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DIPLOMOVÁ PRÁCE

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Struktury určené 3-formami na 7-dimenzionálních varietách

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Rád bych poděkoval svému vedoucímu panu Vanžurovi za to, že mě uvedl do krásného světa algebraické topologie, a za jeho ochotu a pomoc při psaní této práce. Také děkuji Hong Van Le za informace o grupě $\widetilde{G}_2$.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Kladně dne 18.4.2007 Martin Doubek

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Abstrakt : Souběžně rozvíjíme základy teorie oktonionů a split-oktonionů. Získané poznatky využijeme ke studiu topologických vlastností grup G_2 a $\tilde{G}_2$. Tyto pak dále použijeme na odvození některých existenčních vět pro 3-formy typu 8 (G_2) a 3 na 7-dimenzionálních varietách.
Klíčová slova : 3-formy, oktoniony, G_2, $\tilde{G}_2$, Postnikovova věž

Title : Structures defined by 3-forms on 7-dimensional manifolds
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Abstract : We present a parallel development of the basic theory of the octonions and the split-octonions. We use the results thus obtained to invesigate topological properties of the groups G_2 and $\tilde{G}_2$. Finally we obtain some existence results for the 3-forms of type 8 (G_2) and 3 on 7-manifolds.
Keywords : 3-forms, octonions, G_2, $\tilde{G}_2$, Postnikov tower

1 Introduction

In [4] the authors present a classification of 3-forms on a 7-dimensional vector space V and derive some of their properties. In fact $\wedge^3 V$ consists of 14 orbits under the $GL(7)$-action. Two of them are open and represent important forms :

- type 8 - the stabilizer of this form is the compact real form of the exceptional Lie group G_2.
- type 5 - the stabilizer of this form is the noncompact real form of the exceptional Lie group G_2, we denote it $\tilde{G}_2$.

Manifolds carrying forms of these types has recently attracted some attention in physics and the exceptional G_2-geometry is being studied.

In this work we are concerned with the existence of such forms on 7-manifolds. Moreover we study the existence problem for the form of type 3.

The first step is to consider the continuous case only. We apply the methods of the obstruction theory as developped in [11], namely we reformulate the existence problem to a lifting problem and then we attempt to construct a partial lift through the Postnikov tower of the fibration in question. This is very straightforward, but requires hardly accesible information - computing homotopy groups of the fibers and cohomologies of the classifying spaces and namely of succesive stages of the Postnikov tower may turn out to be a problem.

We are able to solve only G_2 case completely. This result is known since 1967 due to Gray, see [7].

For $\tilde{G}_2$ we have obtained only some fibrations which are however not sufficient even to solve the existence problem partially.

For the case of type 3 form on an orientable manifold, we obtain sufficient condition and under certain simplifying hypotheses we obtain necessary condition. These results follow from the sufficient and necessary conditions for the existence of an almost complex structure on a 6-dimensional vector bundle over a 7-manifold which we derive.

On the way we also obtain some classical results on the existence of certain structures on manifolds expressed in terms of characteristic classes, namely orientability and nowhere vanishing vetor field. Few more, e.g. Stiefel theorem on parallelizability of orientable 3-manifolds or existence of Riemannian metric, are easily accesible by the methods used, but they are of no immediate importance for this work and we ommit them.

Substantial part of the paper concerns the algebra of octonions $\mathbb{O}$ and its split-partner $\tilde{\mathbb{O}}$. This is necessary to get some understanding of the group G_2, since it consists of the algebra automorphisms of $\mathbb{O}$. Especially to get the identification $SO(7)/G_2 \cong \mathbb{R}P^7$ takes some effort. There is an alternative approach via *Spin* groups (see e.g. [1]), but we prefer to avoid it here and exploit the knowledge of the octonions as much as possible. From the very beginning we develop the theory for $\mathbb{O}$ and $\tilde{\mathbb{O}}$ parallely, since the proofs are usually formally identical.

With facts gathered in the first 2 chapters we attack the topological existence problems in the last 2 chapters.

2 Octonions and split-octonions

2.1 $*$-Algebras

Here we collect necessary definitions and basic properties of $*$-algebras. From the very beginning we restrict our attention to good algebras (see 2.1). We discuss various types of associativity in some details. We first derive an important associator lemma 2.9 and its consequences : a theorem 2.12 characterising alternativity and an useful formula 2.11 relating scalar product and multiplication. These 2 results are going to be used frequently.

2.1 Definition. Through the text, algebra is always a *real* algebra *with unit* (not necessarily associative or commutative). An algebra morphism is a linear map $f : A \to B$ satisfying

$$f(ab) = f(a)f(b) \text{ and } f(1_A) = 1_B \quad \text{for all } a, b \in A$$

where $1_A, 1_B$ denote the units in the corresponding algebras.

A is called **alternative** iff every subalgebra generated by any 2 elements is associative. A is called **flexible** iff

$$(ab)a = a(ba)$$

for all $a, b \in A$.

A is called **division algebra** iff A has no zero divisors.

We say A has **multiplicative inverses** if for every $a \in A - \{0\}$ there exists $b \in A$ such that $ab = ba = 1$.

$*$-algebra (a.k.a. **algebra with conjugation**) is an algebra A equipped with a map

$$* : A \to A$$

called **conjugation** satisfying

1. $*$ is linear

2. $*(ab) = *(b) * (a)$

3. $** = \mathrm{id}$

$*(a)$ is usually denoted a^* and it is a straightforward generalisation of the notion of complex conjugation.

An algebra morphism $f : A \to B$ is called $*$-algebra morphism iff

$$f(a^*) = f(a)^* \quad \text{for all } a \in A$$

$*$-algebra A is called **real** iff $a^* = a$ for all $a \in A$.

Notice that every algebra comes with the canonical inclusion $\mathbb{R} \hookrightarrow A$ via $r \mapsto r1$, where 1 is the unit in A. This justifies the following definition : $*$-algebra A is called **good** iff $a + a^* \in \mathbb{R}$. Notice that this implies $aa^*, a^*a \in \mathbb{R}$ for all $a \in A$: indeed $aa^* = \frac{1}{2}(aa^* + (aa^*)^*) \in \mathbb{R}$. In case A is good we define the **real part** $\mathrm{Re}(a) = \frac{1}{2}(a + a^*)$ and the **imaginary part** $\mathrm{Im}(a) = \frac{1}{2}(a - a^*)$ of $a \in A$. Denote $\mathrm{Im}(A) := \{a \in A : \mathrm{Im}(a) = a\}$ the subspace of imaginary elements. Notice that $a \in \mathrm{Im}(A)$ is equivalent to $a^* = -a$.

