1$, we say that F is an *indefinite* MC *integral of* f *with respect to* G. We write $MC(G) = MC_1(G)$ and $MC = MC_1$.

Remark II.88. In Definition II.87 the control function φ can be chosen to be bounded. (See [3, Lemma 1].)

Proposition II.89. *Let* $\alpha > 0$ *and let* $f, F, G : \mathbb{R} \to \mathbb{R}$ *be functions, let* G *be continuous. If* F *is an indefinite* MC_α *integral of* f *with respect to* G*, then* F *is continuous.*

Proof. Let us fix $\varepsilon > 0$ and $x \in \mathbb{R}$. We need to find δ such that for every $|h| < \delta$ we have

$$|F(x+h) - F(x)| < \varepsilon.$$

Since G is continuous at x, we can find δ_1 such that for every $|h| < \delta_1$ we have $|G(x+h) - G(x)| < \varepsilon$. Further, since F is the indefinite MC_α integral of f with respect to G, there exists a strictly increasing control function $\varphi : \mathbb{R} \to \mathbb{R}$ and a $\delta < \delta_1$ such that for every $|h| < \delta$ we have

$$\left| \frac{F(x+h) - F(x) - f(x)(G(x+h) - G(x))}{\varphi(x+\alpha h) - \varphi(x)} \right| < \varepsilon.$$

Applying Remark II.88 we can assume that there exists a constant M such that $|\varphi(x)| < M$ for every $x \in \mathbb{R}$.

Hence

$$\begin{aligned} &|F(x+h) - F(x)| \\ &\quad \leq \left| \frac{F(x+h) - F(x) - f(x)(G(x+h) - G(x))}{\varphi(x+\alpha h) - \varphi(x)} (\varphi(x+\alpha h) - \varphi(x)) \right| \\ &\qquad + |f(x)(G(x+h) - G(x))| \\ &\quad < \varepsilon(2M + f(x)). \end{aligned}$$

□

Proposition II.90. *Let* $0 < \alpha < \beta$ *be real numbers,* $f, F, G : \mathbb{R} \to \mathbb{R}$ *be functions and let* G *be continuous. If* F *is an indefinite* MC_α *integral of* f *with respect to* G*, then* F *is also an indefinite* MC_β *integral of* f *with respect to* G*.*

Proof. The proof follows from the fact that for $0 < \alpha < \beta$ we have $|\varphi(x+\alpha h) - \varphi(x)| \leq |\varphi(x+\beta h) - \varphi(x)|$ for $h \in \mathbb{R}$. □

The two following theorems can be found in [2, Theorem 3].

Theorem II.91. *For every $\alpha \geq 2$ there exists a function which is not MC_α integrable but is MC_β integrable for every $\beta > \alpha$.*

Theorem II.92. *Let $\alpha > 2$. Then MC is a proper subspace of MC_α.*

For the proof of the next theorem see [2, Theorem 3].

Theorem II.93. *Let $\alpha \in [1,2]$. Then $MC = MC_\alpha$.*

Theorem II.94. *Let $G, F, f : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that G is continuous. Then F is an indefinite MC integral of f with respect to G if and only if F is an indefinite Henstock-Kurzweil-Stieltjes integral of f with respect to G.*

Proof. For the proof and further details see [3, Theorem 3] and [2, Theorem 17]. □

Theorem II.95. *Let $\alpha \geq 1$, $G, F, f : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that G is continuous. Let F be an indefinite MC_α integral of f with respect to G. Further, let $\mathcal{F}$ and $\mathscr{G}$ be charges induced by F and G in the sense of Example II.24. Then $\mathcal{F}$ is also an indefinite packing $\mathcal{R}$ integral of f with respect to $\mathscr{G}$.*

Proof. First, let us note that F is continuous by Proposition II.89. Hence it is legitimate to use the term charges for the set functions $\mathcal{F}$ and $\mathscr{G}$ constructed as in Example II.24.

Let us set $\tau := 1/\alpha$. Further, let us fix $\varepsilon > 0$ and write $\varepsilon' := \varepsilon^2$. Since f is MC_α integrable, there exists a strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ with the following property: for each $x \in \mathbb{R}$ there exists $\delta(x) > 0$ such that for every $|h| < \delta(x)$ we have

$$|F(x+h) - F(x) - f(x)(G(x+h) - G(x))| < \varepsilon' |\varphi(x + \alpha h) - \varphi(x)|. \quad \text{(II.25)}$$

Moreover, by Remark II.88 we can suppose that there exists $M > 0$ such that $|\varphi| \leq M$.

We need to show that for fixed δ-fine packing $(B(x_i, r_i))_{i=1}^k$, we have

$$\sum_{i=1}^{k} \bar{p}^{\varepsilon}_{x_i, \tau r_i}(\mathcal{F} - f(x_i)\mathscr{G}) < \varepsilon,$$

where $\bar{p}^{\varepsilon}_{x_i,\tau r_i}(\mathcal{F}) = \sup \{ |\mathcal{F}(E)|; E \subset\subset B(x_i, \tau r_i), E \in \boldsymbol{\mathcal{BV}}, (E, x_i) \text{ is } \varepsilon\text{-regular} \}$.

Let us fix $i \in \{1, \ldots, k\}$ and a test set $E_i \in \boldsymbol{\mathcal{BV}}$ such that $E_i \subset\subset B(x_i, \tau r_i)$ and (E_i, x_i) is ε-regular. In other words, $E_i = \bigcup_{j=1}^{l_i} (a^i_j, b^i_j)$ is a finite union of disjoint nondegenerate intervals in $B(x_i, \tau r_i)$ (up to a Lebesgue null set). Moreover, since E_i is ε-regular and $\mathscr{H}^0$ is the counting measure, we estimate

$$\frac{1}{\|E_i\|} \geq \frac{|E_i|}{d(E_i \cup \{x_i\}) \|E_i\|} > \varepsilon$$

and

$$\frac{1}{\varepsilon} \geq \|E_i\| = \mathscr{H}^0(\partial_* E_i) = 2 l_i.$$

Let us set m to be the greatest natural number such that $m \leq 1/(2\varepsilon)$. Then $l_i \leq m \leq 1/(2\varepsilon)$.

Further, since for each a^i_j and b^i_j, $i = 1, \dots, k$ and $j = 1, \dots, l_i$, we have $|a^i_j - x_i| < \delta(x_i)$ and $|b^i_j - x_i| < \delta(x_i)$, by (II.25) and the fact that φ is increasing we have the estimates

$$\left|F\left(b^i_j\right) - \mathcal{F}(x_i) - f(x_i)(G(b^i_j) - G(x_i))\right| < \varepsilon' \left|\varphi\left(x_i + r_i\right) - \varphi(x_i - r_i)\right|$$

and

$$\left|F(a^i_j) - \mathcal{F}(x_i) - f(x_i)(G(a^i_j) - G(x_i))\right| < \varepsilon' \left|\varphi\left(x_i + r_i\right) - \varphi(x_i - r_i)\right|.$$

Moreover, since the system $(B(x_i, r_i))_{i=1}^k$ is pairwise disjoint and φ is strictly increasing and bounded, we have

$$\begin{aligned}
&\sum_{i=1}^k \left|F(b^i_j) - F(a^i_j) - f(x_i)(G(b^i_j) - G(a^i_j))\right| \\
&\qquad\leq \sum_{i=1}^k \left|F(b^i_j) - \mathcal{F}(x_i) - f(x_i)(G(b^i_j) - G(x_i))\right| \\
&\qquad\quad + \left|F(a^i_j) - \mathcal{F}(x_i) - f(x_i)(G(a^i_j) - G(x_i))\right| \\
&\qquad\leq \sum_{i=1}^k 2\varepsilon' \left|\varphi\left(x_i + r_i\right) - \varphi(x_i - r_i)\right| \\
&\qquad< 2\varepsilon'(\varphi(x_k + r_k) - \varphi(x_1 - r_1)) \\
&\qquad< 4\varepsilon' M.
\end{aligned} \tag{II.26}$$

Let us denote $L := \max_i l_i$. For $j \in \{1, \dots, L\}$ let I_j be the set of indices $i \in \{1, \dots, k\}$ for which $l_i \geq j$. Then applying estimates in (II.26) we obtain

$$\begin{aligned}
\sum_{i=1}^k |\mathcal{F}(E_i) - f(x_i)\mathscr{G}(E_i)| &\leq \sum_{i=1}^k \sum_{j=1}^{l_i} \left|F(b^i_j) - F(a^i_j) - f(x_i)(G(b^i_j) - G(a^i_j))\right| \\
&\leq \sum_{i=1}^k \sum_{j=1}^{l_i} \left|F(b^i_j) - \mathcal{F}(x_i) - f(x_i)(G(b^i_j) - G(x_i))\right| \\
&\quad + \left|F(a^i_j) - \mathcal{F}(x_i) - f(x_i)(G(a^i_j) - G(x_i))\right| \\
&\leq \sum_{j=1}^L \sum_{i \in I_j} \left|F(b^i_j) - \mathcal{F}(x_i) - f(x_i)(G(b^i_j) - G(x_i))\right| \\
&\quad + \left|F(a^i_j) - \mathcal{F}(x_i) - f(x_i)(G(a^i_j) - G(x_i))\right| \\
&< \sum_{j=1}^L 4\varepsilon' M = 4L\varepsilon^2 M \leq \frac{4\varepsilon^2 M}{2\varepsilon} = 2M\varepsilon.
\end{aligned}$$

Finally, passing to the supremum we obtain

$$\sum_{i=1}^k \bar{p}^{\varepsilon}_{x_i, r_i}(\mathcal{F} - f(x_i)\mathscr{G}) \leq 2M\varepsilon,$$

which we needed.

$\square$

II.11 Summary of relations

Let $A \subset \mathbb{R}^n$ be an admissible locally BV set. The relation between classes of integrable functions in A is shown in the following diagram.

$$\begin{tikzcd}
\mathcal{IR} \arrow[r, phantom, "\subsetneq"] \arrow[d, phantom, "\parallel"] & \mathscr{G}\mathcal{R} \arrow[r, phantom, "\subsetneq"] & \mathcal{R}^* \arrow[d, phantom, "\rotatebox{-90}{$\subsetneq$}"] \\
\mathcal{R} \arrow[r, phantom, "\subsetneq"] & \mathscr{P}\mathcal{R} \arrow[r, phantom, "\subset"] & \mathscr{P}\mathcal{R}^*
\end{tikzcd}$$

The strictness of the inclusion $\mathcal{IR} \subset \mathscr{G}\mathcal{R}$ holds for $n \geq 2$ and can be found in Theorem II.75(2) and Corollary II.70; the case $n = 1$ is discussed below. The fact that $\mathscr{G}\mathcal{R} \subsetneq \mathcal{R}^*$ is mentioned in Remark II.81. Corollary II.70 shows the equality of $\mathcal{IR}$ and $\mathcal{R}$. The relationship $\mathcal{R} \subsetneq \mathscr{P}\mathcal{R}$ is described in Theorem II.71; Theorems II.86, II.95 and II.94 show that this inclusion is strict. The inclusion $\mathscr{P}\mathcal{R} \subset \mathscr{P}\mathcal{R}^*$ is proved in Theorem II.42. Theorem II.80 proves the inclusion $\mathcal{R}^* \subsetneq \mathscr{P}\mathcal{R}^*$, the fact, that this inclusions is proper follows from Theorems II.95, II.94, II.92 and Proposition II.85.

