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BACHELOR THESIS

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Conformal symmetry and vortices in graphene

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Study programme:	Physics
Study branch:	General Physics

Prague 2019

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In Prague, May 13 Pavel Kůs

Acknowledgment

In the first place, I would like to thank my family for the support of my studies and my further personal development.

I would like to express my gratitude to my supervisor Alfredo Iorio, especially for his endless patience, friendly attitude and dozens of hours spent over my work.

I would like to grasp this opportunity to thank all my teachers that I have ever had, for that they have moved me to where I am today.

Without the support of all, this work would never be written.

Title: Conformal symmetry and vortices in graphene

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Abstract: This study provides an introductory insight into the complex field of graphene and its relativistic-like behaviour. The thesis is opened by an overview to this topic and draws special attention to interesting non-topological vortex solutions of the Liouville equation found by P. A. Horváthy and J.-C. Yéra, which emerge in a context of the Chern-Simons theory [1], [2] and have been put into context of graphene [3], [4]. We introduce the massless Dirac field theory, well describing electronic properties of graphene in the low energy limit, and point to the fact that the action of the massless Dirac field is invariant under Weyl transformations, which has far-reaching consequences. When the graphene membrane is suitably deformed, we assume that the correct description is that of a Dirac field on a curved spacetime. In particular, an important case is that of conformally flat 2+1-dimensional spacetimes. These are obtained when the spatial part of the metric describes a surface of constant intrinsic curvature [3]. In other words, the conformal factor of such spatial metrics has to satisfy the Liouville equation, an important equation of mathematical physics.

In this work, we have identified the kind of surfaces to which the Horváthy-Yéra conformal factors, above recalled, correspond, and have provided the geometrical explanation of the natural number N of such non-topological solutions. We have done that by identifying the appropriate change, from the isothermal coordinates to the canonical coordinates for surfaces of revolution. We found here that, for the generic N, such surfaces are surfaces of positive constant Gaussian curvature of the Bulge type (barrel shaped surfaces, that present singular boundaries), and only for $N = 1$ coincide with the sphere. Finally, we briefly comment on the corresponding 2+1-dimensional spacetimes, and show the possible connection with the Bondi-Lemaitre-Tolman spacetime.

Keywords: gravity analogues, Dirac massless field theory, conformal and Weyl symmetry, Liouville equation, non-topological vortex solutions, 2+1 - dimensional spacetimes, graphene membrane

Contents

Introduction

Graphene's story

Theoretical physics has gone a long journey until present days, but becomes too remote from current experimental physics, which is unable to (in)prove plenty of often curious theoretical predictions. One concrete example is the black hole evaporation. For such kind of problems, when we cannot observe phenomena directly, physicists came up with an interesting concept, following Feynman's motto "*same equation, same solution*". Speaking about gravity, the idea is to replace a relativistic system of our interest (e.g. a black hole) by a relativistic-like system, which shares with the target system some of the relativistic properties, and can be constructed in a laboratory. This concept is known under a name *gravity analogue*. The question is whether such systems exist. For our surprise, the answer is positive. It was discovered in a laboratory that a carbon layer (graphene[1]) is a example of three-dimensional quantum relativistic-like system[2], well described by the Dirac massless[3] field theory (DMFT). It naturally leads to an expectation of an emergence of gravity-like phenomena on the carbon sheet [3], [4].

Next it was discovered that the action of the DMFT enjoys so-called *local Weyl symmetry*

$$g_{\mu\nu}(x) \to \Phi^2(x) g_{\mu\nu}(x), \quad \psi(x) \to \Phi^{-1}(x)\psi(x).$$

The action, which determinates physics of particular system, of the DMFT is invariant under local Weyl transformations. Thus, we say it enjoys the local Weyl symmetry. We emphasize that the Weyl transformation acts on the metric tensor and not on coordinates (see eg. [5]). After the transformation, we deal with a different spacetime, $g'_{\mu\nu} = \Phi^2 g_{\mu\nu}$, usually said "conformal" one another.

With this, if we would like to study a behaviour of a system with spacetime metric tensor denoted $g'_{\mu\nu}$ in laboratory conditions, we can use the Weyl symmetry of the Dirac massless action and set up a experiment with a different system, whose spacetime metric tensor is $g_{\mu\nu}$. Due to the Weyl symmetry, the action is still same.

But what does it mean that the carbon layer is assigned a space-time? Here, the problem starts to be very complicated. Firstly, we note that in this thesis we focus on two-dimensional surfaces instead of layers with complicated structures as carbon layers have. Therefore we will not use "layer" any more and will speak about "membrane" since this moment. Secondly, the effective DMFT forces us to assign a spacetime (in the ordinary sense) to a surface, which represents given a graphene membrane. It makes good sense to expect that the spatial part of the

[1]As well as silicon or germanium, but we will talk only about graphene.

[2]Two dimensions for space, one for time.

[3]Rest mass m or precisely its energy mc^2 has a obvious sense for a true relativistic system, but not in a relativistic-like system. For example, the system might be a carbon layer, where the Fermi velocity v_F replaces the speed of light c, but it is not clear what mv_F^2 means. The requirement of "massless" is therefore natural. Of course, not suprising that it is rather a effective than fundamental theory. We will come back to it later.

spacetime metric tensor should be the metric tensor of the surface. Unfortunately, "should" is on the right place, because the shape of the metric tensor and the meaning of its (matrix) elements depend on used coordinate frame. Moreover, we have not expected anything about the metric tensor of the surface. It is yet an abstract unspecified object. But this is not the end, what about time or even mix elements (space-time) of the spacetime metric tensor? There are still many unanswered questions. Before we introduce the "solution" of these problems, let us realize something very practical.

The reason why we deal with the Weyl symmetry instead of constructing a membrane with spacetime $g'_{\mu\nu}$ right away, is the crucial fact that is not clear how such surface should look like. The spacetime metric tensor is expressed with respect to some coordinates, but finally, we must find the Cartesian (laboratory) coordinates x, y, z of the surface and it means to do coordinate transformations and solve set of non-linear partial differential equations. This can be in general extremely difficult problem (if not impossible). If we decide to work $g'_{\mu\nu}$ instead of $g_{\mu\nu}$, we must deal with same problem, but it can be simpler to solve (find the surfaces corresponding to $g_{\mu\nu}$) which is the key message.

We make a few simplistic assumptions [3]. The first one is that we assume a special shape of the metric tensor

$$g^{(3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -g^{(2)}_{\alpha\beta} \end{pmatrix},$$

where the part of the metric tensor corresponding to the surface is denoted $g^{(2)}_{\alpha\beta}$. The sign 'minus' is given by a standard convention. So we deal with a kind of spacetimes obtained as a product of flat-time and a metric tensor of a specific surface.

The shape of the spacetime metric tensor is suddenly more clear, but we have no information about $g^{(2)}_{\alpha\beta}$. Physicists had taken a crack at it and assumed only *conformally flat* spacetimes, which are richly represented in GR (in $2+1$ dimension). What does it mean? Simply, the spacetime is *conformal* to the Minkowski spacetime: $g_{\mu\nu} = \Phi^2 \eta_{\mu\nu}$. From now, we deal with conformally flat spacetimes of flat-time part of their metric tensors. Of course, it is a loss of generality, but now we have a much clear idea how the metric tensor looks like. Moreover, it is well known that each two-dimensional manifold is conformally flat, hence the surface metric tensor can be written as $g^{(2)}_{\alpha\beta} \equiv e^{\sigma}\delta^{(2)}_{\alpha\beta}$, where σ is a function and $\delta_{\alpha\beta}$ is the Kronecker delta. The necessary and sufficient condition for the conformal flatness in 2+1 dimension is the vanishing of the so-called *Cotton(-York) tensor*. It can be proved (see later), that this means that the conformal factor has to satisfy one famous equation of mathematical physics called *Liouville equation*

$$\Delta\sigma(\tilde{x}, \tilde{y}) = -K e^{2\sigma(\tilde{x}, \tilde{y})},$$

where $\tilde{x}$ and $\tilde{y}$ are so-called *isothermal coordinates*[4]. The constant K is named the *Gaussian curvature.* It is an intrinsic quantity, which can have any real value. In the complex plane $z = \tilde{x} + i\tilde{y}$, $\bar{z} = \tilde{x} - i\tilde{y}$, the solution of the Liouville equation

[4]The infinitesimal line element of a surface in these coordinates is $dl^2 = \phi^2(\tilde{x}, \tilde{y})\left(d\tilde{x}^2 + d\tilde{y}^2\right)$.

for $K \neq 0$ is

$$\phi(z) = \frac{2}{\sqrt{|K|}} \frac{|f'(z)|}{1 \pm |f(z)|^2},$$

where 'plus' or 'minus' refers to the positive or negative constant Gaussian curvature, respectively [3], [6], [7].

As we have learned in previous related work [8], [9], if we allow for certain coordinates redefinitions, the only surface of positive constant Gaussian curvature is the sphere. The sphere is constructable in laboratory conditions and corresponds to the giant fullerene (e.g. [4]). The spacetime related to the sphere will be discussed later.

On the other hand, the number of surfaces of negative constant Gaussian curvature is infinite, hence, new surfaces are discovered over the years. In [3] there are three very well studied surfaces of negative constant Gaussian curvature and it is introduced to which well known spacetimes are conformal. They are: the Beltrami, elliptic and hyperbolic pseudospheres and associated spacetimes are conformal[5] to spacetimes: the Rindler, de Sitter, Bañados - Teitelboim - Zanelli (BTZ) black hole, respectively[4]. We will come back to them later in the thesis. Surely, this is not the end of the story for these particular pseudospheres, because it is still a problem to reproduce them in the laboratory.

At this moment, the situation looks opposite than in the beginning. We know how some particular surfaces (with some required properties) look like, i.e. we know x, y, z coordinates, and we ask ourselves if we can find some spacetime in GR, whose spacetime metric tensor is formally same or is conformal to the metric tensor of the spacetime associated to the surface.

Now, we move a little bit back from the mentioned problem of the reproduction of membranes in the laboratory to the discussion about shapes of surfaces. The introduced pseudospheres were vortex-like surfaces[6]. In the work [3] it was suggested to study in the context of the graphene the *non-topological vortex solutions*, concrete solutions of Liouville equation, which appears in the context of Chern-Simons theory [7] and was introduced P. A. Horváthy and J.-C. Yéra in [1], see also [2]. Although the Liouville equation allows for both signs of K, they are interested in constant $K > 0$ and assume the conformal factor with a *radial symmetric Ansatz* $f(z) = z^{-N}$. In polar coordinates $(z, \bar{z}) \to (\tilde{r}, \tilde{\theta})$ the conformal factor is

$$\phi_+(\tilde{r}) = \frac{2N}{\sqrt{K}} \frac{\tilde{r}^{N-1}}{1 + \tilde{r}^{2N}},$$

where $\tilde{r} = |z|$, $\tilde{r} \in [0, \infty]$, $\tilde{\theta} \in [0, 2\pi]$ and N is a natural number [1]. The sign '+' underlines the positivity of the constant Gaussian curvature. The conformal factor for different N corresponds to different geometries, see Figs 1, 2 and 3.

[5]It is a custom to say "conformal", but more precise is "Weyl related".

[6]Of course, there exist exotic surfaces with constant negative K, which do not look like a vortex, see [4].

[7]In this thesis, we will not deal with the Chern-Simons theory, which is a name for three-dimensional topological quantum field theory. What is quite interesting, is the attribute "non-topological". It has a connection with value of scalar field in the infinity $r \to +\infty$. The vortex is said to be "topological" if the scalar field $|\psi| \to 1$ for $r \to +\infty$ and "non-topological" if $\psi \to 0$ for $r \to +\infty$ [2].

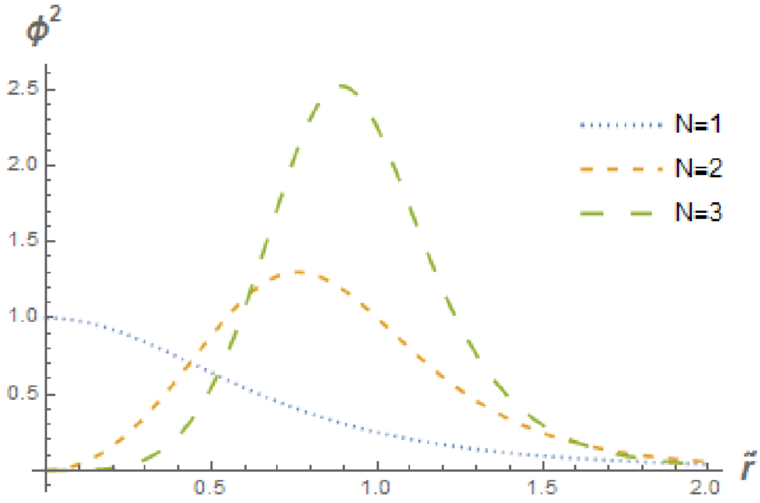


Figure 1: Graph of $\phi_+^2(\tilde{r})$ on $\tilde{r}$ for $N = 1, 2, 3$

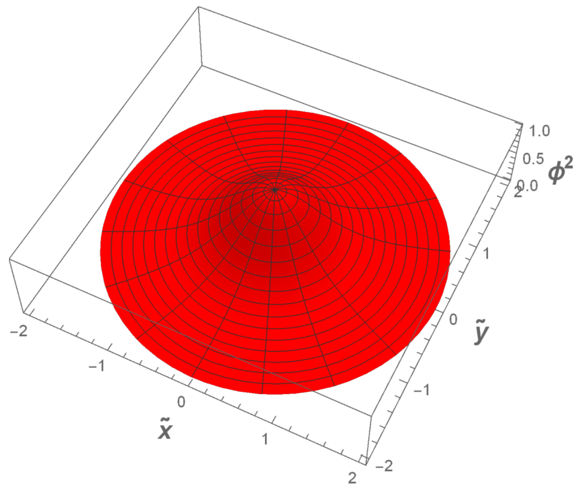
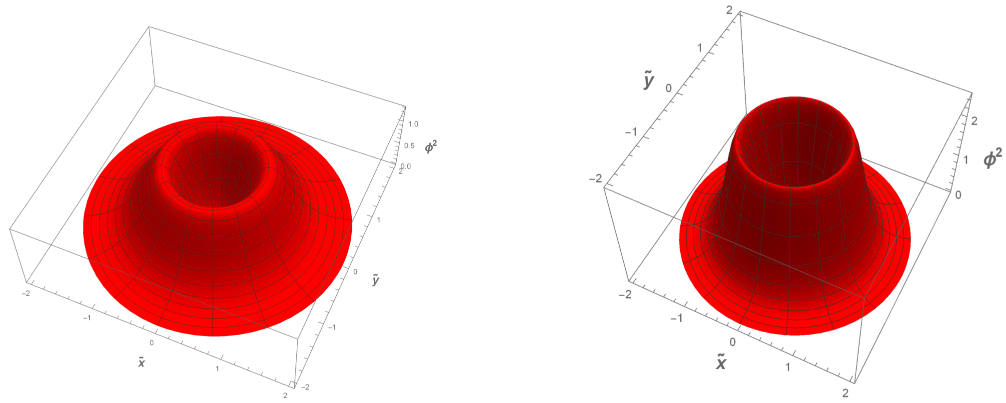


Figure 2: Squared conformal factor $\phi_+^2(\tilde{x}, \tilde{y})$ of the vortex solution for $K = 4$, $\tilde{r} \in [0, +\infty]$, $\tilde{\theta} \in [0, 2\pi]$, $N = 1$

According to our knowledge, the problem of determining spatial coordinates $x, y, z(x, y)$, corresponding to vortex solutions, has not been resolved yet. However, everything what we know is the conformal factor $\phi_+(\tilde{r})$ in polar isothermal coordinates. In general, this is a very difficult problem, as one needs to solve a system of non-linear partial differential equations, coming from changing of coordinates, and that can be tricky in practise. There is one hint, which we mentioned above, coming from the Hilbert theorem, but this will be discussed in details later. Next question is what we can say about the related spacetime (with flat-time part). Is the spacetime formally same/conformally related to some spacetime in GR?

A few words about the thesis

This thesis is divided into four chapters. The first three chapters provide an overview of the necessary background. The first chapter is dedicated to an ex-



(a) $N = 2$ (b) $N = 3$

Figure 3: Squared conformal factor $\phi_+^2(\tilde{x}, \tilde{y})$ of the vortex solution for $K = 4$, $\tilde{\theta} \in [0, 2\pi]$, $\tilde{r} \in [0, +\infty]$, (a) $N = 2$, (b) $N = 3$

planation of the Dirac field theory and its generalization to curved spacetime. Special attention is paid to the Weyl symmetry of the Dirac massless action. The second chapter explains why the conformal factor of any conformally flat surface (membrane) must satisfy the Liouville equation. The third chapter is focused on surfaces of revolution (for both signs of K) and we briefly discuss the spacetimes that are obtained taking the product of flat-time and those two-dimensional surfaces (the Rindler spacetime etc.). The fourth chapter is focused on our own study of presented vortex solutions for $K > 0$. We also discuss the case $K < 0$ to find out whether something new could be given. We identify there surfaces corresponding to vortex solutions (for arbitrary N) and address briefly the spacetimes obtained by taking the product of flat-time and these surfaces.

Finally, we present two appendices, the first is dedicated to conformal symmetry and the second to physics of three particular spacetimes: Rindler, de Sitter and BTZ black hole.

This work indirectly builds on two previous projects [10], [11] supported by two student faculty grants (SFG) on the Faculty of Mathematics and Physics, Charles University, Prague.

"I always do that, get into something and see how far I can go."
Richard P. Feynman

"You never fail until you stop trying."
Albert Einstein

1. Weyl symmetry of the Dirac field

This chapter is divided into three sections. The first one is focused on the Dirac field theory in the Minkowski (flat) spacetime and on how it follows from a combination of quantum mechanics and special relativity. We chose an "intuitive approach": we start with $3 + 1$ dimensions and conclude that it works for arbitrary dimension n (minimum $n = 1 + 1$). In the following section we introduce a concept of *connection*, which is necesary for the generalization of the Dirac action to curved spacetimes. This is done in the last section, where we discuss the Weyl symmetry of the Dirac massless action.

1.1 Dirac field theory in flat spacetime

The DMFT is a backbone of theoretical study of the graphene membrane in low energy excitations, because well describes its electronic properties [3].

We will outline the derivation of the Dirac equation, i.e. relativistic generalization of the Schrödinger equation. In the end, we will discuss the action of the Dirac field theory in flat spacetime [12], [13], [14].

Schrödinger, Klein-Gordon and Dirac equations

Here we begin with well known time-depended Schrödinger equation

$$\hat{H} |\psi\rangle = i\hbar\partial_t |\psi\rangle \,, \tag{1.1}$$

where $|\psi\rangle$ is the ket vector, representing the state of the system, $\hat{H}$ is the hamiltonian, and $\partial_t \equiv \partial/\partial t$.

In the x-representation $\langle x, t|\psi\rangle \equiv \psi(x, t)$, the momentum operator is $\hat{p} = -i\hbar\nabla$ and the Schrödinger equation (1.1) might be rewritten as

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\partial_t\psi. \tag{1.2}$$

That is evidently non-relativistic, as space and time derivatives are of different orders.

