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Helly-type theorems and fractional Helly-type theorems

Katedra aplikované matematiky

Vedoucí diplomové práce
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Rád bych poděkoval profesoru Jiřímu Matouškovi za odborné vedení a věcné připomínky k mé práci.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne Martin Tancer

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Abstrakt:

Simpliciální komplex je d-reprezentovatelný, pokud je nervem souboru konvexních množin v $\mathbb{R}^d$. Klasická Hellyho věta říká, že pokud d-reprezentovatelný komplex obsahuje všechny možné stěny dimenze d, potom se už nutně jedná o plný simplex. Hellyho věta má mnoho zobecnění; uvedeme stručný přehled některých z nich.

Třída d-reprezentovatelných komplexů je podtřídou d-kolabovatelných komplexů, a ta je podtřídou d-Lerayových komplexů. Pro $d \geq 1$ uvedeme příklad komplexů, které jsou $2d$-Lerayovy, ale nejsou $(3d-1)$-kolabovatelné. Pro $d \geq 2$ uvedeme příklad komplexů, které jsou d-Lerayovy, ale nejsou $(2d-2)$-reprezentovatelné. Navíc pro $d \leq 3$ dokážeme, že naposledy zmiňované komplexy jsou také d-kolabovatelné.

Na závěr prezentujeme jednoduchý důkaz kombinatorické Alexandrovy duality. Ta je totiž užitečným topologickým nástrojem pro kombinatoriku, například pro topologické verze Hellyho věty.

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Abstract:

A simplicial complex is d-representable if it is the nerve of a collection of convex sets in $\mathbb{R}^d$. Classical Helly's Theorem states that if a d-representable complex contains all the possible faces of dimension d then it is already a full simplex. Helly's Theorem has many extensions and we give a brief survey of some of them.

The class of d-representable complexes is a subclass of d-collapsible complexes, and the latter is a subclass of d-Leray complexes. For $d \geq 1$ we give an example of complexes that are $2d$-Leray but not $(3d-1)$-collapsible. For $d \geq 2$ we give an example of complexes that are d-Leray but not $(2d-2)$-representable. We show that for $d \leq 3$ the complexes from the last example are also d-collapsible.

We also give a simple proof of the Combinatorial Alexander Duality, which is a useful topological tool for combinatorics, e.g., for topological versions of Helly's Theorem.

Chapter 1

Introduction

In the thesis we study topics related with Helly-type theorems. We briefly introduce the topic; all the terms are defined precisely later on.

The nerve of a family of sets $S_1, S_2, \ldots S_n$ is a simplicial complex $\mathbf{X}$ such that the S_i are vertices of $\mathbf{X}$ and $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$ is a face of $\mathbf{X}$ if $\bigcap_{i=1}^{k} S_{i_k}$ is nonempty. A (finite) simplicial complex is d-representable if it is the nerve of a family of convex sets in $\mathbb{R}^d$. The Helly Theorem states (reformulated) that if a d-representable complex contains all the possible faces of dimension d then it is already a full simplex (a complex that contains all the possible faces of all dimensions). Helly-type theorems can be seen as theorems about properties of d-representable complexes.

A simplicial complex is d-Leray if the homology groups of dimensions greater or equal to d of all of its subcomplexes are zero. It is well known that d-representable complexes are d-Leray. Versions of Helly-type theorems for d-Leray complexes are topological Helly-type theorems. Chapter 4 surveys some Helly-type and topological Helly-type theorems.

Between the class of d-representable complexes and d-Leray complexes there is the class of d-collapsible complexes, precisely defined in Chapter 5. We study differences among these classes of complexes. For $d \geq 1$ we give an example of complexes that are $2d$-Leray but not $(3d-1)$-collapsible. For $d \geq 2$, we also give an example of complexes that are d-Leray but not $(2d-2)$-representable. For $d \leq 3$ we also show that these complexes are d-collapsible as well (and we conjecture that they are d-collapsible for all $d \geq 2$).

In Chapter 6 we are concerned with the Combinatorial Alexander Duality. It is a useful tool for proving combinatorial statements using topology. It is used, e.g., in [9] to prove the Topological Colourful Helly Theorem (Theorem 4.9). Let $\mathbf{X}$ be a simplicial complex with the ground set V. The Alexander dual of $\mathbf{X}$ is the simplicial complex $\mathbf{X}^*$ with the ground set V and its faces are sets $\sigma \subseteq V$ such that $V \setminus \sigma \notin \mathbf{X}$. Letting $n = |V|$, the Combinatorial Alexander Duality states that $\tilde{H}_i(\mathbf{X}) = \tilde{H}_{n-i-3}(\mathbf{X}^*)$, over fields (otherwise, one homology should be replaced by cohomology). We give a quite simple proof of the Combinatorial Alexander Duality. It is based on an idea of Björner [4], but it turned out that the idea had to be modified slightly; some technical details are finished in the thesis.

Chapter 2

Preliminaries

2.1 Sets

In this section we introduce a basic notation from set theory used in the thesis.

Let $k \in \mathbb{N}_0$, let M, N, $M' \subseteq M$, $N' \subseteq N$ be sets and $f : M \to N$ a function. We will use the following notation:

2^M	$= \{K \mid K \subseteq M\}$	the family of all subsets;
$f(M')$	$= \{f(m) \mid m \in M'\}$	the image of M';
$f^{-1}(N')$	$= \{m \in M \mid f(m) \in N'\}$	the preimage of N';
$\binom{M}{k}$	$= \{K \subseteq M \mid \lvert K\rvert = k\}$	the family of subsets of size k;
$\binom{M}{\leq k}$	$= \{K \subseteq M \mid \lvert K\rvert \leq k\}$	the family of subsets of size at most k;
$M \uplus N$	$= M \times \{1\} \cup N \times \{2\}$	the disjoint union of sets M and N.

2.2 Simplicial Complexes

A *simplicial complex* $\mathbf{X}$ is a pair $(V, \mathbf{K})$, where $\mathbf{K} \subseteq 2^V$ is such that if $\sigma \subseteq \tau \in \mathbf{K}$ then $\sigma \in \mathbf{K}$. Unless stated otherwise, we will consider just finite simplicial complexes, i. e., such that V is finite. The set V is the *ground set* of $\mathbf{X}$ and the set $\mathbf{K}$ is the set of *simplices* (also *faces*) of $\mathbf{X}$. The *dimension* of a simplex $\sigma \in \mathbf{K}$ is $\dim \sigma = |\sigma| - 1$. The *dimension* of $\mathbf{X}$ is defined as $\dim \mathbf{X} = \max\{\dim \sigma \mid \sigma \in \mathbf{K}\}$. Simplices of dimension zero are called *vertices*. For simplicity, when $\mathbf{X} = (V, \mathbf{K})$ is a simplicial complex we often write just $\sigma \in \mathbf{X}$ instead of $\sigma \in \mathbf{K}$. Similarly $\mathcal{T} \subseteq \mathbf{X}$ is the notation for $\mathcal{T} \subseteq \mathbf{K}$ and for $\mathcal{T} \subseteq \mathbf{X}$ we define $\mathbf{X} \setminus \mathcal{T}$ as $(V, \mathbf{K} \setminus \mathcal{T})$ supposing that $(V, \mathbf{K} \setminus \mathcal{T})$ is a simplicial complex.

A simplicial complex $\mathbf{Y} = (W, \mathbf{H})$ is a *subcomplex* of $\mathbf{X} = (V, \mathbf{K})$ if $W \subseteq V$ and $\mathbf{H} \subseteq \mathbf{K}$, and it is an *induced subcomplex* if $W \subseteq V$ and $\mathbf{H} = \mathbf{K} \cap 2^W$. We will use notation $\mathbf{Y} \subseteq \mathbf{X}$ for subcomplexes and $\mathbf{Y} \leq \mathbf{X}$ for induced subcomplexes. Moreover, $\mathbf{X}[W]$ denotes the induced subcomplex of $\mathbf{X}$ with ground set W. A simplicial complex $(V, \mathbf{K})$ is a *full simplex* if $\mathbf{K} = 2^V$.

Now, let $\mathbf{X} = (V, \mathbf{K})$ and $\mathbf{Y} = (W, \mathbf{H})$ be simplicial complexes and let $\sigma \in \mathbf{K}$. We introduce several more definitions:

$$\begin{array}{rll}
\mathbf{2}^V & = \left(V, 2^V\right) & \text{the } \textit{full simplex on } V; \\
\mathrm{lk}\,(\mathbf{X},\sigma) & = (V \setminus \sigma, \{\tau \in \mathbf{K} \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \mathbf{K}\}) & \text{the } \textit{link} \text{ of } \sigma; \\
\mathrm{st}\,(\mathbf{X},\sigma) & = (V, \{\tau \in \mathbf{K} \mid \tau \cup \sigma \in \mathbf{K}\}) & \text{the } \textit{star} \text{ of } \sigma; \\
\mathbf{X} \cap \mathbf{Y} & = (V \cap W, \mathbf{K} \cap \mathbf{H}) & \text{the } \textit{intersection} \text{ of } \mathbf{X} \text{ and } \mathbf{Y}; \\
\mathbf{X} \cup \mathbf{Y} & = (V \cup W, \mathbf{K} \cup \mathbf{H}) & \text{the } \textit{union} \text{ of } \mathbf{X} \text{ and } \mathbf{Y}; \\
\mathbf{X} * \mathbf{Y} & = (V \uplus W, \{\sigma \uplus \tau \mid \sigma \in \mathbf{K}, \tau \in \mathbf{H}\}) & \text{the } \textit{join} \text{ of } \mathbf{X} \text{ and } \mathbf{Y}; \\
\mathbf{X}^{(d)} & = (V, \{\sigma \in \mathbf{K} \mid \dim \sigma \leq d\}) & \text{the } d\textit{-skeleton} \text{ of } \mathbf{X}.
\end{array}$$

2.3 Convexity and Affine Independence

A set $K \subseteq \mathbb{R}^d$ is *convex* if $a + \lambda(b - a) \in K$ for every $a, b \in K$ and $\lambda \in (0, 1)$. For a set $A \subseteq \mathbb{R}^d$, the *convex hull* $\mathrm{conv}(A)$ is defined as the intersection of all the convex sets that contain A. It is easy to see that $\mathrm{conv}(A)$ is a convex set. Equivalently,

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{k} \lambda_i a_i \;\middle|\; k \in \mathbb{N}, a_i \in A, \lambda_i \in [0, 1] \right\}.$$

Let p_1, p_2, ..., p_n be points in $\mathbb{R}^d$. They are *affinely dependent* if there exists α_1, α_2, ..., α_n real, not all of them 0, such that $\sum_{i=1}^n \alpha_i = 0$ and $\sum_{i=1}^n \alpha_i p_i = 0$. Otherwise, these points are *affinely independent*.

Let $k \in \mathbb{N}$. A *(geometric) k-simplex* is a convex hull of $k + 1$ affinely independent points. It exists in $\mathbb{R}^d$ for $d \geq k$. If $k \in \mathbb{N}$ is not important we just omit it. A relation between geometric simplices and simplicial complexes will be given in Section 2.5.

In the thesis we will need the following classical theorem [5], [15].

