FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Josef Svoboda

Geometry of Poisson-Lie T-duality

Mathematical Institute of Charles University

Supervisor of the master thesis: doc. Ing. Branislav Jurčo, CSc., DSc.
Study programme: Mathematics
Study branch: Mathematical Structures

Prague 2019

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date .... signature of the author

I would like to thank my supervisor Branislav Jurčo and my coadvisor Jan Vysoký for their inspiring and useful advice. I am very grateful for their patience and the time they devoted to me during the preparation of this thesis. I would like to thank Pavol Ševera for letting us to work on some of his ideas.

Title: Geometry of Poisson-Lie T-duality

Author: Josef Svoboda

Institute: Mathematical Institute of Charles University

Supervisor: doc. Ing. Branislav Jurčo, CSc., DSc., Mathematical Institute of Charles University

Abstract: In this thesis we study geometry of Poisson-Lie T-duality. We develop the language of Lie and Courant algebroids and study generalized metrics on them. Then we use Dirac structures and generalized isometries to formulate a general version of Poisson-Lie T-duality, which comes from string theory.

Keywords: Courant algebroid, Poisson-Lie T-duality, differential geometry

Contents

Introduction

0.1 Motivation

Geometry

This master thesis is about geometry of Poisson-Lie T-duality. Since "geometry" can mean almost anything and "Poisson-Lie T-duality" may seem as an attempt to discourage the reader from opening this text, we should start with few words of motivation.

By geometry, we mean mostly differential geometry, study of smooth manifolds and structures on them, and theory of Lie groups, smooth symmetries, and of course their infinitesimal friends - Lie algebras. While using a lot of usual machinery, we will define and describe structures which are probably not so common. They fit into the program of *higher structures* in differential geometry - the idea comming from category theory.

Category theory teaches us that instead of studying, for example, symmetries of a particular object (manifold, topological space,lattice…), forming a group, we should look at all the (invertible) relations between all different objects, which form a *groupoid*. If the object is a smooth manifold, we usually study smooth symetries, which tend to form Lie group. Now if we have many objects, for example, a smoothly changing family of manifolds (imagine a solid body deforming in time), we want to consider all the (invertible) relations between different elements of the family. This is a Lie groupoid - groupoid which is a smooth manifold at the same time.

When we try to do something similar to Lie algebras, we obtain *Lie algebroids* and (with a slight modification) *Courant algebroids*. As one could expect, these are easier to work with than groupoids. Also the algebroid – groupoid corresponence is a tricky bussiness which has not been fully understood yet (at least not in the broad mathematical community). Nevertheless, both Lie and Courant algebroids have been proved quite useful both in mathematics and theoretical physics.

There is also a different point of view on Courant algebroids, which is comming from some kind of "higher" geometry - namely *graded geometry*. Graded geometry/supergeometry studies spaces equipped with sheaves of *graded* algebras. Courant algebroids fit very naturally to this setting as graded symplectic manifolds. This can give us an idea why they really behave a bit like symplectic manifolds in practice. For example, there is a reduction procedure of Courant algebroids, quite similar to usual symplectic reduction. *Dirac structures*, which play an important role in study of Courant algebroids, are a bit like Lagrangian submanifolds of symplectic manifold.

Our interest in Courant algebroid lies in their applications to theoretical physics, namely string theory. They are especially suitable for studying *σ-models* and string dualities giving equivalences between different σ-models. We will use them to mathematically formulate *Poisson-Lie T-duality* in a very general way.

T-duality

T-duality (target space duality) is a particular symmetry of string theory or more generally a duality between two (type IIA and type IIB) string theories. These theories are described by σ-model (or effective actions) which concerns a manifold M (target space) with an additional structure, so called (*backgrounds*): a Riemannian metric g, two-form B, three-form H and and dilaton field. T-duality enables us to show that two theories with completely different manifolds and backgrounds are equivalent in various senses.

More specifically, a σ-model consists of smooth maps $l : \Sigma \to M$ with the following action:

$$\mathcal{S}_\sigma[l] = \int_\Sigma \langle h, l^*(g) \rangle_h \cdot vol_h + \int_X l^*(B) + \int_X l^*(H), \tag{1}$$

where (Σ, h) is a two-dimensional Lorenzian manifold called *worldsheet* equipped with a Lorenzian metric h, M is a manifold (the *target*) with a metric g, a two-form B, and a closed three-form H (we omit dilaton for simplicity), X is a three-dimensional manifold with Σ as a boundary and l in the last term an arbitrary extension of l on Σ to X. At the classical level, we can find the equations of motion and try to solve them. We can also quantize the σ model which leads consequently to "effective actions" which are functionals of the background fields (so they become dynamic).

T-duality is a tool how to show that two different target spaces lead to equivalent effective actions or even σ- models, which is consequently interpreted as an equivalence of physical theories.

The usual T-duality, also known as Abelian (=commutative) T-duality, concerns target spaces M, which are bundles with tori as fibres (n-dimensional torus $\mathbb{T}^n = U(1)^n$ is the only connected compact Abelian Lie group of dimension n). See [12] for an example of such fibrations. However, physical theories often possess non-abelian symetries (for example, the gauge group of standard model of particles is product of groups of small unitary matrices, namely $U(1) \times SU(2) \times SU(3)$, but it is also often useful to consider theories with $SU(N)$-symmetry, for large N or even $N \to \infty$). Because of this fact, there has been an effort to generalize T-duality to a non-abelian T-duality in past 25 years.

In ninetees Klimčik [1], together with Ševera [2], [3] invented a theory of non-abelian T-duality, called Poisson-Lie T-duality, because they used Poisson-Lie groups (Lie groups with compatible Poisson structure). Ševera then found out that both Poisson-Lie T-duality and Abelian T-duality can be naturally formulated in terms of Courant algebroids. The aim of this thesis is to describe this language and to state a general formulation of T-duality, motivated by thoughts of Ševera.

In this formulation, T-duality becomes "plurality" meaning that it relates several different models, not just two "dual" models. It is worth to say that to get a reasonable physical theory, we also need *dilaton field*. It can be nicely incomporated into the realm of Courant algebroids as a Courant algebroid connection (an analogy of Levi-Civita connection on tangent bundle). See, for example, [4]. However, for our discussion this aspect is not that important and we omit it.

0.2 Outline

In the first chapter, we will recall important notions as principal bundles and their connections. Then we will define *Lie algebroids*, especially Atiyah Lie algebroid, which gives a useful insight into principal bundles. Lie algebroids will also serve as a motivation for more complicated Courant algebroids.

In the second chapter, we start with the theory of *Courant algebroids*. We will give examples and characterize so-called *exact* Courant algebroids. Then we will describe metric aspects of algebroids, namely we define *generalized metric* which encorpotates usual Riemannian metric. Finally we will define Dirac structures and use them to define relations between Courant algebroids similarly to canonical relations in symplectic geometry.

In the third chapter, we will describe an important construction of Courant algebroids - the reduction by a group action. It will give us new interesting examples and also provide an insight how seemingly unrelated Courant algebroids can actually share some properties when they are comming from the same Courant algebroid by reductions by different groups. This is a source of Poisson-Lie T-duality.

In the fourth chapter we will formulate Poisson-Lie T-duality in a general way, without using group actions and reductions. The reduction from the third chapter will serve as a nontrivial example for this fenomenon.

The text is intended for a reader willing to learn basics of Courant algebroids and get an idea about Poisson-Lie T-duality. I tried to present this technical, but beautiful topic in a clear, informal way, which I decided to support by adding several exercises.

The thesis attempts to review the results which are spread around the recent papers of Pavol Ševera, Branislav Jurčo, Jan Vysoký as well as some older, classical results of others. Hovewer all errors and typos in the thesis are done solely by me.

1. Lie algebras and algebroids

1.1 Basic definitions and facts

In this thesis all manifolds and maps between them are smooth (C^∞) otherwise it is explicitly stated. Our base field is $\mathbb{R}$ so all Lie groups and Lie algebras are over $\mathbb{R}$ etc. All Lie algebras and groups are finite dimensional.

I expect the reader to be familiar with smooth manifolds and vector bundles on them and basic Lie theory. If not, I personally recommend the book [Baez].

In this section I will set the notation and remind some basics of fibre (vector, principal) bundles I will need throughout the thesis.

We usually work with a connected Lie group D and its closed Lie subgroup $G \subset D$, with the corresponding Lie algebras $\mathfrak{g} \subset \mathfrak{d}$. Elements of Lie algebras are usually denoted by Greek letters $\xi, \mu, \zeta \ldots$, elements of Lie groups by Latin letters $d, g, h \ldots$.

Fibre bundles

We sometimes omit words "fibre", "vector" and "principal" when it is clear from the context.

We start with recalling the definition of a fibre bundle. These bundles are useful because they contain both vector bundles and principal bundles as special cases. Intuitively, they are manifolds, which locally look like a Cartesian product of two manifolds, but their global topology can be (and usually is) more complicated.

Definition 1.1.1. A *fibre bundle* (P, M, π, F) consists of manifolds P (the *total space*), M (the *base space* or just the *base*), F (the *fibre*) and a surjective map $\pi : P \to M$ (*bundle projection*) such that for every point x of M there exists an open neigbourhood $U \subset M$ and a diffeomorphism (*local trivialization*)

$$\pi^{-1}(U) \cong U \times F. \tag{1.1}$$

We usually omit F in (P, M, π, F) because it can be extracted (up to diffeomorphism) from (P, M, π) as $F \cong \pi^{-1}(x) =: P_x$ for any $x \in M$.

Definition 1.1.2. A *vector bundle* (E, M, π) is a fibre bundle which has a vector space as a fibre and local trivializations are linear with respect to fibres. Dimension of the fibre E_x at a point $x \in M$ is called *rank* of E in x. It is a locally constant function on M so if M is connected, rank of E is just a number.

Example 1.1.A. Tangent and cotangent bundle of a connected manifold M of dimension n are both vector bundles on M with dimension n.

Definition 1.1.3. A *bundle map* $(Q, N, \pi) \to (P, M, \pi')$ consists of a pair of maps $f : N \to M$ and $\phi : Q \to P$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M' \end{array} \tag{1.2}$$

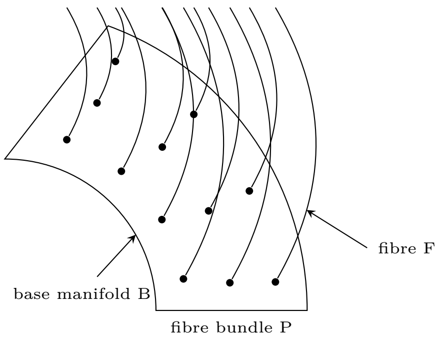


Figure 1.1: A Fibre bundle with the total space P, base space M and fibre F. [5]

commutes. An invertible bundle map is called an *isomorphism* of fibre bundles. For vector bundles we add the assumption that (vector) bundle maps are linear with respect to fibres.

A bundle, which is isomorphic to Cartesian product bundle $P = M \times F$, is called *trivial*. The condition 1.1 is called local triviality, because it describes precisely that P is locally Cartesian product of M with F.

Definition 1.1.4. If (P, M, π) is a fibre bundle, a (smooth) map $s : M \to P$ is a *(global) section* of this bundle, if $\pi \circ s = id_M$. In other words, the section s assigns to every point x of M a point $s(x)$ in the fibre of x (in a smooth way). Space of all sections is denoted by $\Gamma(P)$.

If P is a vector bundle, $\Gamma(P)$ is a natural $C^\infty(M)$-module[1]. For any vector bundle map $E \to F$ there is a corresponding $C^\infty(M)$-linear map between the spaces of sections $\Gamma(E) \to \Gamma(F)$. If there is a $C^\infty(M)$-linear isomorphism $\Gamma(E) \to \Gamma(F)$, then also E and F are isomorphic. In fact, by celebrated Serre-Swan theorem, there is an equivalence of categories of vector bundles over M and finitely generated projective modules over $C^\infty(M)$-modules. We sometimes interchange E and its space of sections $\Gamma(E)$, and similarly with bundle maps.

Exercise 1.1.B. Interpret a global section of a vector bundle as a vector bundle map. What about fibre bundles?

Principal bundles

Definition 1.1.5. Let M be a manifold and D be a Lie group. A *principal D-bundle* is a fibre bundle (P, M, π) together with a right action $P \times D \to P$ such that D preserves fibers of P, it acts freely and transitively on each fibre (for every $x \in M$ and every pair $v, w \in P_x = \pi^{-1}(x)$ there exists a unique element $d \in D$ such that $v \cdot d = w$) and there are some local trivializations $\varphi_U : P \restriction_U \to U \times D$ covering P which respect the action (we say that they are equivariant), in other words, for every $p \in P$ and $d \in D$ we have

$$\varphi(p \cdot d) = \varphi(p) \cdot d.$$

Recall that P is the *total space* and M is the *base space*.

[1] We could also consider the whole sheaf of (local) sections, but in smooth setting we can almost always work with global sections.

For a nice mathematical exposition, see [6]. In physics literature, D is usually called gauge group.

Example 1.1.C. A Lie group D is a principal D-bundle over a one point manifold $M = \{*\}$.

Example 1.1.D. Let (V, M, π) be a vector bundle. Then we can produce a principal bundle called *frame bundle* $\mathcal{F}(V)$ of V. The fibre at a point $x \in M$ consists of all vector space bases of the fibre V_x of V. Lie group $\mathrm{GL}(n)$ of $n \times n$ invertible matrices, where n is rank of V acts on each fibre by matrix multiplication freely and transitively, so $\mathcal{F}$ is a principal $\mathrm{GL}(n)$-bundle.

Exercise 1.1.E. Show that three-sphere S^3 is a principal S^1-bundle over S^2. (Possible hint: if you are familiar with complex projective space $\mathbb{C}P^n$, show that S^{n+1} is a principal S^1-bundle over $\mathbb{C}P^n$ and remember that $\mathbb{C}P^1$ is Riemann sphere.) The projection is the famous Hopf map.

Definition 1.1.6. Let (P, M, π) be a principal D-bundle and V be a vector space with a representation of a Lie group D (a Lie group homomorphism $\rho : D \to \mathrm{GL}(V)$). An *associated bundle* to P with fibre V, is a vector bundle V_P over M whose total space is a set of equivalence classes $[p, v] \sim [pd, d^{-1}v]$ of pairs of elements from $P \times V$. The projection assigns $[p, v]$ to $\pi(p)$.

Analogical construction can be done for fibre bundles if we have a group acting on the fibre and consider left actions on instead of representations.

The following example is so important, that it deserves to be a definition.

Definition 1.1.7. Lie group D has a canonical *adjoint* representation on the underlying vector space of its Lie algebra $\mathfrak{d}$ – take an element $\xi \in \mathfrak{d}$, conjugate its exponential curve $\exp(t\xi)$ by an element d of D and take the derivative at $t = 0$ to get a tangent vector again:

$$\mathrm{Ad}_d(\xi) = \partial_t (d \exp(t\xi) d^{-1})|_{t=0}.$$

So for any principal D-bundle P we can produce the associated bundle $\mathfrak{d}_P$ with respect to this representation, the *adjoint bundle*. It is a special case of a Lie algebra bundle (example 1.2.3). The total space of $\mathfrak{d}_P$ consists of the pairs $[p, \xi] \in P \times \mathfrak{d}$, modulo the equivalence relation $[p \cdot d, \xi] \equiv [p, \mathrm{Ad}_{d^{-1}} \xi]$. The adjoint bundle occurs in the example 1.3.C and theorem 3.3.3.

