THE OHIO STATE UNIVERSITY

Prof. Jan Lang
Department of Mathematics

231 West 18th Avenue
Columbus, OH 43210-1174
USA

614-292-9133 Phone
614-292-5817 Fax

www.math.ohio-state.edu/~lang
lang@math.ohio-state.edu

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Report on master thesis - Anna Dolezalova

Title: Volumes of unit balls of Lorentz spaces

The main focus of this work is to study the volume of the unit balls in n-dimensional sequence spaces equipped with Lorentz sequence norms $l_n^{p,q}$. Results obtained in this work seem to be original, novel and this work certainly satisfies all requirements (i.e originality, quality and thoroughness) expected from a master theses.

In this work the author successfully answer a list of questions related to volume of unit balls in $l_n^{p,q}$ spaces with semi-norms/norms defined by:

$$\|a\|_{p,q} := \left(\frac{p}{q} \sum_{k\geq 1} (a_k^*)^q (k^{q/p} - (k-1)^{q/p}) \right)^{1/q}, \quad \text{for } 0 < p, q < \infty$$

and

$$\|a\|_{p,\infty} := \sum_{k>1} \{k^{1/p} a_k^*\}, \quad \text{for } 0 < p < \infty.$$

Explicit formulas were obtained in the cases $l_n^{p,1}$ and $l_n^{p,\infty}$ and recursive in the case $l_n^{p,\infty}$ with $0 < p < \infty$. These results allowed the author obtain asymptotic behavior for $Vol(B_N^{p,q})^{1/p} \sim n^{-1/p}$ when $n \to \infty$. Another obtained result was about rations of volumes between weak Lebesgue and Lebesgue space $R_{p,n}$. It was shoved that with "small" p this ration is growing exponentially (i.e. $R_{p,n} \succcurlyeq c^n$, for some $c > 1$). Which seems to be interesting and quite surprising.

In the last section sharp estimates were obtained for entropy numbers for embedding $Id : l_n^{1,\infty} \to l_n^1$ and it was observed that, in this case, the rate of decay is quite unusual and peculiar.

It is well known that the volume of the unit ball in n-dimensional space with l_n^p norm is described by formula $2^n\Gamma(1+1/n)^n/\Gamma(1+n/p)$, where Γ is Gamma function. This formula for the volume of the unit ball is used in Operator theory for obtaining quantitative information about behavior of maps between Banach spaces. One of the most typical use of this formula is in estimation of the entropy numbers for operators. Entropy numbers for a given map $T : X \to Y$, which are defined as:

$$e_k(T) := \inf\{r >: \exists y_1, ..., y_{2^{k-1}} \in Y \text{ with } T(B_X) \subset \cup_{j=1}^{2^{k-1}} (y_j + rB_Y)\},$$

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are used to describe the quality of the compact map T, obtaining estimates for eigenvalues and errors of approximation by finite sets. Just recently an interesting behavior of entropy numbers was discovered in the case when operator acted between Banach spaces with Lorentz type norms (see paper Entropy numbers and interpolation by Edmunds and Netrusov). By using information about precise volume of the unit balls for Banach spaces with Lorentz norm, which were obtained in this work, we can expect to acquire new sharp results for decay of Entropy numbers and get better insides on their behavior. The new sharp results about the speed of decay for entropy numbers can instantly translate into better results about eigenvalues which could be employed into study of nuclear operators. I believe that results obtained in this thesis are extremely interesting with quite promising applications and they could be a base for further auspicious work (like: study behavior of the unit balls when a semi-norm is replaced by a norm, application it into study of nuclear operators on Lorentz spaces, etc.)

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