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Faculty of Mathematics and Physics

# DIPLOMA THESIS

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# Lipschitz Functions in Analysis of PDEs

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I declare that I wrote my diploma thesis by myself and only with the use of the cited sources. I agree with lending the thesis.

Prague, April 17, 2007 Veronika Fišerová

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**Abstrakt**

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|---|---|
| *Název práce:* | Lipschitzovské funkce v analýze systémů parciálních diferenciálních rovnic |
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*Abstrakt:* Uvažujeme stacionární proudění homogenní nestlačitelné tekutiny nenewtonovského typu. Předpokládáme, že viskozita tekutiny závisí na střední hodnotě tenzoru napětí (na tlaku) a na rychlosti smyku. Motivací pro tuto závislost může být celá řada z technologického hlediska významných experimentů a studií. Zabýváme se systémem parciálních diferenciálních rovnic, které popisují výše zmíněná proudění, doplněných zároveň o homogenní Dirichletovu (tzv. no-slip) okrajovou podmínku a pro tento systém dokážeme globální existenci slabého řešení za určitých blíže specifikovaných předpokladů kladených na strukturu viskozity. To je provedeno pomocí limitního přechodu od slabého řešení dříve již zavedeného aproximativního systému, pro který je existence slabého řešení rovněž ukázána, a sice pomocí Galerkinových aproximací. Důležitou roli při samotném limitním přechodu poté hraje fakt, že viskozita je v jistém smyslu monotónní. K tomu, abychom ukázali konvergenci tlaku a symetrické části gradientu rychlosti skoro všude, zavedeme rozklad tlaku a použijeme lipschitzovské testovací funkce. Pro tento účel využijeme tzv. Lipschitzovských aproximací Sobolevových funkcí.
*Klíčová slova:* existence, slabé řešení, nestlačitelná tekutina, viskozita závislá na tlaku a na rychlosti smyku, Lipschitzovská aproximace funkcí z $W_0^{1,p}$

**Abstract**

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*Abstract:* We consider a steady flow of a homogeneous incompressible non-Newtonian fluid. We suppose that the viscosity of the fluid depends on the mean normal stress (the pressure) and on the shear rate as this dependence is motivated by many technologically important experiments and studies. We study a system of partial differential equations that govern such flows of fluids subject to the homogeneous Dirichlet (no-slip) boundary condition and establish a global existence of a weak solution under certain specified assumptions on the structure of the viscosity. This is carried out by passing to the limit in the weak solution of a previously introduced approximate system, the existence of which is also shown. The fact that the viscosity is monotone in some sense plays an important role. A decomposition of the pressure and Lipschitz test functions as Lipschitz approximations of Sobolev functions are incorporated in order to obtain almost everywhere convergence of the pressure and the symmetric part of the velocity gradient.
*Keywords:* existence, weak solution, incompressible fluid, pressure- and shear-dependent viscosity, Lipschitz approximation of $W_0^{1,p}$- functions

# 1 Introduction

The most famous model that describes a flow of an incompressible fluid is the well-known Navier-Stokes model which considers the viscosity of the fluid to be a constant. While this model is capable of describing a large class of flows of fluids, it is inadequate to capture the so-called non-Newtonian behavior. The typical non-Newtonian features include for example the dependence of the viscosity on the shear rate, stress relaxation, non-linear creep, the development of normal stress differences in a simple shear flow or yield-like behavior (for details see [26] or [20]). An interesting naturally raised question is whether the viscosity of an incompressible fluid could also depend on the pressure. Such a property would then fall into the class of the non-Newtonian responses as well.

In the year 1845 Stokes was the first to consider this possibility and in his paper [28] he carefully delineates under which conditions the viscosity could be assumed as pressure-independent. Since then the dependence of the viscosity on the pressure has been proven by many experimental studies and we briefly discuss a few of them.

In 1893 Barus in [6] proposed the following relation between the viscosity $\mu$ and the pressure $p$ for liquids

$$\mu(p) = \mu_0 \exp(\alpha p), \;\; \alpha > 0.$$

This expression has been widely used in elastohydrodynamics where the fluid undergoes a wide range of pressures and a significant change in the viscosity occurs (see [29]). There is a great amount of another experimental work prior to 1930 concerning the pressure-dependence of the material coefficients and the related discussion can be found in the book [8] by Bridgman. On the basis of experiments with more than 40 liquids Andrade in [4] suggested the following relationship between the viscosity $\mu$, the pressure $p$, the density $\rho$ and the temperature $\theta$

$$\mu(p, \rho, \theta) = A\rho^{1/2} \exp\left(\frac{B}{\theta}(p + D\rho^2)\right),$$

where $A$, $B$ and $D$ are constants. Notice that also Andrade obtained an exponential dependence on the pressure, however, his expression seems to be more general. A large number of more recent studies concerning other formulas for the variation of the viscosity with pressure are available as well and almost all of these studies more or less involve the exponential dependence. A detailed list of references related to this topic can be found in [21]. Lately, Bair and Kottke (see [5]) have shown that in fact the viscosity can depend even more drastically on the pressure so that the equations above cease to be appropriate.

While the experimential background gives clear evidence for the possible dependence of the viscosity on the pressure, such is not the case in the classical textbooks of continuum mechanics where the models for incompressible fluids with pressure-dependent material coefficients are completely omitted. Let us illustrate the matter by recalling the standard derivation of constitutive equations for a homogeneous compressible and incompressible Newtonian fluid (the second one is referred to as Navier-Stokes fluid). For simplicity, we consider three-dimensional flows.

Ignoring all temperature effects, the standard approach is based on an assumption that the Cauchy stress depends on the density and the velocity gradient and from the so-called principle of material frame-indifference actually only through its symmetric part $\mathbf{D} = \mathbf{D}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$, i.e.,

$$\mathbf{T} = \boldsymbol{f}(\rho, \mathbf{D}).$$

If we require the fluid to be isotropic, then the function $\boldsymbol{f}$ has to satisfy the restriction

$$\boldsymbol{f}(\rho, \mathbf{Q}\mathbf{D}\mathbf{Q}^T) = \mathbf{Q}\boldsymbol{f}(\rho, \mathbf{D})\mathbf{Q}^T \quad \text{for all orthogonal tensors } \mathbf{Q}$$

and the standard representation theorem for isotropic tensor functions implies that the stress has the following form (for representation theorems see for example [30])

$$\mathbf{T} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{D} + \alpha_2 \mathbf{D}^2,$$

where $\alpha_i = \alpha_i(\rho, I_{\mathbf{D}}, II_{\mathbf{D}}, III_{\mathbf{D}})$ and

$$I_{\mathbf{D}} = \operatorname{tr} \mathbf{D}, \quad II_{\mathbf{D}} = \frac{1}{2}((\operatorname{tr} \mathbf{D})^2 - \operatorname{tr} \mathbf{D}^2), \quad III_{\mathbf{D}} = \det \mathbf{D}.$$

The requirement on the stress being linear in $\mathbf{D}$ then yields

$$\mathbf{T} = -p(\rho)\mathbf{I} + \lambda(\rho)(\operatorname{tr} \mathbf{D})\mathbf{I} + 2\mu(\rho)\mathbf{D},$$

which is the Cauchy stress for the homogeneous compressible Newtonian fluid. The pressure $p$ appearing in this relation is a thermodynamic pressure and its relationship with the density $\rho$ is through an equation of state. If this equation is invertible then the bulk and shear viscosities $\lambda$ and $\mu$ can be indeed expressed as functions of the pressure.

On the other hand, an application of the similar procedure to a homogeneous incompressible fluid, i.e., to

$$\mathbf{T} = \boldsymbol{f}(\mathbf{D}) \quad \text{and} \quad \operatorname{tr} \mathbf{D} = 0,$$

leads to

$$\mathbf{T} = \tilde{\alpha}_0 \mathbf{I} + \tilde{\alpha}_1 \mathbf{D} + \tilde{\alpha}_2 \mathbf{D}^2, \tag{1.1}$$

where $\tilde{\alpha}_i$, $i = 0, 1, 2$, depend on

$$-\frac{1}{2} \operatorname{tr} \mathbf{D}^2 = -\frac{1}{2} |\mathbf{D}|^2 \quad \text{and} \quad \det \mathbf{D}.$$

If we again require the stress to be linear in $\mathbf{D}$, the expression (1.1) simplifies to

$$\mathbf{T} = -p\mathbf{I} + 2\mu \mathbf{D},$$

with the viscosity $\mu$ being a positive constant and the pressure $p$ being the mean normal stress, namely

$$p = -\frac{1}{3} \operatorname{tr} \mathbf{T}.$$

Evidently, the viscosity cannot be expressed in terms of the pressure as the viscosity itself is constant. Later on, we shall present an alternative approach that, on the contrary, is capable of describing incompressible fluids with pressure-dependent viscosities as well.

Let us now mention another example. Note that (1.1) also involves the incompressible non-Newtonian fluids with shear rate dependent viscosity, i.e.,

$$\mathbf{T} = -p\mathbf{I} + 2\mu(|\mathbf{D}|^2)\mathbf{D}, \tag{1.2}$$

which include the popular power-law fluids with the viscosity of the form $\mu(|\mathbf{D}|^2) = \mu_0 |\mathbf{D}|^{r-2}$, where $r > 1$ is the power-law exponent and $\mu_0$ is a positive constant, as a special case.

In this thesis we shall be interested in the class of homogeneous incompressible fluids with the viscosity depending on the pressure and the shear rate alike and having the Cauchy stress of the following representation

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p, |\mathbf{D}|^2)\mathbf{D}, \tag{1.3}$$

as this model is considered in various engineering areas, among others in elastohydrodynamics or mechanics of granular and visco-elastic materials.

We would like to remark that we will not focus our attention on the precise dependence of the viscosity on the pressure and the shear rate, but only consider the viscosity to satisfy certain conditions specified later.

Though the models (1.2) and (1.3) look similar, there is a remarkable difference between these two relations. Since the pressure $p$ is the mean normal stress $-\frac{1}{3}\operatorname{tr}\mathbf{T}$, it becomes obvious that the first one is an explicit relationship between $\mathbf{T}$ and $\mathbf{D}$ whereas the second expression is an implicit one and hence cannot be gained by the procedure described above. Nonetheless, there are at least two other concepts that are able to capture models of the type (1.3). The first one is a thermodynamic approach based on the maximization of the rate of dissipation and the second one is the so-called implicit constitutive theory, both methods developed by K. R. Rajagopal and his co-workers (for details see [20] or [21]).

Let us consider the second approach, which shows more similarities with the classical one, and let us start with an implicit equation having the form

$$\boldsymbol{f}(\mathbf{T},\mathbf{D}) = \mathbf{0}.$$

The demand of isotropy now means that

$$\boldsymbol{f}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T,\mathbf{Q}\mathbf{D}\mathbf{Q}^T) = \mathbf{Q}\boldsymbol{f}(\mathbf{T},\mathbf{D})\mathbf{Q}^T \quad \text{for all orthogonal tensors } \mathbf{Q}.$$

In this case, the representation theorem for isotropic tensor functions yields

$$\begin{aligned}\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{D}\mathbf{T}+\mathbf{T}\mathbf{D}) + \alpha_6(\mathbf{T}^2\mathbf{D}+\mathbf{D}\mathbf{T}^2)&\\ + \alpha_7(\mathbf{T}\mathbf{D}^2+\mathbf{D}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{D}^2+\mathbf{D}^2\mathbf{T}^2) = \mathbf{0},&\end{aligned}$$

where $\alpha_i$, $i=1,\dots,8$, depend on the invariants

$$\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{D}, \operatorname{tr}\mathbf{T}^2, \operatorname{tr}\mathbf{D}^2, \operatorname{tr}\mathbf{T}^3, \operatorname{tr}\mathbf{D}^3, \operatorname{tr}(\mathbf{T}\mathbf{D}), \operatorname{tr}(\mathbf{T}^2\mathbf{D}), \operatorname{tr}(\mathbf{D}^2\mathbf{T}), \operatorname{tr}(\mathbf{T}^2\mathbf{D}^2).$$

Consequently, if we choose

$$\begin{aligned}\alpha_0 &= -\frac{1}{3}\operatorname{tr}\mathbf{T},\\ \alpha_1 &= 1,\\ \alpha_2 &= -2\mu(-\frac{1}{3}\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{D}^2),\ \ \mu > 0,\\ \alpha_i &= 0 \quad \text{for } i = 3,\dots,8\end{aligned}$$

and define $p$ to be the mean normal stress

$$p = -\frac{1}{3}\operatorname{tr}\mathbf{T}, \tag{1.4}$$

we obtain the Cauchy stress of the form

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p, |\mathbf{D}|^2)\mathbf{D}. \tag{1.5}$$

Notice that we have derived the model (1.3), and due to (1.4) and (1.5) we have obtained the constraint of incompressibility $\operatorname{tr}\mathbf{D} = \operatorname{div}\boldsymbol{v} = 0$ as a consequence, which is a very interesting feature of the implicit constitutive theory. Thus, (1.3) indeed describes incompressible fluids.

The main aim of the thesis is the mathematical analysis of one such model. To be more precise, we will be interested in steady flows of homogeneous incompressible fluids that have the Cauchy stress of the form (1.3) and are subject to the homogeneous Dirichlet (no-slip) boundary condition. With the help of Lipschitz approximations of Sobolev functions, we shall establish the existence of weak solutions to the system of partial differential equations that govern such flows of fluids. We start discussing the mathematical issues in the next section and begin with a description of the mathematical model.

# 2 Mathematical model

## 2.1 Definition of the problem

In order to derive the governing equations, we consider that the flows take place in an open and bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with a smooth boundary as specified later. On substituting (1.3) into the balance of linear momentum

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \operatorname{div} \mathbf{T} + \rho \boldsymbol{f},$$

where $\boldsymbol{f}$ is the specific body force, we arrive at

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \operatorname{div}(\mu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})) + \nabla p = \rho \boldsymbol{f} \quad \text{in } \Omega. \tag{2.1}$$

To this equation we add the constraint of incompressibility and the boundary condition.

It is also convenient to divide the equation (2.1) by the positive constant value of the density $\rho$. Then, relabelling $\frac{p}{\rho}$, $\frac{\mu}{\rho}$ by $p$ and $\nu$ and remembering that we are interested only in steady flows ($\frac{\partial \boldsymbol{v}}{\partial t} = 0$), we can rewrite the above-mentioned system as

$$\begin{aligned} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div}(\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})) + \nabla p &= \boldsymbol{f} && \text{in } \Omega \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } \Omega \\ \boldsymbol{v} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

where $\otimes$ denotes the standard tensor product.

In addition, we shall suppose that the pressure $p$ meets the following condition

$$\int_\Omega p \, \mathrm{d}x = 0. \tag{2.3}$$

As it is not completely clear why we should assume such a condition, it requires a brief explanation. The pressure in an incompressible fluid is determined to within a constant. In the classical Navier-Stokes equations (or other models with pressure-independent viscosity) only the gradient of the pressure is met and so the choice of a constant that fixes the pressure is irrelevant. In our case, on the other hand, the situation is completely different. We deal with a pressure-dependent viscosity and this means that also the actual value of the pressure is encountered. Therefore, this constant plays an important role and needs to be fixed. From a physical point of view it might be appropriate to prescribe the pressure at some point of $\Omega$ or $\partial\Omega$. Nevertheless, in the context of the mathematical framework, as we consider integrable functions, it would be more preferable to consider the mean value of the pressure over some subdomain of $\Omega$ or over some subpart of the boundary $\partial\Omega$ of non-zero volume and area measure respectively. Currently, there is a work in progress concerning the conditions decribing the normal traction, i.e., $p + (\boldsymbol{v} \cdot \boldsymbol{n})^2 - \nu(p, |\mathsf{D}(\boldsymbol{v})|^2)\mathsf{D}(\boldsymbol{v})\boldsymbol{n} \cdot \boldsymbol{n} = g$, $g$ being a given function, on a part of the boundary having non-zero area measure. Unfortunately, yet no results are available and in view of this fact and for simplicity we restrict ourselves to the condition (2.3) though it is not quite natural.

We denote the system (2.2) together with the condition (2.3) by $(\mathcal{P}^0)$.

## 2.2 Structure of the viscosity and its consequences

As we already mentioned in the introduction, we will not focus our attention on the precise relationship between the viscosity, the pressure and the shear rate.

In the following, we assume that the viscosity $\nu(p, |\mathsf{D}|^2)$ is a $\mathcal{C}^1$-mapping of $\mathbb{R} \times \mathbb{R}_0^+$ into $\mathbb{R}^+$ satisfying for some fixed (but arbitrary) $r \in (1, 2)$ and all $\mathsf{D} \in \mathbb{R}^{d\times d}_{\text{sym}}$, $\mathsf{B} \in \mathbb{R}^{d\times d}_{\text{sym}}$ and $p \in \mathbb{R}$ the following two conditions

$$C_1(1 + |\mathsf{D}|^2)^{\frac{r-2}{2}}|\mathsf{B}|^2 \le \frac{\partial(\nu(p, |\mathsf{D}|^2)\mathsf{D})}{\partial \mathsf{D}} \cdot (\mathsf{B} \otimes \mathsf{B}) \le C_2(1 + |\mathsf{D}|^2)^{\frac{r-2}{2}}|\mathsf{B}|^2 \quad (2.4)$$

$$\left|\frac{\partial(\nu(p, |\mathsf{D}|^2)\mathsf{D})}{\partial p}\right| \le \gamma_0(1 + |\mathsf{D}|^2)^{\frac{r-2}{4}} \le \gamma_0, \quad (2.5)$$

with positive constants $C_1$ and $C_2$ and a sufficiently small constant $\gamma_0 > 0$ on whose value we require that

$$\gamma_0 < \frac{C_1}{C_{\text{div}}(\Omega, 2)(C_1 + C_2)}, \quad (2.6)$$

where the constant $C_{\text{div}}(\Omega, q)$ occurs in the problem of solvability of the equation $\operatorname{div} \boldsymbol{u} = f$ discussed in Appendix (see Lemma A.3).

