CHARLES UNIVERSITY

FACULTY OF SOCIAL SCIENCES

Institute of Economic Studies

Predicting Financial Market Crashes using Log-periodic Oscillation and Critical Slowing Down

Bachelor's thesis

Author: **Daniel Štancl**

Study programme: **Economics and finance**

Supervisor: **doc. PhDr. Ladislav Krištoufek PhD.**

Academic year: **2018/2019**

Declaration of Authorship

I hereby proclaim that I wrote my bachelor thesis on my own under the leadership of my supervisor and that the references include all resources and literature I have used.

I grant a permission to reproduce and to distribute copies of this thesis document in whole or in part.

Prague, May 9, 2019

Signature

Abstract

This bachelor thesis concerns itself with multiple objectives. First, to compare two apparently contradictory frameworks, namely the Log-periodic Power Law model and the Critical Slowing Down, suggested as being able to detect the end of financial bubbles. Second, to enrich current literature dedicated to the Log-periodic Power Law model with a comprehensible description of the non-linear optimization methods in one piece of work. This work, furthermore, aims to compare the performance and the robustness of two versions of this model. Regarding the Critical Slowing down, the correlation across the world market over time prior to a crash is investigated as an addition to two already studied indicators, 1-lag serial correlation and standard deviation of detrended fluctuations. Eventually, both the Log-periodic Power Law models were proved to be able to identify the time of the burst of the financial bubble, while the modified version of the model was found to be more proficient over the initial one in terms of computational efficiency and robustness. In the case of the Critical Slowing Down, obeying auto-correlation of residuals and cross-correlation of intermarket residuals came out to be misleading, and only variance was supported as an appropriate indicator of an imminent tumble, and it was proposed as an aspirant for a potential completion of the Log-periodic Power Law model framework.

Keywords

Financial Markets, Critical Points, Phase Transition, Log-periodic Oscillation, Critical Slowing Down, Non-linear Optimization Methods

Abstrakt

Tato bakalářská práce si klade několik cílů. Zaprvé, snahu porovnat dva zdánlivě protichůdné koncepty, konkrétně model log-periodického mocninného zákonu a kritické zpomalování, kdy oba jsou předpokládány býti schopny detekovat konec finanční bubliny. Dále práce usiluje o doplnění současné literatury věnované modelu log-periodického mocninného zákonu o srozumitelný popis metod používaných k nelineární optimalizaci, a přitom shrnout vše v jedné práci. Krom již zmíněného v textu dále porovnáváme výkon a robustnost obou verzí daného modelu. Co se týče kritického zpomalování, korelace napříč světovými trhy je zkoumána jakožto dodatek ke dvěma již studovaným indikátorům, jimiž jsou autokorelace a standardizovaná odchylka detrendované fluktuace. Ve výsledku, oba modely log-periodické mocninného zákonu prokázaly schopnost detekovat čas prasknutí finanční bubliny, přičemž modifikovaná verze daného modelu se ukázala býti robustnější a komputačně efektivnější metodou. V případě kritického zpomalování, pozorování autokorelace residuí a korelace residuí napříč trhy vyšly jako klamné indikátory blížícího se krachu, a pouze rozptyl je podpořen jako validní ukazatel, který mimo jiné, byl v práci navrhnut jako vhodné doplnění schématu modelu log-periodického mocninného zákonu.

Klíčová slova

Finanční trhy, Kritické body, Přechod fází, Log-periodická oscilace, Kritické zpomalování, Metody nelineární optimalizace

Acknowledgment

I would like to express my sincere gratitude to doc. PhDr. Ladislav Krištoufek for supervising this thesis, sharing finance books and providing me with great ideas for the bachelor thesis and the paper for the course Data Science with R. The programming knowledge gained in that course was highly valuable throughout the whole writing.

Bibliographic record

Štancl, Daniel. *Predicting Financial Market Crashes using Log-periodic Oscillation and Critical Slowing Down.* Prague 2019. Bachelor thesis (Bc.) Charles University, Faculty of Social Sciences, Institute of Economic Studies. Thesis supervisor doc. PhDr. Ladislav Krištoufek, Ph.D.

Bachelor Thesis Proposal

Author:	Daniel Štancl
Supervisor:	doc. PhDr. Jaroslav Krištoufek, Ph.D.
Proposed topic:	Predicting Financial Market Crashes using Log-periodic Oscillation and Critical Slowing down

---

Research question and motiation

As there is still an occurrence of financial market crashes following from deviation of security prices from their fundamental values and subsequent significant drop, a lot of questions have arisen whether and how these unpleasant events are predictable. Since the classic theory is not able to explain plummeting prices, many models from different scientific fields have been adopted.

Contribution

Since large amounts of money are invested in stock portfolios, the presence of crisis and related risk of imprecise recognition represents a true issue. Hence, my overall contribution should be to calibrate these models, apply them on real-life situations and determine their appropriatness of utilization.

In 2003, Sornette et al., proposed the log-periodic power law fitting the stock index prices during pre-crisis phases. Then, in 2009, Scheffer et al., studied the originally biological phenomena Critical Slowing Down describing a period of increasing variance, autocorrelation and slowing recovery of a system preceding the transition phase when the whole environment collapses and thus it can be applied to predict such downfalls.

Methodology

I will conduct time series analysis of important world stock indices such as Standard & Poor's 500, Dow Jones Industrial Average, Nikkei 225 or Europe Stoxx 600. In the case of Critical Slowing Down I will try to control specific characteristics, regarding Log-Periodic Power Law I will test the precision of the model proposed by Sornette, or Brée and Joseph.

Outline

1. Introduction to market crashes, their prediction and basic description of our models
2. Literature review
3. Dataset - choosing data and their description
4. Metodology - Building prediction models, fitting models on dataset
5. Results
6. Discussion and comparing the performance of both models
7. Conclusion

Bibliography

1. Sornette, D. (2003). Why Stock Market Crashes: Critical Events in Complex Financial Systems. Princeton. Princeton University Press
2. Scheffer, M. et al. Early-warning signals for critical transitions. Nature. 461, 53-59 (2009). DOI: 1038/nature08227
3. Scheffer, M. et al. Anticipating Critical Transitions. Science 338, 344-348 (2012). DOI: 10.1126/science.1225244
4. Brée, D. S., Joseph, N. L. Testing for financial crashes using the Log Periodic Power Law model. International Review of Financial Analysis. 30, 289-297 (2013). DOI: 10.1016/j.irfa.2013.05.005
5. Johansen, A. Characterization of large price variations in financial markets. Physica A. 324, 157-166 (2003)

Contents

1 Introduction

Nowadays, trillions of dollars in total are invested[1] in stocks, real estate, and other assets and are managed by various funds, investment banks, and other institutions, or by sole investors. All of them have the very same desire. They want to be able to predict as precisely as possible the future state of an economy, stock market movements, future evolution of real estate prices, and thereby to adjust their investment strategies and appropriately diversify in order to make a profit.

Notwithstanding the efficient market hypothesis, deeply studied by Eugene Fama in his semantic work Fama (1970), which postulates that no investor can beat the market as the stock price movements follow a random walk which is a consequence of market efficiency, financial crashes are of great importance as a lot of endeavour has been devoted to the issue whether there is a possibility to anticipate these events Cecchetti et al. (2009).

Since commonly used methods are often insufficient for predicting drastic shifts in an economy from a tranquil phase Bussiere and Fratzscher (2006), practices from other scientific fields have been adopted, for example from various branches of natural sciences. A vast contribution has been made by the French physicist Didier Sornette whose approach to forecasting financial crashes is comprehensively described in his book *Why Stock Markets Crash: Critical Events in Complex Financial Systems* Sornette (2003). There, the author stressed that huge price drops seen during these events are outliers, and thus may potentially carry a kind of predictability. To tackle this issue Sornette applied the power law enriched by the logarithmically periodic oscillation, formerly rather known from geophysics Saleur et al. (1996).

We try to verify the applicability of two versions of a Log-Periodic Power Law (LPPL) equation, one introduced by Sornette et al. (1996), and the other described by Filimonov and Sornette (2013). An advantage of the latter model suggested by authors of this study is the behaviour of cost function that tends to be dramatically smoother in comparison with the original model, which should signif-

[1]Data underpinning this claim are available on the website of the World bank: https://data.worldbank.org/indicator/cm.mkt.lcap.cd

icantly decrease the complexity of a fitting process Filimonov and Sornette (2013). Nonlinear, and linear parameters are mostly fitted separately in two stages. Here, for fitting nonlinear parameters, the Levenberg-Marquardt algorithm (LMA) is used for the modified model. In the case of the initial model, a heuristic algorithm Taboo Search (TS), described by Cvijović and Klinowski (1995), precedes the LMA due the peculiarity of a loss function. Then for optimization of linear parameters, OLS are to be utilized.

While the LPPL model bets on an accelerating progress towards the crash, there is a conceptually different set of warning indicators of the looming risk of market collapse introduced in the publication by Scheffer et al. (2009), where the Dutch biologist Marten Scheffer followed up on his previous work Van Nes and Scheffer (2007) that shows critical slowing down (CSD) is an appropriate candidate to become a gauge of an imminent phase transition. A phenomenon of CSD stands for a notion of gradual exacerbating system health captured by a progressively more and more sluggish recovery from perturbations to the previous state as a system approaches a critical threshold, usually called a tipping point, when even a very little disturbance may ignite a sudden phase transition. These signs deterioration are suggested to be detectable by increasing variance and 1-lag autocorrelation Scheffer et al. (2009).

Among the major objectives of this thesis is to provide a concise description of the two different aforementioned concepts used for detecting financial bubbles followed by significant tumbles. For this purpose, I also want to provide a theory for both the TS and the LMA to thus obtain appropriate platforms usable for potential future researches. The idea to closely elaborate these methods came to my mind because I had lacked sufficient amount of information precisely describing parts of a nonlinear optimization of the LPPL model in detail (namely details behind TS and LMA) in an economic literature. Hence this work should make this process more palatable and more easily reproducible for beginner economists like me.

Another motivation to compose this text is to compare these two different frameworks for predicting financial market crashes since both of them are based on a little bit different assumptions. Whereas the LPPL model wagers on an accelerating, faster-than-exponential growth decorated by periodic oscillations, thereby an observed system should reach prior peaks in shorter and shorter intervals, the latter concept relies on a continuously vanishing restoration capability. Moreover, the importance of this work is emphasized mainly by the fact there is only a very limited amount of literature, if any, devoted to this comparison.

The rest of this thesis looks as follows. The literature dedicated to these two concepts is reviewed in the following Section 2. Section 3 is devoted to the methodology. In this section, the theory behind crashes and modelling them is outlined. Furthermore, the intuition behind the LPPL model, its derivation and the complex fitting process are described along with the theory underlying the critical slowing down. Data observed in this thesis are enlisted and described in Section 4. Finally, Section 5 evaluates the empirical results, and a brief discussion of examined topics is contained in Section 6.

2 Literature Review

This section is divided into two parts. The first one deals with the literature dedicated to the LPPL model and the other one focuses on the existing endeavours around the phenomenon of CSD. Both the theory and empirical studies are presented.

2.1 Log-periodic Power Law Model

The initial notion of a possible application of the LPPL model was proposed by Sornette et al. (1996) as a reaction to the struggling to explain the large financial crash in October 1987 by a single theory. In those times, the valuation of major market indices in the USA dropped more than 30% during a mere one week. They pointed to self-organization present within the market, the concept inspired by the phenomenon of self-similarity which was proved to be a precursor of extreme events in complex natural systems, such as earthquakes. In the beginning, they suggested a simple model based on a pure power law grasping the acceleration of soaring prices

$$F(t) = A + B(t_c - t)^m, \tag{1}$$

where t_c denotes the moment where the probability of an occurrence of a crash is the highest. Furthermore, in order to explain periodically repeating deviations of the price from the fitted model, they, subsequently, developed the following equation

$$F(t) = A + B(t_c - t)^m + C(t_c - t)^m \cos(\omega \log(t_c - t) - \phi), \tag{2}$$

where t_c again presents the critical time, and the last term of equation (2) grasps oscillations. This newly developed function contains three linear variables A, B, and C and four nonlinear variables t_c, m, ω, and ϕ. This model is then also studied by a group around Sornette Johansen et al. (1999), and nowadays this approach usually referred to as the Johansen-Ledoit-Sornette (JLS) model after the authors of the latter study. Both an after-crash behaviour and log-periodicity for "anti-bubbles" are analyzed in of the cited studies.

While the latter model provides us with a good representation of price movements, it suffers from the overwhelming complexity of a fitting process and multiple local minima of cost function because of interdependence between parameters phase ϕ, and log-frequency ω. Inasmuch as another enhancement of the JLS model through throwing away the aforementioned interdependence so that to diminish an intricacy of the optimization was desired, a further transformation of the then framework was suggested by Filimonov and Sornette (2013). After their modification, they received the following LPPL expression

$$F(t) = A + B(t_c - t)^m + C_1(t_c - t)^m \cos(\omega \log(t_c - t)) + \\ C_2(t_c - t)^m \sin(\omega \log(t_c - t)). \quad (3)$$

Eventually, the currently presented model features only three nonlinear parameters t_c, m, and ω. This change makes the first step of the fitting procedure, which is described in Section 3, noticeably simpler. On the other hand, there are four linear parameters A, B, C_1, and C_2, but the complexity of the whole fitting process remains still quite lower since the cost function of this new version of the LPPL model promises a smoother surface, therefore, finding a global minima should become less tricky.

Another evidence of imitative actions of investor resulting in a bubble was suggested during the Nasdaq's crash on April 14, 2000, by Johansen and Sornette (2000). In this study, Johansen and Sornette also emphasized that the exact time of bursting a bubble is not purely deterministic and rather allows for stochastic influences, which creates a possible window spanning approximately a month for a crash occurrence. In this paper, they also discussed speculative bubbles which could land smoothly. For example, when Dow Jones index dropped significantly on October 15, 1999, after Alan Greenspan's speech, the market subsequently quickly recovered and, therefore, we do not consider that event a crash. Albeit this may seem to be in a contradiction with the hypothesis of successfully predicting crashes using the LPPL model, as Johansen and Sornette admitted, this does not any violate the rational expectations of investors, as one of the key assumptions of this theory, who are willing to undergo the risk of a possible crash since it is not the only outcome of striving markets Johansen et al. (1999). Also, other empirical pieces of evidence of a presence of the pattern in the evolution of share prices preceding to crashes, as described above, not only in the US, but also in Latin-American, and small Asian stock markets as well as in Chinese stock market were detected and summarized in Johansen and Sornette (2001a), and Jiang et al. (2010) respectively.

A critical perspective on the validity of the LPPL model was brought by Bree and Joseph (2010). In their analysis, Bree and Joseph focused on the Hang Seng market between 1970 and 2008 and identified 11 crashes. They concluded the parameters of the model (2) do not fully satisfy the hypothesis of Johansen and Sornette (2001a) that states values of these parameters should range within confined intervals, and actually this condition was met only in seven out of those 11 cases specified above. Furthermore, they suggested that mechanism proposed by Johansen et al. (2000) "must be incorrect as it requires the price to be increasing throughout the bubble", though "the index (or its log) decreases at some point during the bubble" Bree and Joseph (2010).

More concerns regarding the precision of the model were expressed by Laloux et al. (1999) which, in their words, works approximately in one trial out of two. They, moreover, criticized the overwhelming complexity of the model arising from the fitting as many as seven parameters, which eventually can lead to overfitting.

Liberatore (2010) paid attention to the computational efficiency of fitting the LPPL model and stressed the fact that the practical usefulness of an algorithm strongly hinges on its accurate and efficient fitting. Primarily, he further developed the problem-solving methods based on the commonly used LMA that was initially proposed by Levenberg (1944).

A concise summary devoted to the optimization procedure of the LPPL model was published by Pele et al. (2012). In this thesis, Pele provides useful, but quite a brief description of Johansen et al. (2000)'s 2-step nonlinear optimization, which is then applied in the empirical study on Bucharest Stock Exchange. Further inspiration such as utilization of genetic algorithm for nonlinear tasks or considering an application of maximum likelihood approach instead of ordinary least squares in the latter part of tuning the model might be also drawn from this work.

Nowadays, more advanced versions of the LPPL model are still sprung, such as the volatility-confined LPPL model with mean-reverting residuals proposed by Lin et al. (2014) allowing for stochastic conditional expectations of returns and, furthermore, testing residuals for a unit root, which aids to confirm the validity of the model. Filimonov et al. (2017) highly elaborated the application of the modified profile likelihood to estimate the interval for the critical time t_c, and claimed this methods is more suitable compared to OLS as probabilistic nature of crash occurrence hence estimating time intervals is more sensible than single points in time. Both the latter two topics are quite much advanced and they are not to be studied in this thesis.

2.2 Critical Slowing Down

As it was stated in the introduction of this thesis, both concepts have been adopted from natural sciences. Especially, in the case of the CSD, works devoted to this interesting topic in finance started emerging in the last 15 years at the most and not in bulk indeed. Until then, it was earnestly studied by physicists, biologists and other scientists from related fields.

Presence of a slowing recovery in the vicinity of tipping points was shown, for example, on the Ising model Fisher (1986), which is a simplified mathematical representation of ferromagnetism used for identification of a phase transition Salinas (2001). Scholz et al. (1987) found this trait a pivotal feature of phase transitions in complex biological systems. More recently, Dai et al. (2012) conducted a reproducible laboratory experiment to study possible warning indicators of looming population collapse, and proved a CSD to reliably signal a loss of resilience.

Scheffer et al. (2009), finally, described that there are many different complex systems ranging from ecological ones such as ocean circulation, or in medicine there are utterly convoluted systems embodied by human beings where one can observe, for example, epileptic seizures, to financial markets. In all these areas, there might be critical points at which sudden shift from one state to another occurs. Although these systems might look completely different at the first glance, similar patterns prior to tipping points in different cases can actually be observed and, for example, "the collapse of an overharvested population and ancient climatic transitions could be indicated by similar signals" Scheffer et al. (2009). In this cited study, the CSD is defined as a phenomenon when a vector of recovery rate gradually approaches zero after each perturbation till the point where the norm of the regeneration vector equals zero. As soon as this state arrives, a catastrophic bifurcation point is reached and there is a risk of a complete system shift. Furthermore, increased autocorrelation and variance are suggested to be appropriate indicators of this phenomenon and should be observable in time-series data. Nonetheless, this team of scientists admits detection of these pattern in real data is quite challenging and this process may lead to both false positives and false negatives because of their complex nature.

Robustness of foregoing indicators, namely increased serial correlation and elevated variance, were, for example, tested on ecosystem data by Dakos et al. (2012). It has been shown that variance can either increase or decrease when an imminent critical transition is being approached. This inconsistency with an underlying theory was mainly caused by the presence of stochastic behaviour resulting in a state when a system becomes less sensitive to the monitored environmental factors. On the other hand, the validity of rising serial correlation prior to extreme events was successfully confirmed.

Whereas there are solid amounts of theoretical background and mathematical equations behind the bifurcation theory, we are pretty short of real-world clues whether and how a set of early-warning patterns signalizes the critical shifts in an economy. Since financial crashes are outliers Sornette (2003), their occurrence is (fortunately) poor, and therefore, the amount of data to be analyzed is rather insufficient. As a result, there is a lack of financial literature devoted to this topic. However, as Scheffer et al. (2009) emphasised, the similar behaviour close to a critical point is exhibited by different systems regardless of differences among them, therefore, we hopefully can rely, for example, on purely ecological or physical literature as in these research areas, a lot of endeavours to ponder this issue has been dedicated.

During our study, it is important to consider conclusions from thesiss like Guttal et al. (2016) where authors analyzed three major US and two European stock markets, and revealed these financial markets did not exhibit the CSD features prior to crashes over the last century. They further suggested those meltdowns were rather an illustration of stochastic transitions, thereby proposed crashes may occur almost whenever far away from a tipping point if stochastic perturbations amass. This study, however, confirmed a rising variance as a precursor of financial drops albeit researchers must be aware of potential false positives.

3 Methodology

In the first part of this section, we show that crashes are outliers, and thus it is worthwhile to study them independently. Furthermore, we try to reveal any hidden structure in single-trader behaviour and imitative patterns which are triggers of bifurcations.

Then, we focus on the underlying theory behind the LPPL model and provide examples supporting the validity of usage of this scheme. Then, the derivation of the LPPL models utilized in this thesis is conducted. Subsequently, particular elements of our LPPL model are described together with value ranges recommended for individual parameters. All of this is complemented by a description of a complex fitting process of the LPPL model, and both non-linear methods, TS and LMA, are elaborated. Moreover, pseudocode for TS is provided in the main text because there is no in-built function in R or Python which could be directly exploitable.

Another part is dedicated to the major components of the theory of the CSD. This part is accompanied by necessary mathematical foundations and several pieces of intuition in order for the whole concept to be well-understandable.

Finally, this section is concluded by a short review of test methodology followed in this thesis.

3.1 Crashes and Drawdowns

When we want to try to predict an event of some type, then, in the beginning, it is necessary to realize financial crashes and corresponding rebounds are outliers, and they are statistically completely unlikely.

To present this fact, we can simply derive the probability of an occurrence of rebound of +9.7% on October 21, 1987, or even the largest drop ever happened on October 19, 1987, when prices tumbled by 22.6% during one trading day Sornette (2003).

For this purpose, we can rely on a likely obsolete and simplified model assuming daily stock returns follow the Gaussian distribution which was originally used by Louis Bachelier in his pioneering PhD thesis Bachelier (1900).

Considering the time period from January 2, 1986, to February 27, 2018, we obtain a dataset of 8,105 daily returns of the Dow Jones Industrial Average[2]. The mean value is 0.04% with standard deviation equal to 1.10%.

The probability of normal distribution is defined as

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}, \tag{4}$$

μ denotes mean, and σ^2 captures the variance Bartoszynski and Niewiadomska-Bugaj (2007).

Combining everything together, one can derive the following representation of stock returns and their approximation by the Gaussian distribution.

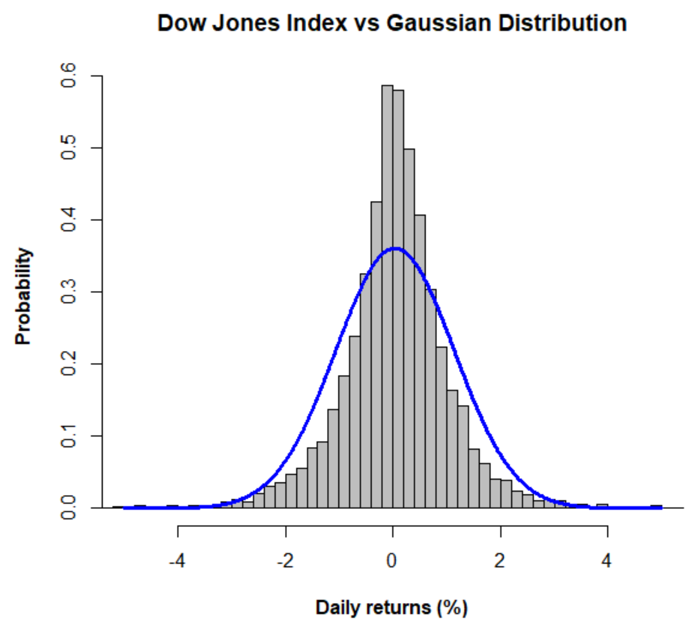

Figure 1: Histogram of daily returns on DJIA and density function of the Gaussian distribution with corresponding parameters
Although the real returns are more concentrated around zero in comparison with generated data, random normal variable may be considered as a relatively good approximation.

We can now reveal probabilities of these monumental events in October 1987. The probability of a rebound of 9.7% is $p(x = +9.7\%) \approx 1.93 \cdot 10^{-14}$. In other words, assuming one year consists of 250 trading days in average, we obtain that such an event should occur once in 772 billion years, which is roughly 170 times of the age of the Earth Patterson (1956). It is easy to realize the occurrence of the preceding price drop is more or less inconceivable under this assumption, though

[2]All the data for these observations is collected through an R package *quantmod*

such events still occur.

There is, however, another component behind the largest drops in history that cannot be simply captured by the frequency distribution of daily returns. If we focus on fourteen largest catastrophes on the DJIA in the twentieth century, only three of them lasted two days or less, whereas, the others stretched out for between four and eleven days Sornette (2003).