Theorem 2.1 (Radon). *Let P be a set of affinely dependent points in $\mathbb{R}^d$. Then there exist two disjoint subsets P_1 and P_2 of P such that* $\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2) \neq \emptyset$.

Lemma 2.2. *Let A and B be finite subsets of $\mathbb{R}^d$. Suppose that there is a point $x \in$* $\mathrm{conv}(A) \cap \mathrm{conv}(B)$*, but $x \notin$* $\mathrm{conv}(A \cap B)$*. Then there exist disjoint sets $C \subseteq A$ and $D \subseteq B$ such that* $\mathrm{conv}(C) \cap \mathrm{conv}(D) \neq \emptyset$.

Proof. Since $x \in \mathrm{conv}(A)$ there exist $\lambda_a \in [0, 1]$, $a \in A$, such that

$$x = \sum_{a \in A} \lambda_a a \qquad \text{and} \qquad \sum_{a \in A} \lambda_a = 1.$$

Similarly, there are $\kappa_b \in [0, 1]$, $b \in B$, such that

$$x = \sum_{b \in B} \kappa_b b \qquad \text{and} \qquad \sum_{b \in B} \kappa_b = 1.$$

Let $K = A \cap B$. From the previous equations

$$0 = \sum_{a \in A} \lambda_a a - \sum_{b \in B} \kappa_b b = \sum_{k \in K} (\lambda_k - \kappa_k) k + \sum_{a \in A \setminus K} \lambda_a a - \sum_{b \in B \setminus K} \kappa_b b.$$

Let $K^+ = \{k \in K \mid \lambda_k - \kappa_k \geq 0\}$ and let $K^- = K \setminus K^+$.

Let

$$y = \sum_{k \in K^+} (\lambda_k - \kappa_k)k + \sum_{a \in A \setminus K} \lambda_a a = \sum_{k \in K^-} (\kappa_k - \lambda_k)k + \sum_{b \in B \setminus K} \kappa_b b.$$

Since $x \notin \operatorname{conv}(A \cap B)$, at least one λ_a for $a \in A \setminus K$ is nonzero. Thus,

$$0 < \alpha = \sum_{k \in K^+} (\lambda_k - \kappa_k) + \sum_{a \in A \setminus K} \lambda_a = 1 - \sum_{k \in K^+} \kappa_k - \sum_{k \in K^-} \lambda_k = \sum_{k \in K^-} (\kappa_k - \lambda_k) + \sum_{b \in B \setminus K \kappa_b} .$$

The definition of y imply that $\alpha^{-1} y \in \operatorname{conv}((A \setminus K) \cup K^+) \cap \operatorname{conv}((B \setminus K) \cup K^-)$. Thus it is sufficient to pick $C = (A \setminus K) \cup K^+$ and $D = (B \setminus K) \cup K^-$ to finish the proof.

□

2.4 Homological Algebra

In this section we give some basic definitions from homological algebra we need in the thesis. More details about homological algebra we use can be found in [16], especially in Chapter 6.

Let R be a commutative ring. A *complex* (also *chain complex*) $\mathcal{A} = (\mathcal{A}, d)$ is a sequence of R-modules A_n and R-homomorphisms d_n

$$\mathcal{A} = \cdots \longleftarrow A_{n-1} \xleftarrow{\;d_n\;} A_n \xleftarrow{\;d_{n+1}\;} A_{n+1} \longleftarrow \cdots, \qquad n \in \mathbb{Z}$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. This condition is equivalent to $\operatorname{im} d_{n+1} \subseteq \ker d_n$. The maps d_n are called *differentiations*.

If $(\mathcal{A}, d)$ is a complex then its *nth homology module* (or *nth homology group*) is defined as

$$H_n(\mathcal{A}) = \ker d_n / \operatorname{im} d_{n+1}.$$

Let $(\mathcal{A}, d)$ and $(\mathcal{A}', d')$ be complexes. A *chain map* $f : \mathcal{A} \to \mathcal{A}'$ is a sequence of R-homomorphisms $f_n : A_n \to A'_n$ such that $f_{n-1} d_n = f_n d'_n$ for every n. It is an *isomorphism* of complexes if all the f_n are R-isomorphisms. Isomorphic complexes have isomorphic homology groups.

If $f : (\mathcal{A}, d^a) \to (\mathcal{B}, d^b)$ is a chain map, we define

$$H_n(f) : H_n(\mathcal{A}) \to H_n(\mathcal{B})$$

by

$$a + \operatorname{im} d^a_{n+1} \to f_n(a) + \operatorname{im} d^b_{n+1}.$$

A sequence (finite of infinite) of R-modules and R-homomorphisms

$$\cdots \xrightarrow{\;f_{n-1}\;} A_n \xrightarrow{\;f_n\;} \cdots$$

is *exact in degree n* if $\ker f_n = \operatorname{im} f_{n-1}$, and it is an *exact* sequence if it is exact in all degrees. Exact sequences of the form

$$0 \xrightarrow{\;0\;} A \xrightarrow{\;\alpha\;} B \xrightarrow{\;\beta\;} C \xrightarrow{\;0\;} 0$$

are called *short exact sequences*. Infinite exact sequences are called *long exact sequences*.

The proof of the following theorem is in [16], Theorem 6.3.

Theorem 2.3 (Long Exact Sequence). *Suppose that in the following diagram,* $(\mathcal{A}, d^a)$, $(\mathcal{B}, d^b)$, $(\mathcal{C}, d^c)$ *are complexes and* f *and* g *are chain maps.*

$$\mathcal{A} \xrightarrow{\quad f \quad} \mathcal{B} \xrightarrow{\quad g \quad} \mathcal{C}$$

Suppose also that for every $n \in \mathbb{Z}$ *the sequence*

$$0 \xrightarrow{\quad 0 \quad} A_n \xrightarrow{\quad f_n \quad} B_n \xrightarrow{\quad g_n \quad} C_n \xrightarrow{\quad 0 \quad} 0$$

is a short exact sequence. Then there is an exact sequence of modules

$$\cdots \quad H_n(\mathcal{A}) \xrightarrow{H_n(f)} H_n(\mathcal{B}) \xrightarrow{H_n(g)} H_n(\mathcal{C}) \xrightarrow{\partial_n^B} H_{n-1}(\mathcal{A}) \xrightarrow{H_{n-1}(f)} H_n(\mathcal{B}) \quad \cdots$$

The homomorphisms ∂_n^B *are called boundary homomorphisms and they are defined as*

$$\partial_n^B : c + \operatorname{im} d_{n+1}^c \to f_{n-1}^{-1} d_n^b g_n^{-1}(c) + \operatorname{im} d_n^a.$$

2.5 Topology

A *topological space* is a pair $(X, \mathcal{G})$ where $\mathcal{G} \subseteq 2^X$ satisfies the following conditions:

- $\emptyset$, $X \in \mathcal{G}$,
- $G_i \in \mathcal{G}$ for $i \in I \Rightarrow \bigcup_{i \in I} G_i \in \mathcal{G}$,
- $G_i \in \mathcal{G}$ for $i \in I$, I is a finite set $\Rightarrow \bigcap_{i \in I} G_i \in \mathcal{G}$.

The set $\mathcal{G}$ is called a *topology* on X and its elements are *open* sets. If it is not necessary to display $\mathcal{G}$ we write just X instead of $(X, \mathcal{G})$.

Let $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ be topological spaces. A map $f : X \to Y$ is continuous if $H \in \mathcal{H}$ implies $f^{-1}(H) \in \mathcal{G}$. In the rest of the text all the maps among topological spaces are supposed to be continuous. Two topological spaces X and Y are *homeomorphic*, if there exists a continuous bijection $f : X \to Y$ such that f^{-1} is also continuous. Such bijection is called a *homeomorphism* of X and Y. Maps $f_0, f_1 : X \to Y$ are *homotopic* if there exists a continuous map $F : X \times [0,1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We write $f_0 \simeq f_1$. Topological spaces X and Y are *homotopy equivalent*, $X \simeq Y$, if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$.

If $(X, \mathcal{G})$ is a topological space and $A \subseteq X$ then A is a *subspace* of X with topology $\{G \cap A \mid G \in \mathcal{G}\}$.

Let $\mathbf{X} = (V, \mathbf{K})$ be a simplicial complex. It has an associated topological space $\|\mathbf{X}\|$ defined in the following way: Suppose that $\{p_v \mid v \in V\}$ is a set of $|V|$ affinely independent points in $\mathbb{R}^d$. Let $P_\sigma = \{p_v \mid v \in \sigma\}$ for $\sigma \subseteq V$. Then

$$\|\mathbf{X}\| = \bigcup_{\sigma \in K} \operatorname{conv}(P_\sigma).$$

Formally, $\|\mathbf{X}\|$ depends on the chosen set $\{p_v \mid v \in V\}$; however, all these spaces are homeomorphic. Informally, $\|\mathbf{X}\|$ is a topological space consisting of simplices in $\mathbf{K}$ glued together.

Two simplicial complexes $\mathbf{X}$ and $\mathbf{Y}$ are *homeomorphic*, if $\|\mathbf{X}\|$ and $\|\mathbf{Y}\|$ are homeomorphic, and they are *homotopy equivalent*, $\mathbf{X} \simeq \mathbf{Y}$, if $\|\mathbf{X}\| \simeq \|\mathbf{Y}\|$. Similarly, a simplicial complex $\mathbf{X}$ is *homotopy equivalent* with a topological space X if $\|\mathbf{X}\| \simeq X$.

A simplicial complex $\mathbf{X}$ is *embedable* into $\mathbb{R}^d$ if $\|\mathbf{X}\|$ is homeomorphic to a subset of $\mathbb{R}^d$.

2.6 Miscellaneous

Let V be a vector space over a field $\mathbb{K}$ and let $M \subseteq V$. Then

$\langle M \rangle = \left\{ \sum_{i=1}^{k} \lambda_i v_i \;\middle|\; \lambda_i \in \mathbb{K}, v_i \in M \right\}$ is the linear hull of M.

If M is written as $M = \{\cdots\}$ we write just $\langle \cdots \rangle$ instead of $\langle \{\cdots\} \rangle$.

Chapter 3

Tools from Algebraic Topology

In this chapter we will introduce the tools from algebraic topology that we will use through the rest of the thesis. We will mention just what we exactly need; much more can be found in [7] or [13]. Throughout this chapter, let $\mathbb{K}$ be a fixed field.

3.1 Reduced Homology of Simplicial Complexes

Let $\mathbf{X}$ be a simplicial complex with the ground set $V = \{1, 2, \dots, n\}$. Let $K_i = K_i(\mathbf{X})$ be the vector space over $\mathbb{K}$ with the basis vectors e_σ for $\sigma \in \mathbf{X}$, $\dim \sigma = i$. The *reduced chain complex* of $\mathbf{X}$ over $\mathbb{K}$ is the complex

$$\tilde{\mathcal{C}}_\circledast(\mathbf{X}; \mathbb{K}) = \cdots \longleftarrow K_{i-1} \xleftarrow{\partial_i} K_i \xleftarrow{\partial_{i+1}} K_{i+1} \longleftarrow \cdots, \qquad i \in \mathbb{Z},$$

where ∂_i are defined as

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} \operatorname{sgn}(j, \sigma) e_{\sigma \setminus j}.$$

Here, for $j \in \sigma \in \mathbf{X}$ we define $\operatorname{sgn}(j, \sigma)$ as $(-1)^{i-1}$, where j is the i-th smallest element of σ. Note that the complex $\tilde{\mathcal{C}}_\circledast(\mathbf{X}; \mathbb{K})$ is formally infinite; however, $K_i = 0$ for $i < -1$ or $i > \dim \mathbf{X}$.