These assumptions deserve a short comment. If $\nu$ is independent of $p$ it is clear that the condition (2.5) is irrelevant and (2.4) is met by the power-law fluids. On the contrary, our assumptions do not allow us to consider any model where the viscosity depends only on the pressure.

For a better illustration, we mention some examples of the forms of viscosities fulfilling the conditions (2.4) and (2.5).

**Example 2.1.** *Consider for* $r \in (1,2)$ *and for a constant* $A \in (0,1]$

$$\nu_i(p, |\mathbf{D}|^2) = \left(A + \gamma_i(p) + |\mathbf{D}|^2\right)^{\frac{r-2}{2}}, \quad i = 1,2,$$

*where* $\gamma_i(p)$ *have the form* $(q \geq 0)$

$$\gamma_1(p) = (1 + \alpha^2 p^2)^{-q/2},$$
$$\gamma_2(p) = \begin{cases} (1 + \exp(\alpha p))^{-q} & \textit{if } p > 0 \\ 1 & \textit{if } p \leq 0. \end{cases}$$

(*Note that* $0 \leq \gamma_i(p) \leq 1$ *for* $i = 1,2$.)

*Then* (2.4) *holds with* $C_1 = 2^{\frac{r-2}{2}}(r-1)$ *and* $C_2 = A^{\frac{r-2}{2}}$ *and* (2.5) *holds with* $\gamma_0 = \frac{2-r}{2}\alpha q$ (see [23] and [9]).

The first very important feature of the viscosity following from the assumptions (2.4) and (2.5) is a certain type of monotonicity. We will see that this property plays a crucial role in mathematical analysis of the problem $(\mathcal{P}^0)$. For simplicity, we set

$$\mathbf{S}(p, \mathbf{D}) := \nu(p, |\mathbf{D}|^2)\mathbf{D}.$$

**Lemma 2.1.** *Let the assumptions* (2.4), (2.5) *hold. For arbitrary* $p^1, p^2 \in \mathbb{R}$ *and* $\mathbf{D}^1, \mathbf{D}^2 \in \mathbb{R}^{d\times d}_{\text{sym}}$ *we set*

$$\mathcal{I}^{1,2} := \int_0^1 (1 + \left|\mathbf{D}^2 + s(\mathbf{D}^1 - \mathbf{D}^2)\right|^2)^{\frac{r-2}{2}} \left|\mathbf{D}^1 - \mathbf{D}^2\right|^2 \mathrm{d}s.$$

*Then*

$$\frac{C_1}{2}\mathcal{I}^{1,2} \leq (\mathbf{S}(p^1, \mathbf{D}^1) - \mathbf{S}(p^2, \mathbf{D}^2)) \cdot (\mathbf{D}^1 - \mathbf{D}^2) + \frac{\gamma_0^2}{2C_1}\left|p^1 - p^2\right|^2. \tag{2.7}$$

The condition (2.4) yields another useful properties (as coercivity and growth) and we summarize them in the next lemma.

**Lemma 2.2.** *Let the assumption* (2.4) *hold for* $r \in (1,2)$. *Then for all* $p \in \mathbb{R}$ *and* $\mathbf{D} \in \mathbb{R}^{d\times d}_{\text{sym}}$

$$\nu(p, |\mathbf{D}|^2)\mathbf{D} \cdot \mathbf{D} \geq \frac{C_1}{2r}(|\mathbf{D}|^r - 1) \tag{2.8}$$

*and*

$$\left|\nu(p, |\mathbf{D}|^2)\mathbf{D}\right| \leq \frac{C_2}{1-(2-r)\lambda}(1+|\mathbf{D}|)^{1-(2-r)\lambda} \quad \textit{for all } \lambda \in [0,1]. \tag{2.9}$$

For the proofs of Lemma 2.1 and Lemma 2.2 see [12] and Lemma 5.1.19 in [22], respectively.

## 2.3 Existence theorem and known results

Before we proceed to the formulation of the existence theorem, we first fix the notation.

We consider that the domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, has Lipschitz boundary $\partial\Omega$ and we write $\Omega \in \mathcal{C}^{0,1}$.

Let $1 \leq q \leq \infty$. In the standard way we denote the Lebesgue spaces $L^q(\Omega)$ equipped with the norm $\|\cdot\|_q$ and the Sobolev spaces $W^{1,q}(\Omega)$ equipped with the norm $\|\cdot\|_{1,q}$ of scalar measurable functions defined on $\Omega \subset \mathbb{R}^d$. $W_0^{1,q}(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{1,q}}$ are Sobolev spaces of functions with zero traces on the boundary $\partial\Omega$. If $X(\Omega)$ is a Banach space then $(X(\Omega))^*$ stands for its dual space and $X(\Omega)^d := \{\boldsymbol{u} : \Omega \to \mathbb{R}^d; u_i \in X(\Omega), i = 1, \dots, d\}$, similarly $X(\Omega)^{d\times d} := \{\mathbf{A} : \Omega \to \mathbb{R}^{d\times d}; \mathbf{A}_{ij} \in X(\Omega), i, j = 1, \dots, d\}$. We also introduce the following subspaces of Lebesgue and Sobolev spaces

$$L_0^q(\Omega) := \left\{ h \in L^q(\Omega) : \int_\Omega h \, \mathrm{d}x = 0 \right\}$$
$$W_{0,\mathrm{div}}^{1,q}(\Omega)^d := \{\boldsymbol{u} \in W_0^{1,q}(\Omega)^d : \operatorname{div} \boldsymbol{u} = 0 \text{ a.e. in } \Omega\}.$$

Let us also denote the norm of the dual space $(W_0^{1,q}(\Omega)^d)^* = W^{-1,q'}(\Omega)^d$ by $\|\cdot\|_{-1,q'}$ and the duality pairing by $\langle\cdot,\cdot\rangle$, $q' = \frac{q}{q-1}$. All the spaces introduced above are Banach spaces. Moreover, if $1 < q < \infty$, then they are also reflexive and separable.

The prime motivation for the thesis was an effort to improve the below-mentioned existence result for the system $(\mathcal{P}^0)$ concerning the parameter $r$ established by Franta, Málek and Rajagopal in [12]. The authors considered the mean value of the pressure to be fixed by an arbitrary constant $p_0 \in \mathbb{R}$. For $p_0 = 0$, the result can be formulated in the form of this theorem.

**Theorem 2.1.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, $d = 2$ or $3$, and let $\Omega \in \mathcal{C}^{0,1}$. Let the assumptions* (2.4) *and* (2.5) *be satisfied with $\frac{3d}{d+2} < r < 2$ and let $\boldsymbol{f} \in (W_0^{1,r}(\Omega)^d)^*$. Then there exists a weak solution $(\boldsymbol{v}, p)$ to the problem $(\mathcal{P}^0)$ such that*

$$\boldsymbol{v} \in W^{1,r}_{0,\mathrm{div}}(\Omega)^d \quad \text{and} \quad p \in L_0^{r'}(\Omega)$$

*and for all $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega)^d$ holds*

$$\int_\Omega \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x - \int_\Omega p \,\mathrm{div}\, \boldsymbol{\varphi} \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle.$$

The lower bound $\frac{3d}{d+2}$ comes from the requirement on integrability of the term $(\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{\varphi}$. Therefore, we are interested in the case when $r > \frac{2d}{d+2}$ since we want the pressure to lie at least in $L^1$. In order to obtain this lower bound, we have to consider test functions from $W_0^{1,\infty}(\Omega)^d$ again to ensure the integrability of the convective term.

The result of the thesis is the following theorem and its proof is the content of Section 4.

**Theorem 2.2. (Existence theorem)** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, $d \geq 2$, and let $\Omega \in \mathcal{C}^{0,1}$. Let the assumptions* (2.4) *and* (2.5) *be satisfied with $\frac{2d}{d+2} < r \leq \frac{3d}{d+2}$ and let $\boldsymbol{f} \in (W_0^{1,r}(\Omega)^d)^*$. Then there exists a weak solution $(\boldsymbol{v}, p)$ to the problem $(\mathcal{P}^0)$ such that*

$$\boldsymbol{v} \in W^{1,r}_{0,\mathrm{div}}(\Omega)^d \quad \text{and} \quad p \in L_0^{\frac{dr}{2(d-r)}}(\Omega)$$

*and for all $\boldsymbol{\varphi} \in W_0^{1,\infty}(\Omega)^d$ holds*

$$\int_\Omega \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x - \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x - \int_\Omega p \,\mathrm{div}\, \boldsymbol{\varphi} \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle.$$

Before we draw up a survey of known mathematical results concerning our model and related problems, we wish to remark that there is a difference between the analysis of the models with $\nu = \nu(p, |\mathbf{D}|^2)$ and with $\nu = \nu(|\mathbf{D}|^2)$.

In the latter case, the standard approach is based on dealing with spaces of divergence-free functions from the very beginning and thus completely eliminating the pressure from the analysis of the problem. The pressure is afterwards reconstructed by using for example de Rham's theorem. Unfortunately, the same method cannot be applied to the problems with pressure-dependent viscosity since we need to have knowledge of the nature of the pressure a priori.

One of the main ingredients of the proof of Theorem 2.2 will be the so-called Lipschitz truncations of Sobolev functions. An application of these Lipschitz approximations can be found in [13] where Frehse, Málek and Steinhauer showed the existence of solutions for steady flows with shear rate dependent viscosity subject to the homogeneous Dirichlet boundary condition for the case $r > \frac{2d}{d+2}$. A simplified version of the proof together with another interesting application of Lipschitz approximations in existence theory of incompressible electro-rheological fluids can be found in the recent study by Diening, Málek and Steinhauer [11]. For the case of pressure and shear rate dependent viscosity, we have already mentioned the existence result established in [12] for exactly our problem for $r$ between $\frac{3d}{d+2}$ and 2. As for unsteady flows, Málek, Nečas and Rajagopal in [23] and Hron, Málek, Nečas and Rajagopal in [17] showed global-in-time existence under spatially periodic boundary conditions and these results were extended to flows in bounded domains subject to the Navier's slip by Bulíček, Málek and Rajagopal in [10]. On the other hand, there is no global existence theory available both for steady and unsteady flows of fluids whose viscosity depends only on the pressure. There are several studies, such as by Renardy [27], Gazzola [14] or Gazzola and Secchi [15], but all of them suffer from the drawback that either the structure of the viscosity is contradicted by experiments or only short-in-time existence of solutions for small data is shown. Recently, some numerical solutions for the flows of fluids with pressure-dependent viscosities in special geometries have been obtained by Hron, Málek and Rajagopal [16].

# 3 Approximate system and existence of its solutions

## 3.1 Introduction of the approximate system

In order to establish the existence of a weak solution to $(\mathcal{P}^0)$, we introduce an approximate system of equations $(\mathcal{P}^{\varepsilon,\eta})$ with the help of the so-called quasi-compressible approximation.

It is based on the fact that no fluid is perfectly incompressible. Therefore, we replace the constraint of incompressibility $\operatorname{div}\boldsymbol{v} = 0$ by a Neumann problem for the pressure and for all $\varepsilon > 0$ of the form

$$\begin{aligned} -\varepsilon\triangle p^\varepsilon + \varepsilon p^\varepsilon + \operatorname{div}\boldsymbol{v}^\varepsilon &= 0 \quad \text{in } \Omega \\ \frac{\partial p^\varepsilon}{\partial \boldsymbol{n}} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Such an approximation also ensures that we have information about the pressure from the very beginning, which is crucial as the viscosity depends on it.

In order to be able to test with the solution itself, we have to make sure that all of the terms appearing in the weak formulation will make a good sense. The trouble arises in the convective term and hence also another level of approximation ($\eta-$approximation) is considered by introducing an extra term to the equation of motion, namely

$$\eta|\boldsymbol{v}^{\varepsilon,\eta}|^{2r'-2}\boldsymbol{v}^{\varepsilon,\eta},$$

where $r' = \frac{r}{r-1}$.

Since $\operatorname{div}\boldsymbol{v}^{\varepsilon,\eta}$ in no longer equal to zero and we would still like to deal easily with the convective term (for preservation of uniform estimates), we modify it as well. For this purpose, we decompose the "approximate" velocity $\boldsymbol{v}^{\varepsilon,\eta}$ in the following way (see Lemma A.3 and below)

$$\boldsymbol{v}^{\varepsilon,\eta} := \mathcal{P}\boldsymbol{v}^{\varepsilon,\eta} + \boldsymbol{g}^{\boldsymbol{v}^{\varepsilon,\eta}},$$

where $\boldsymbol{g}^{\boldsymbol{v}^{\varepsilon,\eta}}$ solves the following problem

$$\begin{aligned} \operatorname{div}\boldsymbol{g}^{\boldsymbol{v}^{\varepsilon,\eta}} &= \operatorname{div}\boldsymbol{v}^{\varepsilon,\eta} \qquad \text{in } \Omega \\ \boldsymbol{g}^{\boldsymbol{v}^{\varepsilon,\eta}} &= \boldsymbol{0} \qquad\qquad \text{on } \partial\Omega. \end{aligned}$$

Notice that from the definition it is obvious that $\operatorname{div} \mathcal{P}\boldsymbol{v}^{\varepsilon,\eta} = 0$ a.e. in $\Omega$ and therefore considering $\boldsymbol{v}^{\varepsilon,\eta} \otimes \mathcal{P}\boldsymbol{v}^{\varepsilon,\eta}$ instead of $\boldsymbol{v}^{\varepsilon,\eta} \otimes \boldsymbol{v}^{\varepsilon,\eta}$ gives

$$\int_\Omega (\boldsymbol{v}^{\varepsilon,\eta} \otimes \mathcal{P}\boldsymbol{v}^{\varepsilon,\eta}) \cdot \nabla \boldsymbol{v}^{\varepsilon,\eta} \, \mathrm{d}x = 0. \tag{3.1}$$

Later on, we will see that thanks to the additional term $\eta|\boldsymbol{v}^{\varepsilon,\eta}|^{2r'-2}\boldsymbol{v}^{\varepsilon,\eta}$ the expression $(\boldsymbol{v}^{\varepsilon,\eta} \otimes \mathcal{P}\boldsymbol{v}^{\varepsilon,\eta}) \cdot \nabla \boldsymbol{v}^{\varepsilon,\eta}$ is indeed an integrable function.

Moreover, from Lemma A.3 we also have the estimates (A.2) and (A.3) for $\boldsymbol{g}^{\boldsymbol{v}^{\varepsilon,\eta}}$ and for $\mathcal{P}\boldsymbol{v}^{\varepsilon,\eta}$, respectively.

Incorporating all of the above-mentioned modifications, we obtain the approximate system $(\mathcal{P}^{\varepsilon,\eta})$ of the following form

$$\left.\begin{aligned} \eta|\boldsymbol{v}^{\varepsilon,\eta}|^{2r'-2}\boldsymbol{v}^{\varepsilon,\eta} + \operatorname{div}(\boldsymbol{v}^{\varepsilon,\eta} \otimes \mathcal{P}\boldsymbol{v}^{\varepsilon,\eta}) \qquad \\ - \operatorname{div}(\nu(p^{\varepsilon,\eta}, |\mathbf{D}(\boldsymbol{v}^{\varepsilon,\eta})|^2)\mathbf{D}(\boldsymbol{v}^{\varepsilon,\eta})) + \nabla p^{\varepsilon,\eta} \end{aligned}\right\} = \boldsymbol{f} \quad \text{in } \Omega \tag{3.2}$$

$$-\varepsilon \triangle p^{\varepsilon,\eta} + \varepsilon p^{\varepsilon,\eta} + \operatorname{div} \boldsymbol{v}^{\varepsilon,\eta} = 0 \quad \text{in } \Omega \tag{3.3}$$

$$\frac{\partial p^{\varepsilon,\eta}}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega$$

$$\boldsymbol{v}^{\varepsilon,\eta} = \boldsymbol{0} \quad \text{on } \partial\Omega.$$

Note that (3.2), (3.3) and the Gauss' theorem imply that

$$\int_\Omega p^{\varepsilon,\eta} \, \mathrm{d}x = 0.$$

First, we will prove that there exists a weak solution to $(\mathcal{P}^{\varepsilon,\eta})$ and then by letting $\varepsilon \to 0$ and $\eta \to 0$ in the weak formulation of $(\mathcal{P}^{\varepsilon,\eta})$, which will be done in Section 4, we shall obtain a weak solution to $(\mathcal{P}^0)$.

