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BACHELOR THESIS

Pricing Options Using Monte Carlo Simulation

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Declaration of Authorship

The author hereby declares that he compiled this thesis independently; using only the listed resources and literature, and the thesis has not been used to obtain a different or the same degree.

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Prague, July 14, 2019

Signature

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Abstract

Monte Carlo simulation is a valuable tool in computational finance. It is widely used to evaluate portfolio management rules, to price derivatives, to simulate hedging strategies, and to estimate Value at Risk. The purpose of this thesis is to develop the mathematical foundation and an algorithmic structure to carry out Monte Carlo simulation to price a European call option, investigate Black-Scholes model to look into the parallel between Monte Carlo simulation and Black-Scholes model, provide a solution for Black-Scholes model using Lognormal distribution of a stock price rather than solving Black-Scholes original partial differential equation, and finally compare the results of Monte Carlo simulation with Black-Scholes closed-form formula.

Author's contribution can be best described as developing the mathematical foundation and the algorithm for Monte Carlo simulation, comparing the simulation results with the Black-Scholes model, and investigating how path-dependent options can be implemented using simulation when closed-form formulas may not be available.

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Keywords	Monte Carlo simulation, Option pricing, Black-Scholes model

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Acronyms

PDE Partial differential equation

MCE Monte Carlo estimate

Bachelor's Thesis Proposal

Author:	Ryan Dutton
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Defense Planned:	September 2019

Proposed Topic:

Pricing Options Using Monte Carlo Simulation

Motivtion:

Derivatives are important financial instruments that can be used for portfolio hedging or as leveraged instruments for trading. Pricing derivatives normally entail computation of integrals, and solving stochastic differential equations with complicated boundary conditions. In many cases, especially for exotic derivatives, where these stochastic differential equations cannot be valued analytically, the derivatives can be evaluated using numerical integration.

Monte Carlo methods offer viable alternative methods to evaluate any kind of derivatives, since they are conceptually numerical integration tools. They are widely used in finance to evaluate portfolio management rules, to simulate hedging strategies, to estimate Value at Risk etc.

We are going to use Monte Carlo simulation for pricing a European call option, since a closed-form Black-Scholes formula is readily available, and it is easy the compare the results. We will derive the Black-Scholes formula assuming risk-neutrality, and the process of derivation will help us to link Monte Carlo simulation mathematically to the Black-Scholes model.

Hypotheses:

1. Hypothesis #1: In a risk-neutral universe the stock price is expected to yield the risk-free interest rate.
2. Hypothesis #2: Simulation of the stock price following lognormal distribution yields option prices that tally with prices generated by Black-Scholes closed-form formula very closely.
3. Hypothesis #3: Path dependent options can be priced by simulation as well. This demonstrates that we can price an option without an analytical solution.

Methodology:

We are going to derive all the mathematical formulas necessary for our simulation as well as the Black-Scholes model in order to carry out the comparison of simulation results with results from the Black-Scholes model. The derivation of the formulas may seem redundant. However, it is by means of derivation of the formulas we clearly see the link between the simulation process and the Black-Scholes model.

The assumption of lognormal distribution of a stock price in a risk-neutral universe is the backbone of our simulation process as well as the Black-Scholes model. After developing the mathematical structure of the simulation process, we carry out all the computation as related to simulation in Matlab.

Expected Contribution:

Author's contribution can be best described as developing the mathematical foundation and the algorithm for Monte Carlo simulation, comparing the simulation results with the Black-Scholes model, and investigating how path-dependent options can be implemented using simulation when closed-form formulas may not be available.

Outline:

1. Introduction: Introduction to the topic of the thesis.
2. Option Basics: Basic information about options. Two types of options, call and put. Determination of payoff based on the stock price and strike price at the expiration.
3. Stochastic Calculus and Itô's Lemma: Derivation of Itô's lemma and establishing the foundation for Monte Carlo simulation and Black-Scholes model.
4. Black-Scholes Model: Representation of stock price by generalized Wiener process.
5. Monte Carlo Simulation for European Call Option: Mathematical formulation of simulation process and its implementation in Matlab. Comparison of the results.
6. Conclusion: Summary of the topic.

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Author **Supervisor**

1 Introduction

Pricing derivatives is an important area in computational finance. Finding a closed-form analytical solution such as the famous Black-Scholes formula for European options always involves solving partial differential equations with different boundary conditions. However, many derivatives such as complex path-dependent options do not yield analytical solutions. This led to the development of numerical methods for estimating derivative prices.

One of the most popular numerical methods in option pricing is Monte Carlo simulation, a technique that was invented by Stansilaw Ulam in late 1940s, while he was working on nuclear weapons project at the Los Alamos National Laboratory. The broad class of Monte Carlo methods is a class of computational algorithm that relies on repeated random sampling to obtain numerical results. In the context of option pricing Monte Carlo simulation entails generation of sample paths of the underlying security by means of random simulation. It then uses these price paths to compute the payoffs. In the end the payoffs are averaged and discounted back to the present value, and this yields the price of the option.

The objective of this thesis is to explore the Black-Scholes model for pricing options, to develop Monte Carlo simulation algorithm, compare the estimates generated by the simulation process with those given by the Black-Scholes model, and to explore how path-dependent options can be priced using Monte Carlo methods.

The thesis is structured as follows: Chapter 2 covers literature review. Chapter 3 covers the theory of stochastic calculus and Black-Scholes model as a foundation to support our methodology. Chapter 4 discusses the methodology by outlining the mathematical structure and the algorithm for the simulation process. It also discusses how the algorithm can be extended to price path-dependent options. Chapter 5 presents the empirical analysis by comparing Monte Carlo estimates with prices obtained from Black-Scholes closed-form formulas. Chapter 6 is the conclusion, summarising what we have accomplished in the thesis and what other interesting research areas that should be studied.

2 Literature Review

The objective of the thesis is not only to find out how options can be priced using Monte Carlo simulation, but also to establish a logical link between the mathematical structure of the simulation algorithm and the formal derivation of the closed-form formula of an option. We, therefore, have reviewed papers and articles for Monte Carlo simulation for pricing derivatives as well as for derivation of the option price in closed-form formulas.

The mathematical models that are most widely used today to evaluate options are the Black and Scholes model (1973), Cox, Ross and Rubinstein model (1985), and the Merton model (1973).

Fischer Black and Myron Scholes in their seminal paper proposed the Black-Scholes model to value options in terms of the price of the stocks. The Cox, Ross and Rubinstein (1979) model is a multi-period binomial model for evaluating real options that derives from the generalization of the one-period binomial model. The model, revised and completed by Cox and Rubinstein (1985) is one of the most effective methods to estimate the value of options.

Despite being the original and popular, the Black-Scholes model is actually built with some simplified assumptions about the market such as the underlying asset follow a lognormal distribution, volatility and interest rate are constant, the options should be exercised only at the expiration, the stock pays no dividends during the life of the option etc. Tereng (2011) discussed some of these limitations.

Due to some of the simplified assumptions and limitations of Black-Scholes model, many models have been proposed to tackle these problems. The most well-known among these are Merton model and KMV-Merton model. Merton (1974) together with Black and Scholes enhanced the original Black-Scholes model and claimed that this model could be used to develop a pricing theory for corporate liabilities. The analysis of their study extended to include also the callable bonds. Under KMV-Merton model, the firm's asset and its volatility are not directly observed. These values can be assessed from the equity's value, its volatility and other observable variables by solving two nonlinear simultaneous equations.

Black and Scholes themselves admitted some biases of the model in their research paper, “The Valuation of Option Contracts and a Test of Market Efficiency”, expressed as “Using the past data to estimate the variance caused the model to overprice options on high variance stocks and under-priced options on low variance stocks. While the model tends to overestimate the value of an option on a high variance security, market tends to underestimate the value, and similarly while the model tends to underestimate the value of an option on a low variance security, market tends to overestimate the value”.

Both the Black-Scholes model and the binomial model of Cox and Rubenstein was simple and exact. Compared to them the Monte Carlo also proved to be simple and offered very good approximations to the exact value of the option, allowing for greater flexibility. Pheilam P. Boyle (1977) was one of the pioneers in using Monte Carlo simulation to study option pricing.