We would like to find a relativistic form of (1.2). Therefore we write down Einstein's relation for energy E, rest mass m and momentum p

$$E^2 = m^2c^4 + p^2c^2 \tag{1.3}$$

and ask ourselves whether (1.2) and (1.3) can be combined to one covariant equation. By applying the so-called *correspondence principle*: $E \to i\hbar\partial_t$ and $p \to -i\hbar\nabla$, we obtain the so-called *Klein-Gordon equation* (e.g. [13], [14])

$$\left(-\frac{1}{c^2}\partial_t^2 + \nabla^2\right)\psi = \frac{m^2c^2}{\hbar^2}\psi. \tag{1.4}$$

The positive result is that the Klein-Gordon equation (1.4) is Lorentz invariant unlike the Schrödinger equation. Is this the relativistic generalization we were looking for? The answer seems to be negative for three reasons.

The first is the Klein-Gordon equation allows for given p both signs of energy: $E = \pm\sqrt{m^2c^4 + p^2c^2}$. The negative sign is usually interpreted as antiparticle.

The second is that the Klein-Gordon equation is linear partial differential equation of second order. It means to solve the equation we need to know ψ as well as $\partial\psi/\partial t$ for initial state. In comparison with the Schrödinger equation there is one more degree of freedom in initial conditions.

The third one is the time-depended probability distribution, which allows the negative value of the probability density. To avoid the negative probability, the highest time derivative cannot be higher than the first order.

Dirac had the original idea to consider the left side of (1.4) as a square of one expression, which contains derivatives of the first order

$$\left(-\frac{1}{c^2}\partial_t^2 + \nabla^2\right)\psi = \left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right)^2\psi,$$

where $\partial_x \equiv \partial/\partial x$ etc. However, this necessarily leads to following conditions

$$AB = BA = \cdots = 0, \qquad A^2 = ... = D^2 = 1. \tag{1.5}$$

These equations make sense if A, B, C, D are matrices. Particularly for 3+1 dimension, we deal with 4×4 matrices and ψ is not one function, but an array of four complex functions. ψ is usually called *spinor*. Then

$$\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right)\psi = \frac{mc}{\hbar}\psi. \tag{1.6}$$

This is called *Dirac equation*, still with unknown matrices. We must write (1.6) in covariant form, therefore we set $A = i\gamma^1, B = i\gamma^2, C = i\gamma^3, D = \gamma^0$. Then the Dirac equation is

$$\left(-i\hbar\gamma^a\partial_a + mc\right)\psi = 0, \tag{1.7}$$

where and $a = 0, 1, 2, 3$ are Lorentz indices and $\partial_0 = \partial_t/c$.

Of course, from the Dirac equation it must be possible to obtain back the Klein-Gordon equation. We multiply the left side of the Dirac equation (1.7) by $\left(i\hbar\gamma^b\partial_b + mc\right)$

$$\left(\gamma^b\gamma^a\partial_b\partial_a + \frac{m^2c^2}{\hbar^2}\right)\psi = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi \equiv \left(\eta^{ab}\partial_a\partial_b + \frac{m^2c^2}{\hbar^2}\right)\psi = 0, \tag{1.8}$$

which gives the requirement[1]

$$\{\gamma^a\gamma^b\} \equiv \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}. \tag{1.9}$$

For instance, for 3+1 dimensional spacetimes, the matrices might have the form [12]:

$$\gamma^0 = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \gamma^0 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \tag{1.10}$$

[1] If choose the signature of the metric tensor as $(+, -, -, -)$.

where $\sigma_x, \sigma_y, \sigma_z$ are the 2×2 Pauli matrices.

For arbitrary dimension $n \geq 2$ equations are formally same up to dimension of matrices and spinors, e.g. for $n = 2$ γ-matrices are 2×2 and spinors have two components.

Action of Dirac (massless) field theory

As we already know the Dirac equation (1.7) is a equation of motion for a relativistic quantum free point particle in the Minkowski spacetime with η_{ab}. On the other hand, there exists an alternative approach how the system can be described. It is usually to work with a functional called *action*, which we already mentioned. Here, the action $\mathscr{A}$ is a functional depended on ψ and its first order derivative

$$\mathscr{A}(\psi, \partial_a \psi) = \int d^n x \mathscr{L}(\psi, \partial_a \psi), \tag{1.11}$$

where n is the dimension of spacetime.

The Euler-Lagrange equations are obtained by extremizing the functional, $\delta \mathscr{A} = 0$, with respect to the conjugate spinor $\bar{\psi}$. This gives

$$\mathscr{A} = \int d^n x \bar{\psi} \left(i\hbar c \gamma^a \partial_a - mc^2 \right) \psi. \tag{1.12}$$

What we introduced is a fundamental theory. The key information for our work is the experimental fact that the massless theory (effectively) describes the low energy behaviour of the conductivity electrons of a graphene membrane, so the spacetime of dimension $n = 3$. Let us remind us that the Minkowski "spacetime", which we have in mind here, is not exactly same as we are used to dealing with in special relativity. The standard three-vector is right now $(v_F t, x_1, x_2)$, where x_1, x_2 are some coordinates covering the surface and v_F denotes the Fermi velocity, the analog to the speed of light in special relativity. The speed of light must be replaced by the Fermi velocity in the whole equation, of course. Meantime, we drop the mass term [3]. Now, we postulate that the action related to the Dirac massless equation in the case of graphene has the form

$$\mathscr{A} = i\hbar v_F \int d^3 x \bar{\psi} \gamma^a \partial_a \psi. \tag{1.13}$$

The components of the spinor are complex functions, therefore each of them has real and imaginary part. Complex conjugated ψ^* is related[2] to $\bar{\psi}$, therefore we can choose whether to vary $\bar{\psi}$ or ψ. Let us choose to vary $\bar{\psi}$, then we get well known results

$$\delta \mathscr{A} = i\hbar v_F \int d^3 x \delta \bar{\psi} \gamma^a \partial_a \psi = 0. \tag{1.14}$$

As explained earlier, we are interested in curved, rather than flat spacetimes. Therefore, the next section is focused on the geometry of a curved spacetime. After that we will be able to write the action of Dirac field theory for curved spacetime.

[2] ψ^* is a column-like array (vector) and $\bar{\psi}$ is row-like array (vector), but include same components, i.e. $\psi^* = \bar{\psi}^T$.

1.2 Geometry of curved spacetime

A fundamental concept in differential geometry is that of differentiable manifold, with its associated metric tensor. These ideas are n-dimensional generalizations of the two-dimensional surfaces (the manifolds), with its Euclidean metric, and various kinds of derivatives and associated differential calculus.

The manifold of our interested is so-called *pseudo-Riemannian manifold*. It is a differentiable manifold with non-degenerated metric tensor, whose positive-definitess is relaxed[3].

The next concept is called *connection*. Simply, some data (vector, ...) are transported along a concrete curve on the manifold and the connection, that takes into account how the manifold departs from flat space, dictates the way to transport such data along the curve. For example, we assume a vector V^a, where a is a Lorentz index in a local (flat) reference frame on the curved manifold. The vector is transported in such a way that it keeps its direction in a local reference frame, i.e. $dV^a/dp = 0$, where p is a parameter of the curve. The equation for the *parallel transport* can be written as

$$\frac{dV^\mu}{dp} + \Gamma^\mu{}_{\rho\sigma} V^\rho \frac{dx^\sigma}{dp} = 0, \tag{1.15}$$

where

$$\Gamma^\mu{}_{\rho\sigma} \equiv \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial^2 \xi^a}{\partial x^\rho \partial x^\sigma} \tag{1.16}$$

are *Christoffel symbols of the second type*, ξ^a denotes local coordinates and x^μ global coordinates, with Greek letters denoting Einstein indices. For smooth neighbourhood along the curve, (1.15) can be rewritten as

$$\nabla_\sigma V^\mu \frac{dx^\sigma}{dp} = 0, \tag{1.17}$$

where ∇_σ denotes the *covariant derivative*, known as the *Levi-Civita connection*[4].

$$\nabla_\sigma V^\mu = \partial_\sigma V^\mu + \Gamma^\mu{}_{\sigma\rho} V^\rho, \tag{1.18}$$

where $\partial_\sigma \equiv \partial/\partial x_\sigma$.

The role of the Christoffel symbols in (1.18) is to compensate for the non-tensorial nature of the standard derivative $\partial_\sigma V^\mu$, giving as a result a derivative that, under general coordinate transformations, indeed is a tensor.

The Christoffel symbols of the first kind can be defined from the Christoffel symbols of the second kind using the metric tensor

$$\Gamma_{\mu\nu\rho} \equiv g_{\alpha\mu} \Gamma^\alpha{}_{\nu\rho}. \tag{1.19}$$

The Christoffel symbols of the first kind are related to the metric tensor as

$$\Gamma_{\mu\nu\rho} = \frac{1}{2} \left(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\rho\mu} - \partial_\mu g_{\nu\rho} \right). \tag{1.20}$$

[3]Examples of such manifolds are Euclidean or Lorentz manifold.
[4]A transport of data is similar to the derivative.

The *Riemann tensor* is the tensor that conveys the complete information about the intrinsic curvature properties of the manifold. It is defined by the following formula[5]

$$R^{\sigma}{}_{\nu\kappa\lambda} \equiv \partial_\lambda \Gamma^{\sigma}{}_{\nu\kappa} - \partial_\kappa \Gamma^{\sigma}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\rho\kappa}\Gamma^{\rho}{}_{\nu\lambda} - \Gamma^{\sigma}{}_{\rho\lambda}\Gamma^{\rho}{}_{\nu\kappa}. \tag{1.21}$$

The tensor can be written with lower indices using the metric tensor

$$R_{\mu\nu\kappa\lambda} = g_{\mu\sigma} R^{\sigma}{}_{\nu\kappa\lambda}. \tag{1.22}$$

The *Ricci tensor*, which is symmetric, is defined as the trace of the Riemann tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}. \tag{1.23}$$

The *Ricci scalar* (or scalar curvature) is the trace of the Ricci tensor

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{1.24}$$

Because $\nabla_\sigma V^\mu$ transforms as a tensor, the metric still may raise or lower indices. It implies that the metric tensor must be invariant under the covariant derivative

$$\nabla_\sigma g_{\mu\nu} = 0, \quad \nabla_\sigma g^{\mu\nu} = 0. \tag{1.25}$$

For a general tensor (with only Einstein indices) $T^{\mu\ldots\nu}{}_{\kappa\ldots\rho}$ the generalization is straightforward

$$\nabla_\sigma T^{\mu\ldots\nu}{}_{\kappa\ldots\rho} = \partial_\sigma T^{\mu\ldots\nu}{}_{\kappa\ldots\rho} + \Gamma^{\mu}{}_{\sigma\lambda} T^{\lambda\ldots\nu}{}_{\kappa\ldots\rho} + \cdots + \Gamma^{\nu}{}_{\sigma\lambda} T^{\mu\ldots\lambda}{}_{\kappa\ldots\rho}$$

$$- \Gamma^{\lambda}{}_{\sigma\kappa} T^{\mu\ldots\nu}{}_{\lambda\ldots\rho} - \cdots - \Gamma^{\lambda}{}_{\sigma\rho} T^{\mu\ldots\nu}{}_{\kappa\ldots\lambda}. \tag{1.26}$$

But how do tensors, with Lorentz flat $a, b, \ldots$ and Einstein curved $\mu, \nu, \ldots$ indices, transform? To find out this, we need to know how to map tensors of flat spacetime, whose metric tensor is denoted by η_{ab}, to tensors of curved spacetime of same dimension, whose metric tensor is denoted by $g_{\mu\nu}$. To be able to do it, we introduce the *Cartan formalism* and define orthogonal vectors $e^{a}{}_{\mu}$, which map flat spacetime on curved spacetime ([14], [17])

$$g_{\mu\nu} = e^{a}{}_{\mu} e^{b}{}_{\nu} \eta_{ab}. \tag{1.27}$$

These vectors, in general are called *vielbein.* For $n = 3$, these vectors are called *dreibein* (or *triad*), for $n = 4$ *vierbein* (or *tetrad*) etc. We denote the inverse triad/vielbein/... as $E_a{}^{\mu}$ and they map the curved spacetime described by $g_{\mu\nu}$ on the Minkowski spacetime η_{ab}

$$\eta_{ab} = E_a{}^{\mu} E_b{}^{\nu} g_{\mu\nu}. \tag{1.28}$$

At the same time

$$T^{a\ldots b}{}_{c\ldots d} = e^{a}{}_{\mu} \ldots e^{b}{}_{\nu} E_c{}^{\lambda} \ldots E_d{}^{\kappa} T^{\mu\ldots\nu}{}_{\lambda\ldots\kappa} \tag{1.29}$$

as well as

$$e^{a}{}_{\mu} E_a{}^{\nu} = \delta^{\nu}_{\mu}, \quad e^{b}{}_{\mu} E_a{}^{\mu} = \delta^{b}_{a}. \tag{1.30}$$

[5]The Riemann tensor enjoys several symmetries, see [16]. The tensor has 20 independent components in 4 dimensions, 6 components in 3 dimensions and 1 component in 2 dimensions.

The orthogonal vectors can be understood as the Jacobi matrix, which corresponds to the transformation of inertial local coordinates, corresponding to local flat frame, to global spacetime coordinates.

Now, we get back to the question how tensors with mixed indices transform. It is clear that the tensor must transform as a flat spacetime tensor with respect to its Lorentz indices and as the curved spacetime tensor with respect to its Einstein indices. The idea now is to introduce a new connection, that plays the role of the Levi-Civita connection (introduced to covariantize the derivative with respect to Einstein indices), in the context of the locally Lorentz frame, with Lorentz indices. Such connection is called *Spin connection*.

We define the covariant derivative D_μ of ${V^a}_b$ simiraly to the covariant derivative ∇_μ (see (1.18))

$$D_\mu {V^a}_b = \partial_\mu {V^a}_b + {\omega_\mu}^a{}_c {V^c}_b - {\omega_\mu}^c{}_b {V^a}_c, \tag{1.31}$$

where a term including the Christoffel symbol is missing because the tensor ${V^a}_b$ does not contein Einstein indices.

From the full metricity condition (analogous to (1.25))

$$D_\mu {e^a}_\mu = 0, \quad D_\mu {E_a}^\nu = 0, \tag{1.32}$$

it follows (e.g. [3],[14] or [17])

$${\omega_\mu}^a{}_b = {e^a}_\nu {E_b}^\lambda {\Gamma^\nu}_{\mu\lambda} - {E_b}^\lambda \partial_\mu {e^a}_\lambda. \tag{1.33}$$

Transformation of the spinor

Now we know how to transform a tensor with mixed indices. But the spinor has specific transformation features, which are not satisfied by D_μ. Hence, we need to define yet another connection Γ_μ [18]

$$\mathscr{D}_\mu = D_\mu + \Gamma_\mu. \tag{1.34}$$

Since $\mathscr{D}_\mu \psi$ must transform as a spinor, than the Dirac matrix γ^a must be covariantly conserved

$$\mathscr{D}_\alpha \gamma^a = 0. \tag{1.35}$$

To obtain the expression for Γ_μ, one considers the effect of the infinitesimal Lorentz transformation: $\xi^a \to \Lambda^a{}_b \xi^b \simeq \xi^a + \epsilon^a{}_b \xi^b$, where $\epsilon_{ab} = -\epsilon_{ba}$, on ψ [18]

$$\psi \to S(\Lambda)\psi \simeq \psi + \frac{1}{2}\epsilon_{ab}\Omega^{ab}\psi, \tag{1.36}$$

where $\Omega^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ are the generators of the transformation.

Since the new connection needs to make covariant the derivative with respect to local Lorentz transformations, the expression we obtain for Γ_μ is dictated by (1.36) (see also [14])

$$\Gamma_\mu = \frac{1}{2}\omega_{\mu ab}\Omega^{ab}. \tag{1.37}$$

1.3 Action in curved spacetime

We already know how to transform any spinor in curved spacetime, hence we introduce the Dirac massless action for curved spacetimes [3]

$$A = i\hbar v_F \int d^3x \sqrt{|\det g_{\mu\nu}|}\bar{\psi}\gamma^a E_a{}^\mu \left(\partial_\mu + \frac{1}{2}\omega_{\mu bc}\Omega^{bc}\right)\psi, \tag{1.38}$$

where we replace ∂_a by D_μ. Because the spinor has no Lorentz or Einstein indices, we can expect only one connection: Γ_μ. Of course, $\gamma^a E_a{}^\mu$ are gamma-matrices expressed in curved spacetime.

Variation of (1.38) with respect to $\bar{\psi}$ gives the Dirac equation in curved spacetime

$$i\hbar v_F \gamma^a E_a{}^\mu \left(\partial_{\ \mu}\psi + \frac{1}{2}\omega_{\mu bc}\Omega^{bc}\psi\right) = 0. \tag{1.39}$$

It is now time to show that the Dirac massless action enjoys the Weyl symmetry. We will indicate the proof specially for a conformally flat spacetime of dimension $n = 3$, because this is relevant for us. The spacetime is conformally flat if

$$g_{\mu\nu} = e^{2\Sigma}\eta_{\mu\nu} \tag{1.40}$$

and so

$$e^a{}_\mu = e^\Sigma \delta^a{}_\mu, \ E_a{}^\mu = e^{-\Sigma}\delta_a{}^\mu, \ \sqrt{|\det g_{\mu\nu}|} = e^{n\Sigma}. \tag{1.41}$$

It implies

$$\omega_{\mu bc} = \delta^a{}_\mu \left(\eta_{ac}\delta_b{}^\nu - \eta_{ab}\delta_c{}^\mu\right)\partial_\nu\Sigma \tag{1.42}$$

and it is straightforward to obtain

$$\gamma^a\Omega_{ab} = \gamma_b, \tag{1.43}$$

where the right side is a special case of $(n-1)\gamma_b/2$ for $n = 3$. With these in mind, it is a routine business to derive the action

$$\mathscr{A} = i\hbar v_F \int d^3x e^{(n-1)\Sigma}\bar{\psi}\gamma^a \left(\partial_a + \frac{n-1}{2}\partial_a\Sigma\right)\psi. \tag{1.44}$$

For $\psi \to e^{-\frac{n-1}{2}\Sigma}\psi$ the action reduces to flat form, which is the end of the proof [3].

2. Weyl symmetry of graphene

In previous chapter, we discussed the Weyl symmetry of the Dirac massless field, although only in the context of conformal flatness. In this chapter we will discuss how the general features discussed earlier apply to the case of graphene. In particular, we will see how the request of conformal flatness for the 2+1-dimensional metric becomes a request on the type of surface the graphene membrane needs to reproduce.

2.1 Flat-time Ansatz

Any two-dimensional surface, embedded in R^3, can be described through the relation $z = z(x, y)$. When we add time, we can distinguish two reference frames: the intrinsic three-dimensional frame $\alpha^\mu = (t, x, y)$ with curved indices $\mu = 0, 1, 2$ and metric tensor $g^{(3)}_{\mu\nu}$, next the four-dimensional extrinsic frame $\beta^a = (t, x, y, z(x, y))$ with flat indices $a = 0, 1, 2, 3$ and metric tensor $\eta^{(4)}_{ab}$. The map from four-dimensional flat spacetime to three-dimensional curved spacetime is

$$g^{(3)}_{\mu\nu} = \eta^{(4)}_{ab} \frac{\partial \beta^a}{\partial \alpha^\mu} \frac{\partial \beta^b}{\partial \alpha^\nu}. \tag{2.1}$$

Notice that the Minkowski four-dimensional metric tensor $\eta^{(4)}_{ab}$ needs to be there for a correct description of the three-dimensional (pseudo-)relativistic spacetime experienced by the conductivity electrons of the graphene membrane [3].

In these coordinates, the metric tensor (of curved spacetime with flat time) must be

$$g^{(3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -g^{(2)}_{\alpha\beta} \end{pmatrix}, \tag{2.2}$$

where $g^{(2)}_{\alpha\beta}$ denotes the spatial part of the metric tensor of the membrane.