In the case $A = \mathbb{R}$, we can compare integrable functions in the following way.

$$\begin{tikzcd}
\mathscr{G}\mathcal{R} \arrow[r, phantom, "="] & \mathcal{R} \arrow[r, phantom, "\subsetneq"] & HK \arrow[r, phantom, "="] \arrow[d, phantom, "\parallel"] & MC \arrow[r, phantom, "\subsetneq"] \arrow[d, phantom, "\parallel"] & MC_\beta \arrow[r, phantom, "\subsetneq"] & \mathscr{P}\mathcal{R} \arrow[r, phantom, "\subset"] & \mathscr{P}\mathcal{R}^* \\
 & & \mathcal{R}^* & MC_\alpha & & &
\end{tikzcd}$$

The equality $\mathcal{GR} = \mathcal{R}$ is described in Theorem II.75(1) and the inclusion $\mathcal{R} \subsetneq HK$ in Theorem II.86. The fact that HK integral coincides with $\mathcal{R}^*$ integral can be found in II.85. Theorem II.94 shows the equality $HK = MC$. Theorem II.93 proves the equality $MC = MC_\alpha$ for $\alpha \in [1, 2]$. The inclusion $MC \subsetneq MC_\beta$ for $\beta > 2$ is proved in Proposition II.90, the fact, that this inclusion is proper is shown in Theorem II.92. The relationship $MC_\beta \subsetneq \mathscr{P}\mathcal{R}$ (not only) for $\beta \geq 2$ is proved in Theorem II.95 and II.91. Finally, the inclusion $\mathscr{P}\mathcal{R} \subset \mathscr{P}\mathcal{R}^*$ is shown in Theorem II.42.

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III. On a generalization of Henstock-Kurzweil integrals

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abstract. We study a scale of integrals on the real line motivated by the MC_α integral by Ball and Preiss and some recent multidimensional constructions of integral. These integrals are non-absolutely convergent and contain the Henstock-Kurzweil integral. Most of results are of comparison nature. Further, we show that our indefinite integrals are a.e. approximately differentiable. An example of approximate discontinuity of an indefinite integral is also presented.

III.1 Introduction

The Riemann approach to integration of a function $f\colon I \to \mathbb{R}$ is based on limits of sums

$$\sum_{i=1}^{m} f(x_i)(b_i - a_i),$$

where $\{[a_i, b_i], x_i\}_{i=1}^m$ is a complete tagged partition of the interval I. By this we mean that the intervals $[a_i, b_i]$ are nonoverlapping, their union is I and $x_i \in [a_i, b_i]$. The improvement by Henstock and Kurzweil consist in the requirement that the partitions are δ-fine for some gage δ. This trick makes the class of integrable functions much wider, in particular, the Henstock-Kurzweil integral extends the Lebesgue integral and integrates all derivatives.

By the Saks-Henstock lemma, the corresponding indefinite integral F of f is characterized by smallness of the sums

$$\sum_{i=1}^{m} |F(b_i) - F(a_i) - f(x_i)(b_i - a_i)|,$$

for this $\{[a_i, b_i], x_i\}_{i=1}^m$ can be an "incomplete" partition, we omit the requirement concerning the union of the intervals $[a_i, b_i]$. Throughout this paper, the term partition will always refer to an incomplete partition.

The aim of this paper is to study a scale of non-absolutely convergent integrals which includes some integrals introduced recently. The common feature of these new integrals is that we estimate the expression $|F(y) - F(x_i) - f(x_i)(y - x_i)|$ on the partition intervals $[a_i, b_i]$ whereas their multiples denoted as $(\bar{a}_i, \bar{b}_i)$ are assumed to be pairwise disjoint.

A multidimensional modification of this idea is to estimate the expression $|F(y) - F(x_i) - f(x_i)(y - x_i)|$ on balls $B(x_i, r_i)$ and to assume that the multiples $B(x_i, \alpha r_i)$ are pairwise disjoint. This leads to the so called packing integrals in [28], [24], [15], [23], investigated in Euclidean or even metric spaces. A natural question arises what happens with these integrals if we consider them in the one-dimensional situation.

On the other hand, we want also to include a scale of one-dimensional monotonically controlled integrals studied by Ball and Preiss in [1]. The monotone control is a descriptive approach introduced in [2] which gives an alternative to Riemann-type constructive definitions.

We introduce the scales of HK_α^p integrals and centered HK_α^p integrals. They are based on partitions $\{[a_i, b_i], x_i\}_{i=1}^m$. The parameter α says that the α-multiples of the partition intervals are assumed to be pairwise disjoint. The parameter p is the Lebesgue exponent of the L^p-norm used to measure the p-oscillation of the expression $|F(y) - F(x_i) - f(x_i)(y - x_i)|$ in $[a_i, b_i]$. If the parameter p is skipped or is equal to the symbol C, it means that the supremum norm is used instead. Precise definitions are in Section III.3.

All integrals considered here are investigated as indefinite integrals. Definite integrals can be introduced as increments of indefinite integrals.

We show that the HK_α integral is exactly the MC_α integral of Ball and Preiss [1] (Theorem III.14). Therefore the results of [1] formulated in terms of MC_α integrals can be applied to the scale of HK_α-integrals as well. The centered HK_α integral is the one-dimensional α-packing BV integral from [23] (Theorem III.42). The centered HK_α^1 integral is the α-packing Lip integral from [24] (Theorem III.35).

Further, we show that the classes of HK_α^p-integrable functions are distinct for different p (Theorem III.20) and that the classes of centered HK_α^p-integrable functions differ from uncentered ones (Theorem III.22).

As shown in [1], the class of HK_α integrable functions contains the class of Henstock-Kurzweil integrable functions, and the inclusion is strict if $\alpha > 2$. Thus, also the classes of HK_α^p integrable functions contain the class of Henstock-Kurzweil integrable functions and the inclusion is strict if $\alpha > 2$, or $p > 1$, or the centered version is considered.

There is a huge variety of non-absolute convergent integrals which also contain the Henstock-Kurzweil integral strictly. The most famous of them is the Denjoy-Khintchine integral [7], [21]. Hence it is interesting to compare the Denjoy-Khintchine integral with integrals of our scale. For the HK_α integrals it has been done in [1], we extend it to the entire scale. The result is that there is no inclusion between (centered or uncentered) HK_α^p-integrable functions and Denjoy-Khintchine integrable function (with the exception of the case of uncentered HK_α for $\alpha \leq 2$). See Theorem III.19.

The new non-inclusion is that the HK_α^p-integral is not contained in the Denjoy-Khintchine integral. But much more is true. There is a variety of so called approximately continuous integrals, with the property that the indefinite integral is approximately continuous, see e.g. [4], [22], [40], [9], [10]. Also these integrals do not contain the packing integral in view of our Theorem III.28. It shows that there is a function f on $\mathbb{R}$ such that its indefinite HK_1^p integral is not approximately continuous at the origin.

Most of our results concern comparison of various classes of integrable functions. To make the list of main results of the present paper complete, let us mention Theorem III.26 which states that each (centered) HK_α^p-integrable function f is at almost every point the approximate derivative of its indefinite (centered) HK_α^p-integral.

The motivation to study nonabsolutely convergent integrals originates from the task to integrate all derivatives and all Lebesgue integrable functions simultaneously. Similarly, the motivation for the multi-dimensional nonabsolutely convergent integrals comes from the task to integrate all divergences or even "generalized divergences" and pass to an application to the divergence theorem. A brief account of the history is postponed to the last section.

III.2 Preliminaries

The open ball in $\mathbb{R}^n$ with center at x and radius r is denoted by $B(x,r)$, whereas $\overline{B}(x,r)$ stands for the corresponding closed ball. If E is a set, χ_E denotes the characteristic function of E. The symbol $|E|$ means the (outer) Lebesgue measure of a set $E \subset \mathbb{R}^n$. The identity function $x \mapsto x$ on an interval I is denoted by Id. If $\Omega \subset \mathbb{R}^n$ is an open set, the symbol $\mathcal{D}(\Omega)$ stands for the set of all infinitely differentiable function with compact support in Ω. A collection of intervals is said to be *nonoverlapping* if their interiors are pairwise disjoint.

III.2.1 Regulated functions

We say that $F\colon [a,b] \to \mathbb{R}$ is a *regulated function* if all one-sided limits of F exist and are finite. The space of all regulated functions equipped with the supremum norm is a Banach space. See [32] for details.

III.2.2 Approximate limit and derivative

We say that $x \in \mathbb{R}$ is a *density point* for a set $E \subset \mathbb{R}$ if

$$\lim_{r\to 0+} \frac{|(x-r,x+r)\setminus E|}{2r} = 0.$$

Let $I \subset \mathbb{R}$ be an open interval. We say that a value $A \in \mathbb{R}$ is an *approximate limit* of a function $F\colon I \to \mathbb{R}$ at a point $x \in I$ if for each $\varepsilon > 0$ there exists a set $E_\varepsilon \subset I$ such that x is a density point of E_ε and $|F - A| < \varepsilon$ on E_ε. Approximate derivative is defined as the approximate limit of difference quotients. See e.g. [41, VII.3] for details.

III.2.3 Denjoy-Khintchine integral

For the description of the Denjoy-Khintchine integral we use the equivalent definition according to [41], which follows the descriptive idea of Luzin [27].

Definition III.1. Let $I = (a,b)$ be an open interval. A function $F : I \to \mathbb{R}$ is said to be AC (*absolutely continuous*) on a set $E \subset I$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite sequence $([a_j,b_j])_{j=1}^m$ of nonoverlapping intervals with endpoints in E we have

$$\sum_{j=1}^m (b_j - a_j) < \delta \implies \sum_{j=1}^m |F(b_j) - F(a_j)| < \varepsilon.$$

We say that F is ACG (*generalized absolutely continuous*) on I if F is continuous on I and there exists a sequence $(E_k)_k$ of subsets of I such that $I = \bigcup_k E_k$ and F is AC on each E_k.

Given a function $f : I \to \mathbb{R}$, we say that $F : I \to \mathbb{R}$ is an *indefinite Denjoy-Khintchine integral* of f if F is ACG on I and f is the approximate derivative of F a.e.

Remark III.2. Every ACG function has an approximate derivative almost everywhere and therefore it acts as its indefinite Denjoy-Khintchine integral, see [41, VII, Theorem 4.3].

III.2.4 Oscillations

Definition III.3 (Oscillations)**.** Let $[a, b] \subset \mathbb{R}$ be a closed interval and $p \in [1, \infty]$. We define the p-oscillation of a measurable function $F\colon [a, b] \to \mathbb{R}$ as

$$\operatorname{osc}_p(F, [a, b]) = (b - a)^{-1/p} \inf\{\|F - c\|_{L^p([a,b])} \colon c \in \mathbb{R}\}. \tag{III.1}$$

Here and in the sequel $1/p = 0$ if $p = \infty$.