### 3.2 Existence of solutions

Our goal is to show that for $\frac{2d}{d+2} < r \le \frac{3d}{d+2}$ and for fixed $\varepsilon$, $\eta > 0$ there is a weak solution $(\boldsymbol{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ to the problem $(\mathcal{P}^{\varepsilon,\eta})$ such that

$$\boldsymbol{v}^{\varepsilon,\eta} \in W_0^{1,r}(\Omega)^d \cap L^{2r'}(\Omega)^d \quad \text{and} \quad p^{\varepsilon,\eta} \in W^{1,2}(\Omega) \tag{3.4}$$

and for all $\xi \in W^{1,2}(\Omega)$

$$\varepsilon \int_\Omega \nabla p^{\varepsilon,\eta} \cdot \nabla \xi \, \mathrm{d}x + \varepsilon \int_\Omega p^{\varepsilon,\eta} \xi \, \mathrm{d}x + \int_\Omega \operatorname{div} \boldsymbol{v}^{\varepsilon,\eta} \xi \, \mathrm{d}x = 0 \tag{3.5}$$

and for all $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega)^d \cap L^{2r'}(\Omega)^d$

$$\eta \int_\Omega |\boldsymbol{v}^{\varepsilon,\eta}|^{2r'-2} \boldsymbol{v}^{\varepsilon,\eta} \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_\Omega \nu(p^{\varepsilon,\eta}, |\mathbf{D}(\boldsymbol{v}^{\varepsilon,\eta})|^2) \mathbf{D}(\boldsymbol{v}^{\varepsilon,\eta}) \cdot \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}x \\ - \int_\Omega (\boldsymbol{v}^{\varepsilon,\eta} \otimes \mathcal{P} \boldsymbol{v}^{\varepsilon,\eta}) \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}x - \int_\Omega p^{\varepsilon,\eta} \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle. \tag{3.6}$$

Note that all the integrals above make sense, including $\int_\Omega \operatorname{div} \boldsymbol{v}^{\varepsilon,\eta} \xi \, \mathrm{d}x$ as $W^{1,2}(\Omega) \hookrightarrow L^{r'}(\Omega)$ for $r > \frac{2d}{d+2}$. The proof of the existence is via Galerkin approximations. Since $\varepsilon$, $\eta > 0$ are fixed, the dependence of the quantities on $\varepsilon$ and $\eta$ is not designated in what follows.

Since all of the considered function spaces are separable, let $\{\boldsymbol{a}^k\}_{k=1}^\infty$ be a basis in $W_0^{1,r}(\Omega)^d \cap L^{2r'}(\Omega)^d$ and $\{\alpha^k\}_{k=1}^\infty$ be a basis in $W^{1,2}(\Omega)$. We look for approximations $p^N$ and $\boldsymbol{v}^N$ of $p$ and $\boldsymbol{v}$ of the form

$$p^N = \sum_{k=1}^N c_k^N \alpha^k \quad \text{and} \quad \boldsymbol{v}^N = \sum_{k=1}^N d_k^N \boldsymbol{a}^k \quad \text{for } N = 1, 2, \ldots, \tag{3.7}$$

where $\boldsymbol{c}^N = (c_1^N, \ldots, c_N^N)$ and $\boldsymbol{d}^N = (d_1^N, \ldots, d_N^N)$ solve the Galerkin system (a system of $2N$ non-linear algebraic equations with $2N$ unknowns)

$$\varepsilon \int_\Omega \nabla p^N \cdot \nabla \alpha^r \, \mathrm{d}x + \varepsilon \int_\Omega p^N \alpha^r \, \mathrm{d}x - \int_\Omega \boldsymbol{v}^N \cdot \nabla \alpha^r \, \mathrm{d}x = 0 \quad r = 1, \ldots, N, \tag{3.8}$$

$$\eta \int_\Omega |\boldsymbol{v}^N|^{2r'-2} \boldsymbol{v}^N \cdot \boldsymbol{a}^s \, \mathrm{d}x + \int_\Omega \nu(p^N, \left|\mathbf{D}(\boldsymbol{v}^N)\right|^2) \mathbf{D}(\boldsymbol{v}^N) \cdot \mathbf{D}(\boldsymbol{a}^s) \, \mathrm{d}x \\ - \int_\Omega (\boldsymbol{v}^N \otimes \mathcal{P} \boldsymbol{v}^N) \cdot \nabla \boldsymbol{a}^s \, \mathrm{d}x + \int_\Omega \nabla p^N \cdot \boldsymbol{a}^s \, \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{a}^s \rangle \quad s = 1, \ldots, N. \tag{3.9}$$

The solvability follows from Lemma A.4, the proof of which is based on Brouwer's fixed point theorem, and uniform estimates. At first, we recall that

$$\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) := \nu(p^N, \left|\mathbf{D}(\boldsymbol{v}^N)\right|^2) \mathbf{D}(\boldsymbol{v}^N).$$

We define a mapping $\boldsymbol{\phi}^N : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ for $s = 1, \ldots, N$ through

$$\boldsymbol{\phi}_s^N(\boldsymbol{d}^N, \boldsymbol{c}^N) := \eta \int_\Omega |\boldsymbol{v}^N|^{2r'-2} \boldsymbol{v}^N \cdot \boldsymbol{a}^s \, \mathrm{d}x - \int_\Omega (\boldsymbol{v}^N \otimes \mathcal{P} \boldsymbol{v}^N) \cdot \nabla \boldsymbol{a}^s \, \mathrm{d}x \\ + \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{a}^s) \, \mathrm{d}x - \int_\Omega p^N \operatorname{div} \boldsymbol{a}^s \, \mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{a}^s \rangle$$

and for $s = N+1, \dots, 2N$ through

$$\boldsymbol{\phi}_s^N(\boldsymbol{d}^N, \boldsymbol{c}^N) := \varepsilon \int_\Omega \nabla p^N \cdot \nabla \alpha^{s-N} \, \mathrm{d}x + \varepsilon \int_\Omega p^N \alpha^{s-N} \, \mathrm{d}x \\ - \int_\Omega \boldsymbol{v}^N \cdot \nabla \alpha^{s-N} \, \mathrm{d}x. \tag{3.10}$$

Notice that $\boldsymbol{\phi}^N$ is a continuous mapping. If $(\boldsymbol{d}_l^N, \boldsymbol{c}_l^N) \to (\boldsymbol{d}^N, \boldsymbol{c}^N)$ in $\mathbb{R}^{2N}$ for $l \to \infty$, then also $(\boldsymbol{v}_l^N, p_l^N) \to (\boldsymbol{v}^N, p^N)$ in $W_0^{1,r}(\Omega)^d \cap L^{2r'}(\Omega)^d \times W^{1,2}(\Omega)$ for $l \to \infty$. Remembering the definition (3.7) of $p^N$ and $\boldsymbol{v}^N$, considering the second property of the viscosity from Lemma 2.2 with $\lambda = 1$ and using Lebesgue's dominated convergence theorem (see Theorem A.4), we can verify

$$\boldsymbol{\phi}^N(\boldsymbol{d}_l^N, \boldsymbol{c}_l^N) \to \boldsymbol{\phi}^N(\boldsymbol{d}^N, \boldsymbol{c}^N) \quad \text{in } \mathbb{R}^{2N} \quad \text{for } l \to \infty.$$

Furthermore, from the definitions of $p^N$, $\boldsymbol{v}^N$ and the mapping $\boldsymbol{\phi}^N$ and with the use of (3.1) we can write

$$\begin{aligned}
\boldsymbol{\phi}^N(\boldsymbol{d}^N, \boldsymbol{c}^N) \cdot (\boldsymbol{d}^N, \boldsymbol{c}^N) &= \sum_{s=1}^{2N} \boldsymbol{\phi}_s^N(\boldsymbol{d}^N, \boldsymbol{c}^N)(\boldsymbol{d}^N, \boldsymbol{c}^N)_s \\
&= \eta \|\boldsymbol{v}^N\|_{2r'}^{2r'} + \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{v}^N) \, \mathrm{d}x \\
&\quad - \int_\Omega p^N \operatorname{div} \boldsymbol{v}^N \, \mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{v}^N \rangle \\
&\quad + \int_\Omega p^N \operatorname{div} \boldsymbol{v}^N \, \mathrm{d}x + \varepsilon \|\nabla p^N\|_2^2 + \varepsilon \|p^N\|_2^2 \\
&= \eta \|\boldsymbol{v}^N\|_{2r'}^{2r'} + \varepsilon \|\nabla p^N\|_2^2 + \varepsilon \|p^N\|_2^2 \\
&\quad + \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{v}^N) \, \mathrm{d}x - \langle \boldsymbol{f}, \boldsymbol{v}^N \rangle.
\end{aligned}$$

After using the growth of the viscosity (2.9) with $\lambda = 1$, Hölder's, Korn's and Poincaré's inequalities, we arrive at

$$\begin{aligned}
\boldsymbol{\phi}^N(\boldsymbol{d}^N, \boldsymbol{c}^N) \cdot (\boldsymbol{d}^N, \boldsymbol{c}^N) &\geq C \|\boldsymbol{v}^N\|_{1,r}^r - \|\boldsymbol{f}\|_{-1,r'} \|\boldsymbol{v}^N\|_{1,r} \\
&\geq C \|\boldsymbol{v}^N\|_{1,r} (\|\boldsymbol{v}^N\|_{1,r}^{r-1} - C).
\end{aligned}$$

Since $r > 1$, Lemma A.5 guarantees that also the second assumption of Lemma A.4 is fulfilled. Therefore, we can apply it to obtain that

$$\exists\, (\boldsymbol{d}^N, \boldsymbol{c}^N) : \boldsymbol{\phi}^N(\boldsymbol{d}^N, \boldsymbol{c}^N) = \mathbf{0}.$$

To derive the uniform estimates, we multiply the $r$-th equation in (3.8) by $c_r^N$ and sum all equations for $r = 1, \dots, N$ and then multiply the $s$-th equation in (3.9) by $d_s^N$ and sum the equations for $s = 1, \dots, N$. Thus,

$$\varepsilon\|\nabla p^N\|_2^2 + \varepsilon\|p^N\|_2^2 + \int_\Omega p^N \operatorname{div} \boldsymbol{v}^N \, \mathrm{d}x = 0,$$

$$\eta\|\boldsymbol{v}^N\|_{2r'}^{2r'} + \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{v}^N) \, \mathrm{d}x - \int_\Omega p^N \operatorname{div} \boldsymbol{v}^N \, \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v}^N \rangle,$$

and after summing these identities

$$\varepsilon\|\nabla p^N\|_2^2 + \varepsilon\|p^N\|_2^2 + \eta\|\boldsymbol{v}^N\|_{2r'}^{2r'} + \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{v}^N) \, \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v}^N \rangle. \quad (3.11)$$

Now, on using the property (2.8) of the viscosity, Korn's, Young's and Poincaré's inequalities, we get

$$\varepsilon\|\nabla p^N\|_2^2 + \varepsilon\|p^N\|_2^2 + C\|\nabla \boldsymbol{v}^N\|_r^r + \eta\|\boldsymbol{v}^N\|_{2r'}^{2r'} \le C < \infty$$

and with the help of (2.9) with $\lambda = 1$ also

$$\|\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N))\|_{r'} \le C < \infty.$$

Therefore, thanks to Theorem A.1 from reflexivity of the function spaces we can find a subsequence (we denote it as the original sequence) such that

$$\begin{aligned}
\boldsymbol{v}^N &\rightharpoonup \boldsymbol{v} && \text{weakly in } W_0^{1,r}(\Omega)^d, \\
\boldsymbol{v}^N &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^{2r'}(\Omega)^d, \\
p^N &\rightharpoonup p && \text{weakly in } W^{1,2}(\Omega), \\
\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) &\rightharpoonup \chi && \text{weakly in } L^{r'}(\Omega)^{d\times d},
\end{aligned}$$

and due to the compact embedding and the interpolation inequality (see Theorem A.3 and Lemma A.2)

$$\begin{aligned}
\boldsymbol{v}^N &\to \boldsymbol{v} && \text{strongly in } L^q(\Omega)^d \text{ for all } q : 1 \le q < \frac{dr}{d-r}, \\
\boldsymbol{v}^N &\to \boldsymbol{v} && \text{strongly in } L^q(\Omega)^d \text{ for all } q : 1 \le q < 2r', \\
\boldsymbol{v}^N &\to \boldsymbol{v} && \text{almost everywhere in } \Omega, \\
p^N &\to p && \text{strongly in } L^2(\Omega).
\end{aligned} \quad (3.12)$$

Owing to the estimate $(A.3)_2$ we also have that

$$\mathcal{P}\boldsymbol{v}^N \to \mathcal{P}\boldsymbol{v} \quad \text{strongly in } L^q(\Omega)^d \text{ for all } q: 1 \le q < 2r'.$$

Moreover, the estimate $\||\boldsymbol{v}^N|^{2r'-2}\boldsymbol{v}^N\|_{\frac{2r'}{2r'-1}} = \|\boldsymbol{v}^N\|_{2r'}^{2r'-1} \le C$ and the fact that $|\boldsymbol{v}^N|^{2r'-2}\boldsymbol{v}^N \to |\boldsymbol{v}|^{2r'-2}\boldsymbol{v}$ a.e. in $\Omega$ imply that

$$|\boldsymbol{v}^N|^{2r'-2}\boldsymbol{v}^N \rightharpoonup |\boldsymbol{v}|^{2r'-2}\boldsymbol{v} \quad \text{weakly in } L^{\frac{2r'}{2r'-1}}(\Omega)^d. \tag{3.13}$$

These convergences allow us to obtain the limit in (3.8)-(3.9). Letting $N$ tend to infinity, for all base functions $\alpha^r$ and $\boldsymbol{a}^s$ we arrive at

$$\varepsilon \int_\Omega \nabla p \cdot \nabla \alpha^r \,\mathrm{d}x + \varepsilon \int_\Omega p\, \alpha^r \,\mathrm{d}x + \int_\Omega \operatorname{div} \boldsymbol{v}\, \alpha^r \,\mathrm{d}x = 0 \quad \text{for all } r \in \mathbb{N}$$

and

$$\eta \int_\Omega |\boldsymbol{v}|^{2r'-2}\boldsymbol{v} \cdot \boldsymbol{a}^s \,\mathrm{d}x + \int_\Omega \chi \cdot \mathbf{D}(\boldsymbol{a}^s) \,\mathrm{d}x - \int_\Omega (\boldsymbol{v} \otimes \mathcal{P}\boldsymbol{v}) \cdot \nabla \boldsymbol{a}^s \,\mathrm{d}x \\ - \int_\Omega p \operatorname{div} \boldsymbol{a}^s \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{a}^s \rangle \quad \text{for all } s \in \mathbb{N}.$$

From the density of linear spans of the base functions we conclude that for all $\xi \in W^{1,2}(\Omega)$ and all $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega)^d \cap L^{2r'}(\Omega)^d$

$$\varepsilon \int_\Omega \nabla p \cdot \nabla \xi \,\mathrm{d}x + \varepsilon \int_\Omega p\, \xi \,\mathrm{d}x + \int_\Omega \operatorname{div} \boldsymbol{v}\, \xi \,\mathrm{d}x = 0$$

and

$$\eta \int_\Omega |\boldsymbol{v}|^{2r'-2}\boldsymbol{v} \cdot \boldsymbol{\varphi} \,\mathrm{d}x + \int_\Omega \chi \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x - \int_\Omega (\boldsymbol{v} \otimes \mathcal{P}\boldsymbol{v}) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x \\ - \int_\Omega p \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle. \tag{3.14}$$

In particular, testing with $\xi = p$ in the first identity and $\boldsymbol{\varphi} = \boldsymbol{v}$ in the second one (note that both of these functions are admissible test functions) and summing them gives us

$$\varepsilon \|\nabla p\|_2^2 + \varepsilon \|p\|_2^2 + \eta \|\boldsymbol{v}\|_{2r'}^{2r'} + \int_\Omega \chi \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{3.15}$$

In order to finish the proof, we need to identify $\chi$ in (3.14). For this purpose, it is enough to show that

$$\mathbf{D}(\boldsymbol{v}^N) \to \mathbf{D}(\boldsymbol{v}) \quad \text{strongly in } L^r(\Omega)^{d\times d}. \tag{3.16}$$

Once we have $(3.12)_4$ and (3.16), we can find another (again not relabelled) subsequence such that

$$p^N \to p \quad \text{a.e. in } \Omega \quad \text{and} \quad \mathbf{D}(\boldsymbol{v}^N) \to \mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in } \Omega. \tag{3.17}$$

Vitali's theorem (see Theorem A.5) then completes the proof by showing that

$$\int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x \to \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x = \int_\Omega \chi \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x.$$

Indeed, knowing (3.17), we have

$$\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{\varphi}) \to \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) \cdot \mathbf{D}(\boldsymbol{\varphi}) \quad \text{a.e. in } \Omega.$$

The second assumption of the theorem is then also satisfied as from (2.9) with $\lambda = 1$ follows that

$$\int_E \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x \le C(1 + \|\mathbf{D}(\boldsymbol{v}^N)\|_{r,E})^{r-1} \|\nabla \boldsymbol{\varphi}\|_{r,E} \le C \|\nabla \boldsymbol{\varphi}\|_{r,E} \le \varepsilon.$$

In order to show (3.16), we recall the monotonicity condition (2.7) from Lemma 2.1. Since $\|\mathbf{D}(\boldsymbol{v}^N)\|_r \le C$ and $\|\mathbf{D}(\boldsymbol{v})\|_r \le C$, we can see that

$$\|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v})\|_r^2 \le C \int_\Omega \mathcal{I}^{\boldsymbol{v}^N, \boldsymbol{v}} \,\mathrm{d}x, \tag{3.18}$$

where

$$\begin{aligned} &\int_\Omega \mathcal{I}^{\boldsymbol{v}^N, \boldsymbol{v}} \,\mathrm{d}x = \\ &= \int_\Omega \int_0^1 (1 + \left|\mathbf{D}(\boldsymbol{v}) + s(\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}))\right|^2)^{\frac{r-2}{2}} \left|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v})\right|^2 \mathrm{d}s \,\mathrm{d}x. \end{aligned}$$

