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Structural Theory of Graph Immersions

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Abstract: Immersion is a notion of graph inclusion related to the notion of graph minors. While the structural theory of graph minors is extensive, there are still numerous open problems in the structural theory of graph immersions. Kuratowski's theorem claims that the class of graphs that do not contain a subdivision of the graphs $K_{3,3}$ and K_5 is exactly the class of planar graphs. The main goal of this thesis is to describe the structure of the graphs that do not contain an immersion of $K_{3,3}$. Such graphs can be separated by small edge cuts into small graphs or planar 3-regular graphs.

Keywords: structural graph theory, graph immersions, decomposition theorems

Contents

Introduction

The concept of graphs immersions is similar to the concept of containment a graph as a subdivision. If a graph G contains a subdivision of a graph H, then we can assign the vertices of H to distinct vertices of G and find pairwise internally vertex-disjoint paths in G connecting these vertices if the corresponding vertices of H are connected by an edge, whereas if G contains an immersion of H, it is sufficient to find an edge-disjoint paths instead of internally vertex-disjoint ones (we will provide a more precise definition later). Observe that containment as a subdivision is a special case of containment as an immersion. Since paths of an immersion can intersect in some internal vertex, it is possible to make a crossing of two such paths; hence, we can find an immersion of non-planar graphs in planar graphs. Actually, every graph is immersed in some planar graph, since we can take a non-planar drawing of a graph and replace every crossing point with a new vertex subdividing crossed edges; therefore, the class of planar graphs cannot be described as a class forbidding some set of graphs as an immersion, while Kuratowski's theorem [1] claims that a graph is planar if and only if it does not contain a subdivision of K_5 (the complete graph on five vertices) or $K_{3,3}$ (the complete bipartite graph with parts of size three). More generally, Wagner [2] proved that a graph does not contain a subdivision of $K_{3,3}$ if and only if it can be obtained from planar graphs and copies of K_5 glued together by 2-clique-sums. The main goal of this thesis is to describe the structure of the graphs that do not contain an immersion of $K_{3,3}$.

While there are many results in the area of graph minors and subdivisions, the area of graph immersions is less developed. Robertson and Seymour [3] proved that graphs ordered by immersion relation form a well-quasi-ordering, confirming a conjencture of Nash-Williams [4]. Wollan [5] proved a structure theorem for graphs without an immersion of a fixed complete graph DeVos et al. [6] have given the minimum degree enforcing an immersion of a fixed complete graph and proved that dense graphs immerse large complete graphs. This bound have been improved by Dvořák and Yepremayen [7].

Giannopoulou, Kamiński, and Thilikos [8] proved that sufficiently connected graphs without an immersion of $K_{3,3}$ and K_5 are either planar and 3-regular or have branch width at most 10. However, branch-width is not a natural parameter for immersion-free graphs, since every graph can be immersed in a graph of branch-width 2. We prove that a graph does not immerse $K_{3,3}$ if and only if it can be obtained by connecting small $K_{3,3}$-immersion-free graphs and planar 3-regular graphs using certain operations such that individual graphs in the composition are separated by small edge cuts. Before this thesis has been finished, DeVos and Malekian [9] presented a precise characterization of graphs without an immersion of $K_{3,3}$. Although their results are more precise, the proof involves a lot of computer verification, whereas our solution is simpler and all proofs are made by hand.

1. Preliminaries

In this thesis graphs are undirected and may contain parallel edges but no loops. If we say that vertices u and v are connected by a simple edge, by a double edge or by a triple edge, we mean that there are exactly one, two or three edges connecting u and v. By a simple graph we mean a graph with no parallel edges. When considering a plane drawing of a graph, parallel edges are considered to be drawn close to each other. That means that if e_1 and e_2 are parallel edges connecting vertices x and y, then the region bounded by the drawing of e_1 and e_2 does not contain any vertex other than x and y.

For $X \subset V(G)$ we use $G - X$ to refer to the subgraph of G induced on $V(G) \setminus X$. For a vertex $v \in V(G)$ we use $G - v$ instead of $G - \{v\}$.

For $X \subset E(G)$ we use $G - X$ to refer to the subgraph of G created by deleting all the edges in X. For an edge $e \in E(G)$ we use $G - e$ instead of $G - \{e\}$. For an edge $e \in E(G)$ we use G/e to refer to the graph created from G by contracting the edge e.

If we talk about a smallest graph from a class of graphs $\mathcal{G}$, we mean a graph such that no graph from $\mathcal{G}$ has fewer vertices and no graph from $\mathcal{G}$ with the same number of vertices has fewer edges.

Definition 1. *We will use K_5 to denote the complete graph on five vertices and $K_{3,3}$ to denote the complete bipartite graph with parts of size three.*

If X is an edge cut in a graph G and $G - X$ has more than two components, it is not necessarily clear into which two parts we consider the graph to be split by the cut. To avoid this issue, we introduce the notion of separation.

Definition 2. *Let G be a connected graph. A* separation *is a pair (A, B) of non-empty disjoint sets $A, B \subset V(G)$ such that $V(G) = A \cup B$. We call A and B the* sides *of the separation. A* cut *for the separation (A, B) is the set $\delta(A, B)$ of the edges of G with one end in A and the other end in B. The* order *of the separation (A, B) is $|\delta(A, B)|$.*

We will occasionally use δH instead of $\delta(V(H), V(G) \setminus V(H))$ when H is an induced subgraph of G and also δA instead of $\delta(A, B)$.

Definition 3. *A graph G* weakly immerses *a graph H if there exist functions $\pi_V : V(H) \to V(G)$ and π_E mapping the edges of H to paths of G such that π_V is an injection, for an edge $e \in E(H)$ with endpoints v_1 and v_2, $\pi_E(e)$ is a path with endpoints $\pi_V(v_1)$ and $\pi_V(v_2)$ and all paths of $\pi_E(E(H))$ are pairwise edge disjoint. A vertex v of G is a* branch vertex *of the immersion if there exists a vertex $v_H \in V(H)$ such that $\pi_V(v_H) = v$. A path P in G is a* composite path *of the immersion if there exists an edge $e_H \in E(H)$ such that $\pi_E(e_H) = P$. A graph G* strongly immerses *a graph H if G weakly immerses H and composite paths of the immersion intersect branch vertices only in their endpoints.*

We will also say that G contains H as a *(weak) immersion*. In this thesis, we will consider only weak immersions; therefore, we will refer to weak immersions simply as immersions.

Definition 4. *Let e_1 and e_2 be edges of a graph such that the e_1 has endpoints x and y, and e_2 has endpoints y and z. By* splitting off *the edges e_1 and e_2 we mean deleting the edges e_1 and e_2 and adding a new edge with endpoints x and z.*

Observe that G immerses H if and only if H can be obtained from G by repeatedly splitting off pairs of edges, deleting edges and deleting isolated vertices.

Definition 5. *Let H_1 and H_2 be graphs with at least three vertices and let $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$ be vertices of the same degree d. A graph G is obtained from graphs H_1 and H_2 by a* join *on v_1 and v_2 if for a bijection π between the edges incident with v_1 and v_2, G is created from the disjoint union of $H_1 - v_1$ and $H_2 - v_2$ by adding the edge between the vertices incident with e and $\pi(e)$ distinct from v_1 and v_2 for each edge e incident with v_1. The separation $(V(H_1 - v_1), V(H_2 - v_2))$ in G is* associated *with the join.*

Definition 6. *A graph G is* internally 4-edge connected *if it is 3-edge connected and every separation of order 3 has a side consisting of a single vertex.*

Definition 7. *Let $\mathcal{C}_4$ be the class of internally 4-edge-connected graphs that do not immerse $K_{3,3}$. Let $\mathcal{P}$ denote the class of internally 4-edge-connected 3-regular planar graphs.*

Observe that if H is immersed in a 3-regular graph G, then a subdivision of H is a subgraph of G; consequently, all graphs immersed in the graphs from $\mathcal{P}$ are planar, and thus $K_{3,3}$ is not immersed in any graph from $\mathcal{P}$.

Lemma 1.1. *Suppose that G is a join of H_1 and H_2 on vertices v_1 and v_2 of degree at most three. If G immerses $K_{3,3}$, then H_1 or H_2 immerses $K_{3,3}$. If $H_1 - v_1$ and $H_2 - v_2$ are connected, then the converse holds as well.*

Proof. Let $(A, B) = (V(H_1 - v_1), V(H_2 - v_2))$ be the separation of G of order at most 3 associated with the join.

Suppose G contains $K_{3,3}$ as an immersion I. The cut $\delta(A, B)$ has size at most 3; therefore it intersects at most 3 composite paths of I. It follows that at most one of A and B contains more than one branch vertex of I since $K_{3,3}$ is internally 4-edge-connected. If say B does not contain any branch vertex, then at most one path of I intersects B. If B contains exactly one branch vertex v, then only the composite paths of I ending in v intersect B. In either case, contracting B to a single vertex transforms G to H_1 and I to an immersion of $K_{3,3}$ in H_1.

Suppose now that say H_1 immerses $K_{3,3}$ and $H_2 - v_2$ is connected. Let d denote the degree of v_2. Since $d \leq 3$, there exists a vertex $w \in B$ such that v_2 and w are connected by d pairwise edge-disjoint paths in H_2. It follows that G immerses H_1 with w and $V(H_1 - v_1)$ as branch vertices. Since the relation of immersion is transitive, G immerses $K_{3,3}$. □

2. Non-planar graphs

The case of non-planar graphs is quite simple, only non-planar 3-edge-connected graphs are those which have less vertices than $K_{3,3}$.

Lemma 2.1. *If G is a 3-edge-connected non-planar graph with at least 6 vertices, then G immerses $K_{3,3}$.*

Proof. By Kuratowski's theorem [1], G contains a subdivision of K_5 or $K_{3,3}$. The subdivision of $K_{3,3}$ is also an immersion of $K_{3,3}$; hence we can assume G contains a subdivision H of K_5.

Suppose first that $H = K_5$ is a subgraph of G. Since $|V(G)| > 5$, there exists a vertex $v \in V(G) \setminus V(H)$, and since G is 3-edge-connected, it contains three pairwise edge-disjoint paths from v to $V(H)$. It is easy to see that the graph G immerses $K_{3,3}$ with $V(H)$ and v as branch vertices.

Therefore, we can assume that at least one edge of K_5 is subdivided. Let P be a path in H corresponding to a subdivided edge of K_5. Let X be the subgraph of G induced by the internal vertices of P. Since G is 3-edge-connected, there is at least one path P' connecting X to the rest of H in $G - E(P)$. Let v_6 denote the endpoint of P' in X. Let v_1 and v_2 denote the endpoints of P, and v_3, v_4 and v_5 denote the other branch vertices of H. If P' intersects any of the paths connecting v_i to v_j where $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$, then G immerses $K_{3,3}$ with v_1, v_2 and v_j as the branch vertices of one part of $K_{3,3}$ and v_k, v_l and v_6 as the branch vertices of the second part of $K_{3,3}$, where $k, l \in \{3, 4, 5\}$ and $j \neq k \neq l \neq j$. Otherwise, P' intersects one of the paths connecting v_i and v_j in an internal vertex v_7, where $i \neq j \in \{3, 4, 5\}$, then G immerses $K_{3,3}$ with v_1, v_2 and v_7 as the branch vertices of one part of $K_{3,3}$ and v_i, v_j and v_6 as the branch vertices of the second part of $K_{3,3}$. □

Since every graph containing at most four vertices is planar, only non-planar 3-edge-connected graphs are K_5 and graphs obtained from K_5 by increasing multiplicity of edges.

