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Martin Černý

Monge property for interval matrices

Department of Applied Mathematics

Supervisor of the bachelor thesis:	doc. Mgr. Milan Hladík, Ph.D.
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Název práce: Mongeova vlastnost intervalových matic

Autor: Martin Černý

Katedra: Katedra aplikované matematiky

Vedoucí: doc. Mgr. Milan Hladík, Ph.D., Katedra aplikované matematiky

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Abstrakt: Tato práce je prvním průzkumem oblasti intervalových matic s Mongeovou vlastností. Zabývá se charakterizacemi a vlastnostmi dvojice tříd matic - třídy intervalových silně Mongeových matic a třídy intervalových slabě Mongeových matic. V práci je představeno několik metod na rozpoznávání a rekonstrukci těchto matic a následně prozkoumána jejich aplikace v problémech kombinatorické optimalizace a v problému související s výpočetní geometrií.

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Author: Martin Černý

Department: Department of Applied Mathematics

Supervisor: doc. Mgr. Milan Hladík, Ph.D., Department of Applied Mathematics

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Abstract: The thesis is a first survey in the field of interval matrices with Monge property. We investigate characteristics and properties of two classes of matrices - interval strong and interval weak Monge matrices. We develop a recognition and reconstruction algorithms and study applications in combinatorial problems and in a problem connected with computational geometry.

Contents

Introduction

In 1781 a French mathematician Gaspard Monge observed a fundamental, but very strong property. He worked on a problem to efficiently tranport soil from one place to another. To transport unit quantities from two points S_1 and S_2 to two points E_1 and E_2 the total distance travelled is always smaller when the route from S_1 do not intersect with the route from S_2. A mathematical formulation of this property laid a foundation of matrices with Monge property. It was shown in the past century that the presence of this property simplifies many optimization problems. The famous NP-complete travelling salesman problem becomes solvable by a linear algorithm. Other optimization problems such as assignment problem, transportation problem or lot-sizing problem can be solved significantly faster using algorithms based on Monge property. Since the property speaks about distances, also many problems from geometry concerning distances of points or areas of polygons become easier to solve. There are also further results in mathematical statistics, linguistics, bioinformatics, graph theory or dynamic programming.

Interval analysis was introduced as an answer for dealing with uncertainty or inaccuracy in data. In almost every area of expertise people encounter a situation where they are limited by the precision of their data. Sometimes the equipment used for measurement is not sophisticated enough, sometimes the data cannot be predicted without a certain error. Sometimes the problem is not in the precision of data but in the machines that manipulate with them. For example real data must be discretized in modern computers. The problem becomes more severe when we use computers to compute abstract problems as part of mathematical proofs. In these problems we cannot allow to neglect errors. In interval analysis we envelope our data into intervals and then perform calculations on these intervals instead of the data itself. The methods of interval analysis ensure that the result is included in the resulting interval. In other typical problem for interval analysis we receive an interval of possible inputs and we want to find the range of all solutions.

In this thesis we interconnected both ideas through an optimization problem - transportation problem. Transportation problem is a subproblem of linear programming. For Monge matrices being cost functions the optimal solution of transportation problem can be found by a greedy algorithm. When studying a family of interval linear programming, it is in general NP-hard to find the "worst case" optimal value. For some subproblems it was shown that the problem is easier to solve, for some problems like interval transportation problem it is still an open question. It appeared that the presence of Monge property could offer a polynomial solution for this problem.

In this thesis we want to formulate the Monge property for interval matrices. We approach this task from two directions. We define a matrix with interval strong Monge property - for which all the realizations of the matrix satisfy the Monge property. This class of matrices proves itself to be easy to work with. The second class of matrices introduced in this thesis are matrices with interval weak Monge property where at least one realization posseses the Monge property. Although harder to work with, these matrices are interesting to study. For example when approaching an NP-hard problem from interval analysis, being able to compute one instance of the problem efficiently might be a good start to formulate an approximation algorithm. We investigate properties of both these classes, ways to recognize them and reconstruct them from almost-Monge matrices and explore their applications in transportation problem and other optimization and geometrical problems.

1. Preliminaries

Before we start with introduction to Monge matrices, we have to fix a notation and introduce basics of interval analysis and interval arithmetics. For further information about interval analysis beyond the basics stated in this text see [1]. Section 1.1 fixes basic notation, Section 1.2 defines interval objects and operations on them and Section 1.3 lists used symbols.

1.1 Notation

By $\mathbb{R}$ we denote the set of real numbers. For intervals of real values we define $\mathbb{IR}$ as the set of all intervals over $\mathbb{R}$. By $\mathbb{T}^{m\times n}$ we denote the matrix of dimension $m \times n$ where the entries of the matrix are from $\mathbb{T}$. Symbol $\mathbb{T}^n$ represents a set of vectors of dimension n with entries from $\mathbb{T}$, $\mathbb{T}^+$ the set of all positive values and $\mathbb{T}_0^+$ the set of all nonnegative values.
For two real matrices $A, B \in \mathbb{R}^{m\times n}$ we say that $A \leq B$ if for all indices i, j it holds that $a_{ij} \leq b_{ij}$.

1.2 Intervals and interval arithmetics

Definition 1. *An interval matrix $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ is defined as*

$$\boldsymbol{A} = \left[\underline{A}, \overline{A}\right] = \left\{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\right\}$$

where $\underline{A}, \overline{A}$ are lower resp. upper bound matrices of $\boldsymbol{A}$.
We can further denote center of $\boldsymbol{A}$ by $A^C = \frac{1}{2}(\underline{A} + \overline{A})$ and a radius of $\boldsymbol{A}$ by $A^\Delta = \frac{1}{2}(\overline{A} - \underline{A})$. Using the center and the radius we can rewrite interval matrix as

$$\boldsymbol{A} = \left\{A \in \mathbb{R}^{m\times n} : A^C - A^\Delta \leq A \leq A^C + A^\Delta\right\}.$$

Definition 2. *An interval vector $\mathbf{v} \in \mathbb{IR}^m$ is defined as*

$$\mathbf{v} = [\underline{v}, \overline{v}] = \{v \in \mathbb{R}^m : \underline{v} \leq v \leq \overline{v}\}.$$

Definition 3. *Let $\boldsymbol{M}, \mathbf{N} \in \mathbb{IR}^{m\times n}$. Then an interval matrix intersection of matrices $\boldsymbol{M}, \boldsymbol{N}$ denoted by $\boldsymbol{M} \cap \mathbf{N}$ is defined as*

$$(\boldsymbol{M} \cap \mathbf{N})_{ij} = \begin{cases} [l, u] & \text{if } l \leq u, \\ \emptyset & \text{if } l > u, \end{cases}$$

where $l = \max\left\{\underline{m}_{ij}, \underline{n}_{ij}\right\}$ and $u = \min\left\{\overline{m}_{ij}, \overline{n}_{ij}\right\}$.

Definition 4. *Let $\boldsymbol{M}, \mathbf{N} \in \mathbb{IR}^{m\times n}$. Then an interval matrix union denoted by $\boldsymbol{M} \cup \mathbf{N}$ is defined as*

$$\boldsymbol{M} \cup \mathbf{N} = \left\{X \in \mathbb{R}^{m\times n} : X \in \boldsymbol{M} \text{ or } X \in \mathbf{N}\right\}.$$

A union of two interval matrices is not always an interval matrix, therefore we define also an *envelope of an interval matrix union.*

Definition 5. *Let $\boldsymbol{M}, \mathbf{N} \in \mathbb{IR}^{m\times n}$. Then an envelope of interval matrix union denoted by* $\square(\boldsymbol{M} \cup \mathbf{N})$ *is defined as*

$$\square(\boldsymbol{M} \cup \mathbf{N}) = \left\{X \in \mathbb{R}^{m\times n} : \min\left\{\underline{M}, \underline{N}\right\} \le X \le \max\left\{\overline{M}, \overline{N}\right\}\right\}.$$

For a binary arithmetic operation $\circ \in \{+, -, \cdot, /\}$ defined on $\mathbb{R}$, we can introduce the corresponding interval operation as follows:

$$\boldsymbol{a} \circ \boldsymbol{b} = \{a \circ b : a \in \boldsymbol{a}, b \in \boldsymbol{b}\}.$$

We can rewrite the definition into an explicit formula:

- $\boldsymbol{a} + \boldsymbol{b} = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right]$,
- $\boldsymbol{a} - \boldsymbol{b} = \left[\underline{a} - \overline{b}, \overline{a} - \underline{b}\right]$,
- $\boldsymbol{a} \cdot \boldsymbol{b} = \left[\min\left\{\underline{a}\cdot\underline{b}, \underline{a}\cdot\overline{b}, \overline{a}\cdot\underline{b}, \overline{a}\cdot\overline{b}\right\}, \max\left\{\underline{a}\cdot\underline{b}, \underline{a}\cdot\overline{b}, \overline{a}\cdot\underline{b}, \overline{a}\cdot\overline{b}\right\}\right]$,
- $\boldsymbol{a}/\boldsymbol{b} = \left[\min\left\{\frac{\overline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\underline{a}}{\underline{b}}\right\}, \max\left\{\frac{\overline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\underline{a}}{\underline{b}}\right\}\right]$ if $0 \notin \boldsymbol{b}$.

Let us note that for interval division there is a known generalization where $0 \in \boldsymbol{b}$.

1.3 List of symbols

$\mathbb{R}$ ... A set of real numbers.
$\mathbb{R}^+$ ... A set of positive real numbers.
$\mathbb{R}_0^+$ ... A set of nonnegative real numbers.
$\mathbb{IR}$ ... A set of all closed interval subsets of $\mathbb{R}$.
$\mathbb{IR}^+$ ... A set of all positive closed interval subsets of $\mathbb{R}$.
$\mathbb{IR}_0^+$ ... A set of all nonnegative closed interval subsets of $\mathbb{R}$.
$\mathbb{IR}^{m\times n}$ ... A set of all interval matrices of dimension $m \times n$.
$\mathbb{IR}^m$ ... A set of all interval m dimension vectors.
$\mathbb{ISM}$ ... A set of all interval strong Monge matrices.
$\mathbb{ISM}^+$ ... A set of all positive interval strong Monge matrices.
$\mathbb{ISM}_0^+$ ... A set of all nonnegative interval strong Monge matrices.
$\mathbb{ISM}^{m\times n}$ ... A set of all interval strong Monge matrices of dimension $m \times n$.
$\mathbb{IWM}$ ... A set of all interval weak Monge matrices.
$\mathbb{IWM}^+$ ... A set of all positive interval weak Monge matrices.
$\mathbb{IWM}_0^+$ ... A set of all nonnegative interval weak Monge matrices.
$\mathbb{IWM}^{m\times n}$ ... A set of all interval weak Monge matrices of dimension $m \times n$.
$\mathbf{A} \in \mathbb{IR}^{m\times n}$ ... An interval matrix of dimension $m \times n$.
$\mathbf{v} \in \mathbb{IR}^m$ ... An interval vector of dimension m.
$\boldsymbol{M} \cap \mathbf{N}$ ... An interval matrix intersection of matrices $\boldsymbol{M}, \mathbf{N}$.
$\boldsymbol{M} \cup \mathbf{N}$ ... An interval matrix union of matrices $\boldsymbol{M}, \mathbf{N}$.
$\square(\boldsymbol{M} \cup \mathbf{N})$ ... An envelope of an interval matrix union of matrices $\boldsymbol{M}, \mathbf{N}$.

2. Real Monge matrices

In this chapter we introduce real Monge matrices and some fundamental results connected with them that we will further use in the text. In study of Monge matrices we followed mostly paper by Burkard, Klinz and Rüdiger [2] which summerizes most of the known results concerning Monge property in optimization. In Section 2.1 we introduce the definition and several different characterizations. In Section 2.2 we deal with closure properties of Monge matrices. In Section 2.3 we present other fundamental properties especially the connection between Monge and totally monotone matrices.

2.1 Definition and characterizations

Let us start with the definition.

Definition 6. *Let $M \in \mathbb{R}^{m \times n}$. Then M is a Monge matrix iff for all i, j, k, ℓ : $1 \leq i < k \leq m, 1 \leq j < \ell \leq n$ it holds $m_{ij} + m_{k\ell} \leq m_{i\ell} + m_{kj}$.*

Since Hoffman rediscovered the Monge property in 1961, several equivalent characterizations have been found. We merge some of the characterizations into a following theorem but first, we define a notion of *submodular* functions.

Definition 7. *Let $\Lambda = (I, \wedge, \vee)$ be a distributive lattice where $I = \{1, ..., m\} \times \{1, ..., n\}$ and join ($\wedge$) and meet ($\vee$) operations are defined as follows:*

- $(x_1, x_2) \wedge (y_1, y_2) = (\min\{x_1, y_1\}, \min\{x_2, y_2\})$ *for all $x, y \in I$,*
- $(x_1, x_2) \vee (y_1, y_2) = (\max\{x_1, y_1\}, \max\{x_2, y_2\})$ *for all $x, y \in I$.*

Function $f : I \to \mathbb{R}$ is said to be submodular on Λ if

$$f(x \vee y) + f(x \wedge y) \leq f(x) + f(y).$$

Theorem 1. *Let $M \in \mathbb{R}^{m \times n}$, then the following are equivalent:*

1. *M is a Monge matrix,*
2. *$m_{ij} + m_{k\ell} \leq m_{i\ell} + m_{kj}$ for all $1 \leq i < k \leq m, 1 \leq j < \ell \leq n$,*
3. *$m_{ij} + m_{i+1,j+1} \leq m_{i,j+1} + m_{i+1,j}$ for all $1 \leq i < m, 1 \leq j < n$,*
4. *Let $\Lambda = (I, \wedge, \vee)$ be a distributive lattice where $I = \{1, ..., m\} \times \{1, ..., n\}$. Then a function $f : I \to \mathbb{R}$ defined by $f(i, j) = m_{ij}$ is submodular on Λ.*

Proof. The equivalence between 1. and 2. is by the definition. The equivalence between 2. and 3. is straightforward by mathematical induction. For the equivalence between 1. and 4. see [3].

□

We present one more characterization of Monge matrices. It connects together the Monge matrices and so called distribution matrices.

Definition 8. *Let $C \in \mathbb{R}^{m \times n}$ be a nonnegative matrix. Then a distribution matrix D generated by a density matrix C is a matrix such that*

$$d_{ij} = \sum_{k=i}^{m} \sum_{\ell=1}^{j} c_{k\ell}.$$

Lemma 2. *Let $M \in \mathbb{R}^{m \times n}$, then M is a Monge matrix if and only if there exists a distribution matrix $D \in \mathbb{R}^{m \times n}$ and two vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ such that*

$$m_{ij} = d_{ij} + u_i + v_j.$$

Proof. See [4].

□

Let us point out that the set of distribution matrices forms an interesting subclass of Monge matrices studied in connection with problems from mathematical statistics [4].

2.2 Closure properties

We present a list of operations under which the Monge matrices are closed.

Lemma 3. *Let $M, N \in \mathbb{R}^{m \times n}$ be Monge. Then the following holds:*

1. *M^T is Monge,*
2. *αM is Monge for $\alpha \geq 0$,*
3. *$M + N$ is Monge,*
4. *a matrix C such that $c_{ij} = m_{ij} + u_i + v_j$ is Monge for any $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$.*

Proof. 1. - 4. are easy to derive from the standard definition of Monge matrices.

□

We further generalized the statement 3. to fully characterize the circumstances under which for a Monge matrix M and a matrix N a sum of $M + N$ remains Monge. For the characterization we need a so called *residual matrix*.

Definition 9. *Let $M \in \mathbb{R}^{m \times n}$. Then for $m > 1, n > 1$ a residual matrix $M^R \in \mathbb{R}^{(m-1)\times(n-1)}$ of matrix M is defined as*

$$m_{ij}^R = m_{i+1,j} + m_{i,j+1} - m_{i+1,j+1} - m_{ij}.$$

Lemma 4. *Let $M \in \mathbb{R}^{m \times n}$ be a Monge matrix and $N \in \mathbb{R}^{m \times n}$. Then $M + N$ is a Monge matrix iff $M^R + N^R$ is a nonnegative matrix.*

Proof. Let $M + N$ be a Monge matrix. For any i, j we get by 3. in Theorem 1 that

$$m_{ij} + m_{i+1,j+1} + n_{ij} + n_{i+1,j+1} \leq m_{i+1,j} + m_{i,j+1} + n_{i+1,j} + n_{i,j+1}.$$

Rearranging the inequality by putting everything to the right side we get

$$0 \leq m_{i+1,j} + m_{i,j+1} - m_{i+1,j+1} - m_{ij} + n_{i+1,j} + n_{i,j+1} - n_{i+1,j+1} - n_{ij} = m^R_{ij} + n^R_{ij}.$$

Let us now suppose that $M^R + N^R$ is a nonnegative matrix. By definition of *residual matrices* it means that for all i, j

$$0 \leq m^R_{ij} + n^R_{ij} = m_{i+1,j} + m_{i,j+1} - m_{i+1,j+1} - m_{ij} + n_{i+1,j} + n_{i,j+1} - n_{i+1,j+1} - n_{ij},$$

which can be rearranged as

$$m_{ij} + n_{ij} + m_{i+1,j+1} + n_{i+1,j+1} \leq m_{i+1,j} + n_{i+1,j} + m_{i,j+1} + n_{i,j+1}.$$

Since this inequality holds for $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$, it also holds that $M + N$ is Monge.

□

Lemma 3.2 and Lemma 3.3 imply that the set of nonnegative Monge matrices forms a convex polyhedral cone. A study of this cone revealed 4 types of 0-1 matrices generating rays of the convex cone. Let H^i denote a 0-1 matrix whose ith row contains all ones while the other entries are zeros and V^j a 0-1 matrix with jth columns set to ones and the rest to zeros. Further, let L^{rs} be a 0-1 matrix where for $l^{rs}_{ij} = 1$ for $i = r, \dots, m$ and $j = 1, \dots, s$. Otherwise $l^{rs}_{ij} = 0$. Similarly let R^{pq} be a 0-1 matrix with $r^{pq}_{ij} = 1$ for $i = 1, \dots, p$ and $j = q, \dots, n$, otherwise $r^{pq}_{ij} = 0$. The following theorem states that a linear combinations of matrices H^i, V^j, L^{rs} and R^{pq} characterizes Monge matrices once again. Although it is possible to characterize all Monge matrices, we focus only on nonnegative matrices since in further chapters we will work with them.

Theorem 5. *Let $M \in \mathbb{R}^{m\times n}$ be a Monge matrix, then there exist nonnegative numbers $\kappa_i, \lambda_j, \mu_{rs}$ and ν_{pq} such that*

$$M = \sum_{i=1}^{m} \kappa_i H^i + \sum_{j=1}^{n} \lambda_j V^j + \sum_{r=2}^{m} \sum_{s=1}^{n-1} \mu_{rs} L^{rs} + \sum_{p=1}^{m-1} \sum_{q=2}^{n} \nu_{pq} R^{pq}.$$

The matrices H^p with $p = 1, \dots, m$, V^q with $q = 1, \dots, n$, L^{rs} with $r = 2, \dots, m$, $s = 1, \dots, n-1$ and R^{pq} with $p = 1, \dots, m-1, q = 2, \dots, n$ generate extreme rays of the cone of nonnegative Monge matrices.

Proof. See [5].

□

Theorem 5 was used to simplify optimality proofs for combinatorial optimization problems concerning Monge matrices [5].

2.3 Other fundamental properties

The following statement lets us set Monge matrices into a larger frame of so called *totally monotone* matrices. The properties of totally monotone matrices are fundamental for many applications of Monge matrices. We will show the proof here instead of giving a reference.