"The purpose of the present paper is to show that Monte Carlo simulation provides a third method of obtaining numerical solutions to option valuation problems. The technique proposed is simple and flexible in the sense that it can easily be modified to accommodate different processes governing the underlying stock returns. This method should provide a useful supplement to the two approaches mentioned above. Furthermore, it has distinct advantages in some specialised situations -e.g. when the underlying stock returns involve jump processes. Essentially the method uses the fact that the distribution of terminal stock prices is determined by the process generating future stock price movements. This process can be simulated on a computer thus generating a series of stock price trajectories. This series determines a set of terminal stock values which can be used to obtain an estimate of the option value. Furthermore, the standard deviation of the estimate can be obtained at the same time so that the accuracy of the results can be established." – Boyle (1977)

It should be noted that the simulation process that we have chosen to develop follows the same mathematical structure and logic as the one developed by Boyle. Boyle provided ample empirical evidence to establish that Monte Carle estimates are unbiased. However, the main disadvantage of Monte Carlo method is its computational inefficiency. According to Boyle, Monte Carlo estimates lie within two standard deviation of the correct answer. However, 95 percent confidence intervals were quite wide. For a Monte Carlo estimate of \$17.19, the 95 percent confidence limits were 17.19 ± 0.958 with 5000 trajectories or trials. To reduce the range of those confidence limits to ± 0.05 the number of trials had to be increased to 1,835,500. This is a very large number and entails a lot of computing.

Since Boyle's pioneering work in 1977 several papers have been written on Monte Carlo simulation for analysing options. One of the important areas has been variance reduction technique to increase the precision and speed up simulation. Boyle, P., Broadie, M. and Glasserman, P. (1998) suggested antithetic method and control variates. Variance reduction by antithetic variates is a simple and widely used methods today to increase the accuracy of the Monte Carlo simulation. It automatically doubles the size of the sample with a minimal increase in computational time. Since in Monte Carlo simulation we generate normally distributed random variables with a mean of zero and a variance of 1, there is an equally likely chance of having drawn the observed value times -1. Thus, for each arbitrary random draw of α, there should be an artificially generated companion observation of $-\alpha$. This is the antithetic variate. Monte Carlo estimates are observed to have reduced errors, when the technique of antithetic variates is used.

One of the important and practical aspects of using Monte Carlo simulation is pricing derivatives, such as exotic options and Asian options, which are path-dependent and for which no closed-form analytical formulas can be found. The method is slow, but very flexible. Fadugba, Nwozo and Babola (2012) discusses two of the primary numerical methods that are currently used by financial professionals for determining the price of an option. These are Monte Carlo method and finite difference method. They also provide comparison of the convergence of methods to the analytical Black-Scholes price of European option. According to the Fadugba and Nwozo Monte Carlo method is good for pricing exotic options while Crank Nicolson finite difference method is stable, more accurate and converges faster than Monte Carlo method, when pricing standard options. The authors conclude that “The Monte Carlo method works well for pricing path dependent options especially Asian Options, approximates every arbitrary exotic option, it is flexible in handling varying and even high dimensional financial problems.” – Nwozo and Fadugba (2012)

We would like to conclude our literature review by saying that we have developed the mathematical structure for the simulation process mainly by following Boyle's work. Hull's "Options, futures and other derivatives" (2012) has been a great source for the theoretical foundation of simulation, while Glasserman's "Monte Carlo methods in financial engineering" (2004) for the basic empirical analysis of the results. While variance reduction technique and pricing exotic options have been studied, we won't be able to delve into those areas, as it would be outside the scope of this thesis.

3 Theory

3.1 Option

An option is a contract that grants the option buyer, also known as the option holder, a right to buy or sell an underlying asset at a specified price on or before a specified date. The specified price is the strike price or exercise price of the option contract and the specified date is known as the expiration date. The option seller, also known as the option writer, grants this right to the option holder in exchange for a certain amount of money which in essence is the option premium or option price.

The asset based on which the option contract is created is called the underlying. The underlying can be an individual stock, a stock index or a futures contract. The option writer can grant the option holder two types of rights. A right is to purchase the underlying is called a call option, whereas a right is to sell the underlying is called a put option.

When the option holder exercises the option, she buys or sells the underlying depending on whether it is a call option or a put option. Depending on the exercise style, an option can also be categorized according to when it is exercised. A European option can only be exercised at the expiration date. An American option can be exercised any time on or before the expiration date.

The terms of transaction are denominated by the contract unit, which is typically 100 shares for an individual stock and multiple times the index value for a stock index. Most option contracts have standardized terms of transaction. By opening the transaction, the option holder enters into the contract. Subsequently, the option holder then has the choice to exercise the option or to sell the option.

3.1.1 Buying Call Option (Long Position in Call)

Suppose, there is a company called ABC with a call option that expires in one month and has a strike price of $100. The option is priced at $3. Suppose that the current or spot price of stock ABC is $100. The profit or loss will depend on the price of stock ABC at the expiration date. The buyer of a call option gains, if the price rises above the strike price. If the price of stock ABC is equal to $103, the buyer of a call option breaks even. The maximum loss is $3, the option price, and there is substantial upside potential if the stock price rises above $103. Using a graph, Figure 3.1 shows the profit/loss profile for the buyer of this call option at the expiration date.

Let us now compare the profit and loss profile of the call option buyer with that of an investor taking a long position in one share of stock ABC. The payoff from the position in the option depends on stock ABC's price at the expiration date. An investor who takes a long position in stock ABC realizes a profit of $1 for every $1 increase in stock ABC's price. As stock ABC's price falls, however, the investor loses, dollar for dollar. If the price drops by more than $3, the long position in stock ABC results in a loss of more than $3. The long call position, in contrast, limits the loss to only the option price of $3 but retains the upside potential. It is, therefore, important to understand the investment portfolio for a call option. Someone with capital of $10,000.00 can take a position in 100 shares of ABC. Alternatively, the person should buy 1 contract that costs $300, and hold the rest the rest of his capital in a money-market account.

3.1.2 Writing Call Option (Short Position in Call)

Let us now look at the option seller's, or writer's, position. We use the same call option we used to illustrate buying a call option. The profit/loss profile at expiration of the short call position, i.e. the position of the call option writer, is just the opposite of the profit and loss profile of the long call position. That is, the profit of the short call position for any given price for stock ABC at the expiration date is the same as the loss of the long call position. Consequently, the maximum profit the short call position can produce is the option price which $3 in our example.

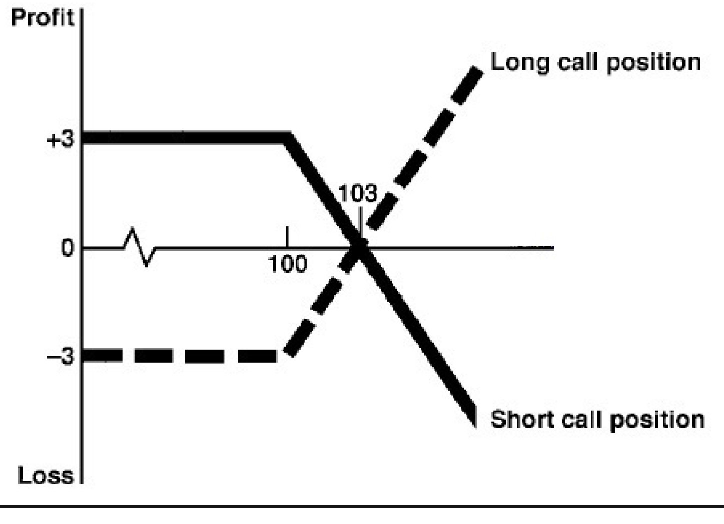

Figure 3.1: Profit/Loss at Expiration of Call Option

The maximum loss is not limited, because it is the highest price reached by stock ABC on or before the expiration date minus the price of the option; this price can be indefinitely high. Using a graph, Figure 3.1 shows the profit/loss profile for the seller of this call option at the expiration date. As with a long position, it is important to understand the investment portfolio with a short in call option. Normally an investor with owns 100 shares of ABC would write a call to lock in a short-term profit. There is no downside protection; however, the investor intends to hold on to his position, even if the stock declines through the expiration.