3. Graphs of small edge-connectivity

The graph $K_{3,3}$ is internally 4-edge connected; therefore, we will describe, how to deal with graphs of weaker edge-connectivity.

Theorem 3.1. *A graph G does not immerse $K_{3,3}$ if and only if G is obtained from graphs in $\mathcal{C}_4$ by repeated application of the following operations:*

- *joins on vertices of degree at most three,*
- *disjoint unions,*
- *subdividing edges, and*
- *adding pendant vertices connected by single or double edges.*

Proof. By Lemma 1.1, joins on vertices of degree at most three cannot create an immersion of $K_{3,3}$, and clearly neither can disjoint unions, edge subdivisions, or adding pendant vertices of degree at most two. Hence, the graphs created as described do not immerse $K_{3,3}$.

Conversely, we will prove by induction on the number of vertices that every $K_{3,3}$-immersion-free graph can be created as described. Suppose G is $K_{3,3}$-immersion-free and the claim holds for all graphs with fewer than $|V(G)|$ vertices. If G is disconnected, then it is a disjoint union of its components, which are $K_{3,3}$-immersion-free; hence, the claim follows by the induction hypothesis. If G contains a vertex of degree one or two, then G is obtained from a smaller $K_{3,3}$-immersion-free graph by either subdividing an edge or adding a pendant vertex connected by a single or double edge, and again the claim follows by the induction hypothesis. Therefore, we can assume that G is connected and has minimum degree at least three. If G is internally 4-edge-connected, then $G \in \mathcal{C}_4$ and the lemma holds. Therefore, we can assume G has a separation (A, B) of order $k \leq 3$, and if $k = 3$, then $|A|, |B| \geq 2$. If $k \leq 2$, note that $|A|, |B| \geq 2$ as well, since G has minimum degree at least three. Consider such a separation with k minimum; then $G[A]$ and $G[B]$ are connected. In this case, G is a join of graphs H_1 and H_2 on vertices of degree k with associated separation (A, B), and by Lemma 1.1, the graphs H_1 and H_2 do not immerse $K_{3,3}$. Since $|A|, |B| \geq 2$, we have $|V(H_1)|, |V(H_2)| < |V(G)|$, and thus the claim follows by the induction hypothesis. □

Hence, we only need to describe the structure of graphs in $\mathcal{C}_4$.

4. Internally 4-edge-connected graphs

The connectivity of $K_{3,3}$ can be described even more precisely, since any separation of $K_{3,3}$ with sides containing three vertices has order at least 5.

Definition 8. *A graph G is* weakly 5-edge-connected *if it is internally 4-edge connected and every separation of order 4 has a side consisting of at most 2 vertices. Let $\mathcal{C}_5$ denote the class of weakly 5-edge-connected graphs that do not immerse $K_{3,3}$.*

Definition 9. *Let A be a subgraph of a graph G such that $\delta A = \{e_1, e_2, e_3, e_4\}$. We say that A* immerses an $\{e_1, e_2, e_3, e_4\}$-vertex *if G contains four pairwise edge-disjoint paths starting in e_1, e_2, e_3, e_4 and ending in the same vertex v of A. We say that A* immerses an (e_1, e_2, e_3, e_4)-span *if there exist distinct vertices $u, w \in A$ and pairwise edge-disjoint paths in G connecting e_1 and e_2 to u, e_3 and e_4 to w, and u to w. The vertices u, v and w are* branch vertices *of the immersion.*

Definition 10. *A graph G is obtained from graphs H_1 and H_2 with at least five vertices by a* special 4-join *if for $i \in \{1, 2\}$, F_i contains a vertex v_i of degree three connected by a triple edge to a vertex w_i of degree seven, and G is obtained from $F_1 - v_1$ and $F_2 - v_2$ by a join on w_1 and w_2. The separation $(V(F_1) \setminus \{v_1, w_1\}, V(F_2) \setminus \{v_2, w_2\})$ in G is* associated *with the special 4-join.*

Definition 11. *Let u and v be vertices connected by a double edge. By* pinching *the double edge uv we mean deleting these two edges and adding a new vertex w connected to u and v by double edges.*

Definition 12. *If H is a subgraph of G, then an* immersion H-bridge *in G is a connected component of the graph $G - E(H)$ with at least two vertices; hence, distinct vertices of G are connected by a path edge-disjoint from H if and only if they belong to the same immersion H-bridge. Note also that immersion H-bridges are pairwise vertex-disjoint. The* attachments *of an immersion H-bridge K are the vertices of $V(H) \cap V(K)$.*

Theorem 4.1. *A graph belongs to $\mathcal{C}_4$ if and only if either it belongs to $\mathcal{P}$ or it is obtained from graphs in $\mathcal{C}_5 \setminus \mathcal{P}$ by repeated application of the following operations:*

- *pinching a double edge between vertices of degree four, and*
- *special 4-joins.*

Proof. First, let us argue that the graphs obtained by the operations are internally 4-edge-connected and do not immerse $K_{3,3}$. Suppose $G \in \mathcal{C}_4$ and G' is a graph obtained from G by pinching a double edge between vertices u and v of degree 4, introducing a new vertex w. If G' is not internally 4-edge-connected, it contains a separation (A, B) of order 3 such that $|A|, |B| \geq 2$. Without loss of generality we can assume that $u, w \in A$. The separation $(A \setminus \{w\}, B)$ in G also has order 3. Since G is internally 4-edge-connected, we have $|A \setminus \{w\}| = 1$, and thus $A = \{u, w\}$. However, since the separation (A, B) in G' has order 3 and wv is a double edge, u has degree 3, which is a contradiction. Hence, G' is internally 4-edge-connected. Suppose now G' immerses $K_{3,3}$. Since this immersion does not appear in G, w is a branch vertex of the immersion. Since $\delta\{u,w,v\}$ has size 4 and any cut separating three vertices of $K_{3,3}$ has size at least 5, at most one of u and v can be a branch vertex of the immersion. We can assume that u is not a branch vertex; therefore, G'/uw also immerses $K_{3,3}$. This is a contradiction since G'/uw is isomorphic to G, and thus $G' \in \mathcal{C}_4$.

Suppose $G_1, G_2 \in \mathcal{C}_4$ and let G' be a graph obtained from G_1 and G_2 by a special 4-join with an associated separation (A, B) of order four. Let us consider the case that G' is not internally 4-edge-connected. Let (C, D) be a separation in G' of order at most three. If $A \cap C = \emptyset$, then $(C, V(G_2) \setminus C)$ is a separation of G_2 contradicting the assumption that G_2 is internally 4-edge-connected. Hence, $A \cap C \neq \emptyset$, and symmetrically $A \cap D, B \cap C, B \cap D \neq \emptyset$. Since G_1 contains at least five vertices, A contains at least three vertices, and since G_1 is internally 4-edge-connected, we have $|\delta(A \cap C)| + |\delta(A \cap D)| \geq 7$. Since $|\delta A| = 4$ and the edges connecting $A \cap C$ to $A \cap D$ are counted twice in the sum, the number of edges of G' connecting $A \cap C$ to $A \cap D$ is at least 2. By symmetry, there are at least two edges connecting $B \cap C$ to $B \cap D$. This is a contradiction, since $|\delta(C, D)| \leq 3$. Hence, G' is internally 4-edge-connected. If G' immerses $K_{3,3}$, then B cannot contain exactly three branch vertices, since any cut separating three vertices of $K_{3,3}$ has size at least 5. Hence, by symmetry we can assume that at most two branch vertices are contained in B. But then the part of the $K_{3,3}$ immersed in $G[B]$ can also be immersed in two vertices connected by a triple edge, and thus $K_{3,3}$ would also be immersed in G_1, which is a contradiction. It follows that $G' \in \mathcal{C}_4$. Thus, all graphs obtained from the described construction belong to $\mathcal{C}_4$.

Conversely, we will prove by induction on the number of vertices that every graph in $\mathcal{C}_4 \setminus \mathcal{P}$ can be obtained by the construction. Consider a graph $G \in \mathcal{C}_4 \setminus \mathcal{P}$. If $G \in \mathcal{C}_5$, then the claim is trivial, and thus we can assume that G contains a separation (A, B) of order four with $|A|, |B| \geq 3$. Since G contains at least 6 vertices, G is planar by Lemma 2.1. Since $G \notin \mathcal{P}$, we can assume A contains a vertex v of degree at least 4.

Consider some fixed plane drawing of G. Since G is 3-edge-connected, $G[B]$ is connected, and thus it is drawn inside a single face f of $G[A]$. Let e_1, e_2, e_3 and e_4 be the edges of $\delta(A, B)$ in the cyclic order according to the drawing of their ends in the boundary walk of f. Let $S = \{e_1, e_2, e_3, e_4\}$. Since G is internally 4-edge-connected, it contains pairwise edge-disjoint paths from v to S, and thus $G[A]$ immerses an S-vertex.

Consider now walks W_1 between e_1 and e_2 and W_2 between e_3 and e_4 in the boundary of f. If W_1 and W_2 share an edge e, then $\{e_1, e_4, e\}$ and $\{e_2, e_3, e\}$ are cuts in G of order 3, and since G is internally 4-edge-connected, A contains two vertices, which is a contradiction. Thus, W_1 and W_2 are edge-disjoint. If $E(G[A]) \cup S = E(W_1) \cup E(W_2)$, then all vertices of A belong to both W_1 and W_2, since G has minimum degree 2; consequently, $G[A]$ is a path of double edges. Let G_0 be a graph obtained from G by contraction of a double edge in $G[A]$; then G can be obtained from a graph G_0 by pinching a double edge. The graph G_0 belongs to $\mathcal{C}_4 \setminus \mathcal{P}$, since contractions preserve the edge-connectivity. The claim follows by the induction hypothesis. Therefore, we can assume there exists an immersion $(W_1 \cup W_2)$-bridge H in $G[A]$. If H is vertex-disjoint from W_1, then the first and the last of the edges of W_2 incident with any vertex of H forms a cut of size 2 in G, which is a contradiction with the 3-edge-connectivity of G. By symmetry, we conclude that both W_1 and W_2 are incident with an edge of H. Suppose there are at least two attachments of H, then H contains distinct vertices $v_1 \in V(W_1)$ and $v_2 \in V(W_2)$. A path in H connecting v_1 and v_2 and the paths W_1 and W_2 forms an immersion of (e_1, e_2, e_3, e_4)-span in $G[A]$ with branch vertices v_1 and v_2. Suppose now that the H has only one attachment v_1 contained in both W_1 and W_2. Let v_2 be a vertex of H distinct from v_1. Since G is 3-edge-connected and v_1 is a cut vertex in G, H contains three edge-disjoint paths between v_1 and v_2; therefore, $G[A]$ immerses (e_1, e_2, e_3, e_4)-span with branch vertices v_1 and v_2. By symmetry, we can also assume that $G[A]$ immerses an (e_1, e_4, e_2, e_3)-span.