Definition 10. *Let* $A \in \mathbb{R}^{m \times n}$ *and let* $j(i)$ *be the index of the column which contains the leftmost minimum of row* i*. Then we say that* A *is monotone if*

$$j(1) \leq j(2) \leq \cdots \leq j(n).$$

Further, A *is totally monotone if such a property holds for all its submatrices.*

Lemma 6. *Monge matrices are totally monotone matrices.*

Proof. Let $M \in \mathbb{R}^{m \times n}$ be a matrix that is not totally monotone. Then there exist row indices $i < k$ and column indices $j < \ell$ such that

$$m_{ij} > m_{i\ell} \text{ and } m_{k\ell} \geq m_{kj}.$$

But these two inequalities imply that

$$m_{ij} + m_{k\ell} > m_{i\ell} + m_{kj},$$

therefore the matrix M is not Monge.

□

3. Interval Strong Monge matrices

In this chapter we introduce the Interval Strong Monge (ISM) matrices. Section 3.1 shows a definition and equivalent characterizations of ISM. In Section 3.2 we discuss closure properties of ISM. In Section 3.3 we deal with interval envelopes of ISM.

3.1 Definition and equivalent characterizations

Definition 11. *An interval matrix $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ is interval strong Monge (ISM) if $\forall M \in \boldsymbol{M}$ it holds that M is Monge. We denote by $\mathbb{ISM}$ the set of interval strong Monge matrices.*

To check whether an interval matrix $\boldsymbol{M}$ is Monge using the definition means to check the Monge property for infinitely many real matrices. The following lemma characterizes ISM matrices via finitely many conditions.

Lemma 7. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$. Then $\boldsymbol{M} \in \mathbb{ISM}$ iff for all i, j, k, ℓ such that $1 \le i < k \le m, 1 \le j < \ell \le n$ it holds that $\overline{m}_{ij} + \overline{m}_{k\ell} \le \underline{m}_{i\ell} + \underline{m}_{kj}$.*

Proof. Let us suppose that $\boldsymbol{M} \in \mathbb{ISM}$. For arbitrary indices i, j, k, ℓ let us suppose a matrix $M \in \boldsymbol{M}$ with entries $m_{ij} = \overline{m}_{ij}$, $m_{k\ell} = \overline{m}_{k\ell}$, $m_{i\ell} = \underline{m}_{i\ell}$, $m_{kj} = \underline{m}_{kj}$. M is Monge since $\boldsymbol{M} \in \mathbb{ISM}$, therefore the inequality

$$\overline{m}_{ij} + \overline{m}_{k\ell} = m_{ij} + m_{k\ell} \le m_{i\ell} + m_{kj} = \underline{m}_{i\ell} + \underline{m}_{k\ell}$$

holds.
Now for the other implication let M be an arbitrary matrix in $\boldsymbol{M}$. The following chain of inequalities holds for all i, j, k, ℓ :

$$m_{ij} + m_{k\ell} \le \overline{m}_{ij} + \overline{m}_{k\ell} \le \underline{m}_{i\ell} + \underline{m}_{kj} \le m_{i\ell} + m_{kj},$$

therefore M is Monge.

□

Lemma 7 gives us $O(m^2n^2)$ conditions to be verified. It can be shown that many of the conditions are redundant and we can limit the problem to $O(mn)$ conditions of *neighbouring quadraples.*

Lemma 8. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$. Then $\boldsymbol{M} \in \mathbb{ISM}$ iff for all indices i, j such that $1 \le i < m, 1 \le j < n$ it holds $\overline{m}_{ij} + \overline{m}_{i+1,j+1} \le \underline{m}_{i,j+1} + \underline{m}_{i+1,j}$.*

Proof. Let $\boldsymbol{M} \in \mathbb{ISM}$. For arbitrary indices i, j let us consider a matrix $M \in \boldsymbol{M}$ with entries $m_{ij} = \overline{m}_{ij}$, $m_{i+1,j+1} = \overline{m}_{i+1,j+1}$, $m_{i,j+1} = \underline{m}_{i,j+1}$ and $m_{i+1,j} = \underline{m}_{i+1,j}$. Matrix M is Monge since $\boldsymbol{M} \in \mathbb{ISM}$, therefore the inequality

$$\overline{m}_{ij} + \overline{m}_{i+1,j+1} = m_{ij} + m_{i+1,j+1} \le m_{i,j+1} + m_{i+1,j} = \underline{m}_{i,j+1} + \underline{m}_{i+1,j+1}$$

holds.
Now for the other implication and for an arbitrary matrix $M \in \boldsymbol{M}$ the following chain of inequalities holds for all indices i, j:

$$m_{ij} + m_{i+1,j+1} \leq \overline{m}_{ij} + \overline{m}_{i+1,j+1} \leq \underline{m}_{i,j+1} + \underline{m}_{i+1,j} \leq m_{i,j+1} + m_{i+1,j}.$$

Therefore by Theorem 1.3 M is Monge.

□

Any ISM matrix can be also defined by a pair of so called *chess matrices.*

Definition 12. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$. Let us denote white resp. black chess matrix as a Monge matrix M^W resp. M^B where*

$$M^W_{ij} = \begin{cases} \overline{m}_{ij}, & \textit{if } i+j \textit{ is even,} \\ \underline{m}_{ij}, & \textit{if } i+j \textit{ odd.} \end{cases}$$

$$M^B_{ij} = \begin{cases} \underline{m}_{ij}, & \textit{if } i+j \textit{ is even,} \\ \overline{m}_{ij}, & \textit{if } i+j \textit{ odd.} \end{cases}$$

Lemma 9. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$. Then $\boldsymbol{M} \in \mathbb{ISM}$ iff M^W and M^B are Monge.*

Proof. Let $\boldsymbol{M} \in \mathbb{ISM}$. Then also M^W and M^B are Monge since both matrices are realizations of $\boldsymbol{M}$.
Let us now suppose that M^W and M^B are Monge. Then for $i+j$ even the entries of M^W satisfy $\overline{m}_{ij} + \overline{m}_{i+1,j+1} \leq \underline{m}_{i,j+1} + \underline{m}_{i+1,j}$ and for $i+j$ odd the entries of M^B give the same inequality. In other words all the conditions from Lemma 8 are satistfied, hence $\boldsymbol{M} \in \mathbb{ISM}$.

□

One of the results in real case connects Monge matrices with submodular functions on lattices (see Theorem 1.3). The transition between these two worlds can be also defined for $\mathbb{ISM}$. For this purpose, we define a generalized submodular function $\mathbf{f} : I \to \mathbb{IR}$.

Definition 13. *Let $\Lambda = (I, \wedge, \vee)$ be a distributive lattice where $I = \{1, ..., m\} \times \{1, ..., n\}$ with join ($\wedge$) and meet ($\vee$) operations. The operations are defined as follows:*

$$(x_1, x_2) \wedge (y_1, y_2) = (\min\{x_1, y_1\}, \min\{x_2, y_2\}) \textit{ for all } x, y \in I$$

and

$$(x_1, x_2) \vee (y_1, y_2) = (\max\{x_1, y_1\}, \max\{x_2, y_2\}) \textit{ for all } x, y \in I.$$

A function $\mathbf{f} : I \to \mathbb{IR}$ is submodular on a lattice I if $\overline{f}(x \vee y) + \overline{f}(x \wedge y) \leq \underline{f}(x) + \underline{f}(y)$ for all $x, y \in I$.

Lemma 10. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ and $\Lambda = (I, \wedge, \vee)$ be a distributive lattice with join ($\wedge$) and meet ($\vee$) operations where $I = \{1, ..., m\} \times \{1, ..., n\}$. Let $\mathbf{f} : I \to \mathbb{IR}$ be defined by $\mathbf{f}(i, j) = \boldsymbol{m}_{ij}$. Then $\boldsymbol{M} \in \mathbb{ISM}$ iff $\mathbf{f}$ is submodular.*

Proof. Let $\boldsymbol{M} \in \mathbb{ISM}$ and let $x = (i, \ell) \in I$ and $y = (k, j) \in I$. W.l.o.g let us suppose that $i < k$ and $j < \ell$. Then

$$\overline{f}(x \wedge y) = \overline{f}((i, \ell) \vee (k, j)) = \overline{f}((i, j)) = \overline{m}_{ij}$$

and

$$\overline{f}(x \vee y) = \overline{f}((i, \ell) \vee (k, j)) = \overline{f}((k, \ell)) = \overline{m}_{k\ell}.$$

Therefore

$$\overline{f}(x \vee y) + \overline{f}(x \wedge y) = \overline{m}_{k\ell} + \overline{m}_{ij} \leq \underline{m}_{i\ell} + \underline{m}_{kj} = \underline{f}(x) + \underline{f}(y)$$

and the inequality holds because $\boldsymbol{M} \in \mathbb{ISM}$.
Let us now consider that the function $\mathbf{f}$ is submodular on the lattice I. Then the condition

$$\overline{m}_{ij} + \overline{m}_{i+1,j+1} \leq \underline{m}_{i+1,j} + \underline{m}_{i,j+1}$$

corresponds with

$$\overline{f}((i+1, j) \wedge (i, j+1)) + \overline{f}((i+1, j) \vee (i, j+1)) \leq \underline{f}((i+1, j)) + \underline{f}((i, j+1)).$$

Since $\mathbf{f}$ is submodular the second inequality really holds, therefore $\boldsymbol{M} \in \mathbb{ISM}$. □

Theorem 11. *Let $\boldsymbol{M} \in \mathbb{IR}^{m \times n}$. Then the following are equivalent:*

1. $\boldsymbol{M} \in \mathbb{ISM}$,
2. $\overline{m}_{ij} + \overline{m}_{k\ell} \leq \underline{m}_{i\ell} + \underline{m}_{kj}$ *for all* $1 \leq i < k \leq m, 1 \leq j < \ell \leq n$,
3. $\overline{m}_{ij} + \overline{m}_{i+1,j+1} \leq \underline{m}_{i,j+1} + \underline{m}_{i+1,j}$ *for all* $1 \leq i < m, 1 \leq j < n$,
4. *Chess matrices M^W and M^B are Monge,*
5. *Let $\boldsymbol{M} \in \mathbb{IR}^{m \times n}$ and $\Lambda = (I, \wedge, \vee)$ be a distributive lattice with join ($\wedge$) and meet ($\vee$) operations where $I = \{1, ..., m\} \times \{1, ..., n\}$. Let $\mathbf{f} : I \to \mathbb{IR}$ be defined by $\mathbf{f}(i, j) = \boldsymbol{m}_{ij}$. Then $\boldsymbol{M} \in \mathbb{ISM}$ iff $\mathbf{f}$ is submodular.*

Proof. See Lemmata 7, 8, 9, 10. □

3.2 Closure properties

The set of nonnegative real Monge matrices forms a convex cone meaning the matrices are closed under linear combinations with nonnegative coeficients. The fact that ISM matrices are convex subsets of the set of real Monge matrices promises similar results for $\mathbb{ISM}$.

Theorem 12. *Let $\boldsymbol{M}, \boldsymbol{N} \in \mathbb{ISM}$ and let $\alpha \in \mathbb{R}_0^+$. Then also $\alpha \boldsymbol{M} \in \mathbb{ISM}$ and $\boldsymbol{M} + \boldsymbol{N} \in \mathbb{ISM}$.*

Proof. Let $\boldsymbol{M} \in \mathbb{ISM}$ and let $\alpha \in \mathbb{R}_0^+$. Then $\alpha\boldsymbol{M} \in \mathbb{ISM}$ because for any pair of indices i, j it holds

$$\overline{m}_{ij} + \overline{m}_{i+1,j+1} \leq \underline{m}_{i,j+1} + \underline{m}_{i+1,j}$$

since $\boldsymbol{M} \in \mathbb{ISM}$. It also holds that

$$\alpha\overline{m}_{ij} + \alpha\overline{m}_{i+1,j+1} \leq \alpha\underline{m}_{i,j+1} + \alpha\underline{m}_{i+1,j}$$

for $\alpha \geq 0$. Therefore, $\alpha\boldsymbol{M} \in \mathbb{ISM}$.
Further, let us consider matrices $\boldsymbol{M}, \boldsymbol{N} \in \mathbb{ISM}$. For any pair of indices i, j we have

$$\overline{m}_{ij} + \overline{m}_{i+1,j+1} \leq \underline{m}_{i,j+1} + \underline{m}_{i+1,j} \text{ and } \overline{n}_{ij} + \overline{n}_{i+1,j+1} \leq \underline{n}_{i,j+1} + \underline{n}_{i+1,j}$$

because both matrices are from $\mathbb{ISM}$. Adding these inequalities together and rearranging the elements we get

$$\overline{m}_{ij} + \overline{n}_{ij} + \overline{m}_{i+1,j+1} + \overline{n}_{i+1,j+1} \leq \underline{m}_{i,j+1} + \underline{n}_{i,j+1} + \underline{m}_{i+1,j} + \underline{n}_{i+1,j}$$

which is a condition of $\boldsymbol{M} + \boldsymbol{N}$. Therefore matrix $\boldsymbol{M} + \boldsymbol{N} \in \mathbb{ISM}$.

□

When it comes to multiplication by interval $\boldsymbol{\alpha} \in \mathbb{IR}_0^+$, ISM matrices are closed only under certain restriction dependent on the lower bound of $\boldsymbol{\alpha}$ and its radius.

Theorem 13. *Let $\boldsymbol{M} \in \mathbb{ISM}_0^+$ and let $\boldsymbol{\alpha} \in \mathbb{IR}_0^+$. Then $\boldsymbol{\alpha M} \in \mathbb{ISM}_0^+$ iff*

$$\frac{\alpha^\Delta}{\alpha^C} \leq \varphi \text{ where } \varphi = \min_{i,j} \left(\frac{\underline{m}_{i,j+1} + \underline{m}_{i+1,j} - \overline{m}_{ij} - \overline{m}_{i+1,j+1}}{\underline{m}_{i,j+1} + \underline{m}_{i+1,j} + \overline{m}_{ij} + \overline{m}_{i+1,j+1}} \right).$$

Proof. For all indices i, j it must hold that

$$\boldsymbol{\alpha}\overline{m}_{ij} + \boldsymbol{\alpha}\overline{m}_{i+1,j+1} \leq \boldsymbol{\alpha}\underline{m}_{i,j+1} + \boldsymbol{\alpha}\underline{m}_{i+1,j}.$$

It holds for all $\alpha \in \boldsymbol{\alpha}$ that

$$\alpha\overline{m}_{ij} + \alpha\overline{m}_{i+1,j+1} \leq \overline{\alpha}\ \overline{m}_{ij} + \overline{\alpha}\ \overline{m}_{i+1,j+1} \leq \underline{\alpha}\ \underline{m}_{i,j+1} + \underline{\alpha}\ \underline{m}_{i+1,j} \leq \alpha\overline{m}_{i,j+1} + \alpha\overline{m}_{i+1,j}.$$

It is clear that the the difference between the sides of inequality is the smallest for

$$\overline{\alpha}\ \overline{m}_{ij} + \overline{\alpha}\ \overline{m}_{i+1,j+1} \leq \underline{\alpha}\ \underline{m}_{i,j+1} + \underline{\alpha}\ \underline{m}_{i+1,j}.$$

Adjusting the inequality, we get

$$\overline{\alpha} \leq \underline{\alpha} \left(\frac{\underline{m}_{i,j+1} + \underline{m}_{i+1,j}}{\overline{m}_{ij} + \overline{m}_{i+1,j+1}} \right).$$

Substituting $\overline{\alpha}$ for $\alpha^C + \alpha^\Delta$, $\underline{\alpha}$ for $\alpha^C - \alpha^\Delta$ and adjusting again the inequality we get the formula

$$\frac{\alpha^\Delta}{\alpha^C} \leq \left(\frac{\underline{m}_{i,j+1} + \underline{m}_{i+1,j} - \overline{m}_{ij} - \overline{m}_{i+1,j+1}}{\underline{m}_{i,j+1} + \underline{m}_{i+1,j} + \overline{m}_{ij} + \overline{m}_{i+1,j+1}} \right). \tag{3.1}$$

It is now clear that the inequality 3.1 holds for all i, j iff it holds for minimum over all indices.

□

Finally, we state two observations. The first one is about matrix transposition and the second one about matrix products.

Observation 14. *For a matrix $\boldsymbol{M} \in \mathbb{ISM}$ the transposition $\boldsymbol{M}^T \in \mathbb{ISM}$.*

Proof. Straightforward from the definition of $\mathbb{ISM}$.

□

Observation 15. *Let us consider matrices*

$$A = \begin{pmatrix} 0.5 & 0.5 \\ 0.01 & 0.01 \end{pmatrix} B = \begin{pmatrix} 1,1 & 0.6 \\ 0.01 & 0.5 \end{pmatrix}.$$

The matrix $A \odot B \notin \mathbb{ISM}$ for $\odot$ representing Standard, Hadamard and Kronecker (tenzor) matrix product.

Proof. It can be easily checked by the definition of all three matrix products that the observation is correct.

□

The closure properties under operations combining $\mathbb{ISM}$ and $\mathbb{IWM}$ (interval weak Monge) matrices are discussed in Section 4.3.2 after we properly introduce the class of $\mathbb{IWM}$.

3.3 Interval envelopes of ISM matrices

In the previous section we showed that many results for real Monge matrices can be easily transformed for $\mathbb{ISM}$. There are two characteristics that failed to be generalized for ISM matrices. One of them is the decomposition of nonnegative Monge matrices into the extreme rays of convex cone (see Theorem 5). The other one is the characterization by distribution matrices (see Theorem 1).

3.3.1 Envelope of decomposition by extreme rays

Any nonnegative real Monge matrix M can be expressed as a nonnegative combination of special matrices

$$M = \sum_{i=1}^{m} \kappa_i H^i + \sum_{j=1}^{n} \lambda_j V^j + \sum_{r=2}^{m} \sum_{s=1}^{n-1} \mu_{rs} L^{rs} + \sum_{p=1}^{m-1} \sum_{q=2}^{n} \nu_{pq} R^{pq}.$$

Matrices H^i, V^j, L^{rs}, R^{pq} form extreme rays of convex cone generated by nonnegative Monge matrices. They are properly defined in Section 2.2.
We proved that the set $\mathbb{ISM}_0^+$ is closed under sum of two matrices and a nonnegative scalar multiplication, therefore it is natural to ask whether there exists a similar decomposition.