We, therefore, need to introduce another concept - drawdowns. "A drawdown is defined as a persistent decrease in the price over consecutive days" Sornette (2003). Therefore, drawdowns can be directly considered as the cumulative loss from a local peak to the next minimum and they also represent how much an investor can potentially lose. A short-lasting correlation can be observed during these events, and these occurrences must be studied solely because they can be neither identified by simple frequency statistics nor detected by two-point correlation which measures an average linear relationship over the whole time period, thereby, these bursts of correlation are to be demeaned. Hence, it is worthwhile to study them independently and ask ourselves whether finding any patterns in the distribution of drawdowns is possible.

It can be again shown that the assumption of fleeting correlation is completely valid. For this empirical proof, we compare distribution of drawdowns derived from historical data of the DJIA[3] in the twentieth century and the beginning of the twenty-first century (more specifically from January 2, 1901, to December 31, 2015[4]; 31,191 stock returns are amassed at total.) with artificially generated drawdowns took as a random sample of the very same size from a random variable $X \sim N(\mu_{DJIA}, \sigma^2_{DJIA})$, where μ_{DJIA} denotes the average daily returns, and σ^2_{DJIA} represents volatility of the DJIA in a given time period measured by variance. We assume X is i.i.d., hence $Corr(x_t, x_{t+1}) = 0$, for any $t \in \mathbb{N}$.

Below (Figure 2 and Figure 3), we show the cumulative distribution of drawdowns for returns generated by our random variable X follows exponential distribution. Thus using a logarithmic transformation we obtain a linear relationship.

[3]All data for this example are downloaded from www.quandl.com through the R package *Quandl* and the function of the same name.

[4]Time period is rather arbitrary and time frame is constrained by the availability of data from one resource. Furthermore, the observed period is intended to be maximized in order to exploit the Law of Large Number thus to obtain the best possible approximation of true distribution.

This exponential distribution is a consequence of the independence of successive price variations. There is a lot of evidence this assumption holds for the majority of trading days Campbell et al. (1997). Nevertheless, the distribution of drawdowns obtained from real data disobeys the outlined relation as soon as the magnitude of price drops becomes statistically less likely.

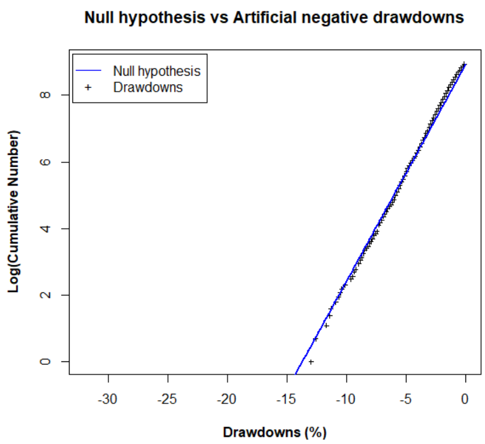

Figure 2: Distribution of drawdowns from i.i.d. normal distribution

Perfectly obeys exponential distribution with decay rate equalling 0.65, which proves independent returns.

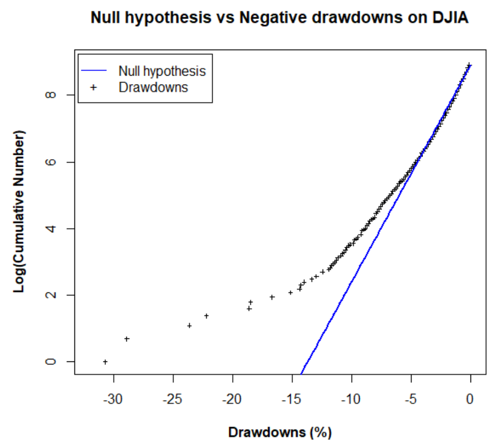

Figure 3: Distribution of drawdowns from DJIA returns

Significantly deviates from proposed exponential distribution (decay rate is 0.65) suggesting transient correlation.

Even though the representation depicted above is simplified and it has been shown the drawdown distribution on stock markets actually rather fits stretched exponential function defined as

$$f(x) = \begin{cases} c\frac{x^{c-1}}{x_0^c} \exp\left[-\left(\frac{x}{x_0}\right)^c\right] & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{5}$$

instead of the simple exponential function, the conclusion remains same as about only 98% of drawdowns are well-represented by this function Johansen and Sornette (2002).

3.2 Crash modelling

To understand what happens inside financial markets during times preceding tremendous price drops, we shed light on two interconnected pieces of this mechanism. Financial crashes are macroscopic phenomenons, but to fathom out how these events spring up, it is necessary to discern decision making of an individual investor who is lured by close counterparts within its network, so a scheme inspired by one historically developed in physics or biology is adopted Liggett et al. (1997). Building on this microworld, a macro model explaining sudden changes in complex systems is derived. Moreover, the hazard rate, a cornerstone for the LPPL model, is introduced as a follow-up to the macroscopic modelling.

3.2.1 Microscopic modelling

Consider a network of agents where each agent is indexed by an integer $i = 1, \ldots, I,\ I \in \mathbb{N}$. Let us define $N(i)$ to be the neighbourhood of agent i denoting those agents who are directly connected to agent i. The agents in $N(i)$ influence each other, and for greater comprehensibility, we can assume they tend to imitate themselves. For simplicity, we further suppose that agent i can take only two possible stands, either to buy, $s_i = 1$, or to sell, $s_i = -1$.

The sum of all agents' states determines a price movement, i.e.

$$\Delta p = \sum_{i=1}^{I} s_i, \tag{6}$$

If the sum in equation (6) is negative, the best decision is to sell as price goes down, an opposite scenario is applicable for positive summation. An equal number of selling and buying orders occurs provided that the sum is zero. However, equation (6) is not known for a single trader, therefore, a given trader must rely on an imitation of its nearest neighbours in hope this sample well represents the whole population.

We can posit the optimal trader's decision is given by

$$s_i(t+1) = \operatorname{sign}\left(K \sum_{j \in N(i)} s_j(t) + \sigma \epsilon_i\right), \tag{7}$$

where K is a positive constant measuring the tendency towards imitation, which is determined by the relative imbalance between buyers and sellers presented in the market and it is inversely proportional to the market depth Sornette (2003). $\epsilon_i \sim N(0, 1)$ represents an idiosyncratic error, and the tendency to the irrational behaviour is governed by σ that resolves imbalance between order and disorder Johansen et al. (2000).

3.2.2 Macroscopic modelling

Since this section will turn around the mass imitative behaviour, let us introduce a useful definition of the term used in the text below.

Definition 3.1 (Self-organization)**.** "In self-organizing systems, pattern formulation occurs through interactions internal to the system, without intervention by extertnal directing influences." Camazine et al. (2003)

The concept of a single agent introduced in Subsection 3.2.1, and its network of neighbours need to be conveyed to the realm of the whole market, which is our representation of the complex system. Most of the time, agents are in disagreements with the others implying the similar numbers of buyers and sellers, thus the market is more or less balanced. Therefore, in those times, no crash occurs.

However, at some point in time, i.e. when a burst of the bubble is being approached, agents starts to be somehow self-organized, and order emanating from imitative behaviour begins to prevail over disorder. One way how to describe this pattern capturing a looming critical transition (it has been shown in various areas the self-organization is a precursor of bifurcations, for example, look at Rietkerk et al. (2004), or Sornette (2006)), is recalling the parameter K from equation (7), and consider the critical value K_c. Then, once a K gets closer to K_c, the probability of crash increases. This relation can be described using a simple power law

$$\chi = A(K_c - K)^{-\gamma}, \tag{8}$$

where A denotes a positive constant, and $\gamma > 0$ is called the critical exponent Johansen et al. (2000). (There is evidence suggesting a behaviour of complex systems following the simple power law or truncated power are valid, e.g. Kéfi et al. (2007).)

Specifically just for the LPPL model, assuming the dynamics of parameter system K evolves smoothly, we can posit an approximation

$$K_c - K \approx \omega \times (t_c - t),\ \omega \in \mathbb{R} \tag{9}$$

and eventually, suggest that the hazard rate delineating the crash probability tends to behave similarly in the vicinity of a crash for $t < t_c$. This consideration yields the following relationship

$$h(t) \approx B \times (t_c - t)^{-\alpha}, \tag{10}$$

where B is a positive constant and $\alpha \in (0, 1)$ for reassuring the price does not go to infinity in case a crash does not occur Johansen et al. (2000).

Integrating expression (10), the probability of crash at time t_c might be derived. However, it is always important to bear in mind, that this value is never equal to one as there exists a non-zero chance of a bubble landing smoothly, i.e.

$$1 - \int_{t_0}^{t_c} h(t) > 0 \tag{11}$$

because the time t_c is not deterministic, but rather stochastic Johansen et al. (1999).

3.3 LPPL Model

To build a reliable framework for an application of the LPPL model, we firstly derive an equation from the hazard rate introduced in Subsection 3.2, that exhibits the desired characteristics of this model. Then, a transformation for this model is proposed in order to obtain a model featuring more robust results. Subsequently, appropriate values for particular parameters of the LPPL model that satisfy its characteristics with further recommendations are presented. At the end of this sections, individual methods incorporated in a fitting procedure, which are more complicated and are not so familiar in the world of economists, are gone through.

3.3.1 Price dynamics

In real markets, there are many factors influencing agents' behaviour encompassing, for example, interest rate, information asymmetry or market clearing conditions which are ignored in our analysis in purpose of a greater simplicity. Further, to streamline our consideration even more, we consider a clearly speculative asset paying out zero dividends and rational expectations are supposed to be held Johansen et al. (2000).

In this setting, agents are assumed to try to maximize their well-being based upon all revealed information. Under the efficient market hypothesis, rational expectations are captured by a well-know martingale condition

$$E_t\left[p(t')\right] = p(t), \quad \forall t' > t, \tag{12}$$

where $p(t)$ denotes the price of an asset at time t, and $E_t\left[p(t')\right]$ states an expected value of asset price at time t' derived from all available information up to time t.

Abandoning price fluctuations elicited by any noise, the solution of an equation (12) is given by $p(t) = p(t_0)$, whereas t_0 indicates some initial time point. Recalling one of our assumptions emphasizing a consideration of a zero-dividend asset, which suggests $p(t_0) = 0$, we finally obtain a relationship expressing that the asset fundamental value is $p(t) = p(t_0) = 0$. Therefore, any positive deviations of p(t) from 0 indicates a speculative bubble Johansen et al. (2000).

Loosening a zero fundamental value premise and instead allowing for any real fundamental value, an observed stock price can be rewritten as a linear combination

$$p = p_0 + p^*, \tag{13}$$

where p_0 and p^* express a fundamental value, and a bubble component respectively. The JLS model, which is a base of ours utilized in this thesis, assumes the bubble component behaves independently of dynamics of the fundamental price. The latter one, p^*, is portrayed by a log-periodic power law structure determined by a mean-reverting behaviour of p_0, and all together work as a robust calibration of an observing price p Ren et al. (2009).

An application of the JLS model requires bubble component dynamics to follow a simple stochastic differential equation with drift and jump

$$\frac{dp}{p} = \mu(t)dt + \sigma dW - \kappa dj, \tag{14}$$

where p denotes stock price, $\mu(t)$ presents a drift term at time t, dW embodies an increment of Wiener process with zero mean and unit variance, with σ corresponding to a tendency towards idiosyncratic behaviour of traders Bree and

Joseph (2010). Furthermore, dj is a jump, with j taking values from a two-element set $\{0, 1\}$, such that $j = 0$ prior to a crash occurrence and $j = 1$ after the crash. The proportion amplitude of a price drop, whose presence is indicated by the value of j, is determined by a parameter κ Sornette et al. (2013). Dynamics of those jumps are dictated by a hazard rate $h(t)$ which captures the probability of a crash occurrence in the region $[t, t + dt]$ assuming it has not happened yet. Therefore, this conditional expectation of dj is defined as $E_t[dj] = 1 \times h(t)dt + 0 \times (1 - h(t)dt)$, hence it holds

$$E_t[dj] = h(t)dt. \tag{15}$$

Under the assumptions of the JLS model, the aggregate impact of a herding behaviour of noise traders is captured by the crash hazard rate in the following form

$$h(t) = B'(t_c - t)^{m-1} + C'(t_c - t)^{m-1} \cos\left(\omega \log(t_c - t) - \phi'\right). \tag{16}$$

In this extension of the hazard rate, the positive feedbacks of agents, driving price above its intrinsic value with growth ending at finite-time singularity Johansen and Sornette (2001b), are captured by the equation (16). The cosine term stated in the right-hand side (RHS) of the given equation is then able to grasp the hierarchical cascades of spurring panic Sornette and Johansen (1997) punctuating price growth, arising in the market due to the pre-existing organization in noise trader sizes Zhou et al. (2005).

The non-arbitrage condition, as a consequence of market efficiency, implies $E_t(dp) = 0$, therefore, using the knowledge from equation (15), and plugging it in equation (14), the following expression for the drift term is obtained

$$\mu(t)dt = E\left[\frac{dp}{p}\right] = \kappa h(t)dt \iff \mu(t) = E\left[\frac{dp/dt}{p}\right] = \kappa h(t). \tag{17}$$

Finally, using equation (17) and substituting the left-hand side (LHS) in equation (16), and taking an integration with respect to t (plus using a chain rule) yields the following equation

$$\int E\left[\frac{dp/dt}{p}\right]dt = \int \Big[\kappa B'(t_c - t)^{m-1} + \\ \kappa C'(t_c - t)^{m-1}\cos\big(\omega\log(t_c - t) - \phi'\big)\Big]dt \quad (18)$$

which gives

$$\log E\left[p(t)\right] = A - \frac{\kappa B'}{m}(t_c - t)^m - \frac{\kappa C'}{m^2+\omega^2}(t_c - t)^m \times \\ \big(m\cos(\phi' - \omega\log(t_c - t)) - \omega\sin(\phi' - \omega\log(t_c - t))\big). \quad (19)$$

Further incorporating a trigonometric identity $\cos(-x) = \cos(x),\ \forall x \in \mathbb{R}$ Hájková et al. (2012), and some approximation, the ultimate result is a representation of the characteristic LPPL equation for detecting financial bubbles

$$\log E\left[p(t)\right] \approx A - \frac{\kappa B'}{m}(t_c - t)^m - \frac{\kappa C'}{m^2+\omega^2}(t_c - t)^m \cos\left(\omega(t_c - t) - \phi\right), \quad (20)$$

By rewriting $B = -\frac{\kappa B'}{m}$ and $C = -\frac{\kappa C'}{m^2+\omega^2}$, we finally obtain

$$\log E\left[p(t)\right] \approx A + B(t_c - t)^m + C(t_c - t)^m \cos\left(\omega(t_c - t) - \phi\right). \quad (21)$$

where A, B, and C are linear parameters and t_c, m, ω, and ϕ are nonlinear ones. The model expressed in equation (20) will be referred in the text below as LPPL1.

The last important information worth to be mentioned is the JLS model does not forecast behaviour after the bursting bubble. For this reason, it is important to realize, that termination time t_c^{true} is predicted with some amount of uncertainty, and its precise value is not exactly known. This point in time may be rather expressed as

$$t_c^{\text{estimated}} = t_c^{\text{true}} + \epsilon, \quad (22)$$

where $t_c^{\text{estimated}}$ is a time returned by our prediction framework and ϵ captures an error term drawn from some distribution Sornette et al. (2013). Generally, it should be considered the time of a crash to lay in a month interval around the predicted critical time t_c Sornette (2003).

Beyond the scope of the thesis, for interested readers, Johansen et al. (1999) suggested after-bubble behaviour can be also described with the LPPL equation, namely it should hold

$$\log E\left[p(t)\right] \approx A - B|t_c - t|^m - C|t_c - t|^m \cos\left(\omega|t_c - t| - \phi\right). \tag{23}$$

3.3.2 Derivation of modified LPPL model

Due to obstacles in the course of the fitting process, the method suggesting to cut the number of nonlinear parameters by one and to substitute the eliminated term by adding a linear parameter, and more importantly, at the same to dispose an interdependence between the phase ϕ and angular log-frequency ω is proposed.

For this purpose, the trigonometry identity, $\cos(x-y) = \cos x \cos y + \sin x \sin y$, $\forall x, y \in \mathbb{R}$, Hájková et al. (2012), is put to use to expand the cosine term in the equation (21) as follows

$$\begin{aligned}\log E\left[p(t)\right] = A + B(t_c - t)^m + C(t_c - t)^m \cos\left(\omega \log(t_c - t)\right) \cos\phi \, + \\ C(t_c - t)^m \sin\left(\omega \log(t_c - t)\right) \sin\phi.\end{aligned} \tag{24}$$

Now, by introducing two new parameters

$$C_1 = C\cos\phi, \quad C_2 = C\sin\phi \tag{25}$$

and plugging (25) to equation (24), we obtain the new form of the LPPL model as

$$\begin{aligned}\log E\left[p(t)\right] = A + B(t_c - t)^m + C_1(t_c - t)^m \cos\left(\omega \log(t_c - t)\right) + \\ C_2(t_c - t)^m \sin\left(\omega \log(t_c - t)\right).\end{aligned} \tag{26}$$

The newly formulated function thus features only three nonlinear parameters t_c, m, ω and four linear ones A, B, C_1, C_2 with C_1 and C_2 absorbing the phase ϕ. This realization of the LPPL model should be more easily and more robustly optimizable as suggested in Filimonov and Sornette (2013). In the text below, we will refer to the model given by equation (26) as LPPL2.

3.3.3 Recommended range of the parameters

The last foregoing cornerstone before we fully impend on the optimization process is to impose reasonable constraints on ranges of individual parameters. There are both logical rationale satisfying general economic concepts, as they are presented below, and another set of reasons based on past experience offering additional specification. The importance of the latter one is emphasized by the necessity of usage of grid search for TS in course of optimization the LPPL1 model to be optimally fitted. Approximate time complexity of the whole process, provided that each of four parameters t_c, m, ω, ϕ, which generate inputs to our grid in the first stage of the optimization, is incremented by the same portion, the running time requirements are estimated roughly in terms of $O(4^n)$ hence the smaller parameter network, the better.

The parameter A in the equation (24) represents logarithm of an observed index price hence there is only one reasonable constraint so that $A > 0$, and it should be mirrored with the current price of an observed asset. Acceleration of the hazard rate is required as time t approaches the critical point t_c thus with respect to equation (16), B always tends to be lower than 0 Filimonov and Sornette (2013). Bothmer and Meister (2003) suggested $|C| < 1$, where C determines the proportion magnitude of price fluctuations about the exponential growth so that the hazard rate remains always positive. This limitation in combination with a shift parameter ϕ belonging to the interval $(0, 2\pi)$ gives us both $|C_1| < 1$ and $|C_2| < 1$.

Generally, it holds that $t_c > 0$, and we further consider t_c is greater than the last available time point t. Besides, Huang et al. (2000) proposed a recommended range for ω to be $[6, 13]$ so that the log-periodic oscillation is neither too fast (as the risk of an arbitrary fitting of the model on a random component looms) nor too slow (otherwise these fluctuations would only contribute to a soaring trend). Subsequently, this boundary was even further narrowed, and it has been shown it is reasonable to consider $\omega \approx 6.36 \pm 1.56$ Johansen (2003). Last but not least, the base condition for the parameter m, which is an exponent of the power law, is $m < 1$ so that an integration of the hazard rate from equation (16) over time up to t_c, determining the probability of a crash occurrence at time

t_c, is to be bounded by 1 (this follows from the basic probability rule, $P(x) \leq 1$, $\forall x \in$ {set of all conceivable events}, Bartoszynski and Niewiadomska-Bugaj (2007)). On the other hand, the log-price in (21) and (24) is aimed to be finite for all $t < t_c$, hence another condition, $m > 0$ is imposed in order this assumption to be satisfied Filimonov and Sornette (2013). An appropriate interval $(0, 1)$ for a parameter m were then further thinned based on various observations, so now is supposed $m \approx 0.33 \pm 0.18$ Bree and Joseph (2010). We will review in Section 5 whether more confined intervals seems to be appropriate or not.

3.3.4 Fitting procedure of modified LPPL Model

The fitting complexity of the LPPL model and robustness of predictions remain still a large issue. In this research thesis, the fitting procedure is mainly based on the one introduced by Filimonov and Sornette (2013) through minimization of the following cost function

$$F(t_c, m, \omega, A, B, C_1, C_2) = \sum_{i=1}^{n} \Big[\log(p_{t_i}) - \big(A + B(t_c - t_i)^m + C_1(t_c - t_i)^m \cos(\omega \log(t_c - t_i)) + C_2(t_c - t_i)^m \sin(\omega \log(t_c - t_i))\big) \Big]^2 \quad (27)$$

in two subsequent steps.

In the first step of the fitting process, we need to solve the non-linear optimization problem

$$\{\hat{t_c}, \hat{m}, \hat{\omega}\} = \arg \min_{t_c, m, \omega} F_1(t_c, m, \omega), \quad (28)$$

where $F_1(t_c, m, \omega)$ represents the following cost function (29) with parameters A, B, C_1, and C_2 are to be given

$$F_1(t_c, m, \omega) = \min_{A,B,C_1,C_2} F(t_c, m, \omega, A, B, C_1, C_2). \quad (29)$$

Different starting values through a grid search are used in an optimization in order to find various local minima, out of them the best one is supposed to be an almost optimal global minimum.

Sometimes, it does not have to be convenient this task to be solved using first-order optimization methods (however, they might be and are put to use) such as gradient descent because $\frac{\partial F_1(t_c,m,\omega)}{\partial t_c}$ is well defined (i.e. in real numbers) if and only if $t_c > t_i$ for any $i \in \{1,\ldots,n\}$. This tool, therefore, would be inapplicable in situations when t_c is considered within a region $[t_1, t_n]$. (As the latter-mentioned situation is not among objective of this thesis, we do not further comment on this.) Here, the LMA exploiting first partial derivatives is used, and it is covered in Subsection 3.3.7.

As determining the critical time t_c is crucial in analyses of this type, special treatment for finding an optimal value of t_c has been developed by Filimonov and Sornette (2013). In this thesis, therefore, after solving the problem given by equation (28), parameters m and ω are subordinated to the critical time t_c. From this, another optimization problem arises in the form

$$\hat{t_c} = \arg\min_{t_c} F_2(t_c), \tag{30}$$

where F_2 is an expression for the following cost function

$$F_2(t_c) = \min_{m,\omega} F_1(t_c, m, \omega). \tag{31}$$

In this problem, again different starting values of parameters m and ω altogether with suitable values of linear parameters from (28) are used to find the best solution. This additionally created function, determined by equations (30) and (31), provides an insight that choices of m and ω are dependent on t_c,

$$\hat{m}(t_c), \hat{\omega}(t_c) = \arg\min_{m,\omega} F_1(t_c, m, \omega). \tag{32}$$

When the first phase of the optimization is completed, only the best values of non-linear parameters are utilized, alternatively a small set of the best fitted parameters can be used, in the second stage of optimization to solve the linear problem defined by the equation

$$\{\hat{A}, \hat{B}, \hat{C_1}, \hat{C_2}\} = \arg\min_{A,B,C_1,C_2} F(t_c, m, \omega, A, B, C_1, C_2). \tag{33}$$

In this thesis, the method of ordinary least squares (OLS) is utilized, where a unique solution is retrieved from satisfying the first-order condition in the following matrix equation

$$\begin{pmatrix} n & \sum f_i & \sum g_i & \sum h_i \\ \sum f_i & \sum f_i^2 & \sum f_i g_i & \sum f_i h_i \\ \sum g_i & \sum f_i g_i & \sum g_i^2 & \sum g_i h_i \\ \sum h_i & \sum f_i h_i & \sum g_i h_i & \sum h_i^2 \end{pmatrix} \begin{pmatrix} \hat{A} \\ \hat{B} \\ \hat{C}_1 \\ \hat{C}_2 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum y_i f_i \\ \sum y_i g_i \\ \sum y_i h_i \end{pmatrix}, \tag{34}$$

where y_i is the notation for logarithm of an observed price, and

$$f_i = |t_c - t|^m, \tag{35}$$

$$g_i = |t_c - t|^m \cos(\omega \log |t_c - t|), \tag{36}$$

$$h_i = |t_c - t|^m, \tag{37}$$

$$g_i = |t_c - t|^m \sin(\omega \log |t_c - t|). \tag{38}$$

3.3.5 Fitting procedure of standard LPPL Model

The optimization process of the LPPL1 model is in the structure more or less identical to the scheme described in Section 3.3.4, therefore, only distinctions are to be mentioned here.