The ordinary oscillation

$$\operatorname{osc}(F, [a, b]) = \operatorname{osc}_C(F, [a, b]) := \tfrac{1}{2} \sup_{x,y \in [a,b]} |F(y) - F(x)|$$

differs from $\operatorname{osc}_\infty$ in the aspect that it does not neglect Lebesgue null sets. The subscript C refers to the space of continuous function and the somewhat unusual factor $\frac{1}{2}$ is an output of the usage of the supremum norm instead of the L^p-norm in (III.1). To simplify the presentation, we consider the symbol C as a possible value of p and $1/p$ is 0 for $p = C$. This convention will be used to include the choices of oscillation all at once.

Remark III.4. Observe the elementary but useful inequality

$$[a, b] \subset [A, B] \implies (b - a)^{1/p} \operatorname{osc}_p(F, [a, b]) \le (B - A)^{1/p} \operatorname{osc}_p(F, [A, B]) \tag{III.2}$$

which holds for a measurable function $F\colon [A, B] \to \mathbb{R}$ and $p \in [1, \infty] \cup \{C\}$.

Definition III.5 (Median)**.** Let F be a measurable function on an interval $[a, b]$ and $\mu \in \mathbb{R}^n$. We say that μ is a *median* of F in $[a, b]$ if there exists a measurable set $M \subset [a, b]$ such that $F \le \mu$ on M, $F \ge \mu$ on $[a, b] \setminus M$ and $|M| = \frac{1}{2}(b-a)$. Each measurable function on $[a, b]$ has a median. On the other hand, its uniqueness is not guaranteed; it holds only under some additional assumptions like continuity of F. Medians give a useful choice of the constant c in (III.1). As shown in the following proposition, they yield a good estimate for all p and for $p = 1$ they are even minimizers.

Proposition III.6. *Let μ be a median of F in $[a, b]$ and $p \in [1, \infty] \cup \{C\}$. Then*

$$\operatorname{osc}_p(F, [a, b]) \le (b - a)^{-1/p} \|F - \mu\|_p \le 2^{1-\frac{1}{p}} \operatorname{osc}_p(F, [a, b])$$

In particular,

$$(b - a)^{-1} \|F - \mu\|_1 = \operatorname{osc}_1(F, [a, b]).$$

Proof. The first inequality is trivial, let us concentrate on the second one. We may assume $\mu = 0$. Consider a measurable set $M_+ \subset [a,b]$ such that $|M_+| = \frac{1}{2}(b-a)$ and $F \geq 0$ on M_+, $F \leq 0$ on $M_- := [a,b] \setminus M_+$. Choose $c \in \mathbb{R}$, e.g. $c \geq 0$. We have

$$\begin{aligned}&|F(x)|^p \leq 2^{p-1}(|F(x)-c|^p + c^p), \qquad x \in M_+\\ &|F(x)|^p \leq |c + |F(x)|\,|^p - c^p \leq 2^{p-1}(|F(x)-c|^p - c^p), \qquad x \in M_-.\end{aligned}$$

Integrating over $[a,b]$ we obtain

$$\int_{[a,b]} |F(x)|^p \, dx \leq 2^{p-1} \int_{[a,b]} |F(x)-c|^p \, dx,$$

as $|M_+| = |M_-|$. Taking infimum over c we obtain

$$\|F\|_p^p \leq 2^{p-1}(b-a)(\operatorname{osc}_p F)^p$$

as required. □

III.3 The definition of integral

Definition III.7. Let $\alpha \geq 1$, $p \in [1,\infty] \cup \{C\}$ and $I \subset \mathbb{R}$ be an open interval. A finite family $([a_i,b_i],x_i)_{i=1}^m$, where $[a_i,b_i] \subset I$ are closed intervals and $x_i \in [a_i,b_i]$, is called an *α-partition* in I if the intervals $(\bar{a}_i, \bar{b}_i)$, where

$$\bar{a}_i - x_i = \alpha(a_i - x_i), \quad \bar{b}_i - x_i = \alpha(b_i - x_i), \tag{III.3}$$

are subsets of I and pairwise disjoint. We say that a partition $([a_i,b_i],x_i)_{i=1}^m$ is *centered* if each x_i is the center of $[a_i,b_i]$. Let $\delta\colon I \to (0,\infty)$ be a *gage* (this means just a strictly positive function). We say that the α-partition is *δ-fine* if $[a_i,b_i] \subset (x_i - \delta(x_i),\, x_i + \delta(x_i))$, $i = 1,\dots,m$.

Definition III.8 (HKS^p_α integrals)**.** Let $I \subset \mathbb{R}$ be an open interval, $p \in [1,\infty] \cup \{C\}$ and $\alpha \geq 1$. Let F, G, f be measurable functions on I. We say that F is an *indefinite HKS^p_α-integral* (HKS refers to Henstock-Kurzweil-Stieltjes) of f with respect to G if for each $\varepsilon > 0$ there exists a gage $\delta\colon I \to (0,\infty)$ such that for each δ-fine α-partition $([a_i,b_i],x_i)_{i=1}^m$ in I we have

$$\sum_{i=1}^m \operatorname{osc}_p(F - f(x_i)G,\ [a_i,b_i]) < \varepsilon.$$

We denote $HKS_\alpha = HKS^C_\alpha$. We reduce the symbol to HK^p_α or HK_α, respectively, if G is the identity $\operatorname{Id}(x) = x$. We call the integral *centered* if only centered α-partitions are taken into account. We say that F is a *free indefinite HKS^p-integral* of f with respect to G if there exists $\alpha \geq 1$ such that F is an *indefinite HKS^p_α-integral* of f with respect to G, similarly to centered versions.

Remark III.9. There are obvious inclusions between the classes of integrable functions. The class of HKS^p_α-integrable functions increases with α and the class of all free HKS^p-integrable functions is the union of the preceding ones over α.

The centered version always leads to a wider class of integrable functions.

Using comparison of L^p norms, we also observe that the class of HKS^p_α-integrable functions decreases with p.

The indefinite integrals to a function f are the same for all choices α, p which make f integrable (with the exception that for $p = C$ only continuous representatives are valid).

Remark III.10. Even if we do not assume that f is measurable in Definition III.8, the measurability of f comes out as a consequence of HKS^p_α-integrability, see [28, Theorem 5.3].

When defining a new notion of indefinite integral, it is desirable to show that this has the expected uniqueness behavior, namely that the indefinite integrals to the same functions differ only by an additive constant.

Theorem III.11. *Let f, F_1, F_2, G be measurable functions on an open interval I. If F_1 and F_2 are indefinite (centered) HKS^p_α-integrals of f with respect to G, then there is a constant $C \in \mathbb{R}$ such that $F_2 - F_1 = C$ a.e.*

Proof. By Remark III.9, it is possible to reduce the question to to uniqueness of centered HKS^α_1. This follows from [24, Theorem 3.10], see Theorem III.35. ☐

Theorem III.12. *Let f, F_1, F_2, G be measurable functions on an open interval I, F_1, F_2, G be regulated. If F_1 and F_2 are indefinite (centered) HKS_α-integrals of f with respect to G, then there is a constant $C \in \mathbb{R}$ such that $F_2 - F_1 = C$.*

Proof. Obviously $F_2 - F_1$ is an indefinite (centered) HK_α-integral of 0 and hence by Theorem III.11, $F_2 - F_1 = C$ a.e. Since $F_2 - F_1$ is regulated, the equality turns to hold everywhere. ☐

Proposition III.13. *Let $I \subset \mathbb{R}$ be an open interval and $F, f, G\colon I \to \mathbb{R}$ be measurable functions. Then F is an indefinite HKS_1-integral of f if and only if F is an indefinite Henstock-Kurzweil-Stieltjes integral of f.*

Proof. Suppose that F is an indefinite Henstock-Kurzweil-Stieltjes integral of f. By the Saks-Henstock lemma ([32, Lemma 6.5.1]), for each $\varepsilon > 0$ there exists a gage $\delta\colon I \to (0, \infty)$ such that for each δ-fine partition $\{([a_i, b_i], x_i)\}_{i=1}^m$ we have

$$\sum_{i=1}^m |F(b_i) - F(a_i) - f(x_i)(G(b_i) - G(a_i))| < \varepsilon.$$

Consider a δ-fine partition $\{([A_i, B_i], x_i)\}_{i=1}^m$. For each j we find $z_i \in [A_i, B_i]$ such that $z_i \neq x_i$ and

$$|F(z_i) - F(x_i) - f(x_i)(G(z_i) - G(x_i))| \geq \frac{1}{2}\operatorname{osc}(F - f(x_i)G, [A_i, B_i]).$$

Set

$$[a_i, b_i] = \begin{cases} [z_i, x_i], & z_i < x_i, \\ [x_i, z_i], & x_i < z_i. \end{cases}$$

Then $\{([a_i, b_i], x_i)\}_{i=1}^m$ is a δ-fine partition and thus

$$\sum_{i=1}^m \operatorname{osc}(F - f(x_i)G, [A_i, B_i]) \leq 2\sum_{i=1}^m |F(b_i) - F(a_i) - f(x_i)(G(b_i) - G(a_i)) < 2\varepsilon.$$

It follows that F an indefinite HKS_1 integral of f. The converse implication is obvious. ☐

III.4 Monotone control

Let $I \subset \mathbb{R}$ be an interval and $p \in [1,\infty] \cup \{C\}$ be fixed. Let f, F, G be measurable functions on I. We say that an increasing function $\varphi\colon I \to \mathbb{R}$ is an *α-control function* for the triple (f, F, G) if for each $x \in I$ we have

$$\lim_{r\to 0+} \frac{\operatorname{osc}_p(F - f(x)G,\ [x, x+r])}{\varphi(x+\alpha r) - \varphi(x)} = \lim_{r\to 0+} \frac{\operatorname{osc}_p(F - f(x)G,\ [x-r, x])}{\varphi(x) - \varphi(x-\alpha r)} = 0.$$

We say that an increasing function $\varphi\colon I \to \mathbb{R}$ is a *centered α-control function* for the triple (f, F, G) and HK_α-integration if for each $x \in I$ we have

$$\lim_{r\to 0+} \frac{\operatorname{osc}_p(F - f(x)G,\ [x-r, x+r])}{\varphi(x+\alpha r) - \varphi(x-\alpha r)} = 0,$$

Following Ball and Preiss [1], we say that F is an indefinite MC_α integral of f if there exists an α-control function for (f, F, Id) and the choice $p = C$. In particular, the MC_1 integral is the MC integral of [2].

Theorem III.14. *Let $I \subset \mathbb{R}$ be an interval and $p \in [1,\infty] \cup \{C\}$ be fixed. Let f, F, G be measurable functions on I. Then F is an indefinite HKS^p_α-integral of f with respect to G if and only if there exists an α-control function for the triple (f, F, G).*

In particular, F is an indefinite HK_α-integral of f if and only if F is an indefinite MC_α-integral of f.