Since $G[A]$ immerses an S-vertex, G contains edge-disjoint paths W_1 between e_1 and e_3 and W_2 between e_2 and e_4; choose such paths with $E(W_1 \cup W_2)$ minimal. Suppose that there exists an immersion $(W_1 \cup W_2)$-bridge H in $G[A]$. If H intersects both W_1 and W_2, then arguing as in the previous paragraph, we conclude that $G[A]$ immerses an (e_1, e_3, e_2, e_4)-span. If H intersects say only W_1, then consider the minimal subpath P of W_1 containing all the attachments of H. Since G is 3-edge-connected, P intersects W_2 in a vertex w. Let W_1' be a path obtained from W_1 by replacing the subpath P with a path connecting the endpoints of P in H. The path P is contained in an immersion $(W_1' \cup W_2)$-bridge H' in $G[A]$. Since H' intersects both W_1' and W_2, again $G[A]$ immerses an (e_1, e_3, e_2, e_4)-span. Finally, suppose that there exists no immersion $(W_1 \cup W_2)$-bridge in $G[A]$, and thus $E(G[A]) \cup S = E(W_1) \cup E(W_2)$. If $G[A]$ is a path of double edges, then G is obtained by pinching a double edge and the claim holds by the induction hypothesis. Otherwise, let xy be an edge of $W_2 - \{e_2, e_4\}$ not parallel to an edge of W_1, let P be the subpath of W_1 between x and y, let W_1' be obtained from W_1 by replacing P with xy and let W_2' be obtained from W_2 by replacing xy with P. Then W_1' and W_2' are edge-disjoint walks from e_1 to e_3 and from e_2 to e_4; but W_2' visits all internal vertices of P twice, and thus there exists a path W_2'' from e_2 to e_4 with $E(W_2'') \subset E(W_2')$, contradicting the assumption $E(W_1 \cup W_2)$ is minimal.

Therefore, we can assume $G[A]$ immerses all $(e_{\pi(1)}, e_{\pi(2)}, e_{\pi(3)}, e_{\pi(4)})$-spans, where π is any permutation of $\{1, 2, 3, 4\}$. If all vertices of B have degree 3 in G, then since $|B| \geq 3$ and G is internally 4-edge-connected, $G[B]$ is 2-edge-connected, and thus also 2-vertex-connected. Therefore, the outer face of $G[B]$ is bounded by a cycle K, and the edges of S are incident with distinct vertices of K. The cycle K together with the (e_1, e_3, e_2, e_4)-span in $G[A]$ gives an immersion of $K_{3,3}$ in G, which is a contradiction.

Therefore, at least one vertex of B has degree at least 4 in G, and thus by symmetry, we can also assume $G[B]$ immerses an S-vertex as well as all possible spans. For $X \in \{A, B\}$, let G_X be the graph obtained from $G[X]$ by adding vertices v_X and w_X such that v_X is connected to w_X by a triple edge and w_X is connected to the ends of edges of S with as many edges as is the number of edges of S incident with that vertex in G. Note that $\{v_X, w_X\}$ immerses a $(\delta\{v_X, w_X\})$-vertex and all possible spans in G_X. Suppose that G_A contains an immersion I of $K_{3,3}$. If at most one of v_A and w_A is a branch vertex of I, then the part of I immersed in $\{v_A, w_A\}$ is also immersed in $G[B]$, since B immerses an S-vertex. If both v_A and w_A are branch vertices of I, then the part of I immersed in $\{v_A, w_A\}$ is also immersed in $G[B]$, since B immerses the corresponding span. It follows that G immerses $K_{3,3}$, which is a contradiction. Symmetrically, G_B does not immerse $K_{3,3}$. Observe furthermore that G_A and G_B belong to $\mathcal{C}_4 \setminus \mathcal{P}$. Since G is a special 4-join of G_A and G_B, the claim follows by the induction hypothesis. $\square$

5. Weakly 5-edge-connected planar graphs

In this chapter, we will prove that the maximum number of vertices of graphs in $\mathcal{C}_5 \setminus \mathcal{P}$ is 20.

Definition 13. *A* bad separation *in a graph that is not weakly* 5*-edge-connected is a separation violating the conditions of weak* 5*-edge-connectivity. That means it is a separation of order at most* 2*, or a separation of order* 3 *with both sides containing at least* 2 *vertices, or a separation of order* 4 *with both sides containing at least* 3 *vertices. If* (A, B) *is a bad separation, we will say that the cut* $\delta(A, B)$ *is a* bad cut.

5.1 Graphs of maximum degree 4

Lemma 5.1. *If* G *is a weakly* 5*-edge-connected graph of maximum degree* 4 *with at least* 6 *vertices, then* G *is* 3*-vertex-connected.*

Proof. Observe that if G contains a cut vertex v, then each component of $G - v$ has to be connected to v by at least three edges making the degree of v at least 6.

Suppose that G contains a vertex cut $\{x, y\}$. Let A, B be the components of $G - \{x, y\}$ such that $|V(A)| \leq |V(B)|$. Assume that the number of the edges connecting x to A is at least the number of the edges connecting y to A. Since both x and y have maximum degree 4, the sum of $|\delta A|$ and $|\delta B|$ is at most 8.

Suppose first that both $|\delta A|$ and $|\delta B|$ are equal to 4. This implies that each of A and B has at most two vertices. The size of $\delta(A \cup \{x\})$ is at most 4; therefore, either A or B contains only one vertex, resulting in maximum 5 vertices in total.

Let us consider the case that $|\delta A|$ is equal to 3. The component A contains only one vertex connected by a double edge to x and by a single edge to y. However, the size of $\delta(A \cup \{x\})$ is at most 3, which is a contradiction.

If $|\delta B| = 3$, then the component B contains only one vertex and since $|V(A)| \leq |V(B)|$, the component A contains one vertex as well. It follows that G has at most 4 vertices.

□

Lemma 5.2. *Let* G *be a plane weakly* 5*-edge-connected graph of maximum degree* 4 *with at least* 6 *vertices. Each vertex* v *of degree* 4 *has either four distinct neighbors or* v *is connected by a double edge to some vertex* w *and* $\{v, w\}$ *has four distinct neighbors. Furthermore, every vertex of degree* 3 *has three distinct neighbors.*

Proof. No vertex can be incident with two double edges as its neighbors would form a vertex cut of size 2 contradicting Lemma 5.1. For the same reason each vertex of degree 3 has three distinct neighbors. Let a, b be vertices connected by a double edge. If a and b have a common neighbor c, then $|\delta\{a, b, c\}| \leq 4$; therefore, G can contain at most 5 vertices, since G is weakly 5-edge-connected.

□

Lemma 5.3. *Let G be a graph of maximum degree 4 such that G contains vertices v_1 and v_2 connected by a double edge. If G/v_1v_2 immerses $K_{3,3}$ then G immerses $K_{3,3}$. Furthermore, if G is weakly 5-edge-connected, then G/v_1v_2 is also weakly 5-edge-connected.*

Proof. The graph G immerses G/v_1v_2. Thus, every graph immersed in G/v_1v_2 is immersed in G as well.

Let (A, B) be a separation in G/v_1v_2 such that the vertex w created by the contraction is contained in A. A separation $(A \setminus \{w\} \cup \{v_1, v_2\}, B)$ in G has the same order as the separation (A, B) in G/v_1v_2 and $|A| < |A \setminus \{w\} \cup \{v_1, v_2\}|$; therefore, if the cut corresponding to the separation $(A \setminus \{w\} \cup \{v_1, v_2\}, B)$ in G is not a bad cut in G, then the cut corresponding to the separation (A, B) is not a bad cut in G/v_1v_2. □

Lemma 5.4. *Let G be a plane weakly 5-edge-connected graph of maximum degree 4 with at least 8 vertices. Suppose there exists a vertex v of degree 4 adjacent to distinct vertices a, b, c and d in the cyclic order. The graph G cannot contain all of the edges ab, bc, cd and da.*

Proof. Let C denote the set $\{a, b, c, d, v\}$. If G contains all of the edges ab, bc, cd and da, then each of the vertices a, b, c and d has three neighbors in C and at most one neighbor in $G - C$. The cut δC has size at most 4; therefore, $G - C$ contains at most two vertices and G contains at most 7 vertices. □

Lemma 5.5. *Let G be a plane weakly 5-edge-connected graph of maximum degree 4. Suppose there exists a vertex v of degree 4 adjacent to distinct vertices a, b, c and d in the cyclic order and let C denote the set $\{a, b, c, d, v\}$. Let F_1 and F_2 be faces of G such that $V(F_1) \cap V(F_2) = v$. If both F_1 and F_2 are incident with at least one vertex that is not contained in C, then G immerses $K_{3,3}$.*

Proof. The 3-vertex-connectivity implies that each pair of faces has at most two vertices in common and such vertices have to be adjacent, and the boundary of every face is a cycle.

Without loss of generality a and b are incident with the face F_1 and c and d are incident with the face F_2. Let $A = V(F_1) \setminus C$ and $B = V(F_2) \setminus C$. Let P_1, P_2, P_3 and P_4 denote the paths between vertices a and b, c and d, a and d, b and c, respectively, such that each path is contained in the boundary of a face incident with v and none of the paths contains v. Observe that the paths intersect only at their endpoints. Let G' be the graph created from G by removing the edges of P_3 and P_4, the edges incident with v and the edges of P_1 and P_2 incident with C. Let A' and B' be the components of G' containing A and B, respectively.

Suppose first that $A' = B'$. Let P_{xy} denote a shortest path connecting A and B in G'. Since $A' = B'$, such path exists. As P_{xy} is a shortest path, $P_{xy} \cap A$ is a single vertex x and $P_{xy} \cap B$ is a single vertex y. The graph G immerses $K_{3,3}$ with one part consisting of the vertices a, b and y, and the second part consisting of the vertices c, d and x.

Suppose now that $A' \neq B'$. Let P'_3 be the path obtained from P_3 by adding the edges incident with a and d in P_1 and P_2, respectively. Let P'_4 be the path obtained from P_4 by adding the edges incident with b and c in P_1 and P_2, respectively. Since G' is plane, we can divide the path P'_3 into two subpaths by removing an edge e_3 such that one subpath does not contain any vertices from B' and the other subpath does not contain any vertices from A'. We can divide the path P'_4 in a similar way by removing an edge e_4. The edges e_3 and e_4 together with one or two edges of C incident with v form a bad cut in G. □

Lemma 5.6. *Let G be a plane weakly 5-edge-connected graph of maximum degree 4 with at least 8 vertices. Suppose there exists a vertex v of degree 4 adjacent to distinct vertices a, b, c and d in the cyclic order. If G contains all of the edges bc, cd, da, then G immerses $K_{3,3}$.*

Proof. By Lemma 5.4, G cannot contain the edge ab.

Let C denote the set $\{a, b, c, d, v\}$. Since $\{a, b\}$ is not a vertex cut, c or d is connected by an edge to $G - C$.

Suppose first that d is connected to $G - C$, while c is not. Each of a and b has to be connected to $G - C$ by two edges, otherwise $|\delta C| \leq 4$ and G contains at most 7 vertices. It follows that c has degree 3.

Suppose that b is connected to two distinct vertices of $G - C$. The vertex b is not adjacent to a and d; thus, G meets assumptions of Lemma 5.5 for the vertex b as v.

Hence, b is connected to some vertex $b' \in G - C$ by a double edge. The vertex b' is not adjacent to d as $\{a, b'\}$ would be a vertex cut. It has to be adjacent to a, otherwise G/bb' meets the assumptions of Lemma 5.5 and by Lemma 5.3 G immerses $K_{3,3}$. The size of $\delta(C \cup b')$ is at most 3; hence G contains at most 7 vertices.

Therefore, we can assume that both c and d are connected to $G - C$. Let F_1 denote the face of G incident with a, b and v. Let F_2 denote the face of G incident with c and d not incident with v.

Consider the case that there exists exactly one vertex x incident with both F_1 and F_2. There have to be edges connecting x to both c and d, otherwise G meets the assumptions of Lemma 5.5 for the vertex x as v since x is not incident with v. There also have to be edges connecting x to both a and b, or else $\{a, x\}$ or $\{b, x\}$ would be a vertex cut of size 2. It follows that G contains at most 6 vertices.