A consequence of this Ansatz is that the Ricci curvature for both metric tensors, $g^{(3)}_{\mu\nu}$ and $g^{(2)}_{\alpha\beta}$, are same

$$R^{(3)} = R^{(2)} \equiv R. \tag{2.3}$$

That means the curvature of spacetime is included only in its spatial part. From that follows that the corresponding Ricci tensor for $g^{(3)}_{\mu\nu}$ is

$$R^{(3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -R^{(2)}_{\alpha\beta} \end{pmatrix}, \tag{2.4}$$

where $R^{(2)}_{\alpha\beta}$ is the Ricci tensor coming from $g^{(2)}_{\alpha\beta}$. We wish $g^{(3)}_{\mu\nu}$ to be *conformally flat* $g^{(3)}_{\mu\nu} = \Phi^2 \eta^{(3)}_{\mu\nu}$. For this reason, we dedicate next section to a brief introduction to the *Cotton tensor* and *Cotton-York tensor*, whose vanishing is necessary and sufficient condition for conformal flatness in three-dimensions.

2.2 *Intermezzo*: Conformal flatness

This section introduces two topics that will meet together later. The first one deals with the *Weyl* tensor, the *Cotton tensor* and *Cotton-York tensor*. The other is focused on the *Liouville* equation, which we have already briefly introduced.

Weyl tensor, Cotton tensor and Cotton-York tensor

The Weyl tensor is a trace-free part of the Riemann tensor [19]

$$C_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} - \frac{1}{n-2}\left(R_{\mu\kappa}g_{\nu\lambda} + R_{\nu\lambda}g_{\mu\kappa} - R_{\mu\lambda}g_{\nu\kappa} - R_{\nu\kappa}g_{\mu\lambda}\right) + \\ + \frac{1}{(n-1)(n-2)}R\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\nu\kappa}g_{\mu\lambda}\right) \tag{2.5}$$

The vanishing of the Weyl tensor is necessary and sufficient condition for the manifold of dimension $n \geq 4$ to be conformally flat [19]. For $n = 3$ the Weyl tensor vanishes identically, hence it cannot be used to check conformal flatness. Fortunately, there is the conformally invariant tensor, the Cotton tensor, defined as

$$C_{\mu\nu\lambda} = \nabla_\mu R_{\nu\lambda} - \nabla_\nu R_{\mu\lambda} - \frac{1}{4}\left(\nabla_\mu R g_{\nu\lambda} - \nabla_\nu R g_{\mu\lambda}\right). \tag{2.6}$$

The manifold of $n = 3$ is locally conformally flat if and only if $C_{\mu\nu\lambda} = 0$.

On the other hand, one can also define the Cotton-York tensor $C_{\mu\nu}$ [20]

$$C_{\mu\nu} \equiv \frac{1}{2} \overset{(3)}{g}_{\mu\lambda} \epsilon^{\lambda\kappa\sigma} C_{\kappa\sigma\nu}, \tag{2.7}$$

which vanishes, if and only if the manifold is conformally flat.

Liouville equation

Each two-dimensional Riemann manifold is locally conformally flat. This means that there exists a set of coordinates, called *isothermal coordinates*, denoted by $\tilde{x}, \tilde{y}$, so that the infinitesimal line element can be expressed as

$$dl^2 = \phi^2(\tilde{x},\tilde{y})\left(d\tilde{x}^2 + d\tilde{y}^2\right), \tag{2.8}$$

where $\phi(\tilde{x},\tilde{y})$ is a smooth function called *conformal factor* [21].

If we rewrite the conformal factor as $\phi(\tilde{x},\tilde{y}) = e^{\sigma(\tilde{x},\tilde{y})}$, then the metric can be expressed as

$$\overset{(2)}{g}_{\alpha\beta} = e^{2\sigma(\tilde{x},\tilde{y})} \overset{(2)}{\delta}_{\alpha\beta}, \tag{2.9}$$

where $\alpha, \beta \in \{\tilde{x}, \tilde{y}\}$. The full information about the curvature is now included in the Ricci scalar

$$R = \overset{(2)}{g}{}^{\alpha\beta} \overset{(2)}{R}_{\alpha\beta} = \overset{(2)}{g}{}^{\tilde{x}\tilde{x}} \overset{(2)}{R}_{\tilde{x}\tilde{x}} + \overset{(2)}{g}{}^{\tilde{y}\tilde{y}} \overset{(2)}{R}_{\tilde{y}\tilde{y}}, \tag{2.10}$$

with $\alpha, \beta \in \{\tilde{x}, \tilde{y}\}$, and we used the Einstein convention for the definition of R, but we do not use it for the explicit expression in terms of components (last expression).

From the metric tensor (2.9) it follows that non-zero components of the Riemann tensor of $n = 2$ are (see also [6])

$$R^{\tilde{x}}{}_{\tilde{y}\tilde{x}\tilde{y}} = R^{\tilde{y}}{}_{\tilde{x}\tilde{y}\tilde{x}} = -\Delta\sigma, \tag{2.11}$$

where $\Delta \equiv \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2$ is the flat Laplace operator.

It means that the non-zero components of the Ricci tensor are: $R_{\tilde{x}\tilde{x}} = R^{\tilde{x}}{}_{\tilde{y}\tilde{x}\tilde{y}}$, $R_{\tilde{y}\tilde{y}} = R^{\tilde{y}}{}_{\tilde{x}\tilde{y}\tilde{x}}$ and we used again here the Einstein convention. Then the Ricci scalar is

$$R = -2e^{-2\sigma(\tilde{x},\tilde{y})}\Delta\sigma(\tilde{x},\tilde{y}). \tag{2.12}$$

Assuming the constant R and defining the Gaussian curvature $K \equiv R/2$, the equation takes the form

$$\Delta\sigma(\tilde{x},\tilde{y}) = -Ke^{2\sigma(\tilde{x},\tilde{y})}. \tag{2.13}$$

Regarded as an equation for σ, this is a famous equation of mathematical physics, *Liouville* equation. In the complex plane $z = \tilde{x} + i\tilde{y}$, $\bar{z} = \tilde{x} - i\tilde{y}$ the Laplace operator is $\Delta \equiv \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 = 4\partial_z\partial_{\bar{z}}$. The Liouville equation might be rewritten with $\phi(z(\tilde{x},\tilde{y}), \bar{z}(\tilde{x},\tilde{y}))$ as

$$\Delta \ln\phi(z,\bar{z}) = -K\phi^2(z,\bar{z}). \tag{2.14}$$

J. Liouville presented the general solution of (2.14) in his article [7]. The solution is

$$\phi_{\pm}(z,\bar{z}) = \frac{2}{\sqrt{|K|}}\frac{|f'(z)|}{1 \pm |f(z)|^2}, \tag{2.15}$$

where f is any meromorphic function, which holds: $f' \equiv df/dz \neq 0$ for all z in the domain of function f (except poles) and $f(z)$ has at most simple poles in its domain of function.

2.3 The key role of the Liouville equation

We know that the three-dimensional manifold is conformally flat if and only if the Cotton-York tensor vanishes. We also know that there exists a set of coordinates, called isothermal coordinates, denoted by $\tilde{x}, \tilde{y}$, that the metric $g^{(2)}_{\alpha\beta}$ is diagonal, i.e.

$$g^{(3)}_{\mu\nu}(\tilde{x},\tilde{y}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -e^{2\sigma(\tilde{x},\tilde{y})} & 0 \\ 0 & 0 & -e^{2\sigma(\tilde{x},\tilde{y})} \end{pmatrix} \tag{2.16}$$

and corresponding formula for the Ricci scalar is computed as [3]

$$\Delta\sigma(\tilde{x},\tilde{y}) = -\frac{1}{2}R(\tilde{x},\tilde{y})e^{2\sigma(\tilde{x},\tilde{y})}. \tag{2.17}$$

Now, we want to apply the condition of conformal flatness $C_{\mu\nu} = 0$, which implies

$$-\partial_{\tilde{x}}\Delta\sigma + 2\left(\partial_{\tilde{x}}\sigma\right)\Delta\sigma = 0, \quad -\partial_{\tilde{y}}\Delta\sigma + 2\left(\partial_{\tilde{y}}\sigma\right)\Delta\sigma = 0. \tag{2.18}$$

The Eqs. 2.17) and (2.18) are compatible if and only if the Ricci scalar is constant. Assuming the Gaussian curvature rather then (constant) Ricci curvature, it follows from that

$$\Delta\sigma = -Ke^{2\sigma}, \tag{2.19}$$

which is the Liouville equation (2.13). It means, the spacetime related to the graphene membrane is conformally flat if the surface's conformal factor satisfies the Liouville equation.

If $K = 0$ the Liouville equation reduces to the Laplace equation $\Delta\sigma = 0$ and surfaces are flat but in general not simply planar and their conformal factors are harmonic functions. In Appendix A we discuss conformal symmetry, and we show that in $n = 2$, this can be naturally done in the complex domain, where the conditions for conformal symmetry become the Cauchy-Riemann conditions for holomorphicity. As well known, harmonic functions are the real counterpart of holomorphic functions. This is not coincidence, as Weyl and conformal symmetries are tightly related. We dedicate Appendix A to this topic, and introduce there the algebra that the conformal generators obey in $n = 2$, that is the *Witt* and *Virasoro* algebras.

This implies two conclusions. Firstly there are infinite ways how to bend the membrane by keeping it intrinsically flat, secondly this family of surfaces are closely related to the Witt algebra and the Virasoro algebra.

3. Surfaces of revolution of constant K and related spacetimes

In the previous chapter we have discussed why we are interested in surfaces with constant Gaussian curvature. In particular, we are interested in *surfaces of revolution*, To them, and to the spacetimes obtained by considering them as the spatial part of the metric (2.2), is dedicated this chapter.

Surfaces of revolution can be defined by this parametrization [8], [23]

$$x(u,v) = R(u)\cos v, y(u,v) = R(u)\sin v, z(u) = \pm \int^{u} \sqrt{1 - [R'(\bar{u})]^2}\mathrm{d}\bar{u}, \quad (3.1)$$

where $v \in [0, 2\pi]$ is the longitude coordinate, $u \in [u_{\min}, u_{\max}]$ is the latitude coordinate and $R^{'} \equiv dR/du$. This implies that the infinitesimal line element

$$dl^2 = dx^2 + dy^2 + dz^2 = du^2 + R^2(u)dv^2 \quad (3.2)$$

The Gaussian curvature satisfies a simple relation [8], [24]

$$K = -\frac{R^{''}(u)}{R(u)} \quad (3.3)$$

At this moment, we must distinguish between constant $K > 0$ and $K < 0$. We will see in next section that up to a redefinition, the sphere is the only surface of constant $K > 0$. In other words, surfaces with singularities, which are a kind of deformations of the sphere, can arise but these singularities are always removable by a suitable redefinition of coordinates and we obtain the (full, undeformed) sphere. Nonetheless, as we will see extensively in the last chapter, these deformations of the sphere are physically distinguishable from the sphere itself, and play a mayor role in the identification of the surfaces corresponding to the Horváthy-Yéra vortices.

On the other hand, the number of surfaces with $K < 0$, called *Lobachevsky* (hyperbolic) surfaces, are infinite and the singularities (like edges or cusps) for $K < 0$ always arise. In comparison with $K > 0$, they are the effect of the fact that only parts of the Lobachevsky plane can be immersed into the Euclidean space as a result of famous *Hilbert's theorem* (see e.g. [9]), which says: "There exists no analytical complete surface of constant negative Gaussian curvature in Euclidean space."

3.1 Positive constant Gaussian curvature

For constant $K = 1/r^2 > 0$, where r is a radius of the surface, by solving the ordinary differential equation (3.3) we obtain

$$R(u) = c\cos\left(\frac{u}{r} + b\right), \quad (3.4)$$

where b, c are integration constants. If we assume: $b = 0$ (this redefines the zero of u), then x, y-coordinates are

$$x = c\cos\frac{u}{r}\cos v,\;\; y = c\cos\frac{u}{r}\sin v \tag{3.5}$$

and the general formula for the z-coordinate is

$$z(u) = \int^{u}\sqrt{1-\left[\bar{R}'(\bar{u})\right]^{2}}\mathrm{d}\bar{u} \tag{3.6}$$

The formula for the z-coordinate depends on a relation between c and r, hence we must distinguish three cases

1. $c = r$

$$z(u) = \int^{u}\sqrt{1-\left[\frac{c^2}{r^2}\sin^2\frac{\bar{u}}{r}\right]}d\bar{u} = r\sin\frac{u}{r} \tag{3.7}$$

The surface is a sphere of a radius r. Ranges are: $v \in [0, 2\pi]$, $u/r \in [-\pi/2, \pi/2]$. The plot is in Fig. 3.1.

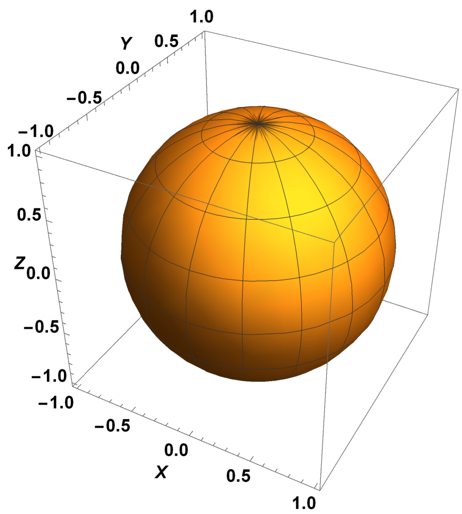


Figure 3.1: $c = 1$, $r = 1$, $v \in [0, 2\pi]$, $u \in [-\pi, \pi]$

2. $c > r$

The z-coordinate can be expressed by the complete elliptic integral of the second kind as:

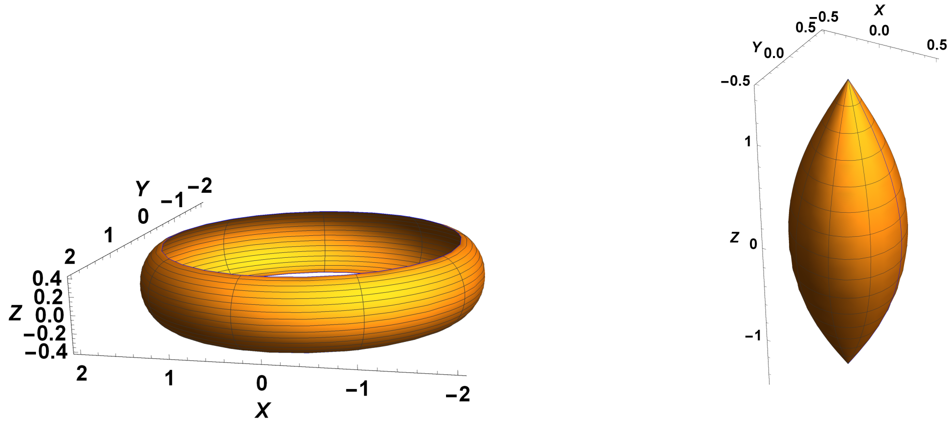
$$z(u) = rE\left[\frac{u}{r}, \frac{c^2}{r^2}\right]. \tag{3.8}$$

The ranges are smaller than in previous case:

$v \in [0, 2\pi]$, $u/r \in [-\arcsin(r/c), \arcsin(r/c)]$. The plot is in 3.2 (a). We call these "Surfaces of the Bulge type".

3. $c < r$

For the z-coordinate we have same formula as previously, see (3.8). The ranges are: $v \in [0, 2\pi]$, $u/r \in [-\pi/2, \pi/2]$. See 3.2 (b). We call these "Surfaces of the Spindle type".



(a) $c = 2, r = 1$ (b) $c = 1/2,\ r = 1$

Figure 3.2: In (a) the case $c > r$, giving surfaces of the Bulge type. In (b) the case $c < r$, giving surfaces of the Spindle type.

The singularities, edges in case 2. and cusps in case 3., can be removed by a simple redefinition

$$v \longmapsto \bar{v} = \frac{c}{r} v, \tag{3.9}$$

which leads to a formal adjustment of (3.2)

$$\mathrm{d}l^2 = \mathrm{d}u^2 + R^2(u)\mathrm{d}v^2 = \mathrm{d}u^2 + \left(\frac{r}{c}R(u)\right)^2 \mathrm{d}\bar{v}^2 = \mathrm{d}u^2 + \bar{R}^2(u)\mathrm{d}\bar{v}^2. \tag{3.10}$$

From (4.12) we obtain the z-coordinate as

$$z(u) = \int^u \sqrt{1 - \left[\bar{R}'(\bar{u})\right]^2} \mathrm{d}\bar{u} = r \sin \frac{u}{r}, \tag{3.11}$$

which obviously gives a sphere.

3.2 Negative constant Gaussian curvature

For constant $K = -1/r^2 < 0$, by solving the ordinary differential equation (3.3) we obtain

$$R(u) = c_1 \sinh \frac{u}{r} + c_2 \cosh \frac{u}{r}, \tag{3.12}$$

where c_1, c_2 are integration constants. We distinguish these cases: *Beltrami pseudosphere* ($c_1 = c_2 = c$), *Hyperbolic pseudosphere* ($c_1 = 0$, $c_2 = c$), *Elliptic pseudosphere* ($c_1 = c$, $c_2 = 0$), where c is any positive real constant.

Beltrami pseudosphere

From conditions $c_1 = c_2 = c > 0$ in (3.12) we obtain

$$R(u) = c \exp \frac{u}{r}, \tag{3.13}$$

which implies

$$z(u) = \int^u \sqrt{1 - \frac{c^2}{r^2} \exp \frac{2\bar{u}}{r}} d\bar{u}. \tag{3.14}$$

Then the z-coordinate is

$$z(u) = r\sqrt{1 - \frac{c^2}{r^2} \exp \frac{2u}{r}} - r \operatorname{arctanh}\left(\sqrt{1 - \frac{c^2}{r^2} \exp \frac{2u}{r}}\right). \tag{3.15}$$

Ranges are: $R(u) \in [0, r]$, $u \in [-\infty, r\ln(r/c)]$. We plot this surface in Fig. 3.3. Therefore the surface is infinite with boundary at $R(u_{\max}) = r$, where $u_{\max} = r\ln(r/c)$. For $R > r$ the surface becomes imaginary.

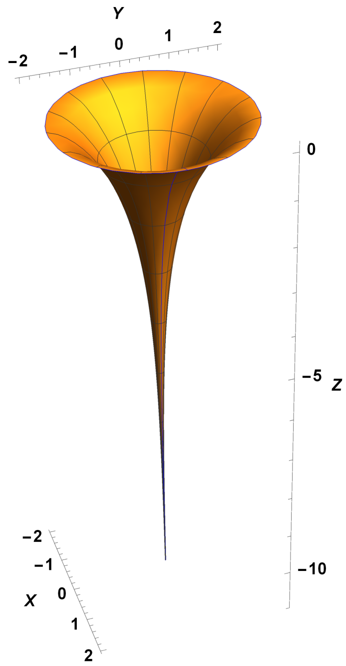


Figure 3.3: Beltrami: Plot: $c = 1$, $r = 2$, $v \in [0, 2\pi]$, $u \in [-5, 2\log 2]$

Elliptical pseudosphere

Conditions $c_1 = c$, $c_2 = 0$ in (3.12) gives us

$$R(u) = c \sinh \frac{u}{r}, \tag{3.16}$$

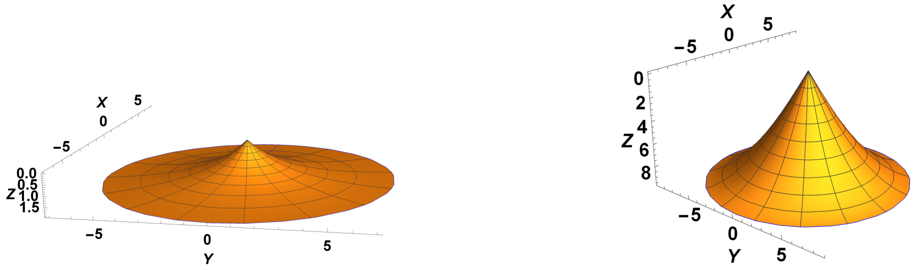
which implies

$$z(u) = \int^u \sqrt{1 - \left[\frac{c^2}{r^2} \cosh^2 \frac{\bar{u}}{r}\right]} d\bar{u}. \tag{3.17}$$

Then the z-coordinate becomes

$$z(u) = \sqrt{c^2 - r^2} E\left(\frac{iu}{r} | \frac{c^2}{c^2 - r^2}\right), \tag{3.18}$$

where $E(\cdot, \cdot)$ denotes the elliptic integral of the second kind.