Definition 14. *Let $M \in \mathbb{R}^{m \times n}$ be a Monge matrix. Then define $C(M)$ a decomposition of M by extreme rays.*

Definition 15. *Let $\boldsymbol{M} \in \mathbb{ISM}^{m \times n}$ and for every $M \in \boldsymbol{M}$ fix one decomposition $C(M)$. Then define the envelope of convex decomposition of $\boldsymbol{M}$ as*

$$\mathbf{C}(\boldsymbol{M}) = \sum_{i=1}^{m} \boldsymbol{\kappa_i} H^i + \sum_{j=1}^{n} \boldsymbol{\lambda_j} V^j + \sum_{r=2}^{m} \sum_{s=1}^{n-1} \boldsymbol{\mu_{rs}} L^{rs} + \sum_{p=1}^{m-1} \sum_{q=2}^{n} \boldsymbol{\nu_{pq}} R^{pq}$$

where

$$\begin{aligned}
\boldsymbol{\kappa_i} = &\quad \square\left\{\kappa_i \in \mathbb{R}_0^+ | \exists M \in \boldsymbol{M} : \kappa_i H^i \text{ is in } C(M)\right\} \\
\boldsymbol{\lambda_j} = &\quad \square\left\{\lambda_i \in \mathbb{R}_0^+ | \exists M \in \boldsymbol{M} : \lambda_j V^i \text{ is in } C(M)\right\} \\
\boldsymbol{\mu_{pq}} = &\quad \square\left\{\mu_{rs} \in \mathbb{R}_0^+ | \exists M \in \boldsymbol{M} : \mu_{rs} L^{rs} \text{ is in } C(M)\right\} \\
\boldsymbol{\nu_{pq}} := &\quad \square\left\{\nu_{pq} \in \mathbb{R}_0^+ | \exists M \in \boldsymbol{M} : \nu_{pq} R^{pq} \text{ is in } C(M)\right\}
\end{aligned}$$

Definition 15 states that for every $M \in \boldsymbol{M}$ we take one convex decomposition and make a union of these decompositions (more precisely, an envelope of this union).
For any real matrix M it holds that $\mathbf{C}(M) = M$. For ISM matrix

$$\boldsymbol{M} = \begin{pmatrix} 0 & [0,5] \\ [0,5] & 0 \end{pmatrix}, \mathbf{C}(\mathbf{M}) = [0,5]\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + [0,5]\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we see that $\boldsymbol{C}(\boldsymbol{M}) = \boldsymbol{M}$ and the decomposition is unique. To see that in general $\boldsymbol{C}(\boldsymbol{M}) \neq \boldsymbol{M}$ let us consider matrix

$$\boldsymbol{M} = \begin{pmatrix} [0,5] & 5 \\ [0,8] & 0 \end{pmatrix}.$$

If $\boldsymbol{C}(\boldsymbol{M}) = \boldsymbol{M}$ then it is not hard to show that $\boldsymbol{C}(\boldsymbol{M})$ must be

$$\boldsymbol{C}(\boldsymbol{M}) = [0,3]\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + [0,5]\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

But for matrix

$$M = \begin{pmatrix} 1 & 5 \\ 6 & 0 \end{pmatrix}$$

there is no decomposition in $\boldsymbol{C}(\boldsymbol{M})$, although $M \in \boldsymbol{M}$. Therefore it must hold that $\boldsymbol{M} \subset \boldsymbol{C}(\boldsymbol{M})$.
The problem to find a tight envelope for a specific matrix $\boldsymbol{M}$ seems to be hard since for a real Monge matrix the decomposition is not unique.
We offer a trivial envelope.

Theorem 16. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ such that $\boldsymbol{M} \in \mathbb{ISM}_0^+$. Then for an envelope*

$$\mathbf{C}'(\boldsymbol{M}) = \sum_{i=1}^{m}[0,h^i]H^i + \sum_{j=1}^{n}[0,v^j]V^j + \sum_{r=2}^{m}\sum_{s=1}^{n-1}[0,l^{rs}]L^{rs} + \sum_{p=1}^{m-1}\sum_{q=2}^{n}[0,r^{pq}]R^{pq}$$

where

- $h^i = \max_j \overline{m}_{ij}$,
- $v^j = \max_i \overline{m}_{ij}$,
- $l^{rs} = \max_{ij} \overline{m}_{ij}$ *such that* $r \leq i$ *and* $j \geq s$,
- $r^{pq} = \max_{i,j} \overline{m}_{ij}$ *such that* $i \leq p$ *and* $q \leq j$

it holds that $\boldsymbol{M} \subset \mathbf{C}'(\boldsymbol{M})$.

Proof. For every matrix $M \in \boldsymbol{M}$ the entry m_{ij} is the sum of some coeficients. Let $M \in \boldsymbol{M}$. For any entry m_{ij} the coefficient of corresponding matrices H^i,V^j,L^{rs} and R^{pq} cannot be in sum larger than m_{ij} which is smaller than $\overline{m}_{ij}$ which is again smaller than the definition of each coefficient.

□

There might be a place for improvement since we envelope all possible decompositions of all matrices $M \in \boldsymbol{M}$. It remains an open question how to fix decompositions close to each other in coefficients.

3.3.2 Envelope of distribution matrix characterization

By Lemma 2 any real Monge matrix M can be expressed as

$$m_{ij} = d_{ij} + u_i + v_j$$

where $D \in \mathbb{R}^{m \times n}$ is a distribution matrix and $u \in \mathbb{R}^m, v \in \mathbb{R}^n$ are real vectors. We would want to generalize this property for interval Monge matrices as well. To have the decomposition in a form $\boldsymbol{m}_{ij} = \mathbf{d}_{ij} + \mathbf{u}_i + \mathbf{v}_j$ where $\boldsymbol{M} \in \mathbb{IR}^{m \times n}, \mathbf{D} \in \mathbb{IR}^{m \times n}, \mathbf{u} \in \mathbb{IR}^m, \mathbf{v} \in \mathbb{IR}^n$ we would have to define the interval distribution matrix which as we show does not seem to be a good approach.

Definition 16. *Let $\mathbf{C} \in \mathbb{IR}^{m \times n}$ be a nonnegative interval matrix. Then an interval distribution matrix $\mathbf{D} \in \mathbb{IR}^{m \times n}$ generated by an interval density matrix $\mathbf{C}$ is a matrix such that*

$$\mathbf{d}_{ij} = \left[\sum_{k=i}^{m} \sum_{\ell=1}^{j} \underline{c}_{k\ell}, \sum_{k=i}^{m} \sum_{\ell=1}^{j} \overline{c}_{k\ell} \right].$$

We show that $\mathbf{D} \notin \mathbb{ISM}$.

Lemma 17. *There exists $\mathbf{D} \in \mathbb{IR}^{m \times n}$ an interval distribution matrix such that $\mathbf{D} \notin \mathbb{ISM}$.*

Proof. Let us define $\mathbf{C} \in \mathbb{IR}^{2 \times 2}$ as

$$\mathbf{C} = \begin{pmatrix} [6,8] & [0,8] \\ [2,3] & [5,8] \end{pmatrix}.$$

The distribution matrix $\mathbf{D}$ generated by $\mathbf{C}$ is therefore

$$\mathbf{D} = \begin{pmatrix} [8,11] & [13,27] \\ [2,3] & [7,11] \end{pmatrix}.$$

Since $11 + 11 > 2 + 13$, $\mathbf{D} \notin \mathbb{ISM}$.

□

Since it is not possible to find a decomposition where $\mathbf{D}$ is an interval matrix we suggest fixing a real distribution matrix and setting interval vectors $\mathbf{u}$ and $\mathbf{v}$ in a way to have a close interval envelope of matrix $\boldsymbol{M}$. Such an envelope might be polynomially computable using linear programming.

4. Interval Weak Monge matrices

In this chapter we introduce interval weak Monge (IWM) matrices. Section 4.1 shows a definition and a polynomial recognizability. In Section 4.2 a necessary conditions and sufficient conditions are discussed and in Section 4.3 closure properties of IWM matrices and closure properties interconnecting ISM and IWM matrices are presented.

4.1 Definition and characterization

Definition 17. *An interval matrix $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ is interval weak Monge (IWM) iff $\exists M \in \boldsymbol{M}$ such that M is Monge. We denote by $\mathbb{IWM}$ the set of all interval weak Monge matrices.*

We start by showing that IWM matrices are polynomially recognizable by a special linear program.

Lemma 18. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ and let $LP(\boldsymbol{M})$ be a linear program defined as*

$$\begin{array}{ll} \textit{minimize} & const. \\ \textit{subject to} & m_{ij} + m_{i+1,j+1} - m_{i,j+1} - m_{i+1,j} \leq 0, \quad (1) \\ & m_{k\ell} \leq \overline{m}_{k\ell}, \quad (2) \\ & -m_{k\ell} \leq -\underline{m}_{k\ell}, \quad (3) \\ & \\ \textit{where} & 1 \leq i < m,\ 1 \leq j < n, \\ & 1 \leq k \leq m,\ 1 \leq \ell \leq n. \end{array}$$

Then the matrix $\boldsymbol{M} \in \mathbb{IWM}$ iff $LP(\boldsymbol{M})$ has a feasible solution.

Proof. Since the cost function is constant $LP(\boldsymbol{M})$ outputs a feasible solution. The solution matrix M satisfies Monge property by (1) and by (2) and (3) every entry $\underline{m}_{ij} \leq m_{ij} \leq \overline{m}_{ij}$, meaning $M \in \boldsymbol{M}$. We see that $LP(\boldsymbol{M})$ has feasible solution iff $\boldsymbol{M} \in \mathbb{IWM}$.

□

Lemma 18 is important because we know that the recognition problem of IWM matrices is polynomial. For $\mathbb{IWM}$ we did not find any other polynomial characterization. Let us note that all of the characterizations of real Monge matrices can be applied for $\mathbb{IWM}$, although none of them can be used without any further modification to construct a recognition algorithm.

4.2 Necessary and sufficient conditions

Although we know the recognition problem of $\mathbb{IWM}$ is polynomial, the only characterization we found was by linear programming which is categorized as one of the hardest problems in the hierarchy of polynomial algorithms (see [6]). Therefore we investigated necessary and sufficient conditions of $\mathbb{IWM}$.

4.2.1 Necessary conditions

The first necessary condition employs *residual* matrices defined in Definition 9 and their interval generalization.

Definition 18. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$. Then an interval residual matrix $\boldsymbol{M}^R \in \mathbb{IR}^{(m-1)\times(n-1)}$ is defined as*

$$\boldsymbol{m}_{ij}^R = \left[\underline{m}_{i+1,j} + \underline{m}_{i,j+1} - \overline{m}_{ij} - \overline{m}_{i+1,j+1}, \overline{m}_{i+1,j} + \overline{m}_{i,j+1} - \underline{m}_{ij} - \underline{m}_{i+1,j+1}\right]$$

Lemma 19. *Let $\boldsymbol{M} \in \mathbb{IWM}$ and $\boldsymbol{M}^R$ be its residual matrix. Then there exists $M^R \in \boldsymbol{M}^R$ such that M^R is nonnegative.*

Proof. If $\boldsymbol{M} \in \mathbb{IWM}$ it means that there exists $M \in \boldsymbol{M}$ such that M is Monge. By Definition 9 the residual matrix M^R of M is nonnegative.

□

Lemma 20. *Let $\boldsymbol{M} \in \mathbb{IWM}^{m\times n}$. Then there exists $M \in \boldsymbol{M}$ such that M is Monge and the number of entries $m_{ij} = \overline{m}_{ij}$ is at least $\max\{m, n\}$.*

Proof. Let $M \in \boldsymbol{M}$ be a Monge matrix. By Theorem 5 we can rewrite M as

$$M = \sum_{i=1}^{m} \kappa_i H^i + \sum_{j=1}^{n} \lambda_j V^j + \sum_{r=2}^{m}\sum_{s=1}^{n-1} \mu_{rs} L^{rs} + \sum_{p=1}^{m-1}\sum_{q=2}^{n} R^{pq}.$$

Let us take M such that the number of entries $m_{ij} = \overline{m}_{ij}$ in M is the highest possible and still lower than $\max\{m, n\}$. Let us suppose that $m > n$. It means that there is a row k in M where $m_{kj} \neq \overline{m}_{kj}$ for every column j. We take $\mu = \min_j \{\overline{m}_{kj} - m_{kj}\}$ and add μH^i to M. The matrix $M + \mu H^i$ is also Monge and the number of upper bounds of intervals in $M + \mu H^i$ is higher than in M. For $n > m$ we employ the matrices of type V^j and the rest of the argument is similar.

□

To show that the bound in Lemma 20 can be achieved we give the following example.

Example. Let $\boldsymbol{M} \in \mathbb{IR}^{4\times 4}$:

$$\boldsymbol{M} = \begin{pmatrix} [3, 1000] & [10, 120] & [17, 20] & [0, 24] \\ [2, 20] & [7, 9] & [0, 12] & [17, 85] \\ [2, 5] & [0, 6] & [10, 14] & [14, 100] \\ [0, 1] & [3, 6] & [5, 21] & [7, 1000] \end{pmatrix}.$$

Matrix $M \in \boldsymbol{M}$:

$$M = \begin{pmatrix} 3 & 10 & 17 & \textcircled{24} \\ 2 & 7 & \textcircled{12} & 17 \\ 2 & \textcircled{6} & 10 & 14 \\ \textcircled{1} & 3 & 5 & 7 \end{pmatrix}$$

is Monge, therefore $\boldsymbol{M} \in \mathbb{IWM}$. Moreover, the blue circles mark 4 values of M that are upper bounds of **M**. It is easy to check that for any $N \in \boldsymbol{M}$ that is Monge, no other entry of N can be an upper bound of **M** since it would violate at least one of neighbouring conditions of Monge property.

4.2.2 Sufficient conditions

The first two sufficient conditions use the decomposition into extreme rays of convex cone (see Theorem 5).

Lemma 21. *Let* $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$. *If it holds for every row* i *that* $\bigcap\limits_j [\boldsymbol{m}_{ij}] \neq \emptyset$ *or for every column* j *that* $\bigcap\limits_i [\boldsymbol{m}_{ij}] \neq \emptyset$, *then* $\boldsymbol{M} \in \mathbb{IWM}$.

Proof. Let us suppose that for every row i it holds that $\bigcap\limits_j [\boldsymbol{m}_{ij}] = [\underline{\alpha}_i, \overline{\alpha}_i]$. Then a matrix

$$M = \alpha_1 H^1 + \alpha_2 H^2 + \cdots + \alpha_n H^n$$

where $\alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i]$ is a Monge matrix by Theorem 5. Since $M \in \boldsymbol{M}$, we conclude that $\boldsymbol{M} \in \mathbb{IWM}$. For nonempty intersections of columns the argument is similar.

□

Lemma 22. *Let* $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$. *If it holds for all indices* i, j *that* $\frac{m^\Delta_{ij}}{|m^C_{ij}|} \geq 1$ *then* $\boldsymbol{M} \in \mathbb{IWM}$.

Proof. Let it hold for all indices i, j that $\frac{m^\Delta_{ij}}{|m^C_{ij}|} \geq 1$. Since $m^\Delta_{ij} \geq |m^C_{ij}|$, the matrix of zeros $0_{m\times n} \in \boldsymbol{M}$. Since $0_{m\times n}$ is Monge, $\boldsymbol{M} \in \mathbb{IWM}$.

□

Lemma 22 cares only for the origin of the coordinate plane. Therefore, it is a question how tight the inequality $\frac{m^\Delta_{ij}}{|m^C_{ij}|} \geq 1$ actually is and whether there exists an ε such that $\frac{m^\Delta_{ij}}{|m^C_{ij}|} \geq \varepsilon > 1$.

Another class of sufficient conditions of $\mathbb{IWM}$ matrices is based on an idea that in a space of matrices we start with M^C and use an easy procedure to move in steps from M^C until we reach a Monge matrix. Depending on the direction and distance of each step we can compute how far we have to move from M^C in each interval entry to achieve a Monge matrix. By this, we can get a sufficient condition dependent on the width of intervals. To determine the necessary width of intervals we employ residual matrices.

Lemma 23. *Let* $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ *and let* $M^R \in \mathbb{R}^{(m-1)\times(n-1)}$ *be the residual matrix of* M^C. *If for all indices* i, j *of* $\boldsymbol{M}$ *it holds that* $m^\Delta_{ij} \geq |\sum\limits_{k=i}^{m-1} \sum\limits_{\ell=j}^{n-1} m^R_{k\ell}|$ *then* $\boldsymbol{M} \in \mathbb{IWM}$.

Proof. Let $M^C \in \mathbb{R}^{m\times n}$ and let $M^R \in \mathbb{R}^{(m-1)\times(n-1)}$ be its residual matrix. In general, the residual matrix M^R will not be nonnegative. Our goal is to eliminate the residues in M^R one by one using a specific elimination order. We see that by subtracting ε from m^C_{ij} the value of m^R_{ij} increases by ε. By this operation, entries $m^R_{i-1,j-1}, m^R_{i-1,j}$ and $m^R_{i,j-1}$ are affected as well (see Figure 4.1). We start from the bottom-right corner of M^R and add the value of $m^R_{m-1,n-1}$ to $m^C_{m-1,n-1}$. This eliminates the residuum $m^R_{m-1,n-1}$ and propagates it into the three neighbouring entries. In next step, we eliminate the residuum of the element $m^R_{m-1,n-2}$ and

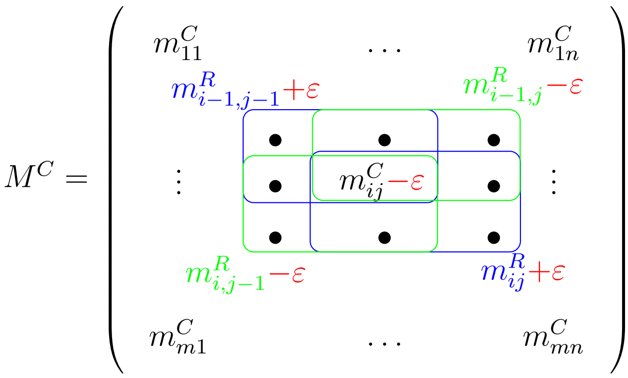

Figure 4.1: Subtracting ε from m^C_{ij} and its effect on entries of M^R.

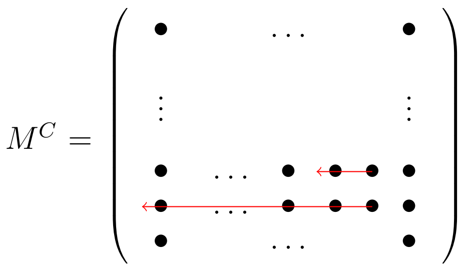

Figure 4.2: The order of changing values in M^C to eliminate the residues of M^R.

continue in the decreasing order of columns until we arrive at the beginning of the row, then proceed with the row above in the same manner (see Figure 4.2). By each step we eliminate one residuum and more importantly, no residuum already eliminated is affected further in the process (see Figure 4.3).

Not only this elimination order yields $0^{(m-1)\times(n-1)}$ residual matrix (therefore

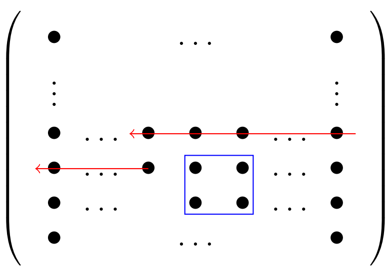

Figure 4.3: Once m^C_{ij} is altered, no further operation changes the value of m^R_{ij}.

a corresponding Monge matrix) but it is also easy to describe the propagation of residual values in M^R. Eliminating the value α from residuum m^R_{ij} adds α to $m^R_{i-1,j}$ and $m^R_{i,j-1}$ and subtracts it from $m^R_{i-1,j-1}$ (see Figure 4.4). Now if the intervals of $\boldsymbol{M}$ are large enough we can move from the center far enough to eliminate the residues. It is now easy to compute by induction the necessary condition for each interval of $\boldsymbol{M}$. For the base step, from the way of propagation (ilustrated by Figure 4.4) it is clear that it must hold that

- $m^\Delta_{m-1,n-1} \geq |m^R_{m-1,n-1}|$

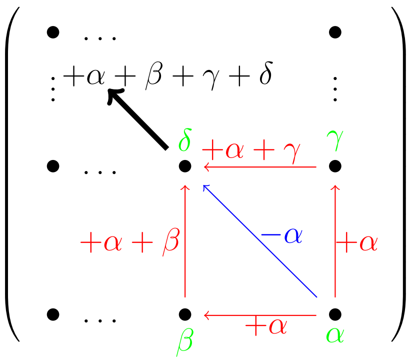

Figure 4.4: The residual propagation in M^R.

- $m^{\Delta}_{m-1,n-2} \geq |m^R_{m-1,n-1} + m^R_{m-1,n-2}|$,
- $m^{\Delta}_{m-2,n-1} \geq |m^R_{m-1,n-1} + m^R_{m-2,n-1}|$,
- $m^{\Delta}_{m-2,n-2} \geq |m^R_{m-2,n-1} + m^R_{m-1,n-2} + 2m^R_{m-1,n-1} - m^R_{m-1,n-1}|$, therefore $m^{\Delta}_{m-2,n-2} \geq |m^R_{m-2,n-1} + m^R_{m-1,n-2} + m^R_{m-1,n-1}|$.