3.1.3 Buying Put Option (Long Position in Put)

To illustrate a long position in a put option, we assume that the current stock price of ABC is $100 a share. Let's assume that the put option is selling for $2 and the strike price is $100. The profit or loss for this position at the expiration date depends on the market price of stock ABC. The option holder gains, if the price drops below the strike price. Using a graph, Figure 3.2 shows the profit/loss profile for the buyer of this put option at the expiration date. The loss is limited to the option price. The profit potential, however, is substantial, depending on how much the stock drops.

To see how an option alters the risk/return profile for an investor, we again compare it with a position in stock ABC. The long put position is compared with a short position in stock ABC, because a short position would also benefit if the price of the stock falls. While the investor taking a short position faces all the downside risk as well as the upside potential, an investor taking the long put position faces limited downside risk.

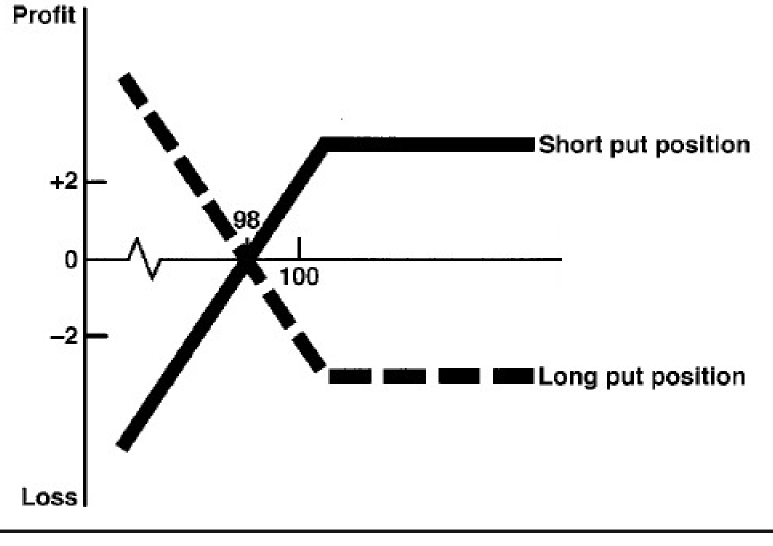

Figure 3.2: Profit/Loss at Expiration of Put Option

3.1.4 Writing Put Option (Short Position in Put)

The profit and loss profile for a short put option is the opposite of the long put option. The maximum profit that the investor can have from this position is the option price. The theoretical maximum loss can be substantial, if the price of the underlying drops. If the price were to fall all the way to zero, the loss would be as large as the strike price minus the option price the seller received. Using a graph, Figure 3.2 shows the profit/loss profile for the seller of this put option at the expiration date.

3.1.5 Factors Influencing Option Price

The following six factors influence the option price:

1. Current price of the underlying instrument

2. Strike price

3. Time to expiration of the option

4. Expected price volatility of the underlying over the life of the option

5. Short-term risk-free rate over the life of the option

6. Anticipated cash dividends on the underlying stock or index over the life of the option.

In the following sections we will look into how options are priced and how each of these factors affect the price. We are going to look at the simplest possible case. Thus, we will consider only a non-dividend-paying stock.

3.2 Itô's Lemma

3.2.1 Stochastic Process

A variable such as a stock price that changes over time in an uncertain way is known to follow a stochastic process, and thus we only know the distribution of the possible values of the process at any point in time. In contrast to a stochastic process, a deterministic process is with an exact value at any point in time. A stochastic process can be discrete-time or continuous-time, depending on whether the change in variable takes place only at a certain point in time or anytime. Stochastic process can also be classified as continuous variable or discrete variable, based on whether the underlying variable can take any value within a range or only certain discrete values.

Even though a stock price is not observed as a continuous-variable, continuous-time process, adopting the continuous-variable, continuous-time process proves to be useful in developing different models, such as Black-Scholes.

3.2.2 Markov Process

A Markov process is a stochastic process where we only need to consider the current value of a variable to predict the future. The path the variable has taken to reach the current state is irrelevant. Stock prices are normally assumed to follow a Markov process. The Markov property implies that the probability distribution of a stock price in the future does not depend on the particular path followed by stock in the past.

3.2.3 Wiener Process

A Wiener process *Z(t)* is in essence a series of normally distributed random variables, and for later points in time the variances of those normally distributed random variables increase to reflect that it is more uncertain to predict the value, as the time period increases.

Defined formally, a variable *Z(t)* follows a Wiener process if it has the following two properties:

1. The change in ΔZ during a small period of time Δt is

$$\Delta Z = \varepsilon\sqrt{\Delta t} \tag{3.1}$$

where $\varepsilon \sim N(0,1)$, the standard normal distribution.

2. The values of ΔZ for any two different intervals of time Δt are independent.

The first property implies that ΔZ follows normal distribution such that

$\mathbf{E}[\Delta Z] = 0$;

$\mathrm{var}[\Delta Z] = \Delta t$;

The second property implies that *Z(t)* follows a Markov process.

The change in *Z(t)* over a period of T would be:

$$Z(T) - Z(0) = \sum_{i=1}^{N} \varepsilon_i\sqrt{\Delta t}\text{ , where } N = \frac{T}{\Delta t} \tag{3.2}$$

The equation (3.2) implies that $[Z(T) - Z(0)]$ also follows normal distribution such that

$\mathbf{E}[Z(T) - Z(0)] = 0$;

$\text{var}[Z(T) - Z(0)] = N * \Delta t = T$

As $N \rightarrow \infty$, Δt converges to 0 and is denoted by *dt*, which means an infinitesimally small interval. Correspondingly, ΔZ is denoted by *dZ*.

3.2.4 Generalized Wiener Process

A generalized Wiener process modifies a Wiener process by incorporating *drift rate* and *variance rate*. The drift rate is the mean change per unit time and the variance rate is the variance per unit time. A generalized Wiener process for a variable X can be written in terms of dZ as

$$dX = adt + bdZ \tag{3.3}$$

where a and b are constants. A generalized Wiener process has a drift rate of a and a variance rate of b^2.

The equation (3.3) implies

$$\boldsymbol{E}(dX) = adt$$

$$var(dX) = b^2 dt$$

$$dX \sim N(adt, b^2 dt)$$

3.2.5 Itô Process

If a generalized Wiener process is modified such that parameters a and b are functions of the underlying variable X and time t, then it becomes an Itô process. An Itô process can be written as

$$dX = a(X,t)dt + b(X,t)dZ \tag{3.4}$$

where dZ is a Wiener process.

The drift and variance rates are no longer constants, and it is no longer so simple to derive $E(dX)$ and $var(dX)$.

The generalized Wiener process as defined by the equation (3.3) and Itô process as defined by the equation (3.4) are called stochastic differential equation (SDE). Itô process was named after Kiyoshi Itô who pioneered the theory of stochastic integration and stochastic differential equations, now also known as Itô calculus. Itô discovered an important result known as Itô's lemma which we will describe in the next section.

3.2.6 Itô's Lemma

The price of a stock option is a function of both underlying stock's price and time. To generalize it, we can say that price of any derivative is a function of the stochastic variables underlying the derivative and time. An important result was formulated by Kiyoshi Itô in 1951, and is known as Itô's lemma.

Let us suppose that a variable X follows Itô process as described by equation (3.4). Itô's lemma states that a function G of X and t follows the process

$$dG = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt + \frac{\partial G}{\partial X}b\,dZ \qquad (3.5)$$

Where dZ is the Wiener process from equation (3.3), and G follows an Itô process with a drift rate of $\left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)$ and a variance rate of $(\frac{\partial G}{\partial X})^2 b^2$.

The equation (3.5) is key to generating the asset paths in our Monte Carlo simulation. While the rigorous proof would be beyond the scope of this paper, we can look at its simple derivation by using a well-known result in differential calculus known as Taylor series expansion.