Thus, from (3.18) and (3.11) we get

$$
\begin{aligned}
&C\|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v})\|_r^2 \\
&\quad \le \int_\Omega (\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))) \cdot (\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}))\,\mathrm{d}x + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 \\
&\quad \le \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{v}^N)\,\mathrm{d}x - \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) \cdot (\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}))\,\mathrm{d}x \\
&\qquad - \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{v})\,\mathrm{d}x + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 \\
&\quad = \langle \boldsymbol{f}, \boldsymbol{v}^N\rangle - \varepsilon\|\nabla p^N\|_2^2 - \varepsilon\|p^N\|_2^2 - \eta\|\boldsymbol{v}^N\|_{2r'}^{2r'} + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 \\
&\qquad - \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) \cdot (\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}))\,\mathrm{d}x - \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{v})\,\mathrm{d}x,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
\varepsilon\|\nabla p^N\|_2^2 + \eta\|\boldsymbol{v}^N\|_{2r'}^{2r'} + C\|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v})\|_r^2 \qquad\qquad& \\
\le \langle \boldsymbol{f}, \boldsymbol{v}^N\rangle - \varepsilon\|p^N\|_2^2 + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 &\\
- \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) \cdot (\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}))\,\mathrm{d}x. &\\
- \int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) \cdot \mathbf{D}(\boldsymbol{v})\,\mathrm{d}x. &
\end{aligned}
$$

Letting $N \to \infty$ and using the weak lower semicontinuity of norms, namely

$$
\begin{aligned}
\liminf_{N\to\infty} \|\nabla p^N\|_2^2 &\ge \|\nabla p\|_2^2, \\
\liminf_{N\to\infty} \|\boldsymbol{v}^N\|_{2r'}^{2r'} &\ge \|\boldsymbol{v}\|_{2r'}^{2r'},
\end{aligned}
$$

we obtain

$$
\limsup_{N\to\infty} \|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v})\|_r^2 \le \langle \boldsymbol{f}, \boldsymbol{v}\rangle - \varepsilon\|\nabla p\|_2^2 - \varepsilon\|p\|_2^2 - \eta\|\boldsymbol{v}\|_{2r'}^{2r'} - \int_\Omega \chi \cdot \mathbf{D}(\boldsymbol{v})\,\mathrm{d}x,
$$

which together with (3.15) implies that

$$
\limsup_{N\to\infty} \|\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v})\|_r^2 \le 0,
$$

and thus (3.16) follows. □

# 4 Proof of the existence theorem

The goal of this section is to provide a proof of Theorem 2.2 and thus to establish the desired existence result. The proof is split into several steps that will be discussed in the following subsections. In order to obtain a weak solution to the problem $(\mathcal{P}^0)$, we first recall the approximate system $(\mathcal{P}^{\varepsilon,\eta})$ and then let $\varepsilon$ and $\eta$ tend to 0 in its weak formulation. In both limits the difficulty occurs in the term with the viscosity as the viscosity itself depends on the pressure and on the shear rate. Therefore, several extra tools are needed such as a decomposition of the pressure or the so-called Lipschitz approximations of Sobolev functions.

## 4.1 Limit $\varepsilon \to 0$

We suppose that for all $\varepsilon$, $\eta > 0$ and $r$ fulfilling $\frac{2d}{d+2} < r \leq \frac{3d}{d+2}$ there is a weak solution $(\boldsymbol{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$ to the problem $(\mathcal{P}^{\varepsilon,\eta})$ satisfying (3.4)–(3.6). For simplicity, we write $(\boldsymbol{v}^{\varepsilon}, p^{\varepsilon})$ instead of $(\boldsymbol{v}^{\varepsilon,\eta}, p^{\varepsilon,\eta})$.

### 4.1.1 Uniform estimates and their consequences

We start with the derivation of uniform estimates for $\boldsymbol{v}^{\varepsilon}$ and $p^{\varepsilon}$. Taking $\xi = p^{\varepsilon}$ in (3.5) and $\boldsymbol{\varphi} = \boldsymbol{v}^{\varepsilon}$ in (3.6) leads to

$$\begin{aligned} \eta\|\boldsymbol{v}^{\varepsilon}\|_{2r'}^{2r'} + \int_{\Omega} \nu(p^{\varepsilon}, |\mathbf{D}(\boldsymbol{v}^{\varepsilon})|^2)\, |\mathbf{D}(\boldsymbol{v}^{\varepsilon})|^2 \,\mathrm{d}x - \int_{\Omega} p^{\varepsilon} \operatorname{div} \boldsymbol{v}^{\varepsilon} \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{v}^{\varepsilon} \rangle \\ \varepsilon\|\nabla p^{\varepsilon}\|_2^2 + \varepsilon\|p^{\varepsilon}\|_2^2 + \int_{\Omega} p^{\varepsilon} \operatorname{div} \boldsymbol{v}^{\varepsilon} \,\mathrm{d}x = 0. \end{aligned} \tag{4.1}$$

On summing these identities, using (2.8), Korn's, Young's and Poincaré's inequalities, we conclude from (4.1) that

$$\varepsilon\|\nabla p^{\varepsilon}\|_2^2 + \varepsilon\|p^{\varepsilon}\|_2^2 + C\|\nabla \boldsymbol{v}^{\varepsilon}\|_r^r + \eta\|\boldsymbol{v}^{\varepsilon}\|_{2r'}^{2r'} \leq C < \infty \tag{4.2}$$

and then from (2.9) with $\lambda = 1$ and (4.2) that

$$\|\nu(p^{\varepsilon}, |\mathbf{D}(\boldsymbol{v}^{\varepsilon})|^2)\mathbf{D}(\boldsymbol{v}^{\varepsilon})\|_{r'} \leq C < \infty. \tag{4.3}$$

In order to obtain an estimate for the pressure $p^\varepsilon$ independent of $\varepsilon$, we take $\boldsymbol{\varphi} = \boldsymbol{\varphi}^\varepsilon$ as a test function in (3.6), where $\boldsymbol{\varphi}^\varepsilon$ solves

$$\begin{aligned} \operatorname{div} \boldsymbol{\varphi}^\varepsilon &= |p^\varepsilon|^{s-2} p^\varepsilon - \frac{1}{|\Omega|} \int_\Omega |p^\varepsilon|^{s-2} p^\varepsilon \, \mathrm{d}x =: h^\varepsilon \quad \text{in } \Omega \\ \boldsymbol{\varphi}^\varepsilon &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned}$$

with $s = \frac{2dr}{(d-2)r+d}$. Note that $\boldsymbol{\varphi}^\varepsilon$ satisfies

$$\|\boldsymbol{\varphi}^\varepsilon\|_{1,q} \le C_{\mathrm{div}}(\Omega, q) \|h^\varepsilon\|_q \quad \text{for all } q : 1 < q \le s' = \frac{2dr}{(d+2)r-d},$$

and for $q = s'$ in particular we have

$$\|\boldsymbol{\varphi}^\varepsilon\|_{1,s'} \le 2C_{\mathrm{div}}(\Omega, s') \|p^\varepsilon\|_s^{s-1}. \tag{4.4}$$

We use the fact that $\int_\Omega p^\varepsilon \, \mathrm{d}x = 0$ and with the help of (2.9), (4.2), (4.4) and Sobolev embeddings, namely $W^{1,s'}(\Omega) \hookrightarrow L^{2r'}(\Omega)$, we can conclude (note that $r \le s'$ and $s \le r'$)

$$\begin{aligned} \|p^\varepsilon\|_s^s &= \int_\Omega \nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2) \mathbf{D}(\boldsymbol{v}^\varepsilon) \cdot \mathbf{D}(\boldsymbol{\varphi}^\varepsilon) \, \mathrm{d}x \; - \langle \boldsymbol{f}, \boldsymbol{\varphi}^\varepsilon \rangle \\ &\quad - \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \mathcal{P}\boldsymbol{v}^\varepsilon) \cdot \nabla \boldsymbol{\varphi}^\varepsilon \, \mathrm{d}x \; + \eta \int_\Omega |\boldsymbol{v}^\varepsilon|^{2r'-2} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi}^\varepsilon \, \mathrm{d}x \\ &\le C(1 + \|\mathbf{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1} \|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \|\boldsymbol{f}\|_{-1,r'} \|\boldsymbol{\varphi}^\varepsilon\|_{1,r} \\ &\quad + \|\boldsymbol{v}^\varepsilon \otimes \mathcal{P}\boldsymbol{v}^\varepsilon\|_s \|\nabla \boldsymbol{\varphi}^\varepsilon\|_{s'} + \eta \|\boldsymbol{v}^\varepsilon\|_{2r'}^{2r'-1} \|\boldsymbol{\varphi}^\varepsilon\|_{2r'} \\ &\le C(\eta) \|\boldsymbol{\varphi}^\varepsilon\|_{1,s'} \le C(\eta) \|p^\varepsilon\|_s^{s-1}, \end{aligned}$$

which leads to

$$\|p^\varepsilon\|_{\frac{2dr}{(d-2)r+d}} \le C(\eta) < \infty. \tag{4.5}$$

According to Theorem A.1, reflexivity of the function spaces and the estimates (4.2), (4.5) and (4.3) allow us to find a (not relabelled) subsequence $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ and $(\boldsymbol{v}, p) \in W_0^{1,r}(\Omega)^d \cap L^{2r'}(\Omega)^d \times L_0^{\frac{2dr}{(d-2)r+d}}(\Omega)$ such that

$$\begin{aligned} \mathbf{D}(\boldsymbol{v}^\varepsilon) &\rightharpoonup \mathbf{D}(\boldsymbol{v}) && \text{weakly in } L^r(\Omega)^{d \times d}, \\ \nabla \boldsymbol{v}^\varepsilon &\rightharpoonup \nabla \boldsymbol{v} && \text{weakly in } L^r(\Omega)^{d \times d}, \\ \boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} && \text{weakly in } W_0^{1,r}(\Omega)^d, \\ \boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^{2r'}(\Omega)^d, \\ p^\varepsilon &\rightharpoonup p && \text{weakly in } L_0^{\frac{2dr}{(d-2)r+d}}(\Omega), \\ \nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2) \mathbf{D}(\boldsymbol{v}^\varepsilon) &\rightharpoonup \overline{\nu \mathbf{D}} && \text{weakly in } L^{r'}(\Omega)^{d \times d}, \end{aligned} \tag{4.6}$$

and due to the compact embedding, the interpolation inequality and by using the same arguments as for (3.13)

$$\begin{aligned}
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{strongly in } L^q(\Omega)^d \text{ for all } 1 \le q < \frac{dr}{d-r}, \\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{strongly in } L^q(\Omega)^d \text{ for all } 1 \le q < 2r', \\
|\boldsymbol{v}^\varepsilon|^{2r'-2}\boldsymbol{v}^\varepsilon &\rightharpoonup |\boldsymbol{v}|^{2r'-2}\boldsymbol{v} && \text{weakly in } L^{\frac{2r'}{2r'-1}}(\Omega)^d.
\end{aligned} \tag{4.7}$$

We would like to pass to the limit in the identities (3.5) and (3.6) of the weak formulation of $(\mathcal{P}^{\varepsilon,\eta})$. Doing so, it follows directly from the first identity and from (4.2) that

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{a.e. in } \Omega. \tag{4.8}$$

This fact helps us to treat the convective term, for (4.8) and $(4.7)_2$ imply that

$$\mathcal{P}\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \quad \text{strongly in } L^q(\Omega)^d \text{ for all } q : 1 \le q < 2r', \tag{4.9}$$

which can be gained from the definition of $\mathcal{P}\boldsymbol{v}^\varepsilon$ and $\boldsymbol{g}^{\boldsymbol{v}^\varepsilon}$ and from estimates (A.2) and (A.3). From that and from $(4.7)_2$ we get

$$\int_\Omega (\boldsymbol{v}^\varepsilon \otimes \mathcal{P}\boldsymbol{v}^\varepsilon) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x \to \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x \quad \forall \boldsymbol{\varphi} \in \mathcal{C}_0^\infty(\Omega)^d.$$

We easily obtain the limit in the term involving the pressure

$$\int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x \to \int_\Omega p \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x \quad \forall \boldsymbol{\varphi} \in \mathcal{C}_0^\infty(\Omega)^d$$

and according to $(4.7)_3$ we also have

$$\eta \int_\Omega |\boldsymbol{v}^\varepsilon|^{2r'-2}\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi} \,\mathrm{d}x \to \eta \int_\Omega |\boldsymbol{v}|^{2r'-2}\boldsymbol{v} \cdot \boldsymbol{\varphi} \,\mathrm{d}x \quad \forall \boldsymbol{\varphi} \in \mathcal{C}_0^\infty(\Omega)^d.$$

To prove the convergence of the term with the viscosity, i.e., to show that

$$\int_\Omega \nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon) \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x \to \int_\Omega \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x, \tag{4.10}$$

we need to know that

$$p^\varepsilon \to p \quad \text{a.e. in } \Omega \quad \text{and} \quad \mathbf{D}(\boldsymbol{v}^\varepsilon) \to \mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in } \Omega, \tag{4.11}$$

at least for a subsequence. Similarly as for the existence for the Galerkin system in Section 3.2., Vitali's theorem (see Theorem A.5) then completes this part of the proof.

The proof of (4.11) is contained in the next subsection.

### 4.1.2 Almost everywhere convergence of $p^\varepsilon$ and $\mathbf{D}(\boldsymbol{v}^\varepsilon)$

In order to show (4.11), we first decompose the pressure $p^\varepsilon$ into two particular pressures. The first one will converge strongly in some Lebesgue space and the other one only weakly but in some "better" Lebesgue space, for example in $L^{r'}(\Omega)$. As a second step, we recall the monotonicity condition (2.7) for the viscosity, with the help of which we then will be able to prove (4.11).

For the decomposition of the pressure, we consider two auxiliary Stokes problems

$$\begin{aligned} -\triangle \boldsymbol{v}_i^\varepsilon + \nabla p_i^\varepsilon &= \boldsymbol{h}_i^\varepsilon && \text{in } \Omega \\ \operatorname{div} \boldsymbol{v}_i^\varepsilon &= 0 && \text{in } \Omega \\ \boldsymbol{v}_i^\varepsilon &= \mathbf{0} && \text{on } \partial\Omega,\ i = 1, 2, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} \boldsymbol{h}_1^\varepsilon &= \operatorname{div}(\nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon)) + \boldsymbol{f} && \in (W_0^{1,r}(\Omega))^* \\ \boldsymbol{h}_2^\varepsilon &= -\operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \mathcal{P}\boldsymbol{v}^\varepsilon) - \eta|\boldsymbol{v}^\varepsilon|^{2r'-2}\boldsymbol{v}^\varepsilon && \in (W_0^{1,s'}(\Omega))^*,\ s' = \frac{2dr}{(d+2)r - d}. \end{aligned}$$

The classical theory for the Stokes system (see for example [3]) implies the existence of solutions $(\boldsymbol{v}_i^\varepsilon, p_i^\varepsilon)$, $i = 1, 2$, with the following estimates for $\nabla \boldsymbol{v}_i^\varepsilon$ and for the pressures $p_i^\varepsilon$ having zero mean value

$$\begin{aligned} \|\nabla \boldsymbol{v}_1^\varepsilon\|_{r'} + \|p_1^\varepsilon\|_{r'} &\le C\|\boldsymbol{h}_1^\varepsilon\|_{(W_0^{1,r}(\Omega))^*} \le C + C\|\nu(p^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon)\|_{r'} \\ \|\nabla \boldsymbol{v}_2^\varepsilon\|_s + \|p_2^\varepsilon\|_s &\le C\|\boldsymbol{h}_2^\varepsilon\|_{(W_0^{1,s'}(\Omega))^*} \le C\|\boldsymbol{v}^\varepsilon \otimes \mathcal{P}\boldsymbol{v}^\varepsilon\|_s + C\eta\|\boldsymbol{v}^\varepsilon\|_{2r'}^{2r'-1} \\ &\le C(\eta)\|\boldsymbol{v}^\varepsilon\|_{2r'}. \end{aligned}$$

From the continuity of the Stokes operator and owing to (4.3) and $(4.7)_2$, these estimates yield that

$$\begin{aligned} \nabla \boldsymbol{v}_1^\varepsilon &\rightharpoonup \nabla \boldsymbol{v}_1 && \text{weakly in } L^{r'}(\Omega)^{d\times d}, \\ p_1^\varepsilon &\rightharpoonup p_1 && \text{weakly in } L^{r'}(\Omega), \\ \nabla \boldsymbol{v}_2^\varepsilon &\to \nabla \boldsymbol{v}_2 && \text{strongly in } L^q(\Omega)^{d\times d}, \\ p_2^\varepsilon &\to p_2 && \text{strongly in } L^q(\Omega), \end{aligned} \tag{4.13}$$

where $q \in [1, \frac{2dr}{(d-2)r+d})$. Moreover, from the uniqueness of solutions of the Stokes system we have

$$p^\varepsilon = p_1^\varepsilon + p_2^\varepsilon \quad \text{and} \quad \boldsymbol{v}_1^\varepsilon = -\boldsymbol{v}_2^\varepsilon \tag{4.14}$$

and therefore also

$$\nabla \boldsymbol{v}_1^\varepsilon \to \nabla \boldsymbol{v}_1 \quad \text{strongly in } L^q(\Omega)^{d\times d}. \tag{4.15}$$