For inductional step let us suppose the residuum in m^R_{ij}. It must hold that

$$m^{\Delta}_{ij} \geq |m^R_{ij} + m^R_{i+1,j} + m^R_{i,j+1} - m^R_{i+1,j+1}|.$$

By induction we know that the residues are equal to

$$m^{\Delta}_{ij} \geq |m^R_{ij} + \sum_{k=i+1}^{m-1} \sum_{\ell=j}^{n-1} m^R_{k\ell} + \sum_{k=i}^{m-1} \sum_{\ell=j+1}^{n-1} m^R_{k\ell} - \sum_{k=i+1}^{m-1} \sum_{\ell=j+1}^{n-1} m^R_{k\ell}|$$

which is equal to the form stated in the lemma.

□

Let us note that the condition we just showed can be checked in $O(mn)$ time using dynamic programming.
What we showed in the previous lemma is one of many modifications of the same condition depending on the order we choose to zero the values in M^R. The advantage of this one-diagonal order is that it is easy to compute the width of intervals. We present one more condition from this class. The previous condition works well when the sum $|\sum_{k=i}^{m-1} \sum_{l=j}^{n-1} m^R_{ij}| \sim 0$ or is at least small for every i, j. If the errors are of the same sign, however, the sum has tendency to grow a lot. This is because we propagate the error only in one direction.
We can choose a point in the matrix and propagate the error in four different (diagonal) directions.

Lemma 24. *Let $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ and let $M^R \in \mathbb{R}^{(m-1)\times(n-1)}$ be the residual matrix of M^C. If there exist indices i, j of $\boldsymbol{M}$ such that*

- $m^{\Delta}_{rs} \geq |\sum_{k=r}^{i-1} \sum_{\ell=s}^{j-1} m^R_{k\ell}|$ *for every* $(r < i) \wedge (s < j)$,
- $m^{\Delta}_{rs} \geq |\sum_{k=r}^{i-1} \sum_{\ell=j}^{s-1} m^R_{k\ell}|$ *for every* $(r < i) \wedge (s > j)$,

- $m^{\Delta}_{rs} \geq |\sum\limits_{k=i}^{r} \sum\limits_{\ell=s}^{j-1} m^R_{k\ell}|$ *for every* $(r > i) \wedge (s < j)$,
- $m^{\Delta}_{rs} \geq |\sum\limits_{k=i}^{r} \sum\limits_{\ell=j}^{s} m^R_{k\ell}|$ *for every* $(r > i) \wedge (s > j)$,

then $\boldsymbol{M} \in \mathbb{IWM}$.

Proof. Let i, j be indices of M^C. Then we can take $m^R_{i-1,j-1}, m^R_{i-1,j+1}, m^R_{i+1,j-1}$ and $m^R_{i+1,j+1}$ as starting points for residual elimination described in Lemma 23. We can see in Figure 4.5 that the residues are not propagated between the blocks of M^R. The conditions follow from Lemma 23.

□

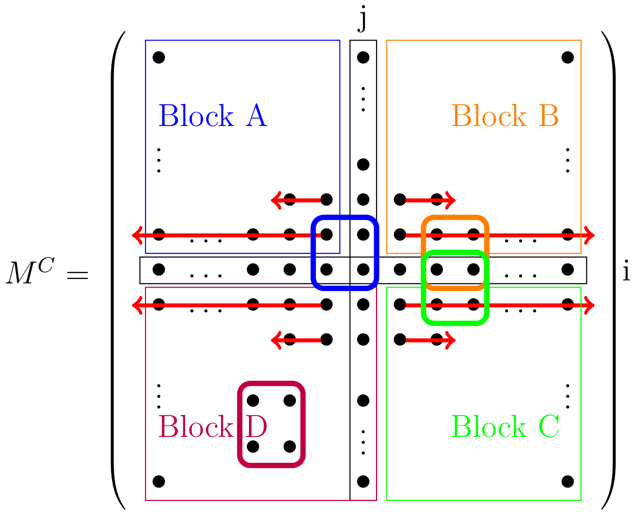

Figure 4.5: The residual propagation does not interfere between the blocks.

Let us now state the list of sufficient conditions in one theorem.

Theorem 25. *Let* $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ *and* $M^R \in \mathbb{R}^{(m-1)\times(n-1)}$ *the residual matrix of* M^C*. If one of the conditions below is true, then* $\boldsymbol{M} \in \mathbb{IWM}$.

1. *For all rows* i *it holds that* $\bigcap\limits_j [\boldsymbol{m}_{ij}] \neq \emptyset$.
2. *For all columns* j *it holds that* $\bigcap\limits_i [\boldsymbol{m}_{ij}] \neq \emptyset$.
3. *For all indices* i, j *of* $\boldsymbol{M}$ *it holds that* $\frac{m^{\Delta}_{ij}}{|m^C_{ij}|} \geq 1$.
4. *For all indices* i, j *of* M^R *it holds that* $m^{\Delta}_{ij} \geq |\sum_{k=i}^{m-1} \sum_{\ell=j}^{n-1} m^R_{k\ell}|$.
5. *If there exist indices* i, j *of* $\boldsymbol{M}$ *such that*
   - $m^{\Delta}_{rs} \geq \sum\limits_{k=r}^{i-1} |\sum\limits_{\ell=s}^{j-1} m^R_{k\ell}|$ *for every* $(r < i) \wedge (s < j)$
   - $m^{\Delta}_{rs} \geq \sum\limits_{k=r}^{i-1} |\sum\limits_{\ell=j}^{s-1} m^R_{k\ell}|$ *for every* $(r < i) \wedge (s > j)$

- $m_{rs}^{\Delta} \geq |\sum_{k=i}^{r-1} \sum_{\ell=s}^{j-1} m_{k\ell}^{R}|$ *for every* $(r > i) \wedge (s < j)$,
- $m_{rs}^{\Delta} \geq |\sum_{k=i}^{r-1} \sum_{\ell=j}^{s-1} m_{k\ell}^{R}|$ *for every* $(r > i) \wedge (s > j)$.

Proof. See Lemmata 21, 22, 23, 24.

□

4.3 Closure properties

We investigated closure properties of several operations on $\mathbb{IWM}$. Most of the results are easy to prove, therefore we state them in one theorem.

4.3.1 Closure properties of IWM

Theorem 26. *Let* $\boldsymbol{P} \in \mathbb{IR}^{m \times n}, \boldsymbol{M}, \boldsymbol{N} \in \mathbb{IWM}^{m \times n}$, $\alpha \in \mathbb{R}_0^+$ *and* $\boldsymbol{\alpha} \in \mathbb{IR}_0^+$. *Then the following holds.*

1. $\boldsymbol{M} + \boldsymbol{N} \in \mathbb{IWM}$,
2. $\boldsymbol{M} + \boldsymbol{P} \in \mathbb{IWM}$ *iff* $\overline{\boldsymbol{M}^R} + \overline{\boldsymbol{P}^R} \geq 0$,
3. $\square(\boldsymbol{M} \cup \boldsymbol{P}) \in \mathbb{IWM}$,
4. $\alpha \boldsymbol{M} \in \mathbb{IWM}$,
5. $\boldsymbol{\alpha M} \in \mathbb{IWM}$.

Proof. All the results are easy to prove from the definition of $\mathbb{IWM}$.

□

4.3.2 Closure properties interconnecting IWM and ISM

Theorem 27. *Let* $\boldsymbol{M} \in \mathbb{ISM}^{m \times n}, \boldsymbol{N} \in \mathbb{IWM}^{m \times n}$, $\alpha \in \mathbb{R}_0^+$ *and* $\boldsymbol{\alpha} \in \mathbb{IR}_0^+$. *Then the following holds.*

1. $\boldsymbol{M} + \boldsymbol{N} \in \mathbb{IWM}$,
2. $\forall i, j$ *it holds that* $\mathbf{m}_{ij} \cap \mathbf{n}_{ij} \neq \emptyset \rightarrow \boldsymbol{M} \cap \boldsymbol{N} \in \mathbb{IWM}$,
3. $\square(\boldsymbol{M} \cup \boldsymbol{N}) \in \mathbb{IWM}$.

Proof. All the results are easy to prove from the definition of $\mathbb{IWM}$ and $\mathbb{ISM}$.

□

5. Reconstruction algorithms

If the recognition algorithms reveal that an interval matrix does not satisfy the Monge property, we might still consider affecting the entries of the matrix in order to reconstruct the property. In this chapter we present a way to reconstruct $\boldsymbol{M}' \in \mathbb{IWM}^{m\times n}$ from $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ and from $\mathbf{N} \in \mathbb{IWM}^{m\times n}$ a matrix $\mathbf{N}' \in \mathbb{ISM}^{m\times n}$. Both methods are based on mathematical programming that gives us the option to encode demands into a cost function. We might prefer to let some entries untouched or change the entries uniformly.
In Section 5.1 we present an Inflation algorithm that for a given interval matrix inflate its entries in order to obtain $\mathbb{IWM}$ matrix. To get $\mathbb{ISM}$ from $\mathbb{IWM}$ we do the opposite - prune the intervals of the interval matrix. The Pruning algorithm is presented in Section 5.2.

5.1 Inflation algorithm

To formulate the Inflation problem we need to employ further definitions.

Definition 19. *Let $U, L \in \mathbb{R}^{m\times n}$ be nonnegative matrices. Then for an interval matrix $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ define an inflation of $\boldsymbol{M}$ as a matrix*

$$\boldsymbol{M}^{UL} \in \mathbb{IR}^{m\times n} \text{ such that } \boldsymbol{m}_{ij}^{UL} = \left[\underline{m}_{ij} - l_{ij}, \overline{m}_{ij} + u_{ij}\right].$$

Call U,L upper resp. lower inflation matrix of matrix $\boldsymbol{M}$.

The problem can be now defined as follows.

Problem (Inflation). Let $\| \bullet \|$ be a matrix norm and $\boldsymbol{M}$ be an interval matrix. Then the Inflation problem is to find a matrix $\boldsymbol{M}^{UL}$ such that $\boldsymbol{M}^{UL} \in \mathbb{IWM}$ and $\|U\| + \|L\|$ is minimal.

Theorem 28. *For some matrix norms there exists a polynomial time algorithm that solves the Inflation problem.*

Proof. Let $\boldsymbol{M}$ and $\| \bullet \|$ be defined. Then derive the following program:

$$\begin{array}{lll} \text{minimize} & \|U\| + \|L\| & \\ \text{subject to} & m_{ij} + m_{i+1,j+1} - m_{i,j+1} - m_{i+1,j} \le 0, & (1) \\ & m_{k\ell} - u_{k\ell} \le \overline{m}_{k\ell}, & (2) \\ & -m_{k\ell} + l_{kl} \le -\underline{m}_{k\ell}, & (3) \\ & & \\ \text{where} & 1 \le i < m,\ 1 \le j < n, & \\ & 1 \le k \le m,\ 1 \le \ell \le n. & \end{array}$$

Conditions of type (1) enforce that any feasible solution forms a matrix that is Monge. The conditions of type (2) and (3) ensure that for the values $u_{k\ell}$ resp. $l_{k\ell}$ it holds that $m_{k\ell} \le \overline{m}_{k\ell} + u_{k\ell}$ and $\underline{m}_{k\ell} - l_{k\ell} \le m_{k\ell}$ meaning that $m_{ij} \in [\underline{m}_{k\ell} - l_{k\ell}, \overline{m}_{k\ell} + u_{k\ell}]$ for all indices i, j. Hence, setting U and L as upper resp. lower inflation matrix of M satisfies that $\boldsymbol{M}^{UL} \in \mathbb{IWM}$. The minimality of $\|U\| + \|L\|$ is guaranteed by the cost function. Notice that the conditions are of form $Ax \leq b$. Hence, depending on the norm this program yields special classes of mathematical optimization problems (e.g. for a linear norm linear programming) which are polynomially solvable (for polynomiality of linear programming see [6]).

□

5.2 Pruning algorithm

For the pruning problem we will employ a definition similar to Definition 19.

Definition 20. *Let $T, B \in \mathbb{R}^{m \times n}$ be nonnegative matrices. Then for an interval matrix $\boldsymbol{M} \in \mathbb{IR}^{m \times n}$ define a pruning of $\boldsymbol{M}$ as a matrix*

$$\boldsymbol{M}^{TB} \in \mathbb{IR}^{m \times n} \text{ such that } \boldsymbol{m}_{ij}^{TB} = \left[\underline{m}_{ij} + b_{ij}, \overline{m}_{ij} - t_{ij}\right].$$

Call T, B upper resp. lower pruning matrix of matrix $\boldsymbol{M}$.

Problem (Pruning). Let $\| \bullet \|$ be a matrix norm and $\boldsymbol{M} \in \mathbb{IWM}$. Then the Pruning problem is to find a matrix $\boldsymbol{M}^{TB}$ such that $\boldsymbol{M}^{TB} \in \mathbb{ISM}$ and $\|T\| + \|B\|$ is minimal.

Theorem 29. *For special matrix norms there exists a polynomial time algorithm that solves the Pruning problem.*

Proof. Let $\boldsymbol{M}$ and $\| \bullet \|$ be defined. Then derive the following program:

$$\begin{array}{ll} \text{minimize} & \|T\| + \|B\| \\ \text{subject to} & -t_{ij} - t_{i+1,j+1} - b_{i+1j} - b_{ij+1} \leq \underline{m}_{i+1,j} + \underline{m}_{i,j+1} - \overline{m}_{ij} - \overline{m}_{i+1,j+1}, \\ & t_{ij} + b_{ij} \leq \overline{m}_{ij} - \underline{m}_{ij}, \\ & \\ \text{where} & 1 \leq i < m,\ 1 \leq j < n. \end{array}$$

Conditions of type $t_{ij} + b_{ij} \leq \overline{m}_{ij} - \underline{m}_{ij}$ ensure that the intervals $\left[\underline{m}_{ij} + b_{ij}, \overline{m}_{ij} - t_{ij}\right]$ are well defined. By Lemma 8 the first type of conditions of the program satisfies that $\boldsymbol{M}^{TB}$ is $\mathbb{ISM}$ since inequalities can be rearranged in a form

$$\overline{m}_{ij} - t_{ij} + \overline{m}_{i+1,j+1} - t_{i+1,j+1} \leq \underline{m}_{i+1,j} + b_{i+1,j} + \underline{m}_{i,j+1} + b_{i,j+1}.$$

The minimality of $\|T\| + \|B\|$ is ensured by the cost function of the program. Notice that the conditions are of form $Ax \leq b$. Hence, similarly to solving the Inflation problem, depending on the norm this program yields special classes of mathematical optimization problems (e.g. for a linear norm linear programming) which are polynomially solvable (for polynomiality of linear programming see [6]).

□

6. Permutation algorithm

In many optimization problems (e.g. travelling salesman problem, transportation problem,...) the optimal solution of the problem is invariant to a row and a column permutation of the cost matrix. Therefore a question whether there exist such permutations is natural. We introduce a generalization of a permutation algorithm by Deineko and Filonenko [7] running in $O(n^2)$ for $\mathbb{ISM}$ square matrices. For matrices from $\mathbb{IWM}$ the algorithm does not seem to have a straightforward generalized and it remains an open problem. In Section 6.1 we present an idea of the algorithm. In Section 6.2 we introduce and prove auxiliary lemmata that we will use in Section 6.3 where the algorithm is derived.

6.1 The idea of the algorithm

The algorithm performs 3 permutations and then checks if the resulting matrix is from $\mathbb{ISM}$. The first one permutes columns upon a necessary condition of two rows. This gives us a prepermutation that divides the columns into so called *ambiguity sets*. Although for two columns from different sets the order is clear, inside the sets it is not. The second permutation is performed on rows. Columns from the first ambiguity set and the last ambiguity set are candidates for the first resp. the last column. A combination of these candidates is used in order to obtain a rule for the second permutation. The necessary conditions from this rule are strong enough to determine the resulting order of rows. Finally, a third permutation permutes columns again by the same necessary condition as was used in the first permutation, although this time it is applied on the first and the last row. By this we deal with those columns inside one ambiguity set, that still need to be switched. After this procedure it remains to check if the resulting matrix is from $\mathbb{ISM}$. If it is not, then there is no way to permute the matrix in order to obtain Monge property. Further in the text we describe the process in detail relying on lemmata that we prove in the next section.

6.2 Lemmata for the derivation of the algorithm

In this section we prove lemmata that are necessary for a derivation of the algorithm. We denote by $\boldsymbol{M}(\sigma, \pi)$ a matrix $\boldsymbol{M}$ permuted by a row permutation σ and a column permutation π. If there exist permutations σ, π such that $\boldsymbol{M}(\sigma, \pi) \in \mathbb{ISM}$ we say that $\boldsymbol{M}$ is *Monge permutable*.
The first lemma states that $\mathbb{ISM}$ is closed under *flipping the matrix upside down and left to right*.

Lemma 30. *Let $\boldsymbol{M} \in \mathbb{ISM}^{m\times n}$. Define $\sigma(i) = m - i + 1$ and $\pi(j) = n - j + 1$. Then $\boldsymbol{M}(\sigma, \pi) \in \mathbb{ISM}$.*

Proof. For every pair of indices i, j we have that

$$\overline{m}_{\sigma(i),\pi(j)} + \overline{m}_{\sigma(i+1),\pi(i+1)} = \overline{m}_{m-i+1,n-j+1} + \overline{m}_{m-i,n-j}.$$

From the Monge property we have

$$\overline{m}_{m-i+1,n-j+1} + \overline{m}_{m-i,n-j} \le \underline{m}_{m-i,n-j+1} + \underline{m}_{m-i+1,n-j},$$

but the righthand side of the inequality is equal to

$$\underline{m}_{m-i,n-j+1} + \underline{m}_{m-i+1,n-j} = \underline{m}_{\sigma(i),\pi(j+1)} + \underline{m}_{\sigma(i+1),\pi(j)}.$$

By Lemma 8 we conclude that $\boldsymbol{M}(\sigma,\pi) \in \mathbb{ISM}$.

□

The following lemma is rather technical. It states for two rows i, k and two columns j, ℓ if certain conditions with upper and lower bounds of intervals hold, than all entries involved in the conditions and all between are real values and we can conclude equality between sums of pairs of these entries.

Lemma 31. *Let $\boldsymbol{M} \in \mathbb{ISM}$ and let row indices $i < k$ and column indices $j < \ell$. If it holds that*

$$\overline{m}_{ij} - \underline{m}_{kj} \le \underline{m}_{i\ell} - \overline{m}_{k\ell} \text{ and } \overline{m}_{i\ell} - \underline{m}_{k\ell} \le \underline{m}_{ij} - \overline{m}_{kj}$$

then for all rows o such that $i < o \le k$ it holds

1. $\boldsymbol{m}_{ij}, \boldsymbol{m}_{oj}, \boldsymbol{m}_{i\ell}, \boldsymbol{m}_{o\ell} \in \mathbb{R}$,
2. $\boldsymbol{m}_{ij} - \boldsymbol{m}_{oj} = \boldsymbol{m}_{i\ell} - \boldsymbol{m}_{o\ell}$.