If G is a continuous and differentiable function of X and t, the Taylor series expansion of ΔG is

$$\Delta G = \frac{\partial G}{\partial X}\Delta X + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\Delta X^2 + \frac{\partial^2 G}{\partial X \partial t}\Delta X \Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \cdots \qquad (3.5)$$

The equation (3.4) can be re-written in the discrete form as

$$\Delta X = a(X,t)\Delta t + b(X,t)\varepsilon\sqrt{\Delta t}$$

Or, if we drop the arguments, we get

$$\Delta X = a\Delta t + b\varepsilon\sqrt{\Delta t} \qquad (3.6)$$

As Δt approaches zero, then in equation (3.5) we can ignore all terms of second or higher orders of Δt. We now turn our attention to ΔX^2 derived from equation (3.6). If we ignore Δt^2, we have

$$\Delta X^2 = b^2\varepsilon^2\Delta t + \text{terms with higher orders of } \Delta t \qquad (3.7)$$

Equantion (3.7) shows that the third term with ΔX^2 in equation (3.5) cannot be ignored, since it has a component that is of order Δt.

The variance of a standard normal distribution is 1.0. Thus,

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

Since $[E(\varepsilon)] = 0$, $E(\varepsilon^2) = 1$. This means that the expected value of ε^2 is 1. Logically, the expected value of $\varepsilon^2 \Delta t$ is Δt. It can be shown that as Δt approaches zero, the term $\varepsilon^2 \Delta t$ can be treated as non-stochastic, and it is equal to its expected value. Thus, ΔX^2 in equation (3.7) becomes non-stochastic and is made equal to $b^2 \Delta t$.

If we substitute ΔX^2 for $b^2 \Delta t$ in equation (3.5) and take the limit, as Δt approaches zero, we get

$$dG = \frac{\partial G}{\partial X} dX + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} b^2 dt$$

By substituting for dX from equation (3.4) we arrive at the celebrated Itô's lemma

$$dG = \left(\frac{\partial G}{\partial X} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} b^2 \right) dt + \frac{\partial G}{\partial X} b \, dZ$$

Itô's lemma is an important equation in stochastic calculus, and in the following section we will use it to simulate geometric Brownian motion for the asset path.

3.3 Geometric Brownian Motion for Stock Price

Since a stock price is a stochastic variable, one may be tempted to consider that it follows a generalized Wiener process, that it has a constant expected drift rate and a constant variance rate. However, there is a fundamental flaw in this model. It does not capture the concept that the percentage return from a stock is independent of the stock price. If investors expect a 10% a year return from a stock, when the stock price is $40, then, ceteris paribus, they will expect a 10% a year return when the stock price is $60.

Obviously, the assumption of a constant drift rate is not appropriate, and we should assume that the expected return is constant. If S is the stock price at time t, then the expected drift rate of the price change in S is μS, where the parameter μ is the expected rate of return on the stock in decimal form. This means that for a short interval Δt the expected rate of change in S is $\mu \Delta t$.

If the co-efficient of dZ is zero, i.e. there is no uncertainty, then we can write

$$\frac{\Delta S}{S} = \mu \Delta t \text{ (drift part only)}$$

As Δt approaches zero, in the limit

$$\frac{dS}{S} = \mu dt \text{ (drift part only)} \tag{3.8}$$

We can also assume the variance of the percentage return of the stock price in a short interval Δt is constant regardless of the price of the stock. In other words, the variability of percentage return is the same when the stock price is \$40 as when it is \$60. The variable part of the rate of change in S in a short interval Δt, as it approaches zero, can be written as

$$\frac{dS}{S} = \sigma dZ \text{ (variable part only)} \tag{3.9}$$

The parameter σ is the volatility of the stock return, and σ^2 is the variance. Combining equation (3.8) and (3.9) we get

$$\frac{dS}{S} = \mu dt + \sigma dZ \tag{3.10}$$

Equation (3.10) denotes the particular behaviour of a stock price, known as geometric Brownian motion. The discrete-time version can be written as

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t} \tag{3.11}$$

Equation (3.10) and (3.11) represent generalized Wiener process, and, thus, $\frac{\Delta S}{S}$ follows normal distribution with mean $\mu \Delta t$ and standard deviation $\sigma \sqrt{\Delta t}$.

3.4 Black-Scholes Model for European Call Option

3.4.1 Expected Return in a Risk-free Universe

The risk-neutral assumption in option pricing means that the expected return from the underlying stock is the risk-free interest. We are now going to rewrite equation (3.11) by substituting μ for r, the risk-free interest rate.

$$\frac{dS}{S} = rdt + \sigma dZ \tag{3.12}$$

Or,

$$dS = rSdt + \sigma SdZ \tag{3.13}$$

This is a generalized Wiener process, where a is rS and b is σS. Applying Itô's lemma it follows that a function G of S and t would yield

$$dG = \left(\frac{\partial G}{\partial S} rS + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\, dZ \tag{3.14}$$

3.4.2 Derivation of Black-Scholes Differential Equation

In this section we are going to derive the famous Black-Scholes formula for a European call option. We first start with a derivative *V(S,t)* of a security *S*. The equation (3.14) can be re-written as

$$dV(S,t) = \left(\frac{\partial V}{\partial S} rS + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial V}{\partial S}\sigma S\, dZ \tag{3.15}$$

In order to set up a risk-neutral portfolio we need to hedge the stock position with a position in option. Since the price of the option changes with respect to the stock price as $\frac{\partial V}{\partial S}$, we need $\frac{\partial V}{\partial S}$ stocks in combination with one option to set up a risk-neutral portfolio. The value of this portfolio, π is given by

$$\pi = V - \frac{\partial V}{\partial S} S \tag{3.16}$$

The change, $d\pi$, in the value of the portfolio in a small time-interval dt is given by

$$d\pi = dV - \frac{\partial V}{\partial S} dS \tag{3.17}$$

Now plugging dV from equation (3.15) and dS from equation (3.13) into equation (3.17) we get

$$d\pi = \left(\frac{\partial V}{\partial S} rS + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial V}{\partial S}\sigma S\, dZ - \frac{\partial V}{\partial S}(rSdt + \sigma SdZ) \tag{3.18}$$

After simplifying equation (3.18) we get

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt \tag{3.19}$$

We arrive at equation (3.19) by creating a risk-free portfolio that holds one option hedged with Δ shares of stocks. It should be noted that equation (3.19) contains no random Brownian motion terms. Since the portfolio is risk-free, it must earn the same return as the short-term risk-free securities like the Treasury. If the portfolio earned more than that, an arbitrageur could make a profit by shorting the risk-free security and using the proceeds to buy this portfolio. If the portfolio earned less, an arbitrageur could make a riskless profit by shorting the portfolio and buying the risk-free security. Since the risk-free portfolio earns the risk-free interest rate, we can write

$$d\pi = r\pi \, dt \tag{3.20}$$

where r is the risk-free interest rate. Substituting for $d\pi$ and π from equation (3.19) and (3.16) we get,

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) dt = r\left(V - \frac{\partial V}{\partial S}S\right) dt \tag{3.21}$$

Simplifying equation (3.21) yields Black-Scholes partial differential equation,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} rS - rV = 0 \tag{3.22}$$

Fischer Black and Myron Scholes solved this PDE with proper boundary conditions in their seminal work that they published in 1973. However, we are not going to solve the PDE. In the next section we will provide an alternative and simpler derivation of the option formula. This derivation and our methodology dovetail much nicer, since both are built on the assumption of risk neutrality and lognormal distribution of a stock price. The concept of lognormal distribution is elaborated later in section 4.1.

3.4.3 Solving for European Call Option

Definition 3.1. The *cumulative distribution function*, F, of a random variable X is defined for all real numbers *b* by

$$F(b) = P\{X \leq b\}$$

We say that X has a probability density function, f, if

$$P\{X \leq b\} = F(b) = \int_{-\infty}^{b} f(x)dx$$

for some non-negative function, $f(x)$.