If there are two vertices x_1 and x_2 incident with both F_1 and F_2, they have to be connected by a double edge, otherwise $\{x_1x_2, av, dv, cd\}$ is a bad cut. By Lemma 5.3, we can contract the edge x_1x_2 and get the previous case. It follows that G contains at most 7 vertices.

Thus, the faces F_1 and F_2 have no common incident vertices. Let $A = V(F_1) \setminus C$ and $B = V(F_2) \setminus C$. Since neither $\{a, b\}$ nor $\{c, d\}$ is a vertex cut, $G - C$ contains a path connecting A to B. Let P_{xy} denote a shortest such path. Since P_{xy} is a shortest path, $P_{xy} \cap A$ is a single vertex x and $P_{xy} \cap B$ is a single vertex y. The graph G immerses $K_{3,3}$ with one part consisting of the vertices a, b and y and the second part consisting of the vertices c, d and x.

□

Lemma 5.7. *Let G be a plane weakly 5-edge-connected graph of maximum degree 4 with at least 9 vertices. Suppose there exists a vertex v of degree 4 adjacent to distinct vertices a, b, c and d in the cyclic order. If G contains both of the edges cd, da then it immerses $K_{3,3}$.*

Proof. We already proved the case that G contains either of the edges ab or bc in Lemma 5.6. Let C denote the set $\{a, b, c, d, v\}$.

Suppose first that the vertex d has degree 3. The vertices a and c have degree 4 and they are not connected by an edge, otherwise $|\delta\{a, c, d, v\}| \leq 4$ and G contains at most 6 vertices. The vertex a is connected by a double edge to some vertex a', or else G meets the assumptions of Lemma 5.5 with the vertex a, since the vertices b and c are not adjacent to a. For the same reason c is connected by a double edge to some vertex c'. The vertices a' and c' are adjacent to b, or else G/aa' and G/cc', respectively, meets the assumptions of Lemma 5.5, resulting in G immersing $K_{3,3}$ by Lemma 5.3. The size of $\delta(C \cup \{a', c'\})$ is at most 3; hence, G can contain at most 8 vertices.

It follows that d has degree 4. Neither ad nor cd is a double edge, otherwise $|\delta\{a, c, d, v\}| \leq 4$ and G contains at most 6 vertices. Let d' denote the neighbor of d distinct from a, c and v. The vertices a and c are not connected by an edge and at least one of them has degree 4, otherwise the size of the $|\delta\{a, c, d, v\}| \leq 4$ and G contains at most 6 vertices. Assume without loss of generality that a has degree 4. If ad' is an edge, G meets the assumption of Lemma 5.6 with the vertex d as v. Thus, the vertex a is connected by a double edge to some vertex a', or else G meets the assumptions of Lemma 5.5 with the vertex a, since the vertices b and d' are not adjacent to a. Consider the graph G/aa'. The new vertex created by the contraction has to be adjacent to b or d', otherwise G/aa' meets the assumptions of Lemma 5.5. In both cases G/aa' meets the assumptions of Lemma 5.6 at v or d; therefore, G/aa' immerses $K_{3,3}$. By Lemma 5.3, G immerses $K_{3,3}$ as well. □

Theorem 5.8. *Let G be a planar weakly 5-edge-connected graph of maximum degree 4 that does not immerse $K_{3,3}$. Then G contains at most 9 vertices.*

Proof. Suppose that G contains a vertex v of degree 4 with distinct neighbors. Let C denote the union of v and its neighbors.

Let us consider the faces incident with v. Each of them either contains a vertex not adjacent to v or not. We proved that all such combinations result in G immersing $K_{3,3}$ or containing at most 8 vertices using Lemmas 5.4, 5.5, 5.6 and 5.7.

If G does not contain a vertex of degree 4 with distinct neighbors, we choose vertices v_1 and v_2 of degree 4 connected by a double edge. Such vertices exist according to Lemma 5.2. By Lemma 5.3, we can use the previous case for G/v_1v_2, resulting in G immersing $K_{3,3}$ or containing at most 9 vertices. □

5.2 Graphs of maximum degree 5

Definition 14. *Let G be a planar weakly 5-edge-connected graph. Let e_1 and e_2 be edges of G incident with a common vertex v. Let v_1 and v_2 denote the endpoints of e_1 and e_2, respectively, distinct from v. A* fragile cut *F for the edges e_1 and e_2 is a cut δX for $X \subset V(G)$ such that $v_1, v_2 \notin X$, $v \in X$ and δX in the graph created by splitting off the edges e_1 and e_2 is a bad cut.*

Observe that if F is a fragile cut for edges e and f, it contains e, f and all edges parallel to them. Furthermore, F contains at most 6 edges, since the corresponding bad cut $F \setminus \{e, f\}$ contains at most four edges.

Lemma 5.9. *Let G be a plane weakly 5-edge-connected graph. If F is a fragile cut for edges e and f, and A and B are sides of F, then both $G[A]$ and $G[B]$ are connected.*

Proof. Suppose that A_1 and A_2 are two components of $G[A]$. If F contains at most 5 edges, at least one of δA_1 and δA_2 contains at most two edges; hence, G is not weakly 5-edge-connected. If F contains 6 edges, both A_1 and A_2 contain only one vertex. Let G' denote the graph created from G by splitting off the edges e and f. The cut $F \setminus \{e, f\}$ in G' contains four edges and A contains two vertices; therefore, $F \setminus \{e, f\}$ is not a bad cut and F is not a fragile cut. □

Lemma 5.10. *Let G be a plane weakly 5-edge-connected graph and let v be a vertex of G of degree 5. Let e_1, e_2, e_3, e_4 and e_5 denote the edges incident with v in the cyclic order. If F is a minimum fragile cut for the edges e_1 and e_3, then $e_2 \in F$ and $e_4, e_5 \notin F$.*

Proof. Let v_1, v_2, v_3, v_4 and v_5 denote the vertices distinct from v incident with the edges e_1, e_2, e_3, e_4 and e_5, respectively. These vertices do not have to be distinct. The cut $\delta\{v\}$ is not a fragile cut, thus F contains at most four edges incident with v.

Suppose that F contains four edges incident with v. Let e denote the edge incident with v such that $e \notin F$ and let w denote the vertex incident with e distinct from v. The cut F contains at most two edges not incident with v. Let us call them e_x and e_y. Observe that $\{e, e_x, e_y\}$ is an edge cut of size 3, and thus e_x and e_y are incident with w; consequently, F is $\delta\{v, w\}$ and it is not a fragile cut. Thus, any minimum fragile cut can contain at most three edges incident with v.

Observe that if $v_1 = v_2$ or $v_2 = v_3$, then $e_2 \in F$; hence, we can assume $v_1 \neq v_2 \neq v_3$.

Let A denote the side of F containing v_1 and v_3 and B denote the side containing v. If $v_2 \in A$, the lemma holds. Suppose that $v_2 \in B$. Since G is a plane graph and $G[A]$ is connected by Lemma 5.9, $\{e_2\}$ is an edge cut of $G[B]$. Let B_1 and B_2 denote the components of $G[B] - \{e_2\}$ containing v_2 and v, respectively. Let E_{AB_1} denote the set of edges connecting A and B_1. The set E_{AB_1} contains at least two edges. At least one of v_4 and v_5 is contained in B_2 as F contains at most three edges incident with v. If exactly one of v_4 and v_5 is contained in A and the other one in B_2, let us assume that $v_4 \in A$, then $\{e_5\}$ is an edge cut in $G[B]$ and at least two edges are connecting A and the component of $G[B] - e_5$ containing v_5; therefore, F has size at least 7; hence F cannot be a fragile cut. If both v_4 and v_5 are contained in B_2, F is not a minimum fragile cut since $F \setminus E_{AB_1} \cup \{e_2\}$ is also a fragile cut.

□

Lemma 5.11. *Let G be a plane weakly 5-edge-connected graph and let v be a vertex of G of degree 5. If there exists a fragile cut for every pair of the edges incident with v, then v is incident with 5 simple edges.*

Proof. Let e_1, e_2, e_3, e_4 and e_5 denote the edges incident with v in the cyclic order. If e_1 and e_2 are parallel edges, a minimum fragile cut for the edges e_2 and e_4 contains e_1, which is a contradiction with Lemma 5.10. □

Lemma 5.12. *Let G be a plane weakly 5-edge-connected graph and let v be a vertex of G of degree 5. Let e_1, e_2, e_3, e_4 and e_5 denote the edges incident with v in the cyclic order. If F is a minimum fragile cut for the edges e_1 and e_3, then $F' = F \setminus \{e_1, e_2, e_3\} \cup \{e_4, e_5\}$ is a fragile cut for e_4 and e_5.*

Proof. Let A and B be sides of F such that $v \in B$. Then $A' = A \cup \{v\}$ and $B' = B \setminus \{v\}$ are sides of F'. By Lemma 5.10, $e_2 \in F$ and $e_4, e_5 \notin F$; therefore $|F'| = |F| - 1$. The cut F' contains at most 5 edges and B' contains at least two vertices by Lemma 5.11; therefore, F' is a fragile cut. ☐

Lemma 5.13. *Let G be a plane weakly 5-edge-connected graph. Let $A \subset V(G)$ such that $|\delta A| = 5$ and $G - A$ contains more than one vertex. If A contains more than three vertices, then $G[A]$ is 2-edge-connected.*

Proof. Suppose $G[A]$ is not 2-edge-connected. Thus, it contains a bridge separating it into subgraphs X and Y. Sum of $|\delta X|$ and $|\delta Y|$ is 7 as they count only the edges of δA and the bridge. Hence, one of the cuts δX and δY has size four and the other one has size 3. Since $G - A$ contains more than one vertex, one of X and Y contains at most two vertices and the other one contains only one vertex; therefore, A contains at most three vertices. ☐

Lemma 5.14. *Let G be a plane weakly 5-edge-connected graph of maximum degree 5. Let $A \subset V(G)$ such that $|\delta A| = 5$ and $G - A$ contains more than one vertex. Let p_1, q_1, q_2, p_2 and r denote the edges of δA in the cyclic order with respect to $G[A]$. If G contains at least four vertices, then:*

1. *$G[A]$ contains distinct vertices v_P and v_Q such that there exists a set of edge-disjoint paths connecting v_P and v_Q, v_P and p_1, v_P and p_2, v_Q and q_1 and either v_Q and q_2, or v_Q and r; and,*
2. *$G[A]$ contains distinct vertices v'_P and v'_Q such that there exists a set of edge-disjoint paths connecting $v_{P'}$ and $v_{Q'}$, $v_{P'}$ and p_1, $v_{Q'}$ and q_1, $v_{Q'}$ and q_2 and either $v_{P'}$ and p_2, or $v_{P'}$ and r.*

Proof. Let us assume that A contains at least four vertices. Let us consider the boundary walk C of the outer face of $G[A]$. Since $G[A]$ is 2-edge-connected by Lemma 5.13, edges do not repeat on C. Without loss of generality we can assume that p_1, q_1, q_2, p_2 and r lie in the outer face of $G[A]$.

If p_1 and q_1 are not incident with a common vertex in C, we can denote the vertex incident with p_1 by v_P or $v_{P'}$, respectively, and the vertex incident with q_1 by v_Q or $v_{Q'}$, respectively, Otherwise, if p_2 and q_2 are not incident with a common vertex, we can denote the vertex incident with p_2 by v_P or $v_{P'}$, respectively, and the vertex incident with q_2 by v_Q or $v_{Q'}$, respectively. If p_2 and r are not incident with a common vertex but p_1 and q_1 are incident with a common vertex, we can denote the vertex incident with p_2 by v_P or $v_{Q'}$, respectively, and the vertex incident with r by v_Q or $v_{P'}$, respectively. Then we can find required paths along the walk C.