Proof. The following sequence of inequalities

$$\overline{m}_{ij} - \underline{m}_{kj} \le \underline{m}_{i\ell} - \overline{m}_{k\ell} \le \overline{m}_{i\ell} - \underline{m}_{k\ell} \le \underline{m}_{ij} - \overline{m}_{kj} \le \overline{m}_{ij} - \underline{m}_{kj}$$

turns into a sequence of equalities since the first and last members are the same. This means that also

$$\underline{m}_{i\ell} - \overline{m}_{k\ell} = \overline{m}_{i\ell} - \underline{m}_{k\ell}.$$

Rearranging the equation we have

$$-2 \cdot m^{\Delta}_{k\ell} = \underline{m}_{k\ell} - \overline{m}_{k\ell} = \overline{m}_{i\ell} - \underline{m}_{i\ell} = 2 \cdot m^{\Delta}_{i\ell},$$

from which

$$-m^{\Delta}_{k\ell} = m^{\Delta}_{i\ell}.$$

But this means that

$$\boldsymbol{m}_{i\ell}, \boldsymbol{m}_{k\ell} \in \mathbb{R}.$$

For $\boldsymbol{m}_{ij}$ and $\boldsymbol{m}_{kj}$ the argument is similar.
Now the inequalities

$$\boldsymbol{m}_{ij} - \boldsymbol{m}_{kj} \le \boldsymbol{m}_{i\ell} - \boldsymbol{m}_{k\ell} \text{ and } \boldsymbol{m}_{i\ell} - \boldsymbol{m}_{k\ell} \le \boldsymbol{m}_{ij} - \boldsymbol{m}_{kj}$$

combined together imply

$$\boldsymbol{m}_{i\ell} + \boldsymbol{m}_{kj} = \boldsymbol{m}_{ij} + \boldsymbol{m}_{k\ell}.$$

Let us now consider o such that $i < o \le k$. Since $\boldsymbol{M} \in \mathbb{ISM}$ it must hold

$$\boldsymbol{m}_{ij} - \boldsymbol{m}_{i\ell} \le \underline{m}_{oj} - \overline{m}_{o\ell} \text{ and } \overline{m}_{oj} - \underline{m}_{o\ell} \le \boldsymbol{m}_{kj} - \boldsymbol{m}_{k\ell}.$$

Connecting both inequalities using the fact that $\underline{m}_{oj} - \overline{m}_{o\ell} \leq \overline{m}_{oj} - \underline{m}_{o\ell}$ we conclude that

$$\underline{m}_{oj} - \overline{m}_{o\ell} = \overline{m}_{oj} - \underline{m}_{o\ell}.$$

This means that $\boldsymbol{m}_{oj}, \boldsymbol{m}_{o\ell} \in \mathbb{R}$ and since

$$\boldsymbol{m}_{ij} - \boldsymbol{m}_{i\ell} \leq \boldsymbol{m}_{oj} - \boldsymbol{m}_{o\ell} \leq \boldsymbol{m}_{kj} - \boldsymbol{m}_{k\ell} = \boldsymbol{m}_{ij} - \boldsymbol{m}_{i\ell}$$

we conclude that $\boldsymbol{m}_{ij} - \boldsymbol{m}_{i\ell} = \boldsymbol{m}_{oj} - \boldsymbol{m}_{o\ell}$.

□

Lemma 32 deals with a combination of conditions of form

$$\overline{m}_{ij} - \underline{m}_{i\ell} \leq \underline{m}_{kj} - \overline{m}_{k\ell} \text{ where } i < k, j < \ell.$$

These conditions are necessary for a matrix to be in $\mathbb{ISM}$. For the combination below first b and last B columns are picked for the combination.

Lemma 32. *Let $\boldsymbol{M} \in \mathbb{IR}^{m \times n}$. If $\boldsymbol{M} \in \mathbb{ISM}$ then for every pair of rows i and k such that $i < k$ it holds*

$$B \cdot \left(\sum_{j=1}^{b} \overline{m}_{ij} \right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \underline{m}_{i\ell} \right) \leq B \cdot \left(\sum_{j=1}^{b} \underline{m}_{kj} \right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \overline{m}_{k\ell} \right)$$

where $1 \leq b < n - B + 1 \leq n$.

Proof. For $i < k$, it holds for every j such that $1 \leq j \leq b$ and every ℓ such that $n - B + 1 \leq \ell \leq n$ that

$$\overline{m}_{ij} - \underline{m}_{i\ell} \leq \underline{m}_{kj} - \overline{m}_{k\ell}.$$

By picking such an inequality for every pair (j, ℓ) where $j \in \{1, \ldots, b\}$ and $\ell \in \{n - B + 1, \ldots, n\}$ and adding all these inequalities together we get the formula above.

□

The following lemma gives an algorithm to compute the permutations of rows and columns.

Lemma 33. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{IR}^n$. Let σ be a permutation of $\{1, \ldots, n\}$ such that whenever*

$$\sigma(i) < \sigma(j) \text{ then } \overline{u}_i - \underline{v}_i \leq \underline{u}_j - \overline{v}_j.$$

Then in $O(n^2)$ we can compute σ or answer that there is no such permutation.

Proof. We construct a directed graph $G = (n, E)$ where $(i, j) \in E$ if $\overline{u}_i - \underline{v}_i \leq \underline{u}_j - \overline{v}_j$. If there is a pair of vertices $i, j \in G$ without an edge between them, it means that

$$\overline{u}_i - \underline{v}_i > \underline{u}_j - \overline{v}_j \text{ and } \overline{u}_j - \underline{v}_j > \underline{u}_i - \overline{v}_i.$$

and by the definition of σ no mutual order of these indices yields the permutation so we stop. From now on, let us suppose that there is at least one edge between all pairs of vertices in graph G.

Now let $c_1, \ldots, c_k$ be strongly connected components of G such that $t(c_1) < \cdots < t(c_k)$, where t is some topological ordering of strongly connected components of

G.

Now define σ as follows. While σ is not defined for all indices $i \in \{1, \ldots, n\}$, pick between indices with unspecified $\sigma(i)$ the one for which the topological number of the strongly connected component containing the vertex i is minimal. Set $\sigma(i)$ as the smallest number from $\{1, \ldots, n\}$ not assigned yet.

To prove that the construction is correct let i, j be indices such that $\sigma(i) < \sigma(j)$. Then either vertices i, j are from the same component or i is from a component with a smaller topological number than the component containing j. If i and j are from the same component of G, it means by the contruction of G that there are edges (i, j) and (j, i) therefore it holds that $\overline{u}_i - \underline{v}_i \leq \underline{u}_j - \overline{v}_j$. If i is in a component with smaller topological number than j it means that there is an edge (i, j). But the edge (i, j) corresponds to the inequality $\overline{u}_i - \underline{v}_i \leq \underline{u}_j - \overline{v}_j$.

There exists an algorithm for finding a topological ordering of strongly connected components of a directed graph running in $O(n+m)$ where n equals the number of vertices and m equals the number of edges (see [8]). Since the number of edges m is in the worst case approximately $m \approx n^2$, the algorithm runs in $O(n^2)$. Defining σ from the topological ordering t takes $O(n)$, therefore the whole construction takes $O(n^2)$.

□

Finally, we prove a lemma about the first step of our algorithm. In this step a pair of rows is determined. The first permutation ρ is based on conditions between these two rows. We demand at least two columns to be strictly ordered otherwise the permutation ρ will have no effect (we want it to prepermute the matrix). A strict order of two columns corresponds to two different *ambiguity sets*. According to logical structure of this chapter we state the lemma in this subsection, however, the notion of *ambiguity sets* necessary in the lemma becomes clear in Subsection 6.3.1. We recommend to the reader to first go through the derivation of the algorithm and the mentioned subsection.

Lemma 34. *Let $\boldsymbol{M} \in \mathbb{IR}^{n\times n}$. Then a problem to decide if there is a row r such that there are two ambiguity sets of columns for rows 1 and r can be computed in $O(n^2)$. If for every row r there is only one ambiguity set of columns, then the matrix is from $\mathbb{ISM}$.*

Proof. For every row k and for all neighbouring pairs of columns (i.e. $j, j+1$ for $1 \leq j \leq n-1$) we check if it holds that

$$\overline{m}_{1j} - \underline{m}_{kj} < \underline{m}_{1,j+1} - \overline{m}_{k,j+1} \text{ or } \overline{m}_{1,j+1} - \underline{m}_{k,j+1} < \underline{m}_{1j} - \overline{m}_{kj}. \tag{6.1}$$

Only one of these inequlities can hold at the same time because otherwise

$$\overline{m}_{1j} - \underline{m}_{kj} < \underline{m}_{1,j+1} - \overline{m}_{k,j+1} \leq \overline{m}_{1,j+1} - \underline{m}_{k,j+1} < \underline{m}_{1j} - \overline{m}_{kj} \leq \overline{m}_{1j} - \underline{m}_{kj}$$

which leads to a contradiction $\overline{m}_{1j} - \underline{m}_{kj} < \overline{m}_{1j} - \underline{m}_{kj}$. If one of the inequalities holds and the other is $=$, then w.l.o.g. consider

$$\overline{m}_{1j} - \underline{m}_{kj} < \underline{m}_{1,j+1} - \overline{m}_{k,j+1} \text{ and } \overline{m}_{1,j+1} - \underline{m}_{k,j+1} = \underline{m}_{1j} - \overline{m}_{kj}.$$

From these two inequalities we can derive that

$$\overline{m}_{1j} - \underline{m}_{kj} < \underline{m}_{1,j+1} - \overline{m}_{k,j+1} \leq \overline{m}_{1,j+1} - \underline{m}_{k,j+1} = \underline{m}_{1j} - \overline{m}_{kj}$$

and therefore $\overline{m}_{1j} - \underline{m}_{kj} < \underline{m}_{1j} - \overline{m}_{kj}$ which is again a contradiction.
This means that if one inequality holds with $<$ the other must hold with $>$, therefore the order of the columns is strict and they cannot be switched. A strict order of two columns means that these columns cannot be in one ambiguity set, therefore we return row k.
It might happen that for every pair $j, j+1$ and for row k neither of the inequalities from 6.1 is strict. It means that

$$\overline{m}_{1j} - \underline{m}_{kj} \geq \underline{m}_{1,j+1} - \overline{m}_{k,j+1} \text{ and } \overline{m}_{1,j+1} - \underline{m}_{k,j+1} \geq \underline{m}_{1j} - \overline{m}_{kj}. \tag{6.2}$$

If both of the inequalities are strict for at least one pair $j, j+1$, it means that no order of columns $j, j+1$ satisfy the Monge property and in that case we stop.
If both of the inequalities hold with equality $=$ for all pairs of columns $j, j+1$ in the row k it means that

$$\overline{m}_{1j} - \underline{m}_{kj} = \underline{m}_{1,j+1} - \overline{m}_{k,j+1} \leq \overline{m}_{1,j+1} - \underline{m}_{k,j+1} = \underline{m}_{1j} - \overline{m}_{kj} \leq \overline{m}_{1j} - \underline{m}_{kj},$$

therefore $\boldsymbol{m}_{1j}\boldsymbol{m}_{kj}$ and also $\boldsymbol{m}_{k,j+1}\boldsymbol{m}_{1,j+1}$ are real values and therefore $\boldsymbol{m}_{1j} - \boldsymbol{m}_{kj} = \boldsymbol{m}_{1,j+1} - \boldsymbol{m}_{k,j+1}$. If this happens for all rows k then the matrix is already Monge because every condition holds with equality.
The last case which remains is when one of the inequalities from 6.2 is strict $>$ and the second one is equal $=$ for at least one row r. Then the order is strict again, because there is only one way to permute these two columns in order to satisfy Monge property. Therefore we return row r.

Applying this procedure to each of $n-1$ rows the number of conditions to check is at most $2(n-1)$ for each row. We conclude that the problem can be computed in $O(n^2)$.

□

6.3 Derivation of the algorithm

With all the lemmata we are prepared to derive the algorithm.

6.3.1 Prepermutation ρ

The first permutation ρ is based upon a property that if $\boldsymbol{M}(\sigma, \pi) \in \mathbb{ISM}^{m \times n}$ then for any two rows $\sigma(i) < \sigma(k)$ it holds from the definition of $\mathbb{ISM}$ that

$$\overline{m}_{\sigma(i),\pi(j)} - \underline{m}_{\sigma(k),\pi(j)} \leq \underline{m}_{\sigma(i),\pi(\ell)} - \overline{m}_{\sigma(k),\pi(\ell)}$$

for all columns $\pi(j) < \pi(\ell)$.
We can choose any two rows i, k from $\boldsymbol{M}$ and permute the matrix by this rule. Notice that such a permutation is not necessarily unique. It might hold for two columns ℓ_1 and ℓ_2 that

$$\overline{m}_{i,\ell_1} - \underline{m}_{k,\ell_1} \leq \underline{m}_{i,\ell_2} - \overline{m}_{k,\ell_2} \text{ and } \overline{m}_{i,\ell_2} - \underline{m}_{k,\ell_2} \leq \underline{m}_{i,\ell_1} - \overline{m}_{k,\ell_1}$$

at the same time. By Lemma 31 we know that

$$\boldsymbol{m}_{i,\ell_1} - \boldsymbol{m}_{k,\ell_1} = \boldsymbol{m}_{i,\ell_2} - \boldsymbol{m}_{k,\ell_2}$$

therefore as far as rows i, k are concerned the mutual position of ℓ_1 and ℓ_2 does not matter. If we choose i, k such that in the permuted matrix $\boldsymbol{M}(\sigma, \pi)$ it holds $\sigma(i) < \sigma(k)$ we get a *prepermutation* of columns. By prepermutation we mean that the permutation divides the columns into sets. The order of these sets is given, although, it might still happen that columns inside one set might need to be switched because for a different pair of columns the conditions are different. Therefore, we call them *ambiguity sets* (see Figure 6.1) as we cannot determine their inner structure using conditions for rows i, k.

The only problem that remains to be determined is how to find rows i, k for

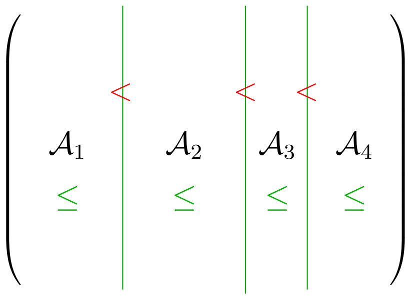

Figure 6.1: Columns cannot be switched between ambiguity sets $\mathcal{A}_i$.

which it holds in $\boldsymbol{M}(\sigma, \pi) \in \mathbb{ISM}$ that $\sigma(i) < \sigma(k)$. Thanks to Lemma 30 we can choose any two rows from matrix $\boldsymbol{M}$ because there are always permutations σ, π where $\sigma(i) < \sigma(k)$. For ρ to have some effect and further in the algorithm it is important for rows i, k to have at least two ambiguity sets of columns. If we have only one ambiguity set for all pairs of rows, the matrix is Monge because all conditions hold with equality $=$. A way to efficiently find such a pair is discribed in Lemma 34.

6.3.2 Row permutation σ

If we knew which columns were the first and the last, we could apply similar rule for the permutation of rows as we did for columns in ρ. In general, however, the first column and the last column are part of the first resp. the last ambiguity sets. We could choose one column randomly from each set but instead we choose a combination of all b columns from the first ambiguity set and all B columns from the last one. Lemma 32 gives us a way to establish a rule from a necessary condition induced by this combination. Permutation σ based on this rule looks as follows. For every pair of rows i, k if $\sigma(i) < \sigma(k)$ then

$$B \cdot \left(\sum_{j=1}^{b} \overline{m}_{ij} \right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \underline{m}_{i\ell} \right) \leq B \cdot \left(\sum_{j=1}^{b} \underline{m}_{kj} \right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \overline{m}_{k\ell} \right). \quad (6.3)$$

After applying σ it might still happen that for two rows i, k the permutation is ambiguous.
In fact, this is not a problem because all entries in these rows are real values and for all columns j, ℓ it holds that

$$\boldsymbol{m}_{ij} + \boldsymbol{m}_{k\ell} = \boldsymbol{m}_{i\ell} + \boldsymbol{m}_{kj}.$$

Let us note that the argument for this is similar to the one in Lemma 31.
From the ambiguity of i, k it holds that

$$B \cdot \left(\sum_{j=1}^{b} \overline{m}_{ij}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \underline{m}_{i\ell}\right) \leq B \cdot \left(\sum_{j=1}^{b} \underline{m}_{kj}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \overline{m}_{k\ell}\right) \quad (6.4)$$

and also

$$B \cdot \left(\sum_{j=1}^{b} \overline{m}_{kj}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \underline{m}_{k\ell}\right) \leq B \cdot \left(\sum_{j=1}^{b} \underline{m}_{ij}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \overline{m}_{i\ell}\right). \quad (6.5)$$

The righthand side of Inequality 6.4 is less or equal to the lefthand side of Inequality 6.5, therefore combining the inequalities together we get

$$B \cdot \left(\sum_{j=1}^{b} \overline{m}_{ij}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \underline{m}_{i\ell}\right) \leq B \cdot \left(\sum_{j=1}^{b} \underline{m}_{ij}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \overline{m}_{i\ell}\right).$$

Similarly, we can derive that

$$B \cdot \left(\sum_{j=1}^{b} \overline{m}_{kj}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \underline{m}_{k\ell}\right) \leq B \cdot \left(\sum_{j=1}^{b} \underline{m}_{kj}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \overline{m}_{k\ell}\right).$$

It is now obvious that all the values $\boldsymbol{m}_{ij}, \boldsymbol{m}_{i\ell}\boldsymbol{m}_{kj}, \boldsymbol{m}_{k\ell}$ are real. From this and the pair of inequalities 6.4, 6.5 we deduce that Inequality 6.3 becomes equality. Now since every column $1 \leq j \leq b$ is a candidate for the first column and every column $n - B + 1 \leq \ell \leq n$ is a candidate for the last column it must hold that $\boldsymbol{m}_{ij} - \boldsymbol{m}_{i\ell} \leq \boldsymbol{m}_{kj} - \boldsymbol{m}_{k\ell}$ for every such j, ℓ if we suppose w.l.o.g. that $i < k$ in the permuted matrix $\boldsymbol{M}(\sigma, \pi)$. If there was a pair of columns where $\boldsymbol{m}_{ij} - \boldsymbol{m}_{i\ell} < \boldsymbol{m}_{kj} - \boldsymbol{m}_{k\ell}$ we would have

$$B \cdot \left(\sum_{j=1}^{b} \boldsymbol{m}_{kj}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \boldsymbol{m}_{k\ell}\right) < B \cdot \left(\sum_{j=1}^{b} \boldsymbol{m}_{kj}\right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \boldsymbol{m}_{k\ell}\right)$$

which is a contradiction with what we proved so far. Therefore

$$\boldsymbol{m}_{ij} - \boldsymbol{m}_{i\ell} = \boldsymbol{m}_{kj} - \boldsymbol{m}_{k\ell}.$$

Lemma 31 tells us that for any row o that is *squeezed* between two rows i, k for which Monge condition holds with an equal sign it must hold that condition between this row o and each of i, k holds with equal sign as well. Applying similar result on columns instead of rows we get for the rest of columns *squeezed* between the sets of candidates equal signs in conditions with candidates, and by this equation between each other as well.
This all means that whenever σ becomes ambiguous for two rows, their order in respect to each other does not matter. We determine the last permutation π in a way that does not interfere with σ, therefore the order of rows is determined with no further need to change it.

6.3.3 Column permutation π

Let us recall that using the necessary conditions, permutation ρ divided the set of columns into ambiguity sets but it might still happen that inside the sets there are columns that need to be permuted. In order to fix this we define column permutation π similarly as we did define permutation ρ only this time on rows $\sigma(1)$ and $\sigma(m)$. For any two columns j, ℓ it should hold that

$$\overline{m}_{\sigma(1),\pi(j)} - \underline{m}_{\sigma(m),\pi(j)} \leq \underline{m}_{\sigma(1),\pi(\ell)} - \overline{m}_{\sigma(m),\pi(\ell)}.$$

Notice that any ambiguity set $\mathcal{A}_\pi$ from π is a subset of ambiguity set $\mathcal{A}_\rho$ from ρ otherwise $\boldsymbol{M}$ is not Monge permutable. This also means that π do not interfere with σ because the sums from Inequality 6.3 do not change (only the order of sum members might change). Moreover, by Lemma 31 it holds that whenever π is determined ambiguously for two columns j, ℓ, the condition

$$\boldsymbol{m}_{ij} - \boldsymbol{m}_{i\ell} = \boldsymbol{m}_{kj} - \boldsymbol{m}_{k\ell}.$$

holds for all rows i, k, therefore the order of these columns does not matter.

6.3.4 Pseudocode of the algorithm

We summerize the algorithm into a pseudocode.