One difference arises in the form of loss function which must be logically rewritten as

$$\begin{aligned} F(t_c, m, \omega, \phi, A, B, C) = \sum_{i=1}^{n} \Big[\log(p_{t_i}) - \big(A + B(t_c - t_i)^m + \\ C(t_c - t_i)^m \cos(\omega \log(t_c - t_i) + \phi)\big)\Big]^2 \end{aligned} \tag{39}$$

It is also necessary to modify the non-linear optimization problem given by equation (28) to the form

$$\{\hat{t_c}, \hat{m}, \hat{\omega}, \hat{\phi}\} = \arg \min_{t_c, m, \omega, \phi} F_1(t_c, m, \omega, \phi), \tag{40}$$

Another distinction can be found in the first part of the fitting process. Here, due to the much more peculiar surface of the loss function, which is investigated and well-described by Filimonov and Sornette (2013), the TS must be applied before the utilization of the LMA in order to find the optimal solution. One potential drawback can be that the execution of TS might be very lengthy.[5] The different starting values are used in accord with the description in Section 3.3.4. For this method, we also omit the corresponding optimization problem given by equation (31) not to further increase the problem complexity.

The linear problem is then defined by the following equation

$$\{\hat{A}, \hat{B}, \hat{C}\} = \arg \min_{A,B,C} F(t_c, m, \omega, \phi, A, B, C), \tag{41}$$

and is also solved by OLS.

3.3.6 Taboo Search

There are more than one heuristic methods applicable to a global minimization problem. Genetic algorithm (GA) described by Holland (1992), and Simulated annealing (SA) introduced by Kirkpatrick et al. (1983) are to be mentioned. In this thesis, nonetheless, TS is chosen to be used. TS has been proved to be superior in time consumed for obtaining a solution for different tasks, moreover, results returned by TS are also better in quality compared with SA Hertz and de Werra (1987), Skorin-Kapov (1990), and Zheng et al. (2005). Besides, TS is conceptually simpler than SA and GA Cvijović and Klinowski (1995), and it has been shown to outperform GA in computational efficiency too Teh and Rangaiah (2003).

TS used for optimization problems in continuous spaces is based on the original work published by Glover (1990) who developed this "higher level" heuristic procedure to find near-optimum global maximum/minimum in complex combinatorial tasks with an ambition to escape the trap of a local extreme. With the rising popularity of TS in discrete space problems Moscato (1993), TS methods applicable for continuous optimization were later introduced by Cvijović and Klinowski (1995).

In general, define the global minimization problem as

$$\min L(x) : x \in X \subset \mathbb{R}^n, \tag{42}$$

[5]This finding is based on own observation conducted during the research. In spite of the function is well vectorized, an execution takes a long time to be completed due to a large number of starting points.

where L is a loss function, and x is a point from X defining the solution space, which is a subspace of an n-dimensional hypercube. TS begins from some staring point and attempts to find a better solution measured by a given loss function. These steps, usually called moves in heuristic procedures Van Laarhoven and Aarts (1987), are defined as a transition from x to x' and further, we specify the move value as a difference $L(x') - L(x)$ thus only moves with negative move values are improving Cvijović and Klinowski (1995).

Before going through individual parts of our procedure, let us introduce some useful definitions.

Definition 3.2 (Interval partitioning)**.** Let $m \in \mathbb{N}$, and $z_1, z_2 \in \mathbb{R}, z_1 < z_2$. Partitioning of an interval $[z_1, z_2]$ on m cells is considered to be

$$z_1 = z_{p_0} < z_{p_1} < \cdots < z_{p_m} = z_2, \tag{43}$$

and $z_{p_j} - z_{p_{j-1}} = c \in \mathbb{R}^+$ for all $j = 1, \ldots, m$.

Definition 3.3 (Taboo move, Cvijović and Klinowski (1995))**.** Let $X \subset \mathbb{R}^n$, and $x, x' \in X$, and $L \in \mathbb{N}$. The move $x \to x'$ is said to be taboo if at least one the following two conditions is met:

i) the move produces a solution, which has been reached in L preceeding steps,

ii) the move results in deterioration greater than allowed by a predefined value.

This definition, however, may be simplified by dropping the first condition. This adjustment makes good sense because from equation (27) we scrutinize the loss function $L : \mathbb{R}^4 \to \mathbb{R}$, where input variables are taken with at least three-decimal-place precision. Then, if parameter bandwidths were arbitrarily 0.1[6], the probability the same values are to be hit and selected at least twice in 10 iterations[7] is rather negligible, only about $9 \cdot 10^{-12}$. Hence, we bring the following modified definition of a taboo move.

Definition 3.4 (Taboo move - modified version)**.** Let $X \subset \mathbb{R}^n$, and $x, x' \in X$. The move $x \to x'$ is said to be taboo if

[6] In our application, ranges are actually a little bit greater in average.

[7] This is a similar value to those one applied in research done by Cvijović and Klinowski (1995).

- the move results in deterioration greater than allowed by a predefined value.

The moves going into consideration in each iterative step of TS are drawn from non-taboo moves from a neighbourhood of a current solution. To define a neighbourhood, let us recall our optimization problem (40) aiming to minimize a loss function (39). In this case, all the linear parameters are given, while $n_s \in \mathbb{N}$ different points for each non-linear parameter t_c, m, ω, and ϕ, are separately drawn from a uniform distribution from $n_c \in \mathbb{N}$ cells. The latter four points define a neighbourhood of the current solution x of the size $n_c n_s$. At the end of each iterative step, the move with the lowest move value is picked to be the next solution. Iterations are executed as long as improvements are being obtained within a limited number of rounds Cvijović and Klinowski (1995).

The whole procedure is summarized by the pseudocode written below. The code in full extent is attached in the Appendix.

Algorithm 1 Taboo Search

```
parameters ← list()
elite_list ← list()
losses ← list()
parameters ← runif(n=30)
losses ← loss_function(parameters)
elite_list ← parameters(argmin(losses, 10))
taboo_condition ← elite_lost[10, 'loss']
if min(losses) < 200 then
    while iter_without_improvement < 100 do
        neighbourhood ← getNeighbourhood()
        non_taboo_solutions ← getNonTaboo(neighbourhood)
        best_sol ← non_taboo_solutions[argmin(self), ]
        if best_sol < elite_list[10, 'loss'] then
            elite_list[10] ← best_sol
            elite_list ← sort(elite_list)
            iter_without_improvement ← 0
        else
            iter_without_improvement += 1
    best_solution ← elite_list[1]
return best_solution
```

3.3.7 Levenberg-Marquardt algorithm

Another method used for solving the non-linear problem is the LMA that was in past independently studied by Levenberg (1944) and Marquardt (1963). The

LMA can be considered a blend of the gradient descent algorithm (GD), invented by the French mathematician L. A. Cauchy (1847), and the Gauss-Newton algorithm (GNA) firstly appeared in Gauss (1809). The choice of the LMA stems from a quicker convergence in comparison with GD which is mainly due to a constant step size in case of GD that needs to be set apriori rather small in order for the algorithm to successfully converge to a local minimum. In areas of a small slope, therefore, this algorithm tends to trudge very slowly to an optimum. Another source of hurdles faced by GD cramping speed efficiency is varying curvature of error function surface in different directions. The latter problem can be mostly overcome by the GNA exploiting the second-order derivatives to set a proper step size, however, this algorithm is prone to divergence in situations when the presence of a quadratic approximation of error function is not reasonable. Hence, the LMA sort of combines these two tools together such as in areas featuring perplexing curvature, LMA exploits GD, and as soon as the function curvature begins to behave more smoothly, a quadratic approximation is wielded to hasten convergence Yu and Wilamowski (2011).

Comprehensive thought of the LMA is not in the scope of this text, and for interested readers that can be found, for example, in Moré (1978), Nocedal and Wright (2006), Kelley (1999), Yu and Wilamowski (2011). We, instead, focus on a brief description of the LMA in order to grasp the substance of the mathematical theory behind this method necessary for a practical implementation.

Same, as in the case of TS, the LMA requires to be fed by an initial solution x_0. Furthermore, grid search is also used to identify different local minima, and the identical minimization problem (42) needs to be figured out. We further suppose to have a point $p \in \mathbb{R}^n$ such that $||p|| < \Delta \in \mathbb{R}^+$, i.e. a point $x + p$ should be in a close vicinity of a current solution x. Then, an application of Taylor Series expansion leads to the approximation

$$L(x + p) \approx L(x) + \mathbb{J}p, \tag{44}$$

where $\mathbb{J}$ is a Jacobian matrix (Jacobian matrix is defined as a matrix of first partial derivatives, i.e. $\mathbb{J} = \frac{\partial f(x)}{\partial x}$, Hájková et al. (2012)), $\mathbb{J} \in \mathbb{R}^{m \times n}$, of some loss function $L : \mathbb{R}^n \to \mathbb{R}^m$. In our case, we are endowed by loss function from a three/four-dimensional real space to a one-dimensional thus for simplicity L will be restricted only on L such that $L : \mathbb{R}^n \to \mathbb{R}$.

In each iterative step, we look for a point p, such that $L(x) + \mathbb{J}p$ is minimized, which is yielded from a normal equation

$$\mathbb{J}^T \mathbb{J}p = \mathbb{J}^T L(x), \tag{45}$$

where $\mathbb{J}^T\mathbb{J}$ is an approximation of the Hessian (Hessian matrix is defined as a matrix of second-order partial derivatives, i.e. an element in the i^{th} row and the j^{th} column is $\mathbb{H}_{ij} = \frac{\partial f(x)}{\partial x_i \partial x_j}$, Hájková et al. (2012)). The LMA slightly adjusts the LHS of equation (45) to secure an invertibility of the Hessian by adding combination coefficient $\mu > 0$ that

$$\mathbb{H} = \mathbb{J}^T\mathbb{J} + \mu\mathbb{I}_n, \tag{46}$$

where $\mathbb{I}_n$ is an identity matrix of size n and thus actually the following statement needs to be solved

$$\mathbb{H}p = \mathbb{J}^T L. \tag{47}$$

.

The validity of this assumed property (invertibility) flows from that in the proximity of any local minimum we always consider a function with a convex surface (this logic stems from a sufficient condition for a local minimum Hájková et al. (2012)). From the field of mathematical analysis, we know the Hessian matrix of such a function is positive semi-definite, which has non-negative elements on a diagonal, hence by adding a non-zero element $\mu > 0$, we are assured to obtain a full-rank Hessian matrix of size n which is by definition invertible. Otherwise, i.e. far from any local minima, the surface of loss function does not have to be convex hence positive semi-definition is not guaranteed. Nevertheless, from the essence of the LMA, μ will be determined such that $\mu \gg 0$, therefore, even in those cases it should hold for $\mathbb{H}$ to be a positive definite matrix.

Now, approaching the theory of the GNA and its update rule that states

$$x_{k+1} = x_k - \left(\mathbb{J}^T\mathbb{J}\right)^{-1} \mathbb{J}^T L, \tag{48}$$

k denotes the current round of optimization process, and by combining equations (46) with (48), an update rule for the LMA can be derived as

$$x_{k+1} = x_k - \left(\mathbb{J}^T \mathbb{J} + \mu \mathbb{I}_n\right)^{-1} \mathbb{J}^T L. \tag{49}$$

In equation (49), depending on the magnitude of the combination coefficient μ, the LMA switches between the GNA and GD. If μ is very small (i.e. close to zero), an approach very similar to GNA is applied. In the opposite scenario, the LMA is an approximation of the GD where each step is defined as

$$x_{k+1} = x_k - \alpha g, \tag{50}$$

α is a learning rate and g denotes the gradient of a utilized loss function.

In each iterative step, a solution obtained from equation (49) is either accepted, which happens when x_{k+1} brings an improvement, i.e. $L(x_{k+1}) \leq L(x_k)$, then the procedure continues with a decreased damping term, otherwise μ is enlarged and the changing step $k+1$ is repeated in an attempt to find an enhancing solution. The LMA terminates as soon as one of the following criterion is met:

i) the magnitude of the update step size defined as $\mathbb{J}^T L$ drops below a pre-specified value ϵ_1,

ii) the relative change in p between two consecutive steps dips below a pre-defined value ϵ_2,

iii) sufficient accuracy has been reached, i.e. $L(x_k) < \epsilon_3$,

iv) maximum number of iterations has been done,

Yu and Wilamowski (2011), Lourakis et al. (2005), and convergence of the LMA is briefly justified in Moré (1978).

3.3.8 Test methodology

The non-negligible pitfall of the LPPL model is the fact that this model is always somehow fitted on time series data regardless of the presence of log-periodic oscillations in them. Therefore, looking at values of R^2 is not generally sufficient. Beside R^2, we also focus on another measurement - root mean-squared error (RMSE). This statistics is an example of an absolute value of errors, which can provide us with a clearer inside into whether the model is truly fitted or not. RMSE is defined as

$$\text{RMSE} = \sqrt{\frac{\sum_{t=1}^{T}(\hat{y}_t - y_t)^2}{T}} \tag{51}$$

and it will be denoted by a sign χ in the remain of this thesis.

We further monitor all parameters to be in recommended confined intervals. Besides, we drop fittings with critical time t_c predicted such as t_c is almost equal to t_n, which is a habit also applied in Sornette (2003).[8]

Moreover, as mentioned above, a crash occurrence is not a purely deterministic event, therefore, in Subsection 5.3 where out-of-sample analysis is to be conducted all results obtained in the particular windows will be taken into consideration and evaluated together. If more than one local minimum with the same goodness-of-fit are found, all of them will be further discussed.

3.4 Critical Slowing Down

Notwithstanding the fact that prediction of critical shifts in various systems is very cumbersome because systems may exhibit only little change before such a transition, it has been shown there exist certain generic symptoms which are detectable in different systems regardless of distinctions of particular systems Schroeder (2009). In the concept of mathematical models, this catastrophic phase transitions are related to bifurcations which have already been heavily studied in mathematics. This topic is quite broad, and deals with various kinds of bifurcation. Nevertheless, in the text below, we always refer only to the ones where a shift from one state of the system to another one suddenly occurs with dramatically changed state conditions. For further study, books like Kuznetsov (2013) or Chow and Hale (2012) can be recommended, though such a theory is beyond the scope of our text.

The CSD has been suggested as one of the most promising barometers of the nearing critical point Wissel (1984). The most direct implication of CSD is gradually slowing recovery rate in the observed system as it approaches the critical

[8] It is a good tactic since these fits would be likely to converge to time smaller than t_c hence they should not be considered as a valid solution.

point Van Nes and Scheffer (2007). Nevertheless, due to the large amount of noise present in data obtained, for example, from financial markets, it may be futile to try to detect this feature. Fortunately, three patterns have been suggested as an indicator of occurrence of CSD in the systems. There are two time-series signals, an increased autocorrelation Dakos et al. (2012), elevated variance Carpenter and Brock (2006), and one cross-sectional change Schroeder (2009) which are detectable. Those three characteristics are elaborated in sections 3.4.2 and 3.4.3, where both intuition and mathematical reasoning are provided.

The distinction in an application in comparison with the LPPL model is that whereas the LPPL model "consumes" the price (or logarithm of the price) of an asset itself, the CSD looks at residuals obtained from detrending.

3.4.1 Detrending

To conduct a proper statistical analysis, it is necessary to obtain a mean-stationary data. The stock price or logarithm of that price generally violates this property due to long-term trends. Detrending is thus necessary and subtracting moving averages from the observed index price is a relatively widely used method Dakos et al. (2008). Here, we modify this scheme by allowing the data nearer to the given time point to have larger weights through the Gaussian kernel smoothing thus we follow the framework applied in Diks et al. (2015).

Consider the case where we always want to probe the last T trading days before the edge of the crisis. Then, the weighted moving average at time t is defined as

$$MA_t = \frac{\sum_{r=1}^{T} G(r-t) y_r}{\sum_{r=1}^{T} G(r-t)},\ t = 1, \dots, T \tag{52}$$

where $G(\cdot)$ denotes a Gaussian kernel function

$$G(s) = \frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{s^2}{2\sigma^2}}, \tag{53}$$

and y_t represents logarithm of index price at time t.

The residual fluctuations, which are then inspected, are obtained by subtracting MA_t from y_t,

$$z_t = y_t - MA_t,\ t = 1, \dots, T. \tag{54}$$

These residuals fluctuate around 0 by their construction thus mean-stationarity is to be satisfied. Further in the text, z_t is referred as residuals or detrended fluctuations.

There is the one parameter needing to be a priori determined, the size of bandwidth σ (here, this parameter can be interpreted in the number of trading days). We face a standard bias-variance trade-off as by choosing σ too high, the resulting time series is over-smoothed and a lot of information is missed. On the other hand, too small bandwidth causes the noisiness of the data not to be sufficiently eliminated and it results in persisting high variance. Therefore, we aim to determine the choice of σ so that the long-term trends are curtailed and at the same time, daily fluctuations around some equilibrium value remain in satisfactory details Diks et al. (2015).

3.4.2 Time-series patterns

Autocorrelation

Generally, an important intuition behind the rise in autocorrelation is that systemic slowing down should imply a gain in serial correlation in an observed data. Intuitively, this can be understood in a way that as an internal speed of regeneration goes down, the state of the system at time $t+1$ is more and more similar to conditions at time t. Mathematically, this fact can grasped by 1-lag autocorrelation as Dakos et al. (2012) pointed out as follows.

For simplicity, consider such a system with one equilibrium state $\bar{x}$ where after each period Δt white noise is present. Furthermore, assume a given system's recovery from perturbations is roughly exponential with some speed λ. All of this can be expressed as

$$x_{t+1} = \exp^{-\lambda \Delta t} x_t + \sigma \epsilon_t, \tag{55}$$

where $\epsilon \sim N(0,1)$, and σ denotes an amplitude of disturbances. Subtracting $\bar{x}$ from equation (55), deviation of x from the equilibrium is given as

$$x_{t+1} - \bar{x} = \exp^{-\lambda \Delta t}(x_t - \bar{x}) + \sigma \epsilon_t \tag{56}$$

and by rewriting $x_t - x = y_t$, we obtain

$$y_{t+1} = \exp^{-\lambda \Delta t} y_t + \sigma \epsilon_t, \tag{57}$$

Assuming independence of λ and Δt of y_t for all $t \in \mathbb{N}$, and restricting λ from being negative, the relationship described by equation (57) can be rewritten as an AR(1) process, i.e. $\exp^{-\lambda \Delta t} = \alpha$, such as

$$y_{t+1} = \alpha y_t + \sigma \epsilon_t \tag{58}$$

This is both mathematically valid and it also corresponds with the hypothesis because we have

$$\lim_{\lambda \to \infty} \exp^{-\lambda \Delta t} = 0, \quad \lim_{\lambda \to 0} \exp^{-\lambda \Delta t} = 1, \tag{59}$$

Scheffer et al. (2009).

This distinguishable increase in autocorrelation that develops long before the phase transition occurs has been proven to hold both in simple models Dakos et al. (2012) and much more complex and realistic systems Lenton et al. (2008).

Variance

The other distinct time-series pattern in fluctuations of system state is a rise in variance. This can be again reasoned using intuition. Consider equation (57), and suppose a system approaching a critical point with slowing speed of recovery. As an eigenvalue of regeneration approaches zero, the system becomes unable to diminish the impact of disturbances. Therefore, the effect of particular perturbations tend to accumulate, which eventually results in a higher variance, as it has been shown, for example, by Carpenter and Brock (2006).

Mathematically, this characteristics can be easily derived using the knowledge from equation of the AR(1) process such as (58). Assuming deviation of a system at some initial time t_0 is equal to zero, i.e. $y_{t_0} = 0$, we can simply compute the variance of this process at time $t \in \mathbb{N}$. For this purpose, recall that $E(\alpha y) = \alpha E(y)$, $E(\epsilon) = 0$ for white noise ϵ, and from linearity of expectation we further have $E(\alpha x + \beta y) = \alpha E(x) + \beta E(y)$. Then the first moment is

$$E(y_t) = \alpha E(y_{t-1}) + \sigma E(\epsilon_{t_1}) = \cdots = \alpha^t E(y_{t_0}) = \alpha^t y_{t_0} = 0 \tag{60}$$

hence, assuming Δt is small enough such as $Var(y_t) - Var(y_{t-1}) < \varepsilon$ for all $\varepsilon \in \mathbb{R},\ \varepsilon > 0$, $Var(y_t)$ is

$$Var(y_t) = E(y_t^2) - (E(y_t))^2 = E(y_t^2) = \alpha E(y_t^2) + \sigma^2 = \frac{\sigma^2}{1-\alpha^2}, \tag{61}$$

provided that $|\alpha| < 1$. However, the last assumption for α is directly satisfied from equation (59) and the monotony of an exponential function.

From the information stated above, it is obvious that as recovery speed decreases, α approaches zero, and consequently variance rises. Therefore, it seems reasonable to observe variance as an indicator of the CSD which is claimed to be a precursor of various types of bifurcations Kuznetsov (2013).

3.4.3 Cross-sectional patterns

As we elaborate in Subsection 3.2.2, there is an imitative behaviour of individual agents behind financial crashes. This pattern, however, can be also considered from different perspective as we can look at, let us say, the U.S. market, whose condition is reflected by, for example, the S&P 500. Thus, instead of examining individual agents we can scrutinize the performance of individual stocks. Therefore, considering this self-organization pattern from Subsection 3.2, we should be able to detect an increase in correlation among individual stocks, or potentially among different financial markets which also affect each other due to high level of globalization nowadays. Evidence of elevated correlation across different stock returns may serve as early-warning signals was, for example, suggested by Hong and Stein (2003).

Due to a huge number of pairwise correlation between individual stocks, we do not investigate this measurement between them, and rather focus only on correlation between particular markets. However, we do not throw this idea away, because if mimic behaviour really emerges prior financial turmoils, this intuition of correlated stocks gives us another justification for scrutinizing variance over time, which simply stems from the mathematical property of variance that states

$$Var(y_t) = Var\left(\sum_{i=1}^{n} w_i r_{it}\right) = \sum_{i=1}^{n} w_i^2 Var(r_{it}) + 2 \sum_{1 \leq i < j \leq n} w_i w_j Cov(r_{it}, r_{jt}) \quad (62)$$

Bartoszynski and Niewiadomska-Bugaj (2007), where y_t denotes detrended fluctuations of any stock index at some time t, r_{it} represents remaining residuals of stock i at time t and w_i stands for a relative weight of an asset involved in an observed gauge.

3.4.4 Test diagnostics

In order to avoid reliance only on the eye inspection when ascertaining leading indicators behave according to patterns described in the subsection above, we test trends over time in gauges, AR(1) and variance, for significance according to scheme described by Dakos et al. (2008) or Diks et al. (2015).

For this purpose, we employ the Kendall rank correlation coefficient introduced by Kendall (1955), which is a non-parametric measurement evaluating the degree of similarity between two sets of variables, in our case between the one of time-varying indicators, and time. Kendall's tau is defined as

$$\hat{\tau} = \frac{C - D}{N} \quad (63)$$

where C is a number of concordant pairs, D denotes the number of discordant pairs, N represents the total number of different pair combinations hence $N = \frac{n(n-1)}{2}$, and n is a number of observations. Kendall's tau takes values from an interval $[-1, 1]$ Abdi (2007). Thus we would expect to obtain values closer to 1, as we suppose a positive development in the observed indicators over the time.

Furthermore, we count on a standardized hypothesis testing. Generally, accepted approximation of variance of Kendall's τ for $n > 10$ is

$$\sigma_\tau^2 = \frac{2(2N + 5)}{9N(N - 1)}. \quad (64)$$

Then

$$Z_\tau = \frac{\tau}{\sigma_\tau} = \frac{\tau}{\sqrt{\frac{2(2N+5)}{9N(N-1)}}} \tag{65}$$

has a normal distribution with zero mean and unit variance. After computing a Z value, corresponding level of significance under which the null hypothesis is rejected may be found Abdi (2007). In our case, null hypothesis represents an insignificant relationship between observed time-varying indicators and time. As the last thing, we will focus on two different long sliding windows, 100 days and 50 days, and accordingly we choose the number of trading days T prior to crash as 200 and 100 days respectively.