Proof. Suppose that the α-control function φ exists. We may assume that $|\varphi|$ is bounded by $\frac{1}{2}$ (otherwise φ can be replaced by $\frac{1}{\pi}\arctan\varphi$). Given $\varepsilon > 0$, for each $x \in I$ we can find $\delta(x) > 0$ such that

$$x \in [a,b] \subset (x - \delta(x), x + \delta(x)) \implies \operatorname{osc}_p(F - f(x)G,\ [a,b]) < \varepsilon(\varphi(\bar{b}) - \varphi(\bar{a})),$$

where

$$\bar{a} = x + \alpha(a - x), \quad \bar{b} = x + \alpha(b - x).$$

Then δ is the desired gage. Indeed, if $([a_i, b_i], x_i)_{i=1}^m$ is a δ-fine α-partition, then

$$\sum_{i=1}^m \operatorname{osc}_p(F - f(x_i)G,\ [a_i, b_i]) \le \varepsilon \sum_{i=1}^m (\varphi(\bar{b}_i) - \varphi(\bar{a}_i)) \le \varepsilon \operatorname{osc}(\varphi, I) \le \varepsilon.$$

For the reverse implication we introduce the following variation depending on an open interval $J \subset I$ and a gage δ:

$$V(J,\delta) = V(J,\delta,f,F,G) = \sup\Big\{ \sum_{i=1}^m \operatorname{osc}_p(F - f(x_i)G,\ [a_i, b_i]) : ([a_i, b_i], x_i)_{i=1}^m \text{ is a } \delta\text{-fine } \alpha\text{-partition in } J \Big\}.$$

For each $k = 1, 2, \dots$ we find a gage $\delta_k\colon I \to (0,\infty)$ such that

$$V(I, \delta_k) < 2^{-k}$$

and set

$$\varphi_k(x) = V(I \cap (-\infty, x),\, \delta_k),$$
$$\varphi(x) = x + \sum_{k=1}^{\infty} k\varphi_k(x).$$

Then $\varphi\colon I \to \mathbb{R}$ is a strictly increasing function. We want to show that φ is an α-control function to the triple (f, F, G). Fix $x \in I$ and choose $\varepsilon > 0$. Find $k \in \mathbb{N}$ such that $1/k < \varepsilon$. If $x \in [a, b] \subset (x - \delta(x), x + \delta(x))$, then for each α-partition $([a_i, b_i], x_i)_{i=1}^m$ in $I \cap (-\infty, \bar{a})$ we observe that $([a_i, b_i], x_i)_{i=1}^{m+1}$ is an α-partition in $I \cap (-\infty, \bar{b})$, where we set $([a_{m+1}, b_{m+1}], x_{m+1}) = ([a, b], x)$. Hence

$$\varphi_k(\bar{a}) + \operatorname{osc}_p(F - f(x)G,\, [a, b]) \le \varphi_k(\bar{b})$$

and thus

$$\frac{\operatorname{osc}_p(F - f(x)G,\, [a, b])}{\varphi(\bar{b}) - \varphi(\bar{a})} \le \frac{\varphi_k(\bar{b}) - \varphi_k(\bar{a})}{\varphi(\bar{b}) - \varphi(\bar{a})} \le \frac{1}{k} < \varepsilon$$

as required. ☐

Theorem III.15. *Let $I \subset \mathbb{R}$ be an interval $p \in [1, \infty] \cup \{C\}$. Let f, F, G be measurable functions on I. Then F is an indefinite centered HKS_α^p-integral of f with respect to G if and only if there exists a centered α-control function for the triple f, F, G.*

Proof. The proof is almost the same as that of Theorem III.14 with obvious modifications. ☐

III.5 Counterexamples

Definition III.16. We denote by $\{0, 1\}^k$ the family of all multiindices $s = (s_1, \ldots, s_k)$, where $s_1, \ldots, s_k \in \{0, 1\}$. The set $\{0, 1\}^0$ contains just one element denoted by o. We simplify the symbols (0), $(1) \in \{0, 1\}^1$ to 0, 1. We denote

$$\mathbb{S} = \bigcup_{k=0}^{\infty} \{0, 1\}^k.$$

If $s = (s_1, \ldots, s_m) \in \{0, 1\}^m$ and $t = (t_1, \ldots, t_n) \in \{0, 1\}^n$, we define the *concatenation* of s and t as

$$s \frown t = (s_1, \ldots, s_m, t_1, \ldots, t_n) \in \{0, 1\}^{m+n}.$$

In particular, if $s \in \{0, 1\}^k$, then $s \frown 0 = (s_1, \ldots, s_k, 0)$ and $s \frown 1 = (s_1, \ldots, s_k, 0)$. The *length* of $s \in \{0, 1\}^k$ is $|s| := k$.

We define the relations $s \prec t$ and $s \succ t$: We write $s \prec t$ if there exists $u \in \mathbb{S}$ such that $t = s \frown u$; the symbol $s \succ t$ means $t \prec s$.

Example III.17. Set

$$\rho = \frac{1}{2 + 4\alpha}. \tag{III.4}$$

We construct a Cantor type set in $[0, 1]$. Let $P_o = [U_o, V_o] = [0, 1]$. Let $s \in \{0, 1\}^k$ and $P_s = [U_s, V_s]$ be an interval of the k-th generation of length ρ^k. We consider the concentric interval $Q_s = (u_s, v_s)$ of length $\rho^k(1-2\rho)$. Also, consider the intervals

$$P_s^* = [u_s,\, u_s + 2\rho^{k+1}],$$
$$Q_s^* = (u_s + \rho^{k+1},\, u_s + 2\rho^{k+1}).$$

The annulus $P_s \setminus Q_s$ splits into two intervals of $(k+1)$-st generation of length ρ^{k+1}, namely

$$P_{s\frown 0} = [U_{s\frown 0}, V_{s\frown 0}] := [U_s, u_s],$$
$$P_{s\frown 1} = [U_{s\frown 1}, V_{s\frown 1}] := [v_s, V_s].$$

Let $\eta\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function with support in $(0,1)$ such that

$$\sup_{x\in(0,1)} \eta(x) = 2 \quad \text{and} \quad \int_0^1 \eta(y)\,dy = 1.$$

Consider sequences $(\lambda_k)_{k=0}^\infty$, $(\sigma_k)_{k=0}^\infty$ of positive real numbers such that $0 < \sigma_k \le 1$, $k = 0, 1, \ldots$. For each $k = 0, 1, \ldots$ and $s \in \{0,1\}^k$ we set

$$F_s\Big(u_s + (1+\sigma_k x)\rho^{k+1}\Big) = \lambda_k\, \eta(x), \qquad x \in \mathbb{R},$$
$$\beta_{k,p} = \operatorname{osc}_p(F_s, P_s^*), \qquad p \in [1,\infty] \cup \{C\}.$$

Then F_s is supported in Q_s^* and

$$\begin{aligned} \beta_{k,C} &= \lambda_k, \\ \beta_{k,p} &\approx \sigma_k^{1/p}\lambda_k, \qquad 1 \le p \le \infty. \end{aligned} \tag{III.5}$$

We define the sets

$$K_k = \bigcup_{s\in\{0,1\}^k} P_s,$$
$$K = \bigcap_{k=1}^\infty K_k$$

and set

$$F = \sum_{s\in\mathbb{S}} F_s.$$

We observe that K is a Cantor type set of measure 0. Further, F is smooth outside K as the intervals Q_s^* are pairwise disjoint and the support of each F_s is in Q_s^*. Finally, we set

$$f(x) = \begin{cases} F'(x), & x \in \mathbb{R}\setminus K, \\ 0, & x \in K. \end{cases}$$

Theorem III.18. *Let $p \in [1,\infty] \cup \{C\}$.*
(a) If

$$\sum_{j=1}^\infty \beta_{j,p} = \infty$$

then f does not have an indefinite centered HK_α^p integral.
(b) If

$$\sum_{j=1}^\infty 2^j \beta_{j,p} = \infty, \tag{III.6}$$

then f does not have an indefinite HK_α^p integral.

Proof. We use the Baire category theorem similarly to the usage for counterexamples in [1]. Consider a gage $\delta\colon \mathbb{R} \to (0,\infty)$ and denote

$$E_n = \{x \in K \colon \delta(x) > 1/n\}.$$

Then, by the Baire category theorem, there exist $n \in \mathbb{N}$ and an open set $\Omega \subset \mathbb{R}$ such that $\Omega \cap K$ is nonempty and $\Omega \cap E_n$ is dense in $\Omega \cap K$. We find $k \in \mathbb{N}$ and a multiindex $t \in \{0,1\}^k$ such that $P_t \subset \Omega$ and $V_t - U_t = \rho^k \le 1/n$. We denote

$$[t,0] = t,\ [t,1] = t \frown 1,\ [t,2] = t \frown (1,1), \dots .$$

Now, we distinguish the cases (a), (b).

(a) Assume that $\tilde{F}$ is an indefinite centered HK^p_α-integral of f. Let δ be chosen such that for each δ-fine centered α-partition $\{([a_i,b_i),x_i\}_{i=1}^m$ with $x_i \in K$ we have

$$\sum_{i=1}^m \operatorname{osc}_p(\tilde{F},[a,b]) < 1. \tag{III.7}$$

The existence of such a gage is clear from the definition of the integral as $f = 0$ on K. For each $j = 0,1,2,\dots$ we find $x_j \in [U_{[t,j]}, u_{[t,j]}] \cap K$ such that $\delta(x_j) > 1/n$ and set

$$[a_j,b_j] = [x_j - 3\rho^{k+j+1},\ x_j + 3\rho^{k+j+1}].$$

Let $\bar{a}_j$, $\bar{b}_j$ be as in (III.3). Since by (III.4)

$$\begin{aligned}\bar{b}_j = x_j + 3\alpha\rho^{k+j+1} \le u_{[t,j]} + 3\alpha\rho^{k+j+1} \le v_{[t,j]} - \alpha\rho^{k+j+1} = U_{[t,j+1]} - \alpha\rho^{k+j+1}\\ \le x_{j+1} - 3\alpha\rho^{k+j+2} = \bar{a}_{j+1},\end{aligned}$$

the intervals $(\bar{a}_j,\bar{b}_j)$, $j = 1,2\dots$, are pairwise disjoint and contained in P_t. Thus, $\{([a_j,b_j],x_j\}_{j=1}^m$ is a δ-fine α-partition for each $m \in \mathbb{N}$. We observe that $[a_j,b_j] \supset P^*_{[t,j]}$. Since F is smooth in $Q_{[t,j]}$, both F and $\tilde{F}$ are indefinite centered HK^p_α-integrals of f in $Q_{[t,j]}$ and thus by uniqueness $\tilde{F} = F + C_j$ on $Q_{[t,j]}$ for some constant C_j. It follows that

$$\operatorname{osc}_p(\tilde{F},P^*_{[t,j]}) = \operatorname{osc}_p(F,P^*_{[t,j]}) = \beta_{k+j,p}. \tag{III.8}$$

Since $b_j - a_j = 6\rho^{k+j+1}$ and the length of $P^*_{[t,j]}$ is $2\rho^{k+j+1}$, by (III.2) we have

$$\operatorname{osc}_p(\tilde{F},[a_j,b_j]) \ge \frac{1}{3^{1/p}}\operatorname{osc}_p(\tilde{F},P^*_{[t,j]}) = \frac{1}{3^{1/p}}\beta_{k+j,p}. \tag{III.9}$$

Since the sum $\sum_j \beta_{j,p}$ diverges, we obtain a contradiction with (III.7).