2.2 Lemma. Let A be a good algebra. Then

$$\langle a, b \rangle := \frac{1}{2}(ab^* + ba^*) = \mathrm{Re}(ab^*) \tag{1}$$

for $a, b \in A$ defines a symmetric bilinear form on A. Define [1]

$$\|a\|^2 := \langle a, a \rangle = aa^* = a^*a$$

Proof. Because A is good we have $\langle a, b \rangle = \frac{1}{2}(ab^* + (ab^*)^*) \in \mathbb{R}$. The symmetry is obvious.

To verify $aa^* = a^*a$ let r be real and i imaginary part of a. Then $aa^* = (r + i)(r - i) = r^2 - i^2 = (r - i)(r + i) = a^*a$. □

2.3 Definition. Let A be a good algebra. We say A has **signature** $(a+, b-, c)$ iff the bilinear form (1) has the signature. We call A **nondegenerate** (resp. **positive definite**) iff $c = 0$ (resp. moreover $b = 0$).

[1]The following expression is formal - generally $\|a\|^2$ doesn't have to be nonnegative and therefore $\|a\|$ is not even defined. On the other hand if A is positive definite, we use $\|a\| := \sqrt{\|a\|^2}$ in the usual sense.

2.4 Lemma. Let A be a positive definite algebra. Then (1) defines a positive definite scalar product, hence $\|\| := \sqrt{\|\|^2}$ is a norm. A has multiplicative inverses explicitly given by

$$a^{-1} := \frac{a^*}{\|a\|^2}$$

2.5 Remark. In the nonassociative case, strange things may occur. For example the existence of multiplicative inverses no longer implies the nonexistence of zero divisors. We give example of such an algebra later in 2.32.

Not even the converse implication is true - there is a division algebra without multiplicative inverses. A simple example can be found in [3].

2.6 Definition. Let A be an algebra. The **commutator** is the (obviously bilinear) map $A^2 \to A$ given by

$$[a,b] := ab - ba \quad \text{for all } a,b \in A$$

The **associator** is the (obviously trilinear) map $A^3 \to A$ given by

$$(a,b,c) := (ab)c - a(bc) \quad \text{for all } a,b,c \in A$$

2.7 Lemma. Let A be a good algebra. For $a,b \in \operatorname{Im}(A)$ we have

$$ab + ba = -2\langle a,b\rangle$$

Namely $a^2 = -\|a\|^2 \in \mathbb{R}$.

Proof. $ab + ba = -ab^* - ba^* = -2\langle a,b\rangle$. □

2.8 Lemma. Let A be a good algebra. For $a,b,c \in A$ we have

1. $[a,b] = -[a^*,b] = -[a,b^*]$
2. $(a,b,c) = -(a^*,b,c) = -(a,b^*,c) = -(a,b,c^*)$

Proof. 1. If one of a,b is real then obviously $[a,b]=0$. Hence

$$[a,b] = [\operatorname{Re}(a),b] + [\operatorname{Im}(a),b] = [\operatorname{Im}(a),b]$$

and therefore

$$[a^*,b] = [\operatorname{Im}(a^*),b] = [-\operatorname{Im}(a),b] = -[a,b]$$

and similarly for the other variable.

2. This is very similar to the previous case of the commutator. If one of a, b, c is real then obviously $(a, b, c) = 0$. Hence

$$(a^*, b, c) = (\operatorname{Im}(a^*), b, c) = (-\operatorname{Im}(a), b, c) = -(a, b, c)$$

2.9 Lemma. Let A be a good algebra. For $a, b, c \in A$ we have

1. $\operatorname{Re}([a, b]) = 0$
2. $\operatorname{Re}((a, b, c)) = 0$ iff A is flexible.

Proof. 1. By 2.8 we get

$$2\operatorname{Re}([a, b]) = ab - ba + b^*a^* - a^*b^* = [a, b] + [b^*, a^*] = [a, b] + [b, a] = 0$$

2. By 2.8 we get

$$\begin{aligned} 2\operatorname{Re}((a, b, c)) &= (ab)c - a(bc) + c^*(b^*a^*) - (c^*b^*)a^* = \\ &= (a, b, c) - (c^*, b^*, a^*) = (a, b, c) + (c, b, a) = 0 \end{aligned}$$

where the flexibility is equivalent to the last equality via polarization (see the proof of 2.12). □

2.10 Remark. Let A be a good algebra and $a \in A$. Then $a, a^*, \operatorname{Re}(a), \operatorname{Im}(a)$ all lie in the subalgebra generated by a (or a^* or $\operatorname{Im}(a)$).

2.11 Theorem. Let A be a good alternative algebra. For $a, b, u \in A$ we have

$$\langle ua, ub \rangle = \langle au, bu \rangle = \|u\|^2 \langle a, b \rangle$$

and

$$\langle ua, b \rangle = \langle a, u^*b \rangle$$
$$\langle au, b \rangle = \langle au^*, b \rangle$$

Proof.

$$\langle au, b \rangle = \operatorname{Re}((au)b^*) = \operatorname{Re}(a(ub^*)) = \langle a, bu^* \rangle$$

by 2.9, 2. Hence

$$\langle au, bu \rangle = \langle a, (bu)u^* \rangle = \langle a, b(uu^*) \rangle = \|u\|^2 \langle a, b \rangle$$

by the previous remark. Next

$$\langle ua, b \rangle = \operatorname{Re}((ua)b^*) = \operatorname{Re}(u(ab^*)) = \operatorname{Re}((ab^*)u) = \operatorname{Re}(a(b^*u)) = \langle a, u^*b \rangle$$

by 2.9, 1. and 2. We get $\langle ua, ub \rangle = \|u\|^2 \langle a, b \rangle$ as before. □

2.12 Theorem. For a good algebra A the following statements are equivalent :

1. A is alternative.
2. The associator on A is skew-symmetric.
3. The following formulas hold for all $a, b \in A$

$$\begin{aligned}(aa)b &= a(ab)\\ (ab)a &= a(ba)\\ (ba)a &= b(aa)\end{aligned}$$

4. Any two of the previous formulas hold.

Proof. We first discuss a simple trick called polarization. Suppose $(aa)b = a(ab)$. Then

$$0 = (a+b, a+b, c) = (a, b, c) + (b, a, c) + (a, a, c) + (b, b, c) = (a, b, c) + (b, a, c)$$

Similarly $(ab)a = a(ba)$ implies $0 = (a, b, c) + (c, b, a)$ and $(ba)a = b(aa)$ implies $0 = (a, b, c) + (a, c, b)$. The converse implications are trivial.