Finally, stationarity of detrended residuals is to be scrutinized by three well-known tests - Augmented Dickey-Fuller test[9], KPSS test[10], and PP test[11], which are well implemented in R and accompanied by a sufficient amount of documentation.

[9]https://www.rdocumentation.org/packages/tseries/versions/0.10-46/topics/adf.test
[10]https://www.rdocumentation.org/packages/tseries/versions/0.10-46/topics/kpss.test
[11]https://www.rdocumentation.org/packages/tseries/versions/0.10-46/topics/pp.test

4 Dataset description

The structure of data used in this thesis can be divided into two parts as the in-sample (calibration) set and the out-of-sample (verification) set. The first one is used to reveal whether two frameworks presented in this thesis are applicable for detecting an end of a financial bubble, or as warning signals of any different imminent crash. The other set is then considered as a way to determine how much prone these concepts are to false positives, verify their power, and hence to discuss their total suitability as warning indicators.

The calibration set consists of data from four past crises - Black Monday in 1987, the Mexican decay at the beginning of 1994 prior to the large national currency devaluation which, subsequently, brought contagion to the whole national economy; Asian financial crisis in 1997, and the burst of the dot-com boom in March 2000.

Regarding the Black Monday, we observe the S&P 500 index, which is a regarded stock index of the largest-cap American equities listed on the New York Stock Exchange, the NASDAQ Stock Market or Chicago Board Options Exchange. The S&P peaked on Friday, October 16, 1987, subsequently on Monday, October 19, a huge drop occurred.

Mexican crisis was ignited by peso devaluation in December 1994, and the complex financial turmoil followed shortly after Sachs et al. (1996). But measurable financial difficulties began even earlier and those ones are the target of our study. We decided to observe IPC Mexico (MXX) which is a stock index gauging the performance of 35 stocks traded on Mexican Stock Exchange. There, we detect MXX peaked on February 8, 1994, and a day later an approximately two-month gradual deterioration started.

To investigate the 1997 Asian financial crisis, we choose to follow the Hang Seng Index (HSI) that contains 50 stocks listed on the Hong Kong Stock Exchange. Multiple points can be selected as the beginning of financial meltdowns as HSI peaked on August 7, but another local maximum was on October 3. Finally, we decided to choose October 20 as a decisive day because then a 5-day decline sparked and HSI dropped by 23.34%.

The dot-com boom was a speculative bubble, motivated by a tremendous rise in the use of the internet, that eventually burst on March 11, 2000, and affected the world economy for more than two years. We use the NASDAQ Composite Index, comprising of stocks and associated securities traded on the Nasdaq Stock Exchange, as a single gauge of this famous event.

For analyzing 1-lag serial correlation and variance for the CSD, we always use time series data consisting of 100 or 200 days (depending on the width of the sliding window) respectively prior to the crash. Specification of time frames utilized in the LPPL analysis is summarized in the table below. Beginning is denoted by time t_1, end is the last date from available time series data, i.e. t_n. Time frame $[t_1, t_n]$ is subsequently used for a prediction of critical time t_c. Finally, only the logarithm of the price is always used.

Index	**Beginning**	**End**	**Time of crash, t_c**
S&P 500	1984-09-17	1987-08-13	1987-10-17
IPC Mexico (MXX)	1993-06-10	1993-12-14	1994-02-09
Hang Seng (HSI)	1995-02-13	1997-08-22	1997-10-20
NASDAQ Composite	1995-01-03	1999-12-29	2000-03-11

Table 1: LPPL: Calibration dataset

For the validation ("out-of-sample") set, we opted to choose two financial crashes as a benchmark for our two methods. The global financial crisis 2007-2008 is the first occasion, the Bitcoin bubble, which burst at the end of the year 2017, is the other one.

In the case of global depression, we select fourteen points in time when predictions are to be made. Time frames for the CSD analysis are conceptually in line with those described for the calibration sets. Regarding the LPPL model, for the seven initial predictions, March 15, 2003, as the starting date is chosen. The beginning is subsequently shifted to October 25, 2005, because of market conditions during years 2004 and 2005 which do not correspond to the bubble-developing behaviour. The last observed trading day was on September 17, 2007. All the time frames are clearly presented in Table 2 accompanied by visualisation.

We follow the Dow Jones Industrial Average, which is a gauge of 30 publicly traded companies on the New York Stock Exchange, and the Nasdaq Exchange.

Beginning	Beginning	End date	End date
2003-03-17	2003.205	2005-03-15	2005.200
2003-03-17	2003.205	2005-06-15	2005.452
2003-03-17	2003.205	2005-09-15	2005.704
2003-03-17	2003.205	2005-12-15	2005.953
2003-03-17	2003.205	2006-03-15	2006.200
2003-03-17	2003.205	2006-06-15	2006.452
2003-03-17	2003.205	2006-09-15	2006.704
2015-10-25	2005.814	2006-12-15	2006.953
2015-10-25	2005.814	2007-03-15	2007.200
2015-10-25	2005.814	2007-05-16	2007.370
2015-10-25	2005.814	2007-06-15	2007.452
2015-10-25	2005.814	2007-07-16	2007.537
2015-10-26	2005.814	2007-08-15	2007.619
2015-10-26	2005.814	2007-09-17	2007.701

Table 2: Financial crisis 2007-2008

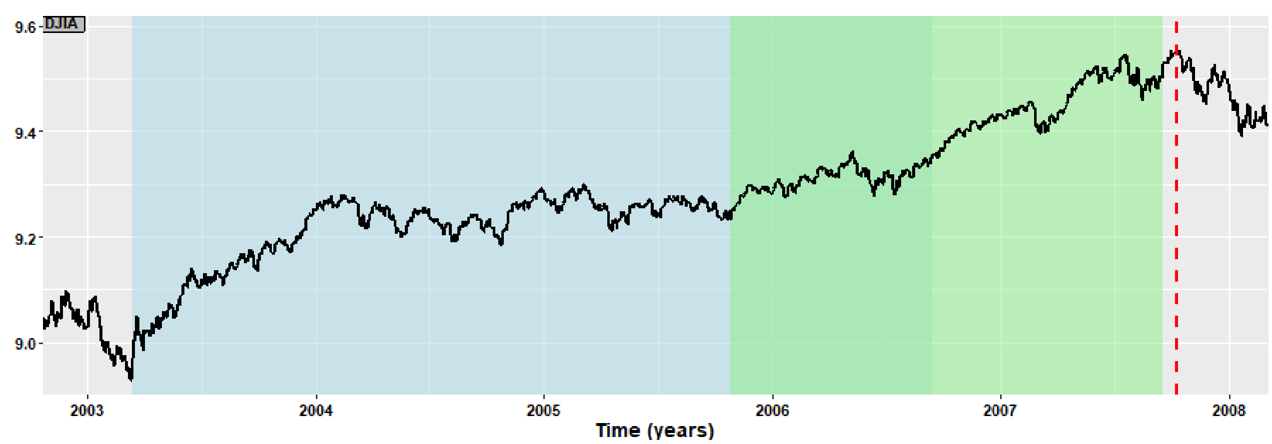

Figure 4: Precursor of the global financial crisis 2007-2008

Red dashed line depicts the time of bursting the bubble; two phases of given time series data with a different beginning date are graphically distinguished.

In the case of Bitcoin, the most famous cryptocurrency which was initially released in January 2009, we start to observe its ascending prices as of August 20, 2015, when the valuation of one coin was mere 232 dollars. Then the Bitcoin price was soaring punctuated by a few tumbles for more than two years and, finally, reached the valuation of 19,345 dollars on December 16, 2017. Subsequently, a substantial depreciation kindled.

For predicting the end of the Bitcoin bubble, we make sixteen predictions in total, the first one was made on August 15, 2016, and the rest of them were made during each consecutive month on the same day. The time window used for prediction are summarized in Table 3 accomplished with a time series plot below.

Start date	**Start date**	**End date**	**End date**
2015-08-20	2015.633	2016-08-15	2016.620
2015-08-20	2015.633	2016-09-15	2016.705
2015-08-20	2015.633	2016-10-15	2016.787
2015-08-20	2015.633	2016-11-15	2016.872
2015-08-20	2015.633	2016-12-15	2016.954
2015-08-20	2015.633	2017-01-15	2017.038
2015-08-20	2015.633	2017-02-15	2017.123
2015-08-20	2015.633	2017-03-15	2017.200
2015-08-20	2015.633	2017-04-15	2017.285
2015-08-20	2015.633	2017-05-15	2017.367
2015-08-20	2015.633	2017-06-15	2017.452
2015-08-20	2015.633	2017-07-15	2017.534
2015-08-20	2015.633	2017-08-15	2017.619
2015-08-20	2015.633	2017-09-15	2017.704
2015-08-20	2015.633	2017-10-15	2017.786
2015-08-20	2015.633	2017-11-15	2017.871

Table 3: Bitcoin bubble

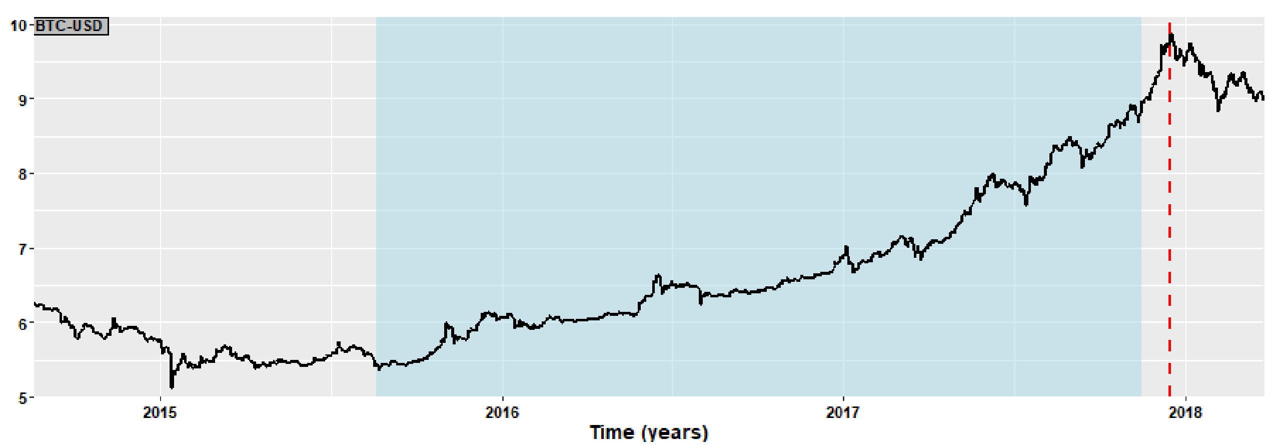

Figure 5: Bitcoin bubble 2015-2017
Red dashed line depicts the time of bursting the bubble

All the data presented in this Section are downloaded using the R package *quantmod*.[12]

[12]https://cran.r-project.org/web/packages/quantmod/index.html

5 Empirical results

This section is divided into three parts. The first two are dedicated to the verification of the capability of both schemes, i.e. whether they are able to detect financial bubbles. The best framework for the LPPL model, and the CSD scheme is then utilized in the last part which concentrates on a more general application since we go gradually through financial time series data, and we try to find out how much each of the chosen models is prone to false alarms. All the tests of the stationarity of detrended fluctuations exploited by the CSD are attached in the Appendix.

5.1 Calibration: LPPL model

The performances of both versions of the model are simultaneously analyzed and compared in the following four subsections. Thus, all of them contain two figures and two tables compiling the quality of fit and the capability of grasping the market dynamics.

5.1.1 Black Monday

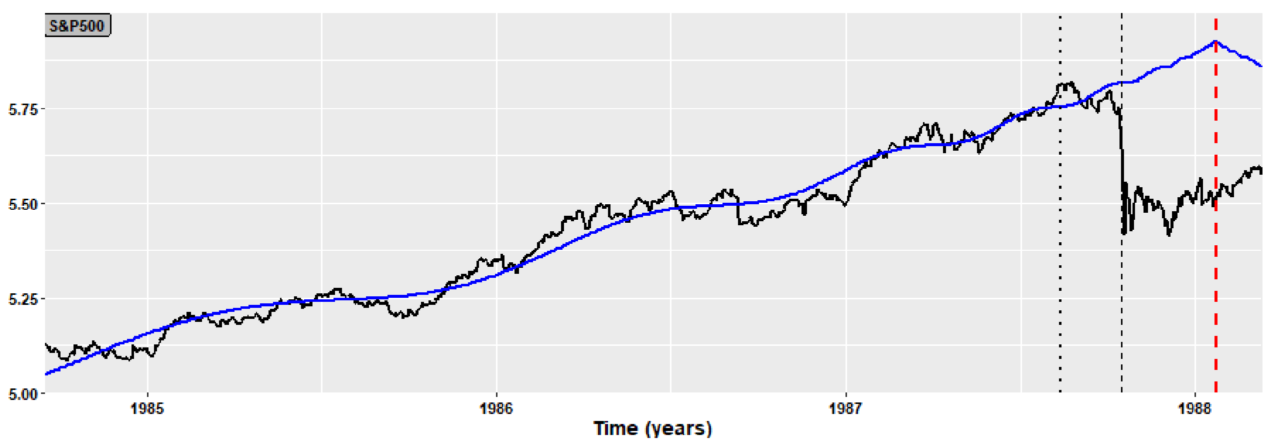

Figure 6: LPPL1: Black Monday 1987 (S&P500)
Dotted line depicts the day of prediction, black dashed line shows the time of crash, and the red dashed line portrays the critical time t_c.

	A	B	C	t_c	m	ω	ϕ	R^2	χ
Fit	5.927	-0.328	0.020	1988.058	0.806	11.193	5.467	.9739	.0317

Table 4: LPPL1: Black Monday

Evaluation of the LPPL scheme on modelling the events of Black Monday can be introduced by a survey of goodness-of-fit. Here, even though the fit measured by R^2 is pretty high, 97.39%, and the fitted curve follows the real price relatively well, the predicted time t_c is quite bad considering the real crash occurred at time $t = 1987.797$. All the parameters, both linear and non-linear ones, are well within the recommended ranges specified in Subsection 3.3.3, though we can say the parameters m and ω are relatively close to the interval boundaries, and do not satisfy the narrower interval proposed by Bree and Joseph (2010). An acceleration parameter m is relatively considerably high which reflects not so explosive rise in the price.

Significantly better work is performed by the modified LPPL model. The figure and the table summarizing the fitting process are provided below.

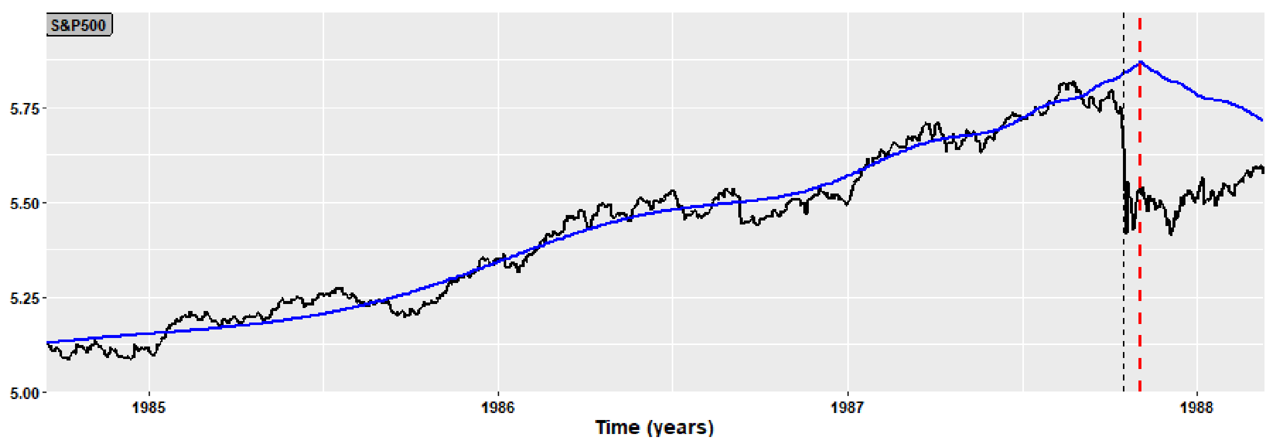

Figure 7: LPPL2: Black Monday 1987 (S&P500)

Dotted line depicts the day of prediction, black dashed line shows the time of crash, and the red dashed line portrays the critical time t_c.

	A	B	C_1	C_2	t_c	m	ω	R^2	χ
Fit	5.873	-0.334	-0.020	0.008	1987.838	0.748	7.321	.9736	.0319

Table 5: LPPL2: Black Monday

Despite the slightly worse R_2, the prediction of the critical time is much more accurate in the case of the LPPL2 model. Again, the value of parameter m is quite high, nevertheless, still well within the boundaries, however, the more confined intervals proposed by Bree and Joseph (2010) are not sufficient. All other parameters again lie well within broader boundaries.

The LPPL1 model and the LPPL2 model can be claimed to be able to fit on the log-periodic price dynamics of the S&P500 prior to the Black Monday, but both predictions are, unfortunately, too late. Nevertheless, the relatively high accuracy of the LPPL2 model must be pointed out as quite promising.

5.1.2 Mexican crisis 1994

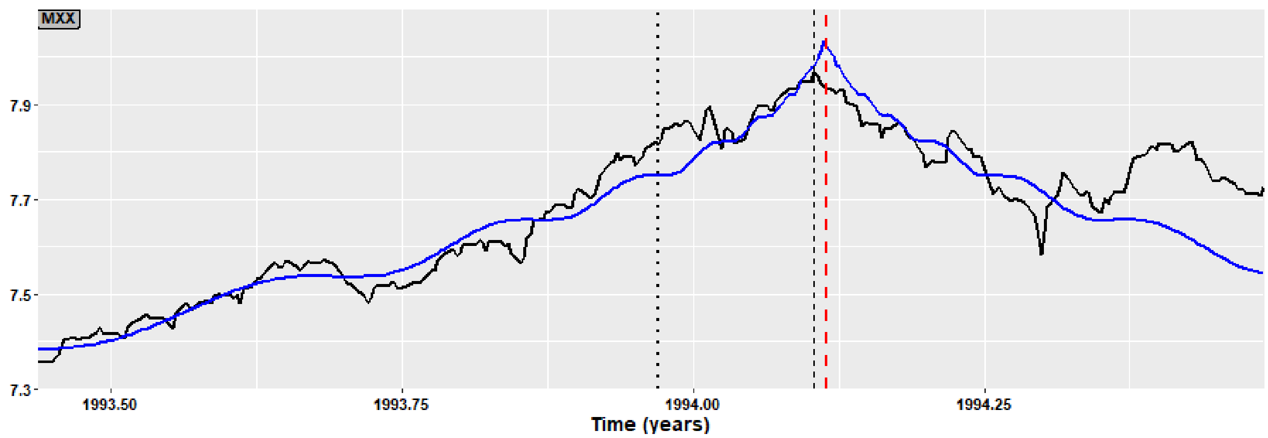

Figure 8: LPPL1: Mexican crisis 1994 (MXX)

Dotted line depicts the day of prediction, black dashed line shows the time of crash, and the red dashed line portrays the critical time t_c.

The Mexican market was substantially decorated by noise prior to the spark of the long-term decline of the Mexican stock index. In spite of these hurdles, the fit of the LPPL1 model is not far from ideal. The prediction of the tipping point is surprisingly quite accurate with respect to a short window utilized for this prediction. Details of fitting are provided in the table below.

	A	B	C	t_c	m	ω	ϕ	R^2	χ
Fit	8.070	-0.808	0.036	1994.114	0.480	11.927	3.827	.9175	.0300

Table 6: LPPL2: Mexican crisis

The value of R^2 is somewhat lower as expected due to noise present, and it flows around 0.92. On the other hand, the RMSE is comparable with the fit on Black Monday hence this appraisal can be considered as a good one. All the parameters, furthermore, satisfy assumptions imposed on the range of parameters. The estimation of the critical time t_c is again a little bit delayed, however, now the prediction is imprecise only by 0.01, which is equivalent to less than four calendar days.

Now, if we look at Figure 9, we can see that the LPPL2 model is much more

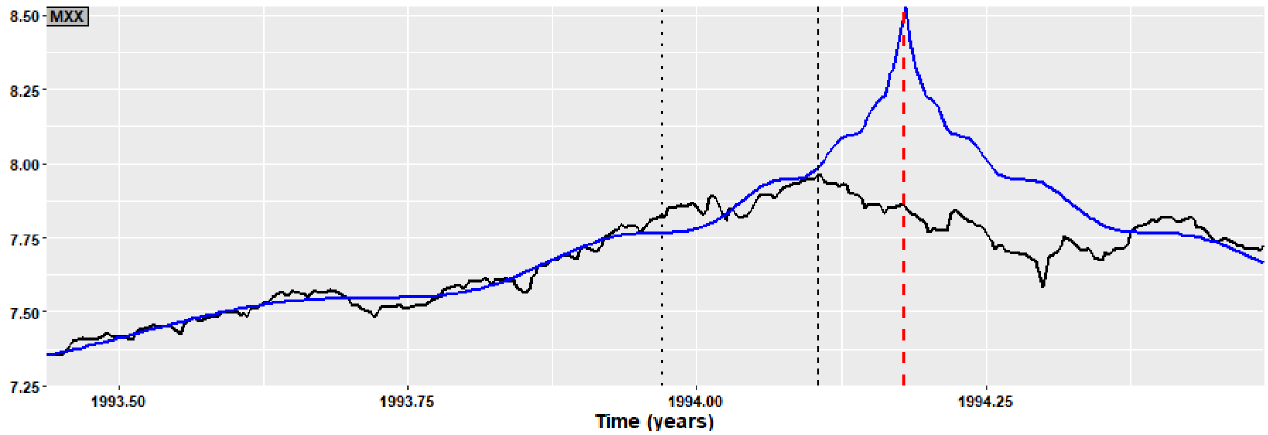

Figure 9: LPPL2: Mexican crisis 1994 (MXX)

Dotted line depicts the day of prediction, black dashed line shows the time of crash, and the red dashed line portrays the critical time t_c.

proficient in tackling fluctuating prices prior to the crash in comparison with the results obtained by the LPPL1 model. This is also mirrored by higher R^2 climbing up to 0.9553. All the parameters are again summarized in the table below.

	A	B	C_1	C_2	t_c	m	ω	R^2	χ
Fit	8.793	-1.519	0.006	0.046	1994.180	0.253	8.206	.9553	.0240

Table 7: LPPL2: Mexican crisis

Nevertheless, the overall performance of the model is poorer since the LPPL2 model let itself be deceived and a further steep climb in price is expected before the drop comes. The soaring predicted price at the end of the bubble is then a consequence of a relatively low value of the parameter m. Parameter ω is lower in this case, but now it satisfies even the confined intervals suggested by Bree and Joseph (2010).

5.1.3 Asian financial crisis 1997

From the eye inspection of Figure 10, the prediction of time t_c can be argued to be quite accurate, though again slightly delayed. The fit looks also quite well except for some major drops in index price, i.e. the one around time $t = 1997.25$, and it is also reflected by a high value of the goodness-of-fit topping 0.95. On the other hand, the highest value of RMSE so far hints local deviations are not captured by this model, yet this statistic does not seem problematic. Besides, all the parameters look reliable and are summarized Table 8 which is also accomplished with a plot of a given time series and following predictions.