(a) (b)

Figure 3.4: Elliptic pseudosphere:
(a) $c = 1$, $r = 2$, $v \in [0, 2\pi]$, $u \in [0, 2\,\mathrm{arccosh}\,2]$;
(b) $c = 5$, $r = 10$, $v \in [0, 2\pi]$, $u \in [0, 10\,\mathrm{arccosh}\,2]$

Because of $c \leq r$, we define β as

$$c \equiv r \sin\beta. \tag{3.19}$$

Ranges are: $R(u) \in [0, r\cos\beta]$, $u \in [0, \mathrm{arcsinh}\cot\beta]$, where β is the angle between the axis of revolution and tangents to the meridians at $R = 0$ [23].

The singular boundaries are a cusp at $R = 0$ ($u = 0$) and a edge at the maximal circle of a radius $R = r\cos\beta$ ($u = u_{\max}$). The plots are in Fig. 3.4.

Hyperbolic pseudosphere

Finally, conditions $c_1 = 0$, $c_2 = c$ in (3.12) lead to

$$R(u) = c\cosh\frac{u}{r}, \tag{3.20}$$

which implies

$$z(u) = \int^u \sqrt{1 - \left[\frac{c^2}{r^2}\sinh^2\frac{\bar{u}}{r}\right]}\, d\bar{u}.$$

Then the z-coordinate becomes

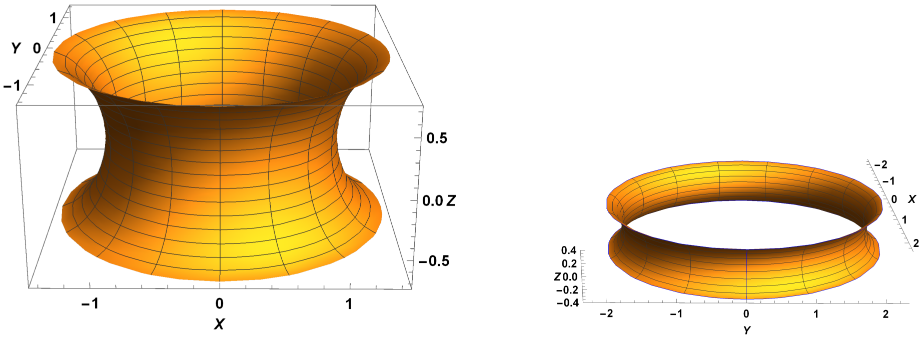
$$z(u) = -irE\left(\frac{iu}{r}\Big| - \frac{c^2}{r^2}\right),$$

where $E(\cdot,\cdot)$ denotes a elliptic integral of the second kind.
Ranges are: $u/r \in [-\,\mathrm{arccosh}\left(\sqrt{1 + (r/c)^2}\right), \mathrm{arccosh}\left(\sqrt{1 + (r/c)^2}\right)]$ and $R(u) \in [c, \sqrt{c^2 + r^2}]$.

Therefore the surface is finite with boundaries at $R(u_{\min})$ and at $R(u_{\max})$ that are two circles. We plot this surface in Figs. 3.5 and 3.6.

3.3 Spacetimes associated to spheres and pseudospheres

Next question is which spacetimes correspond (are conformal) to spacetimes obtained as a product of flat-time and each of the surfaces we discussed above.



(a) (b)

Figure 3.5: Hyperbolic pseudosphere:
(a) $c = 1$, $r = 1$, $v \in [0, 2\pi]$, $u \in [-\operatorname{arccosh}\sqrt{2}, \operatorname{arccosh}\sqrt{2}]$;
(b) $c = 2$, $r = 1$, $v \in [0, 2\pi]$, $u \in [-\operatorname{arccosh}\sqrt{5/4}, \operatorname{arccosh}\sqrt{5/4}]$

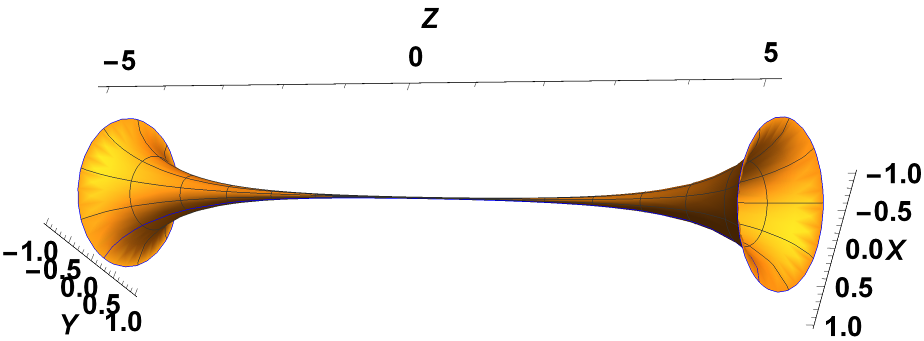


Figure 3.6: Hyperbolic pseudosphere:
$c = 1/100$, $r = 1$, $v \in [0, 2\pi]$, $u \in [-\operatorname{arccosh}\sqrt{10001}, \operatorname{arccosh}\sqrt{10001}]$

We call these spacetimes the *Spherical spacetime*, *Beltrami spacetime*, *Elliptic spacetime* and *Hyperbolic spacetime*.

The Spherical spacetime corresponds to a Einstein static universe. Because this space(time) does not have any singularities, it does not have any horizon and therefore it is not possible to observe phenomena related to the existence of an horizon, like the Hawking effect [4]. From this point of view, it is more interesting to discuss Lobachevsky surfaces. In [4] these results of theoretical studies are summarized:

1. The Beltrami spacetime is conformal[1] to the *Rindler spacetime*,
2. The Elliptic spacetime is conformal to the *de Sitter spacetime*,
3. The Hyperbolic spacetime is conformal to the *BTZ black hole spacetime*.

In next paragraphs, we would like to sketch a proof of each relation 1.-3. A more detailed description of each spacetime (Rindler, de Sitter, BTZ) is in the Appendix B.

Before we begin, we will discuss spacetime horizons related to presented surfaces of revolution of constant $K < 0$. Because a spacetime horizon is a consequence of edge(s) of related Lobachevsky surface, we called it the *Hilbert horizon*

[1] We remind, that more precise term is "Weyl related", but we try to use the same terminology as in [4].

(the definition comes from [4]). On the other hand, the conformally related spacetimes (Rindler etc.) have *event horizons*. The issue is how the corresponding Hilbert and event horizons (e.g. for Beltami and Rindler) are related . They are generally different, but we will see that coincide in a limit.

Beltrami spacetime: conformal to Rindler

We start with the spacetime interval corresponding to the Beltrami spacetime

$$ds_B^2 = dt^2 - du^2 - c^2 e^{2u/r} dv^2 = \frac{c^2 e^{2u/r}}{r^2} \left[\frac{r^2}{c^2} e^{-2u/r} \left(dt^2 - du^2 \right) - r^2 dv^2 \right] \equiv \phi^2 ds_R^2, \tag{3.21}$$

where $\phi \equiv \phi(u) \equiv c/re^{u/r}$ and $ds_R^2 \equiv \frac{r^2}{c^2} e^{-2u/r} \left(dt^2 - du^2 \right) - r^2 dv^2$. If we identify

$$\eta \equiv tr/c \in [-\infty, +\infty], \quad \zeta = -ur/c \in [-(r^2/c) \ln(r/c), +\infty], \tag{3.22}$$

we can easily show that the spacetime interval ds_R is of Rindler type

$$ds_R^2 = e^{2a\zeta} \left(d\eta^2 - d\zeta^2 \right) - r^2 dv^2, \tag{3.23}$$

where $a = c/r^2 > 0$. This corresponds to the right wedge of Minkowski spacetime (i.e. Rindler spacetime). For the right wedge, the Rindler observer reaches the event horizon for $\zeta_{Eh} = -\infty$. If we define $\zeta_{Hh} := -(r^2/c) \ln(r/c)$, then the Hilbert horizon of the Beltrami spacetime and the event horizon of the Rindler spacetime coincide $\zeta_{Hh} \to \zeta_{Eh}$ for $c/r \to 0$ [4].

Elliptic spacetime: conformal to de Sitter

Let us start with the metric of the de Sitter spacetime (dS) [4], [15], [31]

$$ds_{dS_3}^2 = \left(1 - \frac{\mathscr{R}^2}{r^2} \right) dt^2 - \left(1 - \frac{\mathscr{R}^2}{r^2} \right)^{-1} d\mathscr{R}^2 - \mathscr{R}^2 dv^2, \tag{3.24}$$

where $\mathscr{R}$ is the radial coordinate, v is the angular variable, t is time and r is related to the cosmological constant $\Lambda \equiv 1/r^2 > 0$. The event ("cosmological") horizon is $\mathscr{R}_{Eh} = r$.

The counterpart of dS is the *Anti de Sitter spacetime (AdS)*, whose spacetime interval can be found by $r \to ir$, which implies $\Lambda \to -\Lambda$. Thus, the metric of AdS is

$$ds_{AdS_3}^2 = \left(1 + \frac{\mathscr{R}^2}{r^2} \right) dt^2 - \left(1 + \frac{\mathscr{R}^2}{r^2} \right)^{-1} d\mathscr{R}^2 - \mathscr{R}^2 dv^2 \tag{3.25}$$

Obviously the AdS has no event horizon (the spacetime is closed). It makes now more sense to work with dS rather than AdS. But from mathematical point of view, the AdS is Weyl-equivalent to an Einstein Static Universe [4]. By defining

$$\frac{1}{\mathscr{R}^2} \equiv \frac{1}{R^2} - \frac{1}{r^2} = \frac{1}{r^2 \cos^2(u/r)} - \frac{1}{r^2} \tag{3.26}$$

and shifting the angular variable $u \to u + r\pi/2$, we obtain

$$ds_{AdS_3}^2 = \frac{1}{\cos^2(u/r)} \left[dt^2 - du^2 - r^2 \sin^2(u/r) dv^2 \right], \tag{3.27}$$

The line element in squared brackets is the line element of the Spherical spacetime. In our correspondence, it can be obtained by making a giant fullerene. Now we return to the de Sitter spacetime via $r \to ir$, i.e. $-\Lambda \to \Lambda$

$$ds^2_{dS_3} = \frac{1}{\cosh^2(u/r)} \left[dt^2 - du^2 - (r^2 \sin^2 \beta) \sinh^2(u/r) dv^2\right], \tag{3.28}$$

where we consider the orientation of the Elliptic pseudosphere by β, see (3.16) and (3.19). The squared brackets in (3.28) is obviously spacetime interval of the Elliptical spacetime, i.e. the Elliptical spacetime is Weyl related to AdS.

We will not show calculations done in [4], but after more adjustments, using $R(u) = c \sinh(u/r)$ with $c = r \sin \beta$, one can obtain a result (for detailed calculations see [4])

$$ds^2_{Ell} = \left(1 - \frac{\mathscr{R}^2}{r^2 \sin^2 \beta}\right)^{-1} ds^2_{S_3}. \tag{3.29}$$

The edge for a hyperbolic pseudosphere appears for $R_{Hh} \equiv R_{\max} = r \cos \beta$. It can be quite easily shown (based on calculations done in [4]) that this corresponds to $\mathscr{R}_{Hh} = \frac{1}{2} r \sin(2\beta)$.

Because we add more $\sin \beta$ to calculations, the event horizon for the Elliptical spacetime appears for $\mathscr{R}_{Eh} = r \sin \beta$ (the right side is not yet r). Of course $\beta = \arcsin(c/r)$, hence if c/r is very small, the horizons coincide.

Hyperbolic spacetime: conformal to BTZ

The spacetime interval of the BTZ black hole with zero angular moment is [29],[30]

$$ds^2_{BTZ} = \left(\frac{\mathscr{R}^2}{c^2} - M\right) dt^2 - \left(\frac{\mathscr{R}^2}{c^2} - M\right)^{-1} d\mathscr{R}^2 - \mathscr{R}^2 d\theta^2. \tag{3.30}$$

where $-\infty < t < \infty$, $0 < \mathscr{R} < \infty$, $0 \leq \theta \leq 2\pi$, M is a mass, $\mathscr{R}$ a radial function and we assume an Ansatz [3] that $c \equiv l \equiv 1/\sqrt{\Lambda}$, where Λ is the cosmological constant, see Appendix B.3.

We rewrite the spacetime interval as

$$ds^2_{BTZ} = \left(\frac{\mathscr{R}^2}{c^2} - M\right) ds^2, \tag{3.31}$$

where

$$ds^2 = dt^2 - \frac{c^4}{(\mathscr{R}^2 - \mathscr{R}^2_{Eh})^2} d\mathscr{R}^2 - \frac{c^2 \mathscr{R}^2}{\mathscr{R}^2 - \mathscr{R}^2_{Eh}} d\theta^2. \tag{3.32}$$

Here we recognise the event horizon of the BTZ black hole as $\mathscr{R}_{Eh} = c\sqrt{M}$. Now our aim is to show $ds^2 = ds^2_{Hyp}$. If we identify $\theta \equiv v$ as well as

$$du \equiv -\frac{c^2}{\mathscr{R}^2 - \mathscr{R}^2_{Eh}} d\mathscr{R},\ R(\mathscr{R}) \equiv \frac{c\mathscr{R}}{\mathscr{R}^2 - \mathscr{R}^2_{Eh}}, \tag{3.33}$$

then we obtain

$$\mathscr{R}(u) = \mathscr{R}_{Eh} \coth \frac{\mathscr{R}_{Eh} u}{c^2}, \tag{3.34}$$

which implies

$$R(\mathscr{R}(u)) \equiv R(u) = c \cosh \frac{\mathscr{R}_{Eh} u}{c^2}, \tag{3.35}$$

where $r \equiv c^2/\mathscr{R}_{Eh} = c^2/\sqrt{M}$. Therefore the metric tensor of the BTZ black hole is Weyl related to the Hyperbolic spacetime

$$ds^2_{BTZ} = \left(\frac{\mathscr{R}^2}{c^2} - M \right) ds^2_{Hyp}. \tag{3.36}$$

It remains to compare the Hilbert horizon with the event horizon. The edge for the hyperbolic pseudosphere appears for $R_{Hh} \equiv R_{\max} = \sqrt{r^2 + c^2}$, so $u_{Hh} \equiv u_{\max} = r \operatorname{arccosh} \sqrt{1 + r^2/c^2}$ and $\mathscr{R}(u_{Hh}) = \mathscr{R}_{Eh} \coth\left(\operatorname{arccosh} \sqrt{1 + r^2/c^2}\right)$.

Assuming specially $r = 10^n c$, where n is a natural number [4], we obtain

$$\mathscr{R}_{Hh} = \mathscr{R}_{Eh} \times \frac{10^n}{(10^{2n} - 1)^{1/2}}. \tag{3.37}$$

For $n \to \infty$ (i.e. $c/r \to 0$, $M \to 0$) we get $\mathscr{R}_{Hh} \to \mathscr{R}_{Eh}$. In other words, the event horizon of the BTZ black hole coincide with the Hilbert horizon of the Hyperbolic spacetime if the mass of the black hole vanishes.

4. Surfaces corresponding to the Horváthy-Yéra solutions and related spacetimes

Up to now we have provided an overview about the relativistic-like behaviour of graphene and its possible usability in experimental studying of various (conformally flat) spacetimes or phenomena taking place in them (like the Hawking effect on a event horizon of a black hole).

In this chapter we focus on our study of the non-topological vortex solutions presented by P. A. Horváthy and J.-C. Yéra in [1]. In the beginning of this chapter, we will write down the coordinate transformations from $\tilde{r}, \tilde{\theta}$ to x, y, z in an effort to find the actual form of the surfaces in point. The coordinate transformation usually leads to a non-linear set of partial differential equations, that might be very difficult to solve. Generally, there exists no universal way how to find the solution, and analytic solutions might as well not exist, then the numerical methods are in need.

We will conduct a quite general discussion about the infinitesimal line element corresponding to the vortex solutions, which could (and will) reveal us more about the solution of the system of equations.

Because the Liouville equation allows both signs as we already know, we will also discuss the case with a negative sign of K, which no longer corresponds to original Horváthy-Yéra vortex solutions, but it might be a interesting question to ask, which surfaces correspond to them, if they exist.

4.1 Preliminary study of the vortex solutions

We recall here the conformal factors of the vortex solutions

$$\phi_+(\tilde{r}) = \frac{2N}{\sqrt{K}} \frac{\tilde{r}^{N-1}}{\tilde{r}^{2N} + 1}, \tag{4.1}$$

where $\tilde{r} = |z| = \sqrt{\tilde{x}^2 + \tilde{y}^2} \in [0, \infty]$, $\tilde{\theta} \in [0, 2\pi]$ and N is a natural number.

Our main goal is to find what surfaces correspond to ϕ_+ for any given N. We want to find the spatial coordinates x, y, z as functions of $\tilde{r}$, $\tilde{\theta}$. For the infinitesimal line element we can write down

$$dl^2 = dx^2 + dy^2 + dz^2 = \phi_+^2(\tilde{x}, \tilde{y})(d\tilde{x}^2 + d\tilde{y}^2) \equiv \phi_+^2(\tilde{r})(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2), \tag{4.2}$$

where ϕ_+ does not depend on $\tilde{\theta}$ due to radial symmetry. With respect to isothermal polar coordinates $\tilde{r}$, $\tilde{\theta}$ the coordinate transformation gives

$$dl^2 = dx(\tilde{r}, \tilde{\theta})^2 + dy(\tilde{r}, \tilde{\theta})^2 + dz(\tilde{r}, \tilde{\theta})^2 = \left[\left(\frac{\partial x}{\partial \tilde{r}} \right)^2 + \left(\frac{\partial y}{\partial \tilde{r}} \right)^2 + \left(\frac{\partial z}{\partial \tilde{r}} \right)^2 \right] d\tilde{r}^2 +$$

$$+ \left[\left(\frac{\partial x}{\partial \tilde{\theta}} \right)^2 + \left(\frac{\partial y}{\partial \tilde{\theta}} \right)^2 + \left(\frac{\partial z}{\partial \tilde{\theta}} \right)^2 \right] d\tilde{\theta}^2 + 2 \left[\frac{\partial x}{\partial \tilde{r}} \frac{\partial x}{\partial \tilde{\theta}} + \frac{\partial y}{\partial \tilde{r}} \frac{\partial y}{\partial d\tilde{\theta}} + \frac{\partial z}{\partial \tilde{r}} \frac{\partial z}{\partial \tilde{\theta}} \right] d\tilde{r} d\tilde{\theta}, \tag{4.3}$$

which leads to a set of equations, what we would like to solve

$$\phi_+^2(\tilde{r}) = \left(\frac{\partial x}{\partial \tilde{r}}\right)^2 + \left(\frac{\partial y}{\partial \tilde{r}}\right)^2 + \left(\frac{\partial z}{\partial \tilde{r}}\right)^2, \tag{4.4}$$

$$\phi_+^2(\tilde{r})\tilde{r}^2 = \left(\frac{\partial x}{\partial \tilde{\theta}}\right)^2 + \left(\frac{\partial y}{\partial \tilde{\theta}}\right)^2 + \left(\frac{\partial z}{\partial \tilde{\theta}}\right)^2, \tag{4.5}$$

$$0 = \frac{\partial x}{\partial \tilde{r}}\frac{\partial x}{\partial \tilde{\theta}} + \frac{\partial y}{\partial \tilde{r}}\frac{\partial y}{\partial \tilde{\theta}} + \frac{\partial z}{\partial \tilde{r}}\frac{\partial z}{\partial \tilde{\theta}}. \tag{4.6}$$

As we announced, the set of (4.4)- (4.6) is highly non-linear and there is not a straightforward procedure to find solutions.