- "$1 \Rightarrow 2 \Leftrightarrow 3 \Rightarrow 4$" is obvious.
- "$4 \Rightarrow 3$" If only two of the identities hold, say $(a, b, c) + (b, a, c) = 0$ and $(a, b, c) + (c, b, a) = 0$, we get the third as follows

$$(a, c, b) = -(c, a, b) = (b, a, c) = -(a, b, c)$$

- "$\cdots \Rightarrow 1$" Let S be a subalgebra spanned by $a', b' \in A$ and denote a, b their imaginary parts. We claim that S is the linear span L of the elements $1, a, b, ab$. Obviously $1, a, b$ together with all possible "multiplicative combinations" of a, b (this means for example $ab, ba, a(ba), (ab)a, b(ab),\ (ba)b, a(b(ab)), a((ba)b), (ab)(ab), \ldots$) linearly span S. To prove the claim it obviously suffices to check (using 2.7)

$ba, a(ab), b(ab), (ab)^2, (ab)a, (ab)b \in L$:

$$\begin{aligned}
ba &= -ab - 2\langle a, b\rangle \\
a(ab) &= a^2 b = -\|a\|^2 b \\
b(ab) &= b(-ba - 2\langle a, b\rangle) = -b(ba) - 2\langle a, b\rangle b = \\
&= -b^2 a - 2\langle a, b\rangle b = \|b\|^2 a - 2\langle a, b\rangle b \\
(ab)^2 &= (ab)(-ba - 2\langle a, b\rangle) = -(ab)(b^* a^*) - 2\langle a, b\rangle ab = \\
&= -\|ab\|^2 - 2\langle a, b\rangle ab \\
(ab)a &= (-ba - 2\langle a, b\rangle)a = -(ba)a - 2\langle a, b\rangle a = \\
&= -ba^2 - 2\langle a, b\rangle a = \|a\|^2 b - 2\langle a, b\rangle a \\
(ab)b &= ab^2 = -\|b\|^2 a
\end{aligned}$$

To show that S is associative we verify vanishing of the associator on all the spanning elements $1, a, b, ab$. If any of the entries of the associator is 1 then the associator is clearly 0. Now the skew-symmetry implies that the only nontrivial case is

$$(a, b, ab) = 0$$

But we already have

$$\operatorname{Im}((ab)^2) = -2\langle a, b\rangle \operatorname{Im}(ab)$$

and further

$$\begin{aligned}
\operatorname{Im}(a(b(ab))) &= \operatorname{Im}(a(b(-ba - 2\langle a, b\rangle))) = \operatorname{Im}(a(-b^2 a)) - 2\langle a, b\rangle \operatorname{Im}(ab) = \\
&= \operatorname{Im}(\|b\|^2 a^2) - 2\langle a, b\rangle \operatorname{Im}(ab) = -2\langle a, b\rangle \operatorname{Im}(ab)
\end{aligned}$$

so

$$\operatorname{Im}(a, b, ab) = 0$$

and by 2.9 the conclusion follows. □

2.13 Lemma. If A is a good alternative algebra, then

$$\|ab\|^2 = \|a\|^2 \|b\|^2$$

holds for $a, b \in A$. Thus if A is positive definite then it is a division algebra.

Proof. $\|ab\|^2 = (ab)(ab)^* = (ab)(b^* a^*) = a(bb^*)a^* = \|b\|^2 aa^* = \|b\|^2 \|a\|^2$ where the third equality is due to alternativity and the previous theorem. □

2.2 Cayley-Dickson construction

We define the Cayley-Dickson (CD) construction and show that the familiar algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ fit into the general scheme. Finally we construct the octonions. The central theorem 2.18 enables us to get information about a succesor in the CD construction. We use it to derive basic properties of $\mathbb{O}$. It also explains the interesting loss of nice properties for CD constructs.

2.14 Definition. Let A be a $*$-algebra. We are going to construct an $*$-algebra $\mathfrak{CD}(A)$. Let $B := A \oplus A$ as vector spaces. Now equip B with a multiplication

$$(a, b)(c, d) := (ac - d^*b, bc^* + da) \tag{2}$$

and a conjugation

$$(a, b)^* := (a^*, -b)$$

where we use the same symbol for the conjugation on both A and B. This makes B into a $*$-algebra denoted $\mathfrak{CD}(A)$, where the abbreviation CD stays for **Cayley-Dickson**.

2.15 Claim. $\mathfrak{CD}(A)$ is indeed a $*$-algebra. There is a canonical monomorphism $A \hookrightarrow \mathfrak{CD}(A)$ given by $i : a \mapsto (a, 0)$.

Proof. The multiplication on $\mathfrak{CD}(A)$ is indeed linear and distributive and the unit is the element $(1, 0) \in A \oplus A$. The conjugation $*$ is linear with $*^2 = \mathrm{id}$ and $((a, b)(c, d))^* = (c, d)^*(a, b)^*$ follows from a short computation. So $\mathfrak{CD}(A)$ is a $*$-algebra.

It remains to verify that i is injective $*$-algebra-morphism, which is straightforward. □

2.16 Example. Starting with the 1-dimensional algebra $\mathbb{R}$ with $a^* := a$, we apply the CD-construction repeatedly.

Obviously

$$\mathfrak{CD}(\mathbb{R}) = \mathbb{C}$$

with the usual complex conjugation.

Next

$$\mathfrak{CD}(\mathbb{C}) = \mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

is the algebra of quaternions with the usual relations $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. Moreover, $\mathbb{H}$ is (by the CD-construction) equipped with the standard quaternionic conjugation $(a + bi + cj + dk)^* = a - bi - cj - dk$.

The next iteration of CD-construction yields the desired algebra of octonions. Since we don't assume familiarity with the octonions, we define them in this way :

2.17 Definition. The $*$-algebra

$$\mathbb{O} := \mathfrak{CD}(\mathbb{H})$$

is called the algebra of **octonions**.

2.18 Theorem. Let A be a nonzero $*$-algebra.

1. $\mathfrak{CD}(A)$ is never real.
2. A is real iff $\mathfrak{CD}(A)$ is commutative.
3. A is commutative and associative iff $\mathfrak{CD}(A)$ is associative.
4. A is good (positive definite) iff $\mathfrak{CD}(A)$ is good (positive definite).
5. A is associative and good iff $\mathfrak{CD}(A)$ is alternative and good.
6. A is flexible and good iff $\mathfrak{CD}(A)$ is flexible and good.

Proof. 1. $(a,b)^* = (a^*, -b) = (a,b)$ implies $b = 0$ so it cannot hold for every element of $\mathfrak{CD}(A)$.