Let us consider the case the edges p_1 and q_1 are incident with a common vertex w_1 and p_2, q_2 and r are incident with a common vertex w_2. If $w_1 = w_2$, then $G[A]$ contains only one vertex, since G has maximum degree 5, which is a contradiction. Otherwise, since degree of w_2 is at most 5, the edges p_1, q_1 and unnamed edges incident with w_2 forms a cut of size at most 4; therefore, A contains at most three vertices, which is a contradiction. ☐

Lemma 5.15. *Let G be a plane weakly 5-edge-connected graph of maximum degree 5. Let $A \subset V(G)$ such that $G[A]$ is connected and $|\delta A| = 6$. Let q_1, q_2, p_2, p'_2, p'_1 and p_1 denote the edges of δA in the cyclic order with respect to $G[A]$*

such that q_1 and q_2 do not share a common vertex in A. If $G[A]$ contains at least 5 vertices, it contains distinct vertices v_P and v_Q such that there exists a set of edge-disjoint paths connecting v_P and v_Q, v_Q and q_1, v_Q and q_2, either v_P and p_1, or v_P and p'_1 and either v_P and p_2, or v_P and p'_2.

Proof. Suppose first that $G[A]$ is not 2-edge-connected. There is a bridge b separating $G[A]$ into subgraphs X and Y. Sum of $|\delta X|$ and $|\delta Y|$ is 8 as δX and δY contain only edges of δA and b. If $|\delta X| = 4$ and $|\delta Y| = 4$, $G[A]$ contains at most four vertices. Let us assume without loss of generality that $|\delta X| = 3$ and $|\delta Y| = 5$. Since X contains only one vertex, it cannot be incident with both q_1 and q_2. If X is incident with p_1 and p'_1, we can find vertices v_P and v_Q in Y by Lemma 5.14 with p'_2 as r and the b as p_1. The case that X is incident with p_2 and p'_2 is symmetric. If X is incident with p_1 and q_1, we can find vertices v_P and v_Q in Y by Lemma 5.14 with p'_1 as p_1, p'_2 as r and b as q_1. The case that X is incident with p_2 and q_2 is symmetric. If X is incident with p'_1 and p'_2, we can find vertices v_P and v_Q in Y by Lemma 5.14 with b as r.

Suppose that $G[A]$ is 2-edge-connected. Let us consider the boundary walk C of the outer face. Since $G[A]$ is 2-edge-connected, edges do not repeat on C. Without loss of generality we can assume that q_1, q_2, p_2, p'_2, p'_1 and p_1 lie in the outer face of $G[A]$. Let x and y denote the vertices of $G[A]$ incident with q_1 and q_2, respectively. If p_1, p'_1, p_2 and p'_2 are all incident with either x or y, then $G[A]$ contains at most four vertices, as $G[A]$ is 2-edge-connected and G has maximum degree 5. Furthermore, if p'_1 is incident with x then p_1 is also incident with x, and neither p_1 nor p'_1 is incident with y. Symmetrically, if p'_2 is incident with y then p_2 is also incident with y, and neither p_2 nor p'_2 is incident with x. If p'_1 is incident with vertex z distinct from x, we can denote z as v_P and x as v_Q and find required paths along the walk C. The case that p'_2 is not incident with y is symmetric. □

Lemma 5.16. *Let G be a plane weakly 5-edge-connected graph of maximum degree 5 with at least 11 vertices and let v be a vertex of G of degree 5. If there exists a fragile cut for every pair of the edges incident with v, then G immerses $K_{3,3}$.*

Proof. Let e_1, e_2, e_3, e_4 and e_5 denote the edges incident with v in the cyclic order. Let v_1, v_2, v_3, v_4 and v_5 denote the vertices distinct from v incident with the edges e_1, e_2, e_3, e_4 and e_5, respectively. By Lemma 5.11 all the vertices v_1, v_2, v_3, v_4 and v_5 are distinct. Let us consider minimum fragile cuts F_1 and F_2 for e_1, e_3 and e_2, e_4, respectively. Let A_1, B_1 be the sides of F_1 such that v_1, v_2, $v_3 \in A_1$ and v, v_4, $v_5 \in B_1$. Let A_2, B_2 be the sides of F_2 such that v_2, v_3, $v_4 \in A_2$ and v, v_1, $v_5 \in B_2$. Let $AA = A_1 \cap A_2$, $AB = A_1 \cap B_2$, $BA = B_1 \cap A_2$ and $BB = B_1 \cap B_2 \setminus \{v\}$. Since $G[A_1]$, $G[A_2]$, $G[B_1 \setminus \{v\}]$ and $G[B_2 \setminus \{v\}]$ are all connected by Lemma 5.9 and Lemma 5.12, there is at least one edge connecting AA to AB, AA to BA, AB to BB and BA to BB. Let us denote these edges by e_{A_1}, e_{A_2}, e_{B_2} and e_{B_1}, respectively. The cut F_1 contains the edges e_1, e_2, e_3, e_{A_2} and e_{B_2} and the cut F_2 contains the edges e_2, e_3, e_4, e_{A_1} and e_{B_1}. As F_1 and F_2 are both fragile cuts, they contain at most one other edge.

Let us distinguish the following cases:

- Let us consider the case that $|\delta AA| = 4$. The set AA contains only two vertices.

Suppose first that $|\delta BB| = 3$. The set BB contains only one vertex. If there is an edge connecting AB and BA, then $|\delta AB| = |\delta BA| = 4$; hence, the sets AB and BA contain at most two vertices. If there is no such edge, then $|\delta AB| = |\delta BA| = 3$; hence, the sets AB and BA contain only one vertex. In both cases the graph G contains at most 8 vertices.
Suppose that $|\delta BB| = 4$. Without loss of generality there is only one edge connecting BB and AB. The fragile cut F_2 contains only 5 edges. The cut $\delta(B_2 \setminus \{v\})$ has size 4; hence, the sets AB and BB contain only one vertex. The cut δBA has size 3; hence, the set BA contains only one vertex. The graph G then contains at most 6 vertices.
Suppose that $|\delta BB| = 5$. Since the fragile cuts F_1 and F_2 contain at most 6 edges, BB and BA are connected by two edges and BB and AB are connected by two edges. The cut δBA has size 4; hence, the set BA contains at most two vertices. The set $B_2 \setminus \{v\}$ contains more than three vertices, otherwise G contains at most 8 vertices. In $G[B_2 \setminus \{v\}]$ we find vertices $v_{P'}$ and $v_{Q'}$ by Lemma 5.14 with e_{A_1} as p_1, e_1 as q_1, e_5 as q_2, and the edges connecting BA and BB as p_2 and r. Let us label the vertices $v_{P'}$, v and v_3 with P and the vertices $v_{Q'}$ and v_2 with Q. We can then label one vertex of BA with Q in such a way that the graph G immerses $K_{3,3}$ with parts created by equally labeled vertices.

- Let us consider the case that AA is connected to BB. Since fragile cuts F_1 and F_2 contain at most 6 edges, the cuts δAB and δBA have size 3; hence, the sets AB and BA contain only one vertex. The cut δBB has size 4; hence, the set BB contains at most two vertices. The set AA contains more than three vertices, otherwise G contains at most 8 vertices. In $G[AA]$ we find vertices v_P and v_Q by Lemma 5.14 with e_{A_2} as p_1, e_{A_1} as p_2, e_3 as q_1, e_2 as q_2, and the edge connecting AA and BB as r. Let us label the vertices v_P and v with P and vertices v_Q, v_1 and v_4 with Q. We can then label one vertex of BB with P in such a way that the graph G immerses $K_{3,3}$ with parts created by equally labeled vertices.
- Let us consider the case that AA is connected by two edges to BA. The case that AA is connected by two edges to AB is symmetric.
  Suppose first that F_2 contains only 5 edges. The cut δBA has size 4; hence, the set BA contains at most two vertices. The cuts δAB and δBB have size 3; hence, the sets AB and BB contain only one vertex. The set AA contains more than three vertices, otherwise G contains at most 8 vertices. In $G[AA]$ we find vertices $v_{P'}$ and $v_{Q'}$ by Lemma 5.14 with e_{A_1} as p_1, e_2 as q_1, e_3 as q_2, and the edges connecting AA and BA as p_2 and r. Let us label the vertices $v_{P'}$, v and v_5 with P and the vertices $v_{Q'}$ and v_1 with Q. We can then label one vertex of BA with Q in such a way that the graph G immerses $K_{3,3}$ with parts created by equally labeled vertices.
  Suppose that AA is connected to AB by two edges. The cuts δAB and δBA have size 4; hence, the sets AB and BA contain at most two vertices. The cut δBB has size 3; hence, the set BB contains only one vertex. The set AA contains more than four vertices, otherwise G contains at most 10 vertices. In $G[AA]$ we find vertices v_P and v_Q by Lemma 5.15 with e_3 as q_1, e_2 as q_2, the edges connecting AA and AB as p_2 and p_2', and the edges connecting AA and BA as p_1' and p_1. Let us label the vertices v_P, v and v_5 with P and the

vertex v_Q with Q. We can then label one vertex of AB and one vertex of BA with Q in such a way that the graph G immerses $K_{3,3}$ with parts created by equally labeled vertices.

Suppose that BA is connected to BB by two edges. If AA contains more than three vertices, in $G[AA]$ we find vertices $v_{P'}$ and $v_{Q'}$ by Lemma 5.14 with e_{A_1} as p_1, e_2 as q_1, e_3 as q_2, and the edges connecting AA and BA as p_2 and r. Let us label the vertices $v_{P'}$ and v with P and the vertices $v_{Q'}$ and v_1 with Q. We can then label one vertex of BB with P and one vertex of BA with Q in such a way that the graph G immerses $K_{3,3}$ with parts created by equally labeled vertices. Therefore, $G[AA]$ contains at most three vertices. Let v_{A_1} denote the vertex of AA incident with the edge e_{A_1}. If BA contains more than three vertices, in $G[BA]$ we find vertices $v_{P'}$ and $v_{Q'}$ by Lemma 5.14 with e_4 as q_2, the edges connecting AA and BA as p_1 and q_1, and the edges connecting BA and BB as p_2 and r. Let us label the vertices $v_{P'}$, v and v_1 with P and the vertices $v_{Q'}$ and v_{A_1} with Q. We can label one vertex of BB in such a way that the graph G immerses $K_{3,3}$ with parts created by equally labeled vertices. Then BA contains at most three vertices and G contains at most 10 vertices.

In order to find an immersion of $K_{3,3}$ in G, we have to find paths connecting v_{A_1} to all the vertices labeled with P and a path connecting v to $v_{Q'}$ intersecting AA. All these paths have to be pairwise disjoint. Consider the boundary walk W of the outer face of $G[AA]$. Let e_{A_2} and e'_{A_2} denote the edges connecting AA to BA such that e_{A_2} is contained in the boundary of a face incident with v. Let v_{A_2} and v'_{A_2} denote the vertices of AA incident with e_{A_2} and e'_{A_2}, respectively. Let P_1, P_2 and P_3 denote the subwalks of W between v_{A_2} and v_3, v_2 and v_{A_1}, and v_{A_1} and v'_{A_2}, respectively. Observe that the walks P_1, P_2 and P_3 are paths. If $G[AA]$ is 2-edge connected, edges do not repeat on W; therefore, P_1, P_2 and P_3 are pairwise edge disjoint. Suppose that $G[AA]$ is not 2-edge-connected; hence, some of the paths can share an edge. If P_1 and P_2 share an edge b, then $\{b, e_2, e_3\}$ is a bad cut. If P_1 and P_3 share an edge b, then b, e_4 and the edges connecting BA to BB forms a bad cut. If P_2 and P_3 share an edge b, then $\{b, e_{A_1}\}$ is a bad cut. It follows that P_1, P_2 and P_3 are pairwise disjoint; therefore, we can find the paths of immersion such that their intersection with $G[AA]$ is one of the paths P_1, P_2 and P_3.