Algorithm 1. $\mathbb{ISM}$ *permutation algorithm*

Input: $\boldsymbol{M} \in \mathbb{IR}^{nxn}$

Output: *"YES" if $\boldsymbol{M}$ is Monge permutable together with $\boldsymbol{M}(\sigma, \pi) \in \mathbb{ISM}$, "NO" otherwise*

1 *Find a row r such that there are at least two column ambiguity sets for rows $1, r$. If every row has one ambiguity set with row 1 output "YES" with $\sigma = id$ and $\pi = id$. If there is a pair of columns $j, j+1$ which cannot be permuted output "NO".*

2 *Determine permutation ρ such that*

$$\rho(k) < \rho(\ell) \text{ implies that } \overline{m}_{1k} - \underline{m}_{jk} \leq \underline{m}_{1\ell} - \overline{m}_{j\ell}.$$

If no such permutation exists, output "NO".

3 *Determine $b, B \in \{1, ..n\}$ such that b equals to the size of the first ambiguity set of ρ and B equals to the size of the last ambiguity set of ρ.*

4 *Determine row permutation σ such that $\sigma(i) < \sigma(k)$ implies that*

$$B \cdot \left(\sum_{j=1}^{b} \overline{m}_{ij} \right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \underline{m}_{i\ell} \right) \leq B \cdot \left(\sum_{j=1}^{b} \underline{m}_{kj} \right) - b \cdot \left(\sum_{\ell=n-B+1}^{n} \overline{m}_{k\ell} \right).$$

If no such permutation exists, output "NO".

5 *Determine column permutation π such that*

$$\pi(k) < \pi(\ell) \text{ implies that } \overline{m}_{\sigma(1),k} - \underline{m}_{\sigma(n),k} \leq \underline{m}_{\sigma(1),\ell} - \overline{m}_{\sigma(n),\ell}.$$

If no such permutation exists, output "NO".

6 *Check if $\boldsymbol{M}(\sigma, \pi) \in \mathbb{ISM}^{m \times n}$. Output "YES" with σ, π if it does and "NO" otherwise.*

6.3.5 Correctness and complexity of the algorithm

Theorem 35. *Algorithm 1 is correct and for* $\boldsymbol{M} \in \mathbb{IR}^{n \times n}$ *it runs in* $O(n^2)$*.*

Proof. The correctness of Algorithm 1 follows from sections 6.3.1 - 6.3.3. If matrix $\boldsymbol{M}$ is Monge permutable, Algorithm 1 yields one of possible permutations of rows and columns such that the resulting matrix is from $\mathbb{ISM}$. If the procedure fails it means that there is a quadraple of entries that are not ordered correctly. But their correct order violates some condition elsewhere in the matrix.
Step 1 takes $O(n^2)$ by Lemma 34. Steps 2,4 and 5 take $O(n^2)$ by Lemma 33. Step 3 takes $O(n)$ time because we can easily derive b and B from ρ by checking mostly $2n$ conditions. Finally, by Lemma 8 Step 6 takes $O(n^2)$. Altogether, the time complexity of Algorithm 1 is $O(n^2)$.

□

We proved the second part of Theorem 35 for matrices of dimension $n \times n$ in order to simplify the proof of time complexity. Let us note that it can be easily modified to show that for rectangular matrices $\boldsymbol{M} \in \mathbb{IR}^{m \times n}$ it runs in $O(m^2 + mn + n^2)$.

7. Generating IM matrices

In this chapter we deal with generating random IWM and ISM matrices which might be useful for testing hypothesis and understanding the structure of matrices. In Section 7.1 we deal with IWM generating. In Section 7.2 we present a general method with many possible variations. We demonstrate one specific variation and briefly mention others. In Section 7.3 we present a memory friendly method for generating real Monge matrices with $O(m+n)$ memory requirement for storing $\mathbb{R}^{m\times n}$ matrices and in Section 7.4 we present its interval generalization.

7.1 Generating IWM

Generating IWM matrices is an easier task than generating ISM matrices. Since only one realization of the interval matrix has to be Monge, we can use an approach to generate a random real Monge matrix and then randomly inflate the entries into intervals. Therefore, we do not deal with IWM directly, although both methods presented in this chapter can be easily modified for IWM matrices.

7.2 The general method for ISM

The general method can be devided into two steps. First, we generate a random real Monge matrix which will be taken as a special realization of the matrix being generated. This special matrix might be e.g. upper or lower bound matrix or Chess matrix. In the second step, the matrix is inflated into ISM matrix. The entries are inflated one after the other and the order of inflation is either random or arranged according to some key (e.g. rows, columns, ...). We will demonstrate the method using a lower bound matrix. The method to generate the special matrix will be built on a characterization of Monge matrices via distribution matrices. We will use a random inflation order.

Algorithm 2. *A variation of the general method for* $\mathbb{ISM}$

Generate a nonnegative matrix $N \in \mathbb{R}^{m\times n}$
Generate random vectors $u \in \mathbb{R}^m, v \in \mathbb{R}^n$
Set a distribution matrix D *such that* $d_{ij} = \sum_{k=i}^{m} \sum_{\ell=1}^{j} n_{k\ell}$
Set $\underline{m}_{ij} = d_{ij} + u_i + v_j$
Set $\overline{m}_{ij} = \underline{m}_{ij}$
while $\exists \{i,j\}$ *such that* $\overline{m}_{ij}$ *has not been chosen in this loop yet* **do**
    Pick the pair of indices $\{i,j\}$
    $\varepsilon^A := \underline{m}_{i,j-1} + \underline{m}_{i-1,j} - \overline{m}_{i-1,j-1} - \underline{m}_{ij}$
    $\varepsilon^B := \underline{m}_{i,j+1} + \underline{m}_{i+1,j} - \overline{m}_{i+1,j+1} - \underline{m}_{ij}$
    Set $\overline{m}_{ij} \in \left[\underline{m}_{ij}, \underline{m}_{ij} + \min\left\{\varepsilon^A, \varepsilon^B\right\}\right]$
end
Output: $\boldsymbol{M} \in \mathbb{ISM}^{m\times n}$

Theorem 36. *Algorithm 2 is correct.*

Proof. According to Lemma 2 lines 1-4 generate a real Monge matrix. The matrix is then set as a lower bound matrix of $\boldsymbol{M}$. To inflate the matrix we must ensure that the upper bounds of entries are not too large to violate the conditions of Monge property. We start by setting them to values of corresponding lower bounds and we will increase them one by one. Thanks to Theorem 1.3 it suffices for every $\overline{m}_{ij}$ to meet conditions of neighbouring quadraples. Out of four quadraples only upper left quadraple and lower right quadraple put conditions on $\overline{m}_{ij}$ (see Figure 7.1).

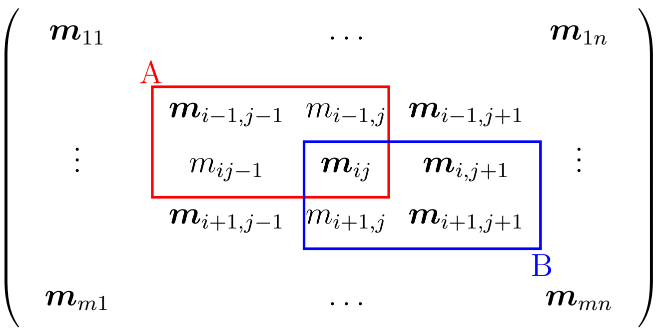

Figure 7.1: The two quadraples A and B that put conditions on $\overline{m}_{ij}$.

For $\overline{m}_{ij}$ the upper left quadraple (labeled A in Figure 7.1) sets condition

$$\overline{m}_{i-1,j-1} + \overline{m}_{ij} \leq \underline{m}_{i-1,j} + \underline{m}_{i,j-1}$$

and the lower right quadraple (labeled B in Figure 7.1) sets condition

$$\overline{m}_{ij} + \overline{m}_{i+1,j+1} \leq \underline{m}_{i,j+1} + \underline{m}_{i+1,j}.$$

When defining $\overline{m}_{ij}$ we can express the tightness of the conditions by $\varepsilon^A, \varepsilon^B$ such that

$$\overline{m}_{i-1,j-1} + \underline{m}_{ij} + \varepsilon^A = \underline{m}_{i-1,j} + \underline{m}_{i,j-1}$$

and

$$\overline{m}_{i+1,j+1} + \underline{m}_{ij} + \varepsilon^B = \underline{m}_{i,j+1} + \underline{m}_{i+1,j}.$$

From this it is obvious we have to choose $\overline{m}_{ij}$ from $\left[\underline{m}_{ij}, \underline{m}_{ij} + \min\left\{\varepsilon^A, \varepsilon^B\right\}\right]$ as described on lines 6-11.

□

7.3 Bost's method

Bost's method in its original form (see [9]) generates matrices with inverse Monge property. Only a little modification is needed to obtain Monge matrices, therefore we keep referring to the method as Bost's even though we are working with Monge matrices. The method is based on taking the values of the matrix along lines with a slope that decreases with an index of a row (see Figure 7.2).

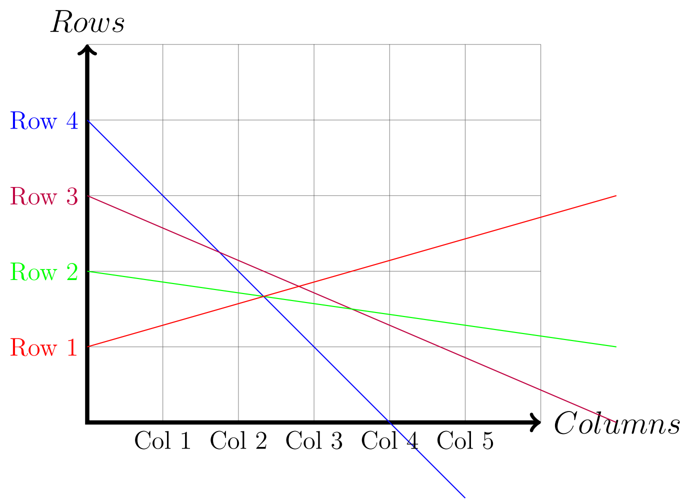

Figure 7.2: Bost's method for generating Monge matrices.

From this idea, $M \in \mathbb{R}^{m \times n}$ is defined as

$$m_{ij} = j \cdot s_i - i.$$

It is easy to prove that such matrices satisfy the Monge property iff $\{s_i\}_{i=1}^{m}$ is a nonincreasing ordered sequence ($\forall i : s_i \geq s_{i+1}$). To avoid regularity of entries in rows given by the generated lines, Bost defines also $n_{ij} = i \cdot r_j - j$ where $\{r_j\}_{j=1}^{n}$ is nonincreasing. Since Monge matrices are closed under transposition, N is also Monge and sum of two Monge matrices is also Monge. Finally, to have a good distribution of slopes, Bost picks an angle α from the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and sets the slope of line to $tan(\alpha)$. To summerize, Bost's method generates two nonincreasing ordered sequences $\{\alpha_i\}_{i=1}^{m}$, $\{\beta_j\}_{j=1}^{n}$ of elements picked uniformly at random from range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and sets m_{ij} to

$$m_{ij} = j \cdot tan(\alpha_i) - i + i \cdot tan(\beta_j) - j.$$

The advantage of this approach is coding $m \cdot n$ entries of the matrix into two vectors of length m and n.

7.4 Generalization of Bost's method

Our generalization is based on a simple idea. Instead of binding one row with slope of one line, we bind an interval of slopes with the row (see Figure 7.3).

Of course, a restriction on intervals has to be made in order to preserve strong Monge property. Our goal is to generate a matrix $\boldsymbol{M}$ of form

$$\boldsymbol{m}_{ij} = j \cdot \mathbf{s}_i - i + i \cdot \mathbf{r}_j - j$$

where $\boldsymbol{s}_i$ and $\boldsymbol{r}_j$ are intervals for all indices i, j of $\boldsymbol{M}$. We present an algorithm followed by an analysis.

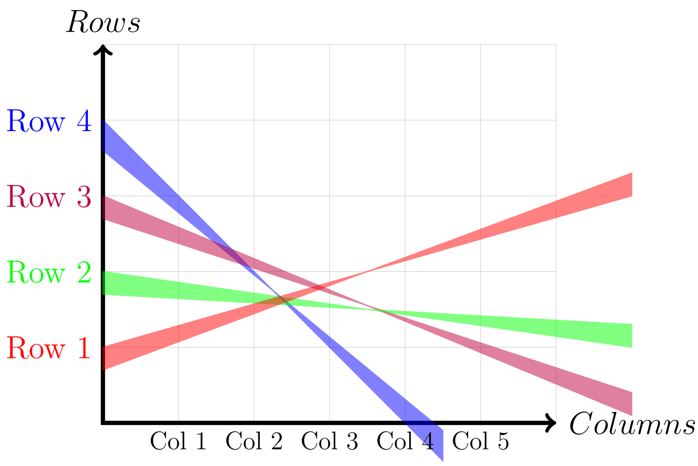

Figure 7.3: Generalized Bost's method for generating $\mathbb{ISM}$ matrices.

Algorithm 3. *Bost's generalized method*

1 *Generate two nonincreasing sequences* $\{\underline{s}_i\}_{i=1}^m$, $\left\{\underline{r}_j\right\}_{j=1}^n$
2 *Set* $\overline{s}_1$ *such that* $\overline{s}_1 \in \left[\underline{s}_1, \underline{s}_1 + \frac{\underline{s}_1 - \underline{s}_2}{m}\right]$
3 *Set* $\overline{s}_i$ *for* $i = 2, \ldots, m$ *such that* $\overline{s}_i \in \left[\underline{s}_i, \underline{s}_{i-1} + \frac{m}{m+1}\left(\underline{s}_i - \overline{s}_{i-1}\right)\right]$
4 *Set* $\overline{r}_1$ *such that* $\overline{r}_1 \in \left[\underline{r}_1, \underline{r}_1 + \frac{\underline{r}_1 - \underline{r}_2}{n}\right]$
5 *Set* $\overline{r}_i$ *for* $i = 2, \ldots, n$ *such that* $\overline{r}_i \in \left[\underline{r}_i, \underline{r}_{i-1} + \frac{n}{n+1}\left(\underline{r}_i - \overline{r}_{i-1}\right)\right]$
6 *Set* $\boldsymbol{M}$ *such that* $\boldsymbol{m}_{ij} = j \cdot \mathbf{s}_i - i + i \cdot \mathbf{r}_j - j$.

Theorem 37. *Algorithm 3 outputs* $\boldsymbol{M} \in \mathbb{ISM}$.

Proof. Step 1 generates two nonincreasing sequences as in the real case. We set these sequences as lower bounds of intervals and all that remains is to set the upper bounds. We derive upper bounds from a characterization of $\mathbb{ISM}$:

$$\overline{m}_{ij} + \overline{m}_{i+1,j+1} \leq \underline{m}_{i,j+1} + \underline{m}_{i+1,j}.$$

Since we want $m_{ij} = j \cdot s_i - i$, we rewrite the inequality above in the terms of sequence $\{s_i\}_{i=1}^m$ as

$$j \cdot \overline{s}_i - i + (j+1)\overline{s}_{i+1} - (i+1) \leq j \cdot \underline{s}_{i+1} - (i+1) + (j+1) \cdot \underline{s}_i - i.$$

Terms i nad $(i+1)$ are redundant on both sides therefore we eliminate them:

$$j \cdot \overline{s}_i + (j+1)\overline{s}_{i+1} \leq j \cdot \underline{s}_{i+1} + (j+1) \cdot \underline{s}_i.$$

We can further rearrange the inequality into form:

$$j(\overline{s}_i - \underline{s}_i + \overline{s}_{i+1} - \underline{s}_{i+1}) \leq \underline{s}_i - \overline{s}_{i+1}.$$

This inequality must hold for all $j \in \{2, \ldots n\}$. Since $(\overline{s}_i - \underline{s}_i + \overline{s}_{i+1} - \underline{s}_{i+1})$ is nonnegative, the inequality holds for all j iff it holds for $j = m$. Considering the case where $j = m$ we can rewrite the inequality in order to get a recurrent formula for determining members of $\{s_i\}_{i=2}^m$:

$$\overline{s}_{i+1} \leq \underline{s}_i + \frac{m}{m+1}\left(\underline{s}_{i+1} - \overline{s}_i\right).$$

All that is left to determine is $\overline{s}_1$ because the formula does not work for $i = 0$. We know that $\overline{s}_1 \geq \underline{s}_1$ but is there an upper bound for the expression? We take the derived formula and set $i = 1$:

$$\overline{s}_2 \leq \underline{s}_1 + \frac{m}{m+1} \left(\underline{s}_2 - \overline{s}_1 \right) = \underline{s}_1 - \frac{m}{m+1} \left(\overline{s}_1 - \underline{s}_2 \right).$$

We can observe that the larger we set the value of $\overline{s}_1$ the stricter we bound the value of $\overline{s}_2$. We can bound $\overline{s}_2$ as $\overline{s}_2 \leq \underline{s}_2 + \epsilon$ and the bound becomes tight when $\overline{s}_2 = \underline{s}_2$, therefore $\overline{s}_1$ can be as big as in the equality

$$\underline{s}_2 = \underline{s}_1 - \frac{m}{m+1}(\overline{s}_1 - \underline{s}_2).$$

Deriving now $\overline{s}_1$ from the equality we get

$$\overline{s}_1 = \underline{s}_2 + \frac{m+1}{m}(\underline{s}_1 - \underline{s}_2) = \underline{s}_1 + \frac{\underline{s}_1 - \underline{s}_2}{m}.$$

This gives us an upper bound for choosing the value of $\overline{s}_1$. It is now clear what Step 2 and Step 3 do. For the sequence $\{r_j\}_{j=1}^{n}$ the argument is similar. Therefore Steps 4 and 5 are correct. Both sequences generate a matrix from $\mathbb{ISM}$. Adding them together yields a matrix from $\mathbb{ISM}$ by Theorem 12 meaning Step 6 is correct.

□

8. Applications of ISM

In this chapter we present results about applications of $\mathbb{ISM}$ namely in Section **??** an interval row-minimization problem, in Section **??** an interval version of transportation problem and in Section **??** an interval travelling salesman problem. For each application we briefly introduce the real version of the problem and the well-known solution considering Monge matrices. After that we present results of our investigation in the interval version of the problem.

8.1 Row-minimization problem

In this section we consider so called *Row-minimization problem* - a problem of computing the minimum entry of all rows of a matrix. One way to define the interval generalization is as follows. For an interval matrix $\boldsymbol{M} \in \mathbb{IR}^{m\times n}$ find a matrix with unspecified entries $\boldsymbol{S} \in \mathbb{IR}^{m\times n}$ such that for every pair of indices i, j and every realization $s_{ij} \in \boldsymbol{s}_{ij}$ there exists a matrix $M \in \boldsymbol{M}$ with $m_{ij} = s_{ij}$ being the minimum value of row i. Matrices with unspecified entries are employed because not all interval entries contain a minumum. Monge matrices with unspecified entries are mentioned in work of Deineko et al. [10]. In this text we will not further deal with them.

Example. Let $\boldsymbol{M} \in \mathbb{ISM}^{3\times 3}$ such that

$$\boldsymbol{M} = \begin{pmatrix} [4,5] & [4,5] & [2,8] \\ [20,22] & [10,12] & [4,6] \\ [20,25] & [14,18] & [0,6] \end{pmatrix}.$$

In the first row, the minimum is from the range $[2, 5]$ because the minimum in $\underline{M}$ is 2 and the minimum of $\overline{M}$ is 5. Therefore we leave the first two entries intact and set the last entry $[2, 5]$.
For the second row no matter what entries we choose from the intervals, the minimum is always in the last column, therefore we set the first and the second entry as unspecified (denoted by ?) and leave the last entry intact. For the third row it is similar as for the second row.
The output matrix $\mathbf{S}$ then looks like this:

$$\mathbf{S} = \begin{pmatrix} [4,5] & [4,5] & [2,5] \\ ? & ? & [4,6] \\ ? & ? & [0,6] \end{pmatrix}.$$

8.1.1 The real version and SMAWK algorithm

For solving the row-minima problem in totally monotone matrices (including Monge matrices) SMAWK algorithm was introduced by P. Shor, S. Moran, A. Aggarwal, R. Wilber and M. Klawe (see [11]). For $m \times n$ matrices the algorithm returns in $O(1(n+log(\frac{m}{n}))$ a vector s where s_i corresponds to the value of minimal entry in row i. The algorithm is fundamental for many other applications e.g. geometrical problems or selecting and sorting problems (see [12]).