2. Let A be real. First note that if a $*$-algebra is real then it is commutative since $ab = (ab)^* = b^*a^* = ba$. Now $(a,b)(c,d) = (ac - d^*b, bc^* + da) = (ca - b^*d, da^* + bc) = (c,d)(a,b)$, i.e. $\mathfrak{CD}(A)$ is commutative.

   Conversly suppose that $\mathfrak{CD}(A)$ is commutative, i.e. $(ac - d^*b, bc^* + da) = (ca - b^*d, da^* + bc)$. For $a = c = 0$, $d = 1$ and b arbitrary we get $(-b, 0) = (-b^*, 0)$, so $b = b^*$.

3. We have

$$\begin{aligned}((a,b)(c,d))(e,f) &= (ac - d^*b, bc^* + da)(e,f) = \\ &= \Big((ac)e - (d^*b)e - f^*(bc^*) - f^*(da), \\ &\qquad (bc^*)e^* + (da)e^* + f(ac) - f(d^*b)\Big) \\ (a,b)((c,d)(e,f)) &= (a,b)(ce - f^*d, de^* + fc) = \\ &= \Big(a(ce) - a(f^*d) - (ed^*)b - (c^*f^*)b, \\ &\qquad b(e^*c^*) - b(d^*f) + (de^*)a + (fc)a\Big)\end{aligned}$$

   If A is associative and commutative, then the associativity of $\mathfrak{CD}(A)$ follows immediately. Conversly if $\mathfrak{CD}(A)$ is associative, then the choice $b = d = f = 0$ gives $((ac)e, 0) = (a(ce), 0)$, i.e. the associativity of A, and the choice $a = c = f = 0, d = 1$ gives $(-be, 0) = (-eb, 0)$ the commutativity.

4. The equalities

$$(a,b)+(a^*,-b)=(a+a^*,0)$$
$$(a,b)(a^*,-b)=(aa^*+b^*b,ba-ba)=(a^*a+b^*b,0)$$

prove all the implications.

5. Let A be good and associative. The previous claim implies that $\mathfrak{CD}(A)$ is good. Due to the theorem 2.12 it suffices to verify

$$((a,b)(a,b))(c,d)=(a,b)((a,b)(c,d))$$
$$((c,d)(a,b))(a,b)=(c,d)((a,b)(a,b))$$

We prove only the first equality, the second one being analogous.

$$\begin{aligned}((a,b)(a,b))(c,d) &= \Big((aa)c-(b^*b)c-d^*(ba^*)-d^*(ba),\\ &\qquad (ba^*)c^*+(ba)c^*+d(aa)-d(b^*b)\Big)\\ (a,b)((a,b)(c,d)) &= \Big(a(ac)-a(d^*b)-(cb^*)b-(a^*d^*)b,\\ &\qquad b(c^*a^*)-b(b^*d)+(bc^*)a+(da)a\Big) \end{aligned} \tag{3}$$

Now $(bb^*)c=(cb^*)b$ because of associativity and goodness - recall $bb^*\in\mathbb{R}$ so it commutes with everything! Next $d^*(ba^*)+d^*(ba)=a(d^*b)+(a^*d^*)b$ because of associativity and goodness - this time recall $a^*+a\in\mathbb{R}$. The remaining terms are treated similarly as well as the other equality $((c,d)(a,b))(a,b)=(c,d)((a,b)(a,b))$.

Now suppose $\mathfrak{CD}(A)$ is alternative and good. (3) holds, so put $c=0$ to obtain

$$d^*(ba^*)+d^*(ba)=a(d^*b)+(a^*d^*)b$$

and further

$$\begin{aligned} d^*(b(a^*+a)) &= a(d^*b)+a^*(d^*b)+(a^*,d^*,b)\\ d^*(b(a^*+a)) &= (a+a^*)(d^*b)+(a^*,d^*,b)\\ 0 &= (a^*,d^*,b) \end{aligned}$$

By the previous claim A is good and therefore we use 2.8 to get $0=(a,d,b)$ for every $a,b,d\in A$.

6. We have

$$((a,b)(c,d))(a,b) = \Big((ac)a - (d^*b)a - b^*(bc) - b^*(da),$$
$$(bc^*)a^* + (da)a^* + b(ac) - b(d^*b)\Big)$$
$$(a,b)((c,d)(a,b)) = \Big(a(ca) - a(b^*d) - (ad^*)b - (c^*b^*)b,$$
$$b(a^*c^*) - b(d^*b) + (da^*)a + (bc)a\Big) \qquad (4)$$

Suppose A is good and flexible. First we observe that

$$(d^*b)a + b^*(da) = a(b^*d) + (ad^*)b \quad \text{for all } a,b,d \in A \qquad (5)$$

Indeed we have equivalently

$$(d^*b)a + (b^*d)a - (b^*,d,a) = a(b^*d) + a(d^*b) + (a,d^*,b)$$
$$(d^*b + b^*d)a - (b^*,d,a) = a(b^*d + d^*b) + (a,d^*,b)$$
$$(b,d,a) = -(a,d,b)$$

where we used 2.8 and $d^*b + b^*d \in \mathbb{R}$. The last equality follows from the flexibility via polarization (see the proof of 2.12).

We see that (5) implies

$$b^*(bc^*) = (c^*b^*)b \quad \text{for all } b,c \in A \qquad (6)$$

simply by setting $d = b$ in (5) and using goodness.

So flexibility and (5) and (6) imply equality of the first slots in (4). For the second slots we observe

$$(bc^*)a^* + b(ac) = b(a^*c^*) + (bc)a$$

Indeed

$$LHS = (bc)a - (bc)2\,\mathrm{Re}(a) - (ba)2\,\mathrm{Re}(c) + b4\,\mathrm{Re}(a)\,\mathrm{Re}(c) + b(ac)$$
$$RHS = b(ac) - (ba)2\,\mathrm{Re}(c) - (bc)2\,\mathrm{Re}(a) + b4\,\mathrm{Re}(a)\,\mathrm{Re}(c) + (bc)a$$

Finally

$$(da)a^* = (da^*)a$$

by similar computations. So (4) holds.

To prove the converse implication of the theorem, set $b = d = 0$ in (4) and again use the claim 4 to obtain goodness. □

As a consequence of 2.18,2.13 and 2.4 we have

2.19 Theorem. $\mathbb{O}$ is 8-dimensional noncommutative, nonassociative, alternative, division and positive definite $*$-algebra with multiplicative inverses.$\square$

We postpone giving relations of generators for this moment, because we will treat them together with split-octonions later.

2.3 2-dimensional $*$-algebras

We enrich our collection of good algebras by some indefinite examples - $\tilde{\mathbb{C}}$ and $\mathbb{C}_0$. To motivate our choice of starting points of the CD construction we prove a classification theorem 2.22 on 2-dimensional $*$-algebras asserting that we miss only certain degenerate algebras.

