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Optimality of function spaces for integral operators

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I would like to thank my supervisor, professor Pick, for always kindly helping with anything I needed.

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Abstract: In this work, we study the behaviour of linear kernel operators on rearrangement-invariant (r.i.) spaces. In particular we focus on the boundedness of such operators between various function spaces. Given an operator and a domain r.i. space Y, our goal is to find an r.i. space Z such that the operator is bounded from Y into Z, and, whenever possible, to show that the target space is optimal (that is, the smallest such space). We concentrate on a particular class of kernel operators denoted by S_a, which have important applications and whose pivotal instance is the Laplace transform. In order to deal properly with these fairly general operators we use advanced techniques from the theory of rearrangement-invariant spaces and theory of interpolation. It turns out that the problem of finding the optimal space for S_a can, to a certain degree, be translated into the problem of finding a "sufficiently small" space X such that a, the kernel of S_a, lies in X.

Keywords: rearrangement-invariant spaces, optimal range, integral operators, Peetre K-functional, Marcinkiewicz space.

Contents

Introduction

The *Laplace transform* is defined by

$$\mathcal{L}f(t) = \int_0^\infty f(s)e^{-ts}\,ds$$

for every function f on $(0,\infty)$ for which the integral makes sense, and for every $t \in (0,\infty)$. It is well known that $\mathcal{L}$ is an important integral operator with plenty of applications for example in the theory of differential equations, probability theory, investigation of spectral properties of pseudo-differential operators or the study of Fredholm integral equations. There exists a vast literature on properties of the Laplace transform and its applications. Of our particular interest is the recent paper [1] in which *optimality* of function spaces on which it acts is studied.

The Laplace transform can be viewed as a particular instance of a fairly more general class of *kernel* integral operators

$$Kf(t) = \int_0^\infty f(s)k(s,t)\,ds,$$

where k is an appropriate measurable function of two variables (in case of the Laplace transform one has, of course, $k(s,t) = e^{st}$).

Kernel operators are known to be very important in various branches of analysis and its applications and they have been widely studied.

In this text we focus on problems concerning action of kernel operators of special type, namely those defined by

$$S_af(t) = \int_0^\infty f(s)a(st)\,ds,$$

that is, operators having kernels of the form $k(s,t) = a(st)$, where a is an appropriate function of one variable. This class of operators, again, contains the Laplace transform as its particular example. Let us point out that various particular types of related operators have been studied by many authors. To name just a few, see, for instance, [2, 3, 4] and the references therein.

We will investigate fine properties of the operators of type S_a on the so-called rearrangement-invariant (r.i. for short) spaces, which are, roughly speaking, those function spaces, in which the decisive parameter is the *size* of a function (see the more precise definitions below). With a little more precision it can be said that the norms in r.i. spaces take into account only the measure of level sets of a given function.

In particular we focus on the question when a given operator is bounded from one r.i. space into another. Furthermore, given an operator of type S_a and having fixed a domain r.i. space Y, we shall find (or construct) a candidate for the target r.i. space, say Z, such that the operator S_a is bounded from Y into Z. Under some rather mild restrictions we show that the target space is optimal, by which we mean the smallest possible space within the given pool of competing spaces.

Let us note that the question of optimality of function spaces for various operators and embeddings has been undergoing a thorough scrutiny during recent years. It was studied in connection with Sobolev-type embeddings (see [5] and the extensive set of references given there) and also in the connection with the Laplace transform (see for example [6] or [1]).

In this text we focus on the question of optimality of function spaces for the operators S_a. Since this class of operators is rather general, we have to develop some new techniques. In particular we have to calculate the Peetre K functional for certain specific pairs of spaces (this has been known only partially) and we have to introduce Marcinkiewicz-type function spaces built on the norm of the dilation operator.

We shall use a combination of techniques from real analysis, functional analysis and the theory of interpolation, some of which we develop here. Our main results are new.

The text is structured as follows. In Chapter 1 we fix notation and collect all the preliminary stuff including all the definitions and basic knowledge about the function spaces, operators and related topics. In Chapter 2 we present background results that will be needed in the proofs of our main results. In particular, we introduce here spaces of endpoint Marcinkiewicz type and study their properties. We also establish important relations concerning the Peetre K-functional. Finally, all the main results are collected in Chapter 3.

1. Preliminaries

In this chapter we recall some definitions and basic properties of rearrangement-invariant spaces. The standard reference is [7].

We denote by m the Lebesgue measure on $(0,\infty)$ and define

$$\mathcal{M} = \{f : (0,\infty) \to [-\infty,\infty] \colon f \text{ is Lebesgue-measurable in } (0,\infty)\},$$

and

$$\mathcal{M}_+ = \{f \in \mathcal{M} : f \geq 0\}.$$

The *distribution function* $f_* : (0,\infty) \to [0,\infty]$ of a function $f \in \mathcal{M}$ is defined as

$$f_*(\lambda) = |\{x \in (0,\infty) : |f(x)| > \lambda\}|, \ \lambda \in (0,\infty),$$

and the *non-increasing rearrangement* $f^* : (0,\infty) \to [0,\infty]$ of a function $f \in \mathcal{M}$ is defined as

$$f^*(t) = \inf\{\lambda \in (0,\infty) : f_*(\lambda) \leq t\}, \ t \in (0,\infty).$$

The operation $f \mapsto f^*$ is monotone in the sense that $|f| \leq |g|$ a.e. in $(0,\infty)$ implies $f^* \leq g^*$. We define the *elementary maximal function* $f^{**} : (0,\infty) \to [0,\infty]$ of a function $f \in \mathcal{M}$ as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s.$$

While the operation $f \mapsto f^{**}$ is subadditive, that is, for any $f, g \in \mathcal{M}$ and $t \in (0,\infty)$ one has

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \tag{1.1}$$

for $f \mapsto f^*$ one only has the following property. Let $s, t \in (0,\infty)$ and $f, g \in \mathcal{M}$, then

$$(f+g)^*(s+t) \leq f^*(t) + g^*(s). \tag{1.2}$$

For reference, see [7, Chapter 2]. We recall that for every $f \in \mathcal{M}_+$ and every $t \in (0,\infty)$, one has

$$|\{s \in (0,\infty) : f(s) > f^*(t)\}| \leq t, \tag{1.3}$$

see [7, Chapter 2, Proposition 1.7]. The *Hardy-Littlewood inequality* asserts that if $f, g \in \mathcal{M}$, then

$$\int_0^\infty |f(t)g(t)|\, dt \leq \int_0^\infty f^*(t)g^*(t)dt. \tag{1.4}$$

We continue by defining rearrangement-invariant norm. We say that a functional $\rho : \mathcal{M} \to [0,\infty]$ is a *Banach function norm*, if for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}$, and every $\lambda \geq 0$, the following properties hold:

(P1) $\rho(f) = 0$ if and only if $f = 0$; $\rho(\lambda f) = \lambda\rho(f)$;

$\rho(f+g) \leq \rho(f) + \rho(g)$;

(P2) $f \leq g$ a.e. implies $\rho(f) \leq \rho(g)$;

(P3) $f_j \nearrow f$ a.e. implies $\rho(f_j) \nearrow \rho(f)$;

(P4) $\rho(\chi_G) < \infty$ for every $G \subset (0,\infty)$ of finite measure;

(P5) for every $G \subset (0,\infty)$ of finite measure there is a constant C_G such that $\int_G f(t)\,dt \le C_G \rho(f)$.