	A	B	C	t_c	m	ω	ϕ	R^2	χ
Fit	9.808	-0.397	0.025	1997.837	0.620	9.024	5.138	.9501	.0389

Table 8: LPPL1: Asian crisis 1997

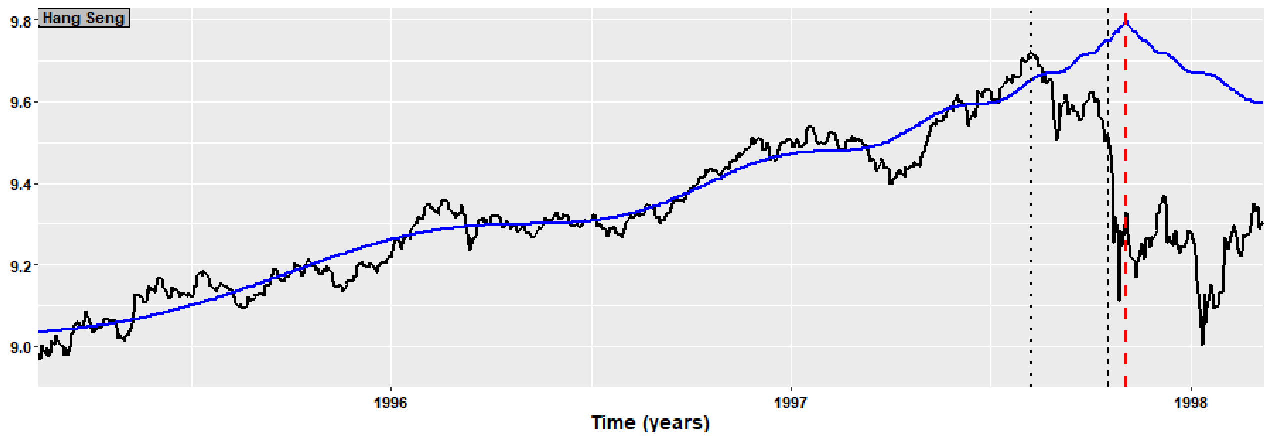

Figure 10: LPPL1: Asian crisis 1997 (Hang Seng)
Dotted line depicts the day of prediction, black dashed line shows the time of crash, and the red dashed line portrays the critical time t_c.

Figure 11 depicts the prediction of the LPPL2 model whose fitted parameters are summarized in the table below.

	A	B	C_1	C_2	t_c	m	ω	R^2	χ
Fit	9.744	-0.346	-0.012	-0.020	1997.780	0.735	6.630	.9502	.0386

Table 9: LPPL2: Asian crisis 1997

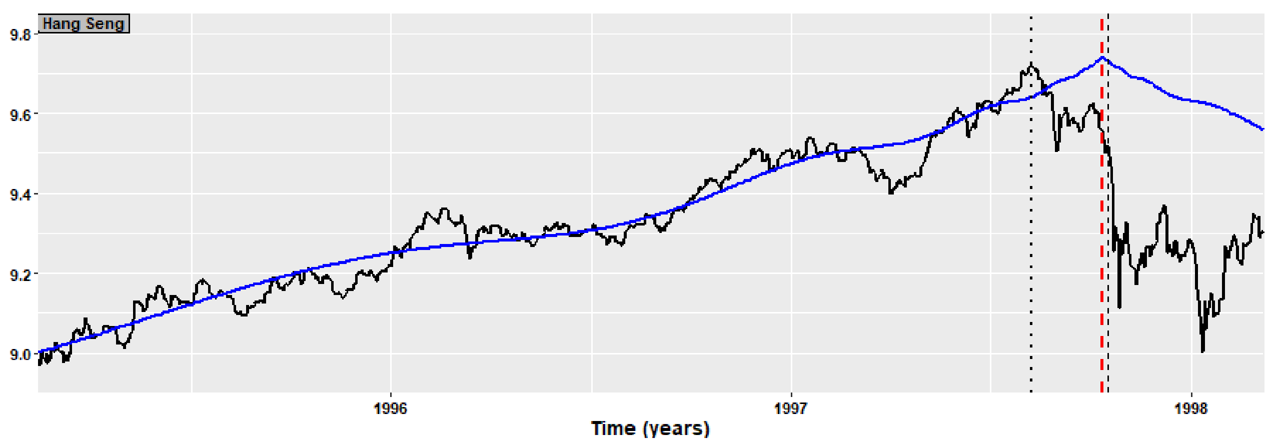

Figure 11: LPPL2: Asian crisis 1997 (Hang Seng)
Dotted line depicts the day of prediction, black dashed line shows the time of crash, and the red dashed line portrays the critical time t_c.

At first sight, the prediction of t_c is much more accurate in this case. All the parameters again satisfy the recommended intervals. Similarly to the LPPL1 model, neither this model is able to fit the price slump around 1997.25, potentially causing a relatively high value of m thereby growth in a terminal phase might seem to be too small.

Importantly, during the fitting of the LPPL2 model, two or three results were obtained with t_c very close to 1997.6, which actually pointed out that the end of the dataset is more or less identical with the time of the burst of a bubble. However, these predictions were dropped in alignment with the intuition provided in Subsection 3.3.8. Generally, the solution returned by the LPPL2 should be preferred to the LPPL1's one.

5.1.4 The dot-com bubble

In this case, the beginning of the burst of the dot-com bubble is predicted by the LPPL1 model quite well. The persistent trend is again well explained by this model, nonetheless, the final acceleration prior to the turbulent outcome is not exactly coped.

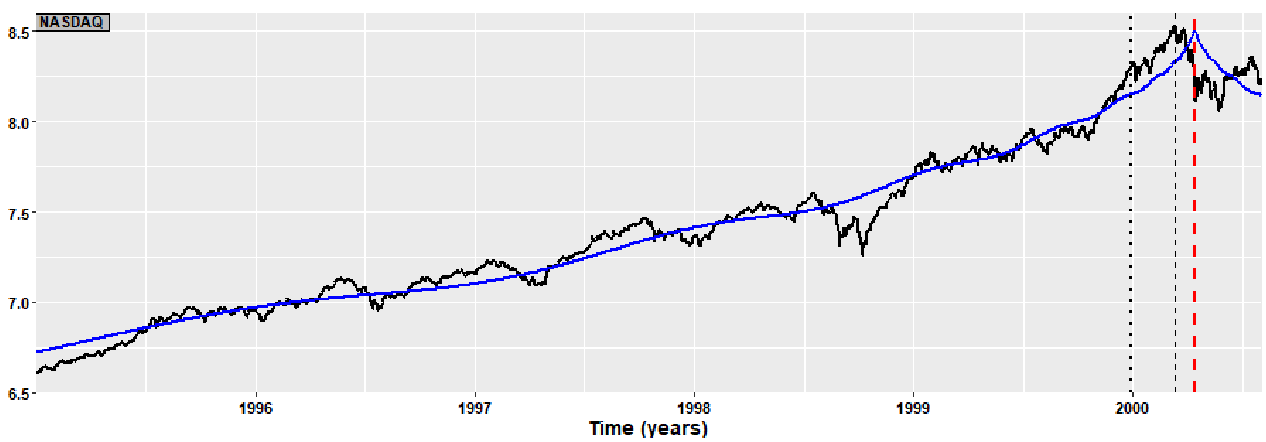

Figure 12: LPPL1: The dot-com bubble (NASDAQ)

Dotted line depicts the day of prediction, black dashed line shows the time of crash, and the red dashed line portrays the critical time t_c.

	A	B	C	t_c	m	ω	ϕ	R^2	χ
Fit	8.541	-0.753	0.020	2000.281	0.524	9.801	1.818	.9685	.0679

Table 10: LPPL1: Dot-com bubble

All the parameters respect the model specifications, and relatively high R^2, almost 97%, confirms the model is quite proficient in explanation of long-term market trends. RMSE, withal, is quite higher suggesting several non-marginal discrepancies between fitted and real log-price. The critical time t_c is not correctly predicted and it is delayed by 0.09 years, which is about 30 calendar days.

On the other hand, taking a deeper look at the graph leads us to a realization there is another peak next to the one occurred on March 10, 2000, which is referred as the day before a burst of the bubble. Subsequently, the major slump happened on March 28, 2010, and this date is about 15-16 calendar days before the predicted t_c, therefore, it is more or less within the 30-day window.

The prediction returned by LPPL2 model on the dot-com bubble is depicted by Figure 13. Visually, it can be seen, there are some chunks of misspecification at the beginning, still, this model is able to grasp the oscillation and acceleration former the jar.

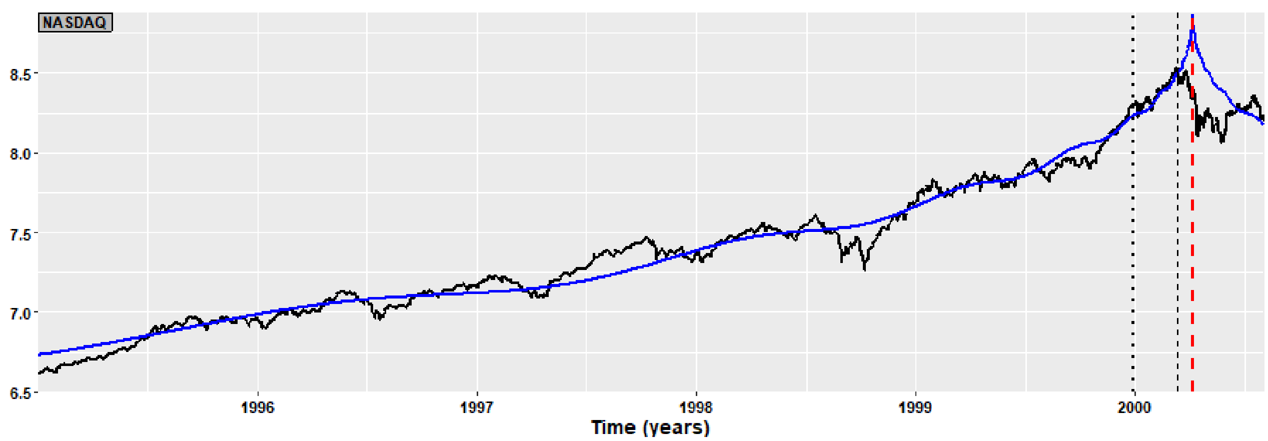

Figure 13: LPPL2: The dot-com bubble (NASDAQ)

Dotted line depicts the day of prediction, black dashed line shows the time of crash, and the red dashed line portrays the critical time t_c.

All the parameters presented in Table 11 are within the specified ranges, and the critical time t_c is predicted with high accuracy. Repeatedly, ω does not satisfy the confined interval proposed by Bree and Joseph (2010), and therefore, these narrower limits do not seem appropriate.

	A	B	C_1	C_2	t_c	m	ω	R^2	χ
Fit	8.926	-1.154	0.026	0.015	2000.262	0.373	9.504	.9670	.0695

Table 11: LPPL2: Dot-com bubble

Overall, despite the slightly lower R^2 in this second case, the decisive moments when the price accelerated in the last weeks before the burst are better captured by the LPPL2 model (for comparison look at Figure 12 and Figure 13). The difference in predictions of the critical time t_c is not so large, it is about 7 days, unfortunately, both estimations are a little bit delayed with respect to the true date of the edge of the crisis.

5.1.5 Summary of LPPL calibration

We analysed performances of both versions of the LPPL model utilized in this thesis on four major stock tumbles in the past thirty years. From the available results, we can conclude none of the models is distinctively superior to the other, however, the LPPL2 seems to be slightly more powerful. Therefore, the modification of the LPPL equation proposed by Filimonov and Sornette (2013) can be concluded to be reasonable because, besides probably higher robustness, the time of fitting is significantly shorter thanks to omitting the application of Tabu Search.

For the reasons stated above, we are confident to conduct further analysis in Section 5.3 only using the LPPL2 model. This model should be able to grasp the evolution of financial bubbles terminating by a burst because it is not any problematic to fit the model in the way so that the parameters satisfy the boundaries of recommended intervals. On the other, we can conclude that the confined ranges proposed by Bree and Joseph (2010) are often insufficient.

5.2 Calibration: Critical Slowing Down

5.2.1 Black Monday

Here, the performance of the CSD scheme is evaluated on the Black Monday 1987. As we can see from Figure 14, smoothing the time series data using our Gaussian kernel functions provides us with residuals distributed roughly around zero.[13] First, the length of sliding windows was selected for 100 days with a 10-day smoothing interval.

There is an apparent increase in the 1-lag autocorrelation supported by the relatively high value of Kendall's τ statistically significant at the 0.01 level. Furthermore, some gain in the standard deviation of residuals is also detectable with sta-

[13] The results of all unit root tests are attached in Appendix.

tistically significant value of Kendall's τ equalling 0.423 at the 0.01 level, though, this fact is not greatly convincing by use of eye inspection. The spike in the variance at the beginning is likely to be explained by worse fitted smoothing line about 200 days before the crash.

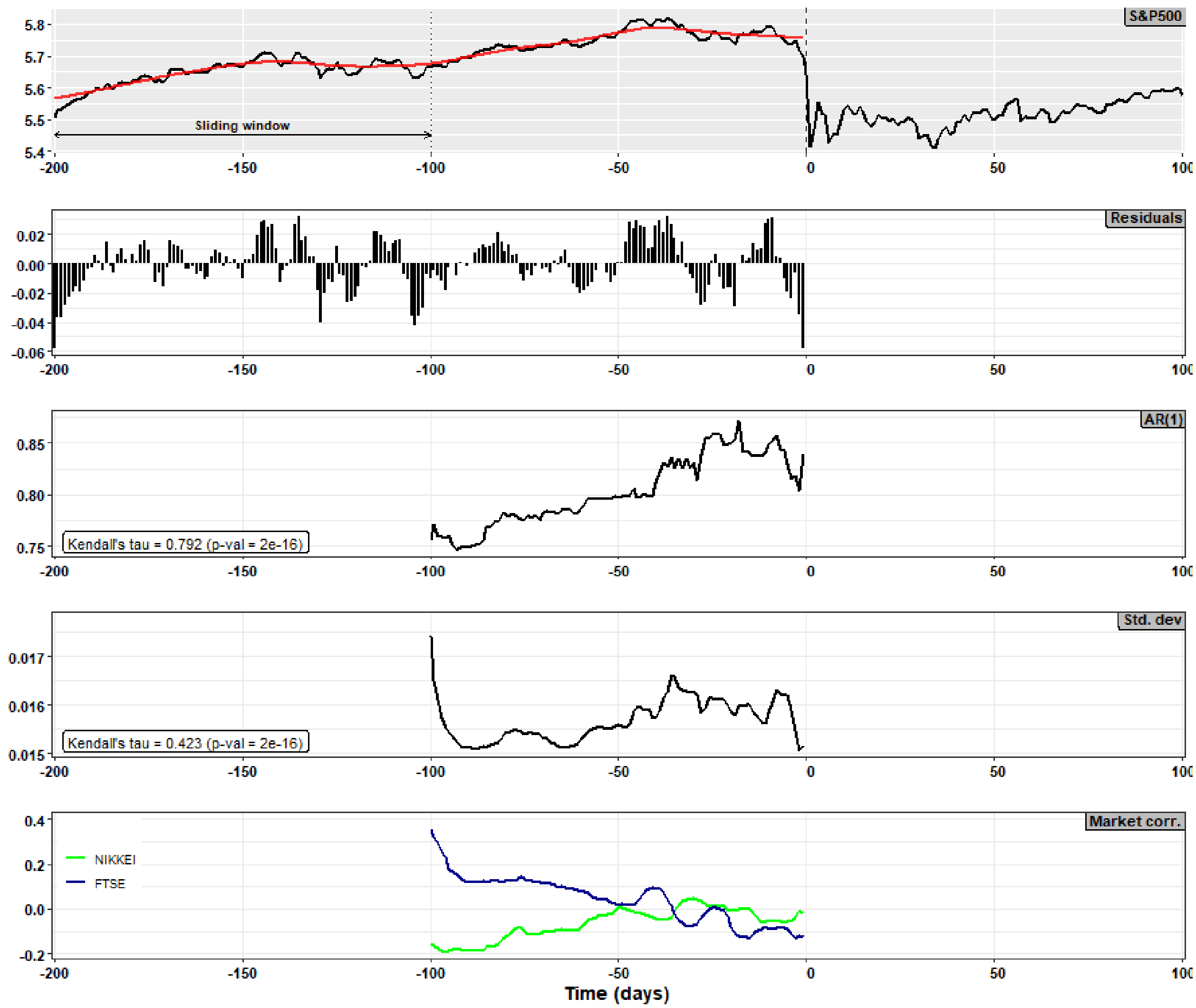

Figure 14: CSD: Black Monday 1987 (S&P500), sliding window = 100

On the other hand, our hypothesis suggesting cross-correlation across the world markets should rise prior to the crash cannot be well approved. Whereas Kendall's τ for the correlation between S&P500's residuals and NIKKEI's residuals is 0.573***, this measurement equals -0.817*** for the correlation between detrended fluctuations of S&P500 and FTSE. Moreover, this premise is also not supported by time-series inference of our plot.

Second, we adjust the smoothing window to 5 trading days, and by narrowing the sliding window to mere 50 days, considerably different results are obtained as suggested by Figure 15. Values of AR(1) a few days before Monday 19, 1987, actually tell nothing about the imminent crash, and this measurement vividly fluctuates still around the similar values, and it is indeed slightly decreasing according to the Kendell's τ which is -0.241, and statistically significant at the 0.01 level. Conversely, the variance of residuals substantially ascends over time, and on this occasion, may serve as a valid indicator. Correlation with other markets again proves to be a kind of an impasse.

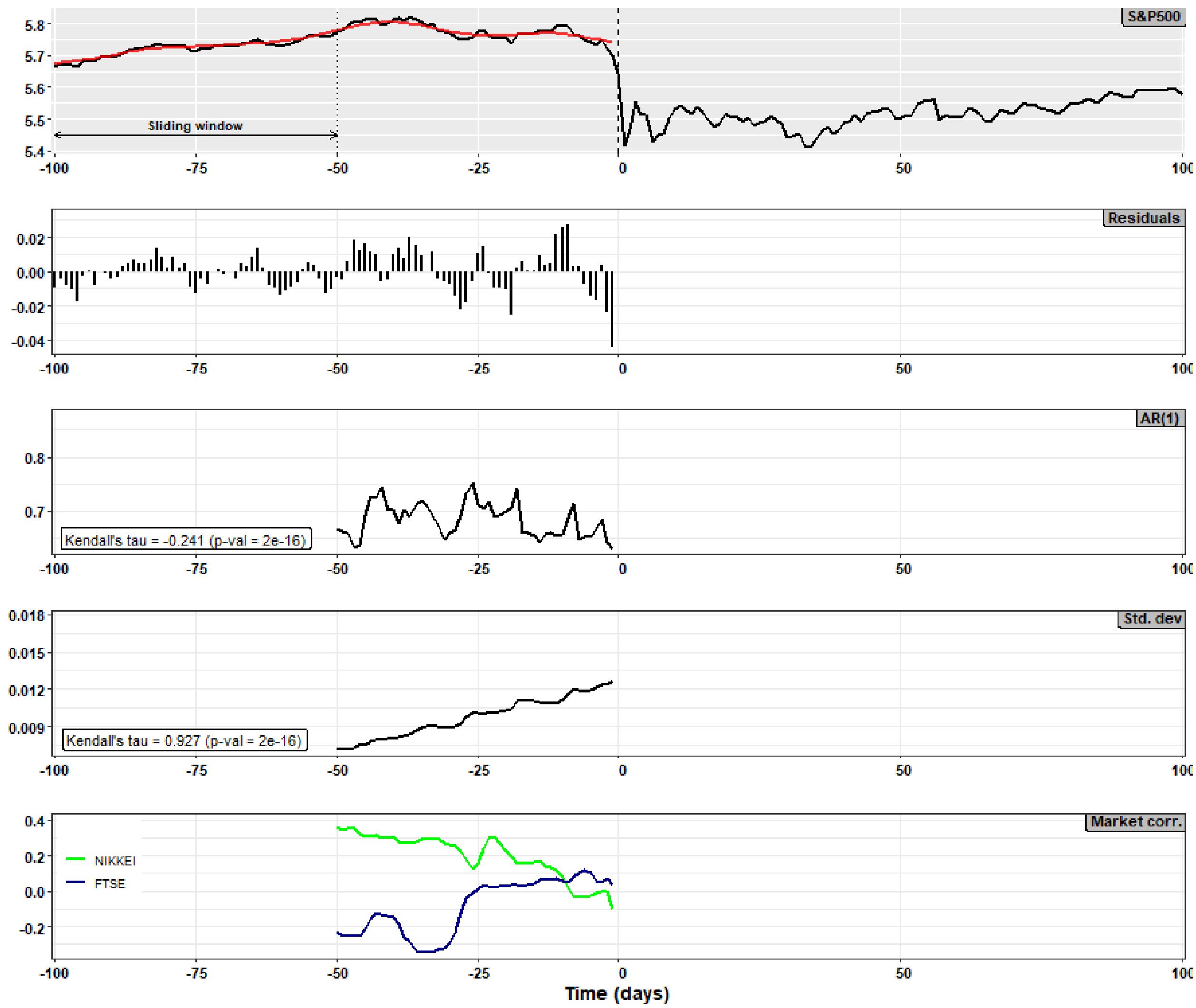

Figure 15: CSD: Black Monday 1987 (S&P500), sliding window = 50

5.2.2 Mexican crisis 1994

An investigation of the Mexican crisis around the initial months of the year

1994 suggests AR(1) of detrended fluctuations could be doubtfully exploited as an indicator of the nearing period of the relatively long downturn because serial correlation was descending over time, which is, beside other things, confirmed by significantly negative Kendall's τ.

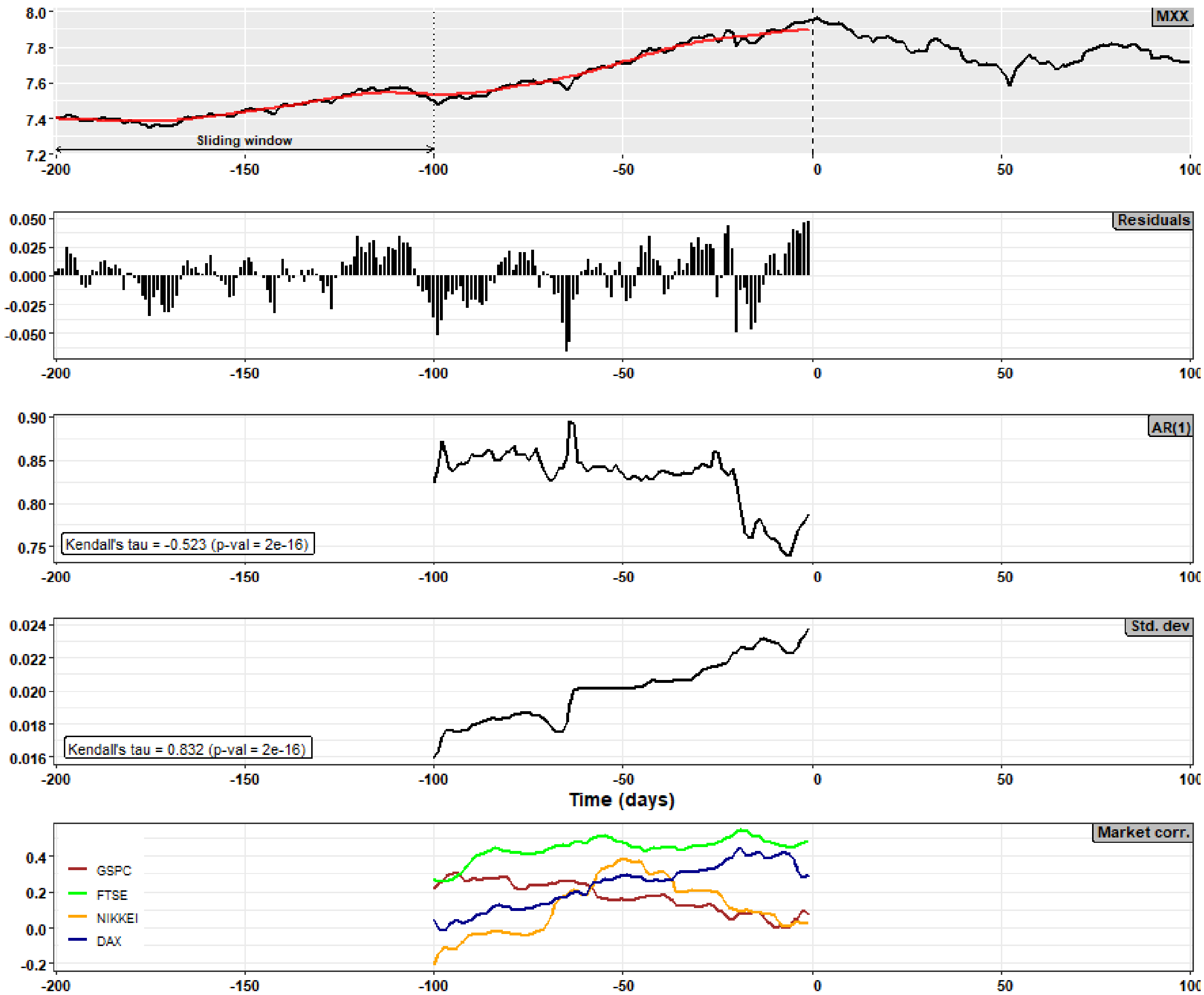

Figure 16: CSD: Mexican crisis 1994 (MMX), sliding window = 100

Fortunately, the development of the variance seems to be more helpful, and it is propped up by the promising value of Kendall's τ equalling to 0.832***. Here, one needs to avoid being misled by a spike in variance roughly 70 days prior to our defined beginning of a serious decline. The aforementioned bump in standard deviation is caused by some different market perturbations deflecting the index valuation for a few days. Eventually, correlation with other markets seems to be rather ambiguous so far hence this gauge might be considered as the least appropriate indicator.[14]

Switching to the more microscopic point of view brings one problem to our analysis arisen from many slumps already occurred prior to the day of a crash. Though we can claim as turbulence in the market amasses, variance goes upward (Kendall's τ = 0.678***). However, the serial correlation of order one of the residuals actually dropped before this more than 50-day decline. This gloomy fact is justifiable by preceding turmoils curbing AR(1). It is, nevertheless, necessary to point out the rise in variance for this microscopic view is not as obvious as for the observation with a 100-day sliding window.