Let us now again remind the definition of

$$\mathbf{S}(p,\mathbf{D}) := \nu(p,|\mathbf{D}|^2)\mathbf{D}$$

and let us integrate the monotonicity condition (2.7) from Lemma 2.1 with $\mathbf{D}^1 = \mathbf{D}(\boldsymbol{v}^\varepsilon)$, $\mathbf{D}^2 = \mathbf{D}(\boldsymbol{v})$, $p^1 = p^\varepsilon$ and $p^2 = p_1 + p_2^\varepsilon$ over the domain $\Omega$. We obtain

$$\begin{aligned}\frac{C_1}{2}\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x \le \int_\Omega (\mathbf{S}(p^\varepsilon,\mathbf{D}(\boldsymbol{v}^\varepsilon)) - \mathbf{S}(p_1+p_2^\varepsilon,\mathbf{D}(\boldsymbol{v})))\cdot(\mathbf{D}(\boldsymbol{v}^\varepsilon)-\mathbf{D}(\boldsymbol{v}))\,\mathrm{d}x \\ + \frac{\gamma_0^2}{2C_1}\|p_1^\varepsilon - p_1\|_2^2,\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}&\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x = \\ &= \int_\Omega\int_0^1 (1+|\mathbf{D}(\boldsymbol{v}) + s(\mathbf{D}(\boldsymbol{v}^\varepsilon)-\mathbf{D}(\boldsymbol{v}))|^2)^{\frac{r-2}{2}}\,|\mathbf{D}(\boldsymbol{v}^\varepsilon)-\mathbf{D}(\boldsymbol{v})|^2\,\mathrm{d}s\,\mathrm{d}x.\end{aligned} \tag{4.17}$$

Our aim is to show that

$$\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x \to 0 \quad \text{for } \varepsilon\to 0.$$

Firstly, since $p_2^\varepsilon$ converges pointwisely a.e., (2.9) with $\lambda = 1$ and Lebesgue's dominated convergence theorem (Theorem A.4) imply that

$$\mathbf{S}(p_1+p_2^\varepsilon,\mathbf{D}(\boldsymbol{v})) \to \mathbf{S}(p,\mathbf{D}(\boldsymbol{v})) \quad \text{strongly in } L^{r'}(\Omega)^{d\times d}. \tag{4.18}$$

Therefore,

$$\int_\Omega \mathbf{S}(p_1+p_2^\varepsilon,\mathbf{D}(\boldsymbol{v}))\cdot(\mathbf{D}(\boldsymbol{v}^\varepsilon)-\mathbf{D}(\boldsymbol{v}))\,\mathrm{d}x \to 0 \quad \text{for } \varepsilon\to 0. \tag{4.19}$$

Next, considering the weak formulation (3.6) with $\boldsymbol{\varphi} = \boldsymbol{v}^\varepsilon - \boldsymbol{v}$, we arrive at

$$\begin{aligned}\int_\Omega \mathbf{S}(p^\varepsilon,\mathbf{D}(\boldsymbol{v}^\varepsilon))\cdot\mathbf{D}(\boldsymbol{v}^\varepsilon-\boldsymbol{v})\,\mathrm{d}x = &\int_\Omega p^\varepsilon\,\mathrm{div}(\boldsymbol{v}^\varepsilon-\boldsymbol{v})\,\mathrm{d}x + \langle \boldsymbol{f},\boldsymbol{v}^\varepsilon-\boldsymbol{v}\rangle \\ &+ \int_\Omega (\boldsymbol{v}^\varepsilon\otimes\mathcal{P}\boldsymbol{v}^\varepsilon)\cdot\nabla(\boldsymbol{v}^\varepsilon-\boldsymbol{v})\,\mathrm{d}x \\ &- \eta\int_\Omega |\boldsymbol{v}^\varepsilon|^{2r'-2}\boldsymbol{v}^\varepsilon\cdot(\boldsymbol{v}^\varepsilon-\boldsymbol{v})\,\mathrm{d}x.\end{aligned}$$

As $\operatorname{div} \boldsymbol{v} = 0$ a.e. in $\Omega$ and $\int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{v}^\varepsilon \,\mathrm{d}x = -\varepsilon\|\nabla p^\varepsilon\|_2^2 - \varepsilon\|p^\varepsilon\|_2^2$, which follows from (3.5) with $\xi = p^\varepsilon$, and since the terms $\varepsilon\|\nabla p^\varepsilon\|_2^2 + \varepsilon\|p^\varepsilon\|_2^2$ are non-negative, we then conclude

$$\begin{aligned}\int_\Omega \mathsf{S}(p^\varepsilon, \mathsf{D}(\boldsymbol{v}^\varepsilon)) \cdot \mathsf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \,\mathrm{d}x \le& \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \mathcal{P}\boldsymbol{v}^\varepsilon) \cdot \nabla(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \,\mathrm{d}x \\ &- \eta \int_\Omega |\boldsymbol{v}^\varepsilon|^{2r'-2} \boldsymbol{v}^\varepsilon \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \,\mathrm{d}x \\ &+ \langle \boldsymbol{f}, \boldsymbol{v}^\varepsilon - \boldsymbol{v} \rangle \\ =& \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \mathcal{P}\boldsymbol{v}^\varepsilon) \cdot \nabla(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \,\mathrm{d}x \\ &- \eta\|\boldsymbol{v}^\varepsilon\|_{2r'}^{2r'} + \eta \int_\Omega |\boldsymbol{v}^\varepsilon|^{2r'-2} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x \\ &+ \langle \boldsymbol{f}, \boldsymbol{v}^\varepsilon - \boldsymbol{v} \rangle.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_\Omega \mathsf{S}(p^\varepsilon, \mathsf{D}(\boldsymbol{v}^\varepsilon)) \cdot \mathsf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \,\mathrm{d}x + \eta\|\boldsymbol{v}^\varepsilon\|_{2r'}^{2r'} \le& \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \mathcal{P}\boldsymbol{v}^\varepsilon) \cdot \nabla(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \,\mathrm{d}x \\ &+ \eta \int_\Omega |\boldsymbol{v}^\varepsilon|^{2r'-2} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x \\ &+ \langle \boldsymbol{f}, \boldsymbol{v}^\varepsilon - \boldsymbol{v} \rangle.\end{aligned} \tag{4.20}$$

Moreover, on letting $\varepsilon$ tend to 0 and using the weak lower semicontinuity of a norm, in our case

$$\liminf_{\varepsilon \to 0} \|\boldsymbol{v}^\varepsilon\|_{2r'}^{2r'} \ge \|\boldsymbol{v}\|_{2r'}^{2r'},$$

we find that

$$\limsup_{\varepsilon \to 0} \int_\Omega \mathsf{S}(p^\varepsilon, \mathsf{D}(\boldsymbol{v}^\varepsilon)) \cdot \mathsf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \,\mathrm{d}x \le g(\varepsilon), \tag{4.21}$$

where $g(\varepsilon) \to 0$ as $\varepsilon \to 0$, since it follows directly from (3.1), $(4.7)_2$, (4.9) and $(4.7)_3$ that the first and the last integral on the right-hand side of (4.20) vanish as $\varepsilon \to 0$ and

$$\lim_{\varepsilon \to 0} \eta \int_\Omega |\boldsymbol{v}^\varepsilon|^{2r'-2} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{v} \,\mathrm{d}x = \eta \int_\Omega |\boldsymbol{v}|^{2r'} \,\mathrm{d}x = \eta\|\boldsymbol{v}\|_{2r'}^{2r'}.$$

In view of (4.21) together with (4.19), the condition (4.16) can be expressed as

$$\frac{C_1}{2}\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x \le g(\varepsilon) + \frac{\gamma_0^2}{2C_1}\|p_1^\varepsilon - p_1\|_2^2, \tag{4.22}$$

with $g(\varepsilon) \to 0$ as $\varepsilon \to 0$.

In order to handle the term $\|p_1^\varepsilon - p_1\|_2^2$, we consider the weak formulation of (4.12), $i = 1$, with a special test function $\boldsymbol{\varphi}^\varepsilon$ satisfying

$$\begin{aligned} \operatorname{div}\boldsymbol{\varphi}^\varepsilon &= p_1^\varepsilon - p_1 \quad \text{in } \Omega \\ \boldsymbol{\varphi}^\varepsilon &= \mathbf{0} \qquad\quad\ \ \text{on } \partial\Omega \\ \|\boldsymbol{\varphi}^\varepsilon\|_{1,q} &\le C_{\mathrm{div}}(\Omega,q)\|p_1^\varepsilon - p_1\|_q \quad \text{for all } q : 1 < q \le r'. \end{aligned} \tag{4.23}$$

Note that $\int_\Omega (p_1^\varepsilon - p_1)\,\mathrm{d}x = 0$. Since $p_1^\varepsilon \rightharpoonup p_1$ weakly in $L^{r'}(\Omega)$, from the continuity and linearity of the Bogovskii operator (Lemma A.3) we see that

$$\boldsymbol{\varphi}^\varepsilon \rightharpoonup \mathbf{0} \quad \text{weakly in } W^{1,r'}(\Omega)^d. \tag{4.24}$$

We have

$$\begin{aligned} \|p_1^\varepsilon - p_1\|_2^2 &= \int_\Omega \nabla \boldsymbol{v}_1^\varepsilon \cdot \nabla\boldsymbol{\varphi}^\varepsilon\,\mathrm{d}x + \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon))\cdot\mathbf{D}(\boldsymbol{\varphi}^\varepsilon)\,\mathrm{d}x \\ &\quad - \langle \boldsymbol{f}, \boldsymbol{\varphi}^\varepsilon\rangle - \int_\Omega p_1(p_1^\varepsilon - p_1)\,\mathrm{d}x \\ &= \int_\Omega \nabla \boldsymbol{v}_1^\varepsilon \cdot \nabla\boldsymbol{\varphi}^\varepsilon\,\mathrm{d}x + \int_\Omega \mathbf{S}(p_1 + p_2^\varepsilon, \mathbf{D}(\boldsymbol{v}))\cdot\mathbf{D}(\boldsymbol{\varphi}^\varepsilon)\,\mathrm{d}x \\ &\quad + \int_\Omega (\mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) - \mathbf{S}(p_1 + p_2^\varepsilon, \mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{\varphi}^\varepsilon)\,\mathrm{d}x \\ &\quad - \langle \boldsymbol{f}, \boldsymbol{\varphi}^\varepsilon\rangle - \int_\Omega p_1(p_1^\varepsilon - p_1)\,\mathrm{d}x. \end{aligned} \tag{4.25}$$

From (4.24) and from $(4.13)_2$ we easily obtain

$$\lim_{\varepsilon\to 0}(\langle \boldsymbol{f}, \boldsymbol{\varphi}^\varepsilon\rangle + \int_\Omega p_1(p_1^\varepsilon - p_1)\mathrm{d}x) = 0. \tag{4.26}$$

Moreover, using the same arguments as in (4.18) and (4.19), we get

$$\int_\Omega \mathbf{S}(p_1 + p_2^\varepsilon, \mathbf{D}(\boldsymbol{v}))\cdot\mathbf{D}(\boldsymbol{\varphi}^\varepsilon)\,\mathrm{d}x \to 0 \quad \text{for } \varepsilon \to 0. \tag{4.27}$$

Since $\nabla\boldsymbol{\varphi}^\varepsilon \rightharpoonup \mathbf{0}$ weakly in $L^{r'}(\Omega)^{d\times d}$ and $r' > 2$, $\nabla\boldsymbol{\varphi}^\varepsilon \rightharpoonup \mathbf{0}$ weakly in $L^2(\Omega)^{d\times d}$ as well. $\nabla\boldsymbol{v}_1^\varepsilon \rightharpoonup \nabla\boldsymbol{v}_1$ weakly in $L^{r'}(\Omega)^{d\times d}$ and according to (4.15) and $q \geq 1$, we also have $\nabla\boldsymbol{v}_1^\varepsilon \to \nabla\boldsymbol{v}_1$ strongly in $L^1(\Omega)^{d\times d}$. The interpolation inequality (Lemma A.2) with $\theta = \frac{2-r}{2} \in (0,1)$ yields that

$$\nabla\boldsymbol{v}_1^\varepsilon \to \nabla\boldsymbol{v}_1 \quad \text{strongly in } L^2(\Omega)^{d\times d}.$$

Consequently,

$$\int_\Omega \nabla\boldsymbol{v}_1^\varepsilon \cdot \nabla\boldsymbol{\varphi}^\varepsilon \,\mathrm{d}x \to 0 \quad \text{for } \varepsilon \to 0. \tag{4.28}$$

From (4.26), (4.27) and (4.28) we then arrive at

$$\|p_1^\varepsilon - p_1\|_2^2 = g(\varepsilon) + \int_\Omega (\mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) - \mathbf{S}(p_1 + p_2^\varepsilon, \mathbf{D}(\boldsymbol{v}))) \cdot \mathbf{D}(\boldsymbol{\varphi}^\varepsilon) \,\mathrm{d}x,$$

where $g(\varepsilon) \to 0$ for $\varepsilon \to 0$. After applying the assumptions (2.4) and (2.5) to the integral on the right-hand side, we obtain

$$\begin{aligned}
&\|p_1^\varepsilon - p_1\|_2^2 \leq g(\varepsilon) + \gamma_0 \int_\Omega |p_1^\varepsilon - p_1| |\mathbf{D}(\boldsymbol{\varphi}^\varepsilon)| \,\mathrm{d}x \\
&+ C_2 \int_\Omega \int_0^1 (1 + |\mathbf{D}(\boldsymbol{v}) + s(\mathbf{D}(\boldsymbol{v}^\varepsilon) - \mathbf{D}(\boldsymbol{v}))|^2)^{\frac{r-2}{2}} |\mathbf{D}(\boldsymbol{v}^\varepsilon) - \mathbf{D}(\boldsymbol{v})| \, |\mathbf{D}(\boldsymbol{\varphi}^\varepsilon)| \mathrm{d}s \,\mathrm{d}x.
\end{aligned}$$

Now, we recall the definition of $\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon, \boldsymbol{v}} \,\mathrm{d}x$ in (4.17) and use the fact that $r < 2$. Hölder's inequality, $r' > 2$ and $(4.23)_3$ with $q = 2$ then imply

$$\begin{aligned}
\|p_1^\varepsilon - p_1\|_2^2 &\leq g(\varepsilon) + \gamma_0 \|p_1^\varepsilon - p_1\|_2 \|\nabla\boldsymbol{\varphi}^\varepsilon\|_2 + C_2 \left( \int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon, \boldsymbol{v}} \,\mathrm{d}x \right)^{\frac{1}{2}} \|\nabla\boldsymbol{\varphi}^\varepsilon\|_2 \\
&\leq g(\varepsilon) + \gamma_0 C_{\mathrm{div}}(\Omega, 2) \|p_1^\varepsilon - p_1\|_2^2 \\
&\quad + C_2 C_{\mathrm{div}}(\Omega, 2) \left( \int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon, \boldsymbol{v}} \,\mathrm{d}x \right)^{\frac{1}{2}} \|p_1^\varepsilon - p_1\|_2.
\end{aligned}$$

Applying Young's inequality gives

$$\begin{aligned}
(1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2)) \|p_1^\varepsilon - p_1\|_2^2 &\leq g(\varepsilon) + \frac{1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2)}{2} \|p_1^\varepsilon - p_1\|_2^2 \\
&\quad + \frac{C_2^2 C_{\mathrm{div}}^2(\Omega, 2)}{2(1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2))} \int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon, \boldsymbol{v}} \,\mathrm{d}x.
\end{aligned}$$

As $1-\gamma_0 C_{\text{div}}(\Omega,2)-\frac{1-\gamma_0 C_{\text{div}}(\Omega,2)}{2}>0$ due to (2.6), we then conclude

$$\|p_1^\varepsilon - p_1\|_2^2 \le g(\varepsilon) + \frac{C_2^2 C_{\text{div}}^2(\Omega,2)}{(1-\gamma_0 C_{\text{div}}(\Omega,2))^2}\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x. \tag{4.29}$$

Coming back to (4.22) and incorporating (4.29), we finally get

$$\frac{C_1}{2}\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x \le g(\varepsilon) + \frac{\gamma_0^2}{2C_1}\frac{C_2^2 C_{\text{div}}^2(\Omega,2)}{(1-\gamma_0 C_{\text{div}}(\Omega,2))^2}\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x,$$

with $g(\varepsilon)\to 0$ for $\varepsilon\to 0$. As $\frac{C_1}{2}-\frac{\gamma_0^2}{2C_1}\frac{C_2^2 C_{\text{div}}^2(\Omega,2)}{(1-\gamma_0 C_{\text{div}}(\Omega,2))^2}>0$, again thanks to (2.6), we indeed have

$$\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x \to 0 \quad \text{for } \varepsilon\to 0. \tag{4.30}$$

The almost everywhere convergence (at least for a subsequence) of the pressure $p^\varepsilon$ in $\Omega$ then follows from (4.14) since (4.30) and (4.29) imply the a.e. convergence of the pressure $p_1^\varepsilon$ and from $(4.13)_4$ also the pressure $p_2^\varepsilon$ converges a.e. in $\Omega$.