□

Lemma 5.17. *Let G be a planar weakly 5-edge-connected graph that does not immerse $K_{3,3}$ such that exactly one vertex v of G has degree 5 and all other vertices have degree 3. Then G contains at most 10 vertices.*

Proof. Suppose first that v is incident with exactly one double edge. If we contract this edge, we get a planar weakly 5-edge-connected graph G' of maximum degree 4 that does not immerse $K_{3,3}$, since G' is immersed in G. By Theorem 5.8, G' has at most 9 vertices; therefore, G has at most 10 vertices. Suppose now that v is incident with two double edges. Let v_1 and v_2 denote the vertices connected by double edges to v. Then $|\delta\{v, v_1, v_2\}| = 3$, hence G contains at most four vertices. Since G is 3-edge-connected, v cannot be incident with a triple edge. Thus, we can assume that all edges incident with v are simple edges.

If $G - v$ is not 2-edge-connected, then it contains a bridge b. Let A be the component of $G-v-b$ in which v has fewer neighbors. If A contains one neighbor of v, then $|\delta A| = 2$, hence δA is a bad cut. If A contains two neighbors of v, then $|\delta A| = 3$ and $|A| \geq 2$, hence δA is again a bad cut. Consequently, $G - v$ is 2-edge-connected.

Let $v_1, v_2, \dots, v_5$ denote the vertices adjacent to v in the cyclic order according to the drawing of G. The boundary of the face of $G-v$ containing all the neighbors of v is a closed tour T. For $i \in \{1, 2, \dots, 5\}$ let T_i denote the subtour of T between v_i and v_{i+1} and let T_5 denote the subtour between v_5 and v_1. We can assume that G is drawn in such a way that each T_i is a path (as otherwise it would contain a cut vertex of G, and we could draw the component attaching to this vertex inside a different face). Since $G - v$ is 2-edge-connected, the tours $T_1, T_2, \dots T_5$ are pairwise edge-disjoint.

Let C be a component of $G-v-E(T)$ such that C contains at least 2 vertices.

Suppose first that for some vertex-disjoint T_i and T_j the component C contains a vertex x_i from T_i and a vertex x_j from T_j. Note that neither x_i nor x_j is an endpoint of T_i or T_j since the endpoints have degree 3. Therefore, G immerses $K_{3,3}$ with branch vertices $v_i, v_{i+1}, v_j, v_{j+1}, x_i$ and x_j.

Suppose that C contains a vertex of only two consecutive paths T_i and T_{i+1} (or T_1 and T_5). Let f_1 be the first edge and f_2 be the last edge of $T_i \cup T_{i+1}$ that is incident with a vertex of C. Then $\{f_1, f_2, e_{i+1}\}$ is a bad cut in G.

Finally, suppose that C contains a vertex of only one of T_i. The first edge and the last edge of T_i that is incident with a vertex of C forms a bad cut in G.

It follows that $G - v - E(T)$ contains no edges; therefore, $G = G[\{v\} \cup V(T)]$ and since G is 3-edge-connected, G contains only 6 vertices. □

Theorem 5.18. *Let G be a planar weakly 5-edge-connected graph of maximum degree 5 that does not immerse $K_{3,3}$. Then G contains at most 10 vertices.*

Proof. Suppose that we have a smallest counterexample graph G and let v be a vertex of G of degree 5. If G contains only one vertex of degree 5 and all other vertices have degree 3, G contains at most 10 vertices by Lemma 5.17. Hence, we can assume G contains another vertex of degree at least 4. By Lemma 5.16, we can split off two edges incident with v and get a weakly 5-edge-connected graph G'. If G' is not a planar graph, it immerses $K_{3,3}$ by Lemma 2.1. If G' has maximum degree 4, it immerses $K_{3,3}$ by Theorem 5.8. The graph G' has the same number of vertices and fewer edges than graph G; therefore, if G' has maximum degree 5, it immerses $K_{3,3}$. Since the relation of being immersed is transitive, G immerses $K_{3,3}$, which is a contradiction. □

5.3 Graphs of maximum degree 6 or more

Definition 15. *Let v be a vertex of degree d. Let $e_1, \dots, e_d$ denote the edges incident with v in the cyclic order. The edges e_i and e_j where $i, j \in \{1, \dots, d\}$, $i < j$ are* opposite *to each other if* $\min\{j-i, d+i-j\} = \left\lfloor \frac{d}{2} \right\rfloor$.

Observe that for even d every edge incident with v has only one opposite edge, and for odd d every edge incident with v has two opposite edges.

Definition 16. *Let v be a vertex of degree d. Let $e_1, \dots, e_d$ denote the edges incident with v in the cyclic order. A fragile cut F is* tight *if for every pair of* *edges $e_i, e_j \in F$ where $i, j \in \{1, \ldots, d\}$ and $i < j$, holds that F contains either all the edges $e_{i+1}, \ldots, e_{j-1}$ or all the edges $e_{j+1}, \ldots, e_d, e_1, \ldots, e_{i-1}$.*

Lemma 5.19. *Let G be a plane weakly 5-edge-connected graph and let v be a vertex of G. If there exists a fragile cut for every pair of the edges incident with v, then v is not a cut vertex.*

Proof. Suppose that X and Y are components of $G - v$. Let e_X and e_Y be edges connecting v to X and Y, respectively. There is no fragile cut for the edges e_X and e_Y since both sides of a minimal such cut would have to be connected by Lemma 5.9. □

Observe that no fragile cut can contain only edges incident with a vertex v, since v is not a cut vertex by Lemma 5.19.

Lemma 5.20. *Let G be a plane weakly 5-edge-connected graph and let v be a vertex of G of degree 7. If there exists a tight fragile cut for every pair of opposite edges incident with v, then G immerses $K_{3,3}$.*

Proof. If a fragile cut F contains 5 edges incident with v, there is a cut of size 3 consisting of the edge of F not incident with v and two edges incident with v not contained in F. Thus, F has a side with two vertices; hence, it is not a fragile cut. Therefore, each fragile cut contains at most four edges incident with v. This implies that all edges incident with v are simple.

If $G - v$ is not 2-edge-connected, then it contains a bridge b. Let A be the component of $G - v - b$ in which v has fewer neighbors. Note that $|A|$ is at least the number of edges of δA incident with v, since all edges incident with v are simple. The cut δA is a bad cut since it contains at most four edges and the number of vertices in A is at least the number of the edges in δA minus 1; consequently, $G - v$ is 2-edge-connected.

The boundary of the face of $G - v$ containing all the neighbors of v is a closed tour; therefore, we can find an immersion of $K_{3,3}$ with 6 neighbors of v as branch vertices. □

Lemma 5.21. *Let G be a planar weakly 5-edge-connected graph. Let v be a vertex of G of degree 9 such that all edges incident with v are simple edges. Then G immerses $K_{3,3}$.*

Proof. Suppose that $G - v$ is not connected. Let A be the component of $G - v$ with minimum $|\delta A|$; hence, $|\delta A| \leq 4$. Since G is weakly 5-edge-connected, either $|\delta A| = 3$ and A contains only one vertex or $|\delta A| = 4$ and A contains at most two vertices. In both cases δA cannot contain only simple edges, which is a contradiction.

If $G - v$ is 2-edge-connected, the boundary of the face of $G - v$ containing all the neighbors of v is a closed tour; therefore, we can find an immersion of $K_{3,3}$ in G with 6 neighbors of v as branch vertices.

It follows that $G - v$ is not 2-edge-connected. Let C_1 and C_2 be two disjoint maximal 2-edge-connected subgraphs of $G - v$ such that there is only one edge connecting C_1 and C_2 to the rest of the graph $G - v$. The cuts δC_1 and δC_2 have size at least 5, otherwise they are bad cuts, since all edges incident with v are simple; hence C_1 contains at least $|\delta C_1| - 1$ vertices, and C_2 contains at least $|\delta C_2| - 1$ vertices. Since both C_1 and C_2 are 2-edge-connected and there is a path connecting C_1 and C_2 in $G - v$, there exists an open tour T in $G - v$ such that it contains four neighbors of v from C_1 and four neighbors from C_2. Let us assume that endpoints of T are neighbors of v. By adding two edges connecting the endpoints of T to v, we get a closed tour; therefore, we can find an immersion of $K_{3,3}$ in G with 6 neighbors of v that are internal vertices of T as branch vertices. □

Lemma 5.22. *Let G be a plane weakly 5-edge-connected graph such that G does not immerse $K_{3,3}$. Let v be a vertex of G of degree at least 8. If there exists a fragile cut for every pair of edges incident with v, then there does not exist a tight fragile cut for every pair of opposite edges incident with v.*

Proof. Suppose for a contradiction that there exists a tight fragile cut for every pair of opposite edges incident with v.

If v has degree at least 10, then any tight fragile cut for opposite edges contains at least 6 edges incident with v. As we noted before, this contradicts Lemma 5.19.

If v has degree 9, then every tight fragile cut contains 5 edges incident with v. For any edge e incident with v we have two opposite edges e' and e''. Let F' be a fragile cut for the edges e and e' and F'' be a fragile cut for the edges e and e''. Let A' and B' denote the sides of F' and A'' and B'' denote the sides of F'' such that $v \in B'$ and $v \in B''$. The cut $\delta(A' \cap A'')$ contains the edge e and at most two other edges since e is the only edge incident with v contained in both F' and F''. Thus, $A' \cap A''$ contains only a single vertex of degree 3. It follows that all neighbors of v are vertices of degree 3 connected to v by simple edges. By Lemma 5.21 such graph immerses $K_{3,3}$.

If v has degree 8, every tight fragile cut contains 5 edges incident with v and one edge not incident with v, since v is not a cut vertex. Let $e_1, e_2, \ldots, e_8$ denote the edges incident with v in the cyclic order. Let F_1 be a tight fragile cut for the edges e_1 and e_5. Without loss of generality we can assume $e_2, e_3, e_4 \in F_1$. Let B_1 denote the side of F_1 such that $v \in B_1$ and let $B'_1 = B_1 \setminus \{v\}$. The cut $\delta B'_1$ has size 4; hence, B'_1 contains at most two vertices. The edge e_7 cannot be a simple edge; hence, we can assume e_7 and e_8 are parallel. Let v_7 denote the vertex incident with e_7 distinct from v. Let F_2 be a tight fragile cut for the edges e_3 and e_7. The cut F_2 contains the edges e_1, e_2 and e_8 since e_8 is parallel to e_7. Let B_2 denote the side of F_2 such that $v \in B_2$ and let $B'_2 = B_2 \setminus \{v\}$. The cut $\delta B'_2$ has size 4; hence, B'_2 contains at most two vertices. The edge e_5 cannot be a simple edge and it cannot be parallel to the edge e_6, since e_5 is contained in F_1, while e_6 is not; therefore e_4 and e_5 are parallel to each other. Let v_5 denote the vertex incident with e_5 distinct from v. The edge e_6 is contained in both $G[B_1]$ and $G[B_2]$; therefore, B'_1 and B'_2 have a common vertex v_6. Since $G[B'_1]$ and $G[B'_2]$ are connected and $B'_1 = \{v_6, v_7\}$ and $B'_2 = \{v_5, v_6\}$, an edge v_5v_6 is contained in F_1 and an edge v_6v_7 is contained in F_2. It follows that $G[B'_1 \cup B'_2]$ has only one neighbor v; therefore, v is a cut vertex which is a contradiction with Lemma 5.19. □

Lemma 5.23. *Let G be a plane weakly 5-edge-connected graph and let v be a vertex of G. Let C be a cycle in G such that $v \in C$. Let X and Y be induced subgraphs of G such that X and Y lie in different parts of plane divided by the cycle C, v is connected to both X and Y by at least two edges, and X and Y are* *connected. If there exists a fragile cut for every pair of the edges incident with v, then v is connected to each of X and Y by exactly two edges.*

Proof. Let n_X and n_Y be the numbers of edges connecting v to X and Y, respectively. Let $e_{X,1}, \ldots, e_{X,n_X}$ and $e_{Y,1}, \ldots, e_{Y,n_Y}$ denote the edges connecting v to X and Y, respectively, in the cyclic order. Let e_X and e_Y denote the edges $e_{X,\lceil \frac{n_X}{2} \rceil}$ and $e_{Y,\lceil \frac{n_Y}{2} \rceil}$, respectively. Let v_X and v_Y denote the vertices incident with e_X and e_Y, respectively, such that $v_X \neq v$ and $v_Y \neq v$.