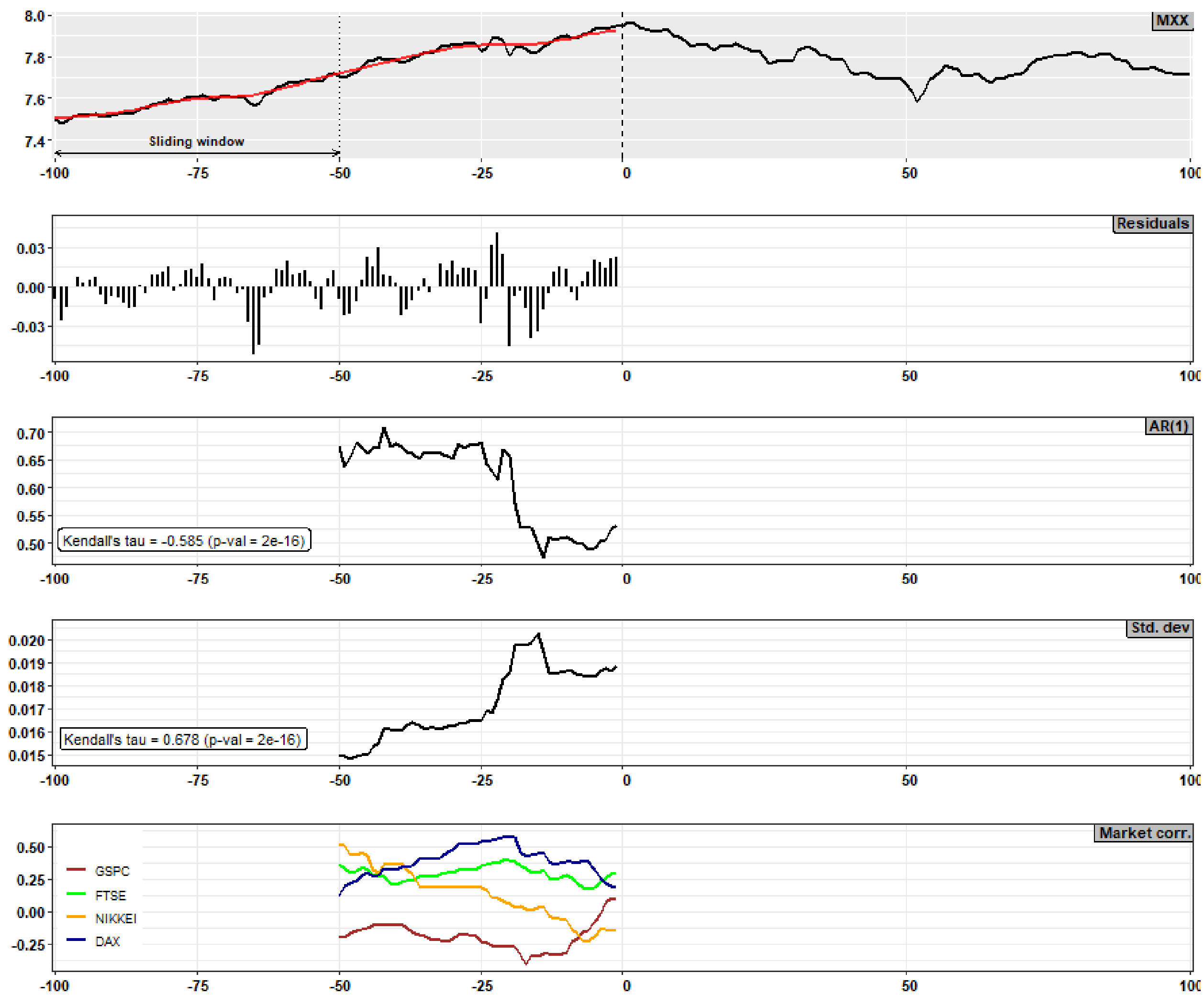

Figure 17: CSD: Mexican crisis 1994 (MMX), sliding window = 50

Finally, correlation with other markets is again very confusing, therefore, it will

[14] In this case, correlation with other markets is likely not to be such determinant because the crisis had no global reach.

not be scrutinized in the occasion of the Asian financial crisis. But this hypothesis will undergo another battery of test in the course of the dot-com bubble since in those times the level of globalization was significantly higher and some new patterns could have emerged.

5.2.3 Asian financial crisis 1997

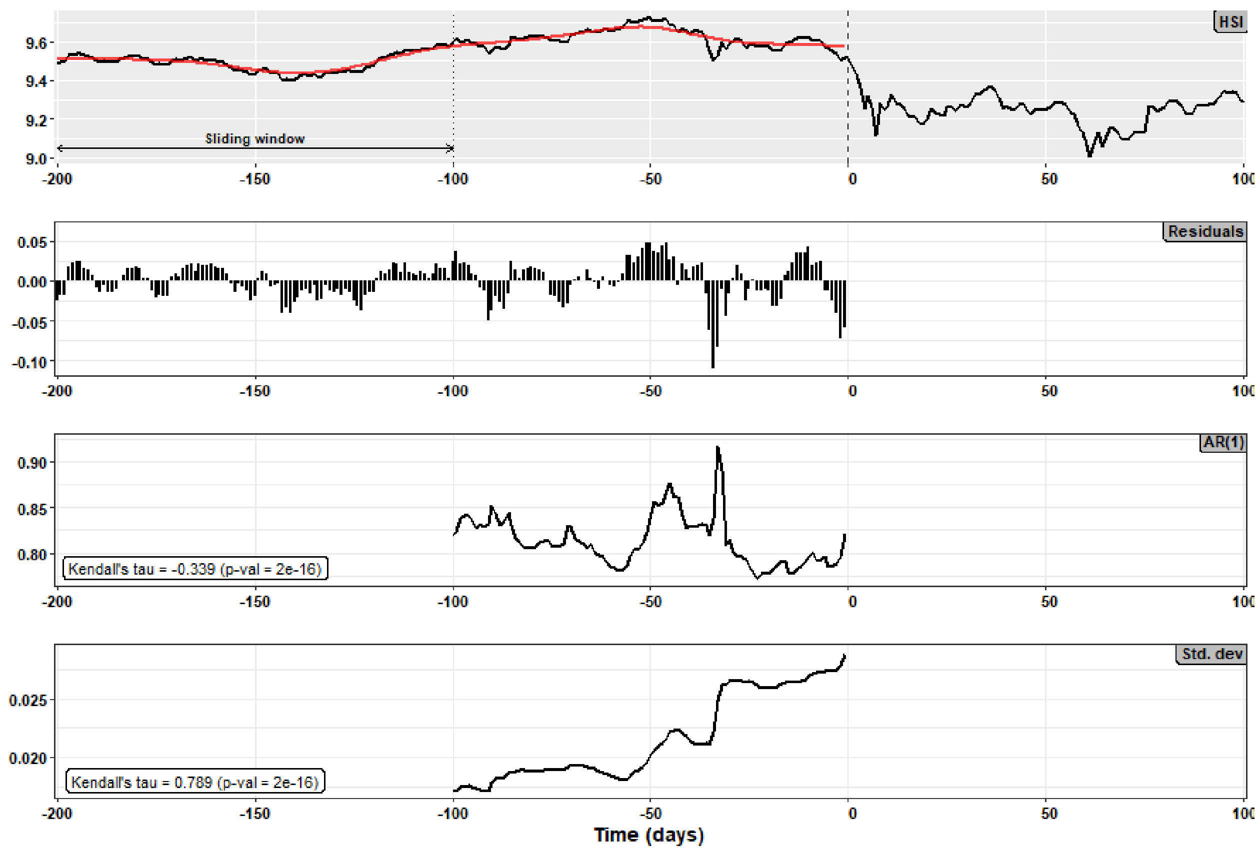

Figure 18: CSD: Asian crisis 1997 (Hang Seng), sliding window = 100

The Asian Market was quite fierce at the beginning of the second half of the year 1997, which is well confirmed by two surges in the volatility of detrended fluctuations. The standard deviation of residuals, interestingly, even after these two jumps, steadily grown until October 17, 1997. Subsequently, on Monday, the huge drop occurred. We, therefore, can claim, that this alarm was quite precise. The Kendall's τ equals 0.789, and this measurement is statistically significant at the 0.01 level.

In contrast, autocorrelation of order one of detrended fluctuations was climbing before the first remarkable drop roughly 35 trading days before the spark of the crisis but this development did not continue afterwards. This is also confirmed by Kendall's τ for serial correlation which is negative and it equals -0.339***.

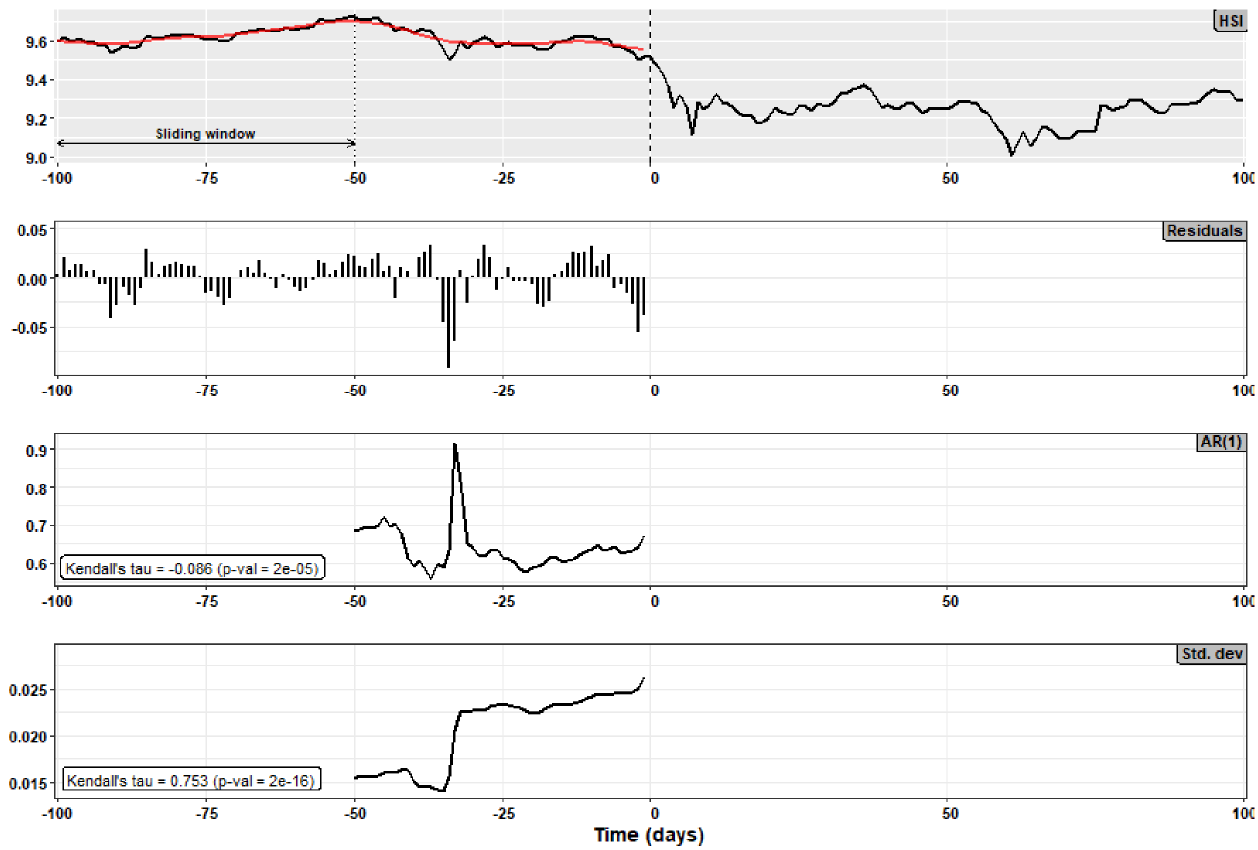

Figure 19: CSD: Asian crisis 1997 (Hang Seng), sliding window = 50

Figure 19 once more confirms the ascending standard deviation of residuals after the substantial hike approximately 30 trading days before the tumble with Kendall's τ equalling to 0.753***. AR(1) of remained fluctuations, unfortunately, is not again proved to be a generic indicator of an imminent bifurcation in the financial market since neither eye inspection nor Kendall's statistics, which is slightly below 0, is any convincing.

5.2.4 The dot-com bubble

Following the Figure 20, we must again be a little bit disappointed by the behaviour of AR(1) of residuals with a 100-day sliding window because it does not exhibit any persistent increase before the burst of the dot-com bubble peaking on Friday, October 3, 2000. Development of a serial correlation of detrended fluctuations is rather wild resulting in an insipid value of Kendall's τ slightly below zero.

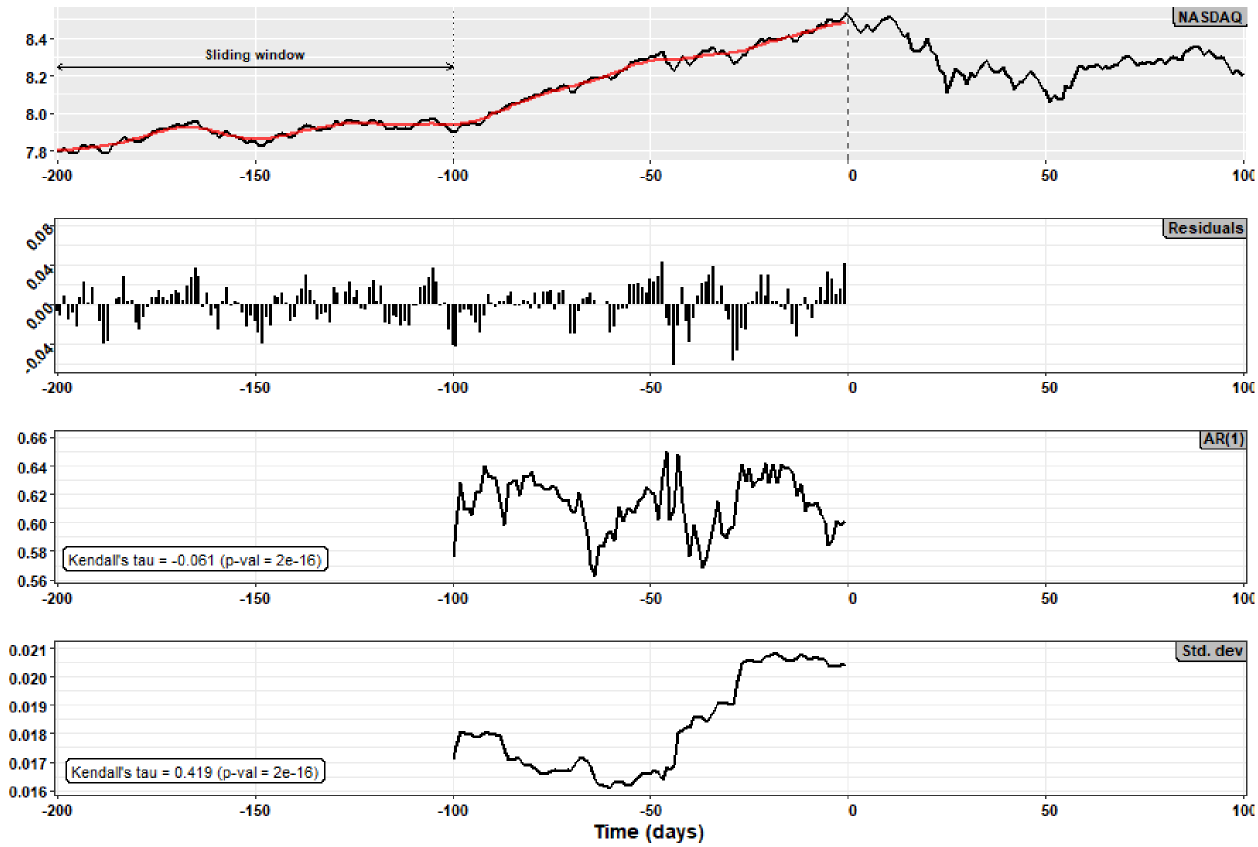

Figure 20: CSD: Burst of the dot-com bubble, sliding window = 100

On the other hand, an increasing variance of detrended fluctuations is detectable before the collapse. Apparently, the variance was rather decreasing in the period between 100 days and 60 days before the crash, nonetheless, the evolution drastically changed approximately 50 days to the burst of the bubble. Moreover, by restricting this measurement only on the last 60 days prior to a dawn of the drop, the value of τ rises to 0.731***. It has been again, therefore, shown that variance of residuals seems to be a much more propitious indicator of impending risk.

Similar feelings are received from an inspection of Figure 21 as in the case of examination of the previous plot. AR(1) of residuals does not exhibit any exactly prevailing patterns before the depression. There is a rather more arbitrary spike visible in the plot roughly 45 trading days prior to the crash that is well justifiable by nine deviations in a row with a positive sign that are distributed around 50 days before the plunge. The ambiguity of this statistic is emphasized by an insignificant

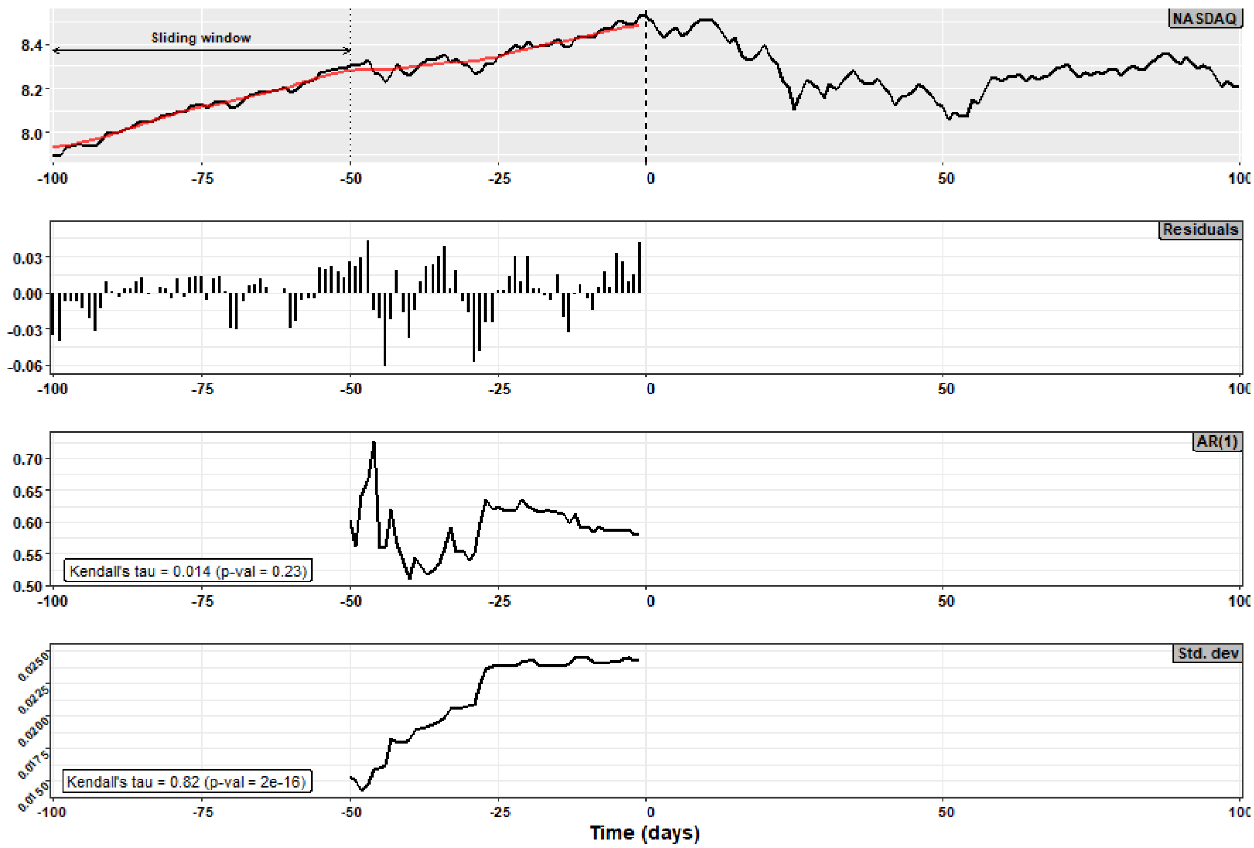

Figure 21: CSD: Burst of the dot-com bubble, sliding window = 50

measurement of Kendall's τ at even the 10% level. Generally, obeying 1-lag serial correlation of residuals is not somehow a reliable technique due to the presence of significant noise on financial markets prior to the collapse, probably sprung up in a consequence of emerging panic.

Variance in this shorter period more or less confirms the hypothesis, based on the truncated 60-day time frame obtained from the long window, thus a positive relationship between the standard deviation of detrended fluctuations and time is apparent. Despite a few spikes caused by two or three non-negligible drops in a phase between 50 and 25 days prior to the burst of the bubble, the variance of fluctuations does not subsequently decrease, and potentially indicates some looming danger.

As we promise in Subsection 5.2.2, the correlation with other markets is also investigated. For greater clarity, the measure is not plotted since five lines in one plot may look confusing. Instead, rather the brief table summarizing Kendall's τ of the correlation between NASDAQ and other five indices is provided. We further restrict only on a 100-day sliding window in order to curb the amount of information to a digestible level.

Corr(Nasdaq, X)	S&P500	FTSE	N225	DAX	MXX
Kendall's τ	.648***	.698***	.163***	.703***	.470***

Table 12: Kendall's τ of correlation between Nasdaq and other indices

On this occasion, a strengthening imitative behaviour across different markets can be proposed to be measurable before the hit since all the values of Kendall's τ for market correlation are quite high except for the correlation between the U.S. Nasdaq and the Japanese Nikkei 225. It is, therefore, not appropriate to reject this hypothesis of rising correlation among the markets prior to turmoils in the current setting of a high level of globalization so far. Hence the analysis of this kind will be also conducted for the world depression ignited in 2007. This investigation is to be omitted for the Bitcoin plummet as the world of cryptocurrency is a very intriguing phenomenon deserving of isolated research.[15]

5.2.5 Summary of CSD calibration

In the previous parts of this subsection, we are able to show that financial markets exhibit some generic patterns prior to the tipping points. We do not fully support the hypothesis that focusing on peer markets can serve as a useful indicator, however, so far we can confirm the rising standard deviation of detrended fluctuations as a reasonably good leading gauge. On the other hand, 1-lag autocorrelation of residuals provides us with rather more ambiguous marks as this statistic gives a clear answer only for the Black Monday with a 100-day time window. Conversely, an ascending trend in variance of remained fluctuations is somehow detectable in all cases but the Black Monday with a same-long window.

The crash in October 1987 can be used as a reference that examination of a shorter sliding window than one lasting 100 days may become worthy as this zooming in reveals a climbing variance of residuals. In Subsection 5.3, therefore, corresponding "zoom" will be provided.[16] The summary of the Kendall rank

[15]Furthermore, I have a lack of knowledge to be confident enough to pick suitable peer cryptocoins.