2.20 Example. Consider
$$\tilde{\mathbb{C}} := \mathbb{R} \oplus \mathbb{R}e$$
with multiplication uniquely given by $e^2 = 1$. This algebra is called algebra of **paracomplex numbers**. We will later treat $\tilde{\mathbb{C}}$ as a $*$-algebra with standard conjugation $(a + be)^* = a - be$ for $a, b \in \mathbb{R}$. Note that for $x = a + be$ the formula $xx^* = x^*x = a^2 - b^2$ no longer defines the square of a norm, but only a quadratic form of signature $(1, 1)$. Obviously $\tilde{\mathbb{C}}$ contains zero divisors : $(1 + e)(1 - e) = 0$.

Since $\tilde{\mathbb{C}}$ is clearly commutative, the identity is a conjugation too.

2.21 Example. Exactly as in the previous example but with $e^2 = 0$, we obtain another example of 2-dimensional algebra denoted $\mathbb{C}_0$. This is not even nondegenerate algebra.

2.22 Theorem. Let A be a 2-dimensional $*$-algebra. Then A is $*$-algebra-isomophic to exactly one of $\mathbb{C}, \tilde{\mathbb{C}}$ or $\mathbb{C}_0$ with either identity conjugation or the standard conjugation $(a + be)^* = a - be$ for $a, b \in \mathbb{R}$.

Proof. We begin by 2 observations :

1. Every ideal I of an algebra[2] A is a vector subspace of A.
   This follows immediately from the invariance of I under multiplication by $\lambda 1$ for $\lambda \in \mathbb{R}$.

2. Every 2-dimensional algebra is commutative.
   Indeed we can always choose a basis containing 1 and it suffices to verify the commutativity for the basis.

[2]The existence of the unit is essential here!

We first classify the algebras without conjugation by discussing ideals in A :

1. A has trivial ideals only.
   Therefore A is a field. Let $1, i$ span A. There are $a, b \in \mathbb{R}$ such that

$$i^2 = a + bi$$

   We have

$$(i - \frac{b}{2})^2 = a + \frac{b^2}{4}$$

   Denote $c := a + \frac{b^2}{4}$. Suppose $c \geq 0$. Then we have the factorization $(i - \frac{b}{2} + \sqrt{c})(i - \frac{b}{2} - \sqrt{c}) = 0$. But then $i - \frac{b}{2} + \sqrt{c} \neq 0$ is a zero divisor which contradicts the assumption that A is a field. Therefore $c < 0$. Set $j := \frac{1}{\sqrt{|c|}}(i - \frac{b}{2})$ to get $j^2 = -1$. Now the algebra-isomorphism $A \to \mathbb{C}$ (for $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}e$ with $e^2 = -1$) is given by

$$1 \mapsto 1$$
$$j \mapsto e$$

2. There is a 1-dimensional ideal $I \subset A$.
   I is necessarily a principal ideal (i) generated by some $i \in A$. Obviously $i \notin \mathbb{R}$ and $i^2 = ai$ for some $a \in \mathbb{R}$.

$$(i - \frac{a}{2})^2 = \frac{a^2}{4}$$

   Let's discuss a :
   If $a = 0$ then the algebra-isomorphism $A \to \mathbb{C}_0$ is given by

$$1 \mapsto 1$$
$$(i - \frac{a}{2})^2 \mapsto e$$

   If $a \neq 0$ then the algebra-isomorphism $A \to \tilde{\mathbb{C}}$ is given by

$$1 \mapsto 1$$
$$(\frac{2}{a}(i - \frac{a}{2}))^2 \mapsto e$$

Obviously the 3 algebras $\mathbb{C}, \tilde{\mathbb{C}}$ and $\mathbb{C}_0$ are not isomorphic.

To finish the proof it now suffices to show that $\mathbb{C}, \tilde{\mathbb{C}}$ and $\mathbb{C}_0$ admit only the conjugations mentioned in the theorem. Let A be one of the above algebras.

We have $A = \mathbb{R} \oplus \mathbb{R}e$ with $e^2 \in \mathbb{R}$. The properties of the conjugation imply $*(a) = a$ for any $a \in \mathbb{R}$. Let $*(e) = a + be$.

$$e = **(e) = *(a + be) = a + ba + b^2 e$$

and therefore $a(b+1) = 0$ and $b^2 = 1$. Now $b = 1$ implies $a = 0$ and so $* = \mathrm{id}$ which is indeed a conjugation. For the second case $b = -1$ we have

$$*(e^2) = *(e) * (e) = (a - e)(a - e) = a^2 + e^2 - 2ae$$

Since $e^2 \in \mathbb{R}$ we get $e^2 = *(e^2) = a^2 + e^2 - 2ae$, so $a = 0$ and $*(\alpha + \beta e) = \alpha - \beta e$. □

2.4 Twisting of $\mathbb{Z}_2^n$

We construct split-analogues of $\mathbb{H}$ and $\mathbb{O}$ and derive their basic properties. We introduce the notion of twisted group algebra and in 2.27 we show how this fits into the CD construction. This enables us to get explicit description of the multiplication in $\mathbb{O}$ and $\tilde{\mathbb{O}}$ with little effort. We exploit this method to investigate an interesting example 2.32 of sedenions which demonstrates some aspects of the behaviour of the CD construction.

2.23 Definition. Applying the CD-construction starting with $\tilde{\mathbb{C}}$, we define

$$\begin{aligned} \mathfrak{CD}(\tilde{\mathbb{C}}) &=: \tilde{\mathbb{H}} \quad \text{split-quaternions} \\ \mathfrak{CD}(\tilde{\mathbb{H}}) &=: \tilde{\mathbb{O}} \quad \text{split-octonions} \end{aligned}$$

The assertion about the signature in the following theorem will be proved a moment later.

2.24 Theorem. $\tilde{\mathbb{O}}$ is 8-dimensional good, noncommutative, nonassociative, alternative $*$-algebra of signature $(4+, 4-)$. □

2.25 Remark. The split-octonions are considerably uglier than the octonions. $\tilde{\mathbb{O}}$ is not division algebra (because $\tilde{\mathbb{C}}$ lacks this property and $\tilde{\mathbb{C}} \hookrightarrow \tilde{\mathbb{O}}$). Therefore $\tilde{\mathbb{O}}$ can't have even multiplicative inverses.

To get an effective tool for working with (split-)octonions, we introduce the notion of twisted group algebra.