If also the property

(P6) $\rho(f) = \rho(g)$ whenever $f^* = g^*$,

holds, we say that ρ is a *rearrangement-invariant Banach function norm*, or just a *rearrangement-invariant norm.* If ρ is a rearrangement-invariant norm, then the collection

$$X = X(\rho) = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$$

is called a *rearrangement-invariant space.* The norm on the space X is given by $\|f\|_X = \rho(|f|)$. Note that $\rho(|f|)$ is defined for every $f \in \mathcal{M}$, and

$$f \in X \quad \Leftrightarrow \quad \rho(|f|) < \infty.$$

For a rearrangement-invariant norm ρ we define its *associate norm* by

$$\rho'(g) = \sup\left\{\int_0^\infty f(t)g(t)\mathrm{d}t : f \in \mathcal{M}_+,\, \rho(f) \le 1\right\} \quad \text{for } g \in \mathcal{M}_+$$

By [7, Chapter 1, Theorem 2.2] ρ' is also a rearrangement-invariant norm. Furthermore, by [7, Chapter 1, Theorem 2.7] it also holds that $\rho'' = \rho$. If $X = X(\rho)$ is a rearrangement-invariant space and ρ' is the norm associate to ρ, then $X(\rho')$ is the *associate space* of X and is denoted by X'.

If X, Y are rearrangement-invariant spaces, we denote by $X \hookrightarrow Y$ the continuous embedding of X into Y and by $T\colon X \to Y$ the boundedness of an operator T from X to Y. By [7, Chapter 1, Proposition 2.10] we have

$$X \hookrightarrow Y \iff Y' \hookrightarrow X'. \tag{1.5}$$

We say that the rearrangement-invariant space Y is *optimal* for the linear operator T and a given domain rearrangement-invariant space X if $T\colon X \to Y$ and for every rearrangement-invariant space Z such that $T\colon X \to Z$ it holds that $Y \hookrightarrow Z$. An operator which will be used extensively throughout this work is the dilation operator E_t defined for any $t \in (0,\infty)$ by the formula

$$E_t f(s) = f\left(\frac{s}{t}\right).$$

We recall that E_t is bounded on every rearrangement-invariant space for any $t \in (0,\infty)$ as stated in [7, Chapter 3, Proposition 5.11].

We define the *fundamental function*, φ_X, of a given rearrangement-invariant space X by $\varphi_X(t) = \|\chi_{(0,t)}\|_X$, $t \in (0,\infty)$. We say that a function $\varphi : (0,\infty) \to (0,\infty)$ is *quasiconcave* if it is non-decreasing, $\varphi(t) > 0, \forall t \in (0,\infty)$ and $\frac{t}{\varphi(t)}$ is non-decreasing. We say that the function φ is satisfies the Δ_2 condition, if it is non-decreasing and there exists a constant $C > 0$ such that $\varphi(2t) \le C\varphi(t)$ for all $t > 0$. By [7, Chapter 2, Corollary 5.3], the fundamental function of any rearrangement-invariant space is quasiconcave. Given a quasiconcave function φ, we define the rearrangement-invariant spaces $M_\varphi, \Lambda_\varphi$ with the rearrangement-invariant norms given by

$$\|f\|_{M_\varphi} = \sup_{t\in(0,\infty)} \varphi(t) f^{**}(t),\ f \in \mathcal{M} \tag{1.6}$$

and

$$\|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(t) \mathrm{d}\varphi(t),\ f \in \mathcal{M}. \tag{1.7}$$

These are indeed rearrangement-invariant norms as proven in [7, Chapter 2, Theorem 5.13]. It is also clear that both M_φ and Λ_φ have a common fundamental function which is equal to φ. For a rearrangement-invariant space X we denote

$$M(X) = M_{\varphi_X},\ \Lambda(X) = \Lambda_{\varphi_X},$$

where φ_X is the fundamental function of X. We recall that by [7, Chapter 2, Theorem 5.13] for a rearrangement invariant space X we have

$$\Lambda(X) \hookrightarrow X \hookrightarrow M(X),$$

with norm of both embeddings equal to 1. In other words, the spaces $M_\varphi, \Lambda_\varphi$ are respectively the largest and the smallest rearrangement-invariant space with the fixed fundamental function equal to φ.

One of the basic examples of rearrangement-invariant spaces are the L^p spaces, where $L^p = X(\rho_p)$ with

$$\rho_p(f) = \begin{cases} (\int_0^\infty f(t)^p \,\mathrm{d}t)^{\frac{1}{p}} & \text{if } 0 < p < \infty, \\ \operatorname{ess\,sup}_{t\in(0,\infty)} f(t) & \text{if } p = \infty \end{cases}$$

Let X_0 and X_1 be quasi-normed spaces, which are *compatible* in the sense that they are embedded in some common Hausdorff topological vector space (throughout this work we are working with $\mathcal{M}$). By $X_0 + X_1$ we denote the set of all functions $f \in \mathcal{M}$ for which there exists a decomposition $f = g + h$ such that $g \in X_0$ and $h \in X_1$. We equip the space $X_0 + X_1$ with the quasinorm

$$\|f\|_{X_0+X_1} = \inf_{f=g+h} (\|g\|_{X_0} + \|h\|_{X_1}),$$

where the infimum is taken over all such decompositions. For $f \in X_0 + X_1$ the *Peetre K-functional* is defined by

$$K(t, f; X_0, X_1) := \inf_{f=g+h} (\|g\|_{X_0} + t\|h\|_{X_1}) \quad \text{for } t > 0.$$

The function K as a function of variable t is increasing and concave on $(0, \infty)$. Furthermore, the function $t^{-1}K(t, f; X_0, X_1)$ is non-increasing on $(0, \infty)$. Observe that

$$\frac{1}{t} K(f, t; X_0, X_1) = K(f, \frac{1}{t}, X_1, X_0). \tag{1.8}$$

Recall that ([7, Chapter 2, Theorem 6.2]) in the case when $X_0 = L^1$ and $X_1 = L^\infty$, an exact formula for the K functional is known, namely,

$$K(f, t; L^1, L^\infty) = \int_0^t f^*(s)\, ds \quad \text{for } t \in (0, \infty) \text{ and } f \in (L^1 + L^\infty). \tag{1.9}$$

2. Background results

In this chapter, we shall establish some results on rearrangement-invariant spaces and K functionals, which will be useful later on when we apply them on a class of linear operators S_a.

Definition 2.1. Let $\varphi : (0,\infty) \to (0,\infty)$ be an increasing, everywhere positive function satisfying the Δ_2 condition. We define the functional $\|\cdot\|_{m_\varphi}$ for $f \in \mathcal{M}$ by the formula

$$\|f\|_{m_\varphi} = \sup_{t\in(0,\infty)} \varphi(t) f^*(t)$$

We define the space m_φ as the set of all functions for which the functional $\|f\|_{m_\varphi}$ is finite.

We shall now show that $\|\cdot\|_{m_\varphi}$ is a quasinorm and that it is equivalent to $\|\cdot\|_{M_\varphi}$ under some additional conditions.