There are at least two ways of thinking about principal bundle (P, M, π) and it is important to change this views frequently.

- We concentrate on M and think about P as a collection of fibres P_x at every point x of M. (Imagine yourself standing on the Earth M and looking around you. The set of all directions you can look to forms a group $D = S^1$ and collection of these S^1 gives a principal S^1-bundle over the Earth M. The action of S^1 turns the head of every terrestrial by a given angle.) Every fibre P_x is isomorphic to group D and we can act by D on each fibre at the same time. This last note is the important distinction between principal and vector bundles - fibres of vector bundle of rank n are also isomorphic to a group - $\mathbb{R}^n$ with usual addition. However, we cannot add a fixed vector $v \in \mathbb{R}^n$ to every vector in each fibre because the meaning of this addition is dependent on the choice of isomorphisms.

- We concetrate on P and think about it as a (huge) manifold equipped with a nice (free) action of a group D. The base space M is then a space of orbits of the action.

  We can happily study additional structures on P (such as vector fields, forms, or even metrics) and keep in mind that if these structures behave well with respect to the action, they can induce similar structures on the space of orbits M. Many structures on M (e.g. principal connections) are complicated just because they are quotients of more natural objects living on P (and taking a quotient is kind of drastic operation).

A fact we will use throughout all the thesis is that D-action induces an action of its Lie algebra $\mathfrak{d}$ by derivations of vector fields. Informal introduction first. There is an advantage and at the same time a source of possible confusement with Lie algebra actions:

Vector fields on M form an (infinite-dimensional) Lie algebra $\Gamma(TM)$ with bracket being usual Lie bracket of vector fields. Although we do not want to get into the wild world of infinite-dimensional manifolds, it is fruitful to think of $\Gamma(TM)$ as a Lie algebra corresponding to the (infinite-dimensional) Lie group of diffeomorphisms $\mathrm{Diff}(M)$ of M (vector fields are "infinitesimal diffeomorphisms" and give rise a flow along them, which can be thought as the exponential map $\Gamma(TM) \to \mathrm{Diff}(M)$).

An usual action of Lie group D on a manifold M is a group homomorphism from D to $\mathrm{Diff}(M)$ so we can expect that there is a corresponding "infinitesimal action", i.e. a Lie algebra homomorphism $\# : \mathfrak{d} \to \Gamma(TM)$. However, it is not an action in the sense that it moves elements of M, rather it describes tangent directions to movements of elements of M.

On the other hand, it is natural to consider an action of $\mathfrak{d}$ on the Lie algebra $\Gamma(TM)$ itself: $\mathfrak{d} \times \Gamma(TM) \to \Gamma(TM)$, $(\xi, X) \mapsto \xi \cdot X$, meaning that for any element ξ we have a linear map from $\Gamma(TM)$ to itself, which is a derivation in the sence that

$$\xi \cdot [X, Y] = [\xi \cdot X, Y] + [X, \xi \cdot Y].$$

In other words, this action is a Lie algebra morphism $\mathfrak{d} \to \mathrm{Der}(\Gamma(TM))$ where $\mathrm{Der}(\Gamma(TM))$ is Lie algebra of all derivations on $\Gamma(TM)$ with commutator as the bracket.

This action takes a vector field X and an element ξ of the Lie algebra, and assings to it the (Lie) derivative of vector field X along the curve, which is given by action of elements of d which lie on the curve tangent to the element ξ of Lie algebra $\mathfrak{d}$.

Both actions are sometimes called "the infinitesimal action of the action of D", because they are completely equivalent. An infinitesimal action $\#$ as a homomorphism from $\mathfrak{d} \to \Gamma(TM)$ induces a homomorphism from $\mathfrak{d}$ to $\mathrm{Der}(\Gamma(TM))$ by $\xi \mapsto [\xi, \cdot]$. On the other hand, every derivation on the Lie algebra of vector fields is of this form, i.e. actually represented by a single vector field[2], so we obtain a Lie algebra homomorphism from $\mathfrak{d}$ to Lie algebra of vector fields.

The moral of the last paragraph is that we can interchange the vector field $\xi^{\#}$ and the map $[\xi^{\#}, \cdot]$ freely and it does not hurt much (but it is good to know

[2] This is a great property of tangent bundle, and one of the complications with Courant algebroids is that they do not possess it.

that we have both). Now we proceed to the precise definition (just in case you have never heard).

Definition 1.1.8. Let (P, M, π) be a principal D-bundle. Let ξ be an element of Lie algebra $\mathfrak{d}$. This means that there is a (smooth) curve $t \mapsto \exp(t\xi)$ in D with ξ as a tangent vector in $t = 0$ (hence at identity element of D). For any element $p \in P$ we consider a curve $p \cdot \exp(t\xi)$ and we call $\xi^\#$ its tangent vector at $t = 0$ (hence at point p). This procedure gives a map

$$\begin{aligned} \# : \mathfrak{d} &\to \Gamma(TP) \\ \xi &\mapsto \xi^\#, \end{aligned}$$

which is injective and preserves the usual Lie bracket of vector fields

$$\#([\xi, \mu]) = [\xi^\#, \mu^\#]$$

for every $\xi, \mu \in \mathfrak{d}$. We call # the *infinitesimal action* of $\mathfrak{d}$ on P. Vector fields of the form $\xi^\#$ for some $\xi \in \mathfrak{d}$ are called *vertical.*

The image of # (as a set of tangent vectors of the sections) is a subbundle of the tangent bundle of P because # is point-wise injective. It is called *vertical bundle* VP. Equivalently (for principal bundles), vertical bundle is the kernel of the tangent map of the projection

$$T\pi : TP \to T(P/D).$$

Its sections are called *vertical* fields. Those are precisely vector fields tangent to orbits of the action. Fundamental vector fields are vertical.

Infinitesimal action gives an isomorphism of VP with the trivial $\mathfrak{d}$-bundle over P. Meaning: if you walk on P and choose a direction of a vertical vector field, you will never leave your fibre, but all vertical vector fields together span all possible directions in the fibre.

We can also reverse the procedure and "integrate" an infinitesimal action of $\mathfrak{d}$ to an action of the Lie group. The proof of this classical theorem can be found in [7].

Theorem 1.1.9 (Lie-Palais)**.** *Every infinitesimal action of a finite-dimensional Lie algebra $\mathfrak{d}$ on a compact manifold M integrates to an action of some Lie group D on M.*

1.2 Lie algebroids

Introduction

Our first task is to explain a technical notion of a Courant algebroid (CA) which we will use to formulate T-duality. For this, we first define a similar but simpler notion of a Lie algebroid and we describe some examples (most notably the tangent bundle (1.2.B) and Atiyah Lie algebroid (1.3.C) of a given principal bundle).

In the next chapter we will proceed to Courant algebroids and we will notify important distinctions between these two concepts, for example (2.1.8) and (2.1.9).

Lie algebroid is a natural generalization of the concept of a Lie algebra.[3] Informally, it is some kind of vector bundle with a structure of Lie bracket on the space of its sections. Lie algebra can then be thought as a Lie algebroid over a single point.

Lie algebroids

Lie algebras play two different roles in our story:

- One is that we aim to generalize them to Lie algebroids, which offer a natural living space and unified language for natural geometric structures which occur as backgrounds for sigma models. Lie algebras or their bundles serve as an example of Lie algebroids. This is the point of view of this section.

- The second role is the usual one, as in the definition 1.1.8 - they consist of infinitesimal symmetries of some object - the object can be a manifold, vector bundle or even an algebroid. We usually have a corresponding Lie group action acting by diffeomorphisms or bundle isomorphisms. This will become important in the chapter 3.

Definition 1.2.1. Let $E \to M$ be a vector bundle with an $\mathbb{R}$-bilinear map $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ and a vector bundle map $\rho : E \to TM$ (called *anchor*) such that the following axioms hold for every $f \in C^\infty(M)$ and $\psi, \psi', \psi'' \in \Gamma(E)$:

1. *Leibniz rule*: $[\psi, f\psi'] = f[\psi,\psi'] + \rho(\psi)(f)\psi'$
2. $(\Gamma(E), [\cdot,\cdot]_E)$ satisfies *Leibniz identity*:

$$[\psi,[\psi',\psi'']]_E = [[\psi,\psi'],\psi'']]_E + [\psi',[\psi,\psi'']]_E$$

Then we call $(E, \rho, [\cdot,\cdot]_E)$ a *Leibniz algebroid*. If $[\cdot,\cdot]_E$ is skew-symmetric, then $(E, \rho, [\cdot,\cdot]_E)$ is called *Lie algebroid*. The space of sections $\Gamma(E)$ then forms a Lie algebra with $[\cdot,\cdot]_E$.

Proposition 1.2.2. *For any Leibniz algebroid* $(E, \rho, [\cdot,\cdot]_E)$*, the anchor* ρ *becomes a bracket homomorphism:*

$$\rho([\psi,\psi']_E) = [\rho(\psi), \rho(\psi)].$$

Proof. I recommend you to do this computation (with use of Leibniz identity and Leibniz rule). It can seem a bit complicated, so we do it here:

We start with Leibniz identity (CA1),

$$[\psi,[\psi',f\psi'']]_E = [[\psi,\psi'],f\psi'']]_E + [\psi',[\psi,f\psi'']]_E$$
$$[[\psi,\psi'],f\psi'']]_E = [\psi,[\psi',f\psi'']]_E - [\psi',[\psi,f\psi'']]_E,$$

[3] There is also a notion of Lie groupoid, which we will not use here.

now we expand everything by Leibnitz rule (CA2)

$$\rho([\psi,\psi'])(f)\psi'' + f[[\psi,\psi'],\psi''] = [\psi, f[\psi',\psi'']]_E + [\psi, \rho(\psi')(f)\psi'']_E \\ - [\psi', f[\psi,\psi'']]_E - [\psi', \rho(\psi)(f)\psi'']_E.$$

On the right hand side, we expand by (CA2) even more,

$$\begin{aligned}\rho([\psi,\psi'])(f)\psi'' + f[[\psi,\psi'],\psi''] = {} & \rho(\psi)(f)[\psi',\psi'']_E + f[\psi,[\psi',\psi'']]_E \\ & + \rho(\psi)(\rho(\psi')(f))\psi'' + \rho(\psi')(f)[\psi,\psi'']_E \\ & - \rho(\psi')(f)[\psi,\psi'']_E - f[\psi',[\psi,\psi'']]_E \\ & - \rho(\psi')(\rho(\psi)(f))\psi'' - \rho(\psi)(f)[\psi',\psi'']_E\end{aligned}$$

Finally, something cancels out and we use Leibnitz identity again.

$$\rho([\psi,\psi']_E)(f)\psi'' = \rho(\psi)(\rho(\psi')(f))\psi'' - \rho(\psi')(\rho(\psi)(f))\psi''.$$

Since f and ψ'' was arbitrary, we obtain the result. ■

Example 1.2.A. If M has only one point $\{*\}$, a Lie algebroid E is just a Lie algebra. The anchor is the trivial map $E \to \{0\} = T\{*\}$.

Example 1.2.B. The tangent bundle TM of a manifold M is a Lie algebroid with Lie bracket of vector fields and the identity of TM as the anchor. This is kind of motivating example of a Lie algebroid. Tangent bundle is not just an ordinary vector bundle, it acts on itself by Lie bracket, which allows us to define Lie derivative on tensors (products of powers of TM and T^*M). Our aim is to study similarly rich structures.

Definition 1.2.3. A vector bundle (E, M, π) is a *Lie algebra bundle* if there exists a Lie algebra $\mathfrak{d}$ such that every fibre E_x has a structure of a Lie algebra isomorphic to $\mathfrak{d}$ and the local trivializations $E_{\restriction U} \cong U \times \mathfrak{d}$ restrict to a Lie algebra homomorphism $E_x \cong \{x\} \times \mathfrak{d}$ for every point x of M.

Example 1.2.C. Any Lie algebra bundle is a Lie algebroid with the constant zero anchor. Notice that a Lie algebroid in general can be something much more complicated then just a bunch of Lie algebras parametrized by points of M, as in the example 1.2.B - Lie bracket of tangent vectors in a given point does not make any sense, because it depends on the behavior of fields at the neighbourhood of the point. This is a translation of the algebraic fact that Lie bracket is not C^∞-linear,

$$[X, fY] \neq f[X, Y],$$

but it satisfies Leibnitz rule instead,

$$[X, fY] = f[X, Y] + X(f) \cdot Y.$$

1.3 Group actions on manifolds and bundles

We will construct an example of a Lie algebroid, which is useful for studying principal bundles. We start with some general comments on group actions on manifolds and bundles.

An action by D on a manifold P can be thought as some artificial degrees of freedom. For example, if P is a classical state space, i.e. a symplectic manifold, two states in the same orbit are interpreted as physically equivalent, so we want to work with the quotient space to get rid of this redundance. The quotient is not necessarily a well defined manifold, but if the action is free and proper, then everything works well.[4]

So let us suppose that we have such a nice action and take the quotient $M = P/D$ of P by D. We would also like to transfer structures as vector bundles, algebroids, metrics etc. from P to M. This is not always possible, for example, a quotient of a symplectic manifold is not symplectic, because we loose the non-degeneracy. Nevertheless we can consider a suitable D-invariant submanifold Q of P such that Q/D is symplectic.

Many structures of our interest have the underlying vector bundle - Lie algebroids for example, but also Riemannian and symplectic structures live naturally on vector bundles (of tensors). So we first describe how to do a quotient of a vector bundle. Bundles which allow us to do it are called equivariant.

Definition 1.3.1. Let $E \xrightarrow{\pi} P$ be a fibre bundle and D be a Lie group with a (right) action on both the base space P and the total space E. Then (E, P, π) is an *equivariant bundle* if π is an equivariant map, i.e. it commutes with actions on P and on E:

$$d \circ \pi = \pi \circ d$$

where d denotes the action of an element $d \in D$ on either P or E. IF E is a vector bundle over P then E is called *equivariant vector bundle* if it is an D-equivariant bundle and D acts on E by vector bundle isomorphisms. In other words, the fibre of p is mapped to the fibre of pd by multiplication by $d \in D$ and for vector bundles we require this multiplication map to be linear.

Remark 1.3.2. For any $d \in D$ we can also act by d^{-1} which maps the fibre of pd to the fibre of p and it is clearly inverse to the multiplication by d, so every multiplication gives an isomorphism of fibres.

Definition 1.3.3. The section $v \in \Gamma(E)$ of an equivariant bundle $E \to P$ is an *invariant*[5] section if $v_p \cdot d = v_{p \cdot d}$.

Exercise 1.3.A (important). If E is a D-invariant vector bundle over P, then E/D is a vector bundle over P/D. Rank of E/D equals rank of E. The space of sections $\Gamma(E/D)$ is in canonical bijection with D-invariant sections $\Gamma(E)^D$ of E, i.e. every D-invariant section induces a unique section of E/D. .

Warning 1.3.4. Invariant sections form a vector space, but not a $C^\infty(P)$-module, so they do *not* form a space of sections of any subbundle of E. However, they are naturally a $C^\infty(P/D)$-module due to the identification with sections of E/D.