Definition 3.2. X is a normal random variable with parameters μ (mean) and σ^2 (variance), if the density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Thus, the cumulative distribution function of random variable following standard normal distribution with mean 0 and variance 1 is given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy$$

Definition 3.3. If X is a continuous random variable having a probability distribution density function $f(x)$, then the expected value of X is given by

$$\boldsymbol{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Definition 3.4. The random variable X is a log-normally distributed if for some normally distributed variable Y, $X = e^Y$. That is, $ln(X)$ is normally distributed.

Definition 3.5. In a risk-neutral universe the value of an asset A, $C(A, 0)$ at $t = 0$ is the expected value of the asset at time t discounted to the present value by risk-free interest rate r.

$$C(A, 0) = e^{-rt}\boldsymbol{E}[C(A, t)]$$

With these definitions we can now proceed to derive Black-Scholes formula for a European call option. In order to be consistent with standard Black-Scholes notations we replace $S(T)$ by S_T and $S(0)$ by S_0. Based on the assumption of lognormal distribution of stock price (explained later in section 4.1), we can write

$$S_T = S_0 \exp[(r - \frac{\sigma^2}{2})T + \sigma\varepsilon\sqrt{T}\,]$$

This equation suggests,

$S_T = S_0 e^Y$, where Y is normally distributed, such that

$$Y \sim N\left(\left(rT - \frac{\sigma^2 T}{2}\right), \sigma^2 T\right)$$

According to definition 3.1 the cumulative distribution of a random variable S_T is

$$F(x) = P\{S_T \leq x\}$$

$$= P\{S_0\, e^y \leq x\}$$

$$= P\{y \leq \ln(\frac{x}{S_0})\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T}} \int_{-\infty}^{\ln(\frac{x}{S_0})} e^{\frac{-(y - rT + \frac{\sigma^2 T}{2})^2}{2\sigma^2 T}} dy$$

Differentiating with respect to x and applying the fundamental theorem of calculus we get the density function of S_T, given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T}x}\, e^{\frac{-(\ln(\frac{x}{S_0}) - rT + \frac{\sigma^2 T}{2})^2}{2\sigma^2 T}}$$

In a risk-neutral universe with an initial stock price S_0 and lognormally distributed stock price S_T at time t, the value *C* of a European call option at time *t* = 0 with strike *K*, and expiration time *T* and *r* being the risk-free interest rate, we have

$$C(S, 0) = e^{-rt}\boldsymbol{E}[C(S, T)]$$

$$= e^{-rt}\boldsymbol{E}[max(S_T - K, 0)]$$

$$= e^{-rT} \int_K^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T}x} (x - K)\, e^{\frac{-(\ln(\frac{x}{S_0}) - rT + \frac{\sigma^2 T}{2})^2}{2\sigma^2 T}} dx$$

$$= e^{-rT} \int_K^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T}x}\, x\, e^{\frac{-(\ln(\frac{x}{S_0}) - rT + \frac{\sigma^2 T}{2})^2}{2\sigma^2 T}} dx$$

$$- e^{-rT} K \int_K^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T}x}\, e^{\frac{-(\ln(\frac{x}{S_0}) - rT + \frac{\sigma^2 T}{2})^2}{2\sigma^2 T}} dx$$

Here we are going to solve two integrals separately. To solve the first integral, we are going to apply the following transformation of variables.

$$z = \frac{(\ln(\frac{x}{S_0}) - rT + \frac{\sigma^2 T}{2})}{\sigma\sqrt{T}}$$

which implies the following.

$$x = S_0 e^{\left(z\sigma\sqrt{T} + rT - \frac{\sigma^2 T}{2} \right)}$$

$$dz = \frac{dx}{x\,\sigma\sqrt{T}}$$

$$dx = x\,\sigma\sqrt{T}\,dz$$

$$dx = \sigma\sqrt{T}\,S_0 e^{\left(z\sigma\sqrt{T} + rT - \frac{\sigma^2 T}{2} \right)}\,dz$$

The lower limit of the integral is transformed as,

$$L = \frac{\left(\ln\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}$$

Writing the first integral in terms of z yields

$$e^{-rT}\int_L^\infty \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T}}\,e^{\frac{-z^2}{2}}\,\sigma\sqrt{T}\,S_0 e^{\left(z\sigma\sqrt{T} + rT - \frac{\sigma^2 T}{2} \right)}\,dz$$

$$= S_0 e^{-rT}\int_L^\infty \frac{1}{\sqrt{2\pi}}\,e^{\left(\frac{-z^2}{2} + z\sigma\sqrt{T} + rT - \frac{\sigma^2 T}{2}\right)}\,dz$$

$$= S_0 e^{-rT}\int_L^\infty \frac{1}{\sqrt{2\pi}}\,e^{-\left(\frac{z^2}{2} - z\sigma\sqrt{T} + \frac{\sigma^2 T}{2}\right) + rT}\,dz$$

$$= S_0 e^{-rT}\int_L^\infty \frac{1}{\sqrt{2\pi}}\,e^{\frac{-(z-\sigma\sqrt{T})^2}{2} + rT}\,dz$$

$$= S_0 e^{-rT} e^{rT}\int_L^\infty \frac{1}{\sqrt{2\pi}}\,e^{\frac{-(z-\sigma\sqrt{T})^2}{2}}\,dz$$

$$= S_0 \int_L^\infty \frac{1}{\sqrt{2\pi}}\,e^{\frac{-(z-\sigma\sqrt{T})^2}{2}}\,dz$$

where $L = \frac{\left(\ln\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}$

Applying another transformation of variable as $y = (z - \sigma\sqrt{T})$, we need to change the limit of integral as

$$A = \frac{(\ln(\frac{K}{S_0}) - rT + \frac{\sigma^2 T}{2})}{\sigma\sqrt{T}} - \sigma\sqrt{T}$$

$$= \frac{(\ln(\frac{K}{S_0}) - rT - \frac{\sigma^2 T}{2})}{\sigma\sqrt{T}}$$

The final form of the integral yields

$$S_0 \int_A^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

A part of this is the cumulative distribution function of the standard normal variable with the limits in a reverse order. The integral can, therefore, be written as

$$S_0(1 - N\left(\frac{\left(\ln\left(\frac{K}{S_0}\right) - rT - \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}\right))$$

$$= S_0 N\left(-\frac{\left(\ln\left(\frac{K}{S_0}\right) - rT - \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}\right)$$

$$= S_0 N\left(\frac{\left(\ln\left(\frac{S_0}{K}\right) + rT + \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}\right)$$

This gives us the first term of Black-Scholes formula.

Now we solve for the second integral which is

$$e^{-rT} K \int_K^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T}x} e^{\frac{-(\ln(\frac{x}{S_0}) - rT + \frac{\sigma^2 T}{2})^2}{2\sigma^2 T}} dx$$

To solve the second integral, we are going to apply the following transformation of variables.

$$z = \frac{(\ln(\frac{x}{S_0}) - rT + \frac{\sigma^2 T}{2})}{\sigma\sqrt{T}}$$

This implies,

$$dz = \frac{dx}{x\,\sigma\sqrt{T}}$$

$$dx = x\,\sigma\sqrt{T}\,dz$$

The lower limit of the integral is transformed as

$$L = \frac{(\ln(\frac{K}{S_0}) - rT + \frac{\sigma^2 T}{2})}{\sigma\sqrt{T}}$$

Writing the second integral in terms of z yields

$$Ke^{-rT}\int_L^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T}x}\, e^{\frac{-z^2}{2}}\, x\,\sigma\sqrt{T}\,dz$$

$$= Ke^{-rT}\int_L^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{\frac{-z^2}{2}}\,dz$$

A part of this is the cumulative distribution function of the standard normal variable with the limits in a reverse order. The integral can, therefore, be written as

$$Ke^{-rT}(1 - N\left(1 - \frac{\left(\ln\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}\right))$$

$$= Ke^{-rT}N\left(-\frac{\left(\ln\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}\right)$$

This gives us the second term of Black-Scholes formula.