To start, we write the line element including the ϕ_-:

$$\phi_-(\tilde{r}) = \frac{2N}{\sqrt{|K|}}\frac{\tilde{r}^{N-1}}{\tilde{r}^{2N}-1}, \tag{4.7}$$

and we make a few adjustments

$$dl^2 = \phi_\pm^2(\tilde{x},\tilde{y})(d\tilde{x}^2 + d\tilde{y}^2) = \phi_\pm^2(\tilde{r})(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2) = \frac{4N^2}{|K|}\frac{\tilde{r}^{2(N-1)}}{(\tilde{r}^{2N}\pm 1)^2}(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2) =$$

$$= \frac{1}{|K|}\frac{N^2}{\left(\frac{\tilde{r}^N \pm \tilde{r}^{-N}}{2}\right)^2}\left[\left(\frac{d\tilde{r}}{\tilde{r}}\right)^2 + d\tilde{\theta}^2\right] = \left\| \begin{array}{l} \text{subst.}: \\ \tilde{R} \equiv \ln \tilde{r}^N \\ \tilde{\omega} \equiv N\tilde{\theta} \\ \frac{1}{a^2} \equiv |K| \end{array} \right\| = \frac{a^2}{\left(\frac{e^{\tilde{R}} \pm e^{-\tilde{R}}}{2}\right)^2}\left(d\tilde{R}^2 + d\tilde{\omega}^2\right) =$$

$$= \begin{cases} \frac{a^2}{\cosh^2 \tilde{R}}\left(d\tilde{R}^2 + d\tilde{\omega}^2\right) & \text{for } K > 0, \\ \frac{a^2}{\sinh^2 \tilde{R}}\left(d\tilde{R}^2 + d\tilde{\omega}^2\right) & \text{for } K < 0. \end{cases} \tag{4.8}$$

In [4] it was shown, that the first line of (4.8), for $K > 0$, is the one of the sphere. On the other hand, in [9] it is shown that the line element for $K < 0$ corresponds to a pseudosphere.

From a local quantity like dl we cannot say much about global properties. We cannot decide whether the surfaces of $K > 0$ are simple spheres or even bulges or spindles. In the case of $K < 0$ we do not know what type of the pseudospheres we deal with. Because we do not know the range for $\tilde{R}$ or $\tilde{r}$ (compare (3.1) and (3.2)), it can even happen that no actual surface exists. Next we must ask what the physical/geometrical meaning of N is. We also realize that for $N \geq 2$ the range of azimuthal angle $\tilde{\theta}$ is N-multiple of $[0, 2\pi]$, but it is not clear what it means physically (except of $N = 1$). For this purpose, it seems that the parametrization (4.8) is not the most useful. We will face all these questions. For the beginning, let us discuss the cases $K > 0$ and $K < 0$ separately.

4.2 Positive constant Gaussian curvature

Sphere with the N-fold rotation

For $K > 0$, one might define new variables based on isothermal coordinates

$$\tilde{\omega} \equiv v, \;\; \tilde{R} \equiv \ln\left(1 + \frac{2}{\cot(u/2a) - 1}\right), \tag{4.9}$$

then the squared infinitesimal line element is (see [4])

$$dl^2 = \frac{a^2}{\cosh^2 \tilde{R}}\left(d\tilde{R}^2 + d\tilde{\omega}^2\right) = du^2 + a^2\cos^2\frac{u}{a}dv^2. \tag{4.10}$$

This is a line element of a surface of revolution (with constant $K = 1/a^2$) with $R(u) = a\cos(u/a)$, see (3.4). We will check whether the parametrization of sphere really satisfies (4.4)-(4.6).

We made an effort to find x, y, z. For $R(\tilde{r}) \equiv R(u(\tilde{R}(\tilde{r})))$ we can write

$$R(u) = a\cos(u/a) \to R(\tilde{r}) = a\cos\left(2\,\mathrm{arccot}\left(\frac{\tilde{r}^N + 1}{\tilde{r}^N - 1}\right)\right) = \frac{2}{\sqrt{K}}\frac{\tilde{r}^N}{\tilde{r}^{2N} + 1}. \tag{4.11}$$

For any surface of revolution, the x, y-coordinates can be expressed as (3.1)

$$x(u, v) = R(u)\cos v, \quad y(u, v) = R(u)\sin v. \tag{4.12}$$

Thus, after the change of coordinates $(u, v) \to (\tilde{r}, \tilde{\theta})$, x, y-coordinates are

$$x(\tilde{r}, \tilde{\theta}) = \frac{2}{\sqrt{K}}\frac{\tilde{r}^N}{\tilde{r}^{2N} + 1}\cos\left(N\tilde{\theta}\right), \;\; y(\tilde{r}, \tilde{\theta}) = \frac{2}{\sqrt{K}}\frac{\tilde{r}^N}{\tilde{r}^{2N} + 1}\sin\left(N\tilde{\theta}\right). \tag{4.13}$$

We need to be cautious with the z-coordinate. Let us put (4.13) into (4.5) and (4.6), respectively

$$\frac{4N^2}{K}\frac{\tilde{r}^{2N}}{(\tilde{r}^{2N} + 1)^2} = \frac{4}{K}\frac{\tilde{r}^{2N}}{(\tilde{r}^{2N} + 1)^2}N^2\left[\sin^2(N\tilde{\theta}) + \cos^2(N\tilde{\theta})\right] + \left(\frac{\partial z}{\partial \tilde{\theta}}\right)^2, \tag{4.14}$$

$$0 = \frac{4N}{K}\frac{\tilde{r}^N}{\tilde{r}^{2N} + 1}\left(\frac{\tilde{r}^N}{\tilde{r}^{2N} + 1}\right)'\left[-\cos\left(N\tilde{\theta}\right)\sin\left(N\tilde{\theta}\right) + \sin\left(N\tilde{\theta}\right)\cos\left(N\tilde{\theta}\right)\right] + \frac{\partial z}{\partial \tilde{r}}\frac{\partial z}{\partial \tilde{\theta}}. \tag{4.15}$$

Both equations are obviously satisfied, if and only if the z-coordinate depends only on $\tilde{r}$, i.e. $z = z(\tilde{r})$. If we put $x(\tilde{r}, \tilde{\theta})$, $y(\tilde{r}, \tilde{\theta})$ of (4.12) into (4.4), we obtain the expression for the z-coordinate

$$z(\tilde{r}) = \int\sqrt{\phi_+^2(\tilde{r}) - [R(\tilde{r})']^2}d\tilde{r} = \int\sqrt{\phi_+^2(\tilde{r}) - \frac{1}{N^2}[(\tilde{r}\phi_+(\tilde{r}))']^2}d\tilde{r} =$$

$$= \int\sqrt{\frac{4N^2}{K}\frac{\tilde{r}^{2N-2}}{\left(\tilde{r}^{2N} + 1\right)^2} - \left(\frac{2N}{\sqrt{K}}\frac{\tilde{r}^{N-1}}{(\tilde{r}^{2N} + 1)} - \frac{4N}{\sqrt{K}}\frac{\tilde{r}^{3N-1}}{\left(\tilde{r}^{2N} + 1\right)^2}\right)^2}d\tilde{r} =$$

$$= \cdots = -\frac{2}{\sqrt{K}} \frac{1}{1 + \tilde{r}^{2N}} \tag{4.16}$$

where dots represent a calculation done with the help of the Wolfram Mathematica.

The expression $\phi_+^2(\tilde{r}) - [R(\tilde{r})']^2$ is always non-negative for $\tilde{r} \in [0, +\infty]$, hence we can enjoy the utmost range $[0, +\infty]$. If we define new radial and angular coordinates: $\eta \equiv \tilde{r}^N$ and $\tilde{\omega} \equiv N\tilde{\theta}$ respectively, then the spatial coordinates can be written as

$$x(\eta, \tilde{\omega}) = \frac{2}{\sqrt{K}} \frac{\eta}{\eta^2 + 1} \cos\tilde{\omega},\ y(\eta, \tilde{\omega}) = \frac{2}{\sqrt{K}} \frac{\eta}{\eta^2 + 1} \sin\tilde{\omega},\ z(\eta) = -\frac{2}{\sqrt{K}} \frac{1}{1 + \eta^2} \tag{4.17}$$

where $\eta \in [0, +\infty]$ and $\tilde{\omega} \in [0, 2\pi N]$.

We do one more substitution $\frac{\eta}{\eta^2+1} \equiv \frac{1}{2} \sin\tilde{\phi}$, where $\tilde{\phi} \in [0, \pi]$. From that we get $\frac{\eta^2}{(\eta^2+1)^2} = \frac{1}{4} \sin^2\tilde{\phi} = \frac{1}{4} - \frac{1}{4} \cos^2\tilde{\phi}$, i.e. $\cos\tilde{\phi} = -\frac{\eta^2-1}{\eta^2+1}$, so $\frac{1}{2} \cos\tilde{\phi} = -\frac{\eta^2-1}{2(\eta^2+1)} = -\frac{1}{2} + \frac{1}{\eta^2+1}$. Finally, assuming $1/\sqrt{K} = a$, we can rewrite the formulae for spatial coordinates as

$$x = a \sin\tilde{\phi} \cos\tilde{\omega},\ y = a \sin\tilde{\phi} \sin\tilde{\omega},\ z = -a \cos\tilde{\phi} - a, \tag{4.18}$$

where $\tilde{\phi} \in [0, \pi]$ and $\tilde{\omega} \in [0, 2\pi N]$. This is a parametrization of a full sphere obtained for arbitrary N, see Fig. 4.1. From mathematical point of view, the set of equations is really solved. On the other hand, the range of $\tilde{\omega}$ is $[0, 2\pi N]$, but this way the geometrical-physical role of N is too implicit. For instance, if one should construct these surfaces with graphene in a laboratory, they would all look just as spheres.

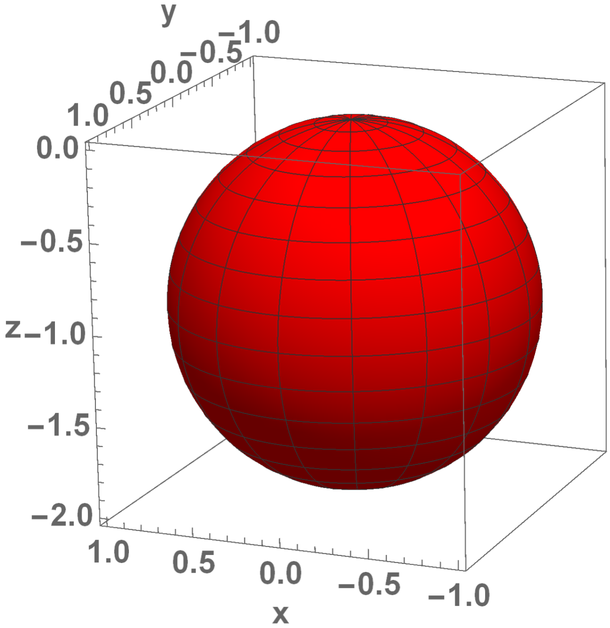


Figure 4.1: Plot of sphere with $K = 1$ (looks same for arbitrary natural N)

Radial symmetric Ansatz

With the above in mind, we will try to solve the system of (4.4)-(4.6) with a different approach, that is, we will use the assumption that the solution has radial symmetry

$$x = F_+(\tilde{r})\cos\tilde{\theta},\; y = F_+(\tilde{r})\sin\tilde{\theta},\; z = z_+(\tilde{r}). \tag{4.19}$$

This Ansatz is almost same as (4.12), but here N does not occur in the argument of sine or cosine. Thus, the range cannot be wider than $[0, 2\pi]$ and it could remedy the problem with the N-fold rotation discussed above. We can immediately see that the Ansatz satisfies (4.6). From (4.5), we get

$$F_+(\tilde{r}) = \tilde{r}\phi_+(\tilde{r}) = \frac{2N}{\sqrt{K}}\frac{\tilde{r}^N}{\tilde{r}^{2N}+1}. \tag{4.20}$$

Obviously $F_+ = NR$ (compare with (4.11)). This can be inserted into (4.4), then we obtain the z-coordinate

$$z_+ = \int \sqrt{\phi_+^2(\tilde{r}) - [F_+'(\tilde{r})]^2}d\tilde{r} = \int \sqrt{\phi_+^2(\tilde{r}) - [(\tilde{r}\phi_+(\tilde{r}))']^2}d\tilde{r} \tag{4.21}$$

and with the help of the Wolfram Mathematica we obtain the z-coordinate as

$$z_+(\tilde{r}) = \frac{\tilde{r}\sqrt{N^2\tilde{r}^{2N-2}\left[\left(\tilde{r}^{2N}+1\right)^2 - N^2\left(\tilde{r}^{2N}-1\right)^2\right]}}{\left(\tilde{r}^{2N}+1\right)N\sqrt{K}}h(\tilde{r}), \tag{4.22}$$

where the function $h(\tilde{r})$ is

$$h(\tilde{r}) \equiv 1 + \frac{g(\tilde{r})f(\tilde{r})}{N^2\left(\tilde{r}^{2N}-1\right)^2 - \left(\tilde{r}^{2N}+1\right)^2} \tag{4.23}$$

and functions $g(\tilde{r})$, $f(\tilde{r})$ are

$$g(\tilde{r}) \equiv i(N-1)\sqrt{\frac{N+1}{1-N}}\left(\tilde{r}^{2N}+1\right)\sqrt{1-\frac{(N-1)\tilde{r}^{2N}}{N+1}}\sqrt{1-\frac{(N+1)\tilde{r}^{2N}}{N-1}}\tilde{r}^{-N}, \tag{4.24}$$

$$f(\tilde{r}) \equiv (N+1)E\left(i\,\mathrm{arcsinh}\left(\sqrt{\frac{N+1}{1-N}}\tilde{r}^N\right)|\frac{(N-1)^2}{(N+1)^2}\right)$$
$$-2F\left(i\,\mathrm{arcsinh}\left(\sqrt{\frac{N+1}{1-N}}\tilde{r}^N\right)|\frac{(N-1)^2}{(N+1)^2}\right), \tag{4.25}$$

where $E(\cdot,\cdot)$, $F(\cdot,\cdot)$ are elliptical integrals.

This parametrization appears quite more complicated than (4.16). We should also avoid inserting $N = 1$ into obtained equations (4.24) and (4.25). Of course, it can be calculated separately and we get, as we already know, the sphere. To emphasize differences between the two parametrizations, let us denote

$$z \equiv \int \sqrt{G(\tilde{r})}d\tilde{r} \tag{4.26}$$

and compare

- the N-fold rotation parametrization:

$$x = \frac{2}{\sqrt{K}} \frac{\tilde{r}^N}{\tilde{r}^{2N}+1} \cos\left(N\tilde{\theta}\right), \qquad y = \frac{2}{\sqrt{K}} \frac{\tilde{r}^N}{\tilde{r}^{2N}+1} \sin\left(N\tilde{\theta}\right),$$

$$G(\tilde{r}) = \frac{4N^2}{K} \frac{\tilde{r}^{2N-2}}{(\tilde{r}^{2N}+1)^2} - \frac{4}{K}\left[\left(\frac{\tilde{r}^N}{\tilde{r}^{2N}+1}\right)'\right]^2,$$

where $\tilde{\theta} \in [0, 2\pi]$ and $\tilde{r} \in [0, \infty]$,

- (at most) 1-fold rotation parametrization:

$$x = \frac{2N}{\sqrt{K}} \frac{\tilde{r}^N}{\tilde{r}^{2N}+1} \cos\tilde{\theta}, \quad y = \frac{2N}{\sqrt{K}} \frac{\tilde{r}^N}{\tilde{r}^{2N}+1} \sin\tilde{\theta},$$

$$G(\tilde{r}) = \frac{4N^2}{K} \frac{\tilde{r}^{2N-2}}{(\tilde{r}^{2N}+1)^2} - \frac{4N^2}{K}\left[\left(\frac{\tilde{r}^N}{\tilde{r}^{2N}+1}\right)'\right]^2,$$

where $\tilde{\theta} \in [0, 2\pi]$ and the range of $\tilde{r}$ (subset of $[0, \infty]$) is still not specified.

In the following paragraphs, we will show how this coordinate redefinition is reflected in surfaces' shapes and ranges of $\tilde{r}$ for various N.

Ansatz for $N = 1$

For $N = 1$, the spatial coordinates are quite simple

$$x = \frac{2}{\sqrt{K}} \frac{\tilde{r}}{\tilde{r}^2+1} \cos\tilde{\theta}, \quad y = \frac{2}{\sqrt{K}} \frac{\tilde{r}}{\tilde{r}^2+1} \sin\tilde{\theta}, \quad z = -\frac{2}{\sqrt{K}} \frac{1}{\tilde{r}^2+1} \tag{4.27}$$

where $\tilde{r} \in [0, +\infty]$, $\tilde{\theta} \in [0, 2\pi]$. As we already know this corresponds to a sphere of radius a, see (4.17) and Fig. 4.1.

Ansatz for $N \geq 2$

The formula for the z-coordinate is complicated when $N \geq 2$. Here we focus on $N = 2, 3$, but we do not show the explicit formula for any N, but is full determined by (4.22), 4.23, (4.24) and (4.25).

We plot surfaces for $N = 2$, $N = 3$, see Figs. 4.2, 4.3, squared conformal factors $\phi^2(\tilde{x}, \tilde{y})$ corresponding to surfaces, see Figs. 4.4. We also plot graph 4.5 for $G(\tilde{r})$ to clearly illustrate that the range of $\tilde{r}$ cannot be $[0, +\infty]$ as for a full sphere, but must be $[\tilde{r}_{\min}, \tilde{r}_{\max}]$, where $0 < \tilde{r}_{\min} < \tilde{r}_{\max} < +\infty$, to get the real z-coordinate.

The surfaces for $N = 2, 3$ look like the surfaces of the Bulge type we already discussed, compare with Fig. 3.2 (a). We now prove that these surfaces we are finding, for any arbitrary N, are really surfaces of the Bulge type. [1].

[1] According to our previous discussion, the other option can only be the Spindle type of surfaces, which clearly we do not see here.