Example 1.3.B (see (1.3.5) and (1.3.8)). Let P be a principal D-bundle, so we have the (free and proper) action of D on P. Tangent bundle TP is always equivariant vector bundle over P, with action on the total space given by tangent maps to

[4]If not, we can still do something, for example work with a so-called ”action Lie groupoid” instead of the (not well defined) quotient. See [8].

[5]It seems to me that the word equivariant would be suitable too, but we keep this notation, because the term ”equivariant differential form” is left for something different

right multiplications by elements of D. The vertical subbundle $VP \subset TP$ is an invariant subbundle and given a principal connection A , its horizontal bundle is also invariant, by the equivariance of the connection.

Atiyah Lie algebroid

We take a principal D-bundle (P, M, π) and look at the tangent bundle TP of P. As we already know, there is a subbundle $VP \subset TP$ of vertical vector fields, is isomorphic to the trivial bundle $P \times \mathfrak{d}$ via the infinitesimal action[6]. But it is not as trivial as it can seem, because it possess the action of D, restricted from TP. If we quotient by this action, we get the adjoint bundle 1.1.7. The bundle TP is an equivariant vector bundle (we have right translations, on $VP \cong \mathfrak{d} \times P$ there is diagonal action), so we can take quotients by the action to get bundles over M.

Moreover, we have a surjective bundle map,

$$\begin{array}{ccc} TP & \xrightarrow{T\pi} & TM \\ \downarrow & & \downarrow \\ P & \xrightarrow{\pi} & M, \end{array} \tag{1.3}$$

which is constant on orbits of the $D-$action on TP (exercise), so it induces a surjective map over the identity of M,

$$\begin{array}{ccc} TP/D & \xrightarrow{T\pi/D} & TM \\ \downarrow & & \downarrow \\ P/D \cong M & \xrightarrow{id} & M. \end{array} \tag{1.4}$$

The kernel of this map is $VP/D \cong \mathfrak{d}_P$.

Example 1.3.C (Atiyah Lie algebroid). For any principal D-bundle P over M, we have *Atiyah algebroid* TP/D which fits into an exact sequence (Atiyah sequence) of vector bundles over M

$$0 \longrightarrow \mathfrak{d}_P \longrightarrow TP/D \xrightarrow{T\pi/D} TM \longrightarrow 0. \tag{1.5}$$

In other words, it is an extension of the Lie algebroid TM by a Lie algebra bundle $\mathfrak{d}_P$. Atiyah algebroid of the principal D-bundle D over a point is its Lie algebra $\mathfrak{d}$, which is easily seen from the sequence - tangent space of a point is just $\{0\}$, and the adjoint bundle is trivial, so $\mathfrak{d} \to TP/D$ must be an isomorphism.

Principal connections

The importance of Atiyah Lie algebroid rises notably from the fact, that principal connections of P are in one-to-one correspondence with splittings of 1.5. We recall the definition of a principal connection and mention most important properties, since they will become an important tool in explicit constructions of reductions of Courant algeboids.

[6] If we pick a basis $(\xi_1, \xi_2, \ldots, \xi_n)$ of $\mathfrak{d}$, then $(\xi_1^\#, \xi_2^\#, \ldots, \xi_n^\#)$ gives a global frame of VP.

Definition 1.3.5. Let (P, M, π) be a principal D-bundle. We denote by

$$\begin{aligned} \# : \mathfrak{d} &\to \Gamma(TP) \\ \xi &\mapsto \xi^{\#} \end{aligned}$$

the infinitesimal action of $\mathfrak{d}$ on P. A *(principal) connection* A on (P, M, π) is a $\mathfrak{d}$-valued one-form on the total space P, $A \in \Omega^1(P, \mathfrak{d}) \cong C^\infty(P, P \otimes \mathfrak{d})$ such that

1. $A(\xi^{\#}) = \xi$ for any $X \in \mathfrak{d}$ (A recovers Lie algebra elements on the fundamental fields of the action).

$$\begin{array}{ccc} P & \xrightarrow{\quad A \quad} & T^*P \otimes \mathfrak{d} \\ & {\scriptstyle \#}\nwarrow \quad \nearrow{\scriptstyle id} & \\ & \mathfrak{d} & \end{array} \tag{1.6}$$

2. A is *D-equivariant*:

$$R_d^* A = Ad_{d^{-1}} A$$

for every $d \in D$ where R_d^* is the pullback of one-forms by the right translation by the element d and Ad is the usual adjoint action (see 1.1.7) of D on its Lie algebra $\mathfrak{d}$.

We can think of A as a bundle map from TP to the trivial bundle $P \times \mathfrak{d}$. The equivariance of A translates to the equivariance of this map, with respect to right action of D on TP by tangent maps TR_d of right translations R_d, and the *right* action of D on $P \times \mathfrak{d}$ being pointwise the usual *left* Ad-action but acting by the inverse element to make it right.

$$D \circlearrowright TP \xrightarrow{\quad A \quad} P \times \mathfrak{d} \circlearrowleft D \tag{1.7}$$

Important remark 1.3.6. For any principal bundle, a principal connection always exists and there are many of them in general. This fact follows from the partition of unity.

Definition 1.3.7. Let A be a connection on a principal D-bundle (P, M, π), then

$$\mathcal{F} = dA + \frac{1}{2}[A \wedge A]_{\mathfrak{d}} \in \Omega^2(P)$$

is its *curvature form* or just *curvature.* The connection A is *flat*, if its curvature vanishes.

The curvature is a two-form on P with values in $\mathfrak{d}$. However, it is D-invariant and horizontal (vanishes whenever it eats at least one vertical field), so it admits a corresponding two-form F on the base space M with values in adjoint bundle $\mathfrak{d}_P$ (1.1.7) (notice that A is not horizontal, so we can not do this globally). [7] This form $F \in \Omega^2(M, \mathfrak{d}_P)$ is often also called curvature.

One of the striking features of the curvature form $\mathcal{F}$ is that it allows us to define some cohomology classes (characteristic classes), such as Pontryiagin

[7]This is why the forms of these two properties together are sometimes called *basic*.

classes, so it gives a link to algebraic topology. Those are elements of de Rham cohomology of M defined with the use of the curvature F and the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ but in the end, they do not depend on the choice of the connection. This is a subject of (much more general) Chern-Weil theory. [9] We will need (and define) just the first Pontryiagin class, which we use to formulate a topological obstruction for reduction of Courant algebroids.

Consider a principal D-bundle $P \xrightarrow{\pi} M$ with a connection $A \in \Omega^1(P, \mathfrak{d})$ and let $F \in \Omega^2(M, \mathfrak{d}_P)$ be the curvature form of A (on the base). Suppose moreover that $\mathfrak{d}$ is equipped with a non-degenerate pairing $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$. Then the first Pontryiagin class of the pair $(A, \langle\cdot,\cdot\rangle_{\mathfrak{d}})$ is the form

$$\langle F \wedge F\rangle_{\mathfrak{d}} \in H^4(M). \tag{1.8}$$

Definition 1.3.8. As we know that elements of Lie algebra $\mathfrak{d}$ correspond to vertical vector fields, we can think of A as a vector bundle map from TP to VP. The first condition says that A equals identity on VP, so it is a projection. We call its kernel *horizontal subbundle* HP and we obtain a vector bundle decomposition $TP \cong VP \oplus HP$ (dependent on A).

The equivariance of the connection 2 implies that the horizontal subbundle is D-invariant. i.e. it is preserved by the action of D. Its section are *horizontal vector fields*. For every vector field X on M there is a unique horizontal field X^h such that $T\pi(X^h) = X$ so we can think of horizontal fields as lifts of vector fields on M (and recall that by definition, we have $A(X^h) = 0$). In fact, principal connections on P are in one-to-one correspondence with D-invariant subbundles HP of TP such that $TP \cong HP + VP$.

Proposition 1.3.9 (Splittings and connections)**.** *Splittings of the sequence 1.5 (vector bundle maps $\sigma : TM \to TP/D$ satisfying $\pi \circ \sigma = id_{TM}$) are in one-to-one correspondence with principal connections of $P \to D$.*

Proof. Given a splitting $\sigma : TM \to TP/D$, the image of σ is D-invariant and complementary to VP, hence it defines a connection. Given a connection A, we define $\sigma(X) := X^h$ where X^h is the horizontal lift of X with respect to the connection. If σ preserves the bracket, we have $[X^h, Y^h] = [X, Y]$ for the corresponding connection A. ∎

Exercise 1.3.D. Splittings, which preserve brackets (*Lie algebroid morphisms*) correspond to connections with zero curvature (flat connections).

2. Courant algebroids

2.1 Courant algebroids

Quadratic Lie algebras

As Lie algebroids generalize Lie algebras, Courant algebroids generalize *quadratic* Lie algebras.

Definition 2.1.1. A quadratic Lie algebra is a Lie algebra $(\mathfrak{d}, [\cdot,\cdot]_{\mathfrak{d}})$ together with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ that is invariant under the adjoint action of $\mathfrak{d}$ on itself, i.e. for every $X, Y, Z \in \mathfrak{d}$

$$\langle [X,Y]_{\mathfrak{d}}, Z\rangle_{\mathfrak{d}} + \langle Y, [X,Z]_{\mathfrak{d}}\rangle_{\mathfrak{d}} = 0$$

Definition 2.1.2. We say that a Lie group D is *quadratic Lie group* if the corresponding Lie algebra $\mathfrak{d}$ is equiped with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ such that this form is D-invariant (with respect to usual Ad action of D on $\mathfrak{d}$). Equivalently, its Lie algebra $\mathfrak{d}$ is a quadratic Lie algebra.

Remark 2.1.3. If $\mathfrak{d}$ is a Lie algebra of a Lie group D, $\mathfrak{d}$ is quadratic if and only if D admits a bi-invariant pseudo-Riemannian structure (smooth fibre-wise symmetric nondegenerate bilinear form on the tangent bundle of D). So informally, quadratic Lie group is a Lie group where we can measure distances and angles, and the same is possible in the corresponding (quadratic) Lie algebra.

Example 2.1.A (Quadratic Lie algebras).

- Every Lie algebra $\mathfrak{d}$ has adjoint representation on itself:

$$\mathrm{Ad} : \mathfrak{d} \times \mathrm{End}(\mathfrak{d}) \tag{2.1}$$
$$\xi \mapsto [\xi, \cdot]_{\mathfrak{d}}, \tag{2.2}$$

  This representation gives us a symmetric bilinear form

$$\langle \xi, \mu\rangle = \mathrm{Tr}(\mathrm{Ad}(\xi)\,\mathrm{Ad}(\mu)),$$

  Killing form, which is non-degenerate if and only if Lie algebra $\mathfrak{d}$ is *semisimple.* Many important algebras are semisimple, for example $\mathfrak{so}_n$ for $n > 2$, $\mathfrak{sl}_n$ for $n > 1$ or $\mathfrak{sp}_{2n}$.

- If moreover $\mathfrak{d}$ is a Lie algebra of a compact group, Killing form is negative definite.

- commutative ones $\mathbb{R}^n$ with the trivial zero bracket and Euclidean inner product.

- We are mostly interested in quadratic Lie algebras of split characteristic (n, n). If we start with Lie algebra $\mathfrak{g}$ we can define (as a vector space) $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$. The vector space $\mathfrak{d}$ admits a natural pairing of characteristic (n, n) where n is the dimension of $\mathfrak{g}$. We could hope that $\mathfrak{d}$ has a compatible structure of a Lie algebra. This is a case just for a special kind of Lie algebras $\mathfrak{g}$, called *Lie bialgebras*. Those are Lie algebras equipped with a "cobracket" $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ compatible in some sence with the Lie algebra bracket. This additional structure induces Lie algebra structure on $\mathfrak{g}^*$ and consequently on $\mathfrak{d}$ making it a quadratic Lie algebra. See [10].

- If we consider $\mathfrak{g}, \mathfrak{g}^*$ and $\mathfrak{d}$ as in the previous example, we have a split exact sequence
$$0 \to \mathfrak{g} \to \mathfrak{d} \to \mathfrak{g}^* \to 0.$$
To generalize this, we can consider just pairs $(\mathfrak{d}, \mathfrak{g})$ where $\mathfrak{d}$ is a quadratic Lie algebra and $\mathfrak{g}$ is a Lagrangian subalgebra, in the sense that $\mathfrak{g} = \mathfrak{g}^\perp$. Note that this condition already forces the characteristic of $\mathfrak{d}$ to be split by (2.1.5). We then have the isomorphism $\mathfrak{d}/\mathfrak{g} \cong \mathfrak{g}^*$, but there is no decomposition like $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ in general as not every exact sequence of Lie algebras splits.

Definition 2.1.4. A pair $(\mathfrak{d}, \mathfrak{g})$ where $\mathfrak{d}$ is a quadratic Lie algebra and $\mathfrak{g}$ is a Lagrangian subalgebra ($\mathfrak{g} = \mathfrak{g}^\perp$) is called *Manin pair*.

When we work with a Manin pair $(\mathfrak{d}, \mathfrak{g})$ we usually assume that $\mathfrak{d}$ integrates to a (quadratic) Lie group D of dimension $2n$ and $\mathfrak{g}$ to a closed subgroup $G \subset D$.

Remark 2.1.5. On the other hand, existence of a subalgebra $\mathfrak{g} \subset \mathfrak{d}$ for which $\mathfrak{g} = \mathfrak{g}^\perp$ implies that the signature is split. If the signature of $\langle \cdot, \cdot \rangle_\mathfrak{d}$ is (p, q), then $\mathfrak{g} \subset \mathfrak{g}^\perp$ implies $\dim(\mathfrak{g}) \leq \min(p, q)$ and if $\mathfrak{g}^\perp \subset \mathfrak{g}$, then $\dim(\mathfrak{g}) \geq \max(p, q)$. We see that this together gives $p = q$.

Nondegeneracy is quite a strong condition. For example, it is the reason why we cannot simply take the quotient of symplectic manifold (M, ω) by symplectic action of a Lie group G - we obtain a manifold M/G with a well defined two form ω', but it will not be non-degenerate. The same problem occurs with Courant algebroids since they are equipped with non-degenerate pairing on sections.

Courant algebroids

Courant algebroids and Dirac structures were introduced by Liu, Weinstein and Xu in [11].

Definition 2.1.6. A *Courant algebroid* (CA) is a vector bundle $E \to M$ with a fibrewise non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ (*the pairing*), with a vector bundle map $\rho : E \to TM$ *the anchor*, and with a $\mathbb{R}$-bilinear map (*the bracket*) $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ such that for all $u, v, w \in \Gamma(E)$ and $f \in C^\infty(M)$

$$[u, [v, w]] = [[u, v], w] + [v, [u, w]] \tag{CA1}$$
$$[u, fv] = f[u, v] + \rho(u)(f)v \tag{CA2}$$
$$\rho(u)\langle v, w \rangle = \langle [u, v], w \rangle + \langle v, [u, w] \rangle \tag{CA3}$$
$$\langle w, [u, v] + [v, u] \rangle = \rho(w)\langle u, v \rangle. \tag{CA4}$$

These axioms surely ask for some explanation. Axioms CA1 and CA2 are the same as for Lie algebroids. Axiom CA3 is compatibility of pairing with bracket. The main difference between Courant and Lie algebroids is that the bracket of CA is not antisymmetric. However, it is not that bad since its symmetric part is ruled by the last axiom CA4.