[16]We also report the results of AR(1), though, it is obvious we cannot fully rely on them as an indicator of looming risk.

correlation coefficient briefly describing indicator performances is presented in the table below.

Event	**Window**	$\tau_{\mathbf{AR(1)}}$	τ_{ρ}
Black Monday	100	.792***	.423***
Black Monday	50	-.241***	.927***
Mexican crisis	100	-.523***	.832***
Mexican crisis	50	-.585***	.678***
Asian crisis	100	-.339***	.789***
Asian crisis	50	-.086***	.753***
Dot-com bubble	100	-.061***	.419***
Dot-com bubble	50	.014	.820***

Table 13: CSD: Summary of Kendall's τ for 1-lag serial correlation and standard deviance
*** denotes the statistical significance of the relationship at the 1% level; Window denotes the length of a sliding window in days.

From Table 13, one can see all measurements except for one are highly statistically significant at the 1% level. Therefore, relying on this computation of p-value seems a little bit unreliable. Because of this reason, some statistical techniques such as bootstrap can be appropriate to be used in order to obtain more robust and trustworthy estimations of statistical significance but bringing another bunch of methodology would be necessary thus we rely only on this "naive" measure and focus mainly on the value of Kendall's τ itself.

5.3 Out-of-sample predictions

In this section, we try to predict crashes on two occasions. We do not study the discussed schemes only separately but we also consider them simultaneously and hence try to exploit advantages of both frameworks to make as the best prediction as possible. As a reminder, only the LPPL2, not the LPPL1, model is used.

5.3.1 Financial crisis of 2007-2008

To forecast the origin of decay of a global economy, fourteen predictions, which are summarized in Table 14, were made. The LPPL model provides us with an early warning for the end of the year 2006. The log-periodic signs, however, then disappeared, and none of the four following predictions fulfilled requirements of the LPPL model's parameters. Then, some predictions satisfying all conditions of

End date	t_c	m	ω	R^2	χ	$\tau_{AR(1)}$	τ_σ	$\tau_{AR(1)}$	τ_σ
2005.200	2006.968	0.243	8.130	.9281	.0197	-.207	-.414	-.313	.275
2005.452	NA	NA	NA	NA	NA	.084	.397	.073	.571
2005.704	NA	NA	NA	NA	NA	-.603	.056	-.787	-.659
2005.953	NA	NA	NA	NA	NA	-.759	-.766	-.413	-.373
2006.200	NA	NA	NA	NA	NA	-.118	-.010	.000	.799
2006.452	2006.819	0.591	10.538	.7936	.0330	.184	.187	-.413	.610
2006.704	2008.540	0.551	11.270	.8017	.0329	.179	.810	.407	-.657
2006.953*	2007.255	0.255	8.343	.8519	.0165	.019	-.819	.507	-.047
2007.200*	2007.969	0.356	8.609	.9205	.0156	.384	-.497	.420	.304
2007.370*	2007.934	0.137	10.925	.9484	.0144	-.500	.579	-.713	.080
2007.452*	2008.068	0.305	8.149	.9281	.0189	-.433	.609	-.160	-.768
2007.537*	2008.205	0.499	12.577	.9683	.0135	-.056	.626	.267	.180
2007.619*	2007.910	0.740	9.363	.9641	.0151	-.267	.639	-.107	.951
2007.701*	2007.767	0.787	8.499	.9493	.0183	-.546	.925	-.460	.847

Table 14: Forecasting of the beginning of the financial crisis 2007-2008
First column represents the date of a prediction, second to fifth columns are dedicated to the result of LPPL model, and the last for columns shows Kendall's tau of particular indicator for 100-day and 50-day sliding window respectively. The graph depicting AR(1) and standard deviation of residuals over time is attached in Appendix B.

the model emerged, however, the goodness-of-fit was quite poor thus we decided to shift the start of the time frame to obtain more suitable time-series data as it is explained in Section 4.

As soon as this adjustment was done, the more promising results began to spring up. First, predictions of the critical time t_c tended to move forward but the last two forecasts exhibited an opposite trend. The last prediction made on September 17, 2007, suggested the most likely time of the crash to be at 2007.767, which is equivalent to October 8, 2007. The DJIA peaked on October 9, 2007, afterwards, the index took a downward trend, therefore, this forecast can be pronounced to be extraordinarily successful.

If we look at the results based on the scheme of CSD, we can conclude 1-lag autocorrelation of detrended fluctuations is more or less meaningless. On the other hand, observing the variance of residuals seems to be again much more promising. A burst of variance during the second measurement (both with the longer and short sliding window) is noticeable, nonetheless, it provides us with the right clue because the DJIA topped two trading days later, afterwards the index was falling for six consecutive trading days and lost more than 3% (although this is not any breathtaking slump, the six-day decline is conspicuous for sure).

Another spike in variance is detectable in the period around predictions at 2006.452 and 2006.704 when, however, no significant events occurred thus it is appropriate these signs be considered as false alarms. Conversely, remarkable values of Kendall's τ of variance can be seen during the last five predictions, and therefore, a potential crash should have been reckoned based on the theory of CSD.

If we combine knowledge from both frameworks together and return to those times, we would have had to be very aware of the looming risk of potential collapse, and furthermore, we would have obtained a quite accurate estimation of a tipping point t_c, whose danger besides the predictions of the LPPL model is emphasized by a persistent rise in standard deviation of detrended fluctuations.

Even though we do not focus on the last 200 days and 100 days respectively as during analyses with the calibration sets, we think this analysis should be considered as sufficient because early warning signals, in the form of a rising variance of residuals, began to emerge quite in advance to the outbreak of the crisis.

Table 15 summarizes results of the Kendall rank correlation coefficient mapping the development of correlation of the DJIA with other indices. At the beginning, some flaws of this analysis need to be mentioned, namely the fact the sole value of Kendall's τ is not a perfect mirror of reality and a positive sign of this measurement does not always imply a positive difference in correlation measured at the end and at the beginning of a given period.

Although tests held during some last weeks prior to the crash are not listed in Table 15, the hypothesis of a rise of imitative behaviour across the world markets cannot be supported as the results do not provide any great insight.[17] In spite of the fact that the conclusion is drawn only from a brief survey, it should be sufficient for our purposes.

[17] Actually, later phases were also investigated, yet no trend emerged.

End date	S&P500	N225	FTSE	MXX	DAX
3/15/2005	-.687*	.846*	.734*	.617*	.797*
6/15/2005	.429*	-.621*	.422*	-.422*	.341*
9/15/2005	.669*	-.133	.474	-.212	-.734
12/15/2005	-.595*	.509*	.637*	.203	.310
3/15/2006	.531*	.362*	-.023*	-.106	-.274*
6/15/2006	-.266	.648*	.526*	.227*	.294*
9/15/2006	-.669*	.613*	.477*	.503*	.668
12/15/2006	-.264	-.774*	-.459	-.522*	-.681*
3/15/2007	-.267	.905*	.102*	.266	-.079
5/16/2007	-.490*	.785*	.506*	.716*	.653*
6/15/2007	-.114	-.717*	-.312*	-.746*	-.747
7/16/2007	-.544*	.390*	.182*	.872	.255*
8/15/2007	-.030*	-.198*	.532*	-.657	-.454*
9/17/2007	.629*	-.271*	.122*	-.189*	-.592

Table 15: Kendall's τ of correlation of DJIA with other indices

* denotes a remarkable change in correlation coefficient at beginning and end of an observed period

5.3.2 Burst of Bitcoin bubble

The results of sixteen predictions of the LPPL2 model made monthly between August 15, 2016, and November 15, 2016 are described in Table 16. First, if we look at values of R^2, it might seem the development of the Bitcoin price is pretty well described by our model, though this statistic can be possibly misleading because investigating the value of errors provides us with a quite different insight as RMSE values are the very highest among other predictions conducted in the thesis. This flaw is partially determined by higher volatility, and further caused by the presence of some short episodes of time of rocketing prices followed by even steeper tumbles, which simply cannot be fitted by the model of ours. Such fierce reversals, moreover, provoke the non-linear parameter to be kind of inconsistent as they significantly vary across particular predictions.

End date	t_c	m	ω	R^2	χ	$\tau_{\mathrm{AR(1)}}$	τ_σ	$\tau_{\mathrm{AR(1)}}$	τ_σ
2016.620	2017.088	0.569	8.887	.9178	.0916	-.651	.918	-.673	-.604
2016.705	2017.155	0.312	12.248	.9335	.0833	-.500	.685	.773	-.605
2016.787	2017.313	0.569	12.326	.9298	.0865	-.600	-.276	.487	-.607
2016.872	2017.276	0.495	12.155	.9120	.0997	.264	-.819	-.660	.249
2016.954	2017.979	0.640	7.000	.9001	.1105	.226	-.763	-.293	.698
2017.038	2017.941	0.766	6.881	.9144	.1096	.409	-.082	-.100	.489
2017.123	2018.089	0.678	7.015	.9281	.1067	.254	.711	-.207	.878
2017.200	2018.093	0.716	6.273	.9418	.1038	.615	.915	.347	-.453
2017.285	2018.058	0.676	7.200	.9432	.1077	-.269	.888	-.160	.600
2017.367	2018.173	0.639	6.108	.9515	.1077	.371	.286	-.047	-.709
2017.452	2017.769	0.302	8.930	.9469	.1306	-.189	-.064	-.180	.913
2017.534	2017.691	0.484	7.249	.9538	.1342	-.466	.224	-.480	.531
2017.619	2018.102	0.358	6.926	.9565	.1435	.060	.672	-.053	.412
2017.704	2018.181	0.289	7.053	.9602	.1525	.131	.580	-.260	-.164
2017.786	2017.926	0.202	11.863	.9678	.1479	-.471	.539	-.420	-.311
2017.871	2018.078	0.523	6.158	.9687	.1590	.448	-.197	.587	-.002

Table 16: Forecasting of the burst of the bitcoin bubble in 2017

First column represents the date of a prediction, second to fifth columns are dedicated to the result of LPPL model, and the last for columns shows Kendall's tau of particular indicator for 100-day and 50-day sliding window respectively. The graph depicting AR(1) and standard deviation of residuals over time is attached in Appendix B.

Elaborating on a discrepancy between the goodness-of-fit and RMSE can serve as a suggestion the LPPL model is capable to cope with the long-term trend materialized by gradually faster and faster, skyrocketing prices, but the real development of the Bitcoin price is not probably decorated by log-periodic oscillations. For this purpose, we can study the very last two fits visually in Figure 22 confirming our hypothesis.[18]

Our hypothesis that an actual fit is not as good as it seems according to the R^2 is confirmed by simple eye inspection. Here, the later prediction looks to be a little bit more plausible approximation of the Bitcoin price prior to the burst, however, estimation of the critical time t_c is delayed as the turmoil began on December 17, 2017, which is equivalent to 2017.959. In this perspective, the preceding prediction is better, nevertheless, the fit does not look optimal hence it

[18]We decided to depict the last two predictions made by the LPPL model since they significantly differ in non-linear parameters

is quite unsure whether relying on this method is advisable in the case of Bitcoin.

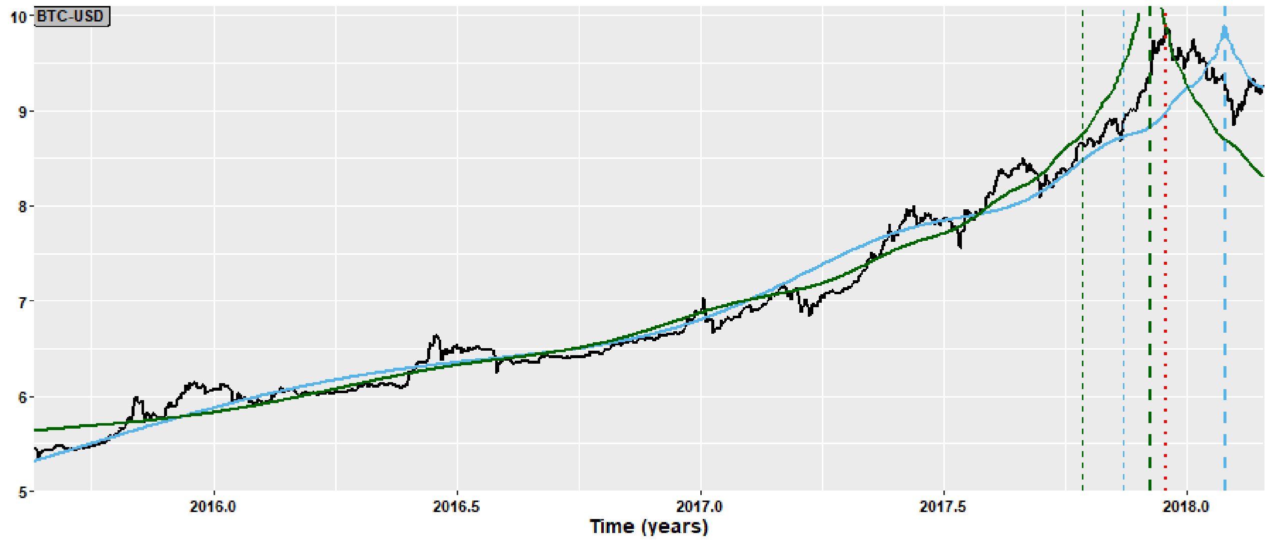

Figure 22: LPPL2: Burst of the Bitcoin bubble

The red dotted line depicts the time of the crash; Green components are assigned to the penultimate prediction, and the blue ones show the result of the very last forecast.

If we focus on the results of the CSD, we can see that an evolution of AR(1) and standard deviation in time is often contradictory. Looking at the turn of years 2016 and 2017 and assuming a 100-day sliding window, a slight increase in 1-lag serial correlation according to Kendall's τ is detected thus correctly notifying of a subsequent drop, however, serial correlation of order one of detrended fluctuations does not look powerful in general.

Conversely, this tumble, which is not of any extraordinary significance in the realm of Bitcoin, was not captured by the variance of detrended fluctuations, which was decreasing prior to this turbulence. Unfortunately, comparing these results with those ones received from the analysis with a 50-day sliding window, completely contradicting results are here provided, thereby making this inference quite doubtful.

Approaching the burst of this bubble at the end of 2017, in case of a 100-day sliding window, the gradual rise in the standard deviation of residuals in the last five consecutive periods but the last one is obtained hence proposing a relatively significant warning signal. The results with a 50-day sliding window are less convincing. We can again state that 1-lag serial correlation of detrended fluctuations does not perform well in this analysis, therefore, it should be considered as an inappropriate indicator of an imminent crash in the analysis of this kind using the similar strategy of obtaining residuals.

Again, even though we do not focus on the last 200 days preceding the peak, we feel this addition is not necessary since early warning signals began to emerge quite in advance in case of the outbreak of the last economic crisis in 2007-2008, hence this analysis should be sufficient, and conclusion that the hypothesis stating the rising variance of residuals is plausible, but that there is no generic pattern in a development of serial correlation of detrended fluctuations over time, is valid.

6 Conclusion

The purpose of the thesis was to individually study two concepts, and subsequently to compare them, which are argued to be suitable frameworks for predicting crashes in financial markets, namely the LPPL model and the CSD. Both of them were calibrated on four historical crises, and then tested for their accuracy, predictability and proneness to false positives. Moreover, the possibilities of exploiting them simultaneously were also discussed.

After this study, we can support, regarding the LPPL model, the modified version of the LPPL equation proposed by Filimonov and Sornette (2013) is capable to detect log-periodic behaviour index prices signalizing thriving bubbles. Importantly, it is significantly computationally much more efficient in comparison with a sort of obsolete model represented by the LPPL1 model, and furthermore, slightly higher accuracy was also reached on the in-sample set.

Needless to say, stochastic nature of an occurrence of crashes must be always borne in mind, thus the construction of some intervals for a possible critical time t_c is suitable. Besides, relying on a single prediction in time is not appropriate, and repeating fitting the discussed model over time is highly recommended not to catch in a trap of a false alarm. Checking the fits for the values of the non-linear parameters is also necessary since the peculiar estimations suggest the violation of an economic theory and thus such fits need to be always dropped.

In general, the world of the LPPL models is quite broad and covering all the related upgrades is unattainable within the scope of a bachelor thesis. We, therefore, focused on the two relatively straightforward models corresponding to the level of a whole work, and from my perspective, studying this intriguing concept in other stages of my study is worthy.

On the other side, the CSD scheme showed not to be so promising even though it has been proved to be well applicable for detecting critical transitions in systems of various kinds regardless of their dissimilarities. Conceivably, the poor performance may have been caused by the way of obtaining detrended fluctuations, nevertheless, as it was discussed, filtering as noise data as stock returns represents the question itself.

Specifically, the 1-lag serial correlation of residuals seemed useless as an indicator of an imminent collapse. Conversely, investigating the progress of variance of detrended fluctuations provided us with promising results, however, as it can be seen in Section 5.3, the standard deviation began to rise quite far in advance the drop hence this theory should be used rather only as an barometer of a looming risk and not as a determinant of the critical time.

Our additional hypothesis stating the correlation across the world markets prior to a price slump came out to be flawed as this phenomenon has been detected only for the dot-com bubble burst. For this purpose, scrutinizing this measure across individual stocks representing particular industries on the same market is proposed as another idea for a future research.

Combining these two frameworks together brought a kind of interesting observation that although these concepts are based on contradictory assumptions, they are kind of exploitable together. In this manner, we showed on the out-of-sample data a relevance of the estimated critical time t_c can be well emphasized by rising variance of residuals and hence making this blended scheme a little bit more robust than they are separately.

To conclude this, further study of advances of the LPPL model is likely to be interesting as a lot of new formulas and corresponding tests have emerged. Finally, complementing already comprehensive framework of the fitting procedure by a careful explanation of non-linear optimization methods packed in a single piece of work further facilitating to take up study this area of finance, together with suggesting an investigation of the variance of detrended residuals brought in from a different concept can be considered as major contributions to an already existing research.

Acronyms

ADF	Augmented Dickey-Fuller test
GA	Genetic algorithm
GD	Gradient/Steepest descent algorithm
GNA	Gauss-Newton algorithm
KPSS	Kwiatkowsi-Philipps-Schmidt-Shin test
LHS	Left-hand side
LMA	Levenberg-Marquardt algorithm
LPPL	Log-periodic power law
MLE	Maximum likelihood estimator
OLS	Ordinary least squares
PP	Phillips-Perron test
RHS	Right-hand side
SA	Simulated annealing
TS	Tabu search

List of Tables

List of Figures

References

H. Abdi. The kendall rank correlation coefficient. *Encyclopedia of Measurement and Statistics. Sage, Thousand Oaks, CA*, pages 508–510, 2007.

L. Bachelier. Théorie de la spéculation. In *Annales scientifiques de l'École normale supérieure*, volume 17, pages 21–86, 1900.

R. Bartoszynski and M. Niewiadomska-Bugaj. *Probability and statistical inference.* John Wiley & Sons, 2007.

H.-C. G. v. Bothmer and C. Meister. Predicting critical crashes? a new restriction for the free variables. *Physica A: Statistical Mechanics and its Applications*, 320: 539–547, 2003.

D. S. Bree and N. L. Joseph. Fitting the log periodic power law to financial crashes: a critical analysis. *arXiv preprint arXiv:1002.1010*, 2010.

M. Bussiere and M. Fratzscher. Towards a new early warning system of financial crises. *Journal of International Money and Finance*, 25(6):953–973, 2006.

S. Camazine, J.-L. Deneubourg, N. R. Franks, J. Sneyd, E. Bonabeau, and G. Theraula. *Self-organization in biological systems.* Princeton university press, 2003.

J. Y. Campbell, J. W. Campbell, A. W.-C. Lo, A. C. MacKinlay, J. J. Champbell, A. A. LO, A. W. Lo, et al. *The econometrics of financial markets.* Princeton University press, 1997.

S. R. Carpenter and W. A. Brock. Rising variance: a leading indicator of ecological transition. *Ecology letters*, 9(3):311–318, 2006.

A. Cauchy. Méthode générale pour la résolution des systemes d´équations simultanées. *Comp. Rend. Sci. Paris*, 25:536–538, 1847.

S. G. Cecchetti, M. Kohler, and C. Upper. Financial crises and economic activity. Technical report, National Bureau of Economic Research, 2009.

S.-N. Chow and J. K. Hale. *Methods of bifurcation theory*, volume 251. Springer Science & Business Media, 2012.

D. Cvijović and J. Klinowski. Taboo search: an approach to the multiple minima problem. *Science*, 267(5198):664–666, 1995.

L. Dai, D. Vorselen, K. S. Korolev, and J. Gore. Generic indicators for loss of resilience before a tipping point leading to population collapse. *Science*, 336 (6085):1175–1177, 2012.

V. Dakos, M. Scheffer, E. H. van Nes, V. Brovkin, V. Petoukhov, and H. Held. Slowing down as an early warning signal for abrupt climate change. *Proceedings of the National Academy of Sciences*, 105(38):14308–14312, 2008.

V. Dakos, E. H. Van Nes, P. D'Odorico, and M. Scheffer. Robustness of variance and autocorrelation as indicators of critical slowing down. *Ecology*, 93(2):264–271, 2012.

C. Diks, C. Hommes, and J. Wang. Critical slowing down as an early warning signal for financial crises? *Empirical Economics*, pages 1–28, 2015.

E. F. Fama. Efficient capital markets: A review of theory and empirical work. *The journal of Finance*, 25(2):383–417, 1970.

V. Filimonov and D. Sornette. A stable and robust calibration scheme of the log-periodic power law model. *Physica A: Statistical Mechanics and its Applications*, 392(17):3698–3707, 2013.

V. Filimonov, G. Demos, and D. Sornette. Modified profile likelihood inference and interval forecast of the burst of financial bubbles. *Quantitative finance*, 17 (8):1167–1186, 2017.

D. S. Fisher. Scaling and critical slowing down in random-field ising systems. *Physical review letters*, 56(5):416, 1986.

C. F. Gauss. *Theoria motus corporum coelestium in sectionibus conicis solem ambientium*, volume 7. Perthes et Besser, 1809.

F. Glover. Tabu search: A tutorial. *Interfaces*, 20(4):74–94, 1990.

V. Guttal, S. Raghavendra, N. Goel, and Q. Hoarau. Lack of critical slowing down suggests that financial meltdowns are not critical transitions, yet rising variability could signal systemic risk. *PloS one*, 11(1):e0144198, 2016.

O. Hájková, O. John, M. Johanis, O. Kalenda, and M. Zelený. *Matematika*. Matfyzpress, 2012.

A. Hertz and D. de Werra. Using tabu search techniques for graph coloring. *Computing*, 39(4):345–351, 1987.

J. H. Holland. Genetic algorithms. *Scientific american*, 267(1):66–73, 1992.

H. Hong and J. C. Stein. Differences of opinion, short-sales constraints, and market crashes. *The Review of Financial Studies*, 16(2):487–525, 2003.

Y. Huang, A. Johansen, M. Lee, H. Saleur, and D. Sornette. Artifactual log-periodicity in finite size data: Relevance for earthquake aftershocks. *Journal of Geophysical Research: Solid Earth*, 105(B11):25451–25471, 2000.

Z.-Q. Jiang, W.-X. Zhou, D. Sornette, R. Woodard, K. Bastiaensen, and P. Cauwels. Bubble diagnosis and prediction of the 2005–2007 and 2008–2009 chinese stock market bubbles. *Journal of economic behavior & organization*, 74 (3):149–162, 2010.

A. Johansen. Characterization of large price variations in financial markets. *Physica A: Statistical Mechanics and its Applications*, 324(1-2):157–166, 2003.

A. Johansen and D. Sornette. The nasdaq crash of april 2000: Yet another example of log-periodicity in a speculative bubble ending in a crash. *The European Physical Journal B-Condensed Matter and Complex Systems*, 17(2):319–328, 2000.

A. Johansen and D. Sornette. Log-periodic power law bubbles in latin-american and asian markets and correlated anti-bubbles in western stock markets: An empirical study. *International Journal of Theoretical and Applied Finace*, 4(6): 853–920, 2001a.

A. Johansen and D. Sornette. Finite-time singularity in the dynamics of the world population, economic and financial indices. *Physica A: Statistical Mechanics and its Applications*, 294(3-4):465–502, 2001b.

A. Johansen and D. Sornette. Large stock market price drawdowns are outliers. *Journal of Risk*, 4:69–110, 2002.

A. Johansen, D. Sornette, and O. Ledoit. Predicting financial crashes using discrete scale invariance. *Journal of Risk*, 1(4):5–32, 1999.