The *n-th reduced homology group* of $\mathbf{X}$ over $\mathbb{K}$ is defined as

$$\tilde{H}_n(\mathbf{X}; \mathbb{K}) = H_n\big(\tilde{\mathcal{C}}_\circledast(\mathbf{X}; \mathbb{K})\big).$$

If $\mathbb{K}$ is understood from context, we write just $\tilde{H}_n(\mathbf{X})$ instead of $\tilde{H}_n(\mathbf{X}; \mathbb{K})$, and similarly for cohomology groups and relative homology groups defined later on.

An important property of homology groups is that they are isomorphic for homotopy equivalent simplicial complexes.

3.2 Reduced Cohomology of Simplicial Complexes

Let $L_i = L_i(\mathbf{X}) = K_i^*(\mathbf{X})$ be the vector space dual of K_i, with basis vectors e_σ^* for $\sigma \in \mathbf{X}$, $\dim \sigma = i$. The *reduced cochain complex* of $\mathbf{X}$ over $\mathbb{K}$ is the complex

$$\tilde{\mathcal{C}}^\circledast(\mathbf{X}; \mathbb{K}) = \cdots \longrightarrow L_{i-1} \xrightarrow{\partial^i} L_i \xrightarrow{\partial^{i+1}} L_{i+1} \longrightarrow \cdots, \qquad i \in \mathbb{Z},$$

where $\partial^i = \partial_i^*$ are dual maps to ∂_i, explicitly stated:

$$\partial^i(e_\sigma^*) = \sum_{\substack{j\notin\sigma \\ \sigma\cup j\in\mathbf{X}}} \operatorname{sgn}(j, \sigma\cup j) e_{\sigma\cup j}^*.$$

The *n-th reduced cohomology group* of $\mathbf{X}$ over $\mathbb{K}$ is defined as

$$\tilde{H}^n(\mathbf{X};\mathbb{K}) = H_n\big(\tilde{\mathcal{C}}^{\circledast}(\mathbf{X};\mathbb{K})\big).$$

It is well known that $\tilde{H}_n(\mathbf{X};\mathbb{K})$ and $\tilde{H}^n(\mathbf{X};\mathbb{K})$ are isomorphic.

Remark 3.1. It is not the task of the thesis to define homology and cohomology when $\mathbb{K}$ is replaced by a general commutative ring (it is defined in a similar way). However, in that case corresponding homology and cohomology groups need not be isomorphic.

3.3 Relative Homology Groups

Suppose that $\mathbf{X}$ is a simplicial complex and $\mathbf{A}$ is a subcomplex of $\mathbf{X}$. Let $R_i = R_i(\mathbf{X},\mathbf{A}) = K_i(\mathbf{X})/K_i(\mathbf{A})$, where K_i was defined in Section 3.1. The *relative reduced chain complex* of $(\mathbf{X},\mathbf{A})$ over $\mathbb{K}$ is the complex

$$\tilde{\mathcal{C}}_{\circledast}(\mathbf{X},\mathbf{A};\mathbb{K}) = \cdots \longleftarrow R_{i-1} \xleftarrow{\partial_i} R_i \xleftarrow{\partial_{i+1}} R_{i+1} \longleftarrow \cdots, \qquad i\in\mathbb{Z},$$

where ∂_i are defined as

$$\partial_i(e_\sigma + K_i(\mathbf{A})) = \sum_{j\in\sigma} \operatorname{sgn}(j,\sigma)\left(e_{\sigma\setminus j} + K_{i-1}(\mathbf{A})\right).$$

The *n-th relative reduced homology group* of $(\mathbf{X},\mathbf{A})$ over $\mathbb{K}$ is defined as

$$\tilde{H}_n(\mathbf{X},\mathbf{A};\mathbb{K}) = \tilde{H}_n\big(\tilde{\mathcal{C}}_{\circledast}(\mathbf{X},\mathbf{A};\mathbb{K})\big).$$

Remark 3.2. When we wish to compute relative homology groups, we can identify $R_i = K_i(\mathbf{X})/K_i(\mathbf{A})$ with the vector space over $\mathbb{K}$ with the basis vectors e_σ for $\sigma\in\mathbf{X}$, $\sigma\notin\mathbf{A}$, $\dim\sigma = i$.

Then ∂_i can be rewritten as:

$$\partial_i(e_\sigma) = \sum_{\substack{j\in\sigma \\ \sigma\setminus j\notin\mathbf{A}}} \operatorname{sgn}(j,\sigma) e_{\sigma\setminus j}.$$

Remark 3.3. Reduced homology groups $\tilde{H}_n(\mathbf{X})$ can be seen as groups $\tilde{H}_n(\mathbf{X},\emptyset)$, where $\emptyset$ stands for the simplicial complex with the ground set same as $\mathbf{X}$, but with no faces.

One of the important properties of relative homology groups is that they fit into a long exact sequence. Suppose that $i_{\star,n} : K_n(\mathbf{A}) \to K_n(\mathbf{X})$ is the map induced by the inclusion $\mathbf{A}\hookrightarrow\mathbf{X}$ and $j_{\star,n} : K_n(\mathbf{X}) \to R_n(\mathbf{X},\mathbf{A})$ is the map induced by the inclusion $(\mathbf{X},\emptyset)\hookrightarrow(\mathbf{X},\mathbf{A})$, in the sense of Remark 3.3. Then $i_\star$ and $j_\star$ are chain maps and there are short exact sequences

$$0 \longrightarrow K_n(\mathbf{A}) \xrightarrow{i_{\star,n}} K_n(\mathbf{X}) \xrightarrow{j_{\star,n}} R_n(\mathbf{X},\mathbf{A}) \longrightarrow 0$$

See e. g. [7] for a proof and more details. Thus Theorem 2.3 implies the following lemma:

Lemma 3.4 (Long Exact Sequence of a Pair). *Suppose that* $\mathbf{X}$ *and* $\mathbf{A}$ *are simplicial complexes* $\mathbf{A} \subseteq \mathbf{X}$. *Then there is a long exact sequence*

$$\cdots \longrightarrow \tilde{H}_n(\mathbf{A}) \longrightarrow \tilde{H}_n(\mathbf{X}) \longrightarrow \tilde{H}_n(\mathbf{X}, \mathbf{A}) \longrightarrow \tilde{H}_{n-1}(\mathbf{A}) \longrightarrow \cdots$$

Chapter 4

Helly-type Theorems

In this Chapter we give a short survey of Helly-type theorems. We also introduce d-representability and Leray number and show their relation to Helly-type theorems.

4.1 Helly's Theorem

Here we mention the well-known Helly Theorem [8].

Theorem 4.1 (Helly). *Let K_1 , K_2, ..., K_n be convex sets in $\mathbb{R}^d$, $n \geq d+1$. Suppose that every $(d+1)$-tuple of them has a nonempty intersection; then all the sets have a nonempty intersection.*

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a family of sets. The *nerve* $\mathbf{N}(\mathcal{F})$ of this family is a simplicial complex $(V, \mathbf{K})$, where $V = \{1, 2, \ldots, k\}$ and for $\sigma \subseteq \{1, 2, \ldots k\}$ we have $\sigma \in \mathbf{K}$ if and only if $\bigcap_{i\in\sigma} F_i \neq \emptyset$. A simplicial complex is *d-representable* if it is isomorphic to the nerve of a family of convex sets in $\mathbb{R}^d$.

Remark 4.2. Suppose that a simplicial complex is d-representable since it is isomorphic the nerve of family $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ of convex subsets of $\mathbb{R}^d$. For every $\sigma \in \mathbf{N}(\mathcal{F})$ let us pick a point $p_\sigma \in \bigcap_{i\in\sigma} F_i$. For $i \in \{1, 2, \ldots, k\}$ let $G_i = \operatorname{conv}\{p_\sigma | \sigma \in \mathbf{N}(\mathcal{F})\}$. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ then it is easy to see that $\mathbf{N}(\mathcal{F}) = \mathbf{N}(\mathcal{G})$. Thus a simplicial complex is d-representable if and only if it is a nerve of family of subsets of $\mathbb{R}^d$ such that each of these sets is a convex hull of finitely many points.

Helly's Theorem may be restated in the following form:

Theorem 4.3. *Let $(V, \mathbf{K})$ be a d-representable simplicial complex, $|V| \geq d+1$. Suppose that $\sigma \in \mathbf{K}$ for every $\sigma \subseteq V$ of dimension d. Then $\mathbf{K} = 2^V$.*

4.2 Fractional and Colourful Helly

Helly's Theorem has a lot of modifications or generalisations. We will mention the Fractional Helly Theorem and the Colourful Helly Theorem here.

Informally, the fractional Helly theorem states that if there are finitely many convex sets in $\mathbb{R}^d$ such that a lot of $(d+1)$-tuples (some fraction of all $d+1$-tuples) among these sets have nonempty intersection, then there is a large number of sets (some fraction among all of them) that have nonempty intersection. Before stating the theorem formally, we define some terms. Let $\mathcal{X}$ be a family of simplicial complexes. We say that $\mathcal{X}$ satisfies the *fractional Helly property* $\mathrm{FH}(k, \alpha, \beta)$ if there is no $\mathbf{X} = (V, \mathbf{K}) \in \mathcal{X}$ on n vertices such that the number of k-tuples $I \subseteq V$, $I \in \mathbf{K}$ is at least $\alpha\binom{n}{k}$ and $\dim \mathbf{X} < \lfloor \beta n \rfloor - 1$. We say that $\mathcal{X}$ has *fractional Helly number* k if for every $\alpha \in (0, 1)$ there exists $\beta = \beta(\alpha) > 0$ such that $\mathcal{X}$ has $\mathrm{FH}(k, \alpha, \beta)$.

The fractional Helly theorem of Katchalski and Liu [10] may be stated in the following form:

Theorem 4.4 (Fractional Helly). *The family of d-representable complexes has fractional Helly number $d + 1$.*

The importance of the fractional Helly theorem may be seen for example in [2] and [1] when proving the so-called (p, q)-theorem, which is an important theorem about nerves of families of convex sets.

The colourful Helly theorem was first proved by Lovász and informally, it says that if there are finitely many convex sets in $\mathbb{R}^d$, each coloured by one among $d + 1$ colours so that every $(d + 1)$-tuple of these sets with pairwise different colours has nonempty intersection, then there exists a colour such that all the sets of that colour have nonempty intersection.

The formal statement in terms of simplicial complexes is the following:

Theorem 4.5 (Colourful Helly). *Let $(V, \mathbf{K})$ be a d-representable complex, where V is a union of sets V_1, V_2, ..., V_{d+1}. Suppose that $\{v_1, v_2, \dots, v_{d+1}\} \in \mathbf{K}$ for all choices $v_1 \in V_1$, $v_2 \in V_2$, ..., $v_{d+1} \in V_{d+1}$. Then exists $i \in \{1, 2, \dots, d + 1\}$ such that $V_i \in \mathbf{K}$.*

Helly theorem can be derived from this theorem when putting $V_1 = V_2 = \cdots = V_{d+1}$.