Combining the two terms we get the complete Black-Scholes formula.

$$C(S,0) = S_0 N\left(\frac{\left(\ln\left(\frac{S_0}{K}\right) + rT + \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}\right)$$

$$+ Ke^{-rT}N\left(-\frac{\left(\ln\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2 T}{2}\right)}{\sigma\sqrt{T}}\right)$$

We will use this formula, as provided by a Matlab function, to compute the price of a European call option. This will be compared with the option price, given by our Monte Carlo simulation. The empirical analysis is provided in chapter 5.

4 Methodology

4.1 Lognormal Distribution for Stock Price

In chapter 3 we developed the concept of Itô's lemma and geometric Brownian motion for a stock price. Now we combine them together to find a functional description of a stock price. We will use this functional description to carry out Monte Carlo simulation. The process will entail generation of a large number of asset paths. Based on these asset paths we will compute the price of the payoff for an option at the expiration. By discounting the payoff with the risk-free interest we will find the present value which will be our option price at present.

Before developing the process of simulation, we need to elaborate an important concept, risk-neutral valuation. While valuing an option we will make the assumption that the investors are risk-neutral. Risk-neutral valuation makes the following two assumptions:

1. The expected return from the underlying stock is the risk-free interest rate.

2. The discount rate used for the expected payoff on an option is the risk-free interest rate.

We are going to re-write the following three equations from section 3.4.1. We need these three equations to develop our model for Monte Carlo simulation.

$$\frac{dS}{S} = rdt + \sigma dZ \tag{4.1}$$

$$dS = rSdt + \sigma SdZ \tag{4.2}$$

$$dG = \left(\frac{\partial G}{\partial S} rS + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial G}{\partial S}\sigma S\, dZ \tag{4.3}$$

Since $\frac{dS}{S}$ in equation (4.1) is $dlog(S)$ in deterministic calculus, we could try $G = \log(S)$ to find a solution to the stochastic differential equation (4.3).

To apply Itô's lemma we first have to compute the partial derivatives:

$$\frac{\partial G}{\partial t} = 0$$

$$\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$$

$$\frac{\partial G}{\partial S} = \frac{1}{S}$$

Based on these results equation (4.3) can be transformed to:

$$dG = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma\, dZ \tag{4.4}$$

Equation (4.4) shows that $G = \log(S)$ follows a generalized Wiener process with a constant drift rate of $\left(r - \frac{\sigma^2}{2}\right)$ and a constant variance of σ^2. The change in $\log(S)$ between time 0 and T is, therefore, normally distributed with mean $\left(r - \frac{\sigma^2}{2}\right)T$ and variance $\sigma^2 T$. This can be written as:

$$\log\big(S(T)\big) - \log(S(0)) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma Z(T)$$

Or,

$$\log\big(S(T)\big) - \log(S(0)) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma Z(T)$$

Or,

$$\log\big(S(T)\big) - \log(S(0)) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma\varepsilon\sqrt{T}$$

Written in terms of *S(t)* we get,

$$S(T) = S(0) \exp[(r - \frac{\sigma^2}{2})T + \sigma\varepsilon\sqrt{T}\,] \tag{4.5}$$

This is a solution to the stochastic differential equation (4.3). Based on this equation we generate multiple asset paths for our Monte Carlo simulation.

4.2 Simulating Asset Path for Stock Price

In order to generate an asset path we discretize (4.5) as follows:

$$S_{t+1} = S_t \exp[(r - \frac{\sigma^2}{2})\Delta t + \sigma\varepsilon\sqrt{\Delta t}\,]$$

where Δt is the time step and ε is a standard normal variable with mean 0 and variance 1. If $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ happen to be a series of normally distributed random numbers, then we can write,

$$S_{t+2} = S_t \exp[(r - \frac{\sigma^2}{2})\Delta t + \sigma\varepsilon_1\sqrt{\Delta t}\,]\exp[(r - \frac{\sigma^2}{2})\Delta t + \sigma\varepsilon_2\sqrt{\Delta t}\,]$$

Extending this equation to n steps we get,

$$S_{t+n} = S_t \exp[(r - \frac{\sigma^2}{2})n\Delta t + \sigma\sqrt{\Delta t}\,(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)] \tag{4.6}$$

Equation (4.6) is the building block for our Monte Carlo simulation. Our objective would be to create a vector, i.e. an asset path, where the number of elements will be the same as the number of steps of the asset path and the kth element will be represented by S_{t+k} in equation (4.6). It should be noted that Black-Scholes model assumes a continuous-time stock price, whereas we are using a discrete-time process.

The asset path is also referred to as a trajectory in the context of simulation, and we complete the simulation process by generating a large number of trajectories in order to compute the expected value of the payoff by taking the average of the payoffs from all the trajectories after a time period of T.

Let us suppose that we are trying to price an option that expires after *T* period. Let *S(T)* denote the price of a stock at the expiration. If the strike price of the option is *K*, then the payoff to the option holder of a call option will be *max{0, S(T) – K}*. To get the present value of the payoff we multiply it by a discount factor e^{-rT}, where *r* is the risk-free continuously compounded interest rate.

If we simulate *n* number of trajectories, we take the average of *n* different $e^{-rT}max\{0, S(T) - K\}$ to compute the expected value of the payoff discounted to its present value, and this gives us the price of a call option from Monte Carlo simulation.

Figure 4.1 demonstrates how equation (4.6) can be implemented in Matlab to yield a matrix of sample asset paths. Each column, i.e. a vector, in matrix s*ample* represents a simulated asset path where the *n-th* row represents *n-th* step of the asset path. We define *step_length* as the smallest unit of time which is Δt in equation (4.6). There are *num_step* number of steps in the entire time period which is given by *TPeriod*. Each column of s*ample*, therefore, has *(1+num_step)* number of elements. We add the stock price S_0 as the first element. The total number of vectors or trajectories is determined by the input argument *num_path*.

```
function [sample, time] =
simulate_asset_path(sigma,TPeriod,num_path,S0,mu,step_length)
% Author: Ryan Dutton. Project: Bachelor Thesis
% Generates the asset paths from simulation, and returns them in
% matrix Sample.  Example:
% [Sample, time] = simulate_asset_path(.12,5,50,100,.05,1/255);

num_step = ceil(TPeriod/step_length);
RandomVectors  = sigma*sqrt(step_length)*cumsum(randn(num_step,
num_path));
Drifts  = repmat((mu - sigma^2/2) *step_length *
(1:num_step)',1,num_path);
sample  = [repmat(S0,1,num_path); S0 *
exp(Drifts + RandomVectors)];
if nargout > 1
   time = [0;step_length * (1:num_step)'];
end
```

Figure 4.1: Matlab code for generating asset paths

```
% option_montecarlo_simulation
% Author: Ryan Dutton. Project: Bachelor Thesis
% The main routine that calculates the payoff from the sample
% asset paths, and computes the price of the call and put options.
% It also prints out the price of call and put options using the
% Black-Scholes formula with the same parameters.

S0       = 100   % Initial value of stock
mu       = .05   % Drift rate of the return, should be the same
                 % as the risk-free interest rate
sigma    = .12   % Standard deviation of the periodic return
TPeriod  =   2    % Time period in year
num_path = 2000 % Number of asset paths for the simulation
step_length    = 1/255 % Time length of one step inside a period
                 % We are using 255 steps
int_rate= .05    % Risk-free interest rate
strike   = 125   % Strike price

[sample, time] =
simulate_asset_path(sigma,TPeriod,num_path,S0,mu,step_length);

figure('Color','white');
plot(time,sample,'linewidth',2);
axis([time(1) time(end) min(sample(:)) max(sample(:))]);

xlabel('Time');ylabel('Stock Price');
title(sprintf('No. of Asset Path Simulated: %d', size(sample,2)));

positive_payoff = (sample(511,:) - strike) > 0;
MCE_call_payoff = (sample(511,:) - strike).*positive_payoff;

negative_payoff = (strike - sample(511,:)) > 0;
MCE_put_payoff = (strike - sample(511,:)).*negative_payoff;

MCE_call = mean(MCE_call_payoff) * exp(-int_rate * TPeriod)
MCE_put = mean(MCE_put_payoff) * exp(-int_rate * TPeriod)

[bs_call, bs_put] = blsprice (S0,strike,int_rate,TPeriod,sigma)
```

Figure 4.2: Matlab code for calculating Monte Carlo estimates

Matrix *sample* represents equation (4.6) which comprises two parts in its exponent. The first is the drift and the second part is the random part. Matrix *RandomVectors* represents the random part which is generated by cumulative totals of random numbers that follow standard normal distribution. Matrix *Drifts* represents the drift part where the *n-th* element is the n times the drift rate.