Remark 2.1.7. The anchor preserves brackets, as in the case of Lie algebroids (with the same argument 1.2.2).

Warning 2.1.8. It is not true that $[fu, v] = f[u, v] - \rho(v)(f)u$ as in case of Lie algebroids. Instead, we get an additional term comming from the lack of antisymmetry of the bracket and by CA4 we have:

$$[fu, v] = f[u, v] - \rho(v)(f)u + d_E(f\langle u, v\rangle),$$

where $d_E f \in \Gamma(E)$ is given by $\langle u, d_E f\rangle = \rho(u)f$ (more on this below). This is a source of many technical problems with Courant algebroids.

Remark 2.1.9. There exist an equivalent definition of CA with antisymmetric bracket, which does not satisfy CA1. We will not use it. The important thing is that if you have both Leibnitz identity CA1 and antisymmetry it forces the anchor to be zero so we do not get much interesting stuff.

Example 2.1.B. A Courant algebroid E over a point is a quadratic Lie algebra. See 1.2.A.

The most important example you should keep in your mind is the generalized tangent bundle equipped with the obvious pairing of vector fields and one-forms and so-called *Courant bracket*, which is a clever extension of Lie bracket of vector fields. This is analogical to usual tangent bundle being the canonical example of a Lie algebroid. Things get a bit more complicated, because there is actually some freedom in choosing the bracket - it can be "twisted" by a three-form, which is one of the first motivations to consider abstract Courant algebroids, to explain this freedom as a "choice of splitting" into a direct sum of some subbundles.

Example 2.1.C. Let M be a manifold of dimension m and consider the (fibrewise) direct sum of its tangent and cotangent bundle $TM \oplus T^*M$. It is sometimes called the *generalized tangent bundle.* This bundle has a natural non-degenerate symmetric bilinear form of the signature (m, m) given by the natural pairing of tangent vectors and one-forms:

$$\langle X + \alpha, Y + \beta\rangle = \beta(X) + \alpha(Y). \tag{2.3}$$

Moreover, $TM \oplus T^*M$ admits a natural Courant bracket

$$[(X, \alpha), (Y, \beta)] = \Big([X, Y], \mathcal{L}_X\beta - i_Y d\alpha\Big) \quad \forall X, Y \in \Gamma(TM), \alpha, \beta \in \Gamma(T^*M). \tag{2.4}$$

It is also useful to thing about the bracket as an action of the space sections of $TM \oplus T^*M$ on itself, which is maybe enlightening in the matrix notation

$$(X + \alpha) \cdot \begin{pmatrix} Y \\ \beta \end{pmatrix} = \begin{pmatrix} \mathcal{L}_X & 0 \\ d\alpha & \mathcal{L}_X \end{pmatrix} \begin{pmatrix} Y \\ \beta \end{pmatrix} \tag{2.5}$$

We see that without α, this would be just the ordinary action of vector fields by Lie derivative on TM and T^*M (recall that $\mathcal{L}_X(Y)$ is precisely $[X, Y]$.)

Proposition 2.1.10. *The bundle $TM \oplus T^*M \to M$ with $\langle\cdot,\cdot\rangle, [\cdot,\cdot]$ defined above and the anchor $\rho : TM \oplus T^*M$ being the projection onto the first factor, is a Courant algebroid.*

Exercise 2.1.D. Prove this to remember CA axioms (hint: it is just Cartan calculus).

Remark 2.1.11. In the literature, you can also find a different convention of the pairing.

$$\langle X+\alpha, Y+\beta\rangle = \frac{1}{2}(\beta(X)+\alpha(Y)),$$

because some people find amusing that then $\langle X+\alpha, X+\alpha\rangle = \alpha(X)$. This is a common source of confusion.

Exact and Transitive Courant algebroids

Nondegeneracy of the pairing $\langle\cdot,\cdot\rangle_E$ means precisely that the map $g_E : E \to E^*$ given by $g_E(v)(w) = \langle v, w\rangle_E$ is an isomorphism. So we can identify E with E^* by this map. The anchor map $\rho : E \to TM$ induces $\rho^T : T^*M \to E^*$ (by composition of linear forms with ρ). Finally, we can use this map and the inverse of g_E to define *adjoint anchor map*:

$$\rho^* = g_E^{-1} \circ \rho^T.$$

By definition, we have the (defining) relation for ρ^*, which is useful in every argument concerning this map:

$$\langle \rho^*(\alpha), u\rangle_E = \alpha(\rho(u)) \tag{2.6}$$

for any $u \in \Gamma(E)$ and $\alpha \in \Gamma(T^*M) = \Omega^1(M)$

and more specifically for exact one-forms

$$\langle \rho^*(df), u\rangle_E = \rho(u)(f). \tag{2.7}$$

The composition $\rho^* \circ d$ is sometimes denoted d_E. Notice that the axiom CA4 gets a form

$$[u,v] + [v,u] = d_E\langle u,v\rangle (=: \rho^* d(\langle u,v\rangle)) \tag{CA4'}$$

Lemma 2.1.12. *For any Courant algebroid $\rho \circ \rho^* = 0$.*

This lemma gives us a sequence (chain complex) of vector bundles over M

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0, \tag{2.8}$$

Definition 2.1.13. A Courant algebroid $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ is *exact*, if the sequence (2.8) is exact. In other words, ρ is surjective and $\mathrm{im}(\rho^*) = \ker(\rho)$. (clearly ρ^* is injective if ρ is surjective).

Every exact CA is, in fact, isomorphic to the generalized tangent bundle $TM \oplus T^*M$ with the usual pairing and some kind of twisted bracket.

Example 2.1.E. Given an exact Courant algebroid, we can always choose an isotropic splitting $\sigma : TM \to E$ of the exact sequence (2.8), meaning that $\rho(\sigma(X)) = X$ and $\langle\sigma(X), \sigma(Y)\rangle = 0$ for every $X, Y \in \Gamma(M)$. This splitting is not unique. For a given σ we can define a three-form H on M ("curvature") by

$$H(X,Y,Z) := \langle[\sigma(X),\sigma(Y)],\sigma(Z)\rangle \quad \forall X,Y,Z \in \Gamma(TM)$$

Notice that the skew-symmetry of H follows because the image of σ is isotropic subspace, and the bracket is skew-symmetric on isotropic subspaces, as we can see from the axiom CA4. With a given splitting σ, we can actually find bundle isomorphism of E with the direct sum $TM \oplus T^*M$.

$$\begin{aligned} TM \oplus T^*M &\cong E \\ X + \alpha &\mapsto \sigma(X) + \rho^*(\alpha). \end{aligned}$$

Exercise 2.1.F. Show that this map preserves the pairing (hint - the images of both σ and ρ^* are isotropic, and use the defining property 2.6 of ρ^*.) Show also that anchors commute with this isomorphism.

By this exercise, we can safely identify E with the generalized tangent bundle $TM \oplus T^*M$ (so a vector field X with the section $\sigma(X)$ of E) and the only task is to compute how the bracket of E behaves in terms of vector fields and one-forms. In terms of this identification, H is as follows:

$$H(X, Y, Z) := \langle [X, Y], Z \rangle \quad \forall X, Y, Z \in \Gamma(TM) \subset \Gamma(E).$$

This form is key to the classification of exact Courant algebroids. It satisfies two properties:

- It is closed: $dH = 0$, so it represents a cohomology class in $H^3(M)$.
- If we change the splitting, the cohomology class of H is preserved.

Proof. We leave the first part as an exercise. Use the invariant formula for exterior derivative and axioms of CA.

$$\begin{aligned} dH(X_0, X_1, X_2, X_3) = &\sum_i (-1)^i X_i H(X_0, \ldots, \hat{X}_i, \ldots, X_3) - \\ &- \sum_{i<j} (-1)^{i+j-1} H([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_4) \end{aligned}$$

Let's take two isotropic splittings σ and σ'. Then $\rho(\sigma(X) - \sigma'(X)) = 0$ for every $X \in \Gamma(TM)$ so $\sigma(X) - \sigma'(X) = \rho^*(\alpha)$ for a unique $\alpha \in T^*M$. We define a two tensor $B \in \Gamma(T^*M^{\otimes 2})$ by $i_X B = \alpha$. It is clearly smooth. By definition of ρ^* we have

$$B(X, \rho(v)) = \langle \sigma(X) - \sigma'(X), v \rangle$$

for any $v \in \Gamma(E)$. We apply this to $v = \sigma(Y)$ and by properties of σ and σ', we get

$$\begin{aligned} B(X, Y) &= -\langle \sigma(X) - \sigma'(X), \sigma(Y) \rangle \\ &= -\langle \sigma'(X), \sigma(Y) \rangle \\ &= -\langle \sigma'(X), \sigma(Y) - \sigma'(Y) \rangle \\ &= -\langle \sigma(Y) - \sigma'(Y), \sigma'(X) \rangle = -B(Y, X) \end{aligned}$$

■

This implies that exact Courant algebroids are classified by third de Rham cohomology group $H^3(M, \mathbb{R})$. Finally, Courant bracket $[\cdot,\cdot]_E$ on $E \cong TM \oplus T^*M$ reads

$$[(X,\alpha),(Y,\beta)] = \Big([X,Y], \mathcal{L}_X\beta - i_Y d\alpha + H(X,Y,\cdot)\Big)$$

for all $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$. We can see that it is the same bracket as of example 2.1.C, except it is "twisted" be the three-form H.

$$H \to H + dB$$

As a consequence, when we integrate the form H over a closed 3-manifold, we can think about the result as an invariant of an exact Courant algebroid, and not just the form H. So when you spot a god-given closed three-form somewhere, you can actually try to find a natural Courant algebroid for it. See for example [12] where (topological) T-duality is formulated in terms of H-twisted de Rham cohomology. We hope this could be generalized (and clarified) in terms of Courant algebroids.

Definition 2.1.14. We say that a Courant algebroid E is *transitive* if its anchor ρ is surjective.

Important remark 2.1.15. Every exact algebroid is transitive, but not the other way round. A transitive Courant algebroid E over M is exact if and only if it has rank $2m$ where $m = \dim(M)$. It follows from the CA short exact sequence (2.8)/rank-nullity theorem (exercise). Therefore it is easy to recognize (non)exactness of CA's.

Example 2.1.G. Transitive algebroids which are not exact come naturally from the "heterotic" reduction 3.3.3 of CA's by an action of a quadratic Lie group.

Example 2.1.H. [13] Let $\mathfrak{d}$ be a Lie algebra with a non-degenerate invariant symmetric pairing $\langle\cdot,\cdot\rangle$ (i.e. $\mathfrak{d}$ is a CA over a point), D a connected Lie group which integrates $\mathfrak{d}$, and G is a Lie subgroup of D with corresponding Lie subalgebra $\mathfrak{g}$ of $\mathfrak{d}$ such that $\mathfrak{g}^\perp = \mathfrak{g}$ (i.e. $\mathfrak{g} \subset \mathfrak{d}$ is a Lagrangian Lie subalgebra). Note that the existence of such algebra already implies that the characteristic of $\langle\cdot,\cdot\rangle$ is split (n, n), as we know from the note 2.1.5. We consider the trivial vector bundle $\mathfrak{d} \times\ D/G$ over D/G, equivalently, the pullback of $\mathfrak{d} \to \{*\}$ by the trivial map $D/G \to \{*\}$.We claim that

$$\mathfrak{d} \times D/G \to D/G$$

is "naturally" an exact Courant algebroid:

- The pairing is easiest to define (by its point-wise nature). We just define the pairing on every fibre as the pairing $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$.
- The anchor of $\mathfrak{d} \to \{*\}$ is trivial (tangent bundle of a point is $\{0\}$) and there is no sensible way how to use it to define a bundle map from $d \times D/G \to T(D/G)$. Fortunately, we can put the anchor by hand by taking the infinitesimal action of $\mathfrak{d}$ on D/G corresponding to canonical (right) action of D on the (left) coset space D/G.
- Once we have defined the anchor, the bracket on $\mathfrak{d} \times D/G$ can be defined on constant sections as the Lie bracket on $\mathfrak{d}$ and then extended by Leibnitz rule.

- It is easy to check that we get a Courant algebroid. Exactness follows from the dimensional argument 2.1.15. If $\mathfrak{g}$ has dimension g, then $\mathfrak{d}$ has dimension $2g$ because $\mathfrak{g}$ is Lagrangian and D/G has dimension g, so the rank of the fibre is twice the dimension of the base space.

We can actually generalize this example for arbitrary transitive Courant algebroid with a non-trivial base at the place of $\mathfrak{d} \to \{*\}$. The CA structure on a pullback will be constructed using the reduction of Courant algebroids. More on this in the next chapter.

Example 2.1.I. Let E be a Courant algebroid with the anchor ρ and take $F := \ker(\rho)$. Then F is a subbundle and both bracket and pairing on E can be restricted to F so that we obtain a Courant algebroid structure (with the new anchor being identically zero). At the same time, it is a Lie algebroid, because the bracket becomes antisymmetric due to the vanishing of the anchor. Notice also that this algebroid is not transitive, and we usually call an algebroid with zero anchor *totally intransitive* algebroid.

Exercise 2.1.J. Find a Courant algebroid E over M, with dimension $2m = 2\dim(M)$ such that it is not exact. Possible hint the example 2.1.I and the theorem 3.3.3.

Generalized metric

By a *metric* we mean Riemannian metric on some manifold M (smooth choice of a positive definite inner product on every tangent space of M). Riemann metric plays a crucial role in general relativity. In strings/supergravity, there are additional fields - so-called B-field, which is for us just a nondegenerate two-form, a closed three-form H (and dilaton φ we omit here).

A generalized metric allows us to take together the Riemannian metric g on M and B-field into one structure (H is given by the splitting of an exact CA). Both g and B can be thought as a nondegenerate bundle map from tangent bundle TM to cotangent bundle T^*M and g (resp. B) is its symmetric (resp. antisymmetric) part.

$$g + B : TM \to T^*M. \tag{2.9}$$

Giving such a map is equivalent to giving its graph

$$(X, (g + B)X) \subset TM \oplus T^*M$$

which is a subbundle on which the restriction of natural pairing of $E = TM \oplus T^*M$ is positive definite. This motivates the following definition:

Definition 2.1.16. *A generalized metric in a Courant algebroid $E \to M$ is a maximal positive-definite subbundle $V^+$of E with respect to $\langle\cdot,\cdot\rangle_E$.*

We also define V^- as an orthogonal complement of V^+ with respect to $\langle\cdot,\cdot\rangle_E$. It is also a subbundle, negative definite with the respect to $\langle\cdot,\cdot\rangle_E$. By nondegeneracy of $\langle\cdot,\cdot\rangle$ we have

$$E = V^+ \oplus V^-.$$

For an element $v \in V$ we define orthogonal projections to V^+ (resp V^-) as v^+ (resp v^-). The vector v^+ is uniquely characterized by $v^+ \in V^+$ and

$$\langle v, v^+\rangle = \langle v, v\rangle. \tag{2.10}$$

Changing the sign of $\langle\cdot,\cdot\rangle_E$ to $-\langle\cdot,\cdot\rangle_E$ does not change the definition of orthogonal complement and projections, which is easy to see from the property 2.10. We also have a formula for the reflection $\mathcal{V}^+ : E \to E$ with respect to the subspace V^+:

$$\mathcal{V}^+(v) = 2v^+ - v.$$

Remark 2.1.17. Notice that in the definition of generalized metric there is no use of the bracket $[\cdot,\cdot]_E$. On the other hand, we can use the bracket to find an elegant formula for Levi-Civita connection of the metric g on M. For more on this, see [14], but beware of the symmetric definition of the bracket there.