4.3 Topological Helly-type Theorems

A simplicial complex $\mathbf{X}$ is *d-Leray* if $\tilde{H}_i(\mathbf{Y}) = 0$ for all induced subcomplexes $\mathbf{Y} \leq \mathbf{X}$ and all $i \geq d$. We define the *Leray number* of $\mathbf{X}$ by the following formula:

$$\lambda(\mathbf{X}) = \min\{k \in \mathbb{N}_0 \mid \mathbf{X} \text{ is } k\text{-Leray}\}.$$

Finiteness of the considered complexes implies that the Leray number is well defined.

A key observation for studying convex sets via homology is that d-representable simplicial complexes are d-Leray. This can be easily derived from the Nerve Theorem (see Theorem 4.6 below) and Remark 4.2, since the homology of subspaces of $\mathbb{R}^d$ is zero in dimensions greater or equal to d. We will see this topic in more detail in the Chapter 5.

The following variant of the Nerve Theorem is an easy consequence of the Theorem 10.7 in Björner's survey [3].

Theorem 4.6 (Nerve Theorem). *Let $\mathcal{F} = \{F_1, F_2, \dots, F_k\}$ be a family of subsets of $\mathbb{R}^d$ such that each F_i is a convex hull of finitely many points in $\mathbb{R}^d$. Then $\bigcup_{F \in \mathcal{F}} F$ is homotopy equivalent to $\mathbf{N}(\mathcal{F})$.*

Helly theorem may be generalised for d-Leray complexes; then it has a very easy proof.

Theorem 4.7 (Topological Helly). *Let $(V, \mathbf{K})$ be a d-Leray simplicial complex, $|V| \geq d+1$. Suppose that $\sigma \in \mathbf{K}$ for every $\sigma \subseteq V$ of dimension d. Then $\mathbf{K} = 2^V$.*

Proof. We will prove the theorem by induction on $|V|$.

For $|V| = d+1$, the assumptions imply that $\mathbf{K} = 2^V$ or $\mathbf{K} = 2^V \setminus \{V\}$. Since the d-dimensional homology has to be 0, we get that $\mathbf{K} = 2^V$.

For $|V| > d+1$, from induction assumption, each proper subset of V forms a full simplex. Similarly as to previous case, to avoid non-zero homology in dimension $|V|-1$ we get that $\mathbf{K} = 2^V$.

□

The topological version of the fractional Helly theorem was proved by Alon et al. [1].

Theorem 4.8 (Topological Fractional Helly). *The family of d-Leray complexes has fractional Helly number $d+1$.*

They proved that a d-representable simplicial complex satisfies $\mathrm{FH}(d+1, \alpha, \beta(\alpha))$ with $\beta(\alpha) = 1-(1-\alpha)^{\frac{1}{d+1}}$ and they also proved that if a d-Leray simplicial complex is a nerve of some family of sets, then the fractional Helly property is also satisfied for the nerve of all the intersections of the family.

The topological version of the colourful Helly theorem was proved by Kalai and Meshulam [9]. They even extended the theorem for matroidal complexes.

A simplicial complex $\mathbf{M}$ with ground set V is a *matroidal complex* if for every $S \subseteq V$ all the maximal faces of $\mathbf{M}[S]$ have the same dimension. Matroidal complexes are complexes such that its simplices are the independent sets of a matroid. The *rank function* $\rho : 2^V \to \mathbb{N}$ is defined by $\rho(S) = \dim \mathbf{M}[S] + 1$ for $S \subseteq V$. If the reader is not familiar with the matroid theory we refer to [14].

Theorem 4.9 (Topological Colourful Helly). *Let $\mathbf{X}$ be a d-Leray complex with ground set V and $\mathbf{M} \subseteq \mathbf{X}$ be a matroidal complex with ground set V and rank function ρ. Then there exists a simplex $\tau \in \mathbf{X}$ such that $\rho(V \setminus \tau) \leq d$.*

Suppose that $\bigcup_{i=1}^{d+1} V_i$ is a partition of V. The corresponding *partition matroid* is defined by $\sigma \in \mathbf{M}$ if and only if $|\sigma \cap V_i| \leq 1$ for all $1 \leq i \leq d+1$. Theorem 4.5 can be derived from Theorem 4.9 by letting $\mathbf{M}$ be the partition matroid.

Chapter 5

Representability, Collapsibility and Leray Number

In this chapter first we introduce d-collapsible complexes, which is a class of simplicial complexes between d-representable and d-Leray ones. Then we study differences among these types of complexes.

5.1 d-Collapsibility

Let $\mathbf{X} = (V, \mathbf{K})$ be a simplicial complex and let $\sigma \in \mathbf{K}$ be a face of dimension at most $d-1$ which is contained in a unique maximal face $\tau \in \mathbf{K}$. The operation $\mathbf{X} \searrow^d \mathbf{Y}$, where $\mathbf{Y} = (V, \mathbf{K} \setminus \{\eta \in \mathbf{K} \mid \sigma \subseteq \eta \subseteq \tau\})$, is called an *elementary d-collapse* (of a face σ). A simplicial complex $\mathbf{X}$ is *d-collapsible* if there exists a sequence of elementary d-collapses

$$\mathbf{X} = \mathbf{X}_1 \searrow^d \mathbf{X}_2 \searrow^d \cdots \searrow^d \mathbf{X}_k = (V, \{\emptyset\}).$$

The class of d-collapsible complexes was first introduced by Wegner [18]. He also showed that d-representable complexes are d-collapsible, and that d-collapsible complexes are d-Leray.

Example 5.1. There is an example of 2-collapsing in Figure 5.1.

Remark 5.2. The notion of d-collapsibility is similar to the well-known collapsibility; however, it differs in some aspects. When d-collapsing, one is allowed either to collapse (in the usual sense) a face of a dimension at most $d-1$ strictly contained in a unique maximal face, or to remove a maximal face of a dimension at most $d-1$. Collapsible simplicial complexes are, of course, d-collapsible for d large enough. The boundary of a d-simplex is d-collapsible; however, it is not collapsible.

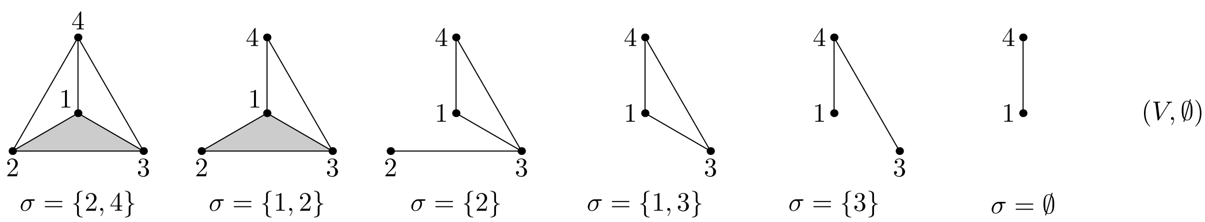

Figure 5.1: An example of 2-collapsing.

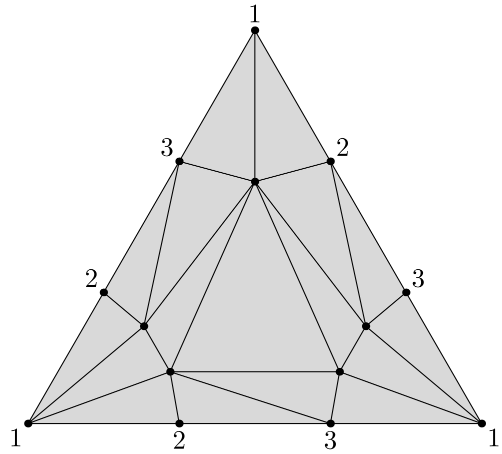

Figure 5.2: The triangulation of the dunce hat [18]; identify vertices with the same numbers.

In [1] Alon et al. asked the following question:

Problem 5.3. *Is there a function $d' = d'(d)$ such that every d-collapsible complex (or even every d-Leray complex) is d'-representable?*

In this context the following problem arises naturally:

Problem 5.4. *Is there a function $d' = d'(d)$ such that every d-Leray complex is d'-collapsible?*

If the answer in Problem 5.4 is positive then it shows that Problem 5.3 is of the same difficulty for d-Leray complexes as for d-collapsible ones. If the answer is negative then also the answer in Problem 5.3 for d-Leray complexes is negative.

For $d \geq 2$, there are known examples of d-Leray complexes that are not d-collapsible [18], like triangulations of contractible but not collapsible spaces. For example the dunce hat in Figure 5.2 or Bing's house [7] in Figure 5.3 for $d = 2$. However, these examples are already $(d+1)$-collapsible. In Section 5.2 we show that this gap can be wider. More precisely, for $d \in \mathbb{N}$ we show examples of complexes that are $2k$-Leray but not $(3k-1)$-collapsible.

For $d = 1$, there is a simple example of a complex that is d-collapsible but not d-representable in Figure 5.4. For $d = 2$ Wegner [18] gave an example of a complex that is d-collapsible but not d-representable; a different example is in Figure 5.5. In Section 5.3 we provide examples of complexes that are d-Leray but not $(2d-2)$-representable; thus, this gap between the representability and the Leray number is wider than the gap implied by the result on the collapsibility and the Leray number. We show that for $d = 2, 3$ that these examples already give a gap between representability and collapsibility and we conjecture that it is so for all $d \geq 2$.

5.2 The Gap between Collapsibility and Leray Number

In this section, for $d \in \mathbb{N}$ we show examples of complexes that are $2d$-Leray but not $(3d-1)$-collapsible.

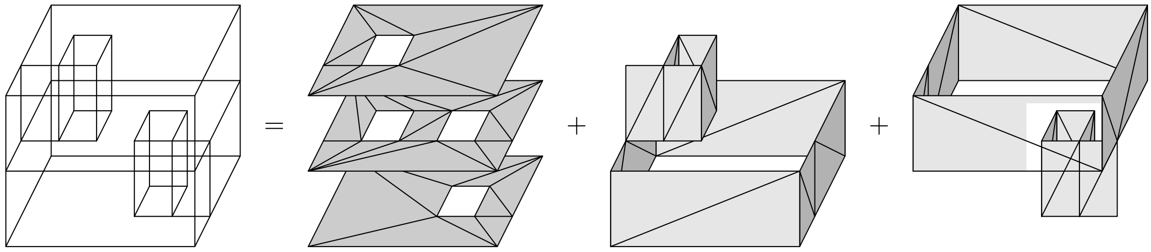

Figure 5.3: A triangulation of Bing's House.

Let $\mathbf{X}$ be a simplicial complex. We say that $\mathbf{X}$ is *weakly d-collapsible* if there exists an elementary d-collapse $\mathbf{X} \searrow^{d} \mathbf{Y}$ for some $\mathbf{Y} \subset \mathbf{X}$. Denote

$$\gamma(\mathbf{X}) = \min\{k \in \mathbb{N} \mid \mathbf{X} \text{ is weakly } k\text{-collapsible}\}.$$

Finiteness of considered complexes implies that $\gamma(\mathbf{X})$ is well defined. Before stating the following lemma let us recall that the term $\lambda(\mathbf{X})$ denotes the Leray number of $\mathbf{X}$ defined in Section 4.3.