Prosposition 2.2. *Let φ be as in Definition 2.1. Then $\|\cdot\|_{m_\varphi}$ is a quasinorm.*

Proof. We need only to show the modified triangle inequality, since the other properties of a quasinorm are clearly satisfied. Since φ satisfies the Δ_2 condition, there exists a $K > 0$ such that $\varphi(2t) \le K\varphi(t)$ for all $t > 0$. Let $f, g \in \mathcal{M}_+$. Then, using (1.2), we obtain

$$\begin{aligned}
\|f+g\|_{m_\varphi} &= \sup_{t\in(0,\infty)} \varphi(t)(f+g)^*(t) \\
&= \sup_{t\in(0,\infty)} \varphi\left(2\cdot\frac{t}{2}\right)(f+g)^*\left(\frac{t}{2}+\frac{t}{2}\right) \\
&\le \sup_{t\in(0,\infty)} K\varphi\left(\frac{t}{2}\right)\left(f^*\left(\frac{t}{2}\right)+g^*\left(\frac{t}{2}\right)\right) \\
&\le K\left(\sup_{t\in(0,\infty)} \varphi\left(\frac{t}{2}\right)f^*\left(\frac{t}{2}\right) + \sup_{t\in(0,\infty)} \varphi\left(\frac{t}{2}\right)g^*\left(\frac{t}{2}\right)\right) \\
&= K(\|f\|_{m_\varphi} + \|g\|_{m_\varphi}).
\end{aligned}$$

□

Definition 2.3. We say that a set $X \subset \mathcal{M}$ together with a linear functional $F : X \to [0,\infty)$ can be equivalently renormed with a rearrangement-invariant norm, if there exists a rearrangement invariant norm ρ and constants C_1 and C_2 such that

$$C_1\rho(|f|) \le F(f) \le C_2\rho(|f|) \quad \text{for } f \in \mathcal{M}$$

If that is the case, we'll consider X to be the rearrangement-invariant space with the norm given by ρ

Theorem 2.4. *Let φ be a quasi-concave function. Then m_φ can be equivalently renormed with a rearrangement-invariant norm $\|\cdot\|_{M_\varphi}$ if and only if there exists a constant $C > 0$ such that*

$$\int_0^t \frac{1}{\varphi(s)}\mathrm{d}s \le C\frac{t}{\varphi(t)} \quad \textit{for every } t \in (0,\infty). \tag{2.1}$$

Proof. Take $g \in \mathcal{M}$. If $\|g\|_{m_\varphi} = \infty$ then obviously $\|g\|_{M_\varphi} \leq \|g\|_{m_\varphi}$. Assume $\|g\|_{m_\varphi} = A \in \mathbb{R}$. Then from definition of m_φ we have

$$g^*(t) \leq \frac{A}{\varphi(t)} \quad \text{for every } t \in (0,\infty).$$

Then by definition of M_φ we have

$$\|g\|_{M_\varphi} = \sup_{t\in(0,\infty)} \frac{\varphi(t)}{t} \int_0^t g^*(s)\mathrm{d}s \leq \sup_{t\in(0,\infty)} \frac{\varphi(t)}{t} \int_0^t \frac{A}{\varphi(s)}\mathrm{d}s \leq CA = C\|g\|_{m_\varphi}.$$

Since $g^* \leq g^{**}$, clearly

$$\|g\|_{m_\varphi} \leq \|g\|_{M_\varphi},$$

and so m_φ can be equivalently renormed with $\|\cdot\|_{M_\varphi}$.

Now assume (2.1) doesn't hold. Clearly $\frac{1}{\varphi} \in m_\varphi$, but

$$\left\|\frac{1}{\varphi}\right\|_{M_\varphi} = \sup_{t\in(0,\infty)} \frac{\varphi(t)}{t} \int_0^t \frac{1}{\varphi(s)}\mathrm{d}s = \infty.$$

□

Now we characterize the K-functional for pair m_φ and L^∞. A similar result can be found in [8]. The author, however, assumes that φ is quasi-concave and that the condition (2.1) holds. On the other hand, his result is more general in a different direction, as he calculated the K-functional for pair m_φ and X for arbitrary rearrangement-invariant space X, so our results overlap somehow.

Theorem 2.5. *Let $\varphi\colon (0,\infty) \to (0,\infty)$ be an increasing, left continuous function satisfying the Δ_2 condition. Then*

$$\|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi} \leq K(f,t;m_\varphi,L^\infty) \leq 2\|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi} \tag{2.2}$$

for every $f \in \mathcal{M}$ and every $t \in (0,\infty)$.

Proof. Let $f \in (m_\varphi + L^\infty)$ and $t > 0$. Both L^∞ and m_φ norms are defined in terms of f^* so it will suffice to prove the assertion assuming that $f \geq 0$. First, decompose $f = f_0 + f_1$, where

$$f_0 = \begin{cases} f - f^*\left(\varphi^{-1}(t)\right) & \text{if } f > f^*\left(\varphi^{-1}(t)\right) \\ 0 & \text{otherwise.} \end{cases}$$

Then since φ is left continuous, we have

$$\begin{aligned} \sup_{0<s<\varphi^{-1}(t)} f^*(s)\varphi(s) &\geq \lim_{s\to\varphi^{-1}(t)_-} f^*(s)\varphi(s) \\ &= \lim_{s\to\varphi^{-1}(t)_-} f^*(s) \lim_{s\to\varphi^{-1}(t)_-} \varphi(s) \\ &\geq f^*(\varphi^{-1}(t))\varphi(\varphi^{-1}(t)) = f^*(\varphi^{-1}(t))t. \end{aligned}$$

And so from the definition of f_0 and the above calculation

$$t\|f_1\|_\infty \leq f^*(\varphi^{-1}(t))t \leq \sup_{0<s<\varphi^{-1}(t)} f^*(s)\varphi(s) = \|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi}. \tag{2.3}$$

We continue by estimating $\|f_0\|_{m_\varphi}$. By definition of f_0,

$$\|f_0\|_{m_\varphi} = \sup_{t\in(0,\infty)} \varphi(t) f_0^*(t) \le \sup_{0<s<\varphi^{-1}(t)} f^*(s)\varphi(s) = \|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi}. \tag{2.4}$$

Combining (2.3) and (2.4) we obtain

$$K(f, t; m_\varphi, L^\infty) \le 2\|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi}, \tag{2.5}$$

establishing the second inequality in (2.2). For the first one, once again, fix $f \in (m_\varphi + L^\infty)$ nonnegative and let $f = g + h$, where $g \in m_\varphi$ and $h \in L^\infty$. Firstly, we shall assert that

$$f^*(t) \le g^*(t) + \|h\|_\infty \quad \text{for every } t \in (0, \infty). \tag{2.6}$$

For $t \in (0,\infty)$, set $\lambda = g^*(t) + \|h\|_\infty$ and $y = |\{s \in (0,\infty), f(s) > \lambda\}|$. Then

$$\begin{aligned} y &= |\{s \in (0,\infty), g(s) + h(s) > \lambda\}| \\ &= |\{s \in (0,\infty), g(s) + h(s) > g^*(t) + \|h\|_\infty\}| \\ &\le |\{s \in (0,\infty), g(s) > g^*(t)\}| + |\{s \in (0,\infty), h(s) > \|h\|_\infty\}| \\ &= |\{s \in (0,\infty), g(s) > g^*(t)\}|, \end{aligned}$$

since the set $\{s \in (0,\infty), h(s) > \|h\|_\infty\}$ obviously has zero measure. By (1.3) we obtain $y \le t$. By definition of the decreasing rearrangement we get (2.6). Consequently, from subadditivity of supremum and because φ is increasing, we obtain

$$\begin{aligned} &\sup_{0<s<\varphi^{-1}(t)} f^*(s)\varphi(s) \le \sup_{0<s<\varphi^{-1}(t)} g^*(s)\varphi(s) + \sup_{0<s<\varphi^{-1}(t)} \|h\|_\infty \varphi(s) \\ &\le \sup_{0<s<\infty} g^*(s)\varphi(s) + \|h\|_\infty \varphi(\varphi^{-1}(t)) = \|g\|_{m_\varphi} + t\|h\|_\infty. \end{aligned} \tag{2.7}$$

Taking infimum over all such representations $f = g + h$, we arrive at

$$\|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi} \le K(f, t; m_\varphi, L^\infty),$$

as desired. The assertion now follows from the combination of (2.5) and (2.7). $\square$

3. Target spaces for one special class of linear operators

Definition 3.1. For $a \in \mathcal{M}_+$ we define the operator S_a by the formula

$$S_a f(t) = \int_0^\infty a(st) f(s) \mathrm{d}s$$

for those $f \in \mathcal{M}$ for which the integral on the right is defined.

Notice that S_a is a generalization of the Laplace transform, since S_a is the Laplace transform for $a(t) = e^{-t}$. In this chapter we will formulate a sufficient condition under which we can find a certain target space for the operator S_a when a domain space is fixed. We will also formulate a sufficient condition for optimality of said target space and then apply the results on S_a with a having some specific properties.

Definition 3.2. Assume that X is a rearrangement-invariant space. We denote by $E(X)$ the function given by

$$E(X)(t) = \frac{t}{\|E_t\|_X} \quad \text{for } t \in (0, \infty).$$

Lemma 3.3. *Let X, Y be rearrangement invariant spaces and $a \in \mathcal{M}$. Then*

$$S_a \colon X \to Y \iff S_a \colon Y' \to X'.$$

Proof. Thanks to the fact, that $X'' = X$ and $Y'' = Y$ it will suffice to show either of the implications. Assume $S_a \colon X \to Y$ and take $f \in Y'$, then by Fubini's theorem, we have

$$\begin{aligned} \|S_a f\|_{X'} &= \sup_{\|g\|_X \le 1} \int_0^\infty S_a(f) g = \sup_{\|g\|_X \le 1} \int_0^\infty S_a(g) f \\ &\le \sup_{\|g\|_X \le 1} \|S_a g\|_Y \|f\|_{Y'} = \|S_a\|_{X \to Y} \|f\|_{Y'}. \end{aligned}$$

□

Theorem 3.4. *If X is a rearrangement-invariant space, then $m_{E(X)}$ is well defined. Let $a \in \mathcal{M}_+$ be non-increasing. If $a \in X'$ then*

$$S_a \colon X \to m_{E(X)}.$$

If $a \notin X'$ then there is no rearrangement invariant space Y such that

$$S_a \colon X \to Y.$$

Proof. Since $E(X)$ is clearly a positive function, to show that $m_{E(X)}$ is well defined, we only need to show that it is non-decreasing and that it satisfies the Δ_2 condition. Fix $t \in (0, \infty)$ and notice that $E_t(f^*) = (E_t f)^*$. Now by definition of norm and from the rearrangement invariance of X we have

$$\|E_t\|_X = \sup_{\|f\|_X \le 1} \|(E_t f)^*\|_X = \sup_{\|f\|_X \le 1} \|E_t(f^*)\|_X. \tag{3.1}$$

Since $X = X''$, the Hardy-Littlewood inequality (1.4) gives

$$\|E_t(f^*)\|_X = \sup_{\|g\|_{X'} \le 1} \int_0^\infty f^*(\tfrac{s}{t}) g^*(s) \mathrm{d}s, \; f \in \mathcal{M} \tag{3.2}$$

Combining (3.1) with (3.2) and using the change of variables $y = \frac{s}{t}$ we obtain

$$\|E_t\|_X = \sup_{\|f\|_X \le 1} \sup_{\|g\|_{X'} \le 1} t \int_0^\infty f^*(y) g^*(ty) \mathrm{d}y,$$

from which it is easy to see that $E(X)$ is non-decreasing. Since $E_t(f^*) = (E_t f)^*$, it is also clear that $\|E_t\|_X$ itself is non-decreasing, which immediately gives

$$\|E_t\|_X \le \|E_{2t}\|_X,$$

therefore

$$E(X)(2t) \le 2E(X)(t),$$

which is the Δ_2 condition. We have shown that $m_{E(X)}$ is well defined. Now assume that $a \in X'$ is non-increasing. Let $f \in \mathcal{M}_+$. We first note that since a is non-increasing, so is $S_a f$. Thus, using the change of variables $st = u$ and the Hölder inequality, we obtain

$$\begin{aligned}
\|S_a f\|_{m_{E(X)}} &= \sup_{t \in (0,\infty)} \frac{t}{\|E_t\|_X} S_a f(t) = \sup_{t \in (0,\infty)} \frac{t}{\|E_t\|_X} \int_0^\infty a(st) f(s) \mathrm{d}s \\
&= \sup_{t \in (0,\infty)} \frac{t}{\|E_t\|_X} \frac{1}{t} \int_0^\infty a(u) E_t f(u) \mathrm{d}u \\
&\le \sup_{t \in (0,\infty)} \frac{1}{\|E_t\|_X} \|E_t f\|_X \|a\|_{X'} \le \|a\|_{X'} \|f\|_X.
\end{aligned}$$

Now assume $a \notin X'$ and that Y is a rearrangement-invariant space such that $S_a \colon X \to Y$. By definition of X' and since a is non-increasing, the Hardy-Littlewood inequality (1.4) gives

$$\infty = \sup \left\{ \int_0^\infty f^* a, \|f\|_X \le 1 \right\}.$$

We can find a sequence f_n such that

$$\int_0^\infty f_n^* a \ge n, \; \|f_n\|_X \le 1, \; n \in \mathbb{N}.$$

Then we have for $t \in [1, 2]$

$$S_a f_n^*(t) = \frac{1}{t} \int_0^\infty E_t(f_n^*) a \ge \frac{1}{t} \int_0^\infty f_n^* a \ge \frac{n}{2}.$$

Applying (P5) from the definition of rearrangement-invariant norm to the set $[1, 2]$ we obtain $\|S_a f_n(t)\|_Y \to \infty$ as n tends to infinity. That is in contradiction with $S_a \colon X \to Y$. □

Thanks to Theorem 2.4 we obtain the following corollary.

Corollary 3.5. *If X is a rearrangement-invariant space such that $a \in X'$ and the condition* (2.1) *holds for $\varphi = E(X)$, then*

$$S_a\colon X \to M_{E(X)}.$$

Lemma 3.6. *Let a be a non-increasing, non-negative function on $(0,\infty)$. Then*

$$(S_a f)^* \leq S_a(f^*) \quad \text{for all } f \in \mathcal{M}.$$

Proof. Taking $t > 0$ and $f \in \mathcal{M}$, we obtain, by the Hardy–Littlewood inequality (recall that a is non-increasing), that

$$\begin{aligned}(S_a f)^*(t) = (|S_a f|)^*(t) \leq (S_a|f|)^*(t) &= \int_0^\infty |f(s)|a(st)\mathrm{d}s \\ &\leq \int_0^\infty f^*(s)a(st)\mathrm{d}s = S_a(f^*)(t).\end{aligned}$$

□

Definition 3.7. Let X be a rearrangement-invariant space such that $\varphi_{X'}$, the fundamental function of X', is strictly increasing and unbounded. Let $\varphi\colon (0,\infty) \to (0,\infty)$ be quasi-concave, unbounded function with $\varphi(0_+) = 0$. Then we define the function ψ with the formula

$$\psi(t) = \varphi_{X'}^{-1}\left(\frac{1}{\varphi(t)}\right) \quad \text{for } t \in (0,\infty). \tag{3.3}$$

For $f \in \mathcal{M}$ and $t \in (0,\infty)$ we define the functions $\alpha(f)$, $\beta(f)$ with the formulas

$$\alpha(f)(t) = \int_0^{\psi(t)} f^*(s)\mathrm{d}s, \tag{3.4}$$

$$\beta(f)(t) = \frac{1}{\varphi(t)}\|f^*\chi_{(\psi(t),\infty)}\|_X, \tag{3.5}$$

and we set $R(f) = \alpha(f) + \beta(f)$.

In the rest of this chapter we will be working in a setting described in the Definition 3.7. This means that, unless stated otherwise, X, φ, ψ, α, β are as in the definition above.

Theorem 3.8. *Let W be a rearrangement-invariant space and set*

$$\rho(f) = \|Rf\|_W,\ f \in \mathcal{M}_+.$$

If the condition

$$\min\left\{\frac{1}{\varphi}, 1\right\} \in W \tag{3.6}$$

holds, then ρ is a rearrangement-invariant Banach function norm.

Proof. To prove the triangle inequality, fix $f_1, f_2 \in \mathcal{M}_+$ and $t \in (0,\infty)$. From the definition of associate norm we have

$$R(f_1+f_2)(t) = \int_0^{\psi(t)} (f_1+f_2)^* + \frac{1}{\varphi(t)} \sup\left\{\int_{\psi(t)}^{\infty} (f_1+f_2)^* g^*,\ \|g\|_{X'} \le 1\right\}$$

Take arbitrary $g \in X'$ with $\|g\|_{X'} \le 1$ and set

$$h(s) = \begin{cases} 1, & s \in (0,\psi(t)) \\ \frac{1}{\varphi(t)} g^*(s) & s \in [\psi(t),\infty) \end{cases}.$$

We know that $X' \hookrightarrow M(X')$ with the norm of the embedding equal to 1. Therefore we have

$$\sup_{s\in(0,\infty)} g^*(s)\varphi_{X'}(s) \le \sup_{s\in(0,\infty)} g^{**}(s)\varphi_{X'}(s) \le \|g\|_{X'} \le 1.$$

In particular for $s = \psi(t)$ we have

$$g^*(\psi(t)) \le \frac{1}{\varphi_{X'}(\psi(t))} = \frac{1}{\varphi_{X'}\varphi_{X'}^{-1}(\frac{1}{\varphi(t)})} = \varphi(t),$$

thus h is non-increasing. Now thanks to the subadditivity of $f \mapsto f^{**}$ we have

$$\int_0^u (f_1+f_2)^* \le \int_0^u f_1^* + f_2^*, \quad u \in (0,\infty).$$

Using Hardy's Lemma [7, Chapter 2, Proposition 3.6] we obtain

$$\int_0^\infty (f_1+f_2)^* h \le \int_0^\infty f_1^* h + f_2^* h. \tag{3.7}$$

From the definition of h it is clear that

$$\int_0^{\psi(t)} (f_1+f_2)^* + \frac{1}{\varphi(t)} \int_{\psi(t)}^\infty (f_1+f_2)^* g^* = \int_0^\infty (f_1+f_2)^* h,$$

which, in combination with (3.7), gives

$$\begin{aligned} &\int_0^{\psi(t)} (f_1+f_2)^* + \frac{1}{\varphi(t)} \int_{\psi(t)}^\infty (f_1+f_2)^* g^* \le \\ &\int_0^{\psi(t)} f_1^* + \frac{1}{\varphi(t)} \int_{\psi(t)}^\infty f_1^* g^* + \int_0^{\psi(t)} f_2^* + \frac{1}{\varphi(t)} \int_{\psi(t)}^\infty f_2^* g^*. \end{aligned}$$

Since this holds for all $g \in X'$ with $\|g\|_{X'} \le 1$, we have

$$R(f_1+f_2)(t) \le R(f_1)(t) + R(f_2)(t) \quad \text{for } t \in (0,\infty),$$

which, using the (P2) property of the norm in W, gives the triangle inequality.

The fact that $\rho(f) = 0 \iff f = 0$ holds trivially. Positive homogeneity is trivial. We've shown that (P1) holds. Next, (P6) holds obviously and (P2) and (P3) are direct consequences of the corresponding properties of $f \mapsto f^*$ and of $\|\cdot\|_X$ and $\|\cdot\|_W$.

To show (P4) we only need to show that $\rho(\chi_{(0,u)}) < \infty$ for any $u \in (0,\infty)$, because ρ is defined in terms of the non-increasing rearrangement and we know, that for a measurable set $E \subset (0,\infty)$ it holds that $(\chi_E)^* = \chi_{(0,|E|)}$. Fix $u \in (0,\infty)$, by the definition of α and β we have for $t \in (0,\infty)$

$$\alpha(\chi_{(0,u)})(t) = \min\{\psi(t), u\} \leq u \min\{\psi(t), 1\}$$

and

$$\beta(\chi_{(0,u)})(t) = \frac{1}{\varphi(t)}\|\chi_{(0,u)\cap(\psi(t),\infty)}\|_X = \frac{1}{\varphi(t)}\varphi_X(\max\{u-\psi(t),0\}) \leq \frac{1}{\varphi(t)}\varphi_X(u),$$

where φ_X denotes fundamental function of X. Furthermore, since $\lim_{t\to 0_+}\psi(t) = \infty$, there exists $\epsilon > 0$ such that $\beta(\chi_{(0,u)})(t) = 0$, for $t < \epsilon$, and $\frac{1}{\varphi(\epsilon)} \geq 1$. Since $\frac{1}{\varphi} \leq \frac{1}{\varphi(\epsilon)}$ on (ϵ,∞) we have

$$\beta(\chi_{(0,u)}) \leq \varphi_X(u) \min\left\{\frac{1}{\varphi}, \frac{1}{\varphi(\epsilon)}\right\} \leq \varphi_X(u)\frac{1}{\varphi(\epsilon)} \min\left\{\frac{1}{\varphi}, 1\right\}.$$

Thus, according to (3.6), $\beta(\chi_{(0,u)}) \in W$. To show that $\alpha(\chi_{(0,u)}) \in W$ we need to only show that there exists a constant C such that $\min\{\psi, 1\} \leq C\min\{\frac{1}{\varphi}, 1\}$, for which it is sufficient to show that there exists $t > 0$ and $C > 0$ such that for all $s > t$

$$\frac{1}{\varphi(s)}C \geq \psi(s) = \varphi_{X'}^{-1}(\frac{1}{\varphi(s)}),$$

which is equivalent to $\varphi_{X'}(\tau C) \geq \tau$, for $\tau \leq \frac{1}{\varphi(t)}$, which follows from quasi-concavity of $\varphi_{X'}$. Indeed, each quasi-concave function dominates the function $\min\{1,\tau\}$, $\tau \in (0,\infty)$, up to a multiplicative constant, and so we have

$$\varphi_{X'}(C\tau) \geq C_1 C\tau,$$

for all C and τ such that $C\tau \leq 1$ and for some C_1. If we set $C = \frac{1}{C_1}$, then we have

$$\varphi_{X'}(C\tau) \geq \tau \quad \text{for } \tau \leq \frac{1}{C}$$

We only need to find t such that $\frac{1}{\varphi(t)} \leq \frac{1}{C}$, which we can do since φ is unbounded. Now we have t and C as we wanted, therefore we have just proven that $\min\{1,\psi\}$ is dominated by $\min\{1,\frac{1}{\varphi}\}$ up to a constant. Now since both $\beta(\chi_{(0,u)}) \in W$ and $\alpha(\chi_{(0,u)}) \in W$, it obviously holds that $\rho(\chi_{(0,u)}) < \infty$ and thus (P4) holds.

It remains to show (P5). Let $E \subset (0,\infty)$ be of finite measure and let $f \in \mathcal{M}_+$. Then, since ψ is non-increasing, one has

$$\begin{aligned}
\rho(f) \geq& \|\alpha(f)\|_W \\
\geq& \|\chi_{(0,\psi(|E|))}(t)\int_0^{\psi(t)} f^*(s)\mathrm{d}s\|_W \\
\geq& \|\chi_{(0,\psi(|E|))}\|_W \int_0^{|E|} f^*(s)\mathrm{d}s \\
\geq& \|\chi_{(0,\psi(|E|))}\|_W \int_E f(s)\mathrm{d}s,
\end{aligned}$$

which establishes (P5). □

In the following theorem we will show that the norm ρ from Theorem 3.8, under some conditions, implicitly defines a space Z such that S_a acts boundedly from Y into Z. Note that R is defined only using the space X and the function φ. If we define $\varphi = E(X)$ then R is defined only using X. Therefore we can translate the problem of finding a target space for S_a with domain space Y fixed, into a problem of finding a space X such that some conditions, specified in the following theorem, are satisfied.

Theorem 3.9. *Let $Y \subset X + L^1$ be a rearrangement-invariant space, assume that φ is strictly increasing and set*

$$\rho(f) = \|Rf\|_{Y'},\ f \in \mathcal{M}_+.$$

Let a be a non-increasing non-negative function on $(0,\infty)$ such that if we set $T = S_a$, then

$$\begin{aligned} T&\colon X \to m_\varphi \\ T&\colon L^1 \to L^\infty. \end{aligned} \tag{3.8}$$

If the condition (3.6) holds for $W = Y'$, then ρ is a rearrangement-invariant Banach function norm. Furthermore, if we set Z to be the rearrangement-invariant space given by ρ', the norm associate to ρ, then

$$T\colon Y \to Z.$$

Proof. The fact that ρ is a rearrangement-invariant norm was proved in the preceding theorem, so we only need to show $T\colon Y \to Z$. To this end, we will need to calculate the K-functionals of spaces (X, L^1) and spaces (m_φ, L^∞). By Theorem 2.5 we have

$$K(f,t;m_\varphi,L^\infty) \approx \|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi},$$

and, by a simple modification of [8, Theorem 5.1], we have

$$K(f,t,L^1,X) \approx \int_0^{\varphi_{X'}^{-1}(t)} f^*(s)\mathrm{d}s + t\|f^*\chi_{(\varphi_{X'}^{-1}(t),\infty)}\|_X.$$

Fix arbitrary $f \in L^1 + L^\infty$ and $t \in (0,\infty)$. By (3.8) and the definition of the K-functional, we have

$$K(Tf,t,L^\infty,m_\varphi) \le CK(f,t,L^1,X)$$

for some constant C. Combining that with the well known equality

$$\frac{1}{t}K(f,t,L^\infty,m_\varphi) = K(f,\frac{1}{t},m_\varphi,L^\infty),$$

we obtain

$$\begin{aligned} &\sup_{0<u<\varphi^{-1}(\frac{1}{t})} (Tf)^*(u)\varphi(u) \le \\ &\le \frac{C}{t}\left(\int_0^{\varphi_{X'}^{-1}(t)} f^*(s)\mathrm{d}s + t\|f^*\chi_{(\varphi_{X'}^{-1}(t),\infty)}\|_X\right). \end{aligned}$$

In particular, since φ is quasiconcave and therefore continuous and $(Tf)^*$ is non-increasing, we can take $u = \varphi^{-1}(\frac{1}{t})$ and obtain

$$\begin{aligned} &\frac{1}{t}(Tf)^*(\varphi^{-1}(\frac{1}{t})) \leq \\ &\leq \frac{C}{t}\left(\int_0^{\varphi_{X'}^{-1}(t)} f^*(s)\mathrm{d}s + t\|f^*\chi_{(\varphi_{X'}^{-1}(t),\infty)}\|_X\right). \end{aligned}$$

Now since φ is a one-to-one mapping on $(0,\infty)$ and the above holds for every $t \in (0,\infty)$, substituting $\varphi^{-1}(\frac{1}{t}) = u$ we obtain, for all $u \in (0,\infty)$,

$$\varphi(u)(Tf)^*(u) \leq C\varphi(u)\int_0^{\psi(u)} f^*(s)\mathrm{d}s + C\|f^*\chi_{(\psi(u),\infty)}\|_X.$$

Dividing by $\varphi(u)$ and changing the variable u to t yields the following result:

$$(\exists C > 0)(\forall f \in X + L^1)(\forall t > 0) : (Tf)^*(t) \leq CRf(t). \tag{3.9}$$

Now we are ready to show $T\colon Y \to Z$. Take $f \in Y$. Then, by the definition of the associate norm and Lemma 3.6, we get

$$\|Tf\|_Z = \|(Tf)^*\|_Z \leq \|T(f^*)\|_Z = \sup_{\rho(g)\leq 1}\int_0^\infty T(f^*)(s)g(s)\mathrm{d}s.$$

Since $a \geq 0$ is non-increasing, so is $T(f^*)$, hence using the Hardy-Littlewood inequality, we arrive at

$$\begin{aligned} \|Tf\|_Z &\leq \sup_{\rho(g)\leq 1}\int_0^\infty T(f^*)(s)g(s)\mathrm{d}s \leq \sup_{\rho(g)\leq 1}\int_0^\infty T(f^*)(s)g^*(s)\mathrm{d}s \\ &= \sup_{\rho(g)\leq 1}\int_0^\infty f^*(s)T(g^*)(s)\mathrm{d}s. \end{aligned}$$

Now (3.9) together with Hölder's inequality gives

$$\begin{aligned} \|Tf\|_Z &\leq C\sup_{\rho(g)\leq 1}\int_0^\infty f^*(s)R(g^*)(s)\mathrm{d}s \leq \\ &\leq C\sup_{\rho(g)\leq 1}\|f\|_Y\|R(g^*)\|_{Y'} = C\|f\|_Y. \end{aligned}$$

□

In conjunction with Theorem 3.4 we can now formulate a corollary of the preceding theorem, which makes the result more manageable.

Corollary 3.10. *Let X be such that, for $\varphi = E(X)$, φ is a quasi-concave unbounded strictly increasing function with $\varphi(0_+) = 0$. Let $Y \subset X + L^1$ be a rearrangement-invariant space and let $a \in X' \cap L^\infty$ be non-increasing. Set*

$$\rho(f) = \|Rf\|_{Y'},\ f \in \mathcal{M}_+$$

and assume that (3.6) holds for $W = Y'$. Then ρ is a rearrangement-invariant norm. Moreover, if we set Z to be the rearrangement-invariant space determined by ρ', then $S_a\colon Y \to Z$.

Proof. Since $a \in L^\infty = (L^1)'$, $E(L^1) \equiv 1$ and $m_1 = L^\infty$, we have $S_a\colon L^1 \to L^\infty$, and since $a \in X'$ and $E(X) = \varphi$, we have $S_a\colon X \to m_\varphi$ thanks to Theorem 3.4. Now we can apply Theorem 3.9. □

The space Z obtained in Theorem 3.9 and Corollary 3.10 is the candidate for the optimal space for S_a and Y. Lemma 3.11 below gives a sufficient condition under which Z is indeed optimal. Note again, that R is defined only in terms of X and therefore the problem of finding the optimal space for S_a and for a fixed domain space Y can be translated into finding X such that the conditions specified in Corollary 3.10 and Lemma 3.11 hold.

Lemma 3.11. *Let Y and ρ be as in Theorem 3.9. If the condition (3.6) holds for $W = Y'$ and also the following condition holds*

$$(\exists C > 0)\,(\forall f \in \mathcal{M}) : \ \|S_a(f^*)\|_{Y'} \geq C\|R(f^*)\|_{Y'}, \tag{3.10}$$

then ρ is a rearrangement-invariant norm, $S_a\colon Y \to Z$ and Z is optimal for S_a and Y.

Proof. All assertions except for the optimality have already been proved in preceding theorems. To show optimality of Z, assume that there is a rearrangement-invariant space W such that $S_a : Y \to W$. By (3.10) there is a constant $C > 0$ such that, for any $g \in \mathcal{M}$,

$$\|g\|_{Z'} = \|R(g^*)\|_{Y'} \leq C\|S_a(g^*)\|_{Y'}.$$

Thanks to Lemma 3.3 we also have $T : W' \to Y'$, therefore we have a constant C_1 such that

$$\|g\|_{Z'} \leq C_1\|g\|_{W'}.$$

In other words, $W' \hookrightarrow Z'$. Therefore, by (1.5), also $Z \hookrightarrow W$, and thus Z is optimal. □

Lemma 3.12. *Let $a : (0,\infty) \to (0,\infty)$ be non-increasing and non-zero on at least some set of non-zero measure. Then there is a constant C such that, for all non-negative $f \in \mathcal{M}$,*

$$S_a(f^*)(t) \geq C \int_0^{\frac{1}{t}} f^*(s)\mathrm{d}s \quad \textit{for all } t \in (0,\infty).$$

Proof. Since a is non-increasing and not zero on at least some set of non-zero measure, there exists $u \in (0,1)$ such that $\inf_{s<u} a(s) = C_1 > 0$. Now if $f \in \mathcal{M}$ is non-negative and $t \in (0,\infty)$, then

$$\begin{aligned} S_a(f^*)(t) &= \int_0^\infty a(st)f^*(s)\mathrm{d}s \geq \int_0^{\frac{u}{t}} a(st)f^*(s)\mathrm{d}s \\ &\geq C_1 \int_0^{\frac{u}{t}} f^*(s)\mathrm{d}s \geq C_1 u \int_0^{\frac{1}{t}} f^*(s)\mathrm{d}s, \end{aligned}$$

where the last inequality follows from the fact that f^{**} is non-increasing. □

Lemma 3.13. *Let $X = L^\infty$ and $\varphi = E(L^\infty)$, that is $\varphi(t) = t$, $t \in (0,\infty)$. Then we have, for all $f \in L^1 + L^\infty$ and $t > 0$,*

$$\int_0^{\frac{1}{t}} f^* \leq Rf(t) \leq 2\int_0^{\frac{1}{t}} f^*.$$

Proof. Take $f \in L^1 + L^\infty$ and $t > 0$. First, we shall observe that $\varphi_{X'}(t) = t$, since $X' = L^1$, therefore $\psi(t) = \frac{1}{t}$. This means that

$$\alpha(f)(t) = \int_0^{\frac{1}{t}} f^*$$

and, since $\beta(f) \geq 0$, we can easily obtain the first inequality from the definition of R. Now, a simple calculation shows that

$$\beta(f)(t) = \frac{1}{t}\|f^*\chi_{(\frac{1}{t},\infty)}\|_\infty = \frac{1}{t}f^*\left(\frac{1}{t}\right) \leq \frac{1}{t}f^{**}\left(\frac{1}{t}\right) = \alpha(f)(t).$$

Therefore

$$Rf(t) = \alpha(f)(t) + \beta(f)(t) \leq \alpha(f)(t) + \alpha(f)(t) = 2\int_0^{\frac{1}{t}} f^*,$$

which establishes the second inequality. □

Now we are ready to generalize the result of E. Buriánková, D. E. Edmunds and L. Pick in [1, Theorem 3.4], where the optimal target space for the Laplace transform $\mathcal{L}$ is found, to the operators S_a. We will show that, in fact, exactly the same result holds for any non-trivial, non-increasing and non-negative function $a \in L^1 \cap L^\infty$.

Theorem 3.14. *Let $a \in L^\infty \cap L^1$ be non-trivial, non-negative and non-increasing. Let Y be a rearrangement-invariant space such that*

$$\min\left\{1, \frac{1}{t}\right\} \in Y'. \tag{3.11}$$

For $f \in \mathcal{M}$ and $t > 0$, set

$$\alpha(f)(t) = \int_0^{\frac{1}{t}} f^*$$

and

$$\rho(f) = \|\alpha(f)\|_{Y'}.$$

Then ρ is a rearrangement-invariant norm such that if we set Z to be the rearrangement-invariant space given by ρ', then

$$S_a \colon Y \to Z,$$

and Y is optimal for S_a and Y. Furthermore, if the condition (3.11) *does not hold, then there is no rearrangement-invariant space Z such that $S_a : Y \to Z$.*

Proof. The proof of the fact that ρ is a rearrangement-invariant norm is almost identical to that of Theorem 3.9, see also [1, Proposition 3.3]. We thus know from Corollary 3.10 and Lemma 3.13 that $S_a : Y \to Z$.

The optimality of Z is a direct consequence of Lemma 3.12, Lemma 3.13 and Lemma 3.11. Indeed, from Lemma 3.12 we have a constant $C > 0$ such that, for all $f \in \mathcal{M}$ and $t \in (0, \infty)$,

$$S_a f(t) \geq C \int_0^{\frac{1}{t}} f^*(s)\mathrm{d}s,$$

and thus from Lemma 3.13 we obtain a possibly different constant $C_1 > 0$ such that

$$S_a(f^*)(t) \geq C_1 R(f^*)(t).$$

Now the (P2) property of the norm in Y' implies

$$\|S_a(f^*)\|_{Y'} \geq C\|R(f^*)\|_{Y'},$$

whence Lemma 3.11 gives the optimality of Z.

It remains to show that if (3.11) does not hold, then there is no rearrangement-invariant space Z such that $S_a : Y \to Z$. To show that, assume there exists such Z. Then, by Lemma 3.3, we have $S_a : Z' \to Y'$, and so there exists a constant C such that

$$\|\chi_{(0,1)}\|_{Z'} \geq C\|S_a\chi_{(0,1)}\|_{Y'} = C\|\int_0^1 a(st)\mathrm{d}s\|_{Y'}.$$

Since a is integrable, bounded and non-zero on some set of non-zero measure and non-increasing, changing variables, we get that there is a constant $C' > 0$ such that

$$\int_0^1 a(st)\mathrm{d}s = \frac{1}{t}\int_0^t a(s)\mathrm{d}s \geq C' \min\left\{1, \frac{1}{t}\right\}.$$

Indeed, suppose first that $t \leq 1$. Then we find $u > 0$ such that $\inf_{s\in(0,u)} a(s) = C_1 > 0$. If $u \geq 1$, then simply

$$\frac{1}{t}\int_0^t a(s)\mathrm{d}s \geq \frac{1}{t}tC_1 = C_1.$$

If $u < 1$, then we have

$$\frac{1}{t}\int_0^t a(s)\mathrm{d}s \geq \frac{1}{t}\int_0^{\min\{t,u\}} a(s)\mathrm{d}s \geq \min\{\frac{1}{t}tC_1, \frac{1}{t}uC_1\} = uC_1.$$

Now, let $t > 1$. Then we set $C_2 = \int_0^1 a(s)\mathrm{d}s$ and observe that

$$\frac{1}{t}\int_0^t a(s)\mathrm{d}s \geq \frac{1}{t}C_2.$$

Combining these estimates, we arrive at

$$\int_0^1 a(st)\mathrm{d}s \geq C' \min\left\{1, \frac{1}{t}\right\},$$

where $C' = \min\{C_1, uC_1, C_2\}$. Therefore

$$\|\chi_{(0,1)}\|_{Z'} \geq CC'\|\min\left\{1, \frac{1}{t}\right\}\|_{Y'} = \infty.$$

This is in contradiction with (P4) property of the norm in Z, therefore Z is not a rearrangement-invariant space. □

Now we will find the optimal space for operators S_a with a not integrable. We will, however, require that a behaves similarly to $t^{-\frac{1}{q}}$ near infinity, for some $q \in (0, \infty)$.