(b) Assume that $\tilde{F}$ is an indefinite $HK_{\alpha,p}$-integral of f. Let δ be chosen such that for each δ-fine (uncentered) α-partition $\{([a_i,b_i),x_i\}_{i=1}^m$ with $x_i \in K$ we have

$$\sum_{i=1}^m \operatorname{osc}_p(\tilde{F},[a_j,b_j]) < 1. \tag{III.10}$$

As in (III.8) we obtain that

$$\operatorname{osc}_p(\tilde{F},P^*_s) = \beta_{k+j,p}, \qquad s \in \{0,1\}^{k+j}.$$

Hence

$$\sum_{j=1}^{\infty} \sum_{s\in\{0,1\}^{k+j},\, s\succ t} \operatorname{osc}_p(\tilde{F}, P_s^*) = \sum_{j=1}^{\infty} 2^j \beta_{k+j,p} = \infty.$$

Find $m \in \mathbb{N}$ such that

$$\sum_{s\in S} \operatorname{osc}_p(\tilde{F}, P_s^*) > 2, \tag{III.11}$$

where

$$S := \bigcup_{j=1}^{m} \{s \in \{0,1\}^{k+j},\ s \succ t\}.$$

Since the intervals $[u_s, v_s]$ are pairwise disjoint and S is finite, we can find $x_s \in [U_s, u_s] \cap K$ such that $\delta(x_s) > 1/n$ and the intervals $[x_s, v_s]$ are still pairwise disjoint when s running through S. Set

$$a_s = x_s, \quad b_s = u_s + 2\rho^{|s|+1}.$$

As in (III.3), write

$$\bar{a}_s = x_s + \alpha(a_s - x_s) = a_s, \quad \bar{b}_s = x_s + \alpha(b_s - x_s) = a_s + \alpha(b_s - a_s).$$

Since

$$\bar{b}_s = a_s + \alpha(b_s - a_s) \leq u_s + \alpha(u_s + 2\alpha\rho^{|s|+1} - U_s) = u_s + 3\alpha\rho^{|s|+1} < v_s,$$

we have $[\bar{a}_s, \bar{b}_s] \subset [x_s, v_s]$, and thus the intervals $(\bar{a}_s, \bar{b}_s)$, $s \in S$, are pairwise disjoint and contained in P_t. It follows that $\{[a_s, b_s], x_s\}_{s\in S}$ is a δ-fine α-partition in P_t. Since $[a_s, b_s]$ contains P_s^* for each $s \in S$, by (III.11) and (III.2) we obtain

$$\sum_{i=1}^{m} \operatorname{osc}_p(\tilde{F}, [a_i, b_i]) > 1,$$

which contradicts (III.10). ☐

III.5.1 Denjoy-Khintchine integrable function which is not HK_α^1-integrable

In [1], it is shown that for $\alpha > 2$ there exists a HK_α-integrable function which is not Denjoy-Khintchine integrable. By Remark III.9, such a function is also HK_α^p-integrable. Also discontinuous HK_α^p-integrable functions serve as examples of HK_α^p-integrable functions which are not Denjoy-Khintchine integrable, see Example III.27. We show that the converse inclusion also fails. We prove this for the widest class of our scale.

Theorem III.19. *For each $\alpha \geq 1$ there exists a Denjoy-Khintchine integrable function which is not centered HK_α^1-integrable.*

Proof. Let F, f be as in Example III.17. Set $\lambda_j = 1/j$ and $\sigma_j = 1$, so that by (III.5) $\beta_{j,\infty} \to 0$ and $\sum_j \beta_{j,1}$ diverges. By Theorem III.18, f does not have an indefinite centered HK_α^1 integral.

We show that F is the indefinite Denjoy-Khintchine integral of f. The function F is smooth, thus AC and the derivative of F is f on each Q_s. Further $F = 0$ is AC on K and $|K| = 0$. It follows that F is ACG and a.e. differentiable in $\mathbb{R}$ and $f = F'$ a.e. Hence F is an indefinite Denjoy-Khintchine integral of f on $\mathbb{R}$. ☐

III.5.2 Comparison of HK^p_α integral and HK^q_α integral

Theorem III.20. *For each $\alpha \geq 1$ and $1 \leq p < q \leq \infty$ there exists a HK^p_1-integrable function which is not centered HK^q_α-integrable.*

Proof. Let F, f be as in Example III.17. Set

$$\lambda_j = 3^{\frac{jp}{q-p}}, \quad \sigma_j = 3^{-\frac{jpq}{q-p}},$$

so that by (III.5) we have $\sum_j \beta_{j,q} = \infty$ and $\sum_j 2^j \beta_{j,p} < \infty$. Then by Theorem III.18, f does not have an indefinite centered HK^q_α integral.

We will show that F is an indefinite HK^p_1-integral of f. Choose $\varepsilon > 0$. Since $F' = f$ in $\mathbb{R} \setminus K$, for each $x \in \mathbb{R} \setminus K$ we can find $\delta(x) > 0$ such that

$$|y-x| < \delta(x) \implies |F(y)-F(x)-f(x)(y-x)| < \varepsilon|\arctan y - \arctan x|. \tag{III.12}$$

Find $k \in \mathbb{N}$ such that

$$\sum_{j>k} 2^j \beta_{j,p} < \varepsilon.$$

If $x \in K$, we can find $\delta(x) > 0$ such that the interval $(x-\delta(x), x+\delta(x))$ does not intersect any of the intervals Q^*_s with $|s| \leq k$. This defines a gage $\delta \colon \mathbb{R} \to (0,\infty)$. Let $\{([a_i,b_i],x_i)\}_{i=1}^m$ be a δ-fine partition. Without loss of generality we may assume that $x_1,\dots,x_n \in K$ and $x_{n+1},\dots,x_m \notin K$. Set

$$\begin{aligned} S_i &= \{s \in \mathbb{S} \colon [a_i,b_i] \cap Q^*_s \neq \emptyset\}, \qquad i = 1,\dots,n, \\ S &= \bigcup_{i=1}^m S_i. \end{aligned}$$

Then $|s| > k$ for each $s \in S$. Fix $i \in \{1,\dots,n\}$. If $[a_i,b_i] \cap Q^*_s \neq \emptyset$, then $[a_i,b_i]$ contains either $[u_s, u_s+\rho^{|s|+1}]$, or $[u_s+2\rho^{|s|+1}, v_s] \supset [v_s-\rho^{|s|+1}, v_s]$. (The last inclusion follows from (III.4).) It follows that the length of $[a_i,b_i]$ is at least $\rho^{|s|+1}$. Observe that 0 is a median of F in P^*_s for each s. In view of Proposition III.6, we have

$$\int_{P^*_s} |F|^p \leq 2^{p-1}\operatorname{osc}_p(F,P^*_s)^p = 2^{p-1}\beta^p_{|s|,p}|P^*_s| = 2^p\rho^{|s|+1}\beta^p_{|s|,p}.$$

Thus

$$\int_{a_i}^{b_i} |F(y)|^p\,dy \leq \sum_{s\in S_i} 2^p\rho^{|s|+1}\beta^p_{|s|,p} \leq 2^p(b_i-a_i)\sum_{s\in S_i}\beta^p_{|s|,p} \leq 2^p(b_i-a_i)\Big(\sum_{s\in S_i}\beta_{|s|,p}\Big)^p,$$

so that

$$\operatorname{osc}_p(F,[a_i,b_i]) \leq 2\sum_{s\in S_i}\beta_{|s|,p}.$$

Since each Q^*_s intersects at most two $[a_i,b_i]$, summing over $i = 1,\dots,n$ we obtain

$$\begin{aligned} \sum_{i=1}^n \operatorname{osc}_p(F-f(x_i)\mathrm{Id},[a_i,b_i]) = \sum_{i=1}^n \operatorname{osc}_p(F,[a_i,b_i]) &\leq 4\sum_{s\in\mathbb{S},\ |s|>k}\beta_{|s|,p} \\ &\leq 4\sum_{j>k}2^j\beta_{|s|,p} < 4\varepsilon. \end{aligned} \tag{III.13}$$

From (III.12) we obtain

$$\begin{aligned}\sum_{i=n+1}^{m} &\operatorname{osc}_p(F - f(x_i)\mathrm{Id}, [a_i, b_i]) \\ &\leq \sum_{i=n+1}^{m} (b_i - a_i)^{-1/p} \Big(\int_{a_i}^{b_i} |F(y) - F(x) - f(x)(y-x)|^p \, dy \Big)^{1/p} \qquad \text{(III.14)} \\ &\leq \varepsilon \sum_{i=n+1}^{m} (\arctan b_i - \arctan a_i) \leq \pi\varepsilon.\end{aligned}$$

From (III.13) and (III.14) we conclude that F is an indefinite HK_1^p-integral of f. $\square$

III.5.3 Centered HK_1-integrable function which is not HK_α^p-integrable

Lemma III.21. *Let $S \subset \mathbb{S}$ be a finite set. For each $s \in S$ denote $T_s^0 = \{t \in \mathbb{S} \colon s \frown 0 \frown t \in S\}$ and $T_s^1 = \{t \in \mathbb{S} \colon s \frown 1 \frown t \in S\}$. Assume the property:*

$$\forall s \in S \quad \text{either } T_s^0 = \emptyset \quad \text{or} \quad T_s^1 = \emptyset. \qquad \text{(III.15)}$$

Then

$$\sum_{s \in S} 2^{-|s|} < 2.$$

Proof. Denote by $\#S$ the number of elements of S. We prove by induction on $\#S$. The statement is true if S consists of one multiindex. Assume that the statement is true when $\#S \leq n$ and consider S with $\#S = n+1$. Consider $k = \min\{|s| \colon s \in S\}$ and $S_k = \{u \in S \colon |u| = k\}$. For each $u \in S_k$, T_u^0 and T_u^1 satisfy the property in consideration and $\#T_u^i \leq n$, $i = 0, 1$. Therefore

$$\sum_{t \in T_u^0} 2^{-|u \frown 0 \frown t|} = 2^{-k-1} \sum_{t \in T_u^0} 2^{-|t|} < 2^{-k}$$

and similarly

$$\sum_{t \in T_u^1} 2^{-|u \frown 1 \frown t|} < 2^{-k}.$$

Since at most one of sets T_u^0, T_u^1 is nonempty, we have

$$\sum_{s \in S \colon s \succ u} 2^{-|s|} < 2^{-|u|} + 2^{-k} = 2^{-k+1}, \qquad u \in S_k.$$

Finally, as $\#S_k \leq 2^k$, we have

$$\sum_{s \in S} 2^{-|s|} < 2.$$

$\square$

Theorem III.22. *Let $p \in [1, \infty] \cup \{C\}$. Then for each $\alpha \geq 1$ there exists a centered HK_1-integrable function which is not HK_α^p-integrable.*

Proof. Let F, f be as in Example III.17. Set

$$\lambda_j = \frac{1}{j} 2^{-j}, \quad \sigma_j = 1,$$

so that by (III.5) $\sum_j 2^j \beta_{j,p}$ diverges. Then by Theorem III.18, f does not have an indefinite HK_α^p integral.