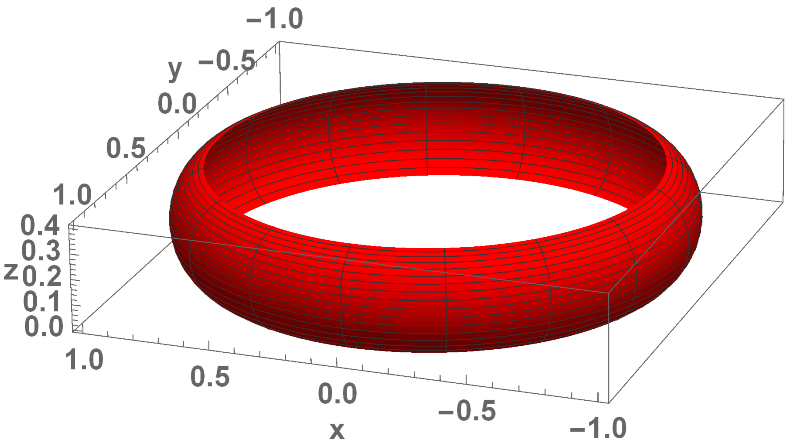


Figure 4.2: Radial symmetric Ansatz of the vortex solution: $N = 2$, $K = 4$, $\tilde{r} \in [1/\sqrt[4]{3}, \sqrt[4]{3}]$, $\tilde{\theta} \in [0, 2\pi]$

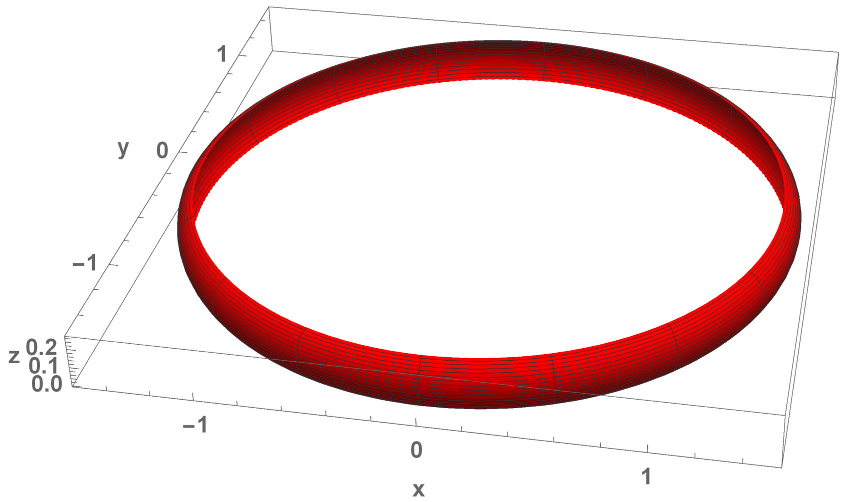
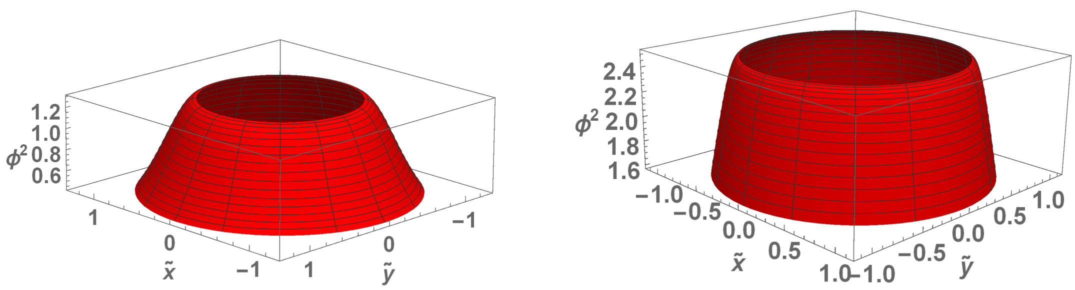


Figure 4.3: Radial symmetric Ansatz of the vortex solution: $N = 3$,$K = 4$, $\tilde{r} \in [1/\sqrt[6]{2}, \sqrt[6]{2}]$, $\tilde{\theta} \in [0, 2\pi]$



(a) (b)

Figure 4.4: Graphs of conformal factors for: $K = 4$, $\tilde{\theta} \in [0, 2\pi]$:
(a) $N = 2$, $\tilde{r} \in [1/\sqrt[4]{3}, \sqrt[4]{3}]$;
(b) $N = 3$, $\tilde{r} \in [1/\sqrt[6]{2}, \sqrt[6]{2}]$

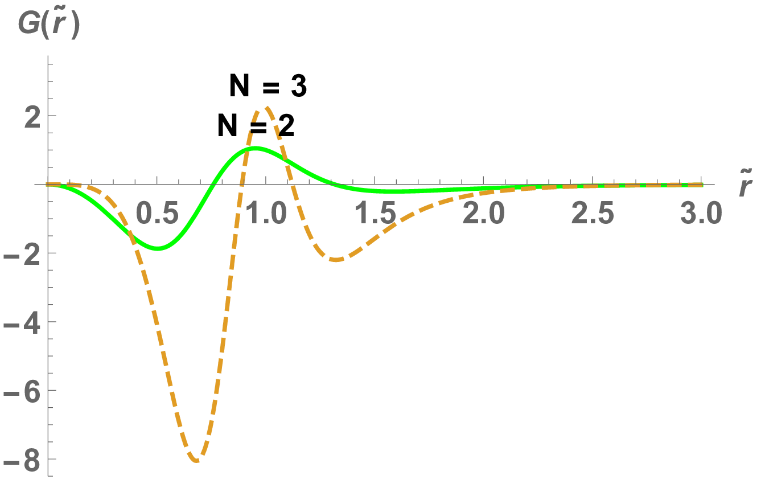


Figure 4.5: Graphs of $G(\tilde{r})$ for $N = 2, 3$

Physical meaning of N

We look at the infinitesimal line element here again, and we would like to understand the physical meaning of N:

$$dl^2 = \frac{4N^2}{K}\frac{\tilde{r}^{2(N-1)}}{(\tilde{r}^{2N}+1)^2}(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2) = \left\| \begin{array}{l} \text{subst. :} \\ \tilde{R} \equiv \ln \tilde{r}^N \end{array} \right\| = \frac{4}{K}\frac{e^{2\tilde{R}}}{(e^{2\tilde{R}}+1)^2}d\tilde{R}^2 +$$

$$+\frac{4N^2}{K}\frac{e^{2\tilde{R}}}{(e^{2\tilde{R}}+1)^2}d\tilde{\theta}^2 = \left\| \begin{array}{l} \text{subst. :} \\ \mathscr{R} \equiv \arctan e^{\tilde{R}} \\ K \equiv 1/a^2 \end{array} \right\| = 4a^2 d\mathscr{R}^2 + \frac{4a^2N^2\tan^2\mathscr{R}}{(\tan^2\mathscr{R}+1)^2}d\tilde{\theta}^2 =$$

$$= 4a^2 d\mathscr{R}^2 + a^2N^2\sin^2(2\mathscr{R})d\tilde{\theta}^2 = \left\| \begin{array}{l} \text{subst. :} \\ u \equiv 2a\mathscr{R} \end{array} \right\| = du^2 + a^2N^2\sin^2\frac{u}{a}d\tilde{\theta} \quad (4.28)$$

where $\tilde{\theta} \in [0, 2\pi]$ and the range for u/a cannot exceed $[0, \pi]$.

This is the infinitesimal line element with $R(u) = aN\sin(u/a)$, but we can perform a transformation $u/a \to u/a - \pi/2$, which implies

$$R(u) = aN\cos(u/a), \quad (4.29)$$

which compared to (3.4) gives

$$c = aN. \quad (4.30)$$

With the result (4.30) we are in the position to fully appreciate the physical role of N: it plays the role of radius (multiplied by a).

Now we need to find the range of u/a. We know it is at most $[-\pi/2, \pi/2]$ and comes from the condition $G \geq 0$, i.e.

$$\frac{4N^2}{K}\frac{\tilde{r}^{2N-2}}{(\tilde{r}^{2N}+1)^2} - \frac{4N^4}{K}\left[\frac{\tilde{r}^{N-1}(1-r^{2N})}{(\tilde{r}^{2N}+1)^2}\right]^2 \geq 0. \quad (4.31)$$

From this inequality, we can obtain $\tilde{r}_{\min}$ and $\tilde{r}_{\max}$ for particular N

$$\tilde{r}_{\min} = \sqrt[2N]{\frac{N-1}{N+1}}, \quad \tilde{r}_{\max} = \sqrt[2N]{\frac{N+1}{N-1}}. \tag{4.32}$$

In previous calculations, we defined $u/a \equiv 2\arctan\left(\tilde{r}^N\right) - \pi/2$, which gives

$$(u/a)_{\min} = 2\arctan\left(\sqrt{\frac{N-1}{N+1}}\right) - \frac{\pi}{2}, \quad (u/a)_{\max} = 2\arctan\left(\sqrt{\frac{N+1}{N-1}}\right) - \frac{\pi}{2}. \tag{4.33}$$

It can be proved that, for any real positive number p, it holds

$$2\arctan\sqrt{p} = \arcsin\left(\frac{p-1}{p+1}\right) + \frac{\pi}{2}, \tag{4.34}$$

which implies

$$(u/a)_{\min} = 2\arctan\sqrt{\frac{N-1}{N+1}} - \pi/2 = -\arcsin(1/N), \tag{4.35}$$

$$(u/a)_{\max} = 2\arctan\sqrt{\frac{N+1}{N-1}} - \pi/2 = +\arcsin(1/N). \tag{4.36}$$

This is a pleasant result, because the range we obtain from (4.35) and (4.36) precisely coincides with the range of the Bulge type of surfaces, as can be seen: $u/r \in [-\arcsin(r/c), \arcsin(r/c)]$ with $r \equiv a$, see (3.8).

This completes our proof that indeed the surfaces corresponding to the vortex solutions are surfaces of the Bulge type, with a clear meaning for the geometrical role of N.

4.3 Negative Gaussian curvature

Here, we ask ourselves whether the Horváthy-Yéra Ansatz $f(z) = z^{-N}$ applied to the Liouville equation with $K < 0$ gives some interesting solution. We recall, that in [9] we found that the infinitesimal line element (4.8) for $K < 0$ corresponds to a pseudosphere for any given N. There is also the problem with N-fold rotation, which we want to avoid. Let us start again with the line element and follow same steps as for (4.28)

$$dl^2 = \frac{4N^2}{|K|}\frac{\tilde{r}^{2(N-1)}}{(\tilde{r}^{2N}-1)^2}(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2) = \left\| \begin{array}{l} \text{subst.}: \\ \tilde{R} \equiv \ln \tilde{r}^N \end{array} \right\| = \frac{4}{K}\frac{e^{2\tilde{R}}}{(e^{2\tilde{R}}-1)^2} d\tilde{R}^2 +$$

$$+\frac{4N^2}{K}\frac{e^{2\tilde{R}}}{(e^{2\tilde{R}}-1)^2} d\tilde{\theta}^2 = \left\| \begin{array}{l} \text{subst.}: \\ \mathscr{R} \equiv \operatorname{arctanh} e^{\tilde{R}} \\ |K| \equiv 1/a^2 \end{array} \right\| = 4a^2 d\mathscr{R}^2 + \frac{4a^2N^2\tanh^2\mathscr{R}}{\left(\tanh^2\mathscr{R}-1\right)^2} d\tilde{\theta}^2 =$$

$$= 4a^2 d\mathscr{R}^2 + a^2N^2\sinh^2(2\mathscr{R})d\tilde{\theta}^2 = \left\| \; u \equiv 2a\mathscr{R} \; \right\| = du^2 + a^2N^2\sinh^2\frac{u}{a}d\tilde{\theta}^2, \tag{4.37}$$

The integration is not simple as before, a problem appears for $e^{2\tilde{R}} = 1$, but we will not deal with this now, because in a moment we will show that the problem is actually there for all values of $\tilde{R}$, not just for $\tilde{R} = 0$.

One can immediately recognise, that this is the line element for the elliptical pseudosphere. On the other hand, we know that the formula for the radial function of the elliptical pseudosphere is $R(u) = c \sinh(u/r)$ and the z-coordinate is not real, if $r > c > 0$. To satisfy this, it is necessary to allow the N-fold rotation, which we denied. This is one reason to believe that no surface of revolution with constant $K < 0$ for any given N can exist.

Let us verify it independently. We will again assume the Ansatz

$$x = F_-(\tilde{r}) \cos \tilde{\theta}, \ y = F_-(\tilde{r}) \sin \tilde{\theta}, \ z = z_-(\tilde{r}) \tag{4.38}$$

and following same steps as before we get

$$F_-(\tilde{r}) = \frac{2N}{\sqrt{|K|}} \frac{\tilde{r}^N}{\tilde{r}^{2N} - 1}, \tag{4.39}$$

which determines the z-coordinate

$$z_- = \int \sqrt{\phi_-^2(\tilde{r}) - [F_-'(\tilde{r})]^2} d\tilde{r} = \int \sqrt{\frac{4N^2}{|K|} \frac{\tilde{r}^{2N-2}}{(\tilde{r}^{2N} - 1)^2} - \frac{4N^2}{|K|} \left[\left(\frac{\tilde{r}^N}{\tilde{r}^{2N} - 1} \right)' \right]^2} d\tilde{r}. \tag{4.40}$$

Again we write $z_- = \int \sqrt{G(\tilde{r})} d\tilde{r}$, where the function $G(\tilde{r})$ is

$$G(\tilde{r}) = \frac{4N^2}{|K|} \frac{\tilde{r}^{2N-2}}{(\tilde{r}^{2N} - 1)^2} - \frac{4N^4}{|K|} \left[\frac{\tilde{r}^{N-1}(r^{2N} + 1)}{(\tilde{r}^{2N} - 1)^2} \right]^2 \leq \frac{4N^2(1 - N^2)}{|K|} \frac{\tilde{r}^{2N-2}}{(\tilde{r}^{2N} - 1)^2}. \tag{4.41}$$

G is always non-positive, which does not allow the real z-coordinates. Therefore, the Horváthy-Yéra solutions cannot be extended, at least in this straightforward manner, to the negative curvature case.

4.4 Vortex solutions and associated spacetimes

We would like to focus on the spacetimes obtained by taking the product of flat time and the surfaces we just discovered to be associated with the vortex solutions.

Case $N = 1$

For simplicity, we begin with $N = 1$ and specific value of Gaussian curvature $K = 4$ to get simple form of the spacetime interval

$$ds^2 = dt^2 - \frac{1}{(1 + \tilde{r}^2)^2} d\tilde{r}^2 - \frac{\tilde{r}^2}{(1 + \tilde{r}^2)^2} d\tilde{\theta}^2. \tag{4.42}$$

If we define $R(\tilde{r}) \equiv \frac{\tilde{r}}{1+\tilde{r}^2}$, then $R'(\tilde{r}) = \frac{1-\tilde{r}^2}{(1+\tilde{r}^2)^2}$ and the spatial line element can be modified as

$$dl^2 = \frac{\left[\frac{1-\tilde{r}^2}{(1+\tilde{r}^2)^2}\right]^2 d\tilde{r}^2}{\left[\frac{1-\tilde{r}^2}{(1+\tilde{r}^2)^2}\right]^2 (1 + \tilde{r}^2)^2} + \frac{\tilde{r}^2}{(1 + \tilde{r}^2)^2} d\tilde{\theta}^2 = \frac{[R'(\tilde{r})]^2 d\tilde{r}^2}{1 + 2\left(-\frac{2\tilde{r}^2}{1+\tilde{r}^2} + \frac{2\tilde{r}^4}{(1+\tilde{r}^2)^2}\right)} + R^2(\tilde{r}) d\tilde{\theta}^2. \tag{4.43}$$

Now, we are in the right position to write the spacetime interval as

$$ds^2 = dt^2 - \frac{[R'(\tilde{r})]^2}{1+2E(\tilde{r})}d\tilde{r}^2 - R^2(\tilde{r})d\tilde{\theta}^2, \tag{4.44}$$

where we denote $E(\tilde{r}) \equiv -\frac{2\tilde{r}^2}{1+\tilde{r}^2} + \frac{2\tilde{r}^4}{(1+\tilde{r}^2)^2}$. The function $E(\tilde{r})$ has one global minimum on $[0,\infty]$: $\tilde{r}_{\rm min} = 1$ and $E(\tilde{r}_{\rm min} = 1) = -1/2$ (equator of the sphere). The maximum value is: $E(\tilde{r}_{\rm max}) = 0$ for $\tilde{r}_{\rm max1} = 0$ or $\tilde{r}_{\rm max2} = \infty$ (both poles of the sphere). Up to the point $\tilde{r}_{\rm min} = 1$ the condition is always satisfied and the point $\tilde{r}_{\rm min} = 1$ behaves like a singularity.

Relation to the Lemaitre-Tolman-Bondi spacetime

Now, we would like to point out on a possible connection with spherical symmetrical dust solution of the Einstein field equation, usually called *Lemaitre-Tolman-Bondi spacetime* [19]. The spacetime interval is

$$ds^2 = dt^2 - \frac{(R')^2}{1+2E}dr^2 - R^2 d\Omega^2 \tag{4.45}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\nu^2$, $R = R(t,r) > 0$, $R' \equiv dR/dr$, $E = E(r) > -1/2$ (to avoid singularity). Quantities r, θ and ν are standard radial distance and spherical angles, respectively. Moreover, the quantity R holds

$$\dot{R}^2 = \frac{2M}{R} + 2E \tag{4.46}$$

where $\dot{R} \equiv dR/dt$ and $M(t)$ is an arbitrary function. On the other hand, we deal with static spacetimes, where $\dot{R} = 0$ and $M = -ER$. More information about the spacetime can be found e.g. in [19].

For $N = 1$, we found a match, but we would like to find similar match for arbitrary N. This is the matter of next paragraph.

Generalization for $N > 1$

For general values of $N, K > 0$ we write down the spacetime interval

$$ds^2 = dt^2 - \frac{4N^2}{K}\frac{\tilde{r}^{2(N-1)}}{(1+\tilde{r}^{2N})^2}d\tilde{r}^2 - \frac{4N^2}{K}\frac{\tilde{r}^{2N}}{(1+\tilde{r}^{2N})^2}d\tilde{\theta}^2. \tag{4.47}$$

We will follow same steps as for (4.44). We introduce a new quantity $R(\tilde{r}) \equiv \frac{\tilde{r}^N}{1+\tilde{r}^{2N}}$, and so $R'(\tilde{r}) = N\frac{\tilde{r}^{N-1}-\tilde{r}^{3N-1}}{(1+\tilde{r}^{2N})^2}$. This leads us to a simple result

$$ds^2 = dt^2 - \frac{4}{K}\frac{[R'(\tilde{r})]^2}{1+2E(\tilde{r})}d\tilde{r}^2 - \frac{4N^2}{K}R^2(\tilde{r})d\tilde{\theta}^2, \tag{4.48}$$

where we denote $E(\tilde{r}) \equiv -\frac{2\tilde{r}^{2N}}{1+\tilde{r}^{2N}} + \frac{2\tilde{r}^{4N}}{(1+\tilde{r}^{2N})^2}$. Finally, let us define $\tilde{w} \equiv 2\tilde{r}/\sqrt{K}$, $\tilde{\vartheta} \equiv 2N\tilde{\theta}/\sqrt{K}$ and insert them into (4.48). Then we get

$$ds^2 = dt^2 - \frac{[R'(\tilde{w})]^2}{1+2E(\tilde{w})}d\tilde{w}^2 - R^2(\tilde{w})d\tilde{\vartheta}^2, \tag{4.49}$$

where $E(\tilde{w}) \equiv -\frac{2(\tilde{w}\sqrt{K}/2)^{2N}}{1+(\tilde{w}\sqrt{K}/2)^{2N}} + \frac{2(\tilde{w}\sqrt{K}/2)^{4N}}{(1+(\tilde{w}\sqrt{K}/2)^{2N})^2}$ and $R(\tilde{w}) \equiv \frac{(\tilde{w}\sqrt{K}/2)^N}{1+(\tilde{w}\sqrt{K}/2)^{2N}}$. For the variable $E(\tilde{w})$ it holds $E(\tilde{w}) \in [-1/2, 0]$ for arbitrary N.