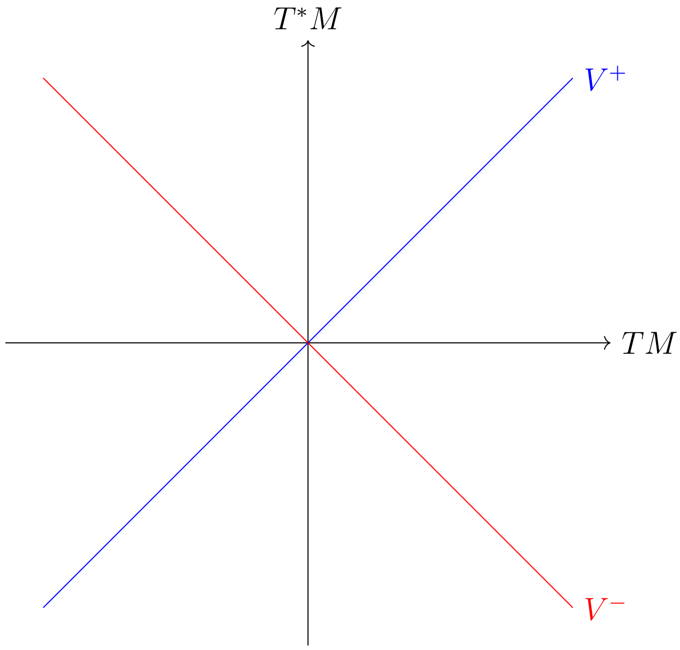


Figure 2.1: Generalized metric

Remark 2.1.18 (Equivalent formulations of generalized metric.). A generalized metric can be also defined as an involutive bundle map $\tau : E \to E$, $\tau^2 = id_E$, for which $\mathcal{G}(\cdot,\cdot) = \langle\cdot,\tau(\cdot)\rangle$ is symmetric and positive definite (point-wise) bilinear form on E. Such a map has two eigenvalues: 1 and -1, and V^+, V^- are corresponding eigenspaces. Conversely, we can define τ as a reflection with respect to the subspace V^+. More on this later in (1). For a detailed treatment of different formulations, see: [15].

The expression $g + B$ (2.9) of a generalized metric on an exact CA E depends on the splitting of E. What happens if we change the splitting?

Lemma 2.1.19. *Let E be an exact Courant algebroid with a given spliting inducing the three form H. Suppose we have also a generalized metric V^+, which has a form $g + B$ in this splitting (g is Riemannian metric and B is a two-form). If we change the splitting by a two-form C:*

$$H \to H + dC$$

the metric V^+ will change its form to $g + B - C$.

Exercise 2.1.K. Prove this (recall that two isotropic splittings always differ by a two-form B in a sense that $\sigma'(X) = \sigma(X) + \rho^*(B(X, \cdot)))$.

So for any exact Courant algebroid E we can take a splitting $E \cong TM \oplus T^*M$ and get a closed three-form H. A generalized metric V^+ is then equivalent to a pair g, B where g is a Riemannian metric on M and B is a two-form on M. Together, an exact CA with a chosen splitting and generalized metric gives us the triple (g, B, H).

But in fact, there is a unique splitting of E such that $B = 0$: In lemma 2.1.19 we just take the splitting with $H := H - dB$. So we deduce that an exact Courant algebroid with a generalized metric is equivalent to a pair g, H, g is a metric, H is a closed 3-form.

Example 2.1.L. Let $\mathfrak{d}$ be a CA over a point $M = \{*\}$ (a quadratic Lie algebra). Generalized metric V^+ is just a maximal positive definite subspace of $\mathfrak{d}$. Notice that since $\mathfrak{d}$ is *not* exact, we can not speak about any g or B (and M is just a point anyway). However, $\mathfrak{d} \to \{*\}$ is transitive, which is important.We can consider Lagrangian subalgebra $\mathfrak{g} \subset \mathfrak{d}$, so we have the exact sequence:

$$0 \to \mathfrak{g} \to \mathfrak{d} \to \mathfrak{d}/\mathfrak{g} \to 0,$$

and the pairing gives us a map

$$\mathfrak{d} \to \mathfrak{g}^* \tag{2.11}$$

$$\xi \mapsto \langle \xi, \cdot \rangle_{\mathfrak{d}} \restriction_{\mathfrak{g}} \tag{2.12}$$

This map has $\mathfrak{g}$ as kernel ($\mathfrak{g}$ is Lagrangian) so we obtain a map $\mathfrak{d}/\mathfrak{g} \to \mathfrak{g}^*$ which is injective and in fact an isomorphism, because $\mathfrak{d}/\mathfrak{g}$ and $\mathfrak{g}^*$ have the same dimension. So we have the exact sequence

$$0 \to \mathfrak{g} \to \mathfrak{d} \to \mathfrak{d}/\mathfrak{g} \to 0.$$

If it splits (this is not possible for every pair $(\mathfrak{g}, \mathfrak{d})$ but for many examples it is), we get a decomposition $\mathfrak{d} \cong \mathfrak{g} + \mathfrak{g}^*$ with the pairing given by canonical pairing of forms and vectors. A generalized metric V^+ is then equivalent to a graph $(v, R(v))$ of a map $R : \mathfrak{g} \to \mathfrak{g}^*$ which is equivalent to a bilinear form B such that $B(v, w) = R(v)(w)$ on $\mathfrak{g}$ with positive definite symmetric part:

$$B(v, w) + B(w, v) = \langle (v, B(v, \cdot)), (w, B(w, \cdot)) \rangle_{\mathfrak{d}} \geq 0.$$

Example 2.1.M. Let V^+ be a generalized metric in $\mathfrak{d}$ as in the previous example. Then $V^+ \times D/G \subset \mathfrak{d} \times D/G$ is a generalized metric in $\mathfrak{d} \times D/G$ since the pairing is defined pointwise. This algebroid is exact, so if we choose a splitting of it (see 2.8), we obtain a Riemannian metric g, a two form B and a closed three-form H on M. The σ-model based on these data is called *Poisson-Lie σ-model.* Poisson-Lie T-duality relates two (or many) such σ-models which live on quotients by different subrgoups G_0, G_1 of the same D and they are given by the same generalized metric on $\mathfrak{d}$.

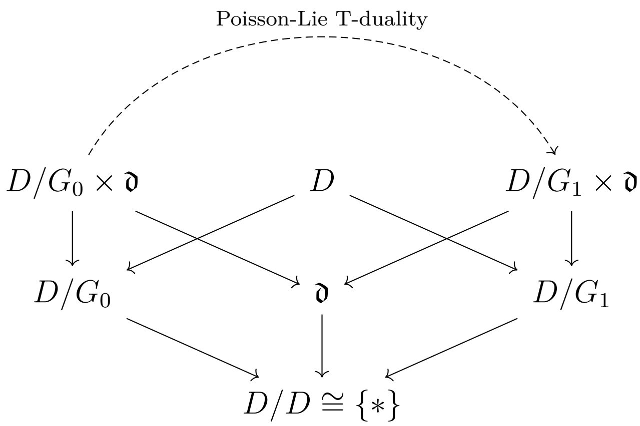


Figure 2.2: Poisson-Lie T-duality scheme over a point.

2.2 Relations between Courant algebroids

It is time to sketch what are we actually going to do (and then continue for some time with definitions and get back in the next chapter). Our goal is to relate somehow two exact Courant algebroids (equipped with generalized metrics) which are seemingly unrelated - they live on different manifolds and metrics look very differently. But in fact, both algebroids and metrics are coming from the same source, which is a third Courant algebroid, but which is just transitive.

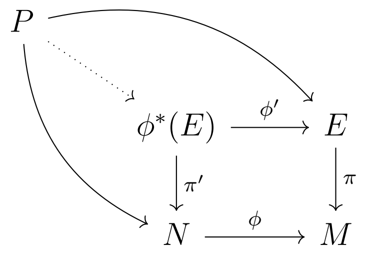
Pullbacks of Courant algebroids

Last two examples form together an example of a "pullback" of Courant algebroid and generalized metric on it. We would like to generalize them to the situation where we have a nontrivial manifold M instead of a point, a transitive Courant algebroid E over M and a surjective submersion $N \to M$. (instead of the trivial map $D/G \to \{*\}$)

Recall that when $N \xrightarrow{\phi} M$ is a map of manifolds and $E \xrightarrow{\pi} M$ is a fibre bundle than we can construct the pullback bundle $\phi^*(E) \xrightarrow{\pi'} N$ such that there is a bundle map

$$\begin{array}{ccc} \phi^*(E) & \xrightarrow{\phi'} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\phi} & M \end{array} \tag{2.13}$$

satisfying the universal property of pullback: For every bundle $F \to N$ and a bundle map $F \to E$, there is a unique bundle map $F \to \phi^*(E)$ such that the following diagram commutes:



(2.14)

The fibre F_p of pullback bundle is isomorphic to $E_{\phi(p)}$ for every point p. Pullback of a vector bundle is a vector bundle too.

If $\phi : N \to M$ is a submersion and E is a CA over M, we can consider the pullback bundle of E by ϕ to obtain a vector bundle E_1 on N of the same rank as E. Now we can ask which structures can be transferred from E to E_1. It is easy to see (not so easy, I should write it down), that we can pull back the pairing to obtain a nondegenerate pairing on E_1, which basically follows from $C^\infty-$bilinearity of the pairing. The problem comes with the anchor and consequently with the bracket because the bracket is not C^∞-bilinear and the lack of this property is measured by the anchor. So we can define bracket on the pullback of sections, but we cannot just simply extend it to other sections without the anchor. It seems that there is no natural anchor, coming from E. Fortunately, as examples above show, we can sometimes define a suitable anchor on M_1 with no obvious relation to the anchor no M.

Example 2.2.A. Courant algebroid $\mathfrak{d} \times D/G$ over D/G from the example 2.1.H is the pullback of (quadratic Lie algebra) $\mathfrak{d}$ over $M = \{*\}$ as a vector bundle by the trivial map $D/G \to \{*\}$. The anchor of $\mathfrak{d}$ is a trivial zero map, while the anchor of $\mathfrak{d} \times D/G$ is nontrivial (infinitesimal action of $\mathfrak{d}$ on D/G).

Pullback of a generalized metric is again a generalized metric.

Dirac structures

Dirac structures were originally defined in [16] and [17] for the generalized tangent bundle $TM \oplus T^*M$ as a far-reaching generalization of symplectic and presymplectic and Poisson structures (so coming from problems of classical mechanics and control theory). Dirac structures were motivated by ideas in Dirac paper [18], hence the name.

Recall that a symplectic form ω on a manifold M is a closed ($d\omega = 0$) nondegenerate two-form on M. A symplectic form $\omega \in \Omega^2(M)$ induces a map $B : TM \to T^*M$ by $X \to i_X\omega = \omega(X, \cdot)$ for any vector field X. A symplectic form is non-degenerate, which means that this map is an isomorphism, and it is closed, which allows us to define Poisson structure (Lie algebra structure on $C^\infty(M)$ which satisfies Leibnitz rule with respect to the product of functions).

We can relax these conditions to get more general structures, and an elegant method how to do it consistently is to consider the graph of B as a subbundle of $TM \oplus T^*M$ and then look to all subbundles with similar properties as those extracted from the graph of B.

This is where Courant algebroids (and their subbundles) naturally come to the game - we are looking for subbundles of generalized tangent bundle and its pairing and bracket are suitable for formulating desired properties. See [19] for a detailed review.

In our point of view, Dirac structures in a Courant algebroid play a similar role as Lagrangian submanifolds do in symplectic geometry.

We always suppose that our CA's have the pairing of split signature. In fact, we know from the remark (2.1.5) that it is forced by the existence of a Lagrangian subbundle, which is equal to its orthogonal complement.

Definition 2.2.1. *Dirac structure* in Courant algebroid $E \to M$ is a submanifold N of M together with a Lagrangian subbundle $L \subset E_{\restriction N}$ with respect to $\langle\cdot,\cdot\rangle_E$ such that $\rho(L) \subset TN$ and if any two sections v, w of E satisfy $v, w_{\restriction N} \in L$, then $[v,w]_{\restriction N} \in L$. The submanifold N is called the *support* of the Dirac structure L (or L is *supported* on N.)

Proposition 2.2.2. *Given an exact Courant algebroid $E \cong TM \oplus T^*M$ with a closed three-form H, a Dirac structure L supported on N with $\rho(L) = TN$ corresponds to a two-form $B \in \Omega^2(N)$ such that $dB = H_{\restriction N}$.*

Remark 2.2.3. If we do not assume that $\rho(L) = TN$, then $\rho(L) \subset TN$ is an integrable distribution (roughly a subset of TN whose sections are locally generated by a set of linearly independent vector fields closed under Lie bracket). Such a distribution is tangent to some submanifolds $K \subset N$, *integral leaves* and Dirac structure corresponds to a two form B_K, on each such leaf K, for which $B_K = H_{\restriction K}$. We will neither prove or use it.

Proof.
1. At every point $n \in N$, the fibre L_n of the subbundle L consists of pairs of tangent vectors and one-forms at n: (X,α) (we use the same notation as for sections). From the surjectivity of $\rho_{\restriction L}$, we know that every tangent vector occurs in such a pair (ρ is just a projection to the first coordinate). If $(X,\alpha),(X,\beta) \in L_n$ then $(0,\alpha-\beta) \in L_n$ and now for every vector $Y \in T_nN$ we can find a pair $(Y,\gamma) \in L_n$ and use that the pairing of E vanishes on L_n (Lagrangian property) to get
$$\langle(0,\alpha-\beta),(Y,\gamma)\rangle_E = (\alpha-\beta)(Y) + \gamma(0) = 0$$
by Lagrangian property of L_n, so $\alpha_{\restriction N} = \beta_{\restriction N}$.

2. We see that for every tangent vector X we have a unique one-form α at the same point such that $(X,\alpha) \in L$. This assigment defines a map $\mathcal{B} : TN \to T^*N$ which is smooth (exercise). Such a map is equivalent to a two-tensor $B \in T^*N^{2\otimes}$ given by $B(X,Y) := \mathcal{B}(X)(Y)$. Moreover, again by Lagrangian property, we obtain
$$B(X,Y) + B(Y,X) = \mathcal{B}(X)(Y) + \mathcal{B}(Y)(X) = \langle(X,\mathcal{B}(X),(Y,\mathcal{B}(Y)\rangle = 0,$$
so B is a two-form on N, $B \in \Omega^2(N)$.