We will show that F is an indefinite centered HK_1-integral of f. Choose $\varepsilon > 0$. Since $F' = f$ in $\mathbb{R} \setminus K$, for each $x \in \mathbb{R} \setminus K$ we can find $\delta(x) > 0$ such that

$$|y - x| < \delta(x) \implies |F(y) - F(x) - f(x)(y - x)| < \varepsilon |\arctan y - \arctan x|. \tag{III.16}$$

Find $k \in \mathbb{N}$ such that $2/k < \varepsilon$. If $x \in K$, we can find $\delta(x) > 0$ such that the interval $(x - \delta(x), x + \delta(x))$ does not intersect any of the intervals Q_s^* with $|s| > k$. This defines a gage $\delta\colon \mathbb{R} \to (0, \infty)$. Let $\{([a_i, b_i], x_i)\}_{i=1}^m$ be a δ-fine partition. Without loss of generality we may assume that $x_1, \dots, x_n \in K$ and $x_{n+1}, \dots, x_m \notin K$. For each $i = 1, \dots, n$ find $s_i \in \mathbb{S}$ such that

$$[a_i, b_i] \cap Q_{s_i}^* \neq \emptyset, \quad \sup_{[a_i,b_i]} F = \sup_{[a_i,b_i] \cap Q_{s_i}^*} F.$$

Set

$$S = \{s_1, \dots, s_n\}.$$

Assume that $s = s_i \in S$. If $x_i \le u_s$, then $[x_i, b_i]$ contains $[u_s, u_s + \rho^{|s|+1}]$, hence $x_i - a_i = b_i - x_i \ge \rho^{|s|+1}$ and

$$a_i = x_i - (x_i - a_i) \le u_s - \rho^{|s|+1} = U_s = U_{s\frown 0}, \quad b_i \ge u_s = V_{s\frown 0}.$$

Therefore none of the intervals $[a_j, b_j]$, $j \neq i$, intersects $P_{s\frown 0}$. Similarly, if $x_i \ge v_s$, then none of the intervals $[a_j, b_j]$, $j \neq i$, intersects $P_{s\frown 1}$. It follows that S satisfies (III.15). We estimate

$$\sum_{i=1}^n \operatorname{osc}(F, [a_i, b_i]) \le \sum_{i=1}^n \beta_{|s_i|,C} = \sum_{s \in S} \frac{1}{|s|} 2^{-|s|}.$$

Since $|s| > k$ for each $s \in S$, using Lemma III.21 we can continue

$$\sum_{i=1}^n \operatorname{osc}(F - f(x_i)\mathrm{Id}, [a_i, b_i]) = \sum_{i=1}^n \operatorname{osc}(F, [a_i, b_i]) \le \frac{1}{k} \sum_{s \in S} 2^{-|s|} \le \frac{2}{k} < \varepsilon. \tag{III.17}$$

From (III.16), as in (III.14) we obtain

$$\sum_{i=n+1}^m \operatorname{osc}(F - f(x_i)\mathrm{Id}, [a_i, b_i]) \le \sum_{i=n+1}^m (\arctan b_i - \arctan a_i) \le \pi\varepsilon. \tag{III.18}$$

From (III.17) and (III.18) we conclude that F is an indefinite centered HK_1-integral of f. ☐

Remark III.23. In all these constructions, the resulting function has the required non-integrability property with a fixed α. The construction can be easily modified to obtain the corresponding free non-integrability. It is enough to propose a function f which fails the $\alpha = n$-integrability property on $[\frac{1}{n+1}, \frac{1}{n}]$, $n = 1, 2, \dots$, and multiply the function f on each $[\frac{1}{n+1}, \frac{1}{n}]$ by an appropriate constant c_n to keep control over the behavior at 0.

III.6 Differentiability and approximate differentiability

Lemma III.24. *Let I be an open interval. Let $\alpha \geq 1$ and $p \in [1, \infty] \cup \{C\}$. Let F be an indefinite centered HK^p_α integral of f on I. Then*

$$\lim_{r \to 0_+} \frac{\operatorname{osc}_p(F - f(x)\operatorname{Id}, [x-r, x+r])}{r} = 0 \tag{III.19}$$

for a.e. $x \in I$.

Proof. From Theorem III.15 we infer that there is a centered α-control function φ for the triple $(f, F, \operatorname{Id})$ and centered HK^p_α-integration. Since φ is monotone, it is a.e. differentiable. If x is a point where φ is differentiable, it is evident that (III.19) holds at x. $\square$

Lemma III.25. *Let I be an open interval and $F\colon I \to \mathbb{R}$ be a measurable function. Let $p \in [1, \infty]$. Let $r_0 > 0$ be such that $(x - r_0, x + r_0) \subset I$ and for each $r \in (0, r_0)$, let $\mu(r)$ be a median of F in $(x - r, x + r)$. Suppose that*

$$\lim_{r \to 0_+} \frac{\operatorname{osc}_p(F, [x-r, x+r])}{r} = 0. \tag{III.20}$$

Then there exists a limit

$$\ell = \lim_{r \to 0_+} \mu(r)$$

and

$$\lim_{r \to 0_+} \frac{\mu(r) - \ell}{r} = 0. \tag{III.21}$$

If, in addition, $F(x) = \ell$ (in particular, if F is approximately continuous at x), then 0 is the approximate derivative of F at x.

Proof. It is enough to consider the case $p = 1$. Pick $s, r \in (0, r_0)$ such that $s < r \leq 2s$. We claim that

$$|\mu(r) - \mu(s)| \leq 8 \operatorname{osc}_1(F, [x-r, x+r]). \tag{III.22}$$

Assume that $\mu(r) \geq \mu(s)$. Find measurable sets E_s, E_r such that $E_r \subset (x-r, x+r)$, $E_s \subset (x-s, x+s)$, $F \leq \mu(s)$ on E_s, $F \geq \mu(r)$ on E_r, $|E_s| = s$ and $|E_r| = r$. Let $c \in \mathbb{R}$. If $c \leq \frac{1}{2}(\mu(s) + \mu(r))$, then

$$\int_{x-r}^{x+r} |F(y) - c| \, dy \geq \int_{E_r} (\mu(r) - c) \geq \frac{r}{2}(\mu(r) - \mu(s)).$$

If $c \geq \frac{1}{2}(\mu(s) + \mu(r))$, then

$$\int_{x-r}^{x+r} |F(y) - c| \, dy \geq \int_{E_s} (c - \mu(s)) \geq \frac{s}{2}(\mu(r) - \mu(s)) \geq \frac{r}{4}(\mu(r) - \mu(s)),$$

as we have assumed $r \leq 2s$. In both cases

$$\mu(r) - \mu(s) \leq \frac{4}{r} \int_{x-r}^{x+r} |F(y) - c| \, dy \leq 8 \operatorname{osc}_1(F, [x-r, x+r]). \tag{III.23}$$

The case $\mu(r) < \mu(s)$ is similar, so that (III.23) is verified. Choose $\varepsilon > 0$ and find $\delta \in (0, r_0)$ such that

$$0 < r < \delta \implies 8 \operatorname{osc}_1(F, [x-r, x+r]) \le \varepsilon r.$$

Find $k \in \mathbb{N}$ such that $2^{-k} < \delta$. Then by (III.23)

$$\sum_{j=k+1}^{\infty} |\mu(2^{-j-1}) - \mu(2^{-j})| \le \sum_{j=k+1}^{\infty} 2^{-j}\varepsilon = 2^{-k}\varepsilon, \tag{III.24}$$

similarly

$$2^{-k-1} \le r \le 2^{-k} \implies |\mu(r) - \mu(2^{-k-1})| \le r\varepsilon. \tag{III.25}$$

We see that the sum

$$\mu(2^{-k}) + \big(\mu(2^{-k-1}) - \mu(2^{-k})\big) + \big(\mu(2^{-k-2}) - \mu(2^{-k-1})\big) + \dots$$

converges absolutely and thus it converges. Set

$$\ell = \lim_{k\to\infty} \mu(2^{-k}).$$

Then ℓ makes a sense. Now, for $r \in (0, \delta)$ we can find $k \in \mathbb{N}$ such that $2^{-k-1} < r \le 2^{-k}$. By (III.24) and (III.25), for $j > k$ we have

$$\begin{aligned} |\mu(r) - \mu(2^{-j})| &\le |\mu(r) - \mu(2^{-k-1})| + |\mu(2^{-k-2}) - \mu(2^{-k-1})| \\ &\qquad + |\mu(2^{-k-2}) - \mu(2^{-k-3})| + \dots \\ &\le r\varepsilon + 2^{-k}\varepsilon \le 3r\varepsilon. \end{aligned}$$

Letting $j \to \infty$ we obtain

$$|\mu(r) - \ell| \le 3r\varepsilon \qquad \text{for } 0 < r < \delta,$$

which verifies (III.21). Now, suppose that $F(x) = \ell$. Using Proposition III.6 we estimate

$$\frac{1}{2r^2}\int_{x-r}^{x+r} |F(y) - F(x)|\, dy \le \frac{1}{2r^2}\int_{x-r}^{x+r} |F(y) - \mu(r))|\, dy + \frac{1}{r}|\mu(r) - \ell| \to 0.$$

It is well known that this property implies that the approximate derivative at x is 0, see the proof of [8, 6.1, Theorem 4]. □

Theorem III.26. *Let $I \subset \mathbb{R}$ be an open interval. Let $\alpha \ge 1$ and $p \in [1, \infty]$. Let F be an indefinite centered HK_α^p integral of f on I. Then f is the approximate derivative of F a.e. If F is an indefinite centered HK_α integral of f on I, then even f is the ordinary derivative of F a.e.*

Proof. Let x be a point where F is approximately continuous and (III.19) holds. (By Lemma III.25 and the Denjoy-Stepanov theorem [41, IV, Theorem 10.6], almost every point $x \in \mathbb{R}$ satisfies these properties.) Set $\tilde{F}(y) = F(y) - f(x)y$ and $\tilde{f}(y) = f(y) - f(x)$. Then $\tilde{F}$ is an indefinite centered HK_α^p integral of $\tilde{f}$, $\tilde{F}$ is approximately continuous at x and (III.19) holds for the pair $(\tilde{F}, \tilde{f})$ at x as well. Thus, by Lemma III.25, 0 is the approximate derivative of $\tilde{F}$ at x, so that $f(x)$ is the approximate derivative of F at x. The ordinary differentiability at a point where (III.19) holds with $p = C$ is obvious. □

III.7 Discontinuity

Example III.27. Let $h : (0,\infty) \to \mathbb{R}$ be a smooth function such that $|h'| \le 1$. Interesting choices are e.g. $h(t) = t$ of $h(t) = \sin t$. Set

$$F(x) = \begin{cases} h\Big(\log\log \frac{1}{|x|}\Big), & 0 < |x| < 1, \\ 0, & x = 0, \end{cases}$$

$$f(x) = \begin{cases} F'(x), & 0 < |x| < 1, \\ 0, & x = 0. \end{cases}$$

Theorem III.28. *Let F, f be as in Example III.27 and $p \in [1,\infty)$. Then*

(a) F is an indefinite HK_1^p integral of f on $(-1,1)$,

(b) F has an approximate limit at 0 if and only if it has the ordinary limit at 0.