Lemma 5.5. *Let $\mathbf{X}$ and $\mathbf{Y}$ be simplicial complexes different from full simplices. Then:*

1. $\lambda(\mathbf{X} * \mathbf{Y}) = \lambda(\mathbf{X}) + \lambda(\mathbf{Y})$.
2. $\gamma(\mathbf{X} * \mathbf{Y}) = \gamma(\mathbf{X}) + \gamma(\mathbf{Y})$.

Proof. 1. The complexes $\mathbf{X}$ and $\mathbf{Y}$ are not full simplices, thus $\lambda(\mathbf{X}) \neq 0$ and $\lambda(\mathbf{Y}) \neq 0$. Let V be the ground set of $\mathbf{X}$ and let W be the ground set of $\mathbf{Y}$. Let $V' \subseteq V$, $W' \subseteq W$ and $Z' = V' \uplus W'$. It is easy to see that $(\mathbf{X} * \mathbf{Y})[Z'] = \mathbf{X}[V'] * \mathbf{Y}[W']$ Thus, according to the Künneth formula:

$$\tilde{H}_k((\mathbf{X} * \mathbf{Y})[Z']) = \bigoplus_{i+j=k-1} \tilde{H}_i(\mathbf{X}[V']) * \tilde{H}_j(\mathbf{Y}[W']).$$

We know that $\tilde{H}_i(\mathbf{X}[V']) = 0$ for $i \geq \lambda(\mathbf{X})$ and $\tilde{H}_j(\mathbf{Y}[W']) = 0$ for $j \geq \lambda(\mathbf{Y})$, hence $\tilde{H}_k((\mathbf{X} * \mathbf{Y})[W']) = 0$ for $k \geq \lambda(\mathbf{X}) + \lambda(\mathbf{Y})$, implying $\lambda(\mathbf{X} * \mathbf{Y}) \leq \lambda(\mathbf{X}) + \lambda(\mathbf{Y})$.

On the other hand, if V' and W' are such that $\tilde{H}_{\lambda(\mathbf{X})-1}(\mathbf{X}[V']) \neq 0$ and also $\tilde{H}_{\lambda(\mathbf{Y})-1}(\mathbf{Y}[W']) \neq 0$ then the Künneth formula gives $\tilde{H}_{\lambda(\mathbf{Y})+\lambda(\mathbf{Y})-1}((\mathbf{X} * \mathbf{Y})[Z']) \neq 0$, implying $\lambda(\mathbf{X} * \mathbf{Y}) \geq \lambda(\mathbf{X}) + \lambda(\mathbf{Y})$.

2. Let $\sigma \in \mathbf{X}$ be a face of dimension $\gamma(\mathbf{X}) - 1$ which is contained in a unique maximal $\sigma' \in \mathbf{X}$. Similarly, let $\tau \in \mathbf{Y}$ be a face of dimension $\gamma(\mathbf{Y}) - 1$ which is contained in a unique maximal $\tau' \in \mathbf{X}$. Then $\sigma' \uplus \tau'$ is easily seen to be the unique maximal face of $\mathbf{X} * \mathbf{Y}$ containing $\sigma \uplus \tau$. The dimension of $\sigma' \uplus \tau'$ is $\gamma(\mathbf{X}) + \gamma(\mathbf{Y}) - 1$. Hence, $\gamma(\mathbf{X} * \mathbf{Y}) \leq \gamma(\mathbf{X}) + \gamma(\mathbf{Y})$.

For the second inequality, suppose that $\theta = \sigma \uplus \tau \in \mathbf{X} * \mathbf{Y}$ is contained in a unique maximal $\theta' = \sigma' \uplus \tau' \in \mathbf{X} * \mathbf{Y}$. Then $\sigma' \in \mathbf{X}$ is the unique maximal face containing $\sigma \in \mathbf{X}$ and $\tau' \in \mathbf{Y}$ is the unique maximal face containing $\tau \in \mathbf{Y}$, and hence $\dim \sigma \geq \gamma(\mathbf{X}) - 1$ and $\dim \tau \geq \gamma(\mathbf{Y}) - 1$. This gives $\dim \theta \geq \gamma(\mathbf{X}) + \gamma(\mathbf{Y}) - 1$, and therefore $\gamma(\mathbf{X} * \mathbf{Y}) \geq \gamma(\mathbf{X}) + \gamma(\mathbf{Y})$.

□

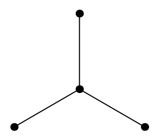

Figure 5.4: A complex that is 1-collapsible but not 1-representable.

Observe that the statement of Lemma 5.5 remains valid even for full simplices if we redefine $\gamma(\mathbf{F}) = 0$ for a full simplex $\mathbf{F}$; however, we are not going to use this lemma for this case.

Let $n \in N$ and $\mathbf{X}$ be a simplicial complex and let

$$\mathbf{X}^{*n} = \underbrace{\mathbf{X} * (\mathbf{X} * \cdots * (\mathbf{X} * (\mathbf{X} * \mathbf{X})) \cdots)}_{n\text{-times } \mathbf{X}}.$$

Note that from the formal definition of the join $\mathbf{X} * (\mathbf{X} * \mathbf{X}) \neq (\mathbf{X} * \mathbf{X}) * \mathbf{X}$; however, these two complexes are isomorphic.

Theorem 5.6. *Let $\mathbf{X}$ be the triangulation of the dunce hat from Figure 5.2 or the triangulation of Bing's house from Figure 5.3 and let $k \in \mathbb{N}$. Then X^{*d} is $2d$-Leray, but it is not $(3d-1)$-collapsible.*

Let us remark that these triangulations are chosen just for concreteness, they could be replaced by others.

Proof. We have $\lambda(X) = 2$, thus due to Lemma 5.5 (1.) X^{*d} is $2d$-Leray. $\gamma(X) = 3$, so Lemma 5.5 (2.) implies that X^{*d} is not weakly $(3d-1)$-collapsible, and hence it cannot be $(3d-1)$-collapsible. □

5.3 The Gap between Representability and Collapsibility (Leray Number)

In this section we give examples of complexes that are d-Leray but not $(2d-2)$-representable. We also show that these examples are d-collapsible for $d \in \{2, 3\}$. These examples are the nerves of simplicial complexes that have dimension $d-1$, but they are not embedable into $\mathbb{R}^{2d-2}$.

We will need several lemmas. Lemma 5.7 is a key tool for showing non-d-representability of possibly d-collapsible complexes. Lemma 5.8 is a technical lemma showing that if we pick some faces of a simplicial complex, then the complex "induced" by these faces is homotopy equivalent to the nerve of these faces. Lemma 5.9 then shows the Leray number of the nerve of faces of a simplicial complex is at most the dimension of the complex exceeded by one.

Lemma 5.7. *Let $\mathbf{X} = (V, \mathbf{K})$ be a simplicial complex such that $\mathbf{N}(\mathbf{K})$ is d-representable. Then $\mathbf{X}$ is embedable into $\mathbb{R}^d$.*

Proof. For $\sigma \in \mathbf{K}$ let $\{C_\sigma \mid \sigma \in K\}$ be a representation of $\mathbf{N}(\mathbf{K})$; $C_\sigma \subseteq \mathbb{R}^d$ are convex sets. For $v \in V$ let p_v be a point belonging to $\bigcap_{\tau; v \in \tau} C_\tau$ (this intersection is nonempty since $\bigcap_{\tau; v \in \tau} \tau$ is nonempty).

For $\sigma \in \mathbf{K}$ let $D_\sigma = \operatorname{conv}\{p_v \mid v \in \sigma\}$. We show that D_σ is a simplex; i.e., the set $\{p_v \mid v \in V\}$ is affinely independent. Let us suppose that this set is affinely dependent. By Radon's Theorem (Theorem 2.1) there are disjoint subsets $\tau, \theta \subseteq \sigma$ such that $D_\tau \cap D_\theta \neq \emptyset$. Then $D_\tau \subseteq C_\tau$ and $D_\theta \subseteq C_\theta$ imply that $C_\tau \cap C_\theta \neq \emptyset$, contradicting the assumption that $\{C_\sigma \mid \sigma \in K\}$ is a representation of $\mathbf{N}(\mathbf{K})$.

If $\sigma, \tau \in \mathbf{K}$, then $D_\sigma \cap D_\tau = D_{\sigma\cap\tau}$: It is clear that $D_{\sigma\cap\tau} \subseteq D_\sigma \cap D_\tau$. To show the second inclusion, let, for contradiction, there be $x \in D_\sigma \cap D_\tau$, but $x \notin D_{\sigma\cap\tau}$. According to Lemma 2.2, there are disjoint $\sigma' \subseteq \sigma$ and $\tau' \subseteq \tau$ such that $D_{\sigma'} \cap D_{\tau'} \neq \emptyset$. However, this implies that $C_{\sigma'} \cap C_{\tau'} \neq \emptyset$, contradicting the assumption that $\{C_\sigma \mid \sigma \in K\}$ is a representation of $\mathbf{N}(\mathbf{K})$. Note that $\sigma', \tau' \in \mathbf{K}$.

From these facts it is easy to see that $\bigcup\limits_{\sigma\in\mathbf{K}} D_\sigma$ is homeomorphic to $\|\mathbf{X}\|$, and thus $\mathbf{X}$ is embedable into $\mathbb{R}^d$.

□

Lemma 5.8. *Let $\mathbf{X} = (V, \mathbf{K})$ be a simplicial complex and let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be a subset of $\mathbf{K}$, $m \in \mathbb{N}$. Let $\mathbf{Z}(S) = (V, \mathbf{L}(S))$ be the simplicial complex, where $\mathbf{L}(S) = \{\sigma \in \mathbf{K} \mid j : 1 \leq j \leq m; \sigma \subseteq \sigma_j\}$. Then $\mathbf{N}(S) \simeq \mathbf{Z}(S)$.*

Proof. Suppose that p_v for $v \in V$ are affinely independent points in $\mathbb{R}^d$, $d \in n$, $d \geq |V|-1$. For $\sigma \in K$ let $D_\sigma = \operatorname{conv}\{p_v \mid v \in \sigma\}$. For simplicity of notation let $D_j = D_{\sigma_j}$ for $1 \leq j \leq m$. Let $\mathcal{D} = \{D_j \mid 1 \leq j \leq m\}$.

As in the proof of Lemma 5.7 we show that $D_\sigma \cap D_\tau = D_{\sigma\cap\tau}$ for $\sigma, \tau \in \mathbf{K}$. It is clear that $D_{\sigma\cap\tau} \subseteq D_\sigma \cap D_\tau$. To show the second inclusion, let, for contradiction, there be $x \in D_\sigma \cap D_\tau$, but $x \notin D_{\sigma\cap\tau}$. According to Lemma 2.2, there are disjoint $\sigma' \subseteq \sigma$ and $\tau' \subseteq \tau$ such that $D_{\sigma'} \cap D_{\tau'} \neq \emptyset$; however, this is a contradiction to the affine independence of the set $\{p_v \mid v \in \sigma' \cup \tau'\}$.

Hence, $\mathbf{N}(S)$ and $\mathbf{N}(\mathcal{D})$ are isomorphic.