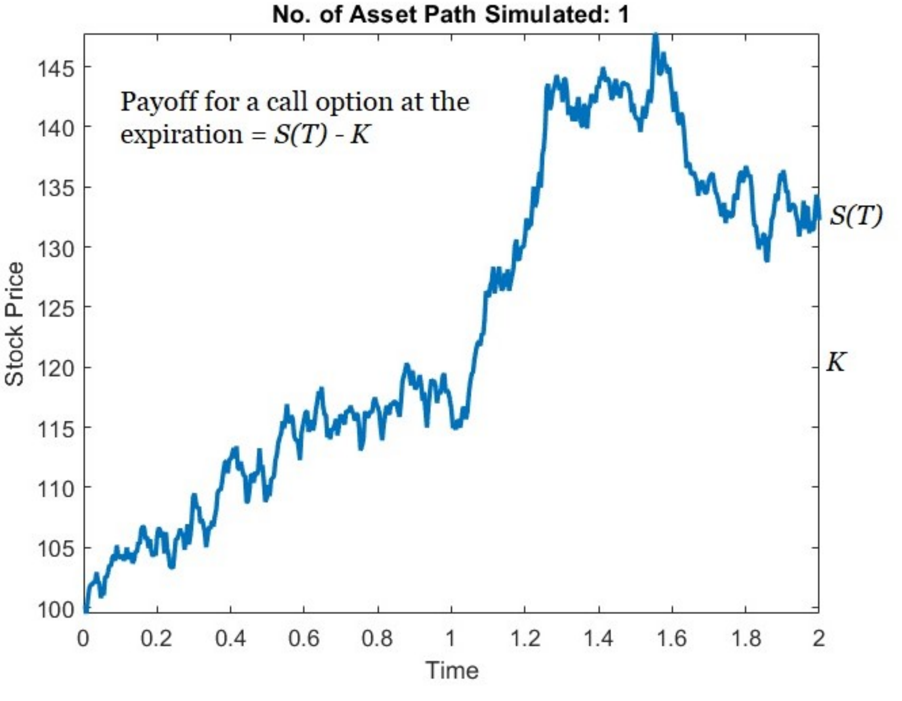

Figure 4.3: A single asset path generated by *simulate_asset_path* (positive payoff)

If we generate one simulated asset path using the function *simulate_asset_path*, with the initial value at 100, the time period being 2 years and the number of steps in each period being 255, we get a path that looks like one in Figure 4.3. This is a random path that takes the stock price to approximately 132.50 in 2 years. As seen in the picture, this particular asset path generates a positive option payoff, since the option price at the expiration is higher than the strike price.

Figure 4.4 shows a different asset path where the option price is lower than the strike price. The payoff in this case is zero. We now extend the idea to a large number of trajectories or different asset paths. If we take the mean of all these payoffs from the entire sample and discount it to the present value by a factor of e^{-rT}, where r is the risk-free interest rate, we get our Monte Carlo estimate. The concept is illustrated in Figure 4.5. In chapter 5 we will compute the estimates with some specific parameters.

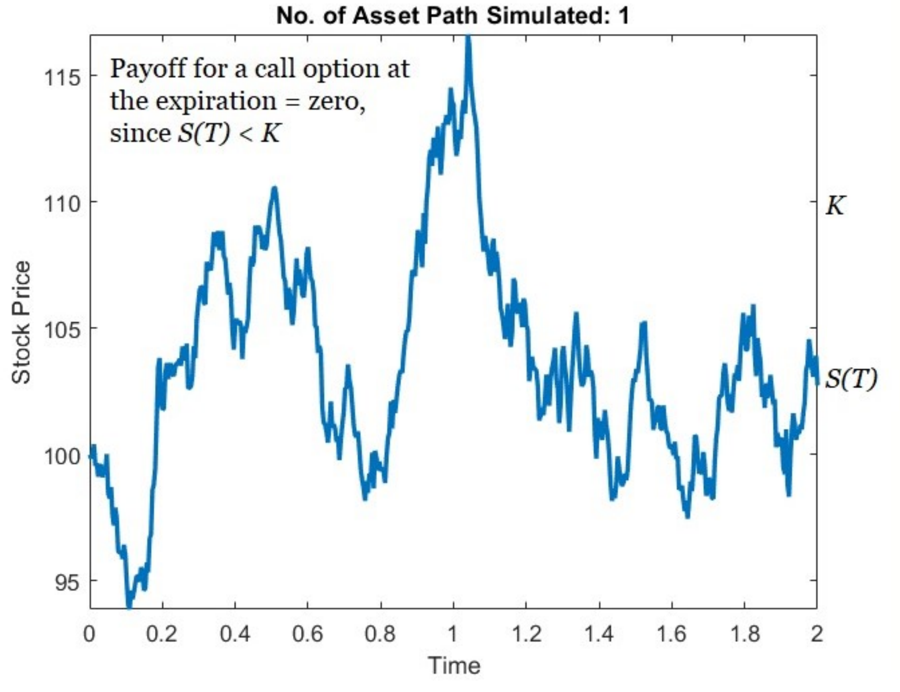

Figure 4.4: A single asset path generated by *simulate_asset_path* (zero payoff)

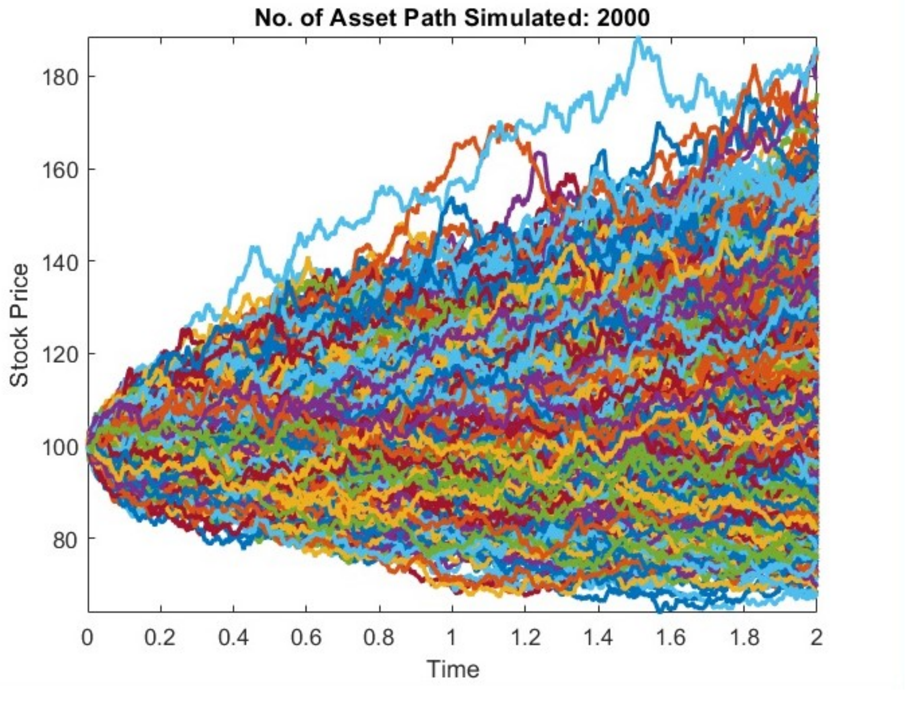

Figure 4.5: 2000 asset paths generated by *simulate_asset_path*

In the following section we are going to look at how Monte Carlo simulation can be applied to pricing some path-dependent options.