Proof. (a) In view of Remark III.9 we can restrict our attention to $p > 1$. Since F is continuously differentiable outside the origin, it is enough to verify that

$$\lim_{r\to 0_+} \operatorname{osc}_p(F,[0,r]) = \lim_{r\to 0_+} \operatorname{osc}_p(F,[-r,0]) = 0.$$

Since

$$\Big(y(\log\log \tfrac{1}{y})^{p-1}\Big)' = \Big(\log\log \tfrac{1}{y}\Big)^{p-1}\Big(1 - (p-1)\frac{1}{\log\frac{1}{y}\log\log\frac{1}{y}}\Big),$$

there exists $\delta \in (0, \frac{1}{e})$ such that

$$\Big(y(\log\log \tfrac{1}{y})^{p-1}\Big)' \ge \frac{1}{2}\Big(\log\log\tfrac{1}{y}\Big)^{p-1}, \quad y \in (0,\delta),$$

and thus

$$\int_0^t \Big(\log\log\tfrac{1}{y}\Big)^{p-1} dy \le 2t\Big(\log\log\tfrac{1}{t}\Big)^{p-1}$$

for each $t \in (0,\delta)$. We can also assume that the function

$$t \mapsto \frac{(\log\log\frac{1}{t})^{p-1}}{\log\frac{1}{t}}$$

is increasing on $(0,\delta)$. Pick $r \in (0,\delta)$. Since the Lipschitz constant of h does not exceed 1, we estimate

$$\begin{aligned} \int_0^r |F(y)-F(r)|^p\,dy &\le \int_0^r \Big(\log\log\tfrac{1}{y} - \log\log\tfrac{1}{r}\Big)^p dy \\ &\le \int_0^r \Big(\log\log\tfrac{1}{y}\Big)^{p-1}\Big(\int_y^r \frac{dt}{t\log\frac{1}{t}}\Big)\,dy \\ &= \int_0^r \Big(\int_0^t \frac{(\log\log\frac{1}{y})^{p-1}}{t\log\frac{1}{t}}\,dy\Big)\,dt \qquad \text{(III.26)} \\ &\le 2\int_0^r \frac{(\log\log\frac{1}{t})^{p-1}}{\log\frac{1}{t}}\,dt \\ &\le 2r\,\frac{(\log\log\frac{1}{r})^{p-1}}{\log\frac{1}{r}}. \end{aligned}$$

Similarly

$$\int_{-r}^{0} |F(y) - F(r)|\,dy \le 2r\,\frac{(\log\log\frac{1}{r})^{p-1}}{\log\frac{1}{r}}.$$

It follows

$$\lim_{r\to 0+} \operatorname{osc}_p(F,[0,r]) = \lim_{r\to 0+} \operatorname{osc}_p(F,[-r,0]) = 0.$$

(b) Assume that F has an approximate limit at 0. Then there exists the limit

$$\lim_{r\to 0+} \frac{1}{er-r}\int_r^{er} F(y)\,dy.$$

Let $y \in [r, er]$. Then

$$|F(y)-F(r)| \le \log\log\frac{1}{r} - \log\log\frac{1}{y} \le \log\frac{\log\frac{e}{r}}{\log\frac{1}{r}} = \log\Bigl(1+\frac{1}{\log\frac{1}{r}}\Bigr) \le \frac{1}{\log\frac{1}{r}} \to 0.$$

It follows that

$$\lim_{r\to 0} F(r) = \lim \frac{1}{er-r}\int_r^{er} F(y)\,dy.$$

□

Remark III.29. The choice $h(t) = t$ shows that the indefinite HK^p_α integral can be unbounded. If F does not have any limit at 0, as if, for example, $h(t) = \sin t$, then the "definite HK^p_α integral" of f does not make sense over any integral with endpoint at 0. The nonexistence of the approximate limit shows that even an attempt to define an "approximate definite integral" fails.

III.8 Notes and problems

III.8.1 The Henstock-Kurzweil integral

The first construction of an integral which integrates all derivatives and includes the Lebesgue integral at the same time was done by Denjoy [6] in 1912, shortly followed by Luzin [27] and Perron [34].

In fifties of the last century, Henstock [12] and Kurzweil [25] discovered independently that the Denjoy-Perron integral can be obtained by a minor, but ingenious, modification of the classical Riemann integral. The advantage of their approach is that it is more comprehensible than the former constructions and opens the possibility of multi-dimensional generalization.

III.8.2 Multi-dimensional analogues and the Pfeffer integral

Both Kurzweil and Henstock considered also multidimensional or abstract versions of their integral, [13], [14], [26]. The fundamental issue in n-dimensional integration is what sets should act as counterpart of intervals in partitions. The choice of all n-dimensional intervals allows straightforward generalization of some one-dimensional ideas but is not suitable for applications. An important step forward has been done by Mawhin [31], who brought the idea of regularity of the partition sets to n-dimensional integration resulting in integrability of all divergences. This idea has been further developed and improved e.g. in [30, 19, 16, 35, 18, 17, 36, 20, 33], see also [3] for a survey.

The most fruitful solution of the problem was to use partitions consisting of regular BV sets. This has been invented by Pfeffer [38], see also presentation in [39], [37] and a generalization in [29]. The Pfeffer integral leads to a very general setting of the Gauss-Green divergence theorem.

III.8.3 General packing integrals

The packing integrals were introduced in [28], [24], [15] to define a class of integrals in $\mathbb{R}^n$ which can be applied to non-absolutely convergent integration with respect to distributions. They can be even generalized to metric measure spaces. One of main motivations was also to prove very general versions of the Gauss-Green divergence theorem.

Definition III.30 (Packing)**.** Let $\alpha \geq 1$. A system $\{B(x_i, r_i)\}_{i=1}^m$ of balls in $\mathbb{R}^n$ is called an *α-packing* if the balls $B(x_i, \alpha r_i)$ are pairwise disjoint. If $\delta\colon \mathbb{R}^n \to [0,\infty)$ is a nonnegative function, we say that $\{B(x_i, r_i)\}_{i=1}^m$ is *δ-fine* if $r_i < \delta(x_i)$ for each $i = 1, \dots, m$. If $\delta(x) = 0$, it has the effect that x cannot be any of x_i for the δ-fine α-packing. If $\mathcal{N}$ is a system of subsets of $\mathbb{R}^n$, we say that $\delta\colon \mathbb{R}^n \to [0,\infty)$ is an $\mathcal{N}$-gage if $\{x\colon \delta(x) = 0\} \in \mathcal{N}$.

Remark III.31. Another application of α-packing related to absolute continuity has been studied by Hencl [11].

Definition III.32 (Packing integral)**.** Let $(\mathcal{X}, \mathbf{p})$ be a structure which associates with any ball $B = B(x,r)$ a normed linear space $(\mathcal{X}(B), \mathbf{p}(\cdot, B))$ of distributions on B. Let $\mathcal{F}, \mathcal{G}$ be distributions on $\mathbb{R}^n$ which belong to $\mathcal{X}(B)$ for each ball $B \subset \mathbb{R}^n$. Let f be a function on $\mathbb{R}^n$. We say that $\mathcal{F}$ is an *indefinite α-packing $(\mathcal{X}, \mathbf{p})$-integral* of f with respect to $\mathcal{G}$ if for each $\varepsilon > 0$ there exists a gage $\delta\colon \mathbb{R}^n \to (0,\infty)$ such that for each δ-fine α-packing $\{B(x_i, r_i)\}_{i=1}^m$ we have

$$\sum_{i=1}^m \mathbf{p}(\mathcal{F} - f(x_i)\mathcal{G}),\ B(x_i, r_i)) < \varepsilon.$$

Remark III.33. In [24], [15] we considered 1-packing and the norm has been read on the balls $B(x_i, \tau r_i)$ with $\tau \leq 1$. This is clearly equivalent to the setting above by the choice $\alpha = 1/\tau$. We have made the change for the purpose of compatibility with the approach of [1].

Remark III.34. This general notion of packing integral opens possibilities of a further research. If we want to apply this general definition to the one-dimensional situation, it is useful to identify a locally integrable function F with its distributional derivative $\mathcal{F}$. We investigated the norms

$$\mathbf{p}(\mathcal{F}, [a,b]) = \operatorname{osc}_p(F, [a,b])$$

defined through the norms of L^p of C. There is a variety of further norms which could be taken into account, like Lorentz norms or Sobolev norms.

III.8.4 Lip-packing integral and centered HK^1_α integral

In [24] we have studied the case of

$$\mathbf{p}_{\mathrm{Lip}}(\mathcal{F}, B(x,r)) = \sup\{\langle \mathcal{F}, \varphi\rangle \colon \varphi \in \mathrm{Lip}_0(B(x,r)),\ \mathrm{Lip}\,\varphi \le \tfrac{1}{r}\} \tag{III.27}$$

where $\mathrm{Lip}_0(B(x,r))$ is the class of all Lipschitz continuous functions on $\mathbb{R}^n$ supported in $\overline{B}(x,r)$ normed by the Lipschitz constant. Let us label the resulting packing integral as the Lip *α-packing integral.* This choice is convenient for generalization to metric spaces and appears to be one of the most natural ones. The right space $\mathcal{X}(B)$ to be used here is the closure of $\mathcal{D}(B(x,r))$ in the dual space $\mathrm{Lip}_0(B(x,r))^*$ to $\mathrm{Lip}_0(B(x,r))$, see [28].

If $n = 1$ and $\mathcal{F}$ is the distributional derivative of a locally integrable function F, we observe that

$$\begin{aligned}
\mathbf{p}_{\mathrm{Lip}}(\mathcal{F}, B(x,r)) &= \sup\Big\{\int_{x-r}^{x+r} F(y)\,\varphi'(y)\,dy \colon \varphi \in \mathrm{Lip}_0(B(x,r)), |\varphi'| \le \tfrac{1}{r}\Big\}\\
&= \sup\Big\{\int_{x-r}^{x+r} (F(y)-c)\,\varphi'(y)\,dy \colon \varphi \in \mathrm{Lip}_0(B(x,r)), |\varphi'| \le \tfrac{1}{r}\Big\}\\
&= \frac{1}{r}\int_{x-r}^{x+r} |F(y)-c|\,dy,
\end{aligned}$$

where c is a median of F in $(x-r, x+r)$. Indeed, the supremum is attained at a function φ with $\varphi'(y) = 1/r$ a.e. on E^+ and $\varphi'(y) = -1/r$ a.e. on E^-, where E_+ and E_- are disjoint, of equal measure, $E_+ \cup E_- = [x-r, x+r]$ and and $F \ge c$ on E_+, $F \le c$ on E_-. By Proposition III.6,

$$\mathbf{p}_{\mathrm{Lip}}(\mathcal{F}, B(x,r)) = 2\,\mathrm{osc}_1(F, [x-r, x+r]).$$

Thus, we have verified the following theorem.