According to the Nerve Theorem (Theorem 4.6),

$$\mathbf{N}(\mathcal{D}) \simeq \bigcup_{j=1}^{m} D_j = \bigcup_{\sigma\in\mathbf{L}(S)} D_\sigma = \|\mathbf{Z}(S)\|.$$

□

Lemma 5.9. *Let $\mathbf{X} = (V, \mathbf{K})$ be a simplicial complex. Then $\mathbf{N}(\mathbf{K})$ is $(\dim \mathbf{K} + 1)$-Leray.*

Proof. Let $k = \dim \mathbf{K}$. We wish to prove that $\tilde{H}_i(\mathbf{Y}) = 0$ for every $i \geq k+1$ and $\mathbf{Y} \leq \mathbf{N}(\mathbf{K})$. Let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be the ground set of $\mathbf{Y}$, $m \in \mathbb{N}$, $\sigma_j \in \mathbf{K}$ for $1 \leq j \leq m$. According to Lemma 5.8, $\tilde{H}_i(\mathbf{Y}) = \tilde{H}_i(\mathbf{N}(S)) = \tilde{H}_i(\mathbf{Z}(S))$, where $\mathbf{Z}(S)$ is defined in the statement of the Lemma 5.8. The dimension of $\mathbf{Z}(S)$ is at most k, and thus $\tilde{H}_i(\mathbf{Z}(S)) = 0$ for $i \geq k+1$. □

Let $d \geq 1$ and let $V = \{1, 2, 3, \ldots, 2d+3\}$. Let $\mathbf{VKF}(d) = \left(\mathbf{2}^V\right)^{(d)}$, i.e., the d-skeleton of the $(2d+2)$-dimensional simplex.

We will need the following non-embedability theorem [6], [11], [17]:

Theorem 5.10 (Van Kampen - Flores). *Let $d \geq 1$. Then $\mathbf{VKF}(d)$ cannot be embedded into $\mathbb{R}^{2d}$.*

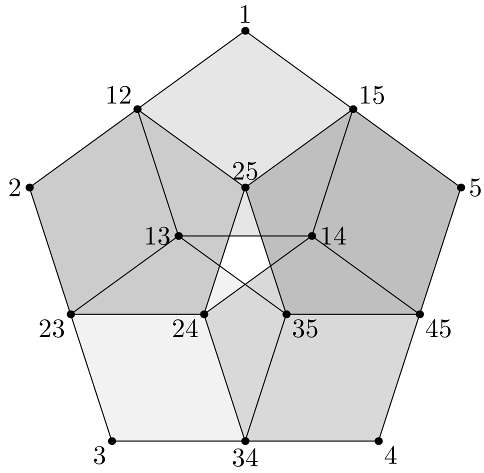

Figure 5.5: The nerve of $\mathbf{VKF}(1)$; each of five smaller pentagons represents a simplex.

Now, we can give an example of a simplicial complex that is d-Leray but not $(2d-2)$-representable.

Theorem 5.11. *Let $d \geq 2$. Let $\mathbf{VKF}(d-1) = (W, \mathbf{L})$. Then the simplicial complex $\mathbf{N}(\mathbf{L})$ is d-Leray but not $(2d-2)$-representable.*

Proof. Lemma 5.7 and Theorem 5.10 imply that the complex $\mathbf{N}(\mathbf{L})$ is not $(2d-2)$-representable. Lemma 5.9 implies that it is d-Leray.

□

Now, we will show that for $d \in \{2, 3\}$ the complex $\mathbf{N}(\mathbf{L})$ from the statement of Theorem 5.11 is d-collapsible.

We introduce some notation. Let $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{Y} \subseteq \mathbf{X}$ be simplicial complexes with a common ground set V. We say that the pair $(\mathbf{X}, \mathbf{Y})$ is *d-collapsible* if there exists a sequence of d-collapses

$$\mathbf{X} = \mathbf{X}_1 \searrow^d \mathbf{X}_2 \searrow^d \cdots \searrow^d \mathbf{X}_k = \mathbf{Y}.$$

We will also need a slightly modified definition of a nerve. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a family of sets. The *nerve′* $\mathbf{N}'(\mathcal{F})$ of this family is a simplicial complex $(V, \mathbf{K})$, where $V = \mathcal{F}$ and for $\sigma \subseteq \{1, 2, \ldots k\}$ we have $\{F_i \mid i \in \sigma\} \in \mathbf{K}$ if and only if $\bigcap_{i \in \sigma} F_i \neq \emptyset$. Let us remark that the complexes $\mathbf{N}(\mathbf{K})$ and $\mathbf{N}'(\mathbf{K})$ are isomorphic, they differ just in the set of vertices.

Lemma 5.12. *Let $\mathbf{X} = (V, \mathbf{K})$ be the 2-skeleton of a full simplex, i.e., $\mathbf{K} = \binom{V}{\leq 2}$. Let $\mathbf{P} = \mathbf{2}^{\mathbf{K}}$, i.e., the full simplex on the ground set $\mathbf{K}$. Then the pair $(\mathbf{P}, \mathbf{N}'(\mathbf{K}))$ is 3-collapsible.*

Proof. Suppose that $V = \{1, 2, \ldots, n\}$. For $\sigma, \tau \in \mathbf{K}$ we say that $\sigma \prec \tau$ if one of the following conditions hold:

- $|\sigma| = |\tau|$ and σ is lexicographically smaller than τ.
- $|\sigma| > |\tau|$.

It is easy to see that $\prec$ is a linear order.

Let $\mathcal{S} = \{\sigma_1, \sigma_2, \sigma_3\}$ and $\mathcal{T} = \{\tau_1, \tau_2, \tau_3\}$ be sets of elements σ_i, $\tau_i \in \mathbf{K}$, $1 \leq i \leq 3$. We say that $\mathcal{S} \lhd \mathcal{T}$ if $\mathcal{S}$ is lexicographically smaller than $\mathcal{T}$ with respect to the order $\prec$. Let

$$\Phi = \{\mathcal{T} \subseteq \mathbf{K} \mid \cap\mathcal{T} = \emptyset, \text{ and } |\mathcal{T}| = 3\}.$$

Suppose

$$\Phi = \{\mathcal{T}_1, \mathcal{T}_2, \dots \mathcal{T}_l\}, \qquad \mathcal{T}_1 \lhd \mathcal{T}_2 \lhd \dots \lhd \mathcal{T}_l.$$

For $i \in \{1, 2, \dots, l\}$ let

$$\Phi_i = \{\mathcal{U} \subseteq \mathbf{K} \mid \exists j \leq i : \mathcal{T}_j \subseteq \mathcal{U}\}.$$

Let $\mathbf{P}_0 = \mathbf{P}$ and let $\mathbf{P}_i = \mathbf{P} \setminus \Phi_i$, for $i \in \{1, 2, \dots, l\}$. We will collapse $\mathbf{P}$ to $\mathbf{P}_l$. We will show that $\mathbf{P}_i$ are simplicial complexes such that $\mathbf{P}_{i-1} \searrow^{d} \mathbf{P}_i$ for $i \in \{1, 2 \dots, l\}$.

Suppose that $i \in \{1, 2, \dots, l\}$; we will show that $\mathcal{T}_i$ is in a unique maximal $\mathcal{V}_i \in \mathbf{P}_{i-1}$ and after 3-collapsing (this face) we get $\mathbf{P}_i$.

We say that $\sigma \in \mathbf{K} \setminus \mathcal{T}_i$ is $\mathcal{T}_i$-*large* if the following condition is satisfied: if $\mathcal{U} \subset \mathcal{T}_i \cup \{\sigma\}$ is such that $|\mathcal{U}| = 3$ and $\cap\mathcal{U} = \emptyset$, then $\mathcal{U} \rhd \mathcal{T}_i$.

Let

$$\mathcal{V}_i = \mathcal{T}_i \cup \{\sigma \in \mathbf{K} \mid \sigma \text{ is } \mathcal{T}_i\text{-large}\}.$$

First, we show that $\mathcal{V}_i$ is a face of $\mathbf{P}_{i-1}$. For contradiction suppose that $\mathcal{V}_i \in \Phi_{i-1}$, i. e., there is $j \leq i-1$ such that $\mathcal{T}_j \subset \mathcal{V}_i$. Let $\mathcal{T}_j = \{\tau_1, \tau_2, \tau_3\}$ and $\mathcal{T}_i = \{\sigma_1, \sigma_2, \sigma_3\}$; $\tau_i \prec \tau_2 \prec \tau_3$, $\sigma_1 \prec \sigma_2 \prec \sigma_3$, and $\mathcal{T}_j \lhd \mathcal{T}_i$. We will distinguish several cases:

- $\tau_1 \prec \sigma_1$.

  In this case $\tau_1 \notin \mathcal{T}_i$, but $\tau_1 \in \mathcal{V}_i$. Hence, τ_1 is $\mathcal{T}_i$-large. Thus, each of the sets $\{\tau_1, \sigma_1, \sigma_2\}$, $\{\tau_1, \sigma_1, \sigma_3\}$, and $\{\tau_1, \sigma_2, \sigma_3\}$ has to have a nonempty intersection, let $x_1 \in \{\tau_1, \sigma_1, \sigma_2\}$, $x_2 \in \{\tau_1, \sigma_1, \sigma_3\}$, and $x_3 \in \{\tau_1, \sigma_2, \sigma_3\}$. Since $\sigma_1 \cap \sigma_2 \cap \sigma_3 = \emptyset$, the points x_1, x_2 and x_3 are pairwise distinct. This is a contradiction, since $|\tau_i| \leq 2$.

- $\tau_1 = \sigma_1$ and $\tau_2 \prec \sigma_2$.

  In this case τ_2 is $\mathcal{T}_i$-large. Since $\{\sigma_1, \tau_2, \sigma_2\} \lhd \mathcal{T}_i$ and $\{\sigma_1, \tau_2, \sigma_3\} \lhd T_i$, these sets have to have a nonempty intersection. Let $x \in \sigma_1 \cap \tau_2 \cap \sigma_2$ and let $y \in \sigma_1 \cap \tau_2 \cap \sigma_3$. Since $\sigma_1 \cap \sigma_2 \cap \sigma_3 = \emptyset$, we have $x \neq y$. Thus, $\tau_1 = \sigma_1 = \tau_2 = \{x, y\}$, a contradiction.

- $\tau_1 = \sigma_1$, $\tau_2 = \sigma_2$ and $\tau_3 \prec \sigma_3$.

  In this case τ_3 is $\mathcal{T}_i$-large. We know that $\mathcal{T}_j \subset \mathcal{T}_i \cup \{\tau_3\}$, $\cap\mathcal{T}_j = \emptyset$, but $\mathcal{T}_j \lhd T_i$, a contradiction.