Since X and Y are connected, there exist closed tours C_X contained in $G[V(X) \cup \{v\}]$ and C_Y contained in $G[V(Y) \cup \{v\}]$ such that $e_{X,1} \in C_X$, $e_{X,n_X} \in C_X$, $e_{Y,1} \in C_Y$, $e_{Y,n_Y} \in C_Y$.

Let us consider a minimal fragile cut F for e_X and e_Y. Let A and B denote the sides of F such that $v \in B$. Since the subgraph $G[A]$ is connected by Lemma 5.9, it intersects C, C_X and C_Y. Since v is contained in B and in all of C, C_X and C_Y, at least two edges from C, C_X and C_Y are contained in F; therefore, $e_X \in C_X$ and $e_Y \in C_Y$, otherwise F contains more than 6 edges. This holds only if $n_X = 2$ and $n_Y = 2$. □

Lemma 5.24. *Let G be a plane weakly 5-edge-connected graph and let v be a vertex of G of degree $d \geq 6$. Let $e_1, e_2, \ldots, e_d$ denote the edges incident with v in the cyclic order. Let F be a minimum fragile cut for a pair of opposite edges e_i and e_j incident with v where $i, j \in \{1, \ldots, d\}$, $i < j$, such that F is not tight. If G contains a fragile cut for every pair of the edges incident with v, then G contains at most 9 vertices.*

Proof. Let A and B denote the sides of F such that $v \in B$. Both $G[A]$ and $G[B]$ are connected by Lemma 5.9; therefore, v is a cut vertex of $G[B]$. Let f denote the number of the edges from F incident with v. By Lemma 5.19 each component of $G[B \setminus \{v\}]$ is connected by at least one edge from F to A; therefore, $G[B \setminus \{v\}]$ has at most $6 - f$ components and f is at most 6 minus the number of components of $G[B \setminus \{v\}]$. In particular, $G[B \setminus \{v\}]$ has at most four components. Every component of $G[B \setminus \{v\}]$ is connected by at least two edges to v, otherwise F is not a minimum fragile cut.

Since $G[A]$ is connected, there exists a cycle C contained in $G[A \cup \{v\}]$ such that $e_i, e_j \in C$. Let X and Y be components of $G[B \setminus \{v\}]$ such that X and Y lie in different parts of plane divided by the cycle C. By Lemma 5.23 both X and Y are connected by two edges to v. It follows that all components of $G[B \setminus \{v\}]$ are connected by two edges to v.

Let us consider the case the subgraph $G[B \setminus \{v\}]$ has exactly three components. Observe that degree of v is 9, since e_i and e_j are opposite to each other, the components of $G[B \setminus \{v\}]$ are connected to v by two edges and $f \leq 6 - 3$. Suppose $i = 1$, $j = 5$ and $e_2 \in F$, all other cases are symmetric. Let X_1, X_2 and X_3 denote the components of $G[B \setminus \{v\}]$ such that X_1 is incident with e_3 and e_4, X_2 is incident with e_6 and e_7, X_3 is incident with e_8 and e_9. All of X_1, X_2 and X_3 are connected to A by one edge; hence, they contain only a single vertex. Let F' be a minimal fragile cut for the edges e_3 and e_8. Since the edges e_3 and e_4 are parallel, the edge e_4 is contained in F'; hence F' is not a tight fragile cut. Let A' and B' denote the sides of F' such that $v \in B'$. By Lemma 5.19 each component of $G[B' \setminus \{v\}]$ is connected by at least one edge from F' to A'; therefore

$G[B' \setminus \{v\}]$ has at most two components. Since F' is not a tight fragile cut and $G[A']$ is connected, $G[B' \setminus \{v\}]$ has exactly two components. Let Z_1 denote the component connected to v by e_1 and e_2 and Z_2 denote the component connected to v by e_5, e_6 and e_7. The component Z_1 contains a single vertex and Z_2 contains at most two vertices since $|\delta Z_1| = 3$ and $|\delta Z_2| = 4$. The edges connecting A to X_1 and X_2 and the edges connecting A' to Z_1 and Z_2 form a cut of size 4; therefore, it has a side of size at most 2. It implies that G has at most 8 vertices.

Let us consider the case the subgraph $G[B \setminus \{v\}]$ has exactly four components. Observe that degree of v is 10, since e_i and e_j are opposite to each other, the components of $G[B' \setminus \{v\}]$ are connected to v by two edges and $f \leq 6-4$. Suppose $i = 1$ and $j = 6$, all other cases are symmetric. Let X_1, X_2, X_3 and X_4 denote the components of $G[B \setminus \{v\}]$ such that X_1 is incident with e_2 and e_3, X_2 is incident with e_4 and e_5, X_3 is incident with e_7 and e_8 and X_4 is incident with e_9 and e_{10}. All of X_1, X_2, X_3 and X_4 are connected to A by one edge; hence, they contain only a single vertex. Let F' be a minimal fragile cut for the edges e_2 and e_7. Let A' and B' denote the sides of F' such that $v \in B'$. By Lemma 5.19, each component of $G[B' \setminus \{v\}]$ is connected by at least one edge from F' to A'; therefore $G[B' \setminus \{v\}]$ has at most two components. Since F' cannot be a tight fragile cut, $G[B' \setminus \{v\}]$ has two components. Let Z_1 denote the component connected to v by e_1, e_9 and e_{10} and Z_2 denote the component connected to v by e_4, e_5 and e_6. Both Z_1 and Z_2 contain at most two vertices since $|\delta Z_1| = 4$ and $|\delta Z_2| = 4$. The edges connecting A to X_1 and X_2 and the edges connecting A' to Z_1 and Z_2 form a cut of size 4; therefore, it has a side of size at most 2. It implies that G has at most 9 vertices.

Therefore, we can assume the subgraph $G[B \setminus \{v\}]$ has exactly two components X_1 and X_2. The vertex v has degree at most 8 since $f \leq 6-2$. Let e_{X_1} and e'_{X_1} denote the edges connecting v to X_1, and e_{X_2} and e'_{X_2} denote the edges connecting v to X_2. The edges are taken in the cyclic order with respect to the vertex v. Let F' be a minimal fragile cut for the edges e_{X_1} and e_{X_2}. Let A' and B' denote the sides of F' such that $v \in B'$. Since $G[A']$ is connected and $v \in B'$, the fragile cut F' contains two edges from the cycle C, as well as the edges e_{X_1} and e_{X_2}; therefore, there are at most two other edges contained in F'. Since X_1 is connected, F' contains either the edge e'_{X_1} or exactly one edge from X_1 and either the edge e'_{X_2} or exactly one edge from X_2. Suppose that F' contains an edge e''_{X_1} from X_1. Let X'_1 be the component of $X_1 - e''_{X_1}$ incident with e'_{X_1}. Since G is 3-edge-connected, there is at least one other edge e_x contained in $\delta X'_1$. Since $V(X'_1) \subset B'$ and $e_x \notin F'$, the edge e_x is contained in $G[B']$ and $e_x \in F$. The graph G is 3-edge-connected; therefore, $X_1 - X'_1$ is connected to A by at least one edge. The analogous statement holds for X_2.

If F' contains an edge from both X_1 and X_2, then F contains the edges e_i, e_j, two edges connecting X_1 to A and two edges connecting X_2 to A; hence, the vertex v has degree 6. The subgraph $G[B \cup B']$ contains four edges of F and four edges of F'; therefore the cut $\delta(B \cup B')$ has size 4 and $A \cap A'$ contains at most two vertices. Observe that $G[B' \setminus \{v\}]$ contains two components with two vertices. The components X_1 and X_2 contain two vertices, one of them shared with a component of $G[B' \setminus \{v\}]$. It follows that G contains at most 9 vertices.

Suppose that F' contains the edges e'_{X_1} and e'_{X_2}. The subgraph $G[B' \setminus \{v\}]$ contains either two components with one vertex or one component with two vertices, since they are connected by at most one edge to A' and by at most three edges to v, and B' contains at least three vertices. It follows that v has degree 8 and X_1 and X_2 are connected by exactly one edge to A; therefore $|X_1| = |X_2| = 1$. The cut $\delta(B \cup B')$ has size 4; therefore, $A \cap A'$ contains at most two vertices. The graph G then contains at most 7 vertices.

Suppose that F' contains an edge from X_1 and the edge e'_{X_2} (the case that F' contains an edge from X_2 and the edge e'_{X_1} is symmetric). Consider a fragile cut F'' for the edges e'_{X_1} and e'_{X_2}. Let A'' and B'' denote the sides of F'' such that $v \in B''$. We can assume that F'' contains exactly one of the edges e_{X_1} and e_{X_2}, since we already proved the case that both e_{X_1} and e_{X_2} are contained in a fragile cut and the case that F'' contains neither of them is symmetric to the proof of F' not containing e'_{X_1} and e'_{X_2}. Thus, F'' contains one edge either from X_1 or from X_2; therefore, both X_1 and X_2 are connected by only one edge to $A \cap A' \cap A''$, since F contains four edges from $G[B \cup B' \cup B'']$. Observe that the separation $(A \cap A' \cap A'', B \cup B' \cup B'')$ has order 4, since $G[B \cup B' \cup B'']$ contains four edges from F and five edges from both F' and F''. therefore, $A \cap A' \cap A''$ contains at most two vertices. Observe that the components X_1 and X_2 either both contain two vertices, or one of them contains three vertices and the other one only one vertex, and the set $A \cap (B' \cup B'')$ contains at most two vertices. It follows that the graph G contains at most 9 vertices.

□

Theorem 5.25. *Let G be a plane weakly 5-edge-connected graph of maximum degree 6 that does not immerse $K_{3,3}$. Then G contains at most 20 vertices.*

Proof. Suppose that we have a smallest counterexample graph G. For every vertex v of G of degree 6, there exists a fragile cut for every pair of edges incident with v, otherwise we could split off a pair of edges to obtain either a smaller counterexample or a graph with maximum degree 4 or 5 contradicting Theorem 5.8 or Theorem 5.18. By Lemma 5.24, there exists a tight fragile cut for every pair of opposite edges incident with v.

The cut containing all the edges incident with v is not a fragile cut. If a fragile cut F contains 5 edges incident with v, there is a cut of size 2 consisting of the edge of F not incident with v and the edge incident with v not contained in F; therefore G is not 3-edge-connected. Thus, every tight fragile cut for the pair of opposite edges incident with v contains four edges incident with v.