Theorem III.35. *Let $f, F, G \colon \mathbb{R} \to \mathbb{R}$ be measurable functions and $\alpha \ge 1$. Then F is an indefinite α-packing* Lip*-integral of f with respect to G if and only if F is the indefinite centered HKS^1_α-integral of f with respect to G.*

III.8.5 BV-packing integral and centered HKS_α integral

The Lip_α packing integral does not include the Pfeffer integral. If we apply the Lip_α packing integral in [28], [24] to generalize the divergence theorem, we obtain new results but we miss the useful features of the Pfeffer integral. To share both advantages of the Pfeffer integral [39] and of the packing approach, in [23] a new integral is introduced. To explain this integral we need first to introduce the notion of charge, which is fundamental also for the Pfeffer integral.

Definition III.36. Recall that the space $BV(\mathbb{R}^n)$ is defined as the space of all L^1 functions u on $\mathbb{R}^n$ such that the distributional derivative Du of u is a $\mathbb{R}^n$-valued Radon measure. Then $\|Du\|$ is defined as the total variation of Du. The BV sets are sets E whose characteristic function χ_E is a BV function; perimeter of a BV-set E is $\|E\| := \|\chi_E\|$. Also, we denote the Lebesgue measure of E by $|E|$ and the diameter of E by $d(E)$. Then the *regularity* of a pair (E, x) is the number

$$r(E,x) = \begin{cases} \frac{|E|}{d(E\cup\{x\})\|E\|} & \text{if } |E| > 0,\\ 0 & \text{if } |E| = 0.\end{cases}$$

Definition III.37 (Charge)**.** Let $\mathcal{F}$ be a linear functional on $\mathcal{D}(\mathbb{R}^n)$. We say that $\mathcal{F}$ is a *charge* if for each $\varepsilon > 0$ there is $\theta > 0$ such that

$$\langle \mathcal{F}, \varphi \rangle \leq \theta \|\varphi\|_1 + \varepsilon(\|\nabla\varphi\|_1 + \|\varphi\|_\infty)$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with support in $B(0, 1/\varepsilon)$, see [5]. We write

$$\mathcal{F}(E) := \mathcal{F}(\chi_E)$$

if E is a BV set.

Definition III.38 (Norms on charges)**.** Let $\mathcal{F}$ be a charge, $B(x, r)$ be a ball in $\mathbb{R}^n$ and $\varepsilon > 0$. We define

$$\mathbf{p}^{\varepsilon}_{BV}(\mathcal{F}, B(x, r)) = \sup\{\mathcal{F}(E)\colon E \subset B(x, r) \text{ is a } BV \text{ set}, \ r(E, x) > \varepsilon\}. \tag{III.28}$$

Definition III.39 (Packing BV integral)**.** Let $\mathcal{N}$ be the class of all sets of σ-finite $(n-1)$-dimensional Hausdorff measure, see [39]. Let $\mathcal{F}, \mathcal{G}$ be charges and $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function. We say that $\mathcal{F}$ is an indefinite BV α-packing integral of f with respect to $\mathcal{G}$ if for each $\varepsilon > 0$ there is an $\mathcal{N}$-gage δ such that for each δ-fine α-packing $\{B(x_i, r_i)\}_{i=1}^m$ we have

$$\sum_{i=1}^{m} \mathbf{p}^{\varepsilon}_{BV}(\mathcal{F} - f(x_i)\mathcal{G}, \ B(x_i, r_i)) < \varepsilon.$$

Remark III.40. The integral defined in Definition III.39 follows the philosophy of packing integrals, but it does not fall to the category of general packing integrals of Definition III.32 as the seminorm depends of ε and the system $\mathcal{N}$ of exceptional sets is considered. However, we did not want to give the general definition more complicated for the sake of one example.

III.8.6 BV packing integral in $\mathbb{R}$ and centered HKS_α integral

In the one-dimensional setting things simplify a lot.

First, BV sets can be represented by figures. These are defined as finite unions of bounded closed intervals. The representation means that the BV set E differs from its representing figure E' only by a Lebesgue null set, thus χ_E and $\chi_{E'}$ represent the same element of the BV function space.

Second, charges are represented by continuous functions: a distribution $\mathcal{F}$ on $\mathbb{R}$ is a charge if and only if there is a continuous function $F\colon \mathbb{R} \to \mathbb{R}$ such that for any closed interval $[a, b]$ we have

$$\mathcal{F}([a, b]) = F(b) - F(a). \tag{III.29}$$

Then $\mathcal{F}$ acts on test functions as the distributional derivative of F.

Third, if $B(x, r) = (x - r, x + r)$ is a ball in $\mathbb{R}$, $E \subset B(x, r)$ is a figure of the form $\bigcup_{j=1}^{k} [u_j, v_j]$, where the intervals $[u_j, v_j]$ are pairwise disjoint, (E, x) is ε-regular and $\mathcal{F}$ is a charge given by (III.29), then $\|E\| = 2k$, $|E| \leq d(E \cup \{x\})$, and thus $2k < 1/\varepsilon$ and

$$\mathcal{F}(E) \leq 2k \operatorname{osc}(F, B(x, r)) < \frac{1}{\varepsilon} \operatorname{osc}(F, B(x, r)).$$

It follows that

$$\mathbf{p}^{\varepsilon}_{BV}(B(x,r)) \leq \frac{1}{\varepsilon}\ \operatorname{osc}(F, B(x,r)). \tag{III.30}$$

On the other hand, the regularity of any interval $x \in [a,b] \subset (x-r, x+r)$ is $r([a,b],x) = \frac{1}{2}$, so that

$$0 < \varepsilon < \frac{1}{2} \implies \operatorname{osc}(F, B(x,r)) \leq \mathbf{p}^{\varepsilon}_{BV}(B(x,r)). \tag{III.31}$$

Fourth, the exceptional sets are just the countable sets. So, we consider $\mathcal{N}$-gages where $\mathcal{N}$ is the family of all countable subsets of $\mathbb{R}$.

We can then reformulate the definition of the BV α-packing integral from [23] for the one-dimensional case as follows:

Definition III.41. Let $f, F, G \colon \mathbb{R} \to \mathbb{R}$ be measurable functions and $\alpha \geq 1$. Assume that F, G are continuous. Let $\mathcal{N}$ be the family of all countable subsets of $\mathbb{R}$. Then F is the indefinite α-packing BV-integral of f with respect to G if for each $\varepsilon > 0$ there exists an $\mathcal{N}$-gage $\delta \colon \mathbb{R} \to [0,\infty)$ such that for each δ-fine centered α-partition $\{([a_i,b_i],x_i\}$ in $\mathbb{R}$ we have

$$\sum_{i=1}^{m} \mathbf{p}^{\varepsilon}_{BV}(F - f(x_i)G, [a_i,b_i]) < \varepsilon.$$

Theorem III.42. *Let $f, F, G \colon \mathbb{R} \to \mathbb{R}$ be measurable functions. Assume that F, G are continuous. Then F is an indefinite α-packing BV-integral of f with respect to G if and only if F is the indefinite centered HKS_{α}-integral of f with respect to G.*

Proof. Let F be an indefinite centered HKS_{α}-integral of f with respect to G. Choose $\varepsilon > 0$. We can find a gage $\delta > 0$ such that for each δ-fine centered α-partition $\{([a_i,b_i],x_i\}$ in $\mathbb{R}$ we have

$$\sum_{i=1}^{m} \operatorname{osc}(F - f(x_i)G,\ [a_i,b_i]) < \varepsilon^2.$$

Using (III.30) we obtain

$$\sum_{i=1}^{m} \mathbf{p}^{\varepsilon}_{BV}(F - f(x_i)G, [a_i,b_i]) \leq \frac{1}{\varepsilon}\sum_{i=1}^{m} \operatorname{osc}(F - f(x_i)G, [a_i,b_i]) < \varepsilon.$$

Thus F is an indefinite α-packing BV-integral of f with respect to G.
Conversely, if F is an indefinite α-packing BV-integral of f with respect to G and $\varepsilon \in (0, \frac{1}{2})$, then there is an $\mathcal{N}$-gage $\delta > 0$ such that for each δ-fine centered α-partition $\{([a_i,b_i],x_i\}$ in $\mathbb{R}$ we have

$$\sum_{i=1}^{m} \mathbf{p}^{\varepsilon}_{BV}(F - f(x_i)G, [a_i,b_i]) < \varepsilon.$$

Let $N = \{x \in \mathbb{R} \colon \delta(x) = 0\}$. Since N is countable, there exists $\xi \colon N \to (0,\infty)$ such that

$$\sum_{x \in N} \xi(x) < \varepsilon.$$

Using continuity of F and G, for each $x \in N$ we find $\hat{\delta}(x) > 0$ such that for each $y \in \mathbb{R}$ we have

$$|y - x| < \hat{\delta}(x) \implies |F(y) - F(x) - f(x)(G(y) - G(x))| < \xi(x). \tag{III.32}$$

We define a gage $\bar{\delta} : \mathbb{R} \to (0, \infty)$ as

$$\bar{\delta}(x) = \begin{cases} \delta(x), & x \in \mathbb{R} \setminus N, \\ \hat{\delta}(x), & x \in N. \end{cases}$$

Now let us fix a $\bar{\delta}$-fine α partition $\{[a_i, b_i], x_i\}_{i=1}^m$ in $\mathbb{R}$. Without loss of generality we may assume that $x_1, \dots, x_k \notin N$ and $x_{k+1}, \dots, x_m \in N$ for some $k \in \{0, 1, \dots, m\}$. Then $\{[a_i, b_i], x_i\}_{i=1}^k$ is δ-fine and thus by (III.31)

$$\sum_{i=1}^{k} \operatorname{osc}(F - f(x_i)G, [a_i, b_i]) < \varepsilon.$$

For $i > k$ we estimate

$$\sum_{i=k+1}^{m} \operatorname{osc}(F - f(x_i)G, [a_i, b_i]) \le \sum_{i=k+1}^{m} \xi(x_i) \le \varepsilon.$$

Together

$$\sum_{i=1}^{m} \operatorname{osc}(F - f(x_i)G, [a_i, b_i]) \le 2\varepsilon.$$

Therefore F is the indefinite centered HKS_α-integral of f with respect to G. ☐

III.8.7 Open problem: dependence on α

In [1], it is shown that each HK_2-integrable function on an open interval $I \subset \mathbb{R}$ is HK_1 integrable, but for each $2 \le \alpha < \beta$ there is a HK_β-integrable function f which is not α-integrable. The characterization of pairs (α, β) such that $HK_\alpha^p = HK_\beta^p$ (here the symbol represents the class of integrable functions) is known neither for $p > 1$, nor for the centered version.

III.8.8 Open problem: approximate nondifferentiability

It would be interesting to know how large can be the set of approximate nondifferentiability of an indefinite (centered) HK_α^p integral.

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