2.26 Definition. Let G be a group and $\mathbb{R}[G]$ its group algebra [3]. Let $\alpha : G \times G \to \{\pm 1\}$ be an arbitrary map. Now consider $A := \mathbb{R}[G]$ equipped with the multiplication $\star$ given by

$$g \star h := \alpha(g,h)gh \quad \text{for } g,h \in G$$

A with $\star$ multiplication is called **α-twisted group algebra** $\mathbb{R}[G]$.

Looking back at the complex numbers as an algebra with generators $1, i$ we see that $\mathbb{C}$ is in fact α-twisted group algebra $\mathbb{R}[\mathbb{Z}_2]$, where the twisting function is given by the table 1 where $\alpha(i,j)$ is in the i^{th} row and j^{th} column and $+, -$ stand for $+1, -1$.

	0	1
0	+	+
1	+	-

Table 1: Twisting of $\mathbb{R}[\mathbb{Z}_2]$ to obtain $\mathbb{C}$

We easily check that the quaternions are the twisted group algebra $\mathbb{R}[\mathbb{Z}_2^2]$. Here and further on $\mathbb{Z}_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ is the product of groups (a.k.a. direct sum), i.e. the group operation is done componentwise. If we're writing the elements of $\mathbb{Z}_2^n$ explicitly, we ommit the brackets, e.g. $(0,0,1) =: 001$ or $011 + 101 = 110$ etc.

	00	01	10	11
00	+	+	+	+
01	+	-	+	-
10	+	-	-	+
11	+	+	-	-

Table 2: Twisting of $\mathbb{R}[\mathbb{Z}_2^2]$ to obtain $\mathbb{H}$

The elements 00,01,10,11 correspond to the standard generators $1, i, j, k$ of $\mathbb{H}$.

The following theorem clarifies the situation.

[3]Group algebra $\mathbb{R}[G]$ is a real vector space with the base G and with multiplicative structure given by

$$(\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) := \sum_{g,h \in G} a_h b_{h^{-1}g} g$$

i.e. polynomial multiplication or so called convolution. Note that all the sums are well defined because only finitely many a_g, b_h's are nonzero.

2.27 Theorem. Let G be a commutative group. Let A be a α-twisted group algebra $\mathbb{R}[G]$ with conjugation given by $g^* = c(g)g$ for $g \in G$ and some map $c : G \to \{\pm 1\}$. Then $\mathfrak{CD}(A)$ is β-twisted group algebra $\mathbb{R}[\mathbb{Z}_2 \times G]$ with β given by the following formulas for $g, h \in G$

$$\begin{aligned}
\beta(\{0\} \times g, \{0\} \times h) &= \alpha(g, h) \\
\beta(\{0\} \times g, \{1\} \times h) &= \alpha(h, g) \\
\beta(\{1\} \times g, \{0\} \times h) &= c(h)\alpha(g, h) \\
\beta(\{1\} \times g, \{1\} \times h) &= -c(h)\alpha(h, g)
\end{aligned}$$

or in the form of the table

$$\begin{pmatrix} \alpha & \alpha^t \\ C(\alpha) & -C(\alpha^t) \end{pmatrix}$$

where the operation C multiplies the column of its argument matrix indexed g by $c(g)$.

Proof. There are bijections

$$\mathfrak{CD}(A) = A \oplus A \cong \{0, 1\} \times A = \mathbb{Z}_2 \times \mathbb{R}[G] \cong \mathbb{R}[\mathbb{Z}_2 \times G]$$

suggesting the obvious isomorphism $\mathfrak{CD}(A) \cong \mathbb{R}[\mathbb{Z}_2 \times G]$ of vector spaces. In this proof we denote by $\star$ the twisted multiplication on a given algebra while we ommit the symbol for the untwisted multiplication. $A \oplus A$ has basis $\{(g, 0) : a \in G\} \cup \{(0, g) : g \in G\}$. The multiplication in twisted $\mathbb{R}[\mathbb{Z}_2 \times G]$ is given by the formula (2), so

$$\begin{aligned}
(g, 0) \star (h, 0) &= (g \star h, 0) = \alpha(g, h)(gh, 0) = \alpha(g, h)(\{0\} \times (gh)) \\
(g, 0) \star (0, h) &= (0, h \star g) = \alpha(h, g)(0, hg) = \alpha(h, g)(\{1\} \times (hg)) \\
(0, g) \star (h, 0) &= (0, g \star c(h)h) = c(h)\alpha(g, h)(0, gh) = c(h)\alpha(g, h)(\{1\} \times (gh)) \\
(0, g) \star (0, h) &= (-c(h)h \star g, 0) = -c(h)\alpha(h, g)(hg, 0) = -c(h)\alpha(h, g)(\{0\} \times (hg))
\end{aligned}$$

Recall that the multiplication in untwisted $\mathbb{R}[\mathbb{Z}_2 \times G]$ is given[4] by

$$\begin{aligned}
(g, 0)(h, 0) &= (\{0\} \times g)(\{0\} \times h) = \{0\} \times (gh) \\
(g, 0)(0, h) &= (\{0\} \times g)(\{1\} \times h) = \{1\} \times (gh) \\
(0, g)(h, 0) &= (\{1\} \times g)(\{0\} \times h) = \{1\} \times (gh) \\
(0, g)(0, h) &= (\{1\} \times g)(\{1\} \times h) = \{0\} \times (gh)
\end{aligned}$$

By the commutativity of G we see the $*$-algebra-isomorphism $\mathfrak{CD}(A) \cong \mathbb{R}[\mathbb{Z}_2 \times G]$ and we also get the desired relations between α and β. $\square$

[4]To avoid any confusion we remind that $\mathbb{Z}_2$ means *additive* group $\{0, 1\}$.

2.28 Remark. One can think of many generalisations of the previous theorem. The restriction on the form of the conjugation on A is essential, because generally it can happen that $\mathfrak{CD}(A)$ is no longer twisted $\mathbb{R}[\mathbb{Z}_2 \times G]$ but rather something more complicated. This situation occurs for example in dimension 4 - just take $A := \mathbb{R}[\mathbb{Z}_2^2]$ with (indeed) conjugation

$$(a00 + b01 + c01 + d11)^* := a00 + c01 + b01 + d11$$

Now $\mathfrak{CD}(A)$ is no longer twisted $\mathbb{R}[\mathbb{Z}_2^3]$ since the conjugation mixes the coordinates in an unpleasant way.

However the restrictive assumption of the theorem is not severe, because we are interested in algebras comming from the CD-construction starting at dimension 2 and in this case the conjugation takes (and continues to have) the desired form due to theorem 2.22.

The theorem 2.27 allows us to quickly recover the twisting for $\mathbb{O}$ and $\tilde{\mathbb{O}}$, see tables 3 and 4.