3. Now we consider two sections (X, i_XB) and (Y, i_YB) of E that restrict to sections of $\Gamma(L)$ on N. The condition of $[\cdot,\cdot]$-preserving of sections of L. is

equivalent to (we restrict everything to N)

$$\begin{aligned}
[X,Y] + i_{[X,Y]}B &= [(X, i_X B), (Y, i_Y B)]_E \\
[X,Y] + i_{[X,Y]}B &= [X,Y] + \mathcal{L}_X i_Y B - i_Y d i_X B + i_X i_Y H \\
i_{[X,Y]}B &= \mathcal{L}_X i_Y B - i_Y i_X B + i_X i_Y H \\
i_{[X,Y]}B &= i_{[X,Y]}B + i_Y \mathcal{L}_X B - i_Y d i_X B + i_X i_Y H \qquad (2.15) \\
0 &= i_Y d i_X B + i_Y i_x dB - i_Y d i_X B + i_X i_Y H \qquad (2.16) \\
i_X i_Y dB &= i_X i_Y H
\end{aligned}$$

This holds for every $X, Y \in \Gamma(E)$, so $dB = H_{\restriction_N}$. where in 2.15 we used the well-known formula

$$[L_X, i_Y] = i_{[X,Y]}$$

and in 2.16 we used Cartan formula

$$\mathcal{L}_X = d i_X + i_X d.$$

Finally, it is easy to verify that given such a two-form B, L defined by

$$L_n = \{X \in T_n N, \alpha \in T_n^* M : B(X, \cdot) = \alpha_{\restriction_{T_n^* N}}\}$$

is a Dirac structure with the support N.

■

Remark 2.2.4. If $H = 0$, we obtain a closed two-form B by preceeding proof. and $\mathcal{B}$ has a constant rank. The manifold M equipped with such a structure is called presymplectic. If $\mathcal{B}$ happens to be an isomorphism (= B is non-degenerate), we obtain a symplectic manifold (M, B). This means that Dirac structures generalize (they relax both non-degeneracy and closedness of B) symplectic and presymplectic structures, which was also the original motivation for their definition. See [19].

Exercise 2.2.B. Let $\mathfrak{g} \subset \mathfrak{d}$ be a Lagrangian Lie subalgebra of a quadratic Lie algebra $\mathfrak{d}$ i.e. $(\mathfrak{d}, \mathfrak{g})$ is a Manin pair. If $\mathfrak{g}'$ is another Lagrangian Lie subalgebra of $\mathfrak{d}$ then the trivial bundle $D/G \times \mathfrak{g}'$ is a Dirac structure (with full support) in Courant algebroid $D/G \times \mathfrak{d}$ from the example 2.1.H.

Generalized isometries

Definition 2.2.5. Let $E_1 \to M_1$ and $E_2 \to M_2$ are Courant algebroids, and denote $\bar{E}_1$ the same CA as E_1 just with the pairing $-\langle , \rangle_{E_1}$ instead of $\langle , \rangle_{E_1}$. *Dirac relation* between E_1 and E_2 is a Dirac structure in $\bar{E}_1 \times E_2 \to M_1 \times M_2$.

Dirac structures play a similar role to canonical structures (Lagrangian submanifolds of products) in symplectic geometry. Namely, they allow us to define generalized maps between Courant algebroids.

Definition 2.2.6. Let V^+ be a generalized metric in E. Reflection $\mathcal{V}$ with respect to V^+ is a usual orthogonal reflection with respect to $\langle , \rangle$, in other words $\mathcal{V}(x) = x^+ - 2x$.

Definition 2.2.7 (Generalized isometry.)**.** Let E_1 and E_2 be Courant algebroids with generalized metrics V_1^+ and V_2^+ respectively. Dirac relation L between E_1 and E_2 is *generalized isometry*, if $(\mathcal{V}_1 + \mathcal{V}_2)L = L$.

We can formulate the definition of generalized isometry in a more familiar way. We start with linear-algebraic lemma:

Lemma 2.2.8. *Let $E_1, \langle\cdot,\cdot\rangle_{E_1}, V_1^+$ and $E_2, \langle\cdot,\cdot\rangle_{E_2}, V_1^+$ be two vector spaces equipped with nondegenerate pairings (=symmetric bilinear forms) and generalized metrics (maximal positive-definite subspace with respect to the pairing) and L be a Lagrangian subspace of $\bar{E}_1 \times E_2$. Then the following are equivalent:*

1. *$(\mathcal{V}_1 + \mathcal{V}_2)L = L$ where $\mathcal{V}_i$ is a reflection of E_i with respect to the subspace V_i.*
2. *L is a graph of a bijection ϕ from E_1 to E_2, which preserves metrics: $\phi(V_1^+) \subset V_2^+$.*

Proof. 1. Assume first 2. This means that L is of the form $(e, \phi(e)), e \in E_1$ and if $e \in V_1^+$ $(e = e^+)$ then $\phi(e) = \phi(e)^+$. We have to show that if $(e, \phi(e)) \in L$ then $(e^+ - 2e, \phi(e)^+ - 2\phi(e) \in L)$. Because L is a linear subspace, this is equivalent to $(e^+, \phi(e)^+) \in L$ which is equivalent to

$$\phi(e^+) = \phi(e)^+.$$

As we have seen, the element e^+ is characterized by two properties: $e^+ \in V^+$ and $\langle e^+, e^+\rangle = \langle e^+, e\rangle$. L is Lagrangian, so ϕ preserves the pairing.

$$\begin{aligned}\langle\phi(e^+), \phi(e^+)\rangle_{E_2} &= \langle e^+, e^+\rangle_{E_1} \\ &= \langle e^+, e\rangle_{E_1} \\ &= \langle\phi(e^+), \phi(e)\rangle_{E_2}\end{aligned}$$

We have assumed that $\phi(e^+) \in V_2^+$ so together we obtain that $\phi(e^+)$ satisfies characteristic properties of $\phi(e)^+$.

2. Suppose now 1. We will show that for every $e \in E_1$ there is a unique $\phi(e) \in E_2$ such that $(e, \phi(e)) \in L$ and ϕ is a linear map preserving the metric. Suppose that there are two elements $g, h \in E_2$ such that $(e, g) \in L$ and $(e, h) \in L$. We can substract them to get $(0, f := g - h) \in L$. Now, by the reflection condition we obtain $0^+ = f^+$, but 0^+ is 0. Similarly $0^- = f^-$ and we have $V^+ \cap V^- = \{0\}$, so $f = 0$ and $g = h$. Similarly, we can show that ϕ is injective and

■

Proposition 2.2.9. *Let E_1,E_2, V_1^+, V_2^+ be as above and L is a Dirac relation between E_1 and E_2. Then L is generalized isometry if and only if L_p is a graph of a bijection $\phi_p : E_{1,\pi_1(p)} \to E_{2,\pi_2(p)}$ and ϕ_p maps V_1^+ to V_2^+.*

Proof. Since this statement is purely pointwise, we can apply the lemma to $L_p, E_{1,\pi_1(p)}, E_{2,\pi_2(p)}$ and we are done. ■

3. Reduction of Courant algebroids

3.1 Why reduction

In this chapter, we describe the reduction of Courant algebroids. It is an important construction which allows us to construct interesting examples of Courant algebroids. We can think about Courant algebroid $E \to P$ as some kind of additional structure on a manifold P, similarly to Riemannian metric or a symplectic form (we know, for example, that an exact Courant algebroid is more or less equivalent to a closed three-form).

If a connected Lie group D acts on P so that also E is equipped with some kind of a D-action (to be defined below), it is natural to ask whether we can define a Courant algebroid on the space of orbits P/D. This task comes with natural difficulties (lack of non-degeneracy of reduced pairing), with similar origins as with group actions in symplectic geometry. Fortunately, these problems can be solved by applying similar methods.

Remark 3.1.1. In fact, as we have mentioned in the introduction, there is a more general procedure of reduction of *graded symplectic manifolds*, *graded symplectic reduction*, which generalizes both usual symplectic reduction and Courant algebroid reduction. You can find more in [20].

3.2 Extended actions

The idea of extended actions comes from [21]. We will not need it in the full generality. Suppose we have a principal D-bundle $P \xrightarrow{\pi} M$. Recall that the action of a Lie group D on P induces the infinitesimal action of Lie algebra $\mathfrak{d}$, which is a Lie algebra homomorphism $\# : \mathfrak{d} \to \Gamma(TP)$ and also an action of $\mathfrak{d}$ on the space $\Gamma(TM)$ by $\xi \cdot X := [\xi^{\#}, X]$ for every $\xi \in \mathfrak{d}$ and $X \in \Gamma(TM)$.

Let E be Courant algebroid over P. Since the space of sections $\Gamma(E)$ is *not* a Lie algebra, we can not talk about a Lie algebra homomorphism $\mathfrak{d} \to \Gamma(E)$, but we can still consider a map $R : \mathfrak{d} \to \Gamma(E)$ which is a morphism of corresponding brackets:

$$R([\xi, \nu]_{\mathfrak{d}}) = [R(\xi), R(\nu)]_E. \tag{3.1}$$

One important feature of the infinitesimal action is preserved - such a map R still induces a true Lie algebra action on the space of sections $\Gamma(E)$, i.e. Lie algebra morphism from $\mathfrak{d} \to \mathrm{Der}(\Gamma(E))$ defined by:

$$\begin{aligned} \mathfrak{d} &\to \mathrm{Der}(\Gamma(E)) \\ \xi &\mapsto [R(\xi), -]_E \end{aligned}$$

so the Lie algebra action induced by R on $\Gamma(E)$ is given by

$$\xi \cdot v = [R(\xi), v]_E. \tag{3.2}$$

Proof. This statement is a direct consequence of Leibnitz identity for E.

$$\begin{aligned}
[\xi,\nu]_{\mathfrak{d}} \mapsto & [R([\xi,\nu]_{\mathfrak{d}}),\cdot]_E \\
&= [[R(\xi),R(\nu)]_E,\cdot]_E && (R \text{ preserves bracket}) \\
&= [R(\xi),[R(\nu),\cdot]_E]_E - [R(\nu),[R(\xi),\cdot]_E]_E && (\text{Leibnitz identity}) \\
&= [[R(\xi),\cdot]_E,[R(\nu),\cdot]_E] && (\text{commutator in } \mathrm{Der}(\Gamma(E))).
\end{aligned}$$

■

Definition 3.2.1 ([21], definition 2.12)**.** Let P be a principal D-bundle over M with the corresponding infinitesimal action $\#$ of $\mathfrak{d}$. A *trivially extended action* of $\mathfrak{d}$ on Courant algebroid $E \to P$ is a map $R : \mathfrak{d} \to \Gamma(E)$, which

1. preserves bracket 3.1,
2. we have $\rho \circ R = \#$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{d} & \xrightarrow{R} & \Gamma(E) \\
& \searrow^{\#} & \downarrow{\rho} \\
& & \Gamma(TM),
\end{array} \tag{3.3}$$

3. the induced action $\xi \cdot v = [R(\xi), v]_E$ integrates to the D-action on the space of sections $\Gamma(E)$ (and hence on total space E.)
4. If moreover $\mathfrak{d}$ is equipped with a pairing $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ and the map R preserves the pairing, the pair (E, R) is called *equivariant Courant algebroid.*

So an equivariant Courant algebroid is an equivariant bundle with respect to some action of D, such that corresponding infinitesimal action comes from the map R which preserves pairing and bracket and it gives the action on base through the anchor.

If E has a surjective anchor (E is transitive) and possesses a trivially extended action, it can be always made to be an equivariant Courant algebroid, by "pulling back" the pairing of E to $\mathfrak{d}$. So let $R : \mathfrak{d} \to \Gamma(E)$ be a trivially extended action. We define pairing on $\mathfrak{d}$ with use of R and $\langle\cdot,\cdot\rangle_E$ as follows:

$$\langle\xi,\nu\rangle_{\mathfrak{d}} := \langle R(\xi), R(\nu)\rangle_E.$$

This is in general a function, so to make the pairing well defined, it has to be constant on P. If P is connected, it is really the case by the following argument:

$$d\langle R(\xi),(R(\nu)\rangle_E = [R(\xi),R(\nu)] + [R(\nu),R(\xi)] = R([\xi,\nu]+[\nu,\xi]) = R(0) = 0.$$

So any extended action R gives us a pairing on the Lie algebra $\mathfrak{d}$ in such a way (tautologically) that R preserves pairings.

Equivariant splitting

The condition that the action of $\mathfrak{d}$ given by R integrates to the action of D is a technical one and it can be easily fullfilled if we have a $\mathfrak{d}$-*invariant isotropic splitting* of E with respect to $\mathfrak{d}$-actions on $\Gamma(TM)$ and $\Gamma(E)$. This means that there is a bundle map $\sigma : TM \to E$ such that $\sigma([\xi, X]) = [R(\xi), \sigma(X)]_E$ and $\rho \circ \sigma = Id_{TM}$.

Proposition 3.2.2. *If E, $\mathfrak{d}$ and R are as above and E has an $\mathfrak{d}$-equivariant isotropic splitting with respect to R, than R is an extended action on E, and E in an equivariant Courant algebroid:*

Proof. See [21]. ■

$$0 \longrightarrow \Gamma(T^*M) \xrightarrow{\rho^*} \Gamma(E) \underset{\sigma}{\overset{\rho}{\rightleftarrows}} \Gamma(TM) \longrightarrow 0 \tag{3.4}$$

We can reformulate this conditions for exact Courant algebroids in terms of the generalized tangent bundle: σ gives us an isomorphism of Courant algebroids $E \cong TM \oplus T^*M$ with bracket twisted by some closed three-form H. The map R is then given by

$$R(\xi) = \xi^\# + \omega_\xi$$

for some one-forms ω_ξ forming together a linear map $\omega : \mathfrak{d} \to \Omega^1(P)$. The action on $E \cong TM \oplus T^*M$ given by R is:

$$[\xi^\# + \omega_\xi, X + \alpha]_E = [\xi^\#, X] + \mathcal{L}_{\xi^\#}\alpha - i_X d\omega_\xi + i_X i_{\xi^\#} H \tag{3.5}$$

or in matrix notation

$$(\xi^\# + \omega_\xi) \cdot \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{\xi^\#} & 0 \\ i_{\xi^\#}H - d\omega_\xi & \mathcal{L}_{\xi^\#} \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix} \tag{3.6}$$

hence we can immidiately see that the equivariance of the splitting can be reformulated as follows:

$$i_{\xi^\#}H = d\omega_\xi. \tag{3.7}$$

This condition has an important consequence - we know that the form H is closed, i.e. $dH = 0$. Now, if we look at Lie derivative along some fundamental vector field $\xi^\#$ and use Cartan formula, we get

$$\mathcal{L}_{\xi^\#}H = di_{\xi^\#}H + i_{\xi^\#}dH = d^2\omega_\xi H = 0 \tag{3.8}$$

We see that H is an invariant form.

We have not used yet that R preserves bracket. Lets see what it means in terms of the splitting. By the definition of R

$$[R(\xi), R(\mu)]_E = [\xi^\#, \mu^\#] + \mathcal{L}_{\xi^\#}\omega_\mu - i_{\mu^\#}d\omega_\xi + i_{\mu^\#}i_{\xi^\#}H \tag{3.9}$$

$$R([\xi, \mu]_\mathfrak{d}) = [\xi, \mu]^\# + \omega_{[\xi,\mu]} \tag{3.10}$$

We know that (3.9) and (3.10) are equal. The map # preserves bracket, so the vector part is fine and we are left with the equality of one-forms, which we reformulate using 3.7.

$$\begin{aligned}\omega_{[\xi,\mu]} &= \mathcal{L}_{\xi^\#}\omega_\mu - i_{\mu^\#}d\omega_\xi + i_{\mu^\#}i_{\xi^\#}H\\ \omega_{[\xi,\mu]} &= \mathcal{L}_{\xi^\#}\omega_\mu - i_{\mu^\#}d\omega_\xi + i_{\mu^\#}d\omega_\xi\\ \omega_{[\xi,\mu]} &= \mathcal{L}_{\xi^\#}\omega_\mu.\end{aligned} \tag{3.11}$$

The resulting equation 3.11 means precisely that $\omega : \mathfrak{d} \to \Omega^1(P, \mathfrak{d})$ is an equivariant map with respect to ad-action on $\mathfrak{d}$ (the derivative of *Ad*-action) and the (usual) action of $\mathfrak{d}$ by Lie derivative of fundamental vector fields on one-forms.