It remains to show the almost everywhere convergence of the symmetric part of the velocity gradient $\mathbf{D}(\boldsymbol{v}^\varepsilon)$. After using the fact that $\|\nabla\boldsymbol{v}^\varepsilon\|_r\le C$, $\|\nabla\boldsymbol{v}\|_r\le C$ and $r<2$ and after applying Hölder's inequality, we arrive at

$$\begin{aligned}
&\int_\Omega |\mathbf{D}(\boldsymbol{v}^\varepsilon-\boldsymbol{v})|\,\mathrm{d}x\\
&\quad\le \int_\Omega (\mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}})^{\frac12}\left(\int_0^1 (1+|\mathbf{D}(\boldsymbol{v})+s(\mathbf{D}(\boldsymbol{v}^\varepsilon-\boldsymbol{v}))|^2)^{\frac{r-2}{2}}\mathrm{d}s\right)^{-\frac12}\,\mathrm{d}x\\
&\quad\le C\int_\Omega (\mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}})^{\frac12}(1+|\mathbf{D}(\boldsymbol{v}^\varepsilon)|+|\mathbf{D}(\boldsymbol{v})|)^{\frac{2-r}{2}}\,\mathrm{d}x\\
&\quad\le C\left(\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x\right)^{\frac12}(1+\|\nabla\boldsymbol{v}^\varepsilon\|_r+\|\nabla\boldsymbol{v}\|_r)^{\frac{2-r}{2}}|\Omega|^{\frac{1}{r'}}\\
&\quad\le C\left(\int_\Omega \mathcal{I}^{\boldsymbol{v}^\varepsilon,\boldsymbol{v}}\,\mathrm{d}x\right)^{\frac12}\overset{(4.30)}{\to} 0
\end{aligned}$$

and at least for a subsequence we conclude that

$$\mathbf{D}(\boldsymbol{v}^\varepsilon)\to\mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in } \Omega. \tag{4.31}$$

Using Vitali's theorem then proves (4.10).

## 4.2 Limit $\eta \to 0$

We have already established the existence of a weak solution to the following system $(\mathcal{P}^\eta)$

$$\begin{aligned} \eta|\boldsymbol{v}^\eta|^{2r'-2}\boldsymbol{v}^\eta + \operatorname{div}(\boldsymbol{v}^\eta \otimes \boldsymbol{v}^\eta) - \operatorname{div}(\nu(p^\eta, |\mathbf{D}(\boldsymbol{v}^\eta)|^2)\mathbf{D}(\boldsymbol{v}^\eta)) + \nabla p^\eta = \boldsymbol{f} &\quad \text{in } \Omega \\ \operatorname{div} \boldsymbol{v}^\eta = 0 &\quad \text{in } \Omega \\ \boldsymbol{v}^\eta = \mathbf{0} &\quad \text{on } \partial\Omega. \end{aligned}$$

In this subsection we would like to pass to the limit in $\eta$ in its weak formulation and obtain a weak solution to the system $(\mathcal{P}^0)$. Similarly as in the preceding subsection, we first derive the uniform estimates. Then in order to be able to pass to the limit in the viscosity term, we show the almost everywhere convergence of the pressure and the symmetric part of the velocity gradient with the help of the Lipschitz approximations.

From technical reasons, we set $\eta := \frac{1}{n}$ and thus if $\eta \to 0$, then $n \to \infty$.

### 4.2.1 Uniform estimates and their consequences

Let us consider $\boldsymbol{v}^n \in W_0^{1,r}(\Omega)^d \cap L^{2r'}(\Omega)^d$ satisfying $\operatorname{div} \boldsymbol{v}^n = 0$ a.e. in $\Omega$ and

$$\begin{aligned} \frac{1}{n}\int_\Omega |\boldsymbol{v}^n|^{2r'-2}\boldsymbol{v}^n \cdot \boldsymbol{\varphi} \,\mathrm{d}x + \int_\Omega \nu(p^n, |\mathbf{D}(\boldsymbol{v}^n)|^2)\mathbf{D}(\boldsymbol{v}^n) \cdot \mathbf{D}(\boldsymbol{\varphi}) \,\mathrm{d}x \\ - \int_\Omega (\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}x - \int_\Omega p^n \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle, \end{aligned} \tag{4.32}$$

for all $\boldsymbol{\varphi} \in \mathcal{C}_0^\infty(\Omega)^d$ or else for all $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega)^d \cap L^{2r'}(\Omega)^d$, $\operatorname{div} \boldsymbol{\varphi} = 0$.

Taking $\boldsymbol{\varphi} = \boldsymbol{v}^n$ in (4.32) and using the same arguments as in (4.2), we verify

$$C\|\nabla \boldsymbol{v}^n\|_r^r + \frac{1}{n}\|\boldsymbol{v}^n\|_{2r'}^{2r'} \le C < \infty \tag{4.33}$$

and then again from (2.9) with $\lambda = 1$ and (4.33) we have that

$$\|\nu(p^n, |\mathbf{D}(\boldsymbol{v}^n)|^2)\mathbf{D}(\boldsymbol{v}^n)\|_{r'} \le C < \infty. \tag{4.34}$$

In order to obtain an estimate for the pressure $p^n$ independent of $n$, we apply the same procedure as for $p^\varepsilon$, namely we take the test function $\boldsymbol{\varphi} = \boldsymbol{\varphi}^n$ in (4.32), where $\boldsymbol{\varphi}^n$ solves

$$\begin{aligned} \operatorname{div} \boldsymbol{\varphi}^n = |p^n|^{s-2}\, p^n - \frac{1}{|\Omega|}\int_\Omega |p^n|^{s-2}\, p^n \,\mathrm{d}x =: h^n &\quad \text{in } \Omega \\ \boldsymbol{\varphi}^n = \mathbf{0} &\quad \text{on } \partial\Omega, \end{aligned}$$

with $s = \frac{dr}{2(d-r)}$. The function $\boldsymbol{\varphi}^n$ then fulfills

$$\|\boldsymbol{\varphi}^n\|_{1,q} \le C_{\mathrm{div}}(\Omega, q)\|h^n\|_q \quad \text{for all } q : 1 < q \le s' = \frac{dr}{(d+2)r - 2d}$$

and for $q = s'$ we have

$$\|\boldsymbol{\varphi}^n\|_{1,s'} \le 2C_{\mathrm{div}}(\Omega, s')\|p^n\|_s^{s-1}. \tag{4.35}$$

Note that $\int_\Omega p^n \,\mathrm{d}x = 0$. With the help of (2.9), (4.33), (4.35), the embedding $W^{1,s_1} \hookrightarrow L^{2r'}(\Omega)$, $s_1 = \frac{2dr}{(d+2)r-d}$, and the fact that $s_1 \le s'$ and $\frac{r+1}{2r} \le 1$ we conclude

$$\begin{aligned}
\|p^n\|_s^s &= \int_\Omega \nu(p^n, |\mathbf{D}(\boldsymbol{v}^n)|^2)\mathbf{D}(\boldsymbol{v}^n)\cdot\mathbf{D}(\boldsymbol{\varphi}^n)\,\mathrm{d}x \,-\langle \boldsymbol{f}, \boldsymbol{\varphi}^n\rangle \\
&\quad - \int_\Omega (\boldsymbol{v}^n \otimes \boldsymbol{v}^n)\cdot\nabla\boldsymbol{\varphi}^n \,\mathrm{d}x \,+ \frac{1}{n}\int_\Omega |\boldsymbol{v}^n|^{2r'-2}\boldsymbol{v}^n\cdot\boldsymbol{\varphi}^n\,\mathrm{d}x \\
&\le C(1 + \|\mathbf{D}(\boldsymbol{v}^n)\|_r)^{r-1}\|\boldsymbol{\varphi}^n\|_{1,r} + \|\boldsymbol{f}\|_{-1,r'}\|\boldsymbol{\varphi}^n\|_{1,r} \\
&\quad + \|\boldsymbol{v}^n \otimes \boldsymbol{v}^n\|_s\|\nabla\boldsymbol{\varphi}^n\|_{s'} + \frac{1}{n}\|\boldsymbol{v}^n\|_{2r'}^{2r'-1}\|\boldsymbol{\varphi}^n\|_{2r'} \\
&\le C\|\boldsymbol{\varphi}^n\|_{1,s'} + C\frac{1}{n}(\|\boldsymbol{v}^n\|_{2r'}^{2r'})^{\frac{r+1}{2r}}\|\boldsymbol{\varphi}^n\|_{1,s_1} \\
&\le C\|\boldsymbol{\varphi}^n\|_{1,s'} \le C\|p^\varepsilon\|_s^{s-1},
\end{aligned}$$

which leads to

$$\|p^n\|_{\frac{dr}{2(d-r)}} \le C < \infty. \tag{4.36}$$

Owing to (4.33), (4.36), (4.34) and Theorem A.1 we can again find a (not relabelled) subsequence $(\boldsymbol{v}^n, p^n)$ and $(\boldsymbol{v}, p) \in W^{1,r}_{0,\mathrm{div}}(\Omega)^d \times L_0^{\frac{dr}{2(d-r)}}(\Omega)$ such that

$$\begin{aligned}
\mathbf{D}(\boldsymbol{v}^n) &\rightharpoonup \mathbf{D}(\boldsymbol{v}) && \text{weakly in } L^r(\Omega)^{d\times d}, \\
\nabla\boldsymbol{v}^n &\rightharpoonup \nabla\boldsymbol{v} && \text{weakly in } L^r(\Omega)^{d\times d}, \\
\boldsymbol{v}^n &\rightharpoonup \boldsymbol{v} && \text{weakly in } W^{1,r}_{0,\mathrm{div}}(\Omega)^d, \\
p^n &\rightharpoonup p && \text{weakly in } L_0^{\frac{dr}{2(d-r)}}(\Omega), \\
\nu(p^n, |\mathbf{D}(\boldsymbol{v}^n)|^2)\mathbf{D}(\boldsymbol{v}^n) &\rightharpoonup \overline{\nu\mathbf{D}} && \text{weakly in } L^{r'}(\Omega)^{d\times d},
\end{aligned} \tag{4.37}$$

and due to the compact embedding

$$\boldsymbol{v}^n \to \boldsymbol{v} \quad \text{strongly in } L^q(\Omega)^d \text{ for all } q : 1 \le q < \frac{dr}{d-r}. \tag{4.38}$$

Especially, $\boldsymbol{v}^n \to \boldsymbol{v}$ strongly in $L^2(\Omega)^d$ (as $r > \frac{2d}{d+2}$).

Thus, after passing to the limit in (4.32), we get

$$\int_\Omega (\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}x \to \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{\varphi} \, \mathrm{d}x \quad \forall \boldsymbol{\varphi} \in W_0^{1,\infty}(\Omega)^d.$$

We can easily obtain the limit in the term involving the pressure

$$\int_\Omega p^n \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x \to \int_\Omega p \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x \quad \forall \boldsymbol{\varphi} \in W_0^{1,\infty}(\Omega)^d,$$

since $r > \frac{2d}{d+2}$, and therefore the pressure $p^n$ converges weakly also in $L^1(\Omega)$. We also have

$$\frac{1}{n} \int_\Omega |\boldsymbol{v}^n|^{2r'-2} \boldsymbol{v}^n \cdot \boldsymbol{\varphi} \, \mathrm{d}x \to 0 \quad \forall \boldsymbol{\varphi} \in L^\infty(\Omega)^d.$$

It remains to prove the convergence of the viscosity term, i.e., to show that for $n \to \infty$ and for all $\boldsymbol{\varphi} \in W_0^{1,\infty}(\Omega)^d$

$$\int_\Omega \nu(p^n, |\mathbf{D}(\boldsymbol{v}^n)|^2) \mathbf{D}(\boldsymbol{v}^n) \cdot \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}x \to \int_\Omega \nu(p, |\mathbf{D}(\boldsymbol{v})|^2) \mathbf{D}(\boldsymbol{v}) \cdot \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}x.$$

For this purpose, we again need to know that

$$p^n \to p \quad \text{a.e. in } \Omega \quad \text{and} \quad \mathbf{D}(\boldsymbol{v}^n) \to \mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in } \Omega, \tag{4.39}$$

at least for a subsequence. Vitali's theorem (Theorem A.5) then completes the whole proof.

The proof of (4.39) is a subject of the following subsection.

### 4.2.2 Almost everywhere convergence of $p^n$ and $\mathbf{D}(\boldsymbol{v}^n)$

In order to show (4.39), we again decompose the pressure $p^n$ into two pressures and recall the monotonicity of the viscosity (2.7). However, we use this condition in a different way as in Section 4.1.2. We apply the so-called Lipschitz approximations of Sobolev functions that (as we are going to see) are essential for our proof.

By an analogous procedure we deal with the decomposition of the pressure. Consider again two auxiliary Stokes problems

$$\begin{aligned} -\triangle \boldsymbol{v}_i^n + \nabla p_i^n &= \boldsymbol{h}_i^n && \text{in } \Omega \\ \operatorname{div} \boldsymbol{v}_i^n &= 0 && \text{in } \Omega \\ \boldsymbol{v}_i^n &= \mathbf{0} && \text{on } \partial\Omega, \ i = 1, 2, \end{aligned} \tag{4.40}$$

this time with the right-hand sides

$$\boldsymbol{h}_1^n = \operatorname{div}(\nu(p^n, |\mathbf{D}(\boldsymbol{v}^n)|^2)\mathbf{D}(\boldsymbol{v}^n)) + \boldsymbol{f} \qquad \in (W_0^{1,r}(\Omega))^*$$
$$\boldsymbol{h}_2^n = -\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) - \frac{1}{n}|\boldsymbol{v}^n|^{2r'-2}\boldsymbol{v}^n \qquad \in (W_0^{1,s'}(\Omega))^*,\ s' = \frac{dr}{(d+2)r-2d}.$$

The existence theory for the Stokes system then yields the existence of solutions $(\boldsymbol{v}_i^n, p_i^n)$, $i = 1, 2$, satisfying the following estimates for $\nabla \boldsymbol{v}_i^n$ and for the pressures $p_i^n$ having zero mean value

$$\|\nabla \boldsymbol{v}_1^n\|_{r'} + \|p_1^n\|_{r'} \le C\|\boldsymbol{h}_1^n\|_{(W_0^{1,r}(\Omega))^*} \le C + C\|\nu(p^n, |\mathbf{D}(\boldsymbol{v}^n)|^2)\mathbf{D}(\boldsymbol{v}^n)\|_{r'}$$
$$\|\nabla \boldsymbol{v}_2^n\|_s + \|p_2^n\|_s \le C\|\boldsymbol{h}_2^n\|_{(W_0^{1,s'}(\Omega))^*} \le C\|\boldsymbol{v}^n \otimes \boldsymbol{v}^n\|_s + C\frac{1}{n^{\frac{1}{2r'}}}\left(\frac{1}{n}\|\boldsymbol{v}^n\|_{2r'}^{2r'}\right)^{\frac{r+1}{2r}}.$$

From these inequalities thanks to (4.34), (4.33) and (4.38) we get

$$\begin{aligned} \nabla \boldsymbol{v}_1^n &\rightharpoonup \nabla \boldsymbol{v}_1 && \text{weakly in } L^{r'}(\Omega)^{d\times d}, \\ p_1^n &\rightharpoonup p_1 && \text{weakly in } L^{r'}(\Omega), \\ \nabla \boldsymbol{v}_2^n &\to \nabla \boldsymbol{v}_2 && \text{strongly in } L^q(\Omega)^{d\times d}, \\ p_2^n &\to p_2 && \text{strongly in } L^q(\Omega), \end{aligned} \tag{4.41}$$

where $q \in [1, \frac{dr}{2(d-r)})$. Furthermore, the uniqueness of solutions of the Stokes problem again implies that

$$p^n = p_1^n + p_2^n \quad \text{and} \quad \boldsymbol{v}_1^n = -\boldsymbol{v}_2^n \tag{4.42}$$

and consequently

$$\nabla \boldsymbol{v}_1^n \to \nabla \boldsymbol{v}_1 \quad \text{strongly in } L^q(\Omega)^{d\times d}.$$

Remembering again the notation $\mathbf{S}(p, \mathbf{D}) := \nu(p, |\mathbf{D}|^2)\mathbf{D}$, we now recall the monotonicity condition (2.7) from Lemma 2.1 with $\mathbf{D}^1 = \mathbf{D}(\boldsymbol{v}^n)$, $\mathbf{D}^2 = \mathbf{D}(\boldsymbol{v})$, $p^1 = p^n$ and $p^2 = p_1 + p_2^n$, but this time we will not consider the integration over the whole domain $\Omega$. Unlike the similar situation (4.16) in Section 4.1.2, we cannot now use $\boldsymbol{\varphi} = \boldsymbol{v}^n - \boldsymbol{v}$ as a test function in order to treat the term

$$\int_\Omega \mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v})\,\mathrm{d}x.$$

The reason is the fact that we are interested in the case when $\frac{2d}{d+2} < r \le \frac{3d}{d+2}$ and for this $\boldsymbol{v}$ is not an admissible test function anymore. The trouble is caused by the convective term since $\int_\Omega (\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \cdot \nabla(\boldsymbol{v}^n - \boldsymbol{v})\,\mathrm{d}x \to 0$ provided that $r > \frac{3d}{d+2}$.