Conclusions

The general research area, where this thesis moves its steps, is the analogue gravity scenarios stemming from the relativistic-like behaviour of graphene.

In the beginning we introduced experimentally tested fact that graphene with lower excited electrons is well described by the massless Dirac field theory (in 2+1 dimensions), where the Fermi velocity replaces the speed of light. Since graphene is effectively described by a quantum field on curved spacetime, provides a good testing laboratory for studying quantum phenomena in relativistic environments.

We provided a introduction to the Dirac field theory, from its birth as a merge of quantum mechanics and special relativity through generalization to curved spacetimes until the discussion about the Weyl symmetry of the Dirac massless action (for truly curved spacetime).

Then we move forward and ask ourselves what the Weyl symmetry means for a graphene membrane. In particular, we assumed conformally flat spacetimes and spacetime metric tensors with a flat-time part. Then the conformal factors, associated to two-dimensional surfaces, have to satisfy the Liouville equation, which includes the constant Gaussian curvature, as follows from the condition that the Cotton tensor vanishes.

In the following chapter, we discussed well known surfaces of revolution with constant Gaussian curvature, positive (simple sphere, surfaces of the Bulge or Spindle type) as well as negative (the Beltrami, elliptic and hyperbolic pseudospheres), and corresponding (flat time) spacetimes. We illustrate that the Beltrami spacetimes is conformal to the Rindler spacetime, similarly the Elliptic spacetime is conformal to the de Sitter spacetimes and the Hyperbolic spacetime is conformal to the BTZ black hole spacetime.

Then we studied the non-topological vortex solutions of the Liouville equation introduced by Horváthy and Yéra. These are given in terms of conformal factors, $\phi(\tilde{r})$, depending on a natural number N. We discovered, which surfaces of positive constant Gaussian curvature correspond to $\phi(\tilde{r})$ and we found that such surfaces are surfaces of revolution. One possible solution is the sphere with the $N-$fold rotation around the z-axis. From the mathematical point of view, it is an expected result because arguments of differential geometry, recalled here, say that if we identify surfaces that differ by a coordinate redefinition, then there is only one surface with constant $K > 0$ and it is the sphere. On the other hand, we have in mind that such configurations might actually be built with graphene in laboratories, giving possibly rise to a Dirac field theory on non-trivial backgrounds. We showed that corresponding surfaces are surfaces of constant positive Gaussian curvature of the Bulge type (barrel or ring shaped surfaces, with boundaries), of varying radii but fixed value of the curvature (the other type of singular surfaces, the Spindle type, is not found).

We also explain why pseudospheres are not suitable generalization of the Horváthy-Yéra vortices, at least in our approach.

Finally, we moved towards spacetimes obtained as a product of flat time and the surfaces we have found, and show that they are formally same as the Lemaitre-Tolman-Bondi spacetimes with fixed meridian (the corresponding azimuthal angle is constant). The correspondence requires that R only depends on $\tilde{r}$, as no time is considered here, because we deal with static spacetimes. We did not consider here the interesting problem of the singularities as it deserves a separate study.

A. Conformal symmetry

This appendix is dedicated to *conformal symmetry*. We introduce several fundamental concepts: *infinitesimal conformal transformations* and their *generators* till the definition of *conformal group* and *conformal algebra* in any dimension. We then show that in $n = 2$, the conformal algebra is the *Witt algebra* and its *central extension* the *Virasoro algebra*. Our goal here is to explain concepts mentioned above in the most natural way and avoid unnecessary mathematical details. This appendix is important to gain insight to the main topic of this work, although the link is only indirect.

A key reference for this appendix is [22], but we also used [26] and [27].

A.1 Infinitesimal conformal transformations

To get an intuitive view of things, the conformal transformation can be described as a transformation that does preserve angles between any two oriented curves going through (arbitrary) same point as well as preserve orientation (non-intersecting curves do not intersect after mapping etc.). On the other hand, lengths of lines or sizes and curvature of infinitesimal samples can be changed, see Fig. A.1.

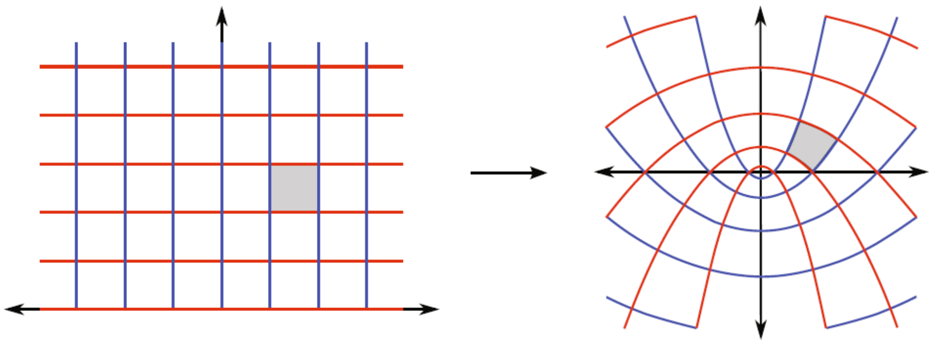


Figure A.1: Conformal transformation in $n = 2$, taken from [22]

To be more precise from a mathematical point of view, let M, M' be two flat vector spaces and $U \in M$ and $V \in M'$ open subsets. Then the conformal transformation is a differentiable map $\xi\colon U \to V$, that $\xi^* g' = \Lambda g$, where g is a general metric tensor defined on M and g' is its image, the *scale factor* Λ is a function of position x.

Let us denote $x' \equiv \xi(x)$, then the transformation can be expressed in the following way

$$g'_{ab}(x')dx'^a dx'^b = g'_{ab}(x')\frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d}dx^c dx^d \equiv \Lambda(x) g_{cd}(x) dx^c dx^d, \tag{A.1}$$

which implies

$$g'_{ab}(x')\frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d} = \Lambda(x) g_{cd}(x). \tag{A.2}$$

For next discussion we confine to one space, i.e. $M = M'$, which implies $g = g'$. We consider a flat space(time) with a constant metric tensor $g_{ab} = \eta_{ab}$. The transformation can be written as

$$\eta_{ab}\frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d} = \Lambda(x)\eta_{cd}, \tag{A.3}$$

where we prefere a signature of the metric tensor as

$$\eta_{ab} = \mathrm{diag}(1, -1, \dots). \tag{A.4}$$

For next discussion, we also restrict ourselves to infinitesimal coordinate transformations. For now, we assume

$$x'^a = x^a + \epsilon^a(x) + \mathbb{O}(\epsilon^2), \tag{A.5}$$

where $\epsilon(x) \ll 1$. Then the left-hand side of (A.3) takes the form

$$\eta_{ab}\left(\delta^a_c + \frac{\partial\epsilon^a}{\partial x^c} + \mathbb{O}(\epsilon^2)\right)\left(\delta^b_d + \frac{\partial\epsilon^b}{\partial x^d} + \mathbb{O}(\epsilon^2)\right) = \eta_{cd} + \left(\frac{\partial\epsilon_c}{\partial x^d} + \frac{\partial\epsilon_d}{\partial x^c}\right) + \mathbb{O}(\epsilon^2), \tag{A.6}$$

where we used the Einstein sum convention for $\epsilon_a = \eta_{ab}\epsilon^b$. From this assumption we have got the modified form of (A.3)

$$\frac{\partial\epsilon_c}{\partial x^d} + \frac{\partial\epsilon_d}{\partial x^c} = (\Lambda(x) - 1)\,\eta_{cd}. \tag{A.7}$$

Let us denote $\partial_i \equiv \partial/\partial x^i$, where i is a flat index, and $K(x) \equiv \Lambda(x) - 1$ for simplicity. Then (A.7) can be rewritten into new form

$$\partial_d\epsilon_c + \partial_c\epsilon_d = K(x)\eta_{cd}. \tag{A.8}$$

We would like to determine the function $K(x)$. This can be done by tracing (A.8) with η^{ab}

$$2\partial^a\epsilon_b = K(x)n. \tag{A.9}$$

Now it is enough to just put it back into (A.8)

$$\partial_d\epsilon_c + \partial_c\epsilon_d = \frac{2}{n}(\partial^a\epsilon_a)\eta_{cd}. \tag{A.10}$$

This result (A.10) is the *conformal Killing equation.* The scale factor $\Lambda(x)$ can be simply written as

$$\Lambda(x) = 1 + \frac{2}{n}\partial^a\epsilon_a + \mathbb{O}(\epsilon^2). \tag{A.11}$$

On the base of previous calculation, it can be derived other useful identities. To get the one useful for us, we apply ∂^c to (A.10) and sum over indices, which leads to

$$\partial_d\left(\partial^a\epsilon_a\right) + \partial^b\partial_b\epsilon_d = \frac{2}{n}\partial^c(\partial^a\epsilon_a)\eta_{cd}. \tag{A.12}$$

Our aim is to combine this result with the conformal Killing equation (A.10). It can be proved that this relation holds

$$2\partial_c\partial_d\left(\partial^a\epsilon_a\right) + \partial^b\partial_b\left(\frac{2}{n}(\partial^a\epsilon_a)\eta_{cd}\right) = \frac{4}{n}\partial_c\partial_d(\partial^a\epsilon_a). \tag{A.13}$$

Contracting by η^{cd} gives us

$$(n-1)\partial^b\partial_b(\partial^a\epsilon_a) = 0. \tag{A.14}$$

This result will be important for us in next section, when we discuss conformal group and algebra of infinitesimal transformation.

A.2 Conformal group and algebra in $n \geq 3$

In previous section A.1, we derived the condition for an infinitesimal transformation to be conformal, see (A.10). Now we will introduce how conformal group and algebra appear in $n \geq 3$. Before that we introduce general definitions of these concepts [22].

The conformal group is the group consisting of globally defined, invertible and finite conformal transformations from the space to itself.

The conformal algebra is the Lie algebra corresponding to the conformal group.

To study the conformal group and algebra, we need to have better idea how ϵ looks like. Looking at (A.14) it is natural to assume an Ansatz to second order of x

$$\epsilon_a = a_a + b_{ab}x^b + c_{abc}x^b x^c, \tag{A.15}$$

where a_a, b_{ab}, c_{abs}, moreover last constant holds $c_{abc} = c_{bac}$.

Now, when we know how the infinitesimal shifts looks like, we can find the generators of the conformal transformations. Detailed analysis, which follows from studying (A.15) and leads to the generators and finite (and not only infinitesimal) conformal transformations, can be found in [22], here we only present results of this study: the generators and finite conformal transformations (translation, dilation, Lorentz rotation and special conformal transformation (SCT)), see Tab. A.1. The dimension of the algebra is equal to the number of all generators $N = n + 1 + n(n-1)/2 + n = (n+1)(n+2)/2$.

Now we could find all commutators $[P_a, D]$, $[P_a, L_{ab}]$ etc., or we can try to define only one generator J_{ab}. The right way is to define

$$J_{a,b} \equiv L_{ab},\ J_{-1,a} \equiv \frac{1}{2}\left(P_a - K_a\right),\ J_{-1,0} \equiv D,\ J_{0,a} \equiv \frac{1}{2}\left(P_a + K_a\right). \tag{A.16}$$

Then $J_{m,q}$ with $m, q = -1, 0, 1, \ldots, n-1$ satisfy

$$[J_{m,q}, J_{r,s}] = i(\eta_{ms}J_{qr} + \eta_{qr}J_{ms} - \eta_{mr}J_{qs} - \eta_{qs}J_{mr}). \tag{A.17}$$

Detailed analysis can be found in [22]. For instance, for n-dimensional Euclidean space $\mathbb{R}^{1,n}$ the metric tensor is $\eta_{mn} \equiv \mathrm{diag}(1, -1, -1, \dots)$ and the commutation relation is (A.17), for $\mathbb{R}^{2,n-2}$ the tensor is $\eta_{mn} \equiv \mathrm{diag}(1, 1, -1\dots)$.

Table A.1: Table of infinitesimal conformal transformations and corresponding generators

transformations		generators
translation	$x'^a = x^a + a^a$	$P_a = -i\partial_a$
dilation	$x'^a = \alpha x^a$	$D = -ix^a\partial_a$
Lorentz rotation	$x'^a = M^a_{\ b}x^b$	$L_{ab} = i\left(x_a\partial_b - x_b\partial_a\right)$
SCT	$x'^a = \frac{x^a-(x\cdot x)b^a}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}$	$K_a = -i\left(2x_a x^b\partial_b - (x\cdot x)\partial_a\right)$

A.3 Conformal group and algebra in $n = 2$

The conformal symmetries in two dimensions is very special case in comparison with higher dimensions $n \geq 3$. Already the discussion about infinitesimal conformal transformations, from commutation relations of generators follow, is completely different. In this section, we will focus on. We restrict only on the Euclidean metric in a flat space.

Infinitesimal conformal transformations

The condition following from (A.10) for two-dimensional space is

$$\partial_0\epsilon_1 = -\partial_1\epsilon_0, \qquad \partial_0\epsilon_0 = \partial_1\epsilon_1. \tag{A.18}$$

On the first look, it points on the fact that (A.18) is formally same with the Cauchy-Riemann conditions. This observation let us use the full power of complex analysis.

For a holomorphic function denoted $f(z)$ of $z = \mathrm{Re}z + i\mathrm{Im}z$, it holds

$$\frac{\partial \mathrm{Im}f(z)}{\partial \mathrm{Re}z} = -\frac{\partial \mathrm{Re}f(z)}{\partial \mathrm{Im}z}, \qquad \frac{\partial \mathrm{Re}f(z)}{\partial \mathrm{Re}z} = \frac{\partial \mathrm{Im}f(z)}{\partial \mathrm{Im}z}. \tag{A.19}$$

Let us define a complex variable z, a complex ϵ and introduce a complex derivative[1] ∂_z:

$$z = x^0 + ix^1, \quad \epsilon = \epsilon^0 + i\epsilon^1, \quad \partial_z = \partial_0 - i\partial_1,$$

$$\bar{z} = x^0 - ix^1, \quad \bar{\epsilon} = \epsilon^0 - i\epsilon^1, \quad \partial_{\bar{z}} = \partial_0 + i\partial_1. \tag{A.20}$$

We define $f(z)$ as an image of the infinitesimal conformal transformation: $f(z) \equiv z' = z + \epsilon(z)$, while $\epsilon(z)$ is expected to be also holomorphic function (satisfying (A.19, and we assume continuous partial derivatives).

A.3.1 Witt algebra

Let us expand the function[2] $\epsilon(z)$ in the Laurent series around $z = 0$. The infinitesimal conformal transformation can be expressed as

$$f(z) \equiv z' = z + \epsilon(z) = z + \sum_{m\in\mathbb{Z}} \epsilon_m(-z^{m+1}), \tag{A.21}$$

[1] Using the complex derivative, an equivalent condition to the Cauchy-Riemann conditions is $\partial f(z)/\partial\bar{z} = 0$.

[2] The function $\epsilon(z)$ is generally meromorphic, i.e. it is holomorphic on some open set D, which has isolated singularities outside this open set.

$$f(\bar{z}) \equiv \bar{z}' = \bar{z} + \epsilon(\bar{z}) = \bar{z} + \sum_{m\in\mathbb{Z}} \bar{\epsilon}_m(-\bar{z}^{m+1}), \tag{A.22}$$

where ϵ_m, $\bar{\epsilon}_m$ are constant infinitesimal parameters.

To find the generators corresponding to the transformation we follow this way: We assume function ψ depending on z and only non-zero m-th term of Laurent series for simplicity. Then we can write

$$\phi(z') = \phi(z - \epsilon_m z^{m+1}) = \sum_{k=0}^{+\infty} \frac{1}{k!}\frac{\partial^k}{\partial z^k}\phi(z)(-\epsilon_m z^{m+1})^k = \exp\Big\{\epsilon_m\left(-z^{m+1}\partial_z\right)\Big\}\phi(z), \tag{A.23}$$

where ϵ_m is a member of group and $l_m \equiv -z^{m+1}\partial_z$ is corresponding generator. For general infinitesimal transformations, the result (A.23) is generalized into

$$\phi(z') = \exp\Bigg\{\sum_{m=-\infty}^{+\infty}\epsilon_m\left(-z^{m+1}\partial_z\right)\Bigg\}\phi(z) = \exp\Bigg\{\sum_{m=-\infty}^{+\infty}\epsilon_m l_m\Bigg\}\phi(z). \tag{A.24}$$

For (A.22) the derivation is same, therefore the m-th generator of both copies (assuming the complex conjugated operator $\bar{l}_m$) are

$$l_m = -z^{m+1}\partial_z, \qquad \bar{l}_m = -\bar{z}^{m+1}\partial_{\bar{z}}. \tag{A.25}$$

Since of $m \in \mathbb{Z}$ the number of independent infinitesimal conformal transformations is infinite, the related algebra has infinite dimension. This is very important observation for conformal field theory in $n = 2$ and it has far-reaching consequences (e.g. [22]). These generators are elements of the Witt algebra, special case of the Lie algebra, with commutation relations

$$[l_k, l_m] = z^{k+1}\partial_z(z^{m+1}\partial_z) - z^{m+1}\partial_z(z^{k+1}\partial_z) = \cdots = (k-m)l_{k+m}. \tag{A.26}$$

Moreover, we can introduce the copy of the Witt algebra complex conjugated generators

$$[\bar{l}_k, \bar{l}_m] = (k-m)\bar{l}_{k+m} \tag{A.27}$$

and the generators of these copies are independent

$$[l_k, \bar{l}_m] = 0. \tag{A.28}$$

It is not too demanding to realize, that not all generators are defined everywhere. For $z = 0$, we can avoid the singularities only for generators with $m \geq -1$. On the other hand, for $z = \infty$, respectively $w \equiv 1/z \to 0$ (for more details see e.g. [22]), generators with $m \leq 1$ are well defined. This simple analysis implies that there are only three globally defined generators $\{l_{-1}, l_0, l_1\}$ satisfying following commutation relations

$$[l_0, l_{\pm 1}] = \mp l_{\pm 1},\ [l, l_{-1}] = 2l_0. \tag{A.29}$$

The generators span a sub-algera, which corresponds to the global conformal group, i.e. infinitesimal conformal transformations, whose generators are $\{l_{-1}, l_0, l_1\}$, creating a sub-algebra of the Witt algebra.

A.3.2 Virasoro algebra

In this short section, we focus on so-called *Virasoro algebra.* We would like to introduce this new algebra in an intuitive way.

As well as the Witt algebra was determined by commutator relations

$$[l_k, l_m] = (k-m)l_{k+m}, \tag{A.30}$$

the Virasoro algebra, whose generators we denote as L_m, is determined by commutators

$$[L_k, L_m] = (k-m)L_{k+m} + \hat{c}p(k,m). \tag{A.31}$$

In this context, $\hat{c}$ is an operator $\hat{c} = c\mathbb{E}$, where $\mathbb{E}$ is a unit matrix and c is a complex number. The complex number c is called central charge and its value depends on physical system (1 for free boson, 1/2 for free fermions etc.). Following commutation relations are obvious

$$[L_m, \hat{c}] = 0,\ [\hat{c}, \hat{c}] = 0 \ \ \forall m \in \mathbb{Z}. \tag{A.32}$$

In [22] it is shown how to determine the function $p(k,m)$. From their study, we can write the commutators

$$[L_k, L_m] = (k-m)L_{k+m} + \frac{1}{12}\hat{c}k(k^2-1)\delta_{m+k,0}. \tag{A.33}$$

where the numerical factor 1/12 is given by convention.