Remark 3.2.3. Both conditions can be put together into this claim: The map $H+\omega : \mathfrak{d} \to \Omega^*(M)$ is a closed *equivariant form* of Cartan model of the equivariant cohomology, see [21], page 8.

Fixing a connection

We have chosen an equivariant isotropic splitting so that we have

$$R(\xi) = \xi^\# + \omega_\xi.$$

We have not really specified these ω's yet, but we have the neccesary condition (3.7),

$$d\omega_\xi = i_{\xi^\#}H. \tag{3.12}$$

There is a nice way for choosing the map ω - we fix a connection $A \in \Omega^1(P, \mathfrak{d})$ and define

$$\omega_\xi = \langle A, \xi\rangle_{\mathfrak{d}}.$$

This is fine because the pairing kills the ”Lie algebra” part and we are left with a one-form on M. In fact (we will not prove this), for a fixed connection A, there is always an isotropic splitting σ such that R is of this form [22]. The condition (3.7) becomes a relation between H and A :

$$\langle dA, \xi\rangle_{\mathfrak{d}} = d\langle A, \xi\rangle_{\mathfrak{d}} = i_{\xi^\#}H. \tag{3.13}$$

Now we reverse the logic and we will analyze what $dH = 0$ means for the connection A. We will obtain that the first Pontryiagin class, which can be constructed from A, vanishes. On the other hand, we know that this class is independent of the connection, so we obtain a topological criterion on the principal bundle P (and the pairing of $\mathfrak{d}$) for the existence of a trivially extended action on the exact Courant algebroid over P, resp. existence of an equivariant exact Courant algebroid.

Proposition 3.2.4 ([22]). *Let $E \overset{\pi}{\to} P$ be an exact Courant algebroid over a principal D-bundle P. Suppose that R is a trivially extended action given by*

$$R(\xi) = \xi^\# + d\langle A, \xi\rangle_{\mathfrak{d}}$$

for some fixed connection A and a non-degenerate pairing on $\mathfrak{d}$.

Then the first Pointryagin class of $(A, \langle\cdot,\cdot\rangle)$ vanishes.

Proof. We start with (3.13),

$$i_{\xi^\#} H = \langle dA, \xi \rangle_{\mathfrak{d}} \tag{3.14}$$

The same property has the Chern-Simons form of A,

$$CS_3(A) := \langle \mathcal{F} \wedge A \rangle - \frac{1}{3} \langle A \wedge [A \wedge A]_{\mathfrak{d}} \rangle_{\mathfrak{d}},$$

namely

$$i_{\xi^\#} CS_3(A) = \langle dA, \xi \rangle_{\mathfrak{d}}.$$

The Chern-Simons form also satisfies

$$dCS_3(A) = \langle \mathcal{F} \wedge \mathcal{F} \rangle = \pi^* \langle F \wedge F \rangle_{\mathfrak{d}}.$$

This together implies that the most general form of the three-form H is

$$H = \pi^* H_0 + CS_3(A) \tag{3.15}$$

where $H_0 \in \Omega^3(M)$ is a three-form on the base M (not necessarily closed),

$$CS_3(A) = \langle \mathcal{F} \wedge A \rangle - \frac{1}{6} \langle A \wedge [A \wedge A]_{\mathfrak{d}} \rangle_{\mathfrak{d}}$$

is the Chern-Simons form of A, and $\mathcal{F}$ is the curvature of A, $\mathcal{F} = dA + \frac{1}{2}[A \wedge A]_d$.

In particular,

$$0 = dH = \pi^* dH_0 + \langle \mathcal{F} \wedge \mathcal{F} \rangle_{\mathfrak{d}}$$

so

$$dH_0 = \langle F \wedge F \rangle_{\mathfrak{d}}$$

hence the first Pontryagin class has to vanish. ∎

3.3 Reduction of Courant algebroids

Heterotic reduction

Heterotic reduction is described in [23], page 17.

Remark 3.3.1. We call the reduction with a non-degenerate pairing on Lie algebra *heterotic* and the resulting Courant algebroid is *heterotic Courant algebroid.* The reason is that there is a connection of this construction with the *heterotic string theory.* See [23].

Suppose now that we have an equivariant exact Courant algebroid E over P as usual.

We consider the image of R

$$K = \mathrm{im}(R) \subset E.$$

Some observations:

- K is a subbundle of E because for every point $p \in P$, the mapping $R : \mathfrak{d} \to E_p$ is injective (the infinitesimal action of a free action is always pointwise injective and we have $\rho \circ R = \#$ so R is injective too). The orthogonal complement $K^\perp$ is a subbundle too.

- K is D-invariant. (D-action is a global version of infinitesimal action of R, so on infinitesimal level it is just the identity $[R(\xi), R(\mu)] = R([\xi, \mu])$.
- $K^\perp$ is D invariant because K is D-invariant and $\langle \cdot, \cdot \rangle$ is D-invariant. Hence $K^\perp / K \cap K^\perp$ is D-invariant.
- The pairing on E restricts to (in general degenerate) pairing on $K^\perp$ and induces a nondegenerate pairing on $K^\perp/(K \cap K^\perp)$.

Remark 3.3.2 (dimension counting). If the dimension of $\mathfrak{d}$ is d and the dimension of M is m, we have the following:

- $\dim K = d$ by the injectivity of R,
- $\dim K^\perp = \dim TP + T^*P - \mathfrak{d} = 2m + 2d - d = 2m+$ by non-degeneracy of $\langle \cdot, \cdot \rangle$,
- $\dim K^\perp / K \cap K^\perp = \dim K^\perp = 2m + d$.

We define the reduced Courant algebroid as

$$\left(\frac{K^\perp}{K \cap K^\perp} \right) / D.$$

Its sections correspond to D-invariant sections of $(K^\perp / K \cap K^\perp)$:

$$\Gamma \left(\left(\frac{K^\perp}{K \cap K^\perp} \right) / D \right) \cong \Gamma \left(\frac{K^\perp}{K \cap K^\perp} \right)^D .$$

We now look at the situation when the pairing is non-degenerate, as in the previous section ($\mathfrak{d}$ is quadratic Lie algebra). In this case, $K^\perp / K \cap K^\perp = K^\perp$ because every subspace is complementary to its orthogonal complement.

Example 3.3.A. Let $P \to M$ be a principal D-bundle, with $\mathfrak{d}$ quadratic, and we fix a connection $A \in \Omega^1(P, \mathfrak{d})$. We take the bundle $TM \oplus \mathfrak{d}_P \oplus T^*M$ (generalized tangent bundle (2.1.C) plus the adjoint bundle (1.1.7)). The space of sections $\Gamma(TM \oplus \mathfrak{d}_P \oplus T^*M)$ consists of triples (X, Ψ, α) where X is a vector field, Ψ is a section of $\mathfrak{d}_P$ and α is a one-form. This bundle can be canonically identified with the reduced bundle $K^\perp / D$. This will give us an explicit description of the reduced bundle and a nice new concrete example of Courant algebroid. We will identify the spaces of sections, in other words, we have to find an isomorphism from $\Gamma(TM \oplus \mathfrak{d}_P \oplus T^*M)$ to $(K^\perp)^D$, the space of D-invariant sections of $K^\perp$.

1. Every vector field X on M has its horizontal lift X^h on P which is a D-invariant section of $K^\perp$: If $\xi \in \mathfrak{d}$ then

$$\langle X^h, \xi^\# + \langle A, \xi \rangle_{\mathfrak{d}} \rangle_E = \langle A(X^h), \xi \rangle_{\mathfrak{d}} = 0$$

because A kills horizontal vector fields.

2. Every section Ξ of $\mathfrak{d}_P$ gives a D-invariant section of $K^\perp$ which at point $p\mathcal{P}$ is an element

$$\Xi^\# + \langle A, \Xi^\# \rangle_{\mathfrak{d}}.$$

3. Every one form $\alpha \in \Omega^1(M)$ has its pullback $\pi^*(\alpha) \in \Omega^1(P)$ by the projection $\pi : P \to M$. It is obviously D-invariant and orthogonal to K.

Theorem 3.3.3 (Heterotic Reduction)**.** *Let $P \to M$ be a principal D-bundle, with $\mathfrak{d}$ equipped with a non-degenerate pairing $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$, and $A \in \Omega^1(P, \mathfrak{d})$ a fixed connection. Denotw $F \in \Omega^1(M, \mathfrak{d}_P)$ its curvature and suppose that the first Pontryiagin class is trivial by $dH_0 = \langle F \wedge F \rangle_{\mathfrak{d}}$ for some $H_0 \in \Omega^3(M)$. We can identify the sections of $\mathfrak{d}_P$ with D-invariant functions $P \to \mathfrak{d}$. The connection A also induces a vector bundle connection ∇ on $\mathfrak{d}_P$ by $\nabla_X(\Psi) = X^h\Psi$.*

Then the reduced Courant algebroid is isomorphic to the bundle

$$E' = TM \oplus \mathfrak{d}_P \oplus T^*M$$

with the pairing given by

$$\langle (X, \Psi, \alpha), (Y, \Psi', \beta) \rangle = \alpha(Y) + \beta(X) + \langle \Psi, \Psi' \rangle_{\mathfrak{d}}, \tag{3.16}$$

the bracket

$$\begin{aligned}[(X, \Psi, \alpha), (Y, \Psi', \beta)] =& ([X, Y], \nabla_X \Psi' - \nabla_Y \Psi - F(X, Y) - [\Psi, \Psi']_{\mathfrak{d}}, \qquad (3.17)\\ & L_X\beta - i_Y\alpha + H_0(X, Y, \cdot)/2 \\ & - \langle F(\cdot, X), \Psi' \rangle_{\mathfrak{d}} + \langle F(\cdot, Y), \Psi \rangle_{\mathfrak{d}} + \langle \nabla\Psi, \Psi' \rangle_{\mathfrak{d}}). \qquad (3.18)\end{aligned}$$

Finally, the anchor is given by the projection to TM. This algebroid is obviously transitive but not exact.

The proof is too technical to write it down. The important thing is that the reduced Courant algebroid exists, it is not exact, because it additionally contains one copy of the adjoint bundle, and we are able to write down explicit formulas if we wish so.

Important remark 3.3.4. It is also important that there is no natural bundle map from $E = TP \oplus T^*P$ to E' because, in the reduction procedure, we take a subspace and then a quotient (similarly as in symplectic reduction). However, we expect there is a "Courant algebroid morphism" (i.e. some correspondence given by Dirac structure of the product) between E and E').

Isotropic reduction

We have just seen what happens if we reduce Courant algebroid by an action of (quadratic) Lie group with non-degenerate pairing on its Lie algebra. Now we are going to describe a different reduction procedure of the same algebroid E over P.

This time, we choose a subgroup G of D such that its Lie algebra $\mathfrak{g}$ is Lagrangian inside $\mathfrak{d}$, i.e. a maximal subalgebra with the property that the pairing of $\mathfrak{d}$ restricts to constant zero on $\mathfrak{g}$. This is kind of oposite extreme to non-degeneracy. While in the non-degenerate situation the image of the action and its orthogonal complement were complementary, $K \cap K^\perp = 0$, this is no longer true for G. We remember that in the (arbitrary) reduction procedure, we have to quotient out the intersection $K \cap K^\perp$. In this case, it is just K because $K \subset K^\perp$ by isotropy of $\mathfrak{g}$. This means that we quotient out one copy of $\mathfrak{g}$ and hence the resulting algebroid will be exact, just by the dimension argument 2.1.15.

We can use the same setting (choice of connection and splitting) as for the heterotic reduction but keep in mind that now we reduce from P to $P/G = N$, so we also want to consider G-related objects. The connection A we chose in the last section has values in $\mathfrak{d}$, not $\mathfrak{g}$, but we keep it (so the naive guess $\langle A, \xi\rangle_{\mathfrak{g}} \upharpoonright_{\mathfrak{g}} = 0$ is not true).

As we said, we are going to obtain an exact Courant algebroid over P/G. Poisson-Lie T-duality (or plurality) relates these algebroids for different subroups of D. We omit details which are similar to the previous treatment and use the same notation with subscript 0 to emphasize that we are working with a subgroup/subalgebra with zero pairing.

So we have $G \subset D$ and corresponding $\mathfrak{g} \subset \mathfrak{d}$ with $\mathfrak{g} = \mathfrak{g}^{\perp}$. The group D acts on P so also G has an action on P with the space of orbits $N := P/G$ and P is a principal G-bundle over N. We also take the restriction of the infinitesimal action $\# : \mathfrak{d} \to \Gamma(TP)$ to subalgebra $\mathfrak{g}$ (denoted by the same symbol). Since $\mathfrak{g}$ is Lagrangian, the pairing is zero $\langle \xi, \mu\rangle_{\mathfrak{d}} \upharpoonright_{\mathfrak{g}} = R(\xi, \mu) = 0$ for all $\xi, \mu \in \mathfrak{g}$.

Finally, we restrict the trivial extended action R to $R_0 : \mathfrak{g} \to \Gamma(E)$ and repeat the construction with the same fixed connection $A \in \Omega^1(P, \mathfrak{d})$ (so it is *not* a principal connection of the G-bundle $P \to P/G = N$) and the same equivariant splitting of E.[1]

We define $K_0 := \mathrm{im}(R_0) \subset \mathrm{im}(R) \subset E$ and denote the orthogonal complement K_0 in E by $K_0^{\perp}$. This subbundle contains $K^{\perp}$ obviously, because K_0, the image of $\mathfrak{g}$, is in the image K of $\mathfrak{d}$. But we can say more:

Lemma 3.3.5.

$$K_0^{\perp} = K^{\perp} \oplus K_0. \tag{3.19}$$

Proof. We know that $K \oplus K^{\perp} = E$. So if $v \in K_0^{\perp}$, we can decompose it into $v = v_K + v_{K^{\perp}}$ with $v_K \in K$ and $v_{K^{\perp}} \in K^{\perp}$. Now v_K is in K, the image of $\mathfrak{d}$-action and it is orthogonal to K_0, the image of $\mathfrak{g}$. But $\mathfrak{g}$ is Lagrangian, so any element of $\mathfrak{d}$ which is orthogonal to $\mathfrak{g}$, lies in $\mathfrak{g}$. Hence $v_K \in K_0$. ∎

Now the reduced Courant algebroid is

$$\left(\frac{K_0^{\perp}}{K_0 \cap K_0^{\perp}}\right)/G = \left(\frac{K_0^{\perp}}{K_0}\right)/G = K^{\perp}/G. \tag{3.20}$$

The space of sections of the reduced CA is isomorphic to

$$(K^{\perp})^G.$$

Theorem 3.3.6 (Reduction by isotropic)**.** *$K^{\perp}/G$ is an exact Courant algebroid over $N = P/G$.*

Proof. See [21]. ∎

It is time for some more dimension counting:

- $\dim \mathfrak{g} = \dim \mathfrak{d}/2 = d/2$ (this is why we like Lagrangian subalgebras),

[1] There is also a possibility to work with a G-connection and a different splitting in which the extended action R_0 becomes just $\#$ on $\mathfrak{g}$, which is useful if we want to compute Ševera form of the reduced CA on N. This is, however, not so important here.