	000	001	010	011	100	101	110	111
000	+	+	+	+	+	+	+	+
001	+	-	+	-	+	-	-	+
010	+	-	-	+	+	+	-	-
011	+	+	-	-	+	-	+	-
100	+	-	-	-	-	+	+	+
101	+	+	-	+	-	-	-	+
110	+	+	+	-	-	+	-	-
111	+	-	+	+	-	-	+	-

Table 3: Twisting of $\mathbb{R}[\mathbb{Z}_2^3]$ to obtain octonions $\mathbb{O}$

The connection with standard formalism is as follows : $e_0 = e_{000} := 000$, $e_1 = e_{001} := 001$, $e_2 = e_{010} := 010$, $e_3 = e_{011} := 011$, $e_4 = e_{100} := 100$, $e_5 = e_{101} := 101$, $e_6 = e_{110} := 110$, $e_7 = e_{111} := 111$. $\mathrm{Im}(\mathbb{O})$ is the linear span of $e_1, e_2, \ldots, e_7$. We use the same symbols for $\tilde{\mathbb{O}}$.

We can now easily identify $\tilde{\mathbb{H}}$ with the $*$-algebra $M_2(\mathbb{R})$ of real 2×2 matrices :

2.29 Claim. Let the conjugation on $M_2(\mathbb{R})$ be given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Define $f : \tilde{\mathbb{H}} \to M_2(\mathbb{R})$ by

$$f(e_0) = \mathrm{id}, \quad f(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad f(e_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

	000	001	010	011	100	101	110	111
000	+	+	+	+	+	+	+	+
001	+	+	+	+	+	+	-	-
010	+	-	-	+	+	+	-	-
011	+	-	-	+	+	+	+	+
100	+	-	-	-	-	+	+	+
101	+	-	-	-	-	+	-	-
110	+	+	+	-	-	+	-	-
111	+	+	+	-	-	+	+	+

Table 4: Twisting of $\mathbb{R}[\mathbb{Z}_2^3]$ to obtain split-octonions $\tilde{\mathbb{O}}$

Then f is a $*$-algebra isomorphism. □

2.30 Claim. For the scalar product on $\mathbb{O}$ given by (1) we get $\langle e_i, e_j \rangle = \delta_{i,j}$. The scalar product on $\tilde{\mathbb{O}}$ given by the same formula satisfies $|\langle e_i, e_j \rangle| = \delta_{i,j}$ and $\langle e_i, e_i \rangle$ is -1 for e_1, e_3, e_5, e_7 and is $+1$ for the other basis vector, i.e. it has signature $(4+, 4-)$. □

2.31 Example. We give a very simple example of the twisted computation in basis to clarify the notation : Let $i, j \in \mathbb{Z}_2^3$. Then

$$e_i e_j = \alpha(i,j) e_{i+j}$$

which is the same as

$$i \cdot j = \alpha(i,j)(i+j)$$

We are now able to give the example of an algebra with multiplicative inverses and zero divisors at the same time, recall 2.5.

2.32 Example. We go bravely one step further in the CD-construction and consider

$$\mathbb{S} := \mathfrak{CD}(\mathbb{O})$$

This algebra is called the algebra of **sedenions**. The theorems 2.4 and 2.18 inform us about the properties of sedenions : $\mathbb{S}$ is 16-dimensional noncommutative, nonalternative, flexible and positive definite $*$-algebra with multiplicative inverses. We notice that the nonassociativity of $\mathbb{S}$ can't be worse. However our concern are the zero divisors. To be able to compute something we use the twisting formulas of the theorem 2.27. We claim

$$(e_1 + e_{10})(-e_4 + e_{15}) = 0 \quad \text{or equivalently} \quad (0001 + 1010)(-0100 + 1111) = 0$$

Indeed

$$\begin{gathered}(0001+1010)(-0100+1111)= \\ =-\beta(0001,0100)0101+\beta(0001,1111)1110+ \\ -\beta(1010,0100)1110+\beta(1010,1111)0101= \\ =(-\alpha(001,100)-c(111)\alpha(111,010))0101+ \\ +(\alpha(111,001)-c(100)\alpha(010,100))1110= \\ =0\end{gathered}$$

by a careful inspection of the twisting table 3 for $\mathbb{O}$.

2.5 Uniqueness theorem

We prove the well known Hurewicz theorem 2.36 asserting a uniqueness of $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$ in a certain sense. We also obtain an analogous theorem 2.37 for the split case. We will later use these theorems to simplify certain computations in the chapter on G_2-structures on manifolds.

2.33 Theorem. Let A be a nondegenerate algebra. Then A satisfies $\|ab\|^2=\|a\|^2\|b\|^2$ iff A is alternative.

Proof. If A is alternative, this is 2.13. Let A satisfy $\|ab\|^2=\|a\|^2\|b\|^2$. We first prove that the formula

$$\langle ua,ub\rangle=\|u\|^2\langle a,b\rangle \quad \text{for } u,a,b\in A$$

from 2.11 is valid even under our current assumptions. First observe

$$\langle a,b\rangle=\frac{1}{2}(\|a+b\|^2-\|a\|^2-\|b\|^2)$$

Then indeed $\langle ua,ub\rangle=\frac{1}{2}(\|u(a+b)\|^2-\|ua\|^2-\|ub\|^2)=\|u\|^2\langle a,b\rangle$ using the assumption.

Hence we have $\langle (u+v)a,(u+v)b\rangle=\|u+v\|^2\langle a,b\rangle$. It means $\langle ua,ub\rangle+\langle ua,vb\rangle+\langle va,ub\rangle+\langle va,vb\rangle=(\|u\|^2+\|v\|^2+2\langle u,v\rangle)2\langle a,b\rangle$ and so

$$\langle ua,vb\rangle+\langle ub,va\rangle=2\langle u,v\rangle\langle a,b\rangle$$

We just remark that we will also obtain this formula later in 3.20 under different assumptions.