Nevertheless, we notice that thanks to $(4.37)_3$ the functions

$$\boldsymbol{u}^n := \boldsymbol{v}^n - \boldsymbol{v}$$

satisfy the assumptions of Theorem A.6 on Lipschitz approximations of functions from $W_0^{1,r}(\Omega)^d$. Therefore, there exists a sequence $\boldsymbol{u}^{n,j} \in W_0^{1,\infty}(\Omega)^d$ possessing the properties (A.5)–(A.9). Of course,

$$\Omega = \{\boldsymbol{u}^n = \boldsymbol{u}^{n,j}\} \cup \{\boldsymbol{u}^n \neq \boldsymbol{u}^{n,j}\} := U_{n,j} \cup \Omega \backslash U_{n,j}.$$

On returning to the monotonicity condition of the form mentioned above and integrating it over the set of coincidence $U_{n,j}$, we obtain ($\chi$ denotes the characteristic function)

$$\begin{aligned} \frac{C_1}{2} \int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}}\,\mathrm{d}x \le \int_{U_{n,j}} &(\mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) - \mathbf{S}(p_1 + p_2^n, \mathbf{D}(\boldsymbol{v}))) \cdot \mathbf{D}(\boldsymbol{u}^n)\,\mathrm{d}x \\ &+ \frac{\gamma_0^2}{2C_1} \|(p_1^n - p_1)\chi_{U_{n,j}}\|_2^2, \end{aligned} \tag{4.43}$$

where

$$\begin{aligned} &\int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}}\,\mathrm{d}x = \\ &= \int_{U_{n,j}} \int_0^1 (1 + |\mathbf{D}(\boldsymbol{v}) + s(\mathbf{D}(\boldsymbol{v}^n) - \mathbf{D}(\boldsymbol{v}))|^2)^{\frac{r-2}{2}} |\mathbf{D}(\boldsymbol{v}^n) - \mathbf{D}(\boldsymbol{v})|^2\,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

Our goal is to show that

$$\limsup_{n\to\infty} \int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}}\,\mathrm{d}x \to 0 \quad \text{for } j \to \infty.$$

Firstly, in the same manner as before we deal with the second integral on the right-hand side, namely we have

$$\int_{U_{n,j}} \mathbf{S}(p_1 + p_2^n, \mathbf{D}(\boldsymbol{v})) \cdot \mathbf{D}(\boldsymbol{u}^n)\,\mathrm{d}x \to 0 \quad \text{for } n \to \infty, \tag{4.44}$$

since Lebesgue's dominated convergence theorem implies that

$$\mathbf{S}(p_1 + p_2^n, \mathbf{D}(\boldsymbol{v})) \to \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) \quad \text{strongly in } L^{r'}(\Omega)^{d\times d}.$$

Secondly, as we already know that $\boldsymbol{u}^n$ is not a suitable test function and since $\boldsymbol{u}^{n,j}$ is not in general divergence-free on the set of non-coincidence $\Omega\backslash U_{n,j}$, we consider the weak formulation (4.32) with a special test function

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^{n,j} := \boldsymbol{u}^{n,j} - \boldsymbol{\psi}^{n,j},$$

where $\boldsymbol{\psi}^{n,j}$ solves the following system of equations

$$\begin{aligned} \operatorname{div} \boldsymbol{\psi}^{n,j} &= \operatorname{div} \boldsymbol{u}^{n,j} = \operatorname{div} \boldsymbol{u}^{n,j} \chi_{\Omega\backslash U_{n,j}} \quad \text{in } \Omega \\ \boldsymbol{\psi}^{n,j} &= \mathbf{0} \quad \text{on } \partial\Omega \end{aligned} \tag{4.45}$$

and satisfies

$$\|\boldsymbol{\psi}^{n,j}\|_{1,r} \le C_{\mathrm{div}}(\Omega, r) \|\operatorname{div} \boldsymbol{u}^{n,j} \chi_{\Omega\backslash U_{n,j}}\|_r.$$

In addition, the properties (A.5)–(A.9) of the sequence $\boldsymbol{u}^{n,j}$ and the continuity and linearity of the Bogovskii operator imply that for $j \in \mathbb{N}$ and $n \to \infty$

$$\begin{aligned} \boldsymbol{\psi}^{n,j} &\to \mathbf{0} && \text{strongly in } L^q(\Omega)^d \quad \forall q \in (1,\infty), \\ \boldsymbol{\psi}^{n,j} &\rightharpoonup \mathbf{0} && \text{weakly in } W_0^{1,q}(\Omega)^d \quad \forall q \in (1,\infty), \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} \limsup_{n\to\infty} \|\boldsymbol{\psi}^{n,j}\|_{1,r} &\le C_{\mathrm{div}}(\Omega, r) \limsup_{n\to\infty} \|\operatorname{div} \boldsymbol{u}^{n,j} \chi_{\Omega\backslash U_{n,j}}\|_r \\ &\le C_{\mathrm{div}}(\Omega, r) \limsup_{n\to\infty} \|\nabla \boldsymbol{u}^{n,j} \chi_{\Omega\backslash U_{n,j}}\|_r \\ &\le C\varepsilon_j, \end{aligned} \tag{4.47}$$

with $\varepsilon_j := K\, 2^{-j/r}$. Note that by $(4.45)_1$ $\operatorname{div} \boldsymbol{\varphi}^{n,j} = 0$ and thanks to (4.46) we also have for $j \in \mathbb{N}$ and $n \to \infty$ that

$$\begin{aligned} \boldsymbol{\varphi}^{n,j} &\to \mathbf{0} && \text{strongly in } L^q(\Omega)^d \quad \forall q \in (1,\infty), \\ \boldsymbol{\varphi}^{n,j} &\rightharpoonup \mathbf{0} && \text{weakly in } W_0^{1,q}(\Omega)^d \quad \forall q \in (1,\infty). \end{aligned} \tag{4.48}$$

Now, considering the weak formulation (4.32) of the problem $(\mathcal{P}^\eta)$ with the test function $\boldsymbol{\varphi} = \boldsymbol{\varphi}^{n,j}$, we can write

$$\begin{aligned}
\int_\Omega \mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{u}^{n,j}) \,\mathrm{d}x &= \langle \boldsymbol{f}, \boldsymbol{\varphi}^{n,j} \rangle - \frac{1}{n} \int_\Omega |\boldsymbol{v}^n|^{2r'-2} \boldsymbol{v}^n \cdot \boldsymbol{\varphi}^{n,j} \,\mathrm{d}x \\
&\quad + \int_\Omega (\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \cdot \nabla \boldsymbol{\varphi}^{n,j} \,\mathrm{d}x \\
&\quad + \int_\Omega \mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{\psi}^{n,j}) \,\mathrm{d}x \\
&:= I^1_{n,j} + I^2_{n,j} + I^3_{n,j} + I^4_{n,j}.
\end{aligned} \tag{4.49}$$

Letting $n \to \infty$ and taking (4.33), (4.38) and (4.48) into account, we see that

$$\lim_{n\to\infty} (I^1_{n,j} + I^2_{n,j} + I^3_{n,j}) = 0.$$

On the other hand, on using Hölder's inequality, (4.34) and (4.47), we get

$$I^4_{n,j} \leq \|\mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n))\|_{r'} \|\mathbf{D}(\psi^{n,j})\|_r \leq C \frac{\gamma_n}{\theta_n} \mu_{j+1} + C\varepsilon_j = g(n) + C\varepsilon_j,$$

where for $j \in \mathbb{N}$ fixed the function $g(n) \to 0$ for $n \to \infty$. Putting everything together, we observe from (4.49) that

$$\int_\Omega \mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{u}^{n,j}) \,\mathrm{d}x \leq g(n) + C\varepsilon_j.$$

Moreover, Hölder's inequality, (4.34) and (A.9) yield that also

$$\left| \int_{\Omega \setminus U_{n,j}} \mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{u}^{n,j}) \,\mathrm{d}x \right| \leq g(n) + C\varepsilon_j,$$

and therefore

$$\begin{aligned}
\int_{U_{n,j}} \mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{u}^{n,j}) \,\mathrm{d}x &\leq \int_\Omega \mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{u}^{n,j}) \,\mathrm{d}x \\
&\quad + \left| \int_{\Omega \setminus U_{n,j}} \mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) \cdot \mathbf{D}(\boldsymbol{u}^{n,j}) \,\mathrm{d}x \right| \\
&\leq g(n) + C\varepsilon_j,
\end{aligned} \tag{4.50}$$

where $g(n) \to 0$ as $n \to \infty$ for $j \in \mathbb{N}$ fixed and $\varepsilon_j \to 0$ as $j \to \infty$.

Coming back to the inequality (4.43), from (4.44) and (4.50) we then conclude

$$\begin{aligned}\frac{C_1}{2}\int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}}\,\mathrm{d}x &\le g(n) + C\varepsilon_j + \frac{\gamma_0^2}{2C_1}\|(p_1^n - p_1)\chi_{U_{n,j}}\|_2^2 \\ &\le g(n) + C\varepsilon_j + \frac{\gamma_0^2}{2C_1}\|p_1^n - p_1\|_2^2.\end{aligned} \tag{4.51}$$

Clearly, we would also like to have that

$$\|p_1^n - p_1\|_2^2 \le g(n) + C\varepsilon_j.$$

For this purpose, we first look at the set $\Omega\backslash U_{n,j} = \{\boldsymbol{u}^n \ne \boldsymbol{u}^{n,j}\}$ and show that its Lebesgue measure is sufficiently small. We recall the Theorem A.6 and since $\lambda_{n,j} \ge 1$, according to (A.4), we observe from (A.9) that

$$\begin{aligned}|\Omega\backslash U_{n,j}| = \|\chi_{\Omega\backslash U_{n,j}}\|_1 &\le c\lambda_{n,j}^{-1}\|\lambda_{n,j}\chi_{\Omega\backslash U_{n,j}}\|_r \\ &\le c\|\lambda_{n,j}\chi_{\Omega\backslash U_{n,j}}\|_r \le g(n) + c\,\varepsilon_j.\end{aligned} \tag{4.52}$$

Next, similarly as in Section 4.1.2, we consider the weak formulation of (4.40), $i = 1$, with a test function $\boldsymbol{\varphi}^n$ fulfilling

$$\begin{aligned}\operatorname{div}\boldsymbol{\varphi}^n &= p_1^n - p_1 \quad \text{in } \Omega \\ \boldsymbol{\varphi}^n &= \mathbf{0} \quad \text{on } \partial\Omega \\ \|\boldsymbol{\varphi}^n\|_{1,q} &\le C_{\mathrm{div}}(\Omega, q)\|p_1^n - p_1\|_q \quad \text{for all } q : 1 < q \le r'.\end{aligned} \tag{4.53}$$

Notice that $\int_\Omega (p_1^n - p_1)\,\mathrm{d}x = 0$. From the properties of the Bogovskii operator and from $(4.41)_2$ we have

$$\boldsymbol{\varphi}^n \rightharpoonup \mathbf{0} \quad \text{weakly in } W^{1,r'}(\Omega)^d.$$

On using the same arguments as for (4.26), (4.27) and (4.28) in Section 4.1.2, we obtain

$$\|p_1^n - p_1\|_2^2 = g(n) + \int_\Omega (\mathbf{S}(p^n, \mathbf{D}(\boldsymbol{v}^n)) - \mathbf{S}(p_1 + p_2^n, \mathbf{D}(\boldsymbol{v}))) \cdot \mathbf{D}(\boldsymbol{\varphi}^n)\,\mathrm{d}x,$$

where $g(n) \to 0$ for $n \to \infty$.

From application of (2.4) and (2.5) to the integral on the right-hand side and Hölder's inequality we get

$$\|p_1^n - p_1\|_2^2 \le g(n) + \gamma_0 C_{\mathrm{div}}(\Omega, 2)\|p_1^n - p_1\|_2^2 + C_2\int_\Omega \mathcal{J}\,\mathrm{d}x, \tag{4.54}$$

where

$$\mathcal{J} := \int_0^1 (1 + |\mathbf{D}(\boldsymbol{v}) + s(\mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v}))|^2)^{\frac{r-2}{2}} \, |\mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v})| \, |\mathbf{D}(\boldsymbol{\varphi}^n)| \, \mathrm{d}s.$$

We consider

$$\int_\Omega \mathcal{J} \, \mathrm{d}x = \int_{U_{n,j}} \mathcal{J} \, \mathrm{d}x \, + \int_{\Omega \setminus U_{n,j}} \mathcal{J} \, \mathrm{d}x$$

and recall the definition of $\mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}}$. On using Hölder's inequality, $(4.53)_3$ with $q = 2$ and Young's inequality, we can write

$$\begin{aligned}
\int_{U_{n,j}} \mathcal{J} \, \mathrm{d}x &\le C_{\mathrm{div}}(\Omega, 2) \left( \int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}} \, \mathrm{d}x \right)^{\frac{1}{2}} \|p_1^n - p_1\|_2 \\
&\le \frac{C_{\mathrm{div}}^2(\Omega, 2) C_2}{2(1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2))} \int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}} \, \mathrm{d}x \\
&\quad + \frac{1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2)}{2C_2} \|p_1^n - p_1\|_2^2,
\end{aligned}$$

and with the help of Hölder's inequality, (4.33), (4.52) and the fact that $r < 2$ we arrive at

$$\begin{aligned}
&\int_{\Omega \setminus U_{n,j}} \mathcal{J} \, \mathrm{d}x \\
&\le \left( \int_{\Omega \setminus U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}} \, \mathrm{d}x \right)^{\frac{1}{2}} \left( \int_{\Omega \setminus U_{n,j}} |\nabla \boldsymbol{\varphi}^n|^{r'} \, \mathrm{d}x \right)^{\frac{1}{r'}} |\Omega \setminus U_{n,j}|^{\frac{r'-2}{2r'}} \\
&\le C |\Omega \setminus U_{n,j}|^{\frac{r'-2}{2r'}} \le g(n) + C(\varepsilon_j)^{\frac{r'-2}{2r'}}.
\end{aligned}$$

Putting these two estimates together with (4.54) gives

$$\begin{aligned}
(1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2)) \|p_1^n - p_1\|_2^2 &\le g(n) + C(\varepsilon_j)^{\frac{r'-2}{2r'}} \\
&\quad + \frac{C_{\mathrm{div}}^2(\Omega, 2) C_2^2}{2(1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2))} \int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}} \, \mathrm{d}x \\
&\quad + \frac{1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2)}{2} \|p_1^n - p_1\|_2^2.
\end{aligned}$$

Thanks to (2.6) is $1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2) - \frac{1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2)}{2} > 0$ and we get for the $L^2$–norm that

$$\|p_1^n - p_1\|_2^2 \le g(n) + C(\varepsilon_j)^{\frac{r'-2}{2r'}} + \frac{C_{\mathrm{div}}^2(\Omega, 2) C_2^2}{(1 - \gamma_0 C_{\mathrm{div}}(\Omega, 2))^2} \int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}} \, \mathrm{d}x. \tag{4.55}$$

Furthermore, as $r' > 2$ and $\frac{C_1}{2} - \frac{\gamma_0^2}{2C_1}\frac{C_{\mathrm{div}}^2(\Omega,2)C_2^2}{(1-\gamma_0 C_{\mathrm{div}}(\Omega,2))^2} > 0$, again due to (2.6), from (4.51) and (4.55) we indeed have

$$\limsup_{n\to\infty}\int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}}\,\mathrm{d}x \to 0 \quad \text{for } j\to\infty. \tag{4.56}$$

The almost everywhere convergence (at least for a subsequence) of the pressure $p^n$ in $\Omega$ then follows from (4.42) as both $p_1^n$ and $p_2^n$ converge a.e. owing to (4.55), (4.56) and $(4.41)_4$.

Finally, considering (4.52) and since $\|\nabla \boldsymbol{v}^n\|_r \le C$, $\|\nabla \boldsymbol{v}\|_r \le C$ and $r < 2$, we obtain (with the help of Hölder's inequality)

$$\begin{aligned}
&\int_\Omega |\mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v})|\,\mathrm{d}x \\
&= \int_{U_{n,j}} |\mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v})|\,\mathrm{d}x + \int_{\Omega\setminus U_{n,j}} |\mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v})|\,\mathrm{d}x \\
&\le \int_{U_{n,j}} (\mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}})^{\frac12}\left(\int_0^1 (1+|\mathbf{D}(\boldsymbol{v}) + s(\mathbf{D}(\boldsymbol{v}^n - \boldsymbol{v}))|^2)^{\frac{r-2}{2}}\mathrm{d}s\right)^{-\frac12}\mathrm{d}x \\
&\quad + \int_{\Omega\setminus U_{n,j}} (|\mathbf{D}(\boldsymbol{v}^n)| + |\mathbf{D}(\boldsymbol{v})|)\,\mathrm{d}x \\
&\le C\int_{U_{n,j}} (\mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}})^{\frac12}(1+|\mathbf{D}(\boldsymbol{v}^n)| + |\mathbf{D}(\boldsymbol{v})|)^{\frac{2-r}{2}}\,\mathrm{d}x \\
&\quad + (\|\nabla\boldsymbol{v}^n\|_r + \|\nabla\boldsymbol{v}\|_r)|\Omega\setminus U_{n,j}|^{\frac{1}{r'}} \\
&\le C\left(\int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}}\,\mathrm{d}x\right)^{\frac12}(1+\|\nabla\boldsymbol{v}^n\|_r + \|\nabla\boldsymbol{v}\|_r)^{\frac{2-r}{2}}|\Omega|^{\frac{1}{r'}} \\
&\quad + g(n) + C(\varepsilon_j)^{\frac{1}{r'}} \\
&\le C\left(\int_{U_{n,j}} \mathcal{I}^{\boldsymbol{v}^n,\boldsymbol{v}}\,\mathrm{d}x\right)^{\frac12} + g(n) + C(\varepsilon_j)^{\frac{1}{r'}} \overset{(4.56)}{\rightarrow} 0,
\end{aligned}$$

which yields that (at least for a subsequence)

$$\mathbf{D}(\boldsymbol{v}^n) \to \mathbf{D}(\boldsymbol{v}) \quad \text{a.e. in } \Omega.$$

Applying Vitali's theorem then finishes the proof of Theorem 2.2. □

# 5 Conclusion

In the preceding section we managed to establish the existence result formulated in Theorem 2.2. To our knowledge, this result seems to be new.