- $\dim N = \dim P/G = \dim P - \dim G = \dim M + \dim G = m + d/2$
- $\operatorname{rank} E = \operatorname{rank} TP \oplus TP = 2n + 2d$
- $\operatorname{rank} K = d/2$
- $\operatorname{rank} K^{\perp} = 2m + 3d/2$ (not $2m + d/2$ as one could expect - K and $K^{\perp}$ are not complementary, rather $K^{\perp}$ consists of all tangent vectors TN, which gives $m + d/2$ (so also elements of K) and such one forms, which anihilate K, but those have dimension d)
- $\operatorname{rank} K^{\perp}/K = 2m + d = m + d/2 + m + d/2 = \dim TN \oplus T^*N$ so we immidiately see (remark 2.1.15) that the result is an exact Courant algebroid.

4. General formulation of Poisson-Lie T-duality

In the last chapter, we state the general formulation of (geometric) Poisson-Lie T-duality and we briefly mention some applications.

4.1 Overview of T-duality

The setting

The whole setting of Poisson-Lie T-duality is a bit complicated, so let's start slowly while looking at the diagram 4.1.

- So far we have met a principal D-bundle P over a space M. We are going to forget about P for some time and we will see it again when we will construct the example.
- On M we take a transitive Courant algebroid E' with the pairing of split signature (we think about the reduced algebroid E' from an exact algebroid on P).
- We consider two manifolds N_0, N_1 with maps $\phi_0 : N_0 \to M$ and $\phi_1 : N_1 \to M$ which are both surjective submersions.
- Suppose moreover that on every N_i there is an exact Courant algebroid E_i which is isomorphic to the pullback of E' by ϕ_i as a vector bundle and has a compatible pairing, for $i = 1, 2$. Notice that this forces the ranks of N_0 and N_1 to be the same.

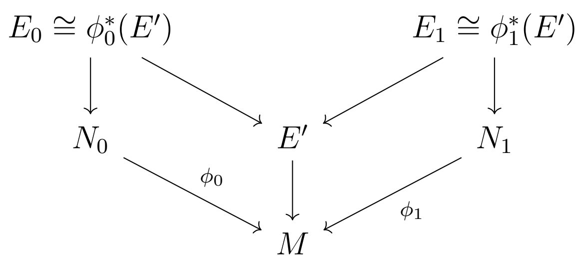


Figure 4.1: Setting

We have seen such a setting with M being a point in the example 2.1.M. This together means that manifolds N_0 and N_1 are quite arbitrary, but their Courant algebroids "have the same origin", which is the algebroid $E' \to M$. It may seem as something artificial, but it can be quite non-trivial to see that such Courant algebroid E' exists when you see just N_0 and N_1 and their algebroids (and also keep in mind that we have to do a step aside from the nice world of exact CA's because E' is not exact).

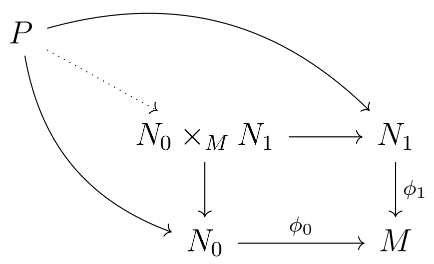


Figure 4.2: The fibered product.

4.2 Poisson-Lie T-duality

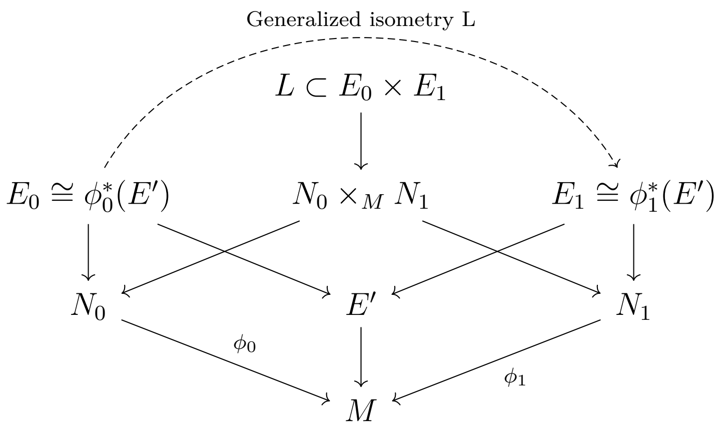


Figure 4.3: General Poisson-Lie T-duality

Theorem 4.2.1 (Poisson-Lie T-duality). *[13] Let $E' \to M$ be a transitive Courant algebroid equipped with a generalized metric V^+. Let $N_0 \xrightarrow{\phi_0} M$ and $N_1 \xrightarrow{\phi_1} M$ be two surjective submersions. Suppose that there is an exact Courant algebroid $E_0 \to N_0$ (resp $E_1 \to N_0$) which is isomorphic to the pullback of E' by ϕ_0 (resp ϕ_1) as a vector bundle with compatible pairing. Then pullback V_0^+ (resp V_1^+) of V^+ by ϕ_0 (resp. ϕ_1) is a generalized metric in E_0 (resp. E_1) and there is a generalized isometry L between E_0 and E_1 sending V_0^+ to V_1^+.*

Proof. The idea is simple - E' sits inside the product of E_0 with E_1 in a diagonal fashion and together with its metric V^+ it provides generalized isometry. This gives a hint that the support of Dirac structure should be a set of pairs of elements $(x, y) \in N_0 \times N_1$ such that $\phi_0(x) = \phi_1(y)$. These pairs form the fibered product

$$N_0 \times_M N_1.$$

This set is actually a smooth manifold, which follows from the fact, that maps ϕ_0, ϕ_1 are *transversal* to each other. This is always true if at least one of them is a surjective submersion.

The "diagonal" $E_0 \times_{E'} E_1$ is a Dirac structure with the support $N_0 \times_M N_1$ and it gives us the generalized isometry between E_0 and E_1. ∎

4.3 The reduction example

As we have said, the fundamental example of Poisson-Lie T-duality comes from the reduction. The setting is described in the diagram 4.4. Courant algebroids E_0 on the left and E_1 on the right side are results of isotropic reductions of the upper algebroid by two subgroups G_0, G_1 of D (with Lagrangian Lie subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ of $\mathfrak{d}$), respectively, and Courant algebroid E' below is the result of the heterotic reduction of E by D.

Dashed arrows symbolize reductions – remember there is no bundle map in general, see (3.3.4).

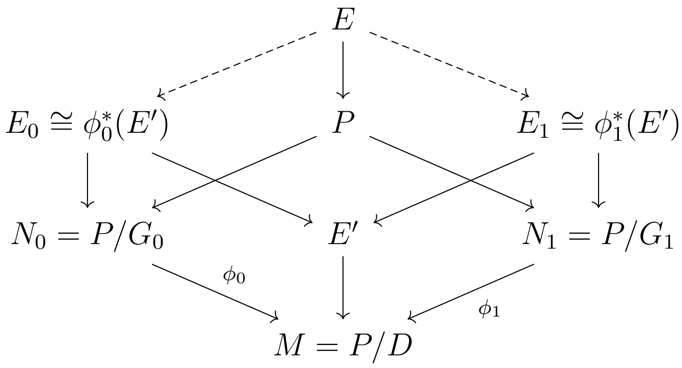
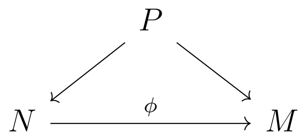


Figure 4.4: Reduction diagram

We define $N_i = P/G_i$, $M = P/D$, then ϕ_i is a natural projection from P/G_i to P/D which sends a G-orbit to a unique D-orbit containing it ($\phi : p\cdot G \mapsto p \cdot D$ for any $p \in P$). The maps ϕ_i are surjective submersions for $i = 1, 2$ (both projections $P \to P/D$ and $P \to P/G$ are surjective submersions hence also the map ϕ is a surjective submersion.



It is not hard to realize that E_0 (resp. E_1) is really isomorphic to the pullback of E' via ϕ_0 (resp. ϕ_1). If we equip E' by a generalized metric, we can pullback it to E_0 and E_1 and get a generalized isometry between E_0 and E_1.

Remark 4.3.1. It is also possible to take a generalized metric on the upper Courant algebroid E and reduce them to E_0, E_1, E'. If we do it, we obtain the generalized geometry between E_0 and E_1. For reductions of metrics see [23].

Exercise 4.3.A. Find another non-trivial example of T-duality 4.2.1.

Conclusion

In the thesis, we were studying geometrical structures which are useful for describing non-abelian Poisson-Lie T-duality. We recalled the language of Lie and Courant algebroids and described how two Courant algebroids on different spaces can be related, which we interpreted as Poisson-Lie T-duality.

We would like to sketch some possible directions for future study.

1. First of all, we should understand better the applications of this geometric picture, namely σ-models arising from it. It also includes understanding, how to incorporate dilation field into it. There are papers [4] and more recently [24] and [25] dealing with that.

2. Courant algebroids are a special case of graded symplectic manifolds. The language of graded geometry is a bit involved, but it provides many new insights into the topic (generalizations, constructions. . .). It would be maybe fruitful to translate as much as possible to this language.

3. There is also topological (abelian) T-duality [12], relating the twisted equivariant cohomology (or K-theory) of two different spaces. We would like to extend it to Poisson-Lie T-duality, with using of this geometric setup.

Bibliography

[1] Ctirad Klimčík. Poisson-Lie T duality. *Nucl. Phys. Proc. Suppl.*, 46:116–121, 1996.

[2] Ctirad Klimčík and Pavol Ševera. Poisson-Lie T duality and loop groups of Drinfeld doubles. *Phys. Lett.*, B372:65–71, 1996.

[3] Ctirad Klimčík and Pavol Ševera. Poisson Lie T duality: Open strings and D-branes. *Phys. Lett.*, B376:82–89, 1996.

[4] Branislav Jurčo and Jan Vysoký. Poisson-Lie T-duality of string effective actions: a new approach to the dilaton puzzle. *J. Geom. Phys.*, 130:1–26, 2018.

[5] Drawing a fiber bundle?

[6] Terence Tao. What is a gauge? *What's new*, 2008.

[7] Richard S. Palais. A global formulation of the Lie theory of transformation groups. *Mem. Amer. Math. Soc. No.*, 22:iii+123, 1957.

[8] nLab authors. action Lie algebroid. `http://ncatlab.org/nlab/show/action%20Lie%20algebroid`, May 2019. Revision 4.

[9] nLab authors. Chern-Weil theory. `http://ncatlab.org/nlab/show/Chern-Weil%20theory`, May 2019. Revision 17.

[10] nLab authors. Lie bialgebra. `http://ncatlab.org/nlab/show/Lie%20bialgebra`, May 2019. Revision 2.

[11] Zhang-Ju Liu, Alan Weinstein, and Ping Xu. Manin triples for Lie bialgebroids. *J. Differential Geom.*, 45(3):547–574, 1997.

[12] Peter Bouwknegt, Jarah Evslin, and Varghese Mathai. T-duality: topology change from H-flux. *Comm. Math. Phys.*, 249(2):383–415, 2004.

[13] Pavol Ševera. Canonical T-dual. *unpublished*, 2018.

[14] Nigel Hitchin. Lectures on generalized geometry. In *Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics*, volume 16 of *Surv. Differ. Geom.*, pages 79–124. Int. Press, Somerville, MA, 2011.

[15] B. Jurčo and J. Vysoký. *Courant algebroid connections and string effective actions*. World Sci. Publ., Hackensack, NJ, 2017.

[16] Ted Courant and Alan Weinstein. Beyond Poisson structures. In *Action hamiltoniennes de groupes. Troisième théorème de Lie (Lyon, 1986)*, volume 27 of *Travaux en Cours*, pages 39–49. Hermann, Paris, 1988.

[17] Theodore James Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319(2):631–661, 1990.

[18] P. A. M. Dirac. Generalized Hamiltonian dynamics. *Canadian J. Math.*, 2:129–148, 1950.

[19] Henrique Bursztyn. A brief introduction to Dirac manifolds. In *Geometric and topological methods for quantum field theory*, pages 4–38. Cambridge Univ. Press, Cambridge, 2013.

[20] Alberto S. Cattaneo and Florian Schätz. Introduction to supergeometry. *Rev. Math. Phys.*, 23(6):669–690, 2011.

[21] Henrique Bursztyn, Gil R. Cavalcanti, and Marco Gualtieri. Reduction of Courant algebroids and generalized complex structures. *Adv. Math.*, 211(2):726–765, 2007.

[22] Pavol Ševera. Poisson–lie t-duality and courant algebroids. *Lett. Math. Phys.*, 105(12):1689, 2015.

[23] Branislav Jurčo and Jan Vysoký. Heterotic reduction of Courant algebroid connections and Einstein-Hilbert actions. *Nuclear Phys. B*, 909:86–121, 2016.

[24] Pavol Ševera and Fridrich Valach. Courant algebroids, poisson-lie t-duality, and type ii supergravities. 2018.

[25] Branislav Jurčo and Jan Vysoký. Effective Actions for σ-Models of Poisson-Lie Type. In *Durham Symposium, Higher Structures in M-Theory Durham, UK, August 12-18, 2018*, 2019.

List of Abbreviations

$\mathbb{C}$	field of complex numbers
$\mathbb{R}$	field of real numbers
$\mathbb{Z}$	ring of rational integers
$C^\infty(M)$	algebra of smooth functions on M
TM	tangent bundle of M
T^*M	cotangent bundle of M
(E, M, π)	fibre/vector bundle (1.1.1)
$\Gamma(E)$	space of sections of bundle E
$\Gamma(TM)$	space (Lie algebra) of vector fields on M
$\Gamma(\bigwedge^k T^*M) \equiv \Omega^k(M)$	space of k-forms on M
i_X	contraction (interior product) by a vector field X
$\mathcal{L}_X$	Lie derivative along X
d	exterior derivative
$H^k(M, \mathbb{R})$	k-th de Rham cohomology
S^n	n-dimensional sphere
$\mathrm{Der}(A)$	Lie algebra of derivations on a (Lie) algebra A
G, D	Lie groups
$\mathfrak{g}, \mathfrak{d}$	corresponding Lie algebras
$\langle \cdot, \cdot \rangle_g$	invariant inner product on g
$\#$	infinitesimal action
R	extended (infinitesimal) action
$\mathrm{GL}(n)$	Lie group of $n \times n$ real invertible matrices
$\langle \cdot, \cdot \rangle_E$	pairing of a Courant algebroid E
$[\cdot, \cdot]_E$	the bracket of a Courant algebroid E
ρ	the anchor map $E \to TM$
σ	an isotropic splitting of the anchor ρ
$f, g, h \ldots$	functions
$u, v, w, \ldots$	sections of algebroids
$X, Y, Z, \ldots$	vector fields
$\alpha, \beta, \gamma, \ldots$	linear forms
$\xi, \nu, \zeta, \ldots$	Lie algebra elements
$End(V)$	set of linear maps $V \to V$
Σ, h	a wordsheet with a Lorentzian metric