Now we show that $\mathcal{V}_i$ is a unique maximal face to which $\mathcal{T}_i$ belongs. If $\sigma \in \mathbf{K}$ and $\sigma \notin V_i$ then $\sigma \notin T_i$ and there exist $\mathcal{U} \subset \mathcal{T}_i \cup \{\sigma\}$ such that $|\mathcal{U}| = 3$, $\cap U = \emptyset$, and $\mathcal{U} \lhd \mathcal{T}_i$. Thus, $\mathcal{U} = \mathcal{T}_j$ for $j \leq i-1$. Hence, $\sigma \cup V_i$ is not a face of $\mathbf{P}_{i-1}$ implying maximality of $\mathcal{V}_i$. Now suppose that $\tau \in \mathbf{K} \setminus \mathcal{T}_i$ be such that $\mathcal{T}_i \cup \{\tau\} \in P_{i-1}$. According to the definition of Φ_{i-1} there is no $j \leq i-1$ such that $\mathcal{T}_j \in \Phi$ (i.e., $\cap\mathcal{T}_j = \emptyset$ and $|\mathcal{T}_j| = 3$), and $\mathcal{T}_j \subset \mathcal{T}_i \cup \{\tau\}$, thus τ is $\mathcal{T}_i$-large, and hence it belongs to $\mathcal{V}_i$. We conclude that V_i is unique.

From the definition of Φ_l we get that the set of faces of $\mathbf{P}_l$ is the set

$$\{\mathcal{U} \subseteq \mathbf{K} \mid \nexists\mathcal{T} \in \Phi; \mathcal{T} \subseteq \mathcal{U}\} = \{\mathcal{U} \subseteq \mathbf{K} \mid \cap\mathcal{U} \neq \emptyset \text{ or } |\mathcal{U}| \leq 2\},$$

since it is easy to see that if $\mathcal{U} \subseteq \mathbf{K}$, $|\mathcal{U}| \geq 3$, and $\cap\mathcal{U} = \emptyset$ then there exists $\mathcal{T} \subseteq \mathcal{U}$ such that $|\mathcal{T}| = 3$ and $\cap\mathcal{T} = \emptyset$.

To get $\mathbf{N}'(\mathbf{K})$ from $\mathbf{P}_l$ it is sufficient to 3-collapse (even 2-collapse) faces $\mathcal{U} \subseteq K$, $\cap\mathcal{U} = \emptyset$, $|\mathcal{U}| = 2$ in any order and as a last one to 3-collapse $\{\emptyset\}$.

□

Proposition 5.13. *Let* $d \in \{2, 3\}$. *Let* $\mathbf{VKF}(d-1) = (W, \mathbf{L})$. *Then the simplicial complex* $\mathbf{N}(\mathbf{L})$ *is* d*-collapsible.*

Proof. We will collapse $\mathbf{N}'(\mathbf{L})$ since it is isomorphic to $\mathbf{N}(\mathbf{L})$. It is easy to 2-collapse $\mathbf{N}'(\mathbf{L})$ for $d = 2$. Thus we prove the statement just for $d = 3$. Let $v \in W$ and let $V_v = W \setminus \{v\}$. Let $\mathbf{J}_v = \{\sigma \mid v \in \sigma \in \mathbf{L}\}$ and let $\mathbf{K}_v = \{\sigma \setminus \{v\} \mid \sigma \in \mathbf{J}_v\}$. The nerve′ $\mathbf{N}'(\mathbf{J}_v)$ is an induced subcomplex of $\mathbf{N}'(\mathbf{L})$; it is a full simplex. Then $\mathbf{K}_v = \binom{V_v}{\leq 2}$ meeting the conditions of the Lemma 5.12. Let $\mathbf{P} = \mathbf{2}^{\mathbf{K}_v}$ and let

$$\mathbf{P} = \mathbf{P}_0 \overset{3}{\searrow} \mathbf{P}_1 \overset{3}{\searrow} \cdots \overset{3}{\searrow} \mathbf{P}_l = \mathbf{N}'(\mathbf{K}_v)$$

be a sequence of 3-collapses obtained by Lemma 5.12.

Let $\mathbf{R}_0 = \mathbf{2}^{\mathbf{J}_v} = \mathbf{N}'(\mathbf{J}_v)$. To each 3-collapse $\mathbf{P}_{i-1} \overset{3}{\searrow} \mathbf{P}_i$ when collapsing $\mathcal{T} \in \mathbf{P}_{i-1}$ let us associate the 3-collapse $\mathbf{R}_{i-1} \overset{3}{\searrow} \mathbf{R}_i$ when collapsing $\mathcal{T}' = \{\sigma \cup \{v\} \mid \sigma \in \mathcal{T}\}$; then the set of the faces of $\mathbf{R}_l$ is the set $\{\mathcal{T} \subseteq \mathbf{J}_v \mid \cap\mathcal{T} \supsetneq \{v\}\} = \{\mathcal{T} \subseteq \mathbf{J}_v \mid 2 \leq |\cap\mathcal{T}|\}$.

The collapsing $\mathbf{N}'(\mathbf{J}_v)$ to $\mathbf{R}_l$ can be seen as a collapsing of the whole of $\mathbf{N}'(\mathbf{L})$, since faces $\sigma \in \mathbf{L} \setminus \mathbf{J}_v$ do not contain v and thus they do not affect the property of faces $\tau \in \mathbf{J}_v$ to belong to a unique maximal face. Moreover, when collapsing $\mathbf{R}_{i-1} \overset{3}{\searrow} \mathbf{R}_i$ just faces $\mathcal{T}$ such that $\cap\mathcal{T} = \{v\}$ are collapsed. Thus, the collapsing of $\mathbf{N}'(\mathbf{J}_v)$ does not affect $\mathbf{N}'(\mathbf{J}_w)$ for $w \neq v$. These collapsings can be performed (in any order) for all $v \in W$ concluding that $\mathbf{N}'(\mathbf{L})$ 3-collapses to the simplicial complex

$$\mathbf{Y} = \left(\mathbf{L}, \{\mathcal{T} \subseteq \mathbf{L} \mid 2 \leq |\cap\mathcal{T}|\}\right).$$

It is already easy to 3-collapse (even 2-collapse) the simplicial complex $\mathbf{Y}$. For example, first 1-collapse all the faces $\{x, y\}$, $x, y \in W$. Then 2-collapse remaining 2-faces, e.g., lexicographically, and finally 1-collapse remaining isolated 1-faces.

□

Conjecture 5.14. *Let* $d \geq 2$. *Let* $\mathbf{VKF}(d-1) = (W, \mathbf{L})$. *Then the simplicial complex* $\mathbf{N}(\mathbf{L})$ *is* d*-collapsible.*

Chapter 6

The Combinatorial Alexander Duality

In this chapter we state the Alexander duality in a form for simplicial complexes and give an easy proof of it. This proof is based on an unpublished slightly modified idea of Björner [4]. An alternative proof, using Tor-functors, is in [12].

Throughout this chapter $\mathbb{K}$ is a fixed field and all the homology groups are over $\mathbb{K}$.

6.1 An Introduction to the Alexander Duality

One of the useful tools for topological proofs of combinatorial statements is the Alexander duality for simplicial complexes; see, e.g., [9] for a nice application. Another tools can be found in [3].

Let $\mathbf{X} = (V, \mathbf{K})$ be a simplicial complex. For $\sigma \in \mathbf{K}$ let $\overline{\sigma} = V \setminus \sigma$. The *Alexander dual* of $\mathbf{X}$ is defined as

$$\mathbf{X}^* = (V, \{\sigma \subseteq V \mid \overline{\sigma} \notin \mathbf{K}\}) .$$

See Figure 6.1 for an example of a simplicial complex and its Alexander dual. It is easy to see that $\mathbf{X}^{**} = \mathbf{X}$. The Alexander duality states that the knowledge of the homology of a simplicial complex gives the knowledge of the homology of its Alexander dual:

Theorem 6.1 (Combinatorial Alexander Duality). *Let* $\mathbf{X}$ *be a simplicial complex with a ground set of the size* n *then*

$$\tilde{H}_i(\mathbf{X}) = \tilde{H}_{n-i-3}(\mathbf{X}^*) .$$

Remark 6.2. The Alexander duality is usually stated for homology on one side of the equality and cohomology on the second one. We state it here (as stated in [9] or [12]) for homology on both sides since homology and cohomology groups over fields are isomorphic.

6.2 The Idea of the Proof

First we present the idea of the proof of Theorem 6.1.

Suppose that $\mathbf{X}$ is a simplicial complex with a ground set V. Let Γ be the lattice of all subsets of V and let $\Gamma_{\mathbf{X}}$ be the sublattice of Γ corresponding to the subsets that are

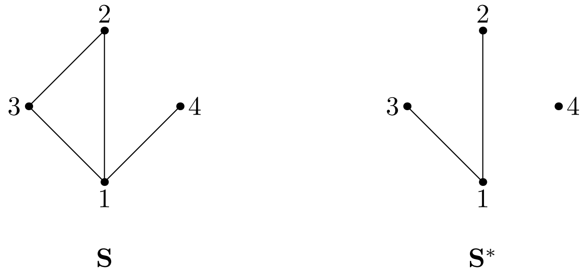

Figure 6.1: Simplicial complex $\mathbf{S}$ and its dual.

in $\mathbf{X}$. Then the n-th homology group of $\mathbf{X}$ depends just on the n-th and $(n+1)$-st levels of $\Gamma_{\mathbf{X}}$. It is easy to see that $\tilde{H}_i(\mathbf{X}) = \tilde{H}_{i+1}\left(\mathbf{2}^V, \mathbf{X}\right)$ (see Lemma 6.4 in the next section). Thus, we restate the problem as computing homologies of the chain complex determined by the complement of $\Gamma_{\mathbf{X}}$ - in the sense of Remark 3.2.

The original idea [4] of the proof is that if we turn the lattice upside down (change $\sigma \subseteq V$ with its complement), then we get a canonical isomorphism between the relative homology of the pair $(\mathbf{2}^V, \mathbf{X})$ and the cohomology of $\mathbf{X}^*$. This idea is basically correct; however, the isomorphism is not canonical — some signs operations are necessary.

Example 6.3. Let $\mathbf{S}$ be the simplicial complex in Figure 6.1. Its ground set is the set $V_{\mathbf{S}} = \{1, 2, 3, 4\}$. The lattices $\Gamma_{\mathbf{S}}$ and $\Gamma_{\mathbf{S}^*}$ are depicted in Figure 6.2, and the left part of the picture also shows the complement of $\Gamma_{\mathbf{S}}$ (bold, dashed) determining the homology of $(\mathbf{2}^{V_{\mathbf{S}}}, \mathbf{S})$.

In the sense of Remark 3.2, the chain complex $\tilde{\mathcal{C}}_{\circledast}\left(\mathbf{2}^{V_{\mathbf{S}}}, \mathbf{S}\right)$ is the complex

$$\cdots \longleftarrow 0 \longleftarrow \langle e_{24}, e_{34}\rangle \xleftarrow{\partial_2} \langle e_{123}, e_{124}, e_{134}, e_{234}\rangle \xleftarrow{\partial_3} \langle e_{1234}\rangle \longleftarrow 0 \longleftarrow \cdots$$

The chain complex $\tilde{\mathcal{C}}^{\circledast}(\mathbf{S}^*)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow \langle e_0^*\rangle \xrightarrow{\partial^0} \langle e_1^*, e_2^*, e_3^*, e_4^*\rangle \xrightarrow{\partial^1} \langle e_{12}^*, e_{13}^*\rangle \longrightarrow 0 \longrightarrow \cdots$$

The map $e_\sigma \to e^*_{\overline{\sigma}}$ is not an isomorphism of these two complexes (if the characteristic of $\mathbb{K}$ is different from two), since $\partial_2(e_{234}) = -e_{24} + e_{34}$, while $\partial^1(e_1^*) = e_{12}^* + e_{13}^*$. Nevertheless, these two complexes are isomorphic, as will be shown in the next section.