The proof itself combines the standard approach usually applied when dealing with mathematical analysis of similar problems with more advanced methods such as the Lipschitz approximations of Sobolev functions, the decomposition of the pressure or the solvability of the divergence equation (according to Bogovskii). Although the uniform estimates give information about the limiting velocity and the limiting pressure, the weak convergence is not enough for passing to the limit in non-linear terms. With the help of the compact embedding for the velocity field we are quite easily able to treat the convective term. The crucial part of the proof then arises in the term with the viscosity as the almost everywhere convergence of the pressure and of the symmetric part of the velocity gradient has to be shown. At this juncture, the importance of assumptions (2.4) and (2.5) on the viscosity is more than obvious since these conditions provide the monotonicity of the viscosity according to Lemma 2.1. Nevertheless, this fact itself is not sufficient and we have to consider the decomposition of the pressure, since we do not have the information about the pressure lying in $L^2$, as well as to use the Lipschitz approximations of the functions $\boldsymbol{v}^n - \boldsymbol{v}$ as test functions in the weak formulation in the second limit procedure, for $\boldsymbol{v}$ is not an admissible test function.

Of course, there still remains a great space for further research concerning the flows of fluids with pressure-dependent viscosities. One possibility could be for example an extension of the existence of weak solutions established in this thesis to unsteady flows. A similar situation only for the case with the viscosity depending on the shear rate is a subject of current research by L. Diening, M. Růžička and J. Wolf and seems to be almost finished. This result would somehow complete the existence theory for fluids with shear rate dependent viscosity initiated by O. A. Ladyzhenskaya and J. L. Lions in the late 1960s and further developed in the 1990s. No doubt a challenge would be mathematical analysis of models wherein the viscosity depends only on the pressure, namely an establishment of some global existence theory for such problems.

# A Needful tools and theorems

In this final section we would like to mention several tools frequently used in the thesis. Proofs of all below-stated lemmas and theorems can be found in the appropriate literature and for this reason we leave them out in what follows.

As we consider the reader to be familiar with the standard Hölder's, Young's and Poincaré's inequalities, we start with another inequality essential for a derivation of uniform estimates. For the proof see for example [24].

**Lemma A.1. (Korn's inequality)** *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary. Then there exists a positive constant $C$ depending only on $p$ and $\Omega$ such that for all $\boldsymbol{u} \in W_0^{1,p}(\Omega)^d$*

$$\|\nabla \boldsymbol{u}\|_p \leq C \|\mathsf{D}(\boldsymbol{u})\|_p.$$

In order to be able to obtain weakly converging subsequences from the uniform estimates, we use the following theorem (see [31]).

**Theorem A.1. (Eberlein-Shmulyan)** *A Banach space $X$ is reflexive if and only if every bounded sequence of $X$ contains a subsequence which converges weakly to an element in $X$.*

It is well-known that the so-called embedding theorems play a crucial role in analysis of such problems as ours and therefore we should not omit them in here (see [18]).

**Theorem A.2. (Sobolev's embedding theorems)** *Let $1 \leq p \leq \infty$ and $k \geq 0$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Then*

1. *if $k < \frac{d}{p}$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q : \frac{1}{q} = \frac{1}{p} - \frac{k}{d}$,*
2. *if $k = \frac{d}{p}$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$,*
3. *if $\frac{d}{p} < k < \frac{d}{p} + 1$ then $W^{k,p}(\Omega) \hookrightarrow C^{0,k-d/p}(\overline{\Omega})$,*
4. *if $k = \frac{d}{p} + 1$ then $W^{k,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$,*
5. *if $k > \frac{d}{p} + 1$ then $W^{k,p}(\Omega) \hookrightarrow C^{0,1}(\overline{\Omega})$.*

**Theorem A.3. (Rellich-Kondrashov's theorems on compact embedding)** *Let $1 \le p \le \infty$ and $k > 0$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Then*

1. *if $k < \frac{d}{p}$ then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [1, p^*)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{d}$,*
2. *if $k = \frac{d}{p}$ then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$,*
3. *if $k > \frac{d}{p}$ then $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$.*

For completeness, we also add the following lemma concerning the interpolation of Lebesgue spaces. The proof is through Hölder's inequality.

**Lemma A.2.** *Let $\Omega \subset \mathbb{R}^d$ be a measurable set, $1 \le p \le r \le q \le \infty$ and $f \in L^p(\Omega) \cap L^q(\Omega)$. Then $f \in L^r(\Omega)$ and*

$$\|f\|_r \le \|f\|_p^\theta \|f\|_q^{1-\theta} \text{ with } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

The following two theorems tell us something about convergences in Lebesgue spaces, namely when the limiting process in an integral is possible. First, we mention the well-known Lebesgue's theorem (see [19]) and then also very important Vitali's theorem (see [2]) that helps us to prove the convergence of the integral with the viscosity.

**Theorem A.4. (Lebesgue's dominated convergence theorem)** *Let $f^n$, $n = 1, 2, \dots,$ be measurable functions in $\Omega$ and let $\lim_{n\to\infty} f^n(x) = f(x)$ for a.a. $x \in \Omega$. If there exists a function $g \in L^1(\Omega)$ such that*

$$|f^n(x)| \le g(x) \text{ for a.a. } x \in \Omega \text{ and for all } n \in \mathbb{N}$$

*then $f \in L^1(\Omega)$ and*

$$\lim_{n\to\infty} \int_\Omega f^n(x)\,\mathrm{d}x = \int_\Omega f(x)\,\mathrm{d}x.$$

**Theorem A.5. (Vitali)** *Let $\Omega$ be a bounded measurable domain in $\mathbb{R}^d$ and $f^n : \Omega \to \mathbb{R}$ be an integrable function for every $n \in \mathbb{N}$. Let*

$$\lim_{n\to\infty} f^n(x) \text{ exist and be finite for a.a. } x \in \Omega,$$

*and let there for all $\varepsilon > 0$ exist $\delta > 0$ such that*

$$\sup_{n\in\mathbb{N}} \int_E |f^n(x)| \, \mathrm{d}x \le \varepsilon \quad \text{for all } E \subset \Omega, \ |E| < \delta.$$

*Then*

$$\lim_{n\to\infty} \int_\Omega f^n(x) \, \mathrm{d}x = \int_\Omega \lim_{n\to\infty} f^n(x) \, \mathrm{d}x.$$

Very often we consider the weak formulation with a test function that solves an equation of the type $\operatorname{div} \boldsymbol{u} = f$. The convenient solution in bounded domains is given by the so-called Bogovskii operator (introduced in [7]). We summarize the features that are important for us and refer the reader to [25], Lemma 3.17, for further details and the proof.

**Lemma A.3. (Bogovskii operator)** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Then there exists a continuous linear operator $\mathcal{B}$ such that $\mathcal{B} : L^p_0(\Omega) \to W^{1,p}_0(\Omega)^d$ for all $1 < p < \infty$ and for all $f \in L^p_0(\Omega)$*

$$\begin{aligned} \operatorname{div}(\mathcal{B}f) &= f \quad \text{in } \Omega \\ \|\mathcal{B}f\|_{1,p} &\le C_{\mathrm{div}}(\Omega, p)\|f\|_p. \end{aligned} \tag{A.1}$$

*Moreover, if $f = \operatorname{div} \boldsymbol{u}$ and $\boldsymbol{u} \in W^{1,p}_0(\Omega)^d \cap L^q(\Omega)^d$ with some $1 < q < \infty$ then there exists a constant $C(\Omega, q)$ such that*

$$\|\mathcal{B}f\|_q \le C(\Omega, q)\|\boldsymbol{u}\|_q.$$

As a certain analogy to the so-called Helmholtz decomposition we introduce the following decomposition. We recall the problem (A.1) with $f := \operatorname{div} \boldsymbol{u}$ and let for $\boldsymbol{u} \in W^{1,p}_0(\Omega)^d \cap L^q(\Omega)^d$ be $\boldsymbol{g}^{\boldsymbol{u}}$ a solution of the problem

$$\begin{aligned} \operatorname{div} \boldsymbol{g}^{\boldsymbol{u}} &= \operatorname{div} \boldsymbol{u} \qquad && \text{in } \Omega \\ \boldsymbol{g}^{\boldsymbol{u}} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned}$$

Then we set

$$\mathcal{P}\boldsymbol{u} := \boldsymbol{u} - \boldsymbol{g}^{\boldsymbol{u}},$$

which is equivalent to $\boldsymbol{u} := \mathcal{P}\boldsymbol{u} + \boldsymbol{g}^{\boldsymbol{u}}$. Note that from the definition of $\mathcal{P}\boldsymbol{u}$ it is obvious that $\operatorname{div} \mathcal{P}\boldsymbol{u} = 0$ a.e. in $\Omega$. Furthermore, from Lemma A.3 we have the following estimates

$$\|\boldsymbol{g}^{\boldsymbol{u}}\|_{1,p} \le C_{\mathrm{div}}(\Omega, p)\|\operatorname{div} \boldsymbol{u}\|_p \qquad \|\boldsymbol{g}^{\boldsymbol{u}}\|_q \le C(\Omega, q)\|\boldsymbol{u}\|_q \tag{A.2}$$

$$\|\mathcal{P}\boldsymbol{u}\|_{1,p} \le (1 + C_{\mathrm{div}}(\Omega, p))\|\boldsymbol{u}\|_{1,p} \qquad \|\mathcal{P}\boldsymbol{u}\|_q \le (1 + C(\Omega, q))\|\boldsymbol{u}\|_q. \tag{A.3}$$

Sobolev functions from $W_0^{1,p}(\Omega)$, $p \geq 1$ can be approximated by Lipschitz functions that coincide with the originals up to sets of small Lebesgue measures. However, such approximations are nothing new. Already in 1984 Acerbi and Fusco in [1] showed their applications in the calculus of variations and since then they have been used by many others in various areas of analysis, for example in the existence theory of partial differential equations or in the regularity theory.

The following theorem mentions the important properties of these approximate functions. More details and the proof can be found in [11].

**Theorem A.6. (Lipschitz approximations)** *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary. Let $\boldsymbol{u}^n \in W_0^{1,p}(\Omega)^d$ be such that $\boldsymbol{u}^n \rightharpoonup \mathbf{0}$ weakly in $W_0^{1,p}(\Omega)^d$ as $n \to \infty$. We set*

$$K := \sup_n \|\boldsymbol{u}^n\|_{1,p} < \infty$$
$$\gamma_n := \|\boldsymbol{u}^n\|_p \to 0 \quad (n \to \infty).$$

*Let $\theta_n > 0$ be such that (e.g. $\theta_n := \sqrt{\gamma_n}$)*

$$\theta_n \to 0 \quad \text{and} \quad \frac{\gamma_n}{\theta_n} \to 0 \quad (n \to \infty).$$

*Let $\mu_j := 2^{2^j}$. Then there exist a sequence $\lambda_{n,j}$ with*

$$\mu_j \leq \lambda_{n,j} \leq \mu_{j+1} \tag{A.4}$$

*and a sequence $\boldsymbol{u}^{n,j} \in W_0^{1,\infty}(\Omega)^d$ such that for all $j, n \in \mathbb{N}$*

$$\|\boldsymbol{u}^{n,j}\|_\infty \leq \theta_n \to 0 \quad (n \to \infty), \tag{A.5}$$
$$\|\nabla \boldsymbol{u}^{n,j}\|_\infty \leq c\lambda_{n,j} \leq c\mu_{j+1} \tag{A.6}$$

*and up to a nullset ($\mathcal{M}$ denotes the Hardy-Littlewood maximal function[1])*

$$\{\boldsymbol{u}^n \neq \boldsymbol{u}^{n,j}\} \subset \Omega \cap (\{\mathcal{M}(\boldsymbol{u}^n) > \theta_n\} \cup \{\mathcal{M}(\nabla \boldsymbol{u}^n) > 2\lambda_{n,j}\}). \tag{A.7}$$

*Moreover, for all $j \in \mathbb{N}$ and $n \to \infty$*

$$\begin{aligned} \boldsymbol{u}^{n,j} &\to \mathbf{0} && \text{strongly in } L^q(\Omega)^d \quad \forall q \in [1,\infty], \\ \boldsymbol{u}^{n,j} &\rightharpoonup \mathbf{0} && \text{weakly in } W_0^{1,q}(\Omega)^d \quad \forall q \in [1,\infty), \\ \nabla \boldsymbol{u}^{n,j} &\overset{*}{\rightharpoonup} \mathbf{0} && \text{weakly* in } L^\infty(\Omega)^{d\times d} \end{aligned} \tag{A.8}$$

[1] For a function $f \in L^1(\mathbb{R}^d)$, we define the Hardy-Littlewood maximal function through $(\mathcal{M}f)(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \mathrm{d}y$.

*and for all $n, j \in \mathbb{N}$*

$$\|\nabla \boldsymbol{u}^{n,j} \chi_{\{\boldsymbol{u}^n \neq \boldsymbol{u}^{n,j}\}}\|_p \leq c \|\lambda_{n,j} \chi_{\{\boldsymbol{u}^n \neq \boldsymbol{u}^{n,j}\}}\|_p \leq c \frac{\gamma_n}{\theta_n} \mu_{j+1} + c\, \varepsilon_j, \tag{A.9}$$

*where $\varepsilon_j := K\, 2^{-j/p}$ vanishes as $j \to \infty$ and the constant $c$ depends on $\Omega$.*

Finally, we mention two lemmas that are useful for the proof of existence of solutions to the Galerkin system.

**Lemma A.4.** *Let $(\mathcal{U}, \cdot, |\cdot|)$ be a finite-dimensional Hilbert space and let the mapping $\phi : \mathcal{U} \to \mathcal{U}$ be continuous and satisfy*

$$\exists \rho > 0 : \phi(u) \cdot u \geq 0 \quad \forall u \in \partial B_\rho \subset \mathcal{U}.$$

*Then $\phi$ has at least one zero in $\overline{B_\rho}$, i.e., $\exists u \in \overline{B_\rho} : \phi(u) = 0$.*

**Proof.** Assume that $\phi$ has no zero in $\overline{B_\rho}$. Then the mapping

$$F : u \longmapsto -\rho \frac{\phi(u)}{|\phi(u)|}, \ u \in \overline{B_\rho}$$

is well-defined, continuous and $F(B_\rho) \subset \partial B_\rho \subset B_\rho$. Since $\mathcal{U}$ is a finite-dimensional space, then according to Brouwer's fixed point theorem there exists $u \in \overline{B_\rho}$ such that $F(u) = u$. Therefore,

$$u = -\rho \frac{\phi(u)}{|\phi(u)|} \in \partial B_\rho.$$

According to the assumption

$$0 \leq \phi(u) \cdot u = -\rho \frac{|\phi(u)|^2}{|\phi(u)|} < 0,$$

which leads to a contradiction. □

Let $\{\boldsymbol{a}^k\}_{k=1}^\infty$ be a basis in some separable space $W_0^{1,p}(\Omega)^d$ and let $N \in \mathbb{N}$ be fixed. Then we can prove the following.

**Lemma A.5.** *For every $K > 0$ there exists $\rho > 0$ such that for all $|\boldsymbol{d}^N| = \rho$ holds $\| \sum_{s=1}^{N} d_s^N \boldsymbol{a}^s \|_{1,p} \geq K$.*

**Proof.** We denote $j := \inf\limits_{|\boldsymbol{d}^N|=1} \| \sum\limits_{s=1}^{N} d_s^N \boldsymbol{a}^s \|_{1,p}$ and first show that $j > 0$.
Considering that $j = 0$ yields existence of a (sub)sequence $\boldsymbol{d}_k^N$ ($|\boldsymbol{d}_k^N| = 1$) such that

$$\lim_{k\to\infty} \sum_{s=1}^{N} d_{s,k}^N \boldsymbol{a}^s = 0 \quad \text{a.e. in } \Omega.$$

This, however, also means that

$$\sum_{s=1}^{N} \lim_{k\to\infty} d_{s,k}^N \boldsymbol{a}^s = 0 \quad \text{a.e. in } \Omega.$$

Since $\{\boldsymbol{a}^k\}_{k=1}^{\infty}$ is a basis, then necessarily $\lim\limits_{k\to\infty} d_{s,k}^N = 0$, $s = 1, \ldots, N$, which leads to a contradiction with $|\boldsymbol{d}_k^N| = 1$.
We set $\rho := \frac{K}{j}$ and finally conclude that

$$\begin{aligned} \inf_{|\boldsymbol{d}^N|=\rho} \| \sum_{s=1}^{N} d_s^N \boldsymbol{a}^s \|_{1,p} &= \inf_{|\boldsymbol{d}^N|=\frac{K}{j}} \| \sum_{s=1}^{N} d_s^N \boldsymbol{a}^s \|_{1,p} = \inf_{|\boldsymbol{d}^N|=\frac{K}{j}} \frac{K}{j} \| \sum_{s=1}^{N} \frac{j}{K} d_s^N \boldsymbol{a}^s \|_{1,p} \\ &= \frac{K}{j} \inf_{|\tilde{\boldsymbol{d}}^N|=1} \| \sum_{s=1}^{N} \tilde{d}_s^N \boldsymbol{a}^s \|_{1,p} = K. \end{aligned}$$

□

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