4.3 Monte Carlo Estimates for Path-dependent Options

There are complex options such as path-dependent options for which there are no closed-form analytical solutions. Monte Carlo simulation is often used to price such options. As explained in chapter 3, the price of an option is the discounted value of the expected payoff at the expiration. The price of the option is independent of the path that a stock may follow until the expiration. If the price at expiration is 125 and the strike price is 100, the payoff is 25. It makes no difference, if the stock has been at 60 or 160 before it reaches the expiration date. However, if the payoff were calculated based on the path that the stock has taken, it would make the payoff path-dependent. As a result, the option price is also path-dependent. The three most common path-dependent options are Lookback option, Asian option and Barrier Option. We are going to explain how Monte Carlo methods are applied to Lookback and Barrier Options.

A Lookback call option is created by replacing the strike price *K* with the minimum value the security has attained over the life of the option. And the opposite for a Lookback put option. For a Lookback put option the strike price *K* is replaced by the maximum value of the security over the life of the option. Lookback options are either at the money or in the money. A Lookback option can be interpreted as an option that is always sold at the highest possible value. For this reason, the Lookback options are more expensive than the standard options.

In order to simulate a Lookback option by Monte Carlo method, we can start with the function *simulate_asset_path*, as described in section 4.1. To calculate the payoff, we look at each asset path vector in matrix *sample*, and compute $\max(0, S_T - S_{min})$ rather than $\max(0, S_T - K)$. The average payoff is then calculated and discounted to get the Monte Carlo estimates.

A Barrier option is an option for which the payoff depends on the price reaching a certain barrier during the life of the option. Four basic types of Barrier options are down-and-out, down-and-in, up-and-out and up-and-in. A down-and-out option goes out of existence, if the price of the stock ever falls to the barrier. An up-and-out option goes out of existence, if the price of the stock ever rises to the barrier. A down-in-option comes into existence, if the price of the stock ever falls to the barrier. An up-and-in option comes into existence, if the price of the stock ever rises to the barrier.

Now we are going to look at how down-and-out barrier option can be priced using simulation. In order to simulate a down-and-out barrier option by Monte Carlo method, we can start with the function *simulate_asset_path*, as before, and proceed as follows:

We extract each vector from matrix *sample*, and check if any element of the vector is less than or equal to barrier. In Matlab this can be accomplished by creating a logical variable *crossed* and then creating an array of the logical indicators by using *any* function, *crossed = any(sample(:, i) <= barrier)*, where *i* indicates the path number. If *crossed* is true, we write off the asset path completely. The average payoff is calculated by considering only those asset paths for which *crossed* is false. By discounting the average, we get the Monte Carlo estimates.

Path-dependent options have been developed for a number of reasons, such as, hedging, taxes, legal and regulatory reasons. While closed-form formulas exist for some of them, many don't have analytical solutions. Monte Carlo methods are general enough to accommodate various path-dependent options, since we have a sample asset path with hypothetical data that we can manipulate that to fit our logic and determine a payoff at the expiration.

5 Empirical Analysis

5.1 Monte Carlo Estimates

In order to generate Monte Carlo estimates for European call and put options we ran our simulation process with the following parameters.

Initial value of stock S_0 = 100

Strike price K= 125

Period T = 2 (years)

Standard deviation of returns σ= .12

Risk-free interest rate r= .05 (5% APR)

With 2000 trajectories in each trial, we ran 30 trials. The estimates, as generated in each trial, are presented in Table 5.1. Taking the average of 30 trials, the results can be summarised as

Monte Carlo call estimate: 2.49 with a standard error of 0.16

Monte Carlo put estimate: 15.53 with a standard error of 0.36

Considering a sample size of 30, 95% confidence intervals would be

Monte Carlo call estimate: $2.49 \pm 1.96\,(\frac{0.16}{\sqrt{30}})$ or, 2.49 ± 0.06

Monte Carlo put estimate: $15.53 \pm 1.96\,(\frac{0.36}{\sqrt{30}})$ or, 15.53 ± 0.13

Using the same set of parameters, Black-Scholes closed-form formulas would yield

European call option: 2.46

European put option: 15.57

The true values, given by Black-Scholes model, lie within the 95% confidence interval of the Monte Carlo estimates.

Table 5.1: Results from Monte Carlo simulation

Trial no.	MCE (call)	MCE (put)
1	2.65	15.47
2	2.39	15.40
3	2.52	15.47
4	2.60	15.50
5	2.43	16.40
6	2.51	15.11
7	2.59	15.30
8	1.98	16.13
9	2.58	15.76
10	2.38	15.90
11	2.22	15.76
12	2.30	15.87
13	2.62	15.53
14	2.36	15.43
15	2.63	14.89
16	2.67	15.41
17	2.45	15.66
18	2.56	15.48
19	2.52	15.38
20	2.52	15.76
21	2.41	15.12
22	2.49	16.14
23	2.57	15.06
24	2.57	15.39
25	2.38	15.67
26	2.72	14.83
27	2.63	15.38
28	2.33	15.82
29	2.80	15.18
30	2.46	15.57

5.2 Efficiency of Monte Carlo Estimators

It can be proven formally to show that Monte Carlo estimators are unbiased. If $\widehat{C_n}$ is a Monte Carlo estimator with *n* number of trajectories, then we can write

$$E\left[\widehat{C_n}\right] = C$$

where *C* is the true value of an option. The estimator $\widehat{C_n}$ is the mean of *n* independent and identically distributed variables so that

$$\widehat{C_n} = \frac{1}{n}\sum_{i=1}^{n} C_i$$

with $\boldsymbol{E}[C_i] = C$ and $var[C_i] = {\sigma_C}^2 < \infty$. The central limit theorem asserts that as the number of replication (trajectories in the case of simulation) increases, the standardized estimator $(\widehat{C_n} - C)/(\sigma_C/\sqrt{n})$ converges in distribution to the standard normal variable. Written formally,

$$\frac{(\widehat{C_n} - C)}{(\sigma_C/\sqrt{n})} \rightarrow N(0,1)$$

where $\rightarrow$ denotes convergence in distribution. The rate of convergence also depends on the variance of the estimates. The problem is, we never know what the true variance is. The variance is always estimated from the sample. In our analysis above we had a relatively small variance. It should be noted that the variance increases as the time period increases. Increasing variance will require larger replication to lower the range of an interval estimate. Boyle (1977) in his first paper on Monte Carlo simulation claimed that for a particular option priced at 17.19, to reduce the 95% confidence interval from ±0.958 to ±0.05, the number of sample size had to be increased from 5000 to 1,835,500. This highlights a major drawback of Monte Carlo simulation and the computational burden associated with it.

5.3 Discretization Error

Monte Carlo methods are essentially a numerical integration tool. Any numerical integration requires some form of discretization. In the case of our simulation process the discretization comes in the form of steps. This introduces discretization bias in the estimate, as it results from time-discretization of the continuous-time dynamics of the theoretical model. Obviously, higher the number of steps, greater is the convergence of the estimates to the true value. However, increased number of steps entails computational burden. Discretization error is a major limitation of Monte Carlo methods.

6 Conclusion

We have looked at the derivation of Black-Scholes model to understand how the lognormal distribution of a stock price can be applied to arrive at the same results without solving Black-Scholes PDE. This derivation is very similar to Cox, Ross and Rubenstein binomial model. We have demonstrated how Monte Carlo method can be applied to simulate asset paths in a risk-neutral universe, and based on such simulated sample how we can compute Monte Carlo estimates for an option. We have provided statistical analysis to show that Monte Carlo estimates can be reliable with 95 percent confidence intervals. We have also explained how the simulation can be extended to price path-dependent options.

Monte Carlo simulation, in our judgement, is an important technique that will always find its application in pricing derivatives in various asset classes. The most important application of Monte Carlo simulation will always be pricing complex derivatives that do not have any analytical solutions in closed-form formulas.

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Appendix A: Contents of Enclosed Files

The file enclosed to this thesis contains Matlab source code for:

- option_montecarlo_simulation.m
- simulate_asset_path.m