Let $e_1, \ldots, e_6$ denote the edges incident with v in the cyclic order. Consider minimum tight fragile cuts F_1, F_2 and F_3 for the edges e_1 and e_4, e_2 and e_5, e_3 and e_6, respectively. For $i \in \{1, 2, 3\}$ let A_i and B_i denote the sides of F_i such that $v \in B_i$ and let F'_i be the cut with the sides $A'_i = A_i \cup \{v\}$ and $B'_i = B_i \setminus \{v\}$. F'_1, F'_2 and F'_3 are cuts of size at most 4. For $i \in \{1, 2, 3\}$ the side A'_i contains more than two vertices since A_i intersects both sides of the cut F_j where $j \in \{1, 2, 3\}$, $i \neq j$; therefore, B'_1, B'_2 and B'_3 contain at most two vertices. Moreover, if F_i has size 6, then B'_i contains 2 vertices, otherwise F_i is not a fragile cut, and if F_i has size 5, then B'_i contains only one vertex, otherwise F'_i is a bad cut. Only one of B'_1, B'_2 and B'_3 can contain only a single vertex, otherwise G contains a bad cut or at most 7 vertices.

For every tight fragile cut there are two options how to choose the edges incident with v; however, there are only two non-equivalent options how to choose the edges incident with v in F_1, F_2 and F_3.

In the first case $e_2, e_3 \in F_1$, $e_3, e_4 \in F_2$ and $e_4, e_5 \in F_3$. Since each of B'_1, B'_2 and B'_3 intersects both sides of at least one of the cuts F'_1, F'_2 and F'_3, B'_1, B'_2 and B'_3 consist of two vertices connected by an edge. The cut $\delta(B_1 \cup B_2 \cup B_3)$ contains four edges; therefore, $G - B_1 - B_2 - B_3$ contains at most two vertices and G contains at most 7 vertices.

In the second case $e_2, e_3 \in F_1$, $e_1, e_6 \in F_2$ and $e_4, e_5 \in F_3$. Let us assume that if any of B'_1, B'_2 and B'_3 contains only a single vertex, then it is B'_1. Let G' be the graph created from G by contracting the subgraph B_1 to a vertex v'. Observe that G immerses G'. The degree of vertex v' is equal to the size of the cut F_1. Suppose there is a tight fragile cut F''_1 for the edges e_1 and e_4 in G'. If F''_1 contains the edges e_1, e_2, e_3, e_4 and two edges e_x and e_y not incident with v, then $\{e_5, e_6, e_x, e_y\}$ is a bad cut in G. If F''_1 does not contain the edges e_2 and e_3, it contains an edge e'_2 from $G[B'_2]$ and an edge e'_3 from $G[B'_3]$. Since $\{e_2, e'_2, e_3, e'_3\}$ is a cut of size 4, it has a side with two vertices connected by an edge. It follows that the cut $\delta(B'_2 \cup B'_2 \cup \{v\})$ has size 4; therefore it has a side of size at most 2 and G has at most 7 vertices. If there is a fragile cut that is not tight for the edges e_1 and e_4 in G', then it would also be a fragile cut in G.

Let us perform a sequence of operations on G to get a weakly 5-edge-connected graph G'' of maximum degree 4 or 5. For every vertex u of degree 6, if there is a pair of edges incident with u such that there is no fragile cut for these edges, then we split off these edges. Otherwise, we perform a contraction described earlier and then if the contracted vertex has degree 6, we split off two edges incident with the contracted vertex such that there is no fragile cut for these edges. If the graph become non-planar or has at most 7 vertices, we stop performing the operations. The graph G'' does not immerse $K_{3,3}$, since G'' is immersed in G and G does not immerse $K_{3,3}$. By Lemma 2.1, G'' is a planar graph or has at most 5 vertices.

For every vertex u of the graph G, let $ch(u)$ denote the number of edges connecting u to a vertex of degree 6. We say $ch(u)$ is the charge of the vertex u. Let a be the number of vertices of degree 6, b be the number of vertices of degree at most 5, and $c = \sum_{u \in V(G)} ch(u)$. Observe that $c = 6a$ and $a + b = |V(G)|$.

Let w be a vertex of G of degree 6. There is a tight fragile cut for every pair of opposite edges incident with w. Let X_1, X_2 and X_3 be the sides of these fragile cuts containing w such that X_1 is the smallest. Let $X'_1 = X_1 \setminus \{w\}$, $X'_2 = X_2 \setminus \{w\}$ and $X'_3 = X_3 \setminus \{w\}$. As we proved before, $|X'_2| = |X'_3| = 2$ and X'_1 contains at most two vertices.

Every vertex in X'_1, X'_2 and X'_3 has degree at most 5. Suppose for a contradiction that $X \in \{X'_1, X'_2, X'_3\}$ contains a vertex v_X of degree 6. Clearly, if $X = X'_1$ and $|\delta X| = 3$, then v_X has degree 3. Let v'_X denote the vertex of X distinct from v_X. Since $|X| = 2$, $|\delta X| = 4$ and G is 3-edge-connected, either v_X is connected by three edges to v'_X and v'_X has degree 4 or v_X is connected by four edges to v'_X and v'_X has degree 6. Let G^- be the graph obtained from G by removing one of these edges. Suppose that G^- contains a bad separation (C, D). Since G does not contain a bad separation, the vertices v_X and v'_X are not contained in a same side of the separation; therefore $\delta(C, D)$ contains at least two edges connecting v_X to v'_X. Let us assume that $v_X \in C$ and $v'_X \in D$. The vertex v'_X has degree 3 or 5 in G^-, hence (C, D) has order at least 3. Suppose that (C, D) has order 3 in G^-. Since (C, D) is not a bad separation in G, it has a side with two vertices. Suppose that D contains vertex x_D distinct from v'_X. The vertex x_D is incident with at most one edge of $\delta(C, D)$; therefore, it is connected by at least two edges to v'_X. It follows that $\delta(D \cup v_X)$ contains at most three edges, which is a contradiction. The case that C contains two vertices is symmetric. Finally suppose that (C, D) has order 4 in G^-. If v_X is connected by two edges to v'_X in G^- and v'_X has degree 3, then the cut $\delta(D \setminus v'_X)$ contains at most three edges. If v_X is connected by three edges to v'_X in G^- and v'_X has degree 5, then again the cut $\delta(D \setminus v'_X)$ contains at most three edges. Since (C, D) is a bad separation in G^-, D contains at least three vertices and $D \setminus v'_X$ contains at least two vertices; therefore $\delta(D \setminus v'_X)$ is a bad cut in G, which is a contradiction. It follows that G^- is a plane weakly 5-edge-connected graph that does not immerse $K_{3,3}$ with fewer edges than G, which is a contradiction.

Since $|\delta X'_2| = 4$, a sum of the charges of the vertices of X'_2 is at most 4. The same holds for X'_3 and for X'_1 in the case that $|X'_1| = 2$. Every vertex in $X \in \{X'_1, X'_2, X'_3\}$ has charge at most 2. If $X = X'_1$ contains only one vertex x_1, then the only vertex of degree 6 adjacent to x_1 is w since each vertex x adjacent to vertex of degree 6 is either connected by a double edge with this vertex or is adjacent to another vertex y such that $|\delta\{x, y\}| = 4$. Therefore, $ch(x_1) = 2$. If X contains a vertex x such that $ch(x) = 4$, then x is a cut vertex and has degree at least 7, which is a contradiction. Suppose that X contains a vertex x such that $ch(x) = 3$. Let y denote the second vertex of X. Suppose that x is adjacent to three vertices of degree 6. Since x has degree at most 5, it cannot be a cut vertex, hence it is incident with only three edges of δX and each of these edges is a simple edge connected to a vertex of degree 6. Since each vertex of degree 6 is adjacent to a neighbor of x that has degree at most 5 and x has only one such neighbor y, all three vertices of degree 6 are adjacent to y; therefore $|\delta X| = 6$, which is a contradiction. It follows that the vertex x cannot be adjacent to three vertices of degree 6; therefore it is connected to a vertex z of degree 6 by a double edge. The vertex x is connected by at least two edges to y. Suppose that G contains a fragile cut F_x for the edges xy and xz. Since the edges xy and xz are double edges, F_x contains at most two other edges. These two edges together with an edge incident with x not incident with y and z forms a bad cut. Thus, we can split off one of the edges connecting x and y and one of the edges connecting x and z and get a plane weakly 5-edge-connected graph that does not immerse $K_{3,3}$ with fewer edges than G, which is a contradiction. It follows that every vertex of G has charge at most 2; consequently, $c \leq 2b$.

Let us count the number of vertices of G''. After every operation, the number of vertices of degree 6 decreases by one and the number of vertices of degree at most 5 decreases by at most one. The graph G'' has at least $b - a$ vertices. We have $a + b \geq |V(G)|$, $a - b \leq |V(G'')|$, $c = 6a$ and $c \leq 2b$; hence, $a \leq \frac{b}{3}$, $\frac{4}{3}b \geq |V(G)|$ and $\frac{2}{3}b \leq |V(G'')|$; therefore, $|V(G'')| \geq \frac{|V(G)|}{2}$. Since we assume that G contains at least 21 vertices, G'' contains at least 11 vertices, which is a contradiction with Lemma 5.8 or Lemma 5.18. □

Lemma 5.26. *Let G be a plane weakly 5-edge-connected graph with at least 10 vertices and let v be a vertex of G of degree at least 7. If there exists a fragile cut for every pair of the edges incident with v, then G immerses $K_{3,3}$.*

Proof. If there exists a tight fragile cut for every pair of the opposite edges, then G immerses by Lemma 5.20 or by Lemma 5.22. Otherwise, there are two opposite edges e and e' such that every fragile cut for the edges e and e' is not tight; consequently, G has at most 9 vertices by Lemma 5.24, which is a contradiction. □

Theorem 5.27. *Let G be a graph from $\mathcal{C}_5 \setminus \mathcal{P}$. Then G contains at most* 20 *vertices.*

Proof. If G is non-planar, then G contains at most 5 vertices by Lemma 2.1; therefore, we can assume G is planar.

We have already proved the case that maximum degree of G is 4, 5 or 6 in Theorem 5.8, Theorem 5.18 and Theorem 5.25.

Let us consider the case that G has maximum degree at least 7. Suppose that we have a smallest counterexample graph G and let v be a vertex of G of degree at least 7.

By Lemma 5.26, we can split off two edges incident with v to get a weakly 5-edge-connected graph G'. If G' is not a planar graph, it immerses $K_{3,3}$ by Lemma 2.1. If G' has maximum degree 5 or 6, it immerses $K_{3,3}$ by Theorem 5.18 or by Theorem 5.25. The graph G' has the same number of vertices and fewer edges than graph G; therefore, if G' has maximum degree at least 7, it immerses $K_{3,3}$. Since the relation of being immersed is transitive, G immerses $K_{3,3}$, which is a contradiction. □

Conclusion

We have proved that every graph that does not immerse $K_{3,3}$ can be created from small planar weakly 5-edge-connected graphs and 3-regular planar graphs by joining them together with operations join and special 4-join, subdividing edges, pinching double edges and adding pendant vertices. We have proved that the small graphs have at most 20 vertices; although this bound can be improved to 8 vertices as DeVos and Malekian [9] proved. They used computer-assisted enumeration to precisely describe these small graphs. It is unclear whether a reasonably simple direct argument to list them exists.

In this thesis, we have dealt with weak immersions, but it could be worth finding similar characterization also for strong immersions.

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