Set $v=1$ to get $\langle ua,b\rangle+\langle ub,a\rangle=\langle u,1\rangle\langle a,b\rangle$ and so $\langle ua,b\rangle=\langle a,(-u+\langle u,1\rangle)b\rangle$. We get

$$\langle ua,b\rangle=\langle a,u^*b\rangle$$

and similarly

$$\langle au, b\rangle = \langle a, bu^*\rangle$$

Finally we prove $a(ab) = a^2b$ and $ba^2 = (ba)a$. Notice $2\langle a, 1\rangle = a^* + a$.

$$\begin{aligned}\langle a(ab), t\rangle &= \langle ab, a^*t\rangle = \langle ab, (-a + 2\langle a, 1\rangle)t\rangle = -\langle ab, at\rangle + 2\langle a, 1\rangle\langle ab, t\rangle = \\ &= \langle(-\|a\|^2 + 2a\langle a, 1\rangle)b, t\rangle = \langle(-aa^* + a^2 + aa^*)b, t\rangle = \langle a^2b, t\rangle \\ \langle(ba)a, t\rangle &= \langle ba, ta^*\rangle = -\langle ba, ta\rangle + 2\langle a, 1\rangle\langle ba, t\rangle = \\ &= \langle b(-\|a\|^2 + 2a\langle a, 1\rangle), t\rangle = \langle b(-aa^* + a^2 + aa^*), t\rangle = \langle ba^2, t\rangle\end{aligned}$$

2.34 Lemma. Let A be a nondegenerate $*$-algebra satisfying $\|ab\|^2 = \|a\|^2\|b\|^2$. Let $B \subset A$ be a proper $*$-subalgebra such that $B^\perp := \{x \in A : \langle x, b\rangle = 0 \text{ for all } b \in B\}$ contains an element $i \in B^\perp$ with $\|i\|^2 = 1$. Then A contains $\mathfrak{CD}(B)$ as a $*$-subalgebra.

2.35 Remark. Note that for positive definite algebras the condition on the existence of norm one $i \in B^\perp$ is automatically satisfied for any subalgebra.

Proof (of 2.34). Let $b \in B$. We have $ib^* + bi^* = 0$. Especially for $b = 1$ we get $i + i^* = 0$ and therefore

$$ib^* = bi$$

We use this observation in the further computations. A is alternative by 2.33. We also use various formulas from the proof of 2.33 (we could alternativly deduce those from the alternativity, e.g. see 3.20). We claim

$$(a + bi)(c + di) = (ac - d^*b) + (bc^* + da)i \quad \text{for all } a, b, c, d \in A$$

We have to verify 3 things :

1. $(bi)c = (bc^*)i$?
   Let $t \in A$ be arbitrary.

   $$\langle(bi)c, t\rangle = \langle bi, tc^*\rangle = -\langle bc^*, ti\rangle + 2\langle b, t\rangle\langle i, c^*\rangle = \ldots$$

   Now $i \in B^\perp$ and $c \in B$ imply $\langle i, c^*\rangle = 0$.

   $$\ldots = -\langle(bc^*)i^*, t\rangle = \langle(bc^*)i, t\rangle$$

2. $a(di) = (da)i$?

   $$\langle a(di), t\rangle = \langle di, a^*t\rangle = \langle id^*, a^*t\rangle = -\langle it, a^*d^*\rangle + 2\langle i, a^*\rangle\langle d^*, t\rangle = \ldots$$

   Now $i \in B^\perp$ and $a \in B$ imply $\langle i, a^*\rangle = 0$.

   $$\ldots = -\langle it, a^*d^*\rangle = \langle t, i(a^*d^*)\rangle = \langle t, i(da)^*\rangle = \langle t, (da)i\rangle$$

3. $(bi)(di) = -d^*b$?

$$\langle (bi)(di), t\rangle = \langle bi, t(i^*d^*)\rangle = \langle ib^*, t(i^*d^*)\rangle = -\langle i(i^*d^*), tb^*\rangle + 2\langle i, t\rangle\langle b^*, i^*d^*\rangle = \ldots$$

Now $\langle b^*, i^*d^*\rangle = \langle b^*d, i^*\rangle = 0$ by the condition $i \in B^\perp$. Use alternativity in the first slot of the first scalar product.

$$\ldots = -\langle d^*, tb^*\rangle = -\langle d^*b, t\rangle$$

So we finally see that $B \oplus Bi$ is indeed $\mathfrak{CD}(B)$, because $(a+bi)^* = a^* + i^*b^* = a^* - ib^* = a^* - bi$. □

2.36 Theorem (Hurewicz). Let A be a positive definite $*$-algebra satisfying $\|ab\| = \|a\|\|b\|$. Then A is isomorphic to one of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Proof. A contains $\mathbb{R}$ as $*$-subalgebra. By the previous lemma A contains a chain $\mathbb{R} \subset B_1 \subset \ldots \subset B_n = A$, where $B_1 = \mathfrak{CD}(\mathbb{R})$ and $B_{i+1} = \mathfrak{CD}(B_i)$. By 2.18 and 2.33 we see that $n \leq 3$ - $B_1 = \mathbb{C}$ is not real, $B_2 = \mathbb{H}$ is not commutative, $B_3 = \mathbb{O}$ is not associative and therefore B_4 is not alternative, i.e. equivalently doesn't satisfy $\|ab\| = \|a\|\|b\|$. □

2.37 Theorem. Let A be a 8-dimensional $*$-algebra of signature $(p+, q-)$ with $p \geq 3$ and $q \geq 1$ satisfying $\|ab\|^2 = \|a\|^2\|b\|^2$. Then A is isomorphic to $\tilde{\mathbb{O}}$.

Proof. By the assumptions there is a element $e \in A - \mathbb{R}$ with $\|e\|^2 = -1$, therefore $e^2 = 1$. Since $1, e$ are linearly independent, they span a $*$-algebra which is obviously isomorphic to $\tilde{\mathbb{C}}$. Now apply the same argument as in the proof of the Hurewicz theorem, this time starting with $B_1 = \tilde{\mathbb{C}}$, which is not real. By the assumption about the signature, there is $i_1 \in B_1^\perp$ with $\|i_1\|^2 = 1$ as required by 2.34 and therefore $B_2 = \mathfrak{CD}(B_1) = \tilde{\mathbb{H}}$ is contained in A. The signature of $\tilde{\mathbb{H}}$ is obviously $(2+, 2-)$ and so again there is $i_2 \in B_2^\perp$ with $\|i_2\|^2 = 1$ and consequently $B_3 = \tilde{\mathbb{O}}$ is contained in A. But B_3 is already 8-dimensional, so $A = \tilde{\mathbb{O}}$. □

2.38 Remark. By considering a modification of the CD-construction, in [8] one obtains that there are no nondegenerate algebras satisfying $\|ab\|^2 = \|a\|^2\|b\|^2$ other than $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \tilde{\mathbb{C}}, \tilde{\mathbb{H}}, \tilde{\mathbb{O}}$.

3 The groups G_2 and $\tilde{G}_2$

3.1 Definitions

We define G_2 and $\tilde{G}_2$ as the groups consisting of algebra automorphisms of the (split-)octonions. We show that these automorphisms are in fact orthogonal and preserve the subspace spanned by the unit element. We also present Lie algebras as derