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Andrea Srnáková

Seasonal mortality and its application in life insurance

Department of Probability and Mathematical Statistics

Supervisor of the bachelor thesis:	prof. RNDr. Tomáš Cipra, DrSc.
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Author: Andrea Srnáková

Department: Department of Probability and Mathematical Statistics

Supervisor: prof. RNDr. Tomáš Cipra, DrSc., Department of Probability and Mathematical Statistics

Abstract: Assumptions like uniform distribution, constant force of mortality and the Balducci assumption frequently used for modeling mortality data do not reflect the variability of monthly death rates. Often a phenomenon of winter excess mortality occurs, which is not respected by these assumptions. We shall apply a seasonal mortality assumption, which uses non-negative trigonometric sums for modeling the distribution of monthly death rates. We then apply our findings to the Czech mortality data. We calculate monthly premiums in a short-term life insurance policy and compare the result with results given by the classical assumptions.

Keywords: life insurance, trigonometric sums, circular distribution, fractional ages, seasonal mortality

Contents

Introduction

The uncertainty of human life is something many institutions have to take into consideration. This topic is important primarily for insurance companies and banks. There is a number of models being used, however with the mortality data changing, new models are constantly being developed in the hopes of maximizing profit.

Reasons for contracting insurance are to protect one self against unexpected events and their negative consequences. Short term life insurance may not be as common as long term life insurance, but there is a number of situations where it's important. Examples include: traveling, taking part in a risky event, business loans and personal loans. In case of death of the insured individual, the insurance should cover the costs of the loss events.

The goal for this thesis is to look at various principles being used to model the mortality behavior and apply our findings to specific data in the case of a short term life insurance policy.

The first chapter summarizes theoretical principles used in life insurance as well as commonly used notations and formulas. We mention various types of common life insurance policies and ways of calculating corresponding net premiums.

In the second chapter we introduce the uniform distribution assumption, constant force of mortality assumption, Balducci assumption and deduce corresponding distribution functions and probability density functions. We predict an inadequate reflection of the variability of mortality data throughout the year. Therefore we then introduce the seasonal mortality assumption developed by Fernández-Durán [2004]. This assumption uses non-negative trigonometric sums (shortly NNTS) for modeling circular distribution, for which the distribution function and density function is also derived. We elaborate on fitting the NNTS distribution using the method of maximum likelihood. The equations for the calculation of net single premiums under these assumptions are also deduced in this chapter.

In the third chapter we work with data provided by the Czech Statistical Office. We analyze the behavior of monthly mortality rates throughout the year. Graphically comparing the empirical data with the introduced typical assumptions on fractional ages confirms the previous hypothesis about the variability of the data not being reflected when using these assumptions. We estimate the parameters for fitting the data using a probability density function of the distribution based on non-negative trigonometric sums. We are thereby able to calculate monthly premiums using the seasonal mortality assumption and compare them with results using classical assumptions.

1. Actuarial Principles of Life Insurance

1.1 Mortality in Life Insurance

Life insurance is a form of insurance in which a person makes regular payments to an insurance company, in return for a sum of money to be paid to this person after a period of time, or to other beneficiaries if this person dies.

The definition of mortality we will be using is the number of deaths in a given time and place. Mortality rate is the proportion of deaths to population. These statistical data are used for constructing life tables. Life tables are standard tools used in life insurance. They contain probabilities of a person aged x dying within one year. When underwriting a life insurance policy, the insurer has to take into consideration mainly the age of the individual and the probabilities of death within a time period.

1.1.1 Probabilities in Life Insurance

We would like to examine the future lifetime of a person aged x years (see Gerber [1990]). This is a random variable which we'll denote as *T(x)*, shortly *T*.
T is usually not an integer. Let's consider

$$T(x) = K(x) + S(x), \tag{1.1}$$

where K is the amount of integer years the person lives and S is a fraction of the last year of their life.
The probability distribution function of T is given by the probability that the lifetime of a x aged person will be less than or equal to t years. We well denote this distribution function as

$$G(t) = P(T \leq t) \text{ for any } t \geq 0. \tag{1.2}$$

We can expect G to be continuous and available from suitable life tables of a population.
We will denote the probability density of T as the probability that the person will die between the time $x+t$ and $x+t+dt$ for infinitesimal dt:

$$g(t)dt = P(t < T < t + dt). \tag{1.3}$$

We will be using the following usual notation. In the case t=1, the index t is not written:
The probability of a person aged x dying within t years:

$$_tq_x = G(t). \tag{1.4}$$

The probability of a person aged x surviving at least t years:

$$_tp_x = P(T > t) = 1 - {_tq_x}. \tag{1.5}$$

The probability of a x aged person surviving s years and dying withing the next t years:

$$ {}_{s|t}q_x = P(s < T < s+t) = G(s+t) - G(s) = {}_{s+t}q_x - {}_sq_x. \tag{1.6} $$

The conditional probability of a person dying within t years, after surviving s years:

$$ {}_tp_{x+s} = P(T > s+t|T > s) = \frac{1 - G(s+t)}{1 - G(s)}. \tag{1.7} $$

The conditional probability of a person surviving t more years, after surviving s years before:

$$ {}_tq_{x+s} = P(T \leq s+t|T > s) = \frac{G(s+t) - G(s)}{1 - G(s)}. \tag{1.8} $$

The expected lifetime of a x aged person:

$$ {}^oe_x = E[T] = \int_0^\infty tg(t)dt. \tag{1.9} $$

The force of mortality is defined as

$$ \mu_{x+t} = \frac{g(t)}{1 - G(t)} = -\frac{d}{dt}\ln(1 - G(t)) = -\frac{d}{dt}\ln({}_xp_t) \tag{1.10} $$

which implies

$$ {}_xp_t = e^{-\int_0^t \mu_{x+s}ds}. \tag{1.11} $$

The probability of dying between t and t+dt:

$$ P(t < T < t+dt) = {}_tp_x\mu_{x+t}dt. \tag{1.12} $$

The random variable $K(x)$ in (1.1) is defined as the curated future lifetime of a person aged x. The probability distribution of $K(x)$ is

$$ P(K(x) = k) = P(k \leq T < k+1) = {}_kp_xq_{x+k}. \tag{1.13} $$

For $K(x)$ and $S(x)$, the following equation holds

$$ P(S(x) \leq s|K(x) = k) = \frac{{}_sq_{x+k}}{q_{x+k}}. \tag{1.14} $$

1.1.2 The Principle of Equivalence

The insurer of a life insurance wants to avoid generating loss L. Loss is the difference between the present values of benefits and premiums. The principle of equivalence states that the present value of the benefits and premiums should be equal, making the expected value of the loss equal to 0.

$$ E[L] = 0. \tag{1.15} $$

1.2 Actuarial formulas

1.2.1 Standard Life Insurance Formulas

Firstly let's consider a fixed annual effective interest rate i, and an equivalent annual nominal interest rate $i^{(m)}$ compounded m times per year.

$$(1+\frac{i^{(m)}}{m})^m = 1+i \tag{1.16}$$

If we consider continuous compounding, we get the force of interest δ.

$$\delta = \lim_{m\to\infty} i^{(m)} \tag{1.17}$$

The following relations hold:

$$\delta = \ln(1+i)$$
$$e^{\delta} = 1+i$$

$$e^{-\delta} = v. \tag{1.18}$$

Let d, defined as

$$d = \frac{i}{1+i} \tag{1.19}$$

be interpreted as interest in advance(i discounted). We get the following equations for $d^{(m)}$ which is the equivalent nominal interest in advance compounded m times per year:

$$d^{(m)} = \frac{i^{(m)}}{1+i^{(m)}/m} \implies \frac{1}{d^{(m)}} = \frac{1}{m} + \frac{1}{i^{(m)}} \tag{1.20}$$

from which we obtain

$$\lim_{m\to\infty} d^{(m)} = \lim_{m\to\infty} i^{(m)} = \delta. \tag{1.21}$$

We denote Z as the present value of the sum insured. The payment is discounted using a fixed interest rate i, so-called technical interest rate. The net single premium is *E(Z)*, the expected present value of the sum insured. We will firstly look at some insurance products with the assumption that the insured sum is payable at the end of the year, if not stated otherwise. (Gerber [1990])

- Whole life insurance
  In the case of a whole life insurance, a fixed monetary unit is payable at the end of the year of death (K+1). The present value of the sum insured is

  $$Z = v^{(K+1)}. \tag{1.22}$$

  Since the time of death is a random variable, Z is also random. To calculate the net single premium, we must know the distribution of Z.

  $$P(Z = v^{(K+1)}) = P(K = k) = {}_kp_x q_{x+k} \text{ for } k = 0, 1, 2.... \tag{1.23}$$

  We denote the net single premium as A_x,

  $$A_x = E[Z] = E[v^{(K+1)}] = \sum_{k=0}^{\infty} v^{(K+1)}{}_kp_x q_{x+k}. \tag{1.24}$$

- Term insurance of duration n
  The sum insured is payable only in the case of death occuring within n years and nothing is payed if the person survives n years. The present value of the insured sum is

$$Z = \begin{cases} v^{(K+1)} & \text{for } K = 0, 1, ..., n-1 \\ 0 & \text{for } K = n, n+1, ... \end{cases} \tag{1.25}$$

  We denote the net single premium as $A^1_{x:\overline{n}|}$:

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{(k+1)}{}_k p_x q_{x+k}.$$

- Pure endowment of duration n
  The payment occurs if the person is alive after n years. Otherwise no payment is made.

$$Z = \begin{cases} 0 & \text{for } K = 0, 1, ..., n-1 \\ v^n & \text{for } K = n, n+1, ... \end{cases} \tag{1.26}$$

  We denote the net single premium as $A_{x:\overset{1}{\overline{n}|}}$:

$$A_{x:\overset{1}{\overline{n}|}} = v^n{}_n p_x. \tag{1.27}$$

- Endowment
  The payment occurs at the end of the year of death if $K < n$. Otherwise, if the person survives n years, it occurs at the end of the nth year. This insurance type is a combination of a term insurance of duration n and a pure endowment of duration n. Giving us the present value of the sum insured as

$$Z = \begin{cases} v^{K+1} & \text{for } K = 0, 1, ..., n-1 \\ v^n & \text{for } K = n, n+1, ... \end{cases} \tag{1.28}$$

  The net single premium is denoted as$A_{x:\overline{n}|}$. Since Z can be expressed as the sum of the two previously mentioned insurance types, the net single premium can be calculated as

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}. \tag{1.29}$$

- m year deferred whole life insurance
  The payment occurs at the end of the year of death if $K \geq m$, otherwise nothing is payed.

$$Z = \begin{cases} 0 & \text{for } K = 0, 1, ..., m-1 \\ v^{K+1} & \text{for } K = m, m+1, ... \end{cases} \tag{1.30}$$

  The net single premium is

$${}_{m|}A_x = {}_m p_x v^m A_{x+m}. \tag{1.31}$$

- Insurance payable at time T

$$Z = v^T. \tag{1.32}$$

The net single premium $\bar{A}_x$ must be calculated continuously as follows

$$\bar{A}_x = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt. \tag{1.33}$$

where ${}_tp_x\mu_{x+t}$ is the probability of death occurring between t and $t+dt$ mentioned in (1.12). From the equation $T = K + S$ we can express the net single premium as

$$\bar{A}_x = E[v^{K+1}]E[(1+i)^{1-S}] = \frac{i}{\delta} A_x. \tag{1.34}$$

1.2.2 Standard Life Annuities Formulas

We will firstly mention some formulas, which will be used in specific life annuity products. (Gerber [1990])
A series of n payments of 1 unit made at the beginning of each time period is called an annuity-due. The present value is

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + ... + v^n = \frac{1 - v^n}{d}. \tag{1.35}$$

A different approach to represent $\ddot{a}_{\overline{n}|}$ could be as a difference of a perpetuity starting at time 0 and a perpetuity starting at time 1. We obtain the following relationship: $\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{\infty}|} - v^n \ddot{a}_{\overline{\infty}|}$
A series of n payments made at the end of each period is called an immediate annuity. The present value is

$$a_{\overline{n}|} = v + v^2 + ... + v^n = \frac{1 - v^n}{i}. \tag{1.36}$$

We denote Y as the present value of a stream of payments the beneficiary obtains while alive. Y depends on the random variable T, making Y a random variable. The net single premium is E(Y).

- Whole life annuity-due
  Beneficiary receives one unit annually at the beginning of a period until he dies.

$$Y = 1 + v + v^2 + ... + v^K = \ddot{a}_{\overline{K+1}|} = \sum_{k=0}^{\infty} v^k \mathbb{1}_{\mathrm{K \geq k}}. \tag{1.37}$$

To calculate the net single premium, we must first express the distribution of Y.

$$P(Y = \ddot{a}_{\overline{K+1}|}) = P(K = k) = {}_kp_x q_{x+k}. \tag{1.38}$$

Then the net single premium $\ddot{a}_x$ is

$$\ddot{a}_x = E[Y] = E[\ddot{a}_{\overline{K+1}|}] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{K+1}|} {}_kp_x q_{x+k}. \tag{1.39}$$

If we consider

$$Y = \frac{1 - v^{K+1}}{d} = \frac{1 - Z}{d}, \tag{1.40}$$

then the net single premium can be also written as

$$\ddot{a}_x = \frac{1 - A_x}{d}. \tag{1.41}$$

- n-year life annuity-due
  The beneficiary receives annually in advance one unit while surviving the next n years. The present value of Y is

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & \text{for } K = 0, 1, ..., n-1 \\ \ddot{a}_{\overline{n}|} & \text{for } K = n, n+1, ... \end{cases} \tag{1.42}$$

  The net single premium is

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|\,n}p_x q_{x+k} + \ddot{a}_{\overline{n}|\,n}p_x. \tag{1.43}$$

- Immediate life annuity
  In case of immediate annuities, the beneficiary makes payments at the end of time periods:

$$Y = v + v^2 + v^3 + ... + v^K = a_{\overline{K}|}. \tag{1.44}$$

  The net single premium is

$$a_x = \ddot{a}_x - 1. \tag{1.45}$$

- m year deferred life annuity-due
  The beneficiary receives annual payments in advance starting at year m.

$$Y = \begin{cases} 0 & \text{for } K = 0, 1, .., m-1 \\ v^m + v^{m+1} + ... + v^K & \text{for } K = m, m+1, ... \end{cases} \tag{1.46}$$

  The net single premium is

$$_{m|}\ddot{a}_x = {}_np_x v^m \ddot{a}_{x+m}. \tag{1.47}$$

- Payments made m times per year
  Payments $1/m$ are made at the beginning of each time segment. We can denote the net single premium due to (1.41)

$$\ddot{a}_x^{(m)} = \frac{1}{d^{(m)}} - \frac{1}{d^{(m)}} A_x^{(m)}. \tag{1.48}$$

1.2.3 Net Premium Calculation

We define a *net premium* as the premium satisfying the principle of equivalence. The insured can pay a single net premium at the beginning, constant periodic *level* premiums, or varying periodic premiums. In the case of a single premium, we have already described the net single premium calculations in the previous section. Let's list ways of calculating constant annual premiums in various types of insurance. (Gerber [1990])

- Whole Life Insurance
  The insured pays net annual premiums at the beginning of each year and a receives an insured sum of 1 at time $K+1$. We denote the annual premiums as P_x :
  $$L = v^{K+1} - P_x \ddot{a}_{\overline{K+1}|} \implies P_x = \frac{A_x}{\ddot{a}_x}. \quad (1.49)$$

- Term Insurance of duration n
  The insured pays net annual premiums at the beginning of each year for n years. He receives the insured amount at time $K+1$ in case of death within n years:
  $$L = \begin{cases} v^{K+1} - P^1_{x:\overline{n}|} \ddot{a}_{\overline{K+1}|} & \text{for } K = 0, 1, ..., n-1 \\ -P^1_{x:\overline{n}|} & \text{for } K = n, n+1, ... \end{cases} \quad (1.50)$$
  Then the net annual premium is:
  $$P^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}. \quad (1.51)$$

- Pure Endowment of duration n
  If the insured is alive after n years, the insured amount is payed:
  $$L = \begin{cases} -P_{x:\overset{1}{\overline{n}|}} \ddot{a}_{\overline{K+1}|} & \text{for } K = 0, 1, ...n-1 \\ v^n - P_{x:\overset{1}{\overline{n}|}} \ddot{a}_n & \text{for } K = n, n+1, ... \end{cases} \quad (1.52)$$
  Then the net annual premium is
  $$P_{x:\overset{1}{\overline{n}|}} = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}. \quad (1.53)$$

- Endowment
  The annual net premium in this insurance type explained in 1.2.1 is
  $$P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}. \quad (1.54)$$

2. Seasonal mortality in life insurance

The seasonal mortality effect can be observed in many countries, with usual reports of a 5% to 30% winter excess mortality (Healy [2003]). What interest us is the mathematics in this occurrence, specifically applied to short life insurance. Short term life insurance usually covers a period less than one year, with monthly premiums or a single premium payed in advance.

One of the possible approaches to this problem is described in Fernández-Durán and Gregorio-Domínguez [2015]. This paper addresses the problem of calculating net premiums in short term life insurance, taking into consideration seasonal mortality. The authors approach this problem by using circular distributions based on non-negative trigonometric sums (NNTS circular distributions) to model monthly mortality data.

2.1 Typical assumptions for fractional ages

The random variables which we will be working with $T(x), K(x), S(x)$ have been defined in section 1.1.1. $S(x) = T(x) - K(x)$ being the fraction of a person's last year. The seasonal mortality assumption will be expressed using the probability density function of $S(x)$ conditional on $K(x) = k$ for a person born in month b

$$f^b_{S(x)|K(x)=k}(s) \text{ for } s \in (0,1) \tag{2.1}$$

defined for k an integer and $s \in (0,1)$ as

$$f^b_{S(x)|K(x)=k}(s) = \frac{d}{ds} F^b_{S(x)|K(x)=k}(s), \tag{2.2}$$

where the distribution function is defined as

$$F^b_{S(x)|K(x)=k}(s) = P(S(x) \leq s | K(x) = k) =$$

$$= \frac{P(S(x) \leq s \cap K(x) = k)}{P(K(x) = k)} = \frac{P(k < T(x) \leq k + s)}{P(k < T(x) \leq k + 1)} = \frac{F_{T(x+k)}(s)}{F_{T(x+k)}(1)}. \tag{2.3}$$

Let's consider three assumptions often used for fractional ages (see Fernández-Durán and Gregorio-Domínguez [2015] or Gerber [1990]) which use life tables to compute necessary probabilities. All these assumptions imply the probability of death that do not dependent on b:

- Uniform distribution of deaths

  Let's assume uniform distribution for fractional ages. From (2.2) and (2.3) we can express the distribution function of $T(x + s)$ and the probability density function for the fractional ages as follows

$$F^b_{S(x)|K(x)=k}(s) = \frac{F_{T(x+k)}(s)}{F_{T(x+k)}(1)} = s,$$

i.e.

$$F_{T(x+k)}(s) = sF_{T(x+k)}(1). \tag{2.4}$$

One obtains obviously

$$f^b_{S(x)|K(x)=k}(s) = f_{S(x)|K(x)=k}(s) = 1. \tag{2.5}$$

We can conclude that the probability of death in this assumption does not depend on the month of birth and random variables $S(x)$ and $K(x)$ are independent.

- Constant force of mortality

Assuming for $0 < s < 1$ that μ_{x+u} is constant, let's denote it by μ. From (1.10) and (1.11) we can express the force of mortality and the probability of an individual surviving a fraction of a year s (see Gerber [1990]):

$$\mu = -\ln p_x, \tag{2.6}$$

$$_sp_x = e^{-\mu s} = (p_x)^s. \tag{2.7}$$

From the equations above and (1.14) we obtain the following relationship

$$P(S(x) \leq s|K(x) = k) = \frac{1 - {_sp_{x+k}}}{q_{x+k}} = \frac{1 - (p_{x+k})^s}{q_{x+k}}. \tag{2.8}$$

The distribution function $F_{T(x+s)}(1)$ represents the probability of a person aged $x+k$ surviving one year at most. This can also be also expressed as $1 - p_{x+k} = q_{x+k}$. Applying this to our desired distribution function we obtain

$$\frac{F_{T(x+k)}(s)}{F_{T(x+k)}(1)} = \frac{1 - (p_{x+k})^s}{q_{x+k}} = \frac{1 - (1 - F_{T(x+k)}(1))^s}{F_{T(x+k)}(1)},$$

i.e.

$$F_{T(x+k)}(s) = 1 - (1 - F_{T(x+k)}(1))^s. \tag{2.9}$$

Substituting $F_{T(x+s)}(1)$ with $1 - p_{x+k}$ and q_{x+k} results in

$$F^b_{S(x)|K(x)=k}(s) = \frac{1 - (p_{x+k})^s}{q_{x+k}}. \tag{2.10}$$

We can easily calculate the derivative, considering $p^s_{x+k} = e^{-\mu s}$

$$f^b_{S(x)|K(x)=k}(s) = \frac{d}{ds}\frac{1 - e^{-\mu s}}{q_{x+k}} = \frac{1}{q_{x+k}}\mu e^{-\mu s}. \tag{2.11}$$

We see that $S(x)$ and $K(x)$ are not independent, but the probability of death in this assumption again does not depend on the month of birth.

- Balducci assumption

  The Balducci assumption assumes the linearity of the reciprocal of the survival function (see Gerber [1990] or Hossain [2011]) .

$$ {}_{1-s}q_{x+s} = (1-s)q_x. \tag{2.12} $$

  Hence we can write

$$ {}_s p_x = \frac{p_x}{{}_{1-s}p_{x+s}} = \frac{1-q_x}{1-(1-s)q_x}. \tag{2.13} $$

  To calculate the force of mortality, we need the following derivative

$$ \frac{d}{ds} \ln {}_s p_x = -\frac{1-(1-s)q_x}{1-q_x} \frac{1-q_x}{(1-(1-s)q_x)^2} q_x. \tag{2.14} $$

  Then we use (1.10) to obtain

$$ \mu_{x+s} = \frac{q_x}{1-(1-s)q_x}. \tag{2.15} $$

  The following relationship

$$ {}_s q_{x+k} = 1 - {}_s p_{x+k} = 1 - \frac{1-q_{x+k}}{1-(1-s)q_{x+k}} = \frac{s q_{x+k}}{1-(1-s)q_{x+k}} \tag{2.16} $$

  and (1.14) lead to

$$ P(S(x) \leq s | K(x) = k) = \frac{s}{1-(1-s)q_{x+k}} = \frac{F_{T(x+k)}(s)}{F_{T(x+k)}(1)}. \tag{2.17} $$

  We can express the distribution function of $T(x+k)$ for fractional ages as

$$ F_{T(x+k)}(s) = \frac{sF_{T(x+k)}(s)}{1-(1-s)F_{T(x+k)}(1)}. \tag{2.18} $$

  The derivative of the distribution function (3.17) gives the probability density function

$$ f^b_{S(x)|K(x)=k}(s) = f_{S(x)|K(x)=k}(s) = \frac{1-q_{x+k}}{(1-(1-s)q_{x+k})^2}. \tag{2.19} $$

  In conclusion, the random variables $S(x)$ and $K(x)$ are not independent in the Balducci assumption and the probabilities of death does not depend on b.

When these assumptions are used to calculate mortality rates for each month, the variability typical in data collected throughout the year is not reflected. To reflect the varying mortality rates, a seasonal mortality assumption was introduced.

2.2 Seasonal mortality assumption

Fernández-Durán [2004] has developed circular distributions based on non-negative trigonometric sums, following research done by Fejér [1915] and Dimitrov [2002]. We will be using these circular distributions (shortly NNTS) to model mortality. The random variable $S(x)|K(x) = k$ will be modeled as a circular random variable (see Fernández-Durán and Gregorio-Domínguez [2015]). The distribution for a circular random variable θ measured in radians is expressed by Fernández-Durán [2004] as a non-negative trigonometric sum

$$f_\theta(\theta; \underline{c}, M) = || \sum_{k=0}^{M} c_k e^{ik\theta} ||^2, 0 \leq \theta \leq 2\pi, \tag{2.20}$$

which can also be written as

$$f_\theta(\theta; \underline{c}, M) = \sum_{k=0}^{M} \sum_{m=0}^{M} c_k \bar{c}_m e^{i(k-m)\theta}, 0 \leq \theta \leq 2\pi, \tag{2.21}$$

where $\underline{c}$ and M are parameters, (c_k are complex numbers), M is the order of the trigonometric sum and $i = \sqrt{-1}$.

To express parameters of the probability density function given the complex numbers c_k, we use the following theorem by Dimitrov [2002]:

Theorem **:** The trigonometric polynomial

$$T(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos_k \theta + b_k \sin_k \theta) \tag{2.22}$$

of order n is non-negative for every real θ if and only if there exist complex numbers c_k, $k = 0, 1, ..., n$ such that

$$a_0 = \sum_{k=0}^{n} |c_k|^2, \tag{2.23}$$

$$a_k - ib_k = 2 \sum_{v=0}^{n-k} c_{k+v} \bar{c}_v \text{ for } k = 1, ..., n. \tag{2.24}$$

Since the trigonometric sum must integrate to one in order to be a probability density function, we add the following constraint:

$$a_0 = \sum_{k=0}^{n} |c_k|^2 = \frac{1}{2\pi}. \tag{2.25}$$

We can write the probability density function for our circular random variable defined by Fernández-Durán [2004] as

$$f(\theta; \underline{c}, M) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{M} (a_k \cos(k\theta) + b_k \sin(k\theta)), 0 \leq \theta \leq 2\pi. \tag{2.26}$$

2.3 Fitting of NNTS distribution

In Fernández-Durán and Gregorio-Domínguez [2015] one suggests the estimation of NNTS distribution using the method of maximum likelihood. Let's define $\underline{c} = (c_0, ..., c_M)'$ as the vector of complex parameters in the sum of order M. Let's define the vector of observations as $\underline{N} = (N_1, ..., N_{12})$, where N_k is the number of deaths during the k-th month for an age group we are examining, for $k = 1, ..., 12$. The likelihood function is (Fernández-Durán and Gregorio-Domínguez [2015])

$$L(\underline{c}, M|\underline{N}) = \prod_{r=1}^{12} (F(2\pi u_r; \underline{c}, M) - F(2\pi u_{r-1}; \underline{c}, M))^{N_r}, \tag{2.27}$$

where $u_r = r/12$ represents the proportional part of the year at the end of month r, and $F(\theta; \underline{c}, M)$ is the accrued NNTS distribution function.
NNTS distribution can be realized using R package CircNNTS, Fernández-Durán and Gregorio-Domínguez [2016] available online at
`https://cran.r-project.org/package=CircNNTSR`.

2.4 Calculation of net premiums in short term life insurance

We are interested in calculating monthly premiums for one year term life insurance payed at the time od death (1.33), where we have an annual force of interest δ (1.18). In our case, we are considering an individual who was born in month b and has already survived x years.
Firstly let's express the net single premium for one year, and then the corresponding monthly premiums. The net single premium can be expressed as

$$\bar{A}^{1}_{x:\overline{1|}}{}^{b} = E[e^{-\delta T(x)}] = \int_0^1 e^{-\delta t} f^b_{T(x)}(t) dt. \tag{2.28}$$

In the equation above, $f^b_{T(x)}(t)$ represents the probability of death at time t of a person aged x born in month b.
The net premium can be equivalently expressed as a sum within particular months as follows:

$$\bar{A}^{1}_{x:\overline{1|}}{}^{b} = \sum_{h=0}^{11} \int_{\frac{h}{12}}^{\frac{h+1}{12}} e^{-\delta t} f^b_{T(x)}(t) dt. \tag{2.29}$$

Here h represents the number of survived whole months. Let's use the substitution $t = \frac{h}{12} + s$, where s ($s \in [0, 1]$) represents the number of survived days in the last month. Using this substitution we obtain:

$$\begin{aligned} \bar{A}^{1}_{x:\overline{1|}}{}^{b} &= \sum_{h=0}^{11} \int_0^{\frac{1}{12}} e^{-\delta(\frac{h}{12}+s)} f^b_{S(x)|K(x)=0} \left(\frac{h}{12} + s | K(x) = 0 \right) P(K(x) = 0) ds \\ &= \sum_{h=0}^{11} e^{-\delta \frac{h}{12}} q_x \int_0^{\frac{1}{12}} e^{-\delta s} f^b_{S(x)|K(x)=0} \left(\frac{h}{12} + s | K(x) = 0 \right) ds. \end{aligned} \tag{2.30}$$

The net premium can also be written as a sum of discounted monthly premiums multiplied by a corresponding probability of surviving a given month:

$$\bar{A}^{\;\;b}_{\underset{x:\overline{1}|}{1}} = \sum_{h=0}^{11} e^{-\delta \frac{h}{12}} \, {}_{\frac{h}{12}}p^b_x \, \bar{A}^{\;\;b}_{\underset{x+\frac{h}{12}:\overline{\frac{1}{12}}|}{1}}. \tag{2.31}$$

When we put equations (2.30) and (2.31) together, we can express the monthly premium given that the individual has survived x years and h months as

$$\bar{A}^{\;\;b}_{\underset{x+\frac{h}{12}:\overline{\frac{1}{12}}|}{1}} = \frac{q_x}{{}_{\frac{h}{12}}p^b_x} \int_0^{\frac{1}{12}} e^{-\delta s} f^b_{S(x)|K(x)=0}\left(\frac{h}{12} + s|K(x) = 0\right) ds. \tag{2.32}$$

In our model, S represents the fraction of the individual's last year. The circular random variable θ was introduced in 2.2. Since S should lie in the interval [0,1], we will use the following transformation: $t : \theta \to \frac{\theta}{2\pi}$. From the theorem on the transformation of continuous random variables, we obtain the expression for the distribution function of S:

$$t : \theta \to \frac{\theta}{2\pi} = S, \qquad t^{-1} : S \to 2\pi S, \qquad \frac{\partial t^{-1}(s)}{\partial s} = 2\pi,$$

so that

$$f_S(s) = f_\theta(t^{-1}(s)) \left| \frac{\partial t^{-1}(s)}{\partial s} \right| = f_\theta(2\pi s) 2\pi. \tag{2.33}$$

Let's explain the principle reflecting the birth month on a simple example. Let's consider an individual born in February (b=2/12). Assuming this individual survives 65 years and 3 months (S=3/12), death happens in May. The month of birth is reflected, but the time the policy was issued is irrelevant. Since we can imagine the year as a circle split into 12 months, there is a certain periodicity occurring. Hence we express the probability density function of $S(x)$ depending on the month of birth as

$$f^b_{S(x)}(s) = 2\pi f_\theta(2\pi(s + b)). \tag{2.34}$$

Using (2.21) we express the density of NNTS distribution

$$\begin{aligned} f^b_{S(x)|K(x)=0}(s|K(x) = 0) &= 2\pi f_\theta(2\pi(s + b)) \\ &= 2\pi \sum_{k=0}^{M} \sum_{m=0}^{M} c_k \bar{c}_m e^{i(k-m)2\pi(s+b)}. \end{aligned} \tag{2.35}$$

Due to the constraint (2.25) for $k = m$, we obtain

$$\begin{aligned} f^b_{S(x)|K(x)=0}(s|K(x) = 0) &= 2\pi \left[\frac{1}{2\pi} + \sum_{k=0}^{M} \sum_{\substack{m=0 \\ m \neq k}}^{M} c_k \bar{c}_m e^{i(k-m)2\pi(s+b)} \right] \\ &= 1 + 2\pi \sum_{k=0}^{M} \sum_{\substack{m=0 \\ m \neq k}}^{M} c_k \bar{c}_m e^{i(k-m)2\pi(s+b)}. \end{aligned} \tag{2.36}$$

We use this result to calculate the integral in (2.32)

$$\begin{aligned}
&\int_0^{\frac{1}{12}} e^{-\delta s} f^b_{S(x)|K(x)=0}\left(\frac{h}{12}+s|K(x)=0\right)ds \\
&\quad= \int_0^{\frac{1}{12}} e^{-\delta s}\left[1+2\pi\sum_{k=0}^{M}\sum_{\substack{m=0\\m\neq k}}^{M} c_k\bar{c}_m e^{i(k-m)2\pi(\frac{h}{12}+s+b)}\right]ds \\
&\quad= \int_0^{\frac{1}{12}} e^{-\delta s}ds + 2\pi\sum_{k=0}^{M}\sum_{\substack{m=0\\m\neq k}}^{M} c_k\bar{c}_m e^{i(k-m)2\pi(\frac{h}{12}+b)}\int_0^{\frac{1}{12}} e^{i(k-m)2\pi s-\delta s}ds.
\end{aligned} \tag{2.37}$$

The particular integral in (2.37) can be calculated separately as follows:

$$\int_0^{\frac{1}{12}} e^{-\delta s}ds = -\frac{1}{\delta}\left[e^{-\delta s}\right]_0^{\frac{1}{12}} = \frac{1}{\delta}(1-e^{-\frac{\delta}{12}}),$$

$$\begin{aligned}
\int_0^{\frac{1}{12}} e^{s(i(k-m)2\pi-\delta)}ds &= \frac{1}{i(k-m)2\pi-\delta}\left[e^{s(i(k-m)2\pi-\delta)}\right]_0^{\frac{1}{12}} \\
&= \frac{1}{\delta-i(k-m)2\pi}\left(1-e^{\frac{1}{12}(i(k-m)2\pi-\delta)}\right).
\end{aligned}$$

Hence we obtain the equation

$$\begin{aligned}
&\int_0^{\frac{1}{12}} e^{-\delta s} f^b_{S(x)|K(x)=0}\left(\frac{h}{12}+s|K(x)=0\right)ds \\
&\quad= \frac{1}{\delta}(1-e^{-\frac{\delta}{12}}) + 2\pi\sum_{k=0}^{M}\sum_{\substack{m=0\\m\neq k}}^{M} c_k\bar{c}_m e^{i(k-m)2\pi(\frac{h}{12}+b)}\left[\frac{1-e^{\frac{1}{12}(i(k-m)2\pi-\delta)}}{\delta-i(k-m)2\pi}\right].
\end{aligned} \tag{2.38}$$

To calculate the monthly premium, the last thing we need is the probability that an individual aged x born in month b will survive h months. This probability is denoted as ${}_{\frac{h}{12}}p^b_x$ and calculated as the complementary probability to dying within h months:

$$\begin{aligned}
{}_{\frac{h}{12}}p^b_x &= 1-\int_0^{\frac{h}{12}} f^b_{T(x)(t)dt} \\
&= 1-P(K(x)=0)\int_0^{\frac{h}{12}} f^b_{S(x)|K(x)=0}(s|K(x)=0)ds \\
&= 1-q_x\int_0^{\frac{h}{12}}\left[1+2\pi\sum_{k=0}^{M}\sum_{\substack{m=0\\m\neq k}}^{M} c_k\bar{c}_m e^{i(k-m)2\pi(s+b)}\right]ds \\
&= 1-q_x\left[\frac{h}{12}+2\pi\sum_{k=0}^{M}\sum_{\substack{m=0\\m\neq k}}^{M} c_k\bar{c}_m e^{i(k-m)2\pi b}\left(\frac{1-e^{i(k-m)2\pi\frac{h}{12}}}{-i(k-m)2\pi}\right)\right].
\end{aligned} \tag{2.39}$$

In section 2.1 we introduced typically used assumptions for fractional ages. For comparison we will calculate the monthly premiums using the probability density functions obtained in the section 2.1 under these assumptions in order to compare the results with the ones applying the NNTS distribution.

- Uniform distribution

$$ {}_{\frac{h}{12}}p_x^b = 1 - q_x \int_0^{\frac{h}{12}} 1ds = 1 - \frac{h}{12} q_x, $$

$$ \bar{A}^{\ b}_{\overset{1}{x+\frac{h}{12}}:\overline{\frac{1}{12}}|} = \frac{q_x}{1 - \frac{h}{12} q_x} \int_0^{\frac{1}{12}} e^{-\delta s} 1ds = \frac{q_x}{1 - \frac{h}{12} q_x} \left(\frac{1 - e^{\frac{1}{12}}}{\delta} \right). \tag{2.40} $$

- Constant force of mortality

$$ {}_{\frac{h}{12}}p_x^b = 1 - q_x \int_0^{\frac{h}{12}} \frac{\mu e^{-\mu s}}{q_x} ds = e^{-\mu \frac{h}{12}}, $$

$$ \bar{A}^{\ b}_{\overset{1}{x+\frac{h}{12}}:\overline{\frac{1}{12}}|} = \frac{q_x}{e^{-\mu \frac{h}{12}}} \int_0^{\frac{1}{12}} e^{-\delta s} \frac{\mu e^{-\mu s}}{q_x} ds = \frac{\mu}{e^{-\mu \frac{h}{12}}} \left(\frac{1 - e^{-\frac{1}{12}(\mu + \delta)}}{\delta + \mu} \right). \tag{2.41} $$

- Balducci Assumption

$$ \begin{aligned} {}_{\frac{h}{12}}p_x^b &= 1 - q_x \int_0^{\frac{h}{12}} \frac{1 - q_x}{(1 - (1 - s) q_x)^2} ds = 1 - q_x \left[\frac{s}{1 - (1 - s) q_x} \right]_0^{\frac{h}{12}} \\ &= \frac{12 \left[1 - \left(1 - \frac{h}{12} \right) q_x \right] - h q_x}{12 \left[1 - \left(1 - \frac{h}{12} \right) q_x \right]} = \frac{1 - q_x}{1 - \left(1 - \frac{h}{12} \right) q_x}, \end{aligned} $$

$$ \begin{aligned} \bar{A}^{\ b}_{\overset{1}{x+\frac{h}{12}}:\overline{\frac{1}{12}}|} &= \frac{q_x \left[1 - \left(1 - \frac{h}{12} \right) q_x \right]}{1 - q_x} \int_0^{\frac{1}{12}} e^{-\delta s} \frac{1 - q_x}{(1 - (1 - s) q_x)^2} ds \\ &= q_x \left[1 - \left(1 - \frac{h}{12} \right) q_x \right] \int_0^{\frac{1}{12}} \frac{e^{-\delta s}}{(1 - (1 - s) q_x)^2} ds. \end{aligned} $$

After inserting into Wolfram Mathematica Inc., we obtain:

$$ \begin{aligned} &\bar{A}^{\ b}_{\overset{1}{x+\frac{h}{12}}:\overline{\frac{1}{12}}|} \\ &= \frac{1}{(1 - q_x) q_x (12 - 11 q_x)} (e^{-\frac{\delta}{12}} (1 - \frac{h q_x}{12}) (12(-1 + q_x) q_x \\ &- e^{\frac{\delta}{12}} q_x (-12 + 11 q_x) - e^{(-\frac{11}{12} + \frac{1}{q_x})\delta} (12 - 23 q_x + 11 q_x^2) \delta \\ &Ei \left[\left(\frac{11}{12} - \frac{1}{q_x} \right) \delta \right] + e^{(-\frac{11}{12} + \frac{1}{q_x})\delta} (12 - 23 q_x + 11 q_x^2) \delta Ei \left[\frac{(-1 + q_x)\delta}{q_x} \right])), \end{aligned} \tag{2.42} $$

where

$$ Im[q] \neq 0, 0 < Re[q] < 1, Re[1/q] < 11/12. $$

3. Application to Czech data

We are considering a one-year term life insurance where the insured individual pays monthly premiums. We will be analyzing data collected by the Czech Statistical Office, containing information about the age at death, gender and month of death. The gender is not important in our case, since gender doesn't pay a role in insurance policies according to Directives of the EU. This data had been collected throughout the years 2008-2017. Assuming the death rates are stable throughout the years, we can calculate average monthly death rates. We calculated the rates for 5 year interval age groups (see Table 3). Since the target age group for short term life insurance policies is older than 20 years, we will be working with these data from now on.

	Month											
Age at death	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Total	0.093	0.086	0.090	0.083	0.081	0.077	0.080	0.080	0.077	0.084	0.080	0.088
<1	0.088	0.082	0.081	0.078	0.094	0.083	0.086	0.084	0.079	0.079	0.083	0.082
1-4	0.104	0.079	0.078	0.081	0.097	0.080	0.088	0.083	0.068	0.090	0.076	0.076
5-9	0.096	0.082	0.073	0.065	0.080	0.076	0.111	0.091	0.093	0.078	0.071	0.083
10-14	0.108	0.082	0.095	0.076	0.071	0.077	0.093	0.085	0.062	0.084	0.065	0.101
15-19	0.089	0.069	0.077	0.074	0.079	0.098	0.088	0.096	0.094	0.081	0.079	0.076
20-24	0.085	0.074	0.083	0.084	0.080	0.092	0.101	0.098	0.081	0.077	0.069	0.074
25-29	0.077	0.076	0.088	0.080	0.086	0.087	0.095	0.089	0.084	0.084	0.083	0.072
30-34	0.093	0.078	0.088	0.079	0.087	0.081	0.091	0.090	0.089	0.079	0.075	0.070
35-39	0.094	0.083	0.080	0.077	0.080	0.083	0.095	0.087	0.077	0.080	0.081	0.083
40-44	0.091	0.081	0.090	0.081	0.080	0.082	0.085	0.084	0.078	0.078	0.080	0.091
45-49	0.088	0.081	0.089	0.083	0.082	0.084	0.085	0.085	0.078	0.082	0.076	0.085
50-54	0.094	0.085	0.086	0.079	0.082	0.079	0.083	0.079	0.084	0.083	0.080	0.085
55-59	0.094	0.082	0.090	0.083	0.083	0.081	0.080	0.081	0.078	0.083	0.080	0.085
60-64	0.092	0.084	0.091	0.082	0.081	0.079	0.082	0.080	0.078	0.085	0.080	0.086
65-69	0.091	0.084	0.089	0.083	0.082	0.078	0.082	0.081	0.079	0.084	0.081	0.088
70-74	0.091	0.085	0.090	0.082	0.081	0.078	0.079	0.079	0.079	0.084	0.082	0.089
75-79	0.093	0.087	0.090	0.085	0.082	0.077	0.081	0.078	0.076	0.083	0.079	0.088
80-84	0.095	0.088	0.090	0.084	0.080	0.076	0.079	0.079	0.077	0.083	0.081	0.088
>85	0.094	0.088	0.091	0.083	0.080	0.076	0.078	0.078	0.075	0.085	0.081	0.091
Average 20+	0.091	0.083	0.088	0.082	0.082	0.081	0.086	0.083	0.080	0.082	0.079	0.084

Source: Czech Statistical Office

Table 3.1: Average Czech monthly death rates

To obtain a more representative image of the mortality rates throughout the months, we multiplied the averages by 30 and divided by the number of days in the particular month, which is a reasonable normalization of the data.

Figure 3.1 shows the normalized empirical data compared with typical assumptions for fractional ages mentioned in 2.1. For specific calculations we used $q_x = 0.01$ and $\delta = 0.04$. It can be observed that the higher mortality rates at the beginning of the year are not reflected under the typical assumptions.

Computing in R (R Development Core Team [2018]) using the CircNNTS package (Fernández-Durán and Gregorio-Domínguez [2016]), we fitted our data as shown in Figure 3.2.

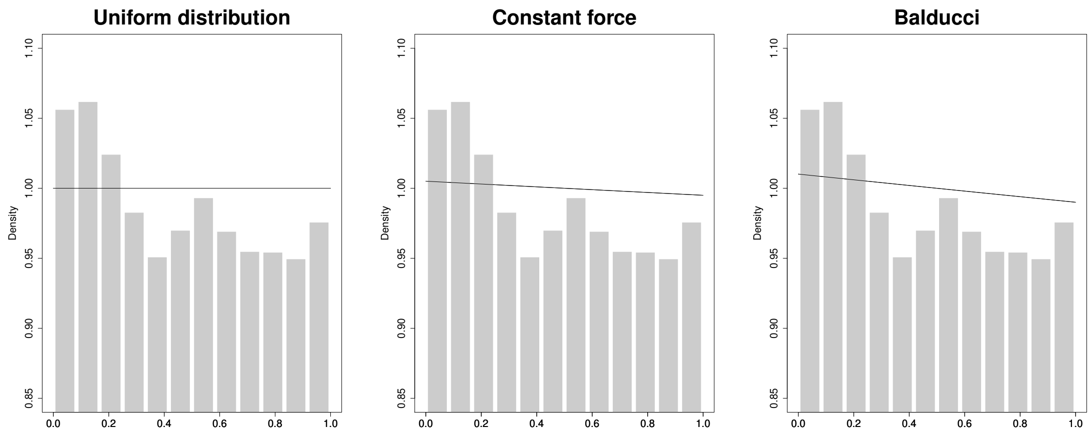

Figure 3.1: Empirical average death rates for individuals older than 20 years compared with monthly dead rates calculated using typical assumptions mentioned in 2.1

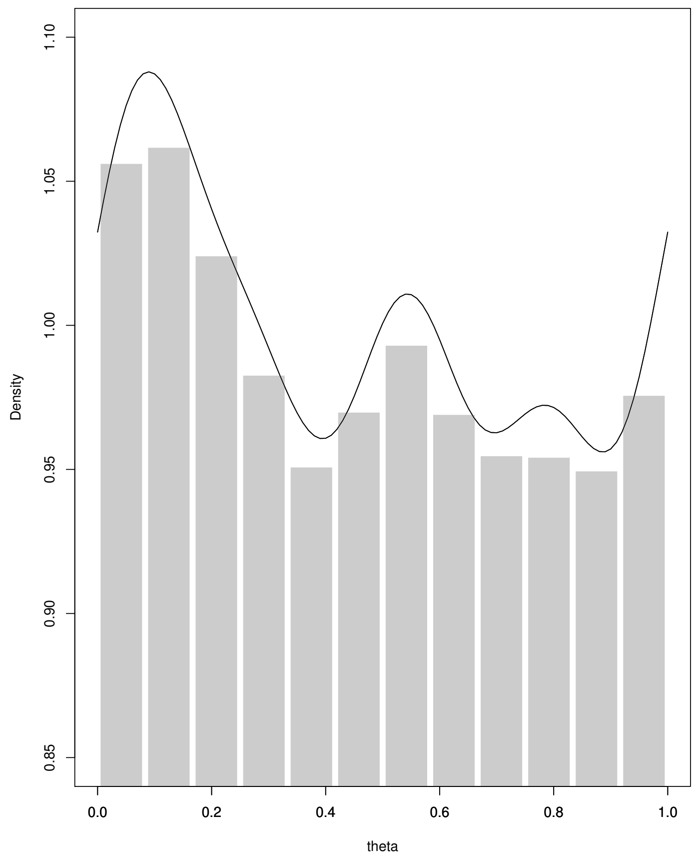

Figure 3.2: Empirical Czech mortality data fitted by NNTS probability density function

We observed a best fit for $M = 4$ and the vector $\underset{\sim}{c}$ where

$$\begin{aligned} c_0 =& 0.008260735, \\ c_1 =& 0.014057275 - 0.02284671i, \\ c_2 =& - 0.047886442 - 0.08647993i, \\ c_3 =& - 0.362284320 + 0.08710416i, \\ c_4 =& 0.001954026 - 0.09876413i. \end{aligned}$$

Higher mortality rates during winter months as well as slightly higher rates during summer months are reflected in the fitted NNTS density function, see Figure 3.2.

3.1 Monthly Net Premium Calculations

In section 2.1 we concluded that the month of birth and the probability of death are independent for the uniform distribution, constant force of mortality and Balducci assumption. Hence we calculated the monthly premiums for 100000 monetary units under these assumptions using (2.40),(2.41),(2.42), where $h = 0, ..., 11$ and we obtained:

	h=0	h=1	h=2	h=3	h=4	h=5	h=6	h=7	h=8	h=9	h=10	h=11
Uniform	83.19	83.26	83.33	83.40	83.47	83.54	83.61	83.68	83.75	83.82	83.89	83.96
Constant	83.58	83.65	83.72	83.79	83.86	83.92	84.00	84.07	84.14	84.21	84.28	84.35
Balducci	83.96	84.04	84.11	84.18	84.25	84.32	84.39	84.46	84.53	84.60	84.67	84.74

Table 3.2: Monthly premiums using typical assumptions for fractional ages

Under our introduced seasonal mortality assumption 2.2, we must consider the month of birth. Assuming that the behavior of death rates during the year does not depend on the number of survived years, (i.e. $f^b_{S(x)|K(x)} = f^b_{S(x)}$ for every x) we use the fitted NNTS density shown in Figure 3.2 for calculating the monthly premiums shown in Table 3.3.

	h=0	h=1	h=2	h=3	h=4	h=5	h=6	h=7	h=8	h=9	h=10	h=11
b=0	88.58	90.08	86.14	82.53	80.60	81.60	83.58	82.01	80.23	80.56	80.03	82.39
b=1/12	90.16	86.22	82.60	80.67	81.67	83.65	82.07	80.29	80.62	80.10	82.46	88.66
b=2/12	86.30	82.67	80.74	81.74	83.71	82.14	80.36	80.69	80.16	82.52	88.73	90.24
b=3/12	82.75	80.81	81.81	83.78	82.21	80.43	80.76	80.23	82.59	88.80	90.31	86.37
b=4/12	80.88	81.88	83.86	82.28	80.49	80.82	80.30	82.66	88.87	90.38	86.44	82.82
b=5/12	81.95	83.93	82.35	80.56	80.89	80.36	82.73	88.95	90.46	86.51	82.88	80.95
b=6/12	84.01	82.42	80.63	80.96	80.43	82.79	89.02	90.53	86.58	82.95	81.01	82.02
b=7/12	82.50	80.70	81.03	80.49	82.86	89.09	90.61	86.65	83.02	81.08	82.09	84.08
b=8/12	80.77	81.10	80.56	82.93	89.17	90.68	86.72	83.08	81.15	82.15	84.15	82.56
b=9/12	81.17	80.64	83.00	89.24	90.76	86.79	83.15	81.21	82.22	84.21	82.63	80.84
b=10/12	80.71	83.08	89.32	90.83	86.86	83.22	81.28	82.29	84.28	82.70	80.91	81.24
b=11/12	83.15	89.40	90.91	86.94	83.29	81.35	82.36	84.35	82.77	80.97	81.31	80.78

Table 3.3: Monthly premiums using fitted NNTS density

We can do a number of conclusions from Table 3.3. Let's take any month, for example March. The premiums for March calculated for an individual born in January (b=0 and h=2) and for an individual born in November (b=10/12 and h=4) are very similar. This can be generalized for all b and h giving the same sum, which indicates the same considered month. In case $12b + h \geq 12$, we can observe a similar value for $12b + h - 12$, since the survived part of the year was defined as a circular random variable.

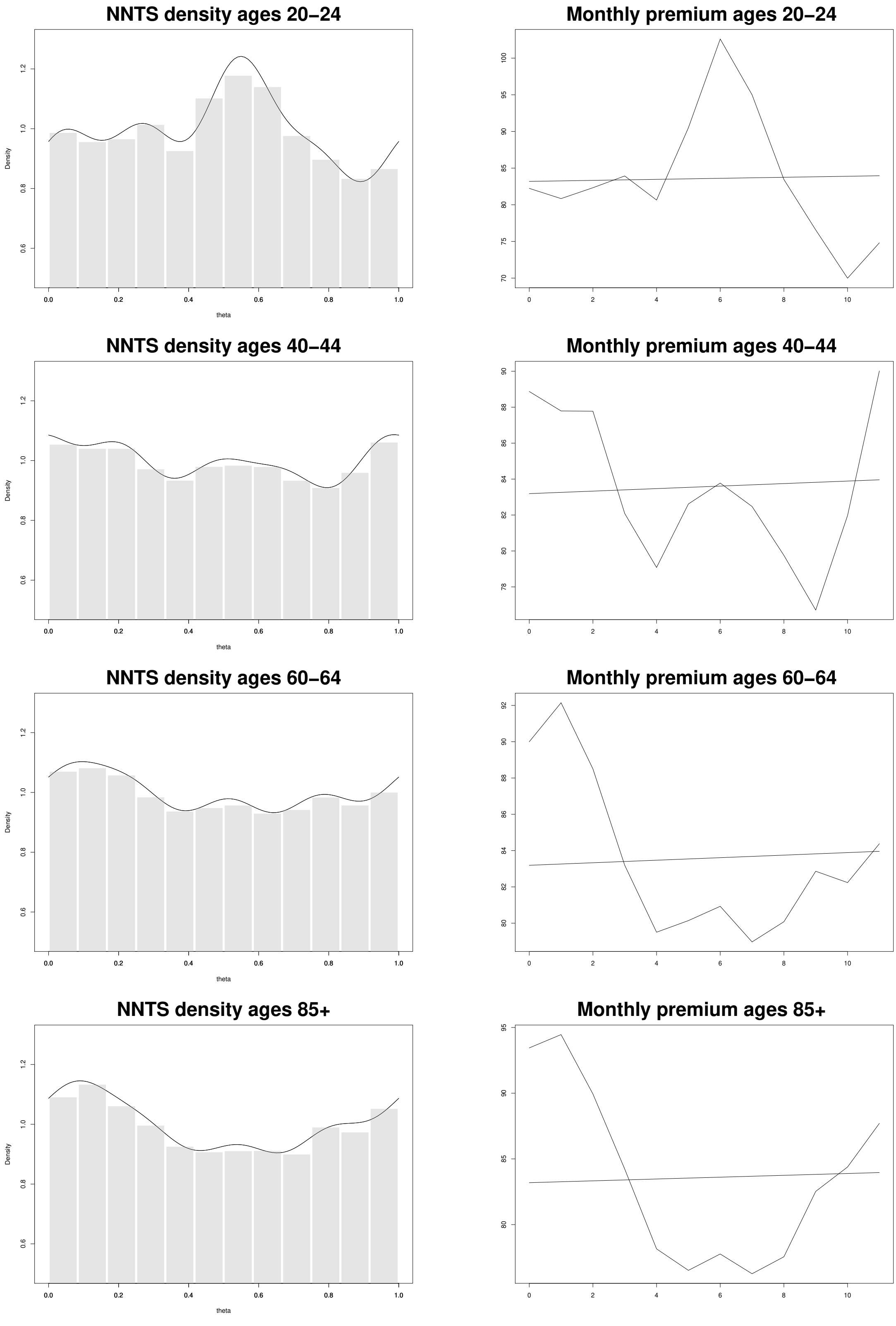

Figure 3.3: Fitted NNTS density for age groups 20-24, 40-44, 60-64 and 85+ along with corresponding monthly premium amounts.

The fact that mortality rates in the Czech Republic are higher during winter months and slightly higher during summer months, in July in particular, is reflected by higher premiums during these months.

We can also analyze the patterns of monthly premiums for different age shown in Figure 3.1, assuming that the behavior depends on the number of survived years. We studied the age groups 20-24, 40-44, 60-64 and people aged 85 years or more. For simplicity, we compared the monthly premiums for people born in January. These results are shown in Figure 3.1.

It can be concluded that within the age group 20-24, an excess mortality occurs during summer months rather than during winter months. With increasing age, the the higher mortality rates during summer months decrease and mortality rates in the winter months increase. This behavior is reflected in the monthly premium amounts as well.

In comparison with Fernández-Durán and Gregorio-Domínguez [2015], we do not observe such a strong winter excess mortality. As a consequence of this, our results are different. Slightly different computation results, for example in the calculations of monthly premiums using the Balducci assumption, can consist in using different mathematical tools for computation. However, the differences are relatively small.

Conclusion

In this paper we firstly addressed the problematic of life insurance. We introduced common life insurance products and formulas. We analyzed typically used assumptions for modeling mortality behavior during the year: the uniform distribution of deaths, constant force of mortality and Balducci assumption.

Analyzing data about mortality in the Czech Republic, we observed the highest mortality rates during the months of January and February as well as an increase during the summer months, specifically July. This was not reflected under the typical assumption for fractional ages. For this reason we introduced the seasonal mortality assumption which uses trigonometric sums for modeling the distribution. We were able to deduce formulas for calculating monthly premiums and compare results under various assumptions and thus to observe the effect of seasonal mortality.

We fitted our data using the NNTS distribution, which indicated the variability of the data throughout the year. After calculating the net premiums, we were able to observe a significant difference between results when using the seasonal mortality assumption and the classical assumptions. The month of birth was not relevant in the latter, while being an important parameter in the season mortality assumption.

Finally, the following findings are important from the practical point of view: when insurance companies use classical assumptions for fractional ages, where the month of birth and the month of the policy issue is not reflected, then the monthly premiums are either undervaluated or overvaluated.

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