B. Spacetimes of interest

In this appendix we discuss important spacetimes relevant to our study: the Rindler, de Sitter and Bañados - Teitelboim - Zanelli (BTZ) black hole spacetimes. When we discuss the Rindler spacetime, we will discuss the geometry of the Schwarzschild black hole in the vicinity of its horizon. We have one important reason to do this. The geometry of the Schwarzschild black hole tends to the geometry of the Rindler spacetime as we get closer to the horizon. In fact, the Rindler spacetime shares the near horizon properties of many spacetimes with horizons, including the cosmological horizon.

Then we will focus on the de Sitter spacetime. To understand it well we will introduce how the de Sitter spacetime is related to the de Sitter universe, and we will show some properties of the de Sitter spacetime for dimension $n \geq 2$.

Finally, we will briefly introduce a very interesting and surprising solution of the Einstein field equations with negative cosmological constant $\Lambda < 0$ in $n = 3$.

In what follows we set the speed of light is $\nu_{light} \equiv 1$ and the gravitational constant is $G \equiv 1$.

B.1 Rindler spacetime

Accelerated observer

In this part we remind the problem of the accelerated observer in the Minkowski spacetime in $n = 2$ [16] [25] [28]. Let us consider an inertial system with the Minkowski frame of reference (T,X). There is an observer of rest mast m_0 subject to an external constant force f, hence the observer moves with a constant proper acceleration $\alpha = f/m_0$. Therefore one can write

$$f = \frac{dp}{dT} = \text{const} \Rightarrow p = fT, \text{ i.e. } \frac{m_0 v}{\sqrt{1-v^2}} = fT \Rightarrow v = \frac{fT}{\sqrt{m_0^2 + f^2T^2}}, \tag{B.1}$$

where we chose the constant of integration $p_0 = 0$. Because of $X = \frac{m_0}{f}\sqrt{1 + \frac{f^2T^2}{m_0^2}}$, it holds

$$X^2 - T^2 = \frac{m_0^2}{f^2} \equiv \frac{1}{\alpha^2}, \tag{B.2}$$

where we assume that the initial position of the observer is $X(T = 0) = \frac{m_0}{f} = \frac{1}{\alpha}$. From (B.2) it is obvious to see that the worldline of the accelerated observer is a hyperbola with asymptotes $X = \pm T$.

It is interesting to write the equation for X and T with respect to the proper time τ of the observer

$$\frac{d\tau}{dT} = \sqrt{1-v^2} = \frac{m_0}{\sqrt{m_0^2 + f^2T^2}} \Rightarrow T = \frac{m_0}{f}\sinh\frac{f\eta}{m_0}, X = \frac{m_0}{f}\cosh\frac{f\eta}{m_0}. \tag{B.3}$$

Therefore the Minkowski coordinates of the observer moving at constant acceleration α are

$$T = \frac{1}{\alpha}\sinh(\alpha\tau), \ X = \frac{1}{\alpha}\cosh(\alpha\tau). \tag{B.4}$$

Let us define a new coordinate system (τ, x) such that

$$T = x \sinh(\alpha\tau), \; X = x \cosh(\alpha\tau) \tag{B.5}$$

The coordinates (τ, x) are called *Rindler coordinates* and the metric in these coordinates is

$$ds^2 = \alpha^2 x^2 d\tau^2 - dx^2 \tag{B.6}$$

For the observer, moving a constant acceleration α, holds: $x = 1/\alpha$, then the Minkowski coordinates are (B.4) and the metric is

$$ds^2 = d\tau^2 - dx^2 \tag{B.7}$$

Such observer who has constant spatial Rindler coordinates and only τ varies as time passes [1] is called *Rindler observer.*

Let us introduce a less intuitive, but very useful coordinate system called *Lass coordinates*

$$T = \frac{1}{a} e^{a\eta} \sinh(a\tau), \; X = \frac{1}{a} e^{a\eta} \cosh(a\tau) \tag{B.8}$$

Comparing them to Rindler coordinates, only one coordinate and one constant are new[2].

Because of this identity

$$X^2 - T^2 = \frac{e^{2a\eta}}{a^2}, \tag{B.9}$$

we obtain from (B.2) that the relation between a and α for given η is

$$\alpha = a e^{-a\eta}. \tag{B.10}$$

The hyperbolic motion means strictly $\eta = \text{const}$. The result (B.8) implies

$$X = T \tanh(\alpha\tau). \tag{B.11}$$

Therefore X and T are constantly proportional for $\tau = \text{const}$.

In Lass coordinates the metric takes the form

$$ds^2 = e^{2a\eta} \left(d\tau^2 - d\eta^2 \right). \tag{B.12}$$

Especially this coordinate system is very useful when we focus on its associated spacetime and its relation to te Beltrami spacetime, see (3.21). The generalization to $n \geq 3$ is straightforward: the other dimensions, than two involved, are just spectators, e.g. in Lass coordinates

$$ds^2 = e^{2a\eta} \left(d\tau^2 - d\eta^2 \right) - dx_2^2 - \cdots - dx_{n-1}^2. \tag{B.13}$$

[1]This observer is at rest in Rindler coordinates.

[2]Since the observer is at rest in Rindler coordinates, e.g. the x-position is constant, his Minkowski coordinates depend only on τ, see (B.4). If we choose a different coordinate system, let it denote by (τ, η), then $T = T(\tau, \eta)$, $X = X(\tau, \eta)$ and the spatial coordinate of Rindler observer is not constant.

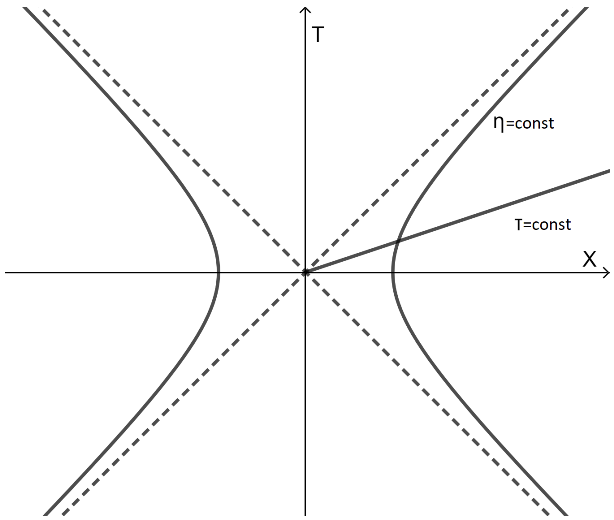


Figure B.1: The right Rindler wedge is a right quadrant of the Minkowski space constituted by asymptotes: $X = \pm T$, where $X \in [0, +\infty]$.

Rindler horizon

As we mentioned above the worldline of the accelerated observer is the hyperbola with the asymptotes $X = \pm T$. It implies that $\tau \to \pm\infty$ and $\eta = \pm\infty$. These asymptotes are the horizon, called *Rindler horizon* and the quadrant of Minkowski space, where the observer can move and is described by (τ, η), is called *Rindler wedge*. The observer's spacetime is called *Rindler spacetime*. The accelerated observer will never reach the asymptotes and observe a light signal sent from outside of the Rindler wedge.

Schwarzschild black hole: Rindler in a neighbourhood of the event horizon

The Schwarzschild solution describes the metric of curved spacetime outside of a spherically symmetrical, electrically neutral, spineless body of mass M. Next assumption is that the cosmological constant is equal to zero $\Lambda = 0$, i.e. one must solve the reduced Einstein field equations

$$G_{\mu\nu} = 0, \tag{B.14}$$

where $G_{\mu\nu}$ is the Einstein tensor [15].

The metric written in the *Schwarzschild coordinates* for $n = 4$ is

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{B.15}$$

where r is the radial coordinate, t is the time coordinate and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, where ϕ is the longitude and θ is the colatitude.

Let us focus on a spacetime patch on the horizon [32]. The interval of proper distance ρ is related to dr by $\rho = \sqrt{g_{rr}}dr$. Then the total proper distance holds

$$\rho = \int_{2GM}^{r} \sqrt{g_{rr}}dr = 2\sqrt{2GM(r - 2MG)}. \tag{B.16}$$

If we express the metric in ρ instead of r and define dimensionless time $\omega \equiv dt/(4GM)$, we can rewrite the metric as

$$ds^2 = \rho^2 d\omega^2 - d\rho^2 - r(\rho)^2 d\Omega^2. \tag{B.17}$$

Due to the symmetry, we can set $\theta = \pi/2$ and focus only on the case $d\Omega = d\phi$. If the angle ϕ is small, we can write the spatial coordinates x and y as $x = 2GM \cos\phi$, $y = 2GM \sin\phi$. We have obtained

$$ds^2 = \rho^2 d\omega^2 - d\rho^2 - dx^2 - dy^2. \tag{B.18}$$

The metric is expressed in cylindrical hyperbolic coordinates. If we define $T = \rho \cosh\omega$, $Z = \rho \sinh\omega$, we get the Minkowskian form

$$ds^2 = dT^2 - dZ^2 - dx^2 - dy^2. \tag{B.19}$$

This coordinate system is the Rindler coordinate system and covers just one wedge of the Minkowski spacetime, as explained in the previous paragraphs.

B.2 de Sitter spacetime

To start we will introduce the concept of homogeneous and isotropic universe. The model is based on assumptions, which are supported by observations: The universe seems to be homogeneous and isotropic on the huge scales. This is known as the *cosmological principle* and its consequence is that no point in the universe is preferred and any observer, staying in any point of the universe, should observe same things (e.g. local quantities like pressure and density have to be same in all locations of size like Mpc [3]). From the condition of homogeneity it follows that the spatial curvature must be same for all points of the manifold[4]. Therefore there are only three possibilities: a sphere (surface of positive constant curvature), plane (flat surface) and hyperboloid (surface of negative constant curvature). On the base of the cosmological principle one can derive the metric of such universe (homogeneous and isotropic), called *Friedmann–Lemaître–Robertson–Walker metric* (FLRW metric) [15], [19], [31].

FLRW metric

To study the metric of the curved space with constant curvature, let us obtain the metric of two-dimensional sphere. The sphere can be embedded into three-dimensional Euclidean space, where the equation of the sphere is

$$\sum_{i=1}^{3} x_i^2 = r^2, \tag{B.20}$$

where $r > 0$ is the curvature radius.

[3] We cannot look at the discrete structure of the universe, e.g. the size of the galaxies or less.

[4] Otherwise, we would be able to distinguish between each pair of points and the space would not be homogeneous.

The line element in such space is

$$dl^2 = \sum_{i=1}^{3} dx_i^2, \tag{B.21}$$

where x_i is ith-component. From (B.20) it can be directly obtained

$$\sum_{i=1}^{3} x_i dx_i = 0. \tag{B.22}$$

At this moment, one can express e.g. dx_3 from (B.22) and put into the equation for the line element (B.21). From (B.20) we can obtain x_3 and put into (B.21), which now depends on x_1, x_2. Because the sphere is two-dimensional object, two parameters are enough for its description, so we succeeded.

For curved three-dimensional space of constant curvature the idea is the same. The equation of three-dimensional sphere/pseudosphere embedded into four-dimensional Euclidean/Minkowski space is

$$x_4^2 \pm \sum_{i=1}^{3} x_i^2 = a^2, \tag{B.23}$$

its element line σ is

$$d\sigma^2 = dx_4^2 \pm \sum_{i=1}^{3} dx_i^2 \tag{B.24}$$

and from (B.23) it follows a useful relation

$$\sum_{i=1}^{4} x_i dx_i = 0. \tag{B.25}$$

Here '+' is for the sphere and '-' for the pseudosphere, a plays the role of scale factor. We emphasize that x_4 is only an abstract coordinate, which must be assumed for working in four-dimensional Euclidean spaces.

If we do a standard coordinate transformation from Cartesian coordinates to spherical coordinates

$$x_1 = r \sin\theta \cos\phi, \;\; x_2 = r \sin\theta \sin\phi, \;\; x_3 = r\cos\theta \tag{B.26}$$

and follow same steps, which were done for two-dimensional sphere, we can obtain the spatial part of the metric. If we add time coordinate called *cosmic time* (proper time of cosmic fluid), we will get the FLRW metric

$$ds^2 = dt^2 - \frac{dr^2}{1 - K\frac{r^2}{a^2}} - r^2 d\Omega^2. \tag{B.27}$$

where $\Omega = d\theta^2 + \sin^2\theta d\phi^2$ and $K = 1$ for positive constant curvature (sphere), $K = 0$ for zero curvature (flat space) and $K = -1$ for negative constant curvature (hyperboloid).

Let us do a transformation from radial distance r to radial angular distance ζ: $r = a\zeta$, where $\zeta = \sin\kappa$ for $K = +1$, $\zeta = \kappa$ for $K = 0$ and $\zeta = \sinh\kappa$ for

$K = -1$, where κ is called *comoving distance* and it is independent of time. Then we can rewrite the metric

$$ds^2 = dt^2 - a^2(d\kappa^2 + \zeta^2 d\Omega^2). \tag{B.28}$$

Let us make one generalization: We have defined a as the scale factor, which has to be independent of spatial coordinates, of course. But generally there is not any reason why it could not be time-dependent. Therefore we assume a time-dependence of a:

$$ds^2 = dt^2 - a(t)^2(d\kappa^2 + \zeta^2 d\Omega^2). \tag{B.29}$$

However, how can be the scale factor $a(t)$ which describes the dynamics of the universe determined? To find the scale factor one must put the FLRW metric into the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{B.30}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is scalar curvature, $g_{\mu\nu}$ is the metric, Λ is the cosmological constant and $T_{\mu\nu}$ is the Stress–energy tensor of cosmic fluid. The (B.28) solves the Einstein equations if $a(t)$ satisfies the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi\rho}{3} + \frac{\Lambda}{3}, \tag{B.31}$$

where ρ is the mass density of matter and radiation [15], [33].

In a universe with no matter, neither ordinary, nor dark matter, the dark energy dominates. If the cosmological constant is positive and the spatial curvature is zero, then the universe is called *de Sitter universe*. In this particular case the scale factor is

$$a(t) = e^{\sqrt{\frac{\Lambda}{3}}t} = e^{Ht}, \tag{B.32}$$

where H is the Hubble constant, which is defined as $H \equiv \dot{a}/a$. Now we can rewrite the metric as

$$ds^2 = dt^2 - e^{2Ht}(d\kappa^2 + \zeta^2 d\Omega^2). \tag{B.33}$$

We have now all the elements to define the de Sitter spacetime as the maximally symmetric spacetime of constant positive curvature [31]. To find the metric we can repeat the process, which we have done for two-dimensional sphere. If we embed the n-dimensional de Sitter spacetime dS_n into the $n+1$-dimensional Minkowski spacetime, then the equation of the hypersurface of the curvature $\alpha > 0$ is (we must be careful with the signature, we chose sign $= (1, -1, \dots)$)

$$x_0^2 - \sum_{i=1}^{n} x_i^2 = -\alpha^2 \tag{B.34}$$

and the line element is

$$ds^2 = dx_0^2 - \sum_{i=1}^{n} dx_i^2. \tag{B.35}$$

The dimension of our universe is $n = 4$ (time plus three spatial dimensions). For such universe, let us assume a transformation from abstract coordinates

$(x_0, x_1, x_2, x_3, x_4)$ to coordinates called *flat slicing* (t, y_2, y_3, y_4), which describe dS_4 [31]

$$x_0 = \alpha \sinh \frac{t}{\alpha} + \frac{1}{2\alpha} r^2 e^{t/\alpha}, \; x_1 = \alpha \cosh \frac{t}{\alpha} - \frac{1}{2\alpha} r^2 e^{t/\alpha}, \; x_2 = e^{t/\alpha} y_2, x_3 = e^{t/\alpha} y_3, \tag{B.36}$$

where $\sum_{i=2}^{4} y_i^2 = r^2$.

In coordinates (t, y_2, y_3, y_4) the spacetime interval is [31]

$$ds^2 = dt^2 - e^{2t/\alpha} dy^2, \tag{B.37}$$

where $dy^2 = \sum_{i=2}^{4} dy_i^2$.

From comparison of (B.33) and (B.37) one can easily obtain $\alpha = 1/H = \sqrt{3/\Lambda}$.

Let us consider another transformation from $(x_0, x_1, x_2, x_3, x_4)$ to *static coordinates* (t, r, z_2, z_3, z_4) [31]

$$x_0 = \sqrt{\alpha^2 - r^2} \sinh \frac{t}{\alpha}, \; x_1 = \sqrt{\alpha^2 - r^2} \cosh \frac{t}{\alpha}, \; x_2 = r z_2, \; x_3 = r z_3, \; x_4 = r z_4, \tag{B.38}$$

where r is the radial coordinate, z_i gives the standard embedding the two-dimensional sphere in R^3. The metric in the static coordinates is

$$ds^2 = \left(1 - \frac{r^2}{\alpha^2}\right) dt^2 - \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{B.39}$$

Let us return to the result $\alpha = 1/H = \sqrt{3/\Lambda}$. This relation between α and Λ, valid for $n = 4$, can be obtained from the relation for general dimension n (e.g. [31])

$$\Lambda = \frac{(n-1)(n-2)}{2\alpha^2}. \tag{B.40}$$

Valid for $n \geq 2$. The line element of space with its dimension n is

$$ds^2 = \left(1 - \frac{r^2}{\alpha^2}\right) dt^2 - \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 - r^2 d\Omega_{n-2}^2. \tag{B.41}$$

From (B.39) it is easy to see that there is a horizon for $r_{hor} = \alpha$. This makes sense, because $\alpha = 1/H$, so $1 = H r_{hor}$. This is Hubble's law $v = Hr$, where v is the recessional velocity, in this particular case $v = 1$. If any object lies in a distance $r > r_{hor}$, then the observer, who stands at the origin, can never see the object.

B.3 BTZ black hole

The BTZ black hole is a solution of the Einstein equations, including the cosmological constant Λ, in (2+1) dimensions [29], [30]. The BTZ black hole in "Schwarzschild" coordinates is described by the metric [30]

$$ds^2 = f^2 dt^2 - f^{-2} d\mathscr{R}^2 - \mathscr{R}^2 (N^\theta dt + d\theta)^2, \tag{B.42}$$

where $-\infty < t < \infty$, $0 < \mathscr{R} < \infty$, $0 \leq \theta \leq 2\pi$ and

$$f \equiv \left(-M + \frac{\mathscr{R}^2}{l^2} + \frac{J^2}{4\mathscr{R}^2}\right)^{1/2}, \; N^\theta(\mathscr{R}) \equiv -\frac{J}{2\mathscr{R}^2}, \; |J| < Ml, \tag{B.43}$$

where M and J are constants of integration and their physical interpretations are the mass and the angular momentum, respectively.

It is straightforward to check that this metric satisfies the Einstein equations in (2+1) dimensions

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{l^2}g_{\mu\nu}, \tag{B.44}$$

where $\Lambda = -1/l^2$ is the cosmological constant. The metric (B.42) is stationary and axially symmetric, with Killing vectors ∂_t and ∂_θ and there are no other symmetries. Generally, in general relativity in (2+1) dimensions there are no Newtonian limit or propagating degrees of freedom [30].

The line element of BTZ black hole with zero angular momentum is (see also [3])

$$ds^2_{BTZ} = \left(\frac{\mathscr{R}^2}{l^2} - M\right)dt^2 - \left(\frac{\mathscr{R}^2}{l^2} - M\right)^{-1} d\mathscr{R}^2 - \mathscr{R}^2 d\theta^2. \tag{B.45}$$

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