A. Johansen, O. Ledoit, and D. Sornette. Crashes as critical points. *International Journal of Theoretical and Applied Finance*, 3(02):219–255, 2000.

S. Kéfi, M. Rietkerk, C. L. Alados, Y. Pueyo, V. P. Papanastasis, A. ElAich, and P. C. De Ruiter. Spatial vegetation patterns and imminent desertification in mediterranean arid ecosystems. *Nature*, 449(7159):213, 2007.

C. T. Kelley. *Iterative methods for optimization*. SIAM, 1999.

M. G. Kendall. *Rank correlation methods*. Hafner Publishing Co, 1955.

S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi. Optimization by simulated annealing. *science*, 220(4598):671–680, 1983.

Y. A. Kuznetsov. *Elements of applied bifurcation theory*. Springer Science & Business Media, 2013.

L. Laloux, M. Potters, R. Cont, J.-P. Aguilar, and J.-P. Bouchaud. Are financial crashes predictable? *EPL (Europhysics Letters)*, 45(1):1, 1999.

T. M. Lenton, R. J. Myerscough, R. Marsh, V. N. Livina, A. R. Price, S. J. Cox, and G. team. Using genie to study a tipping point in the climate system. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 367(1890):871–884, 2008.

K. Levenberg. A method for the solution of certain non-linear problems in least squares. *Quarterly of applied mathematics*, 2(2):164–168, 1944.

V. Liberatore. Computational lppl fit to financial bubbles. *arXiv preprint arXiv:1003.2920*, 2010.

T. M. Liggett et al. Stochastic models of interacting systems. *The Annals of Probability*, 25(1):1–29, 1997.

L. Lin, R. E. Ren, and D. Sornette. The volatility-confined lppl model: A consistent model of 'explosive'financial bubbles with mean-reverting residuals. *International Review of Financial Analysis*, 33:210–225, 2014.

M. I. Lourakis et al. A brief description of the levenberg-marquardt algorithm implemented by levmar. *Foundation of Research and Technology*, 4(1):1–6, 2005.

D. W. Marquardt. An algorithm for least-squares estimation of nonlinear parameters. *Journal of the society for Industrial and Applied Mathematics*, 11(2): 431–441, 1963.

J. J. Moré. The levenberg-marquardt algorithm: implementation and theory. In *Numerical analysis*, pages 105–116. Springer, 1978.

P. Moscato. An introduction to population approaches for optimization and hierarchical objective functions: A discussion on the role of tabu search. *Annals of Operations Research*, 41(2):85–121, 1993.

J. Nocedal and S. Wright. *Numerical optimization*. Springer Science & Business Media, 2006.

C. Patterson. Age of meteorites and the earth. *Geochimica et Cosmochimica Acta*, 10(4):230–237, 1956.

D. T. Pele et al. An lppl algorithm for estimating the critical time of a stock market bubble. *Journal of Social and Economic Statistics*, 1(2):14–22, 2012.

R. Ren, D. Sornette, et al. A consistent model of'explosive'financial bubbles with mean-reversing residuals. *CCSS Working Paper*, 2009.

M. Rietkerk, S. C. Dekker, P. C. De Ruiter, and J. van de Koppel. Self-organized patchiness and catastrophic shifts in ecosystems. *Science*, 305(5692):1926–1929, 2004.

J. Sachs, A. Tornell, and A. Velasco. The mexican peso crisis: Sudden death or death foretold? *Journal of international economics*, 41(3-4):265–283, 1996.

H. Saleur, C. G. Sammis, and D. Sornette. Discrete scale invariance, complex fractal dimensions, and log-periodic fluctuations in seismicity. *Journal of Geophysical Research: Solid Earth*, 101(B8):17661–17677, 1996.

S. Salinas. *Introduction to statistical physics*. Springer Science & Business Media, 2001.

M. Scheffer, J. Bascompte, W. A. Brock, V. Brovkin, S. R. Carpenter, V. Dakos, H. Held, E. H. Van Nes, M. Rietkerk, and G. Sugihara. Early-warning signals for critical transitions. *Nature*, 461(7260):53–59, 2009.

J. Scholz, J. Kelso, and G. Schöner. Nonequilibrium phase transitions in coordinated biological motion: critical slowing down and switching time. *Physics Letters A*, 123(8):390–394, 1987.

M. Schroeder. *Fractals, chaos, power laws: Minutes from an infinite paradise*. Courier Corporation, 2009.

J. Skorin-Kapov. Tabu search applied to the quadratic assignment problem. *ORSA Journal on computing*, 2(1):33–45, 1990.

D. Sornette. *Why stock markets crash: critical events in complex financial systems*. Princeton University Press, 2003.

D. Sornette. *Critical phenomena in natural sciences: chaos, fractals, selforganization and disorder: concepts and tools*. Springer Science & Business Media, 2006.

D. Sornette and A. Johansen. Large financial crashes. *Physica A: Statistical Mechanics and its Applications*, 245(3-4):411–422, 1997.

D. Sornette, A. Johansen, and J.-P. Bouchaud. Stock market crashes, precursors and replicas. *Journal de Physique I*, 6(1):167–175, 1996.

D. Sornette, R. Woodard, W. Yan, and W.-X. Zhou. Clarifications to questions and criticisms on the johansen–ledoit–sornette financial bubble model. *Physica A: Statistical Mechanics and its Applications*, 392(19):4417–4428, 2013.

Y. Teh and G. P. Rangaiah. Tabu search for global optimization of continuous functions with application to phase equilibrium calculations. *Computers & chemical engineering*, 27(11):1665–1679, 2003.

P. J. Van Laarhoven and E. H. Aarts. Simulated annealing. In *Simulated annealing: Theory and applications*, pages 7–15. Springer, 1987.

E. H. Van Nes and M. Scheffer. Slow recovery from perturbations as a generic indicator of a nearby catastrophic shift. *The American Naturalist*, 169(6):738–747, 2007.

C. Wissel. A universal law of the characteristic return time near thresholds. *Oecologia*, 65(1):101–107, 1984.

H. Yu and B. M. Wilamowski. Levenberg-marquardt training. *Industrial electronics handbook*, 5(12):1, 2011.

R. T. Zheng, N. Ngo, P. Shum, S. Tjin, and L. Binh. A staged continuous tabu search algorithm for the global optimization and its applications to the design of fiber bragg gratings. *Computational Optimization and Applications*, 30(3): 319–335, 2005.

W.-X. Zhou, D. Sornette, R. A. Hill, and R. I. Dunbar. Discrete hierarchical organization of social group sizes. *Proceedings of the Royal Society B: Biological Sciences*, 272(1561):439–444, 2005.

Appendix A - Critical slowing down - Unit root tests

Calibration set

Event	Window	ADF	KPSS	PP
Black Monday	100	0.03	0.10	0.01
Black Monday	50	0.02	0.10	0.01
Mexican crisis	100	0.01	0.10	0.01
Mexican crisis	50	0.01	0.10	0.01
Asian crisis	100	0.01	0.10	0.01
Asian crisis	50	0.01	0.10	0.01
Dot-com bubble	100	0.01	0.10	0.01
Dot-com bubble	50	0.01	0.10	0.01

Table: Results of unit root tests

Financial crisis of 2007-2008

End date	ADF	KPSS	PP	ADF	KPSS	PP
2005.200	0.01	0.10	0.01	0.03	0.10	0.01
2005.452	0.01	0.10	0.01	0.01	0.10	0.01
2005.704	0.01	0.10	0.01	0.01	0.10	0.01
2005.953	0.01	0.10	0.01	0.01	0.10	0.01
2006.200	0.01	0.10	0.01	0.01	0.10	0.01
2006.452	0.01	0.10	0.01	0.01	0.10	0.01
2006.704	0.01	0.10	0.01	0.01	0.10	0.01
2006.953	0.01	0.10	0.01	0.01	0.10	0.01
2007.200	0.01	0.10	0.01	0.01	0.10	0.01
2007.370	0.01	0.10	0.01	0.01	0.10	0.01
2007.452	0.01	0.10	0.01	0.01	0.10	0.01
2007.537	0.01	0.10	0.01	0.01	0.10	0.01
2007.619	0.01	0.10	0.01	0.01	0.10	0.01
2007.701	0.01	0.10	0.01	0.01	0.10	0.01

Results of unit root tests; The left half of the table is dedicated to the analysis with a 100-day sliding window, the rest of the table then corresponds to the test conducted on residuals from a 50-day sliding window.

Burst of bitcoin bubble

End date	ADF	KPSS	PP	ADF	KPSS	PP
2016.620	0.01	0.10	0.01	0.01	0.10	0.01
2016.705	0.01	0.10	0.01	0.01	0.10	0.01
2016.787	0.01	0.10	0.01	0.01	0.10	0.01
2016.872	0.01	0.10	0.01	0.01	0.10	0.01
2016.954	0.01	0.10	0.01	0.01	0.10	0.01
2017.038	0.01	0.10	0.01	0.38	0.10	0.01
2017.123	0.01	0.10	0.01	0.08	0.10	0.01
2017.200	0.01	0.10	0.01	0.04	0.10	0.01
2017.285	0.01	0.10	0.01	0.01	0.10	0.01
2017.367	0.01	0.10	0.01	0.02	0.10	0.01
2017.452	0.01	0.10	0.01	0.01	0.10	0.01
2017.534	0.01	0.10	0.01	0.04	0.10	0.01
2017.619	0.01	0.10	0.01	0.01	0.10	0.01
2017.704	0.01	0.10	0.01	0.01	0.10	0.01
2017.786	0.01	0.10	0.01	0.01	0.10	0.01
2017.871	0.01	0.10	0.01	0.01	0.10	0.01

Results of unit root tests; The left half of the table is dedicated to the analysis with a 100-day sliding window, the rest of the table then corresponds to the test conducted on residuals from a 50-day sliding window.

Appendix B - Critical slowing down - Out-of-sample predictions

Financial crisis of 2007-2008

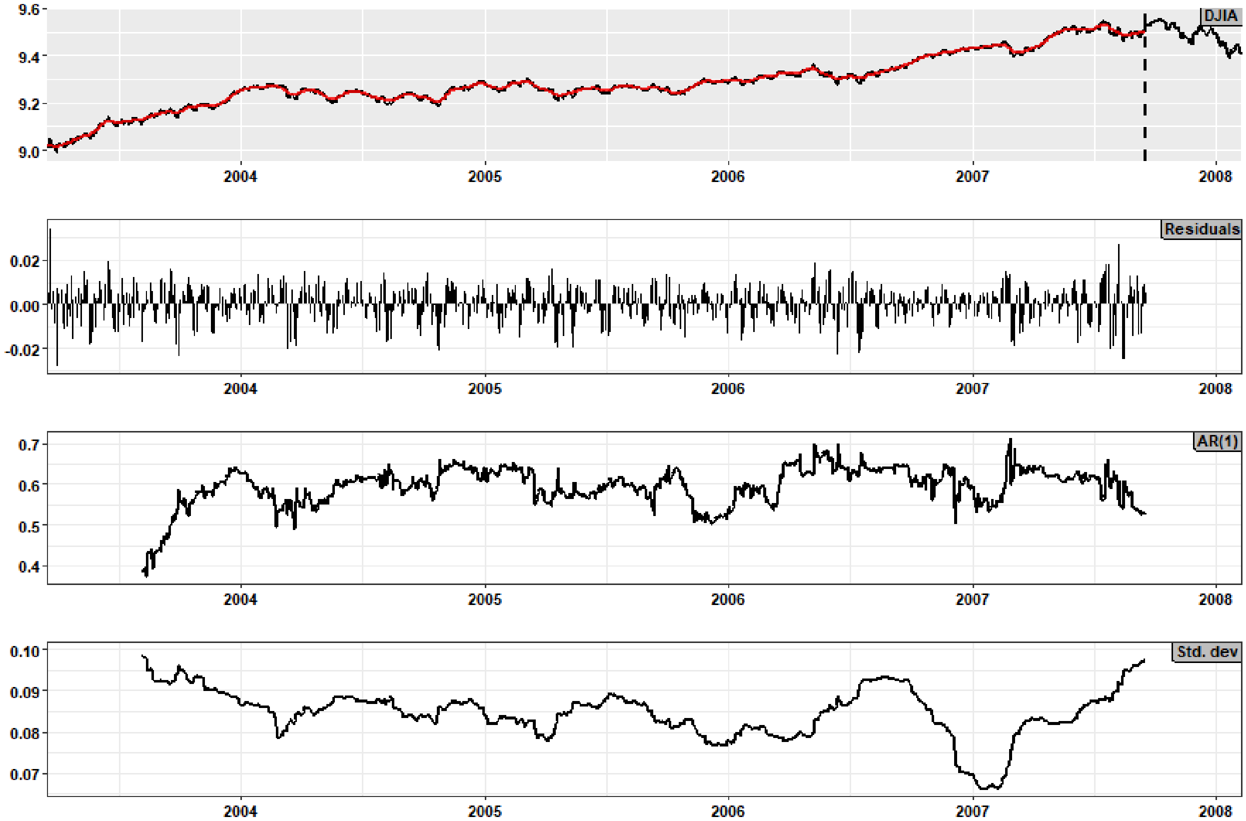

Figure 23: CSD: Financial crisis of 2007-2008, sliding window = 100

Burst of Bitcoin bubble

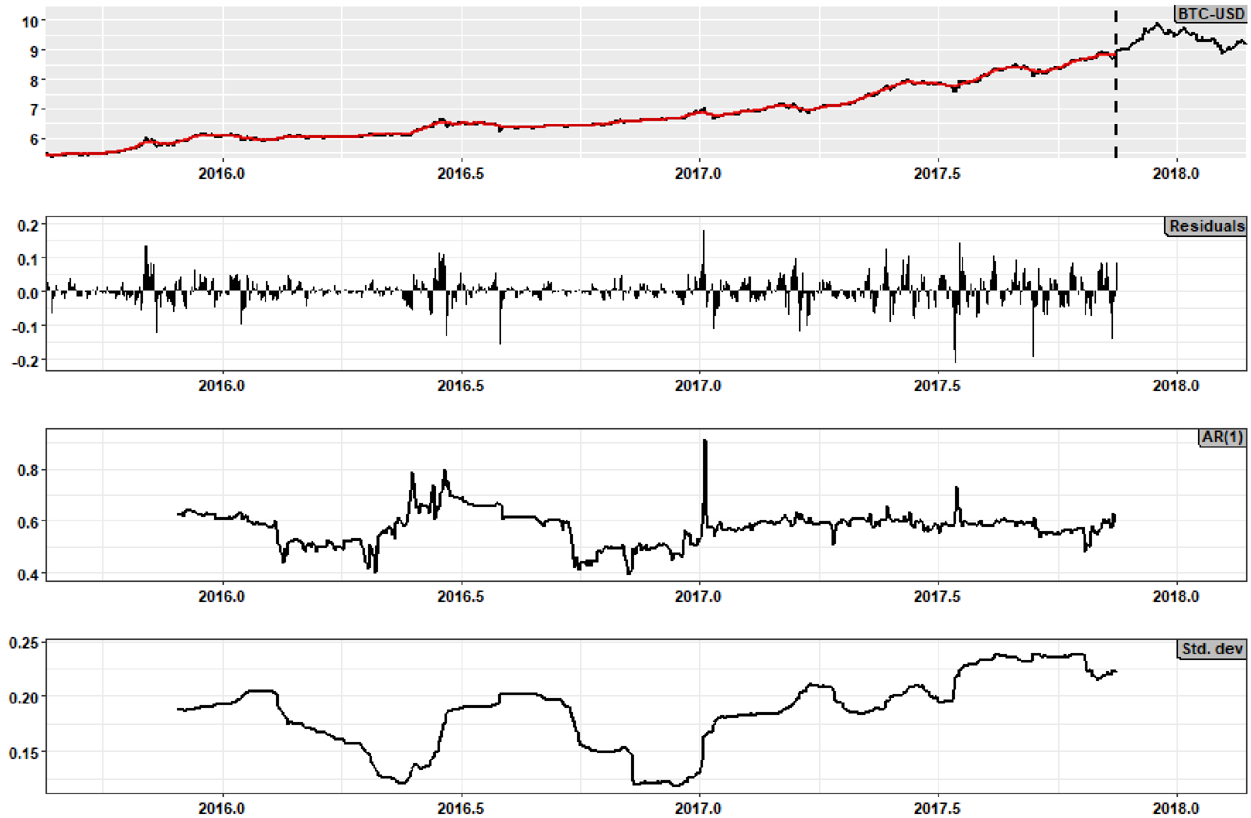

Figure 24: CSD: Burst of Bitcoin bubble, sliding window = 100

Appendix C - Tabu Search

```
# Loss Function
L = function(A, B, C, t_c, m, omega, phi, price, t){
  loss <- sum((price - (
    A + B*abs(t_c - t)^m + (
      (C*abs(t_c-t)^m)*cos(omega * log(abs(t_c - t)) - phi))))^2)
}


# Tabu Search
TabuSearch <- function(A, B, C){
  S <- hash() # S represents current solution
  iter_without_improvement <- 0

  # Generating a set of initial 10 * 3 parameters
  t_c_set <- runif(n = 40, min = max(base_data$t),
                   max = max(base_data$t) + 0.6)
                  # adding 0.6 year representing time horizon
  m_set <- runif(n = 40, min = 0.1, max = 0.9)
  omega_set <- runif(n = 40, min = 6, max = 13)
  phi_set <- runif(n = 40, min = 0, max = 2*pi)

  # Looking for the 10 elite solutions out of 30 initial points,
  #the best one is then our starting point. Furthermore, we choose
  #the worst solution to set the taboo condition.
  elite_list <- vector()
  random_solutions <- cbind(t_c_set, m_set, omega_set, phi_set)
  losses <- apply(random_solutions, 1,
                  function(x) L(A = A, B = B, C = C,
                               t_c = x[1],m = x[2],
                               omega = x[3], phi = x[4],
                               price = base_data$price,t = base_data$t))
  losses <- as.data.frame(cbind('loss' = losses, 't_c' = t_c_set,
```

```r
                              'm' = m_set, 'omega' = omega_set,
                              'phi' = phi_set))
losses <- losses %>%
  arrange(loss)
elite_list <- losses[1:10,]
taboo_condition <- losses[nrow(losses),1]
S[['loss']] <- losses[1,'loss']
S[['t_c']] <- losses[1,'t_c']
S[['m']] <- losses[1,'m']
S[['omega']] <- losses[1,'omega']
S[['phi']] <- losses[1,'phi']

if (min(losses, na.rm = TRUE) < 200){
  # Partitioning and setting parameters for the number
  #of randomly drawn cells and points within them
  partitions <- c(6,6,6,6)
  n_c <- 2
  n_s <- 6
  t_c.partitions <- seq(from = max(base_data$t),
                        to = max(base_data$t) + 4,
                        length.out = partitions[1] + 1)
  m.partitions <- seq(from = 0.1, to = 0.9,
                      length.out = partitions[2] + 1)
  omega.partitions <- seq(from = 6, to = 13,
                          length.out = partitions[3] + 1)
  phi.partitions <- seq(from = 0, to = 2*pi,
                        length.out = partitions[4] + 1)
  partitions_matrix <- rbind(t_c.partitions, m.partitions,
                             omega.partitions,phi.partitions)

  # Searching procedure
  while (iter_without_improvement < 100){
```

```
# Drawing n_c * n_s points for looking for new solutions
chosen_cells <- t(sapply(partitions,
                         function(x) sample(1:x, size = n_c,
                                            replace = FALSE)))
drawn_points <-
  sapply(1:nrow(partitions_matrix),
         function(row)
           sapply(chosen_cells[row,],
                  function(x)
                    as.vector(sapply(x,function(y)
                      runif(n = n_s,
                            min = partitions_matrix[row,y],
                            max = partitions_matrix[row,y + 1])))))
colnames(drawn_points) <- c('t_c', 'm', 'omega', 'phi')

# Computing the value of loss function for new points,
#dropping points returning losses in a taboo region
losses <- apply(drawn_points, 1,
                function(x) L(A = A, B = B, C = C, t_c = x[1],
                              m = x[2], omega = x[3], phi = x[4],
                              price = base_data$price,
                              t = base_data$t))
drawn_points <- cbind('loss' = losses, drawn_points)
# dropping all points with non-defined loss function
drawn_points <- drawn_points[complete.cases(drawn_points),,
                             drop = FALSE]

losses <- losses[complete.cases(losses)]
non.taboo <- ifelse(losses < taboo_condition, TRUE, FALSE)
drawn_points <- drawn_points[non.taboo,,drop = FALSE]

# Picking the nontaboo point with the lowest move value
```

```
if (nrow(drawn_points) > 0){
  S[['loss']] <- drawn_points[
    which.min(drawn_points[,'loss'] - S$loss), 'loss'
    ]
  S[['t_c']] <- drawn_points[
    which.min(drawn_points[,'loss'] - S$loss), 't_c'
    ]
  S[['m']] <- drawn_points[
    which.min(drawn_points[,'loss'] - S$loss), 'm'
    ]
  S[['omega']] <- drawn_points[
    which.min(drawn_points[,'loss'] - S$loss), 'omega'
    ]
  S[['phi']] <- drawn_points[
    which.min(drawn_points[,'loss'] - S$loss), 'phi'
    ]

  # Executing elite_list modification in case of the loss
  #of a new points is lower than the loss of the 10th element
  #in the elite_list
  if (S$loss < elite_list[10,'loss']){
    elite_list <- rbind(elite_list[1:9,],
                        drawn_points[
                          which.min(drawn_points[,'loss'] - S$loss),
                          ]) %>%
      arrange(loss)
    iter_without_improvement <- 0
  } else {
    iter_without_improvement <- iter_without_improvement + 1
  }
} else {
  iter_without_improvement <- iter_without_improvement + 1
```

```
      }
    }
  }
  # Filling results to the Grid
  grid <- grid %>%
    filter(a == A, b == B, c == C) %>%
    mutate(loss = elite_list[1,'loss'], t_c = elite_list[1,'t_c'],
           m = elite_list[1,'m'], omega = elite_list[1,'omega'],
           phi = elite_list[1,'phi'])
  print(elite_list)
  return(grid)
}
```

Appendix D - LPPL1 model: Partial derivatives

$$(1)\ \frac{\partial y(t)}{\partial t_c} = Bm(t_c - t)^{m-1} + Cm(t_c - t)^{m-1} \cos\left(\omega \log(t_c - t) - \phi\right) - C(t_c - t)^m \sin\left(\omega \log(t_c - t) - \phi\right) \frac{\omega}{t_c - t}$$

$$(2)\ \frac{\partial y(t)}{\partial m} = B(t_c - t)^m \log(t_c - t) + C(t_c - t)^m \log(t_c - t) \cos\left(\omega \log(t_c - t) - \phi\right)$$

$$(3)\ \frac{\partial y(t)}{\partial \omega} = -C(t_c - t)^m \sin\left(\omega \log(t_c - t) - \phi\right) \log(t_c - t)$$

$$(4)\ \frac{\partial y(t)}{\partial \phi} = C(t_c - t)^m \sin\left(\omega \log(t_c - t) - \phi\right)$$

Appendix E - LPPL2 model: Partial derivatives

$$(1)\ \frac{\partial y(t)}{\partial t_c} = Bm(t_c-t)^{m-1} + C_1 m(t_c-t)^{m-1}\cos(\omega\log(t_c-t)) - C_1(t_c-t)^m \sin(\omega\log(t_c-t)))\frac{\omega}{t_c-t} + C_2 m(t_c-t)^{m-1}\sin(\omega\log(t_c-t)) + C_2(t_c-t)^m\cos(\omega\log(t_c-t)))\frac{\omega}{t_c-t}$$

$$(2)\ \frac{\partial y(t)}{\partial m} = B(t_c-t)^m\log(t_c-t) + C_1(t_c-t)^m\log(t_c-t)\cos(\omega\log(t_c-t)) + C_2(t_c-t)^m\log(t_c-t)\sin(\omega\log(t_c-t))$$

$$(3)\ \frac{\partial y(t)}{\partial \omega} = -C_1(t_c-t)^m\sin(\omega\log(t_c-t))\log(t_c-t) + C_2(t_c-t)^m\cos(\omega\log(t_c-t))\log(t_c-t)$$