8.1.2 The interval version

Having the method for the real version, the interval version is easy to solve using a simple observation. Let us take row minima $(m_1, m_2, \ldots, m_n)$ of the upper bound matrix $\overline{M}$ (where m_i is the minimum of row i). For any matrix realization $M \in \boldsymbol{M}$ the row minima will be less or equal to $(m_1, m_2, \ldots, m_n)$. Furthermore, for all values $m_{ik} \in \boldsymbol{m}_{ik}$ less or equal to minimum of row i a matrix realization can be chosen in order to represent this value as the minimum of row i.
Let $\overline{m}_{ij}$ be the minimum of row i in the upper bound matrix $\overline{M}$ and let $m_{ik} \leq \overline{m}_{ij}$ for some $1 \leq k \leq n$. Then choose matrix M^{ik} such that $(M^{ik})_{rs} = (\overline{M})_{rs}$ for all $r \neq i$ or $s \neq k$ and $(M^{ik})_{ik} = m_{ik}$. The entry m_{ik} is obviously the row minimum of the row i in the matrix M^{ik} since for all ℓ such that $\ell \neq k$ it holds that $m_{ik} \leq m_i \leq \overline{m}_{i\ell}$.We can therefore derive an algorithm as follows.

Algorithm 4. *Interval Row Minima algorithm*

Input: *An interval matrix* $\boldsymbol{M} \in \mathbb{ISM}^{m \times n}$

1 $(m_1, m_2, \ldots, m_n) \leftarrow$ *Run SMAWK algorithm on the upper bound matrix* $\overline{M}$
2 *For every row i:*
3 *For every entry j:*
4 **if** $\underline{m}_{ij} \leq m_i < \overline{m}_{ij}$ **then**
5 | *set* $\boldsymbol{S}_{ij} := \left[\underline{m}_{ij}, m_i\right]$
6 **end**
7 **else if** $\overline{m}_{ij} \leq m_i$ **then**
8 | *set* $\boldsymbol{S}_{ij} := \boldsymbol{M}_{ij}$
9 **end**
10 **else if** $m_i < \underline{m}_{ij}$ **then**
11 | *set* $\boldsymbol{S}_{ij}$:=?
12 **end**

Output: *An interval matrix* $\boldsymbol{S}$ *of dimension* $m \times n$ *with unspecified entries*

In the first phase the algorithm computes the row minima of the upper bound matrix and in the second phase it uses them as tresholds for row prunning.
The asymptotical time complexity is $O(mn)$ since step 1 takes $O(1(n + log(\frac{m}{n}))$ (see[12]) and steps $2 - 12$ take $O(mn)$.

8.2 Transportation problem (TP)

In 1961 Hoffman presented a greedy algorithm for solving Hitchcock transportation problem where the cost function was a Monge matrix (actually, a more general matrix with a Monge sequence). The algorithm employed the famous NWC rule for finding an initial feasible solution of a linear program. This meant a significant improvement over general LP algorithm.

8.2.1 The real version of TP

In transporation problem we have m producers and n consumers with information about the amount of commodity that they produce resp. require. In addition a value c_{ij} represents the cost of transporting one unit of a commodity from a producer i to a consumer j. We add a condition that the amount of the produced commodity over all producers equals the amount of the commodity that is desired by all costumers. The goal is to transport the commodity between the producers and the costumers with a minimal transporting cost. Mathematically, the problem has the following form.

Problem (Transportation).

$$
\begin{array}{lll}
\text{minimize} & \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} c_{ij}x_{ij} & \\
\text{subject to} & \sum\limits_{j} x_{ij} = a_i & \text{for } 1 \leq i \leq m, \\
& \sum\limits_{i} x_{ij} = b_j & \text{for } 1 \leq j \leq n, \\
& \sum\limits_{i} a_i = \sum\limits_{j} b_j, & \\
& x_{ij} \geq 0 & \text{for } 1 \leq i \leq m, 1 \leq j \leq n.
\end{array}
$$

The last condition $\sum\limits_{i} a_i = \sum\limits_{j} b_j$ is crucial for Theorem **??**. We call this condition a *balance condition.*

Theorem 38. *The north-west corner rule produces an optimal solution of the transportation problem if the cost matrix C is a Monge matrix.*

Proof. See [13].

□

To explain how the algorithm works we define properly the north-west corner rule.

North-west corner rule (NWC rule)

The north-west corner rule is a method to obtain an initial feasible solution of a linear program. In a table where first m rows represent producers and first n columns represent consumers the cell at position (i, j) represents the cost of transporting one unit of commodity from producer i to consumer j. We add another row of total demands of each consumer and another column of total supply of each producer (see Figure **??**). We start assigning the commodity greedily from the upper-left corner and continue either right if the demand of the current producer is satisfied, down if the supply is exhausted or diagonally if both (see Figure **??**). It is obvious that this procedure yields a feasible solution.

8.2.2 The optimal value range of ITP

We can generalize the transportation problem as a family of linear programs where we call a specific realization of interval values a *scenario.*

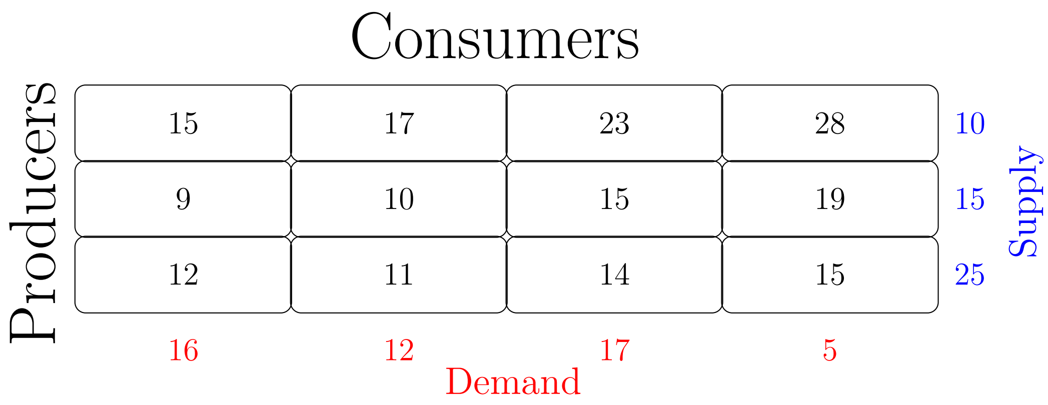

Figure 8.1: The table of NWC rule.

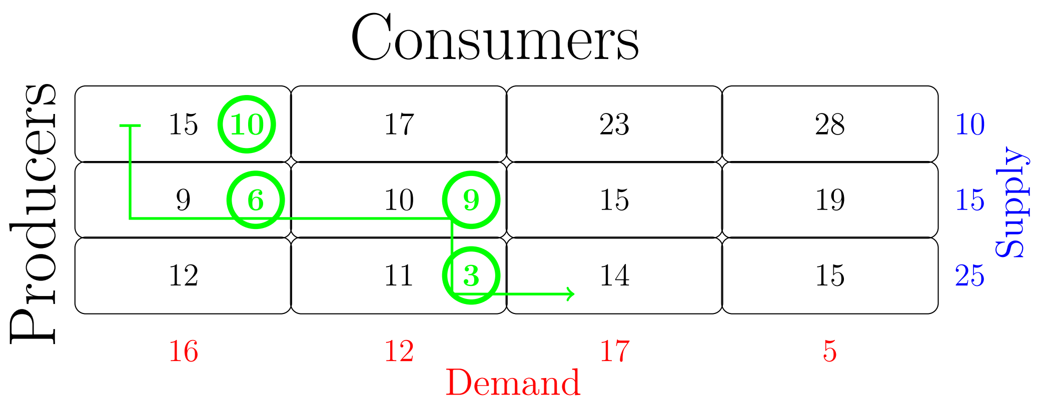

Figure 8.2: The application of NWC rule in progress.

Definition 21. *Let* $\boldsymbol{C} \in \mathbb{IR}^{m \times n}, \boldsymbol{a} \in \mathbb{IR}^m$ *and* $\boldsymbol{b} \in \mathbb{IR}^n$ *be given. Then the interval transportation problem (ITP) is a family of linear programs*

$$\begin{aligned} &\textit{minimize} && \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ &\textit{subject to} && \sum_{j} x_{ij} = a_i && \textit{for } 1 \leq i \leq m, \\ & && \sum_{i} x_{ij} = b_j && \textit{for } 1 \leq j \leq n, \\ & && \sum_{i} a_i = \sum_j b_j, \\ & && x_{ij} \geq 0 && \textit{for } 1 \leq i \leq m, 1 \leq j \leq n. \end{aligned}$$

denoted by $(\mathbf{a}, \mathbf{b}, \mathbf{C})$*, where* $c_{ij} \in \boldsymbol{C}_{ij}, a_i \in \boldsymbol{a}_i$ *and* $b_j \in \boldsymbol{b}_j$*.*

For further usage let us properly define a scenario.

Definition 22. *A scenario is an ordered triplet* (a, b, C) *where* $a \in \mathbf{a}$*,* $b \in \mathbf{b}$ *are vectors of producers resp. consumers and* $C \in \mathbf{C}$ *is a cost matrix of some ITP.*

In this thesis we study the optimal value range of ITP.

Definition 23. *Let $f(a,b,C)$ be the optimal value of scenario (a,b,C). Then an optimal value range of ITP is* $\mathbf{f} = \left[\underline{f}, \overline{f}\right]$ *where*

$$\underline{f} = \min_{(a,b,C)} \{f(a,b,C)\},$$

$$\overline{f} = \max_{(a,b,C)} \{f(a,b,C)\}.$$

The optimal value range problem might be solved by an interval linear programming (ILP). Although it is easy to compute the lower bound $\underline{f}$, it is known that for the upper bound $\overline{f}$ the general ILP is NP-hard (see [14]). For the interval transportation problem, the complexity is still an open problem. We will therefore focus on computing the upper bound $\overline{f}$.

8.2.3 NWC rule based algorithm

We present an algorithm which solves the optimal value range problem of ITP by exponentional number of linear programs of reduced size. The algorithm is not polynomial, however, it is easy to parallelize which can make the algorithm faster than the methods known for solving ITP.
The algorithm is based on the NWC rule. For a specific scenario, the NWC rule defines a *route* in the cost matrix. The route is defined for every scenario. It consists of positions in the cost matrix for which the corresponding variable of the optimal solution vector is non-zero.

Definition 24. *Let x be the optimal solution of a scenario (a,b,C) of an ITP. Then a route R of a scenario (a,b,C) is a lexicographically ordered sequence of pairs (i,j) such that $x_{ij} \neq 0$.*

Definition 25. *Let $R = (r_1, r_2, \ldots, r_n)$ be a route of some scenario (a,b,C). Then a subroute S of R is an ordered subsequence of R such that there exist $k \leq \ell$ such that $S = (r_k, r_{k+1}, \ldots, r_\ell)$. We denote the subsequence S of route R by $S \subseteq R$ and $(i,j) \in R$ if a sequence $((i,j)) \subseteq R$.*

From the NWC rule we see that for a Monge matrix C any route R contains $(1,1)$ and for every (i,j) in R it either contains $(i+1,j)$, $(i,j+1)$ or $(i+1,j+1)$. We can merge together scenarios that have the same NWC route. For every route in the cost matrix we generate a linear program (see Lemma **??**). Each linear program corresponds to a route in a way that every feasible solution of this program corresponds to a scenario with optimal solution on this route. The maximum of this LP corresponds to maximal optimal value over all scenarios with this NWC-route. Therefore, the maximum over all LPs is the upper bound $\overline{f}$ of the optimal value range problem.
This devides the problem into polynomially solvable subproblems, however, the number of subproblems is still exponential.
To prove all this we start with few lemmata to further use in a proof of a main theorem.

Lemma 39. *Let* $\boldsymbol{C} \in \mathbb{IR}^{m\times n}$ *be a nonnegative matrix and let* $\boldsymbol{f} = \left[\underline{f}, \overline{f}\right]$ *be the optimal value range of an ITP* $(\mathbf{a}, \mathbf{b}, \mathbf{C})$*. Then*

$$\underline{f} := \min_{(a,b,\underline{C})} \{f(a,b,\underline{C})\},$$

$$\overline{f} := \max_{(a,b,\overline{C})} \left\{f(a,b,\overline{C})\right\}.$$

Proof. Let us suppose that there exists a scenario (a', b', C) of ITP such that $C \neq \overline{C}$ and $f(a', b', C) > \max_{(a,b,\overline{C})} \left\{f(a,b,\overline{C})\right\}$. Since $C \in \mathbf{C} \geq 0$ it holds

$$f(a', b', C) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \leq \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{c}_{ij}x_{ij} = f(a', b', \overline{C}) \leq \max_{(a,b,\overline{C})} \left\{f(a,b,\overline{C})\right\}.$$

For $\underline{f}$ the argument is similar.

□

From the previous lemma, when looking for $\overline{f}$ the only condition that is necessary for our algorithm is that $\overline{C}$ is Monge. However, when dealing with the set of all optimal routes it is still necessary to consider $\boldsymbol{C} \in \mathbb{ISM}$.

Lemma 40. *The number of routes in the cost matrix is at least* $3^{\min\{m-1,n-1\}}$ *where* m *and* n *are the numbers of producers resp. consumers.*

Proof. We can construct a ternary tree representing all the routes in the cost matrix. The root of the tree has label $(1, 1)$ and for every node with label (i, j) the corresponding children are $(i + 1, j), (i, j + 1)$ and $(i + 1, j + 1)$ for $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$. We see that up to the level $\min\{m - 1, n - 1\}$ the tree is full, therefore the number of leaves which represents the number of routes in the cost matrix is at least $3^{\min\{m-1,n-1\}}$.

□

Lemma 41. *For every route R of ITP $(\mathbf{a}, \mathbf{b}, \mathbf{C})$ a linear program*

$$
\begin{array}{lll}
\textit{maximize} \sum\limits_{(i,j)\in R} \overline{c}_{ij} x_{ij} & & \\
\textit{subject to} & & \\
\forall (i,j) \in R: & & \\
\sum\limits_{k\leq i} a_k < \sum\limits_{\ell \leq j} b_\ell & \textit{if } ((i,j)(i+1,j)) \subseteq R, & (1)\\
\sum\limits_{k\leq i} a_k > \sum\limits_{\ell \leq j} b_\ell & \textit{if } ((i,j),(i,j+1)) \subseteq R, & (2)\\
\sum\limits_{k\leq i} a_k = \sum\limits_{\ell \leq j} b_\ell & \textit{if } ((i,j),(i+1,j+1)) \subseteq R, & (3)\\
x_{ij} = b_j & \textit{if } ((i,j-1),(i,j),(i,j+1)) \subseteq R, & (4)\\
x_{ij} = b_j & \textit{if } ((i,j-1),(i,j),(i+1,j+1)) \subseteq R, & (5)\\
x_{ij} = b_j & \textit{if } ((i-1,j-1),(i,j),(i,j+1)) \subseteq R, & (6)\\
x_{ij} = b_j & \textit{if } ((i-1,j-1),(i,j),(i+1,j+1)) \subseteq R, & (7)\\
x_{ij} = a_i & \textit{if } ((i-1,j-1),(i,j),(i+1,j)) \subseteq R, & (8)\\
x_{ij} = a_i & \textit{if } ((i-1,j),(i,j),(i+1,j+1)) \subseteq R, & (9)\\
x_{ij} = a_i & \textit{if } ((i-1,j),(i,j)(i+1,j)) \subseteq R, & (10)\\
x_{ij} = a_i - \sum\limits_{\ell \neq j} x_{i\ell} & \textit{if } ((i,j-1),(i,j),(i+1,j) \subseteq R, & (11)\\
x_{ij} = b_j - \sum\limits_{k \neq i} x_{kj} & \textit{if } ((i-1,j),(i,j),(i,j+1)) \subseteq R, & (12)\\
x_{11} = b_1 & \textit{if } ((1,1),(1,2)) \subseteq R, & (13)\\
x_{11} = b_1 & \textit{if } ((1,1),(2,2)) \subseteq R, & (14)\\
x_{11} = a_1 & \textit{if } ((1,1),(2,1)) \subseteq R, & (15)\\
\textit{with variables } x_{ij}, a_i, b_j: & & \\
\underline{a}_i \leq a_i \leq \overline{a}_i & \textit{for all } i, & (16)\\
\underline{b}_j \leq b_j \leq \overline{a}_i & \textit{for all } j, & (17)\\
x_{ij} \geq 0 & \textit{for all } i,j, & (18)
\end{array}
$$

finds the maximal optimal value over the route R.

Proof. Conditions (1) - (3) set constraints on vectors a and b to ensure the route R. We prove this by an induction on elements of R ordered by the lexicographical ordering.

The pair $(1,1)$ is in every route R. For $(1,1)$ let us suppose that $(2,1) \in R$ is the next element. But by the NWC rule it must hold that $a_1 < b_1$ because going south means that the producer 1 was not fully exhausted. For cases $(1,2) \in R$ and $(2,2) \in R$ the argument is similar.

Now let us suppose that for $(i,j) \in R$ the sub-route from $(1,1)$ to (i,j) is ensured through the conditions by induction hypothesis and let us suppose that $(i,j+1)$ is the next in the route R. Then no matter what the actual structure of the route is, $\sum\limits_{k\leq i} a_k < \sum\limits_{\ell\leq j} b_\ell$ because consumer j was not completely satisfied. For cases where $(i+1,j)$ and $(i+1,j+1)$ are next in R the argument is similar.

Conditions (4) - (12) encode the amount of comodity transported from producer i to consumer j into variable x_{ij}. Every condition should be obvious from the picture (see Figure **??**). Conditions (13) - (15) have to be specifically written because x_{11} does not have any predecessor on the route R.

Conditions (16) and (17) ensure that variables a_i and b_j are from the intervals $\mathbf{a}_i$,$\mathbf{b}_j$ and finally condition (18) for the nonnegativity of x_{ij} is obvious.