Definition 3.15. For $p \in (0,\infty)$, we define the functions φ_p and a_p by formulas

$$\varphi_p(t) = t^{\frac{1}{p}}, \quad a_p = \min\{1, \frac{1}{\varphi_p}\} \quad \text{for } t \in (0,\infty).$$

Prosposition 3.16. *Let $q \in (0,\infty)$ and p be such that $\frac{1}{p} + \frac{1}{q} = 1$. Further, set $X = \Lambda_{\varphi_p}$ and assume that $Y \subset X + L^1$ and $a_q \in Y'$. Then $E(X) = p\varphi_q$. Further, if we set $\varphi = \varphi_q$ and*

$$\rho(f) = \|Rf\|_{Y'},\ f \in \mathcal{M}_+$$

and denote by Z the rearrangement-invariant space determined by ρ', then Z is optimal for S_{a_q} and Y.

Proof. First, fix $q \in (0,\infty)$, $t \in (0,\infty)$, $f \in \mathcal{M}$. It is easy to see that $(E_t(f))^* = E_t(f^*)$, therefore we have

$$\begin{aligned}
\|E_t f\|_X &= \int_0^\infty E_t(f^*)(s)\mathrm{d}\varphi_p(s) \\
&= \frac{1}{p}\int_0^\infty f^*\left(\frac{s}{t}\right)\frac{1}{s^{\frac{1}{q}}}\mathrm{d}s \\
&= \frac{t}{p}\int_0^\infty f^*(u)(tu)^{-\frac{1}{q}}\mathrm{d}u \\
&= \frac{t^{\frac{1}{p}}}{p}\|f\|_X,
\end{aligned}$$

where the last but one equality is obtained using the change of variables $s = tu$. This easily implies that $E(X) = p\varphi_q$. Set $\varphi = \varphi_q$ and $a_q = a$. We notice that $a \in X'$ which implies $a \in M_{\varphi_q} = m_{\varphi_q}$ (note that the equality is set-wise, the norms in the mentioned spaces are, however, equivalent), since (2.1) holds for φ_q. Thus, we have, by Theorem 3.4,

$$S_a : X = \Lambda_{\varphi_p} \to m_{E(X)} = m_{\varphi_q} = M_{\varphi_q}$$

and

$$S_a : L^1 \to L^\infty.$$

Now Theorem 3.9 gives $S_{a_q} : Y \to Z$. To prove optimality of Z, we only need to show that $S_a f^* \geq CRf^*$ for every $f \in X + L^1$ and some constant $C > 0$ and use Lemma 3.11. To that end, fix $f \in X + L^1$ and $t \in (0,\infty)$. From the definition of X and φ it is easy to calculate that $\psi(t) = \frac{1}{t}$. Then we have

$$\begin{aligned}
Rf^*(t) &= \int_0^{\frac{1}{t}} f^*(s)\mathrm{d}s + t^{-\frac{1}{q}}\int_{\frac{1}{t}}^\infty f^*(s)\mathrm{d}(\varphi_p(s)) \\
&= \int_0^{\frac{1}{t}} f^*(s)\mathrm{d}s + t^{-\frac{1}{q}}\frac{1}{p}\int_{\frac{1}{t}}^\infty f^*(s)s^{-\frac{1}{q}}\mathrm{d}s \\
&\geq \frac{1}{p}\left(\int_0^{\frac{1}{t}} f^*(s)\mathrm{d}s + \int_{\frac{1}{t}}^\infty f^*(s)(st)^{-\frac{1}{q}}\mathrm{d}s\right) \\
&= \frac{1}{p}\left(\int_0^\infty \min\left\{1,(ts)^{-\frac{1}{q}}\right\} f^*(s)\mathrm{d}s\right) \\
&= \frac{1}{p}S_a f^*(t),
\end{aligned}$$

which gives $S_a f^* \geq pRf^*$. Consequently, the assertion follows. $\square$

The preceding proposition can be now reformulated in a more accessible way.

Corollary 3.17. *Let* $a\colon (0,\infty) \to (0,\infty)$ *be non-increasing and bounded and let* $q \in (0,\infty)$*. Assume that* $Y \subset \Lambda_{\varphi_p} + L^1$ *is a rearrangement-invariant space such that* $a_q \in Y'$*. For* $f \in \mathcal{M}$*, set*

$$\rho(f) = \|S_{a_q}(f^*)\|_{Y'}.$$

If

$$0 < \lim_{t\to\infty} \varphi_q(t)a(t) < \infty,$$

then ρ *is a rearrangement-invariant norm such that the space determined by* ρ' *is optimal for* S_a *and* Y*.*

Proof. It is easy to show that a is equivalent to a_q in the sense that there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 a(t) \le \min\{1, \frac{1}{\varphi_q}\} \le C_2 a(t), \quad t \in (0,\infty).$$

Then we use Proposition 3.16 and realize that we can omit the multiplicative constant $\frac{1}{p}$ from the second term in Rf^*, with $X = \Lambda_{\varphi_p}$ and $\varphi = E(X)$, without losing properties of a rearrangement-invariant norm of ρ. Indeed, all the properties still hold since S_{a_q} and R are equivalent in the common sense, except for perhaps triangle inequality. But since a_q is non-increasing, Hardy's Lemma in conjunction with subadditivity of $f \mapsto f^{**}$ gives the following pointwise estimate for each $f_1, f_2 \in \mathcal{M}$ and $t \in (0,\infty)$

$$S_{s_q}((f_1 + f_2)^*)(t) \le S_{a_q}(f_1^*)(t) + S_{a_q}(f_2^*)(t),$$

whence the (P2) property of the norm in Y' gives the desired triangle inequality. □

As one can see from Corollary 3.17, if we set $a = a_q$ then the optimal space for S_a and Y is defined using the norm associate to

$$\rho(f) = \|S_a(f^*)\|_{Y'}. \tag{3.12}$$

A natural question arises. Why not define it as such for any a? Since if a is non-increasing, it is not hard to prove that ρ defined as in (3.12) is a rearrangement invariant norm under the following condition

$$S_a(\chi_{(0,1)}) = a^{**} \in Y'. \tag{3.13}$$

Notice that boundedness of S_a from Y into Z (defined in terms of R) is a consequence of $(S_a f)^* \le CRf$ for $f \in X + L^1$ and the optimality of Z is a consequence of $\|S_a(f^*)\|_{Y'} \ge C\|Rf\|_{Y'}$ for $f \in \mathcal{M}$. Both of these inequalities obviously hold if we replace Rf with $S_a(f^*)$. Therefore if we defined Z as the rearrangement-invariant space determined by the norm associate to ρ (defined in terms of S_a as in (3.12)), we would obtain the optimal space rather easily. This course of action is much more straightforward and requires a lot less work. However, it defines the optimal space using the function a, which is very problematic. The principal difficulty consists in the fact that under such definition it would be next to impossible to nail down the optimal range partner space for a given particular domain space. In this work we have avoided using a in definition of the target (and sometimes optimal) space and we instead used X, which can be very helpful. For example, in Theorem 3.14, we defined the optimal space using the function

$$\int_0^{\frac{1}{t}} f^*(s)\mathrm{d}s = \frac{1}{t} f^{**}(\tfrac{1}{t}),\ f \in \mathcal{M}.$$

A great deal of theory is known about the elementary maximal function f^{**}, not so much, however, is known about $S_a(f^*)$.

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