6.3 The Proof of the Alexander Duality

Lemma 6.4. *Let $\mathbf{X}$ be a simplicial complex with the ground set V. Then*

$$\tilde{H}_i(\mathbf{X}) = \tilde{H}_{i+1}\left(\mathbf{2}^V, \mathbf{X}\right).$$

Proof. The proof follows from Lemma 3.4. There is the long exact sequence of the pair $(\mathbf{2}^V, \mathbf{X})$:

$$\cdots \longrightarrow \tilde{H}_{i+1}\left(\mathbf{2}^V\right) \longrightarrow \tilde{H}_{i+1}\left(\mathbf{2}^V, \mathbf{X}\right) \longrightarrow \tilde{H}_i(\mathbf{X}) \longrightarrow \tilde{H}_i\left(\mathbf{2}^V\right) \longrightarrow \cdots$$

The groups $\tilde{H}_{i+1}\left(\mathbf{2}^V\right)$ and $\tilde{H}_i\left(\mathbf{2}^V\right)$ are zero, implying that the groups $\tilde{H}_{i+1}\left(\mathbf{2}^V, \mathbf{X}\right)$ and $\tilde{H}_i(\mathbf{X})$ are isomorphic.

□

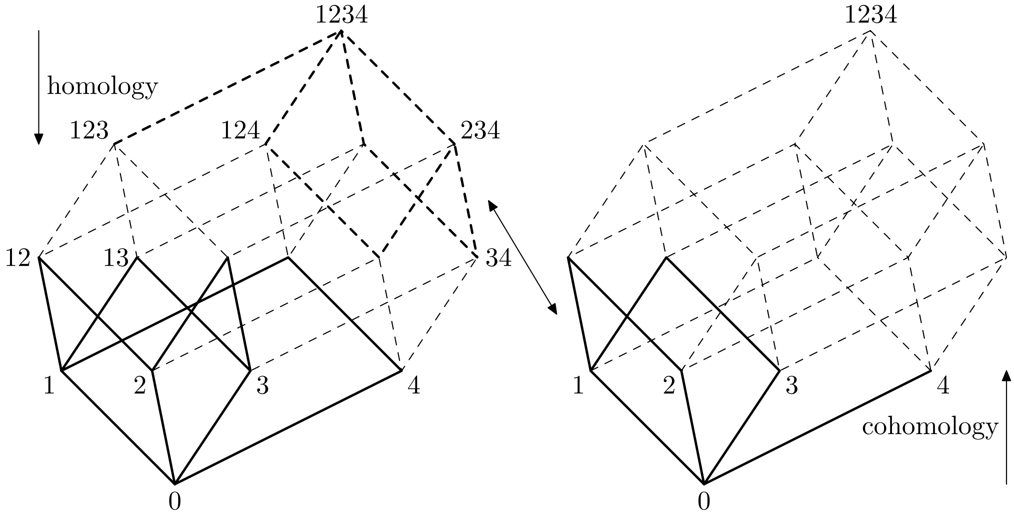

Figure 6.2: The lattices $\Gamma_{\mathbf{S}}$ (left) and $\Gamma_{\mathbf{S}^*}$ (right).

Now we extend the definition of $\operatorname{sgn}(k,\sigma)$ for $k \in \sigma \subseteq \{1,2,\ldots,n\}$. Suppose $\tau \subseteq \sigma \subseteq \{1,2,\ldots,n\}$. For $i \in \tau$, let $o(i)$ be such that i is the $o(i)$-th smallest element of σ. We define

$$\operatorname{sgn}(\tau,\sigma) = \prod_{i\in\tau} (-1)^{o(i)-i}.$$

Observe that $\operatorname{sgn}(k,\sigma) = \operatorname{sgn}(\{k\},\sigma)$ for $k \in \sigma \subseteq V$.

Lemma 6.5. *Let $V = \{1,2,\ldots,n\}$, $\sigma \subseteq V$ and $k \in \sigma$. Then*

$$\operatorname{sgn}(k,\sigma)\operatorname{sgn}(\sigma \setminus k, V) = (-1)^{|\sigma|+1}\operatorname{sgn}(k,\overline{\sigma}\cup k)\operatorname{sgn}(\sigma,V).$$

Proof. Suppose that $\sigma = \{p_1,p_2,\ldots,p_j,k,p_{j+1},p_{j+2},\ldots,p_t\}$, where $p_i < p_j$ for $i < j$ and $p_j < k < p_{j+1}$. Then

$$\begin{aligned}
\operatorname{sgn}(k,\sigma) &= (-1)^j,\\
\operatorname{sgn}(\sigma\setminus k, V) &= \prod_{i=1}^{t}(-1)^{p_i-i},\\
\operatorname{sgn}(k,\overline{\sigma}\cup k) &= (-1)^{p_1-1}\cdot\left(\prod_{i=2}^{j}(-1)^{p_i-p_{i-1}-1}\right)\cdot(-1)^{k-j-1} = (-1)^{k-(j+1)},\\
\operatorname{sgn}(\sigma,V) &= \left(\prod_{i=1}^{j}(-1)^{p_i-i}\right)\cdot(-1)^{k-(j+1)}\cdot\left(\prod_{i=j+1}^{t}(-1)^{p_i-(i+1)}\right) =\\
&= (-1)^{k+t-1}\left(\prod_{i=1}^{t}(-1)^{p_i-i}\right).
\end{aligned}$$

These equalities immediately imply the lemma (note that $|\sigma| = t+1$).

□

Lemma 6.6. *Let $\mathbf{X}$ be a simplicial complex with the ground set V with n elements. Then*

$$\tilde{H}_{i+1}\left(\mathbf{2}^V,\mathbf{X}\right) = \tilde{H}_{n-i-3}\left(\mathbf{X}^*\right).$$

Proof. The idea of the proof is to describe the chain complex for reduced homology of the pair $(\mathbf{2}^V,\mathbf{X})$ and the chain complex for reduced cohomology of $\mathbf{X}$ and to show that these two complexes are isomorphic.

Suppose that $V = \{1, 2, \dots, n\}$. The chain complex for reduced homology of the pair $(\mathbf{2}^V, \mathbf{X})$ is the complex

$$\cdots \xleftarrow{\partial_{j-1}} R_{j-1} \xleftarrow{\partial_j} R_j \xleftarrow{\partial_{j+1}} \cdots, \qquad j \in \mathbb{Z},$$

where

$$R_j = \langle e_\sigma \mid \sigma \subseteq V, \sigma \notin \mathbf{X}, \dim \sigma = j \rangle$$

and ∂_j are the unique homomorphisms satisfying

$$\partial_j(e_\sigma) = \sum_{\substack{k \in \sigma \\ \sigma \setminus k \notin \mathbf{X}}} \operatorname{sgn}(k, \sigma) e_{\sigma \setminus k}.$$

The chain complex for reduced cohomology of $\mathbf{X}$ is the complex

$$\cdots \xrightarrow{\partial^{j-1}} L_{j-1} \xrightarrow{\partial^j} L_j \xrightarrow{\partial^{j+1}} \cdots, \qquad j \in \mathbb{Z},$$

where

$$L_j = \langle e^*_\sigma \mid \sigma \subseteq V, \dim \sigma = j, \sigma \in \mathbf{X}^* \rangle = \langle e^*_\sigma \mid \sigma \subseteq V, \dim \overline{\sigma} = n - j - 2, \overline{\sigma} \notin \mathbf{X} \rangle$$

and ∂^j are the unique homomorphisms satisfying

$$\partial^j\,(e^*_\sigma) = \sum_{\substack{k \notin \sigma \\ k \cup \sigma \in \mathbf{X}^*}} \operatorname{sgn}(k, \sigma \cup k) e^*_{\sigma \cup k} = \sum_{\substack{k \in \overline{\sigma} \\ \overline{\sigma} \setminus k \notin \mathbf{X}}} \operatorname{sgn}(k, \sigma \cup k) e^*_{\overline{\overline{\sigma} \setminus k}}.$$

Let $c(\sigma) = (-1)^{\frac{|\sigma|(|\sigma|-1)}{2}}$ for $\sigma \subseteq V$. Let $\phi_j : R_j \to L_{n-j-2}$ be the isomorphisms generated by formula

$$\phi_j(e_\sigma) = c(\sigma) \operatorname{sgn}(\sigma, V) e^*_{\overline{\sigma}}$$

for $\sigma \notin \mathbf{X}$, $\dim \sigma = j$ (note that these two conditions are equivalent to $\dim \overline{\sigma} = n - j - 2$, $\overline{\sigma} \in \mathbf{X}^*$).

$$\begin{array}{ccccccc}
\cdots & \xleftarrow{\partial_{j-1}} & R_{j-1} & \xleftarrow{\partial_j} & R_j & \xleftarrow{\partial_{j+1}} & \cdots \\
 & & \downarrow \phi_{j-1} & \circ & \downarrow \phi_j & & \\
\cdots & \xleftarrow{\partial^{n-j}} & L_{n-j-1} & \xleftarrow{\partial^{n-j-1}} & L_{n-j-2} & \xleftarrow{\partial^{n-j-2}} & \cdots
\end{array}$$

We check that $\phi_{j-1} \circ \partial_j = \partial^{n-j-1} \circ \phi_j$. Let $\sigma \subseteq V$, $\sigma \notin \mathbf{X}$, $\dim \sigma = j$. Let us compute

$$\phi_{j-1} \circ \partial_j(e_\sigma) = \phi_{j-1}\left(\sum_{\substack{k \in \sigma \\ \sigma \setminus k \notin \mathbf{X}}} \operatorname{sgn}(k, \sigma) e_{\sigma \setminus k} \right) = \sum_{\substack{k \in \sigma \\ \sigma \setminus k \notin \mathbf{X}}} \operatorname{sgn}(k, \sigma) c(\sigma \setminus k) \operatorname{sgn}(\sigma \setminus k, V) e^*_{\overline{\sigma \setminus k}},$$

$$\partial^{n-j-1} \circ \phi_j(e_\sigma) = \partial^{n-j-1}\left(c(\sigma) \operatorname{sgn}(\sigma, V) e^*_{\overline{\sigma}}\right) = \sum_{\substack{k \in \sigma \\ \sigma \setminus k \notin \mathbf{X}}} c(\sigma) \operatorname{sgn}(k, \overline{\sigma} \cup k) \operatorname{sgn}(\sigma, V) e^*_{\overline{\sigma \setminus k}}.$$

These two terms are equal due to Lemma 6.5 and the easy fact $c(\sigma) = (-1)^{|\sigma|+1} c(\sigma \setminus k)$.

Thus ϕ is an isomorphism of the complexes implying

$$\tilde{H}_{i+1}\left(\mathbf{2}^V, \mathbf{X}\right) = \tilde{H}^{n-i-3}(\mathbf{X}^*) = \tilde{H}_{n-i-3}\left(\mathbf{X}^*\right).$$

□

Proof of Theorem 6.1. Use Lemma 6.4 and Lemma 6.6. □

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