By Lemma **??** and the fact that every feasible solution of the LP is an optimal

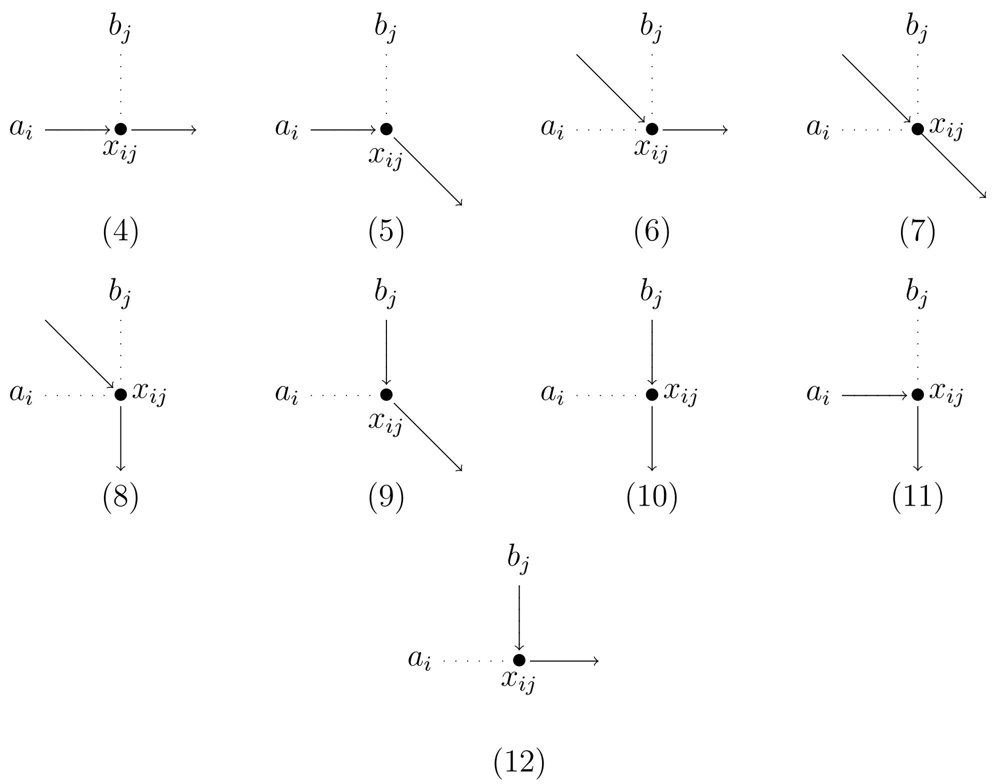

Figure 8.3: A depiction of conditions (4) - (12).

solution of corresponding scenario of ITP $(a, b, \overline{C})$ the optimal solution of the LP is the highest optimal solution over all scenarios of route R.

□

Finally, we can describe the algorithm in pseudocode.

Algorithm 5. *The NWC-based algorithm for optimal value range of ITP*

Input: *An instance* $(\mathbf{a}, \mathbf{b}, \mathbf{C})$ *of ITP*

For every route R_i *in* $\boldsymbol{C}$*:*

$\overline{r}_i \leftarrow$ *Solve a linear program for* R_i *described in Lemma* **??**

$\overline{f} = \max_i \overline{r}_i$

Output: *The upper bound optimal value range* $\overline{f}$

Theorem 42. *The NWC-based algorithm for optimal value range of ITP is correct and computes exponentially many subroutines of polynomial time complexity.*

Proof. Let $S_{max} = (a, b, \overline{C})$ be a scenario such that the optimal value of S_{max} is $\overline{f}$. By Lemma **??** S_{max} exists. Because $\boldsymbol{C} \in \mathbb{ISM}$, the optimal values of the scenario forms a route R_{max}. By Lemma **??** $\overline{f}$ can be computed by a linear program of R_{max}. Since the algorithm picks maximum over all routes' maxima, it is easy to see that it is correct.
A linear program is solvable in polynomial time and since the number of LPs correspond to the number of routes, by Lemma **??** the number of LPs is exponential.

□

8.2.4 Reducing the number of routes

The main downside of our algorithm is the exponential number of possible routes that we have to investigate. It would be convenient to reduce the number of routes or at least specify conditions under which we can omit most of them.
We cannot employ Monge property for restricting the number of possible routes since it strictly depends on vectors $\mathbf{a}, \mathbf{b}$. Unless further restricting $\mathbf{a}, \mathbf{b}$ we can find an instance of ITP where all possible routes are covered by at least one scenario.

Example. Let $(\mathbf{a}, \mathbf{b}, \mathbf{C})$ be an instance of ITP such that

- $\mathbf{a} \in \mathbb{IR}^2$ and for all i it holds that $\mathbf{a_i} = [1, 3]$,
- $\mathbf{b} \in \mathbb{IR}^3$ and for all j it holds that $\mathbf{b_j} = [1, 3]$,
- $\mathbf{C} \in \mathbb{ISM}^{2\times 3}$.

The following scenarios

1. $S_1 = ((3, 1), (1, 1, 2), C)$,
2. $S_2 = ((2, 1), (1, 1, 1), C)$,
3. $S_3 = ((2, 2), (1, 2, 1), C)$,
4. $S_4 = ((1, 2), (1, 1, 1), C)$,
5. $S_5 = ((1, 3), (2, 1, 1), C)$,

cover all the possible routes of $(\mathbf{a}, \mathbf{b}, \mathbf{C})$ (see Figure **??**).

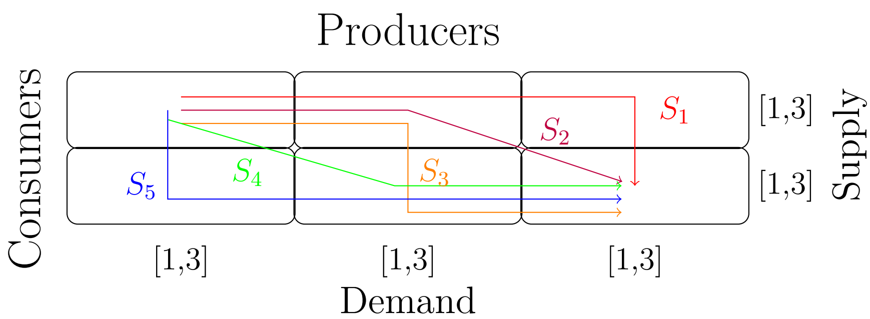

Figure 8.4: A visualisation of an instance of ITP with all 5 possible routes.

Let us note that it is possible to construct a class of infinitely many instances with all possible routes covered. We omit this construction here. Although all routes are possible in general, we can easily specify condition under which there is only one possible route in the ITP.

Lemma 43. *Let* $(\mathbf{a}, \mathbf{b}, \mathbf{C})$ *be an instance of ITP and let* R *be a route of* $(\mathbf{a}, \mathbf{b}, \mathbf{C})$ *such that*

$$\sum_{k\leq i} a_k < \sum_{\ell\leq j} b_\ell \quad \text{for all } i, j \text{ such that } ((i, j), (i + 1, j)) \subseteq R,$$
$$\sum_{k\leq i} a_k > \sum_{\ell\leq j} b_\ell \quad \text{for all } i, j \text{ such that } ((i, j), (i, j + 1)) \subseteq R,$$
$$\sum_{k\leq i} a_k = \sum_{\ell\leq j} b_\ell \quad \text{for all } i, j \text{ such that } ((i, j), (i + 1, j + 1)) \subseteq R,$$

If it holds that

$\sum\limits_{k\le i}\overline{a}_k < \sum\limits_{\ell\le j}\underline{b}_\ell$ *for all* i,j *such that* $((i,j),(i+1,j))\subseteq R$,

$\sum\limits_{k\le i}\underline{a}_k > \sum\limits_{\ell\le j}\overline{b}_\ell$ *for all* i,j *such that* $((i,j),(i,j+1))\subseteq R$,

$\sum\limits_{k\le i}\underline{a}_k = \sum\limits_{\ell\le j}\overline{b}_\ell$ *for all* i,j *such that* $((i,j),(i+1,j+1))\subseteq R$,

$\sum\limits_{k\le i}\overline{a}_k = \sum\limits_{\ell\le j}\underline{b}_\ell$ *for all* i,j *such that* $((i,j),(i+1,j+1))\subseteq R$,

then R *is the only possible route of* $(\mathbf{a},\mathbf{b},\mathbf{C})$.

Proof. No matter what realization of $a\in\mathbf{a}, b\in\mathbf{b}$ is chosen, the conditions still hold in the form that describes the route R. Therefore R is the only possible route.

□

From Lemma **??** we can deduce a straightforward corrolary.

Corollary 44. *Let* $(\mathbf{a},\mathbf{b},\mathbf{C})$ *be an instance of ITP. If there is only one possible route* R *with a condition* $\sum\limits_{k\le i} a_k = \sum\limits_{\ell\le j} b_\ell$ *for some* i,j*, then the values* $a_1,\dots a_i$ *and* $b_1,\dots b_j$ *are real.*

Proof. Let R be the only possible tour of $(\mathbf{a},\mathbf{b},\mathbf{C})$. For i,j by Lemma **??** the condition $\sum\limits_{k\le i} a_k = \sum\limits_{\ell\le j} b_\ell$ implies that

$$\sum_{k\le i}\underline{a}_k = \sum_{\ell\le j}\overline{b}_\ell \text{ and } \sum_{k\le i}\overline{a}_k = \sum_{\ell\le j}\underline{b}_\ell.$$

But since

$$\sum_{k\le i}\underline{a}_k \le \sum_{k\le i}\overline{a}_k = \sum_{\ell\le j}\underline{b}_\ell \le \sum_{\ell\le j}\overline{b}_\ell = \sum_{k\le i}\underline{a}_k,$$

it follows that

$$\sum_{k\le i}\underline{a}_k = \sum_{k\le i}\overline{a}_k$$

and by a similar conclusion

$$\sum_{\ell\le j}\underline{b}_\ell = \sum_{\ell\le j}\overline{b}_\ell.$$

But this means that $a_1,\dots a_i$ and $b_1,\dots b_j$ are real values.

□

8.3 Travelling salesman problem (TSP)

The travelling salesman problem is probably one of the most famous problems in combinatorial optimization. An instance of TSP consists of n cities with distances d_{ij} denoting the distance from city i to city j. The goal is to find the shortest closed tour through all n cities. Although NP-hard in general, for a distance matrix satisfying the Monge property (actually the matrix has to be Monge and symmetric) the problem becomes easily solvable by a so called pyramidal tour. We study the interval version of the problem. More precisely we are interested in the optimal value range of all possible scenarios of TSP within an interval distance matrix. We show that for a subset of $\mathbb{ISM}$ we achieve a well-solvable case of the problem.

8.3.1 The real version of TSP

We start with a mathematical definition of the problem.

Definition 26. *Let $D \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then the Travelling Salesman problem (TSP) is to find a permutation Φ of elements $\{1, 2, \dots, n\}$ that minimizes*

$$\left(\sum_{i=1}^{n-1} d_{\Phi(i),\Phi(i+1)}\right) + d_{\Phi(n),\Phi(1)}.$$

The person to prove the following theorem was Fred Supnick, therefore a symmetric Monge matrix is called a *Supnick* matrix.

Theorem 45. *Let $D \in \mathbb{R}^{n\times n}$ be a Supnick matrix. Then the shortest TSP tour is given by permutation σ^{min} where*

$$\sigma^{min} = \langle 1, 3, 5, \dots, n-3, n-1, n, n-2, n-2, \dots, 6, 4, 2\rangle \textit{ for } n \textit{ even,}$$

$$\sigma^{min} = \langle 1, 3, 5, \dots, n-4, n-2, n, n-1, n-3, \dots, 6, 4, 2\rangle \textit{ for } n \textit{ odd}$$

and the longest TSP tour is given by the permutation

$$\sigma^{max} = \langle n, 2, n-2, 4, n-4, 6, \dots, n-5, 5, n-3, 3, n-1, 1\rangle$$

where a permutation is defined as $\Phi = \langle \Phi(1), \Phi(2), \dots, \Phi(n)\rangle$.

Proof. See[15]. □

An interesting fact is that σ^{min} and σ^{max} are the same for all instances of TSP with Supnick distance matrix. We will use this fact in the interval version of TSP.

8.3.2 The interval TSP

Much of the work on TSP is not motivated by real-world applications but rather on fact that it is a decent platform for study of general methods that can be applied elsewhere. In real-world applications the distance matrix might represent not only distances between the cities but also time spent on the road which differs depending on the traffic or the fuel spent while travelling etc. It is therefore convenient to employ interval matrices.

Definition 27. *Let $\mathbf{D} \in \mathbb{IR}^{n\times n}$. Then the interval travelling salesman problem (ITSP) is a family of problems*

$$\min_{\Phi} \left\{ \left(\sum_{i=1}^{n-1} d_{\Phi(i),\Phi(i+1)}\right) + d_{\Phi(n),\Phi(1)} \right\}$$

where $D \in \mathbf{D}$.

Matrix D is called a *scenario* of the *ITSP*. We are interested in a problem where the goal is to compute the optimal value range of *ITSP* which means we want to find a lower and an upper bound

$$\underline{f}(\mathbf{D}) = \min_{D\in\mathbf{D}} \left\{ \min_{\Phi} \left\{ \left(\sum_{i=1}^{n-1} d_{\Phi(i),\Phi(i+1)} \right) + d_{\Phi(n),\Phi(1)} \right\} \right\}$$

and

$$\overline{f}(\mathbf{D}) = \max_{D\in\mathbf{D}} \left\{ \min_{\Phi} \left\{ \left(\sum_{i=1}^{n-1} d_{\Phi(i),\Phi(i+1)} \right) + d_{\Phi(n),\Phi(1)} \right\} \right\}.$$

We will show a way to compute the optimal value range of *ITSP* by two instances of $\mathbf{D}$.

Lemma 46. *Let* $\mathbf{D} \in \mathbb{IR}^{n\times n}$ *be a nonnegative interval matrix and let*

$$\underline{f} = \min_{\Phi} \left\{ \left(\sum_{i=1}^{n-1} \underline{d}_{\Phi(i),\Phi(i+1)} \right) + \underline{d}_{\Phi(n),\Phi(1)} \right\}$$

and

$$\overline{f} = \min_{\Phi} \left\{ \left(\sum_{i=1}^{n-1} \overline{d}_{\Phi(i),\Phi(i+1)} \right) + \overline{d}_{\Phi(n),\Phi(1)} \right\}.$$

Then the optimal value range of the ITSP is $\mathbf{f} = \left[\underline{f}, \overline{f}\right]$.

Proof. Let $\underline{\Phi}$ and $\overline{\Phi}$ be the optimal tours of scenarios $\underline{D}$, resp. $\overline{D}$. Then for any $D \in \mathbf{D}$ and its optimal tour Φ

$$\sum_{i=1}^{n-1} \underline{d}_{\underline{\Phi}(i),\underline{\Phi}(i+1)} + \underline{d}_{\underline{\Phi}(n),\underline{\Phi}(1)} \le \sum_{i=1}^{n-1} \underline{d}_{\Phi(i),\Phi(i+1)} + \underline{d}_{\Phi(n),\Phi(1)} \le \sum_{i=1}^{n-1} d_{\Phi(i),\Phi(i+1)} + d_{\Phi(n),\Phi(1)}$$

and conversely

$$\sum_{i=1}^{n-1} d_{\Phi(i),\Phi(i+1)} + d_{\Phi(n),\Phi(1)} \le \sum_{i=1}^{n-1} d_{\overline{\Phi}(i),\overline{\Phi}(i+1)} + d_{\overline{\Phi}(n),\overline{\Phi}(1)} \le \sum_{i=1}^{n-1} \overline{d}_{\overline{\Phi}(i),\overline{\Phi}(i+1)} + \overline{d}_{\overline{\Phi}(n),\overline{\Phi}(1)}$$

thanks to the nonnegativity of $\mathbf{D}$.

□

The result of Lemma **??** simply states that the optimal value range problem is solvable for nonnegative matrices by solving the scenarios of $\underline{D}$ and $\overline{D}$. By Theorem **??** it is easy to see that if $\underline{D}$ and $\overline{D}$ are both symmetric Monge matrices, then the problem is solvable in $O(n)$ time. We show that the tour σ_{min} is optimal for slightly larger class of Monge matrices than Supnick matrices.

Lemma 47. *Let* $M \in \mathbb{R}^{n\times n}$ *be a Monge matrix such that*

$$M = S + \sum_{i=1}^{m} \kappa_i H^i + \sum_{j=1}^{n} \lambda_j V^j + \sum_{r,s} \mu_{rs} L^{rs} + \sum_{p,q} \nu_{pq} R^{pq}$$

where S *is a Supnick matrix,* $\kappa_i, \lambda_j, \mu_{rs}, \nu_{pq} \ge 0$,

$$(r-s) \begin{cases} \ge 3, & \text{if } r \text{ is even,} \\ \ge 2, & \text{if } r \text{ odd,} \end{cases} \quad \text{and } (q-p) \begin{cases} \ge 3, & \text{if } q \text{ is odd,} \\ \ge 2, & \text{if } q \text{ even.} \end{cases}$$

Then the optimal tour of M *is* σ_{min}.

Proof. For any S the optimal tour is σ_{min} by Theorem **??**. Adding $\kappa_i H^i$ to S means to add κ_i to every column in a row i. From the point of view of the optimal tour the entry d_{ij} represents the distance from city i to city j. Since every tour has to leave city i at some point, the length of every tour in $S+\kappa_i H^i$ is the length of the tour in S plus the value κ_i, therefore the tour σ_{min} remains optimal.
For matrices of type $\lambda_j V^j$ the line of reasoning is analogous. Every tour has to enter city j at some point and the distance d_{kj} will increase by λ_j for every k.
Finally, observe that for matrices L^{rs} and R^{pq} the restrictions on r, s, p, q select only those matrices which do not interfere with the optimal tour σ_{min}. It means that the entries of the tour σ_{min} in $S + \sum_{r,s} \mu_{rs} L^{r,s} + \sum_{p,q} \nu_{pq} R^{pq}$ are the same as the entries of σ_{min} in S and other entries in $S + \sum_{r,s} \mu_{rs} L^{r,s} + \sum_{p,q} \nu_{pq} R^{pq}$ are only greater or equal to entries in S, respectively.

□

For matrices L^{rs}, R^{pq} which violate the condition on r, s, p, q we can observe that the optimal tour might be different from σ_{min}.

Observation 48. *Let*

$$D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \textit{and } \mu L^{22} = \mu \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

We see that $(r-s) = 2-2 = 0$ violates the condition $(r-s) \geq 3$. The sum of distances of a tour $\Phi = \langle 1, 2, 3, 4 \rangle$ in $D + \mu L^{22}$ is $4 + \mu$, however, the sum of distances in σ_{min} is $4 + 2\mu$. For $\mu > 0$ permutation Φ yields a better solution.

We note that for some more general Monge matrices than stated in Lemma **??** it might be still possible to characterize the optimal tour, however, the optimal tour of the matrix might not be σ_{min}. We did not study this problem, therefore we state it as an open question.

Conclusion

The main task of this thesis was to examine interval Monge matrices - to find different characterizations or at least necessary and sufficient conditions, analyze algorithmic aspects of these conditions, investigate closure properties of operations applied on the matrices and finally, investigate applications of interval Monge matrices in optimization.

Results of this thesis

We introduced two classes of interval Monge matrices - $\mathbb{ISM}$ and $\mathbb{IWM}$. For $\mathbb{ISM}$ following mostly results of real Monge matrices we generalized several characterizations and in Sections 3.3.1 and 3.3.2 we showed that some characterizations of real Monge matrices lead to inflation of interval Monge matrices.
For $\mathbb{IWM}$ we showed several necessary and sufficient conditions. For the condition from Lemma 23 we indicated that the condition is from a larger class of conditions that might be interesting to further investigate.
We presented lists of closure properties under operations on $\mathbb{ISM}$ and $\mathbb{IWM}$ and under operations combining both classes of matrices.
We investigated applications in two interval optimization problems - ITP and ITSP. For ITP we studied the optimal value range problem. We divided the problem into possibly exponential number of polynomially solvable subproblems and showed a sufficient condition under which only one subproblem is needed to solve the problem. For ITSP we studied the optimal value range problem and showed that the problem is solvable by solving two realizations of the interval cost matrix. Then we extended a result by Supnick by characterizing which interval matrices are solvable by Supnick's greedy algorithm.
We also generalized a famous SMAWK algorithm for row minima searching in Monge matrices. The algorithm is fundamental for some geometrical applications of real Monge matrices. For these applications a generalized version was not investigated in this thesis.
In addition we introduced three algorithms for reconstruction of Monge matrices. Two based on interval modification and one on rows and columns permutation.
We also generalized Bost's method for generating real Monge matrices and presented own method for more general generating of interval matrices where we can control the specifics of generating.

Future work

The study of $\mathbb{IWM}$ offers a lot of open questions that still remains to be answered. In near future author wants to deeper comprehend these matrices. For $\mathbb{ISM}$ there is still plenty of work in applications, namely in geometrical applications of SMAWK algorithm. In ITP polynomiality of computing the upper bound of optimal value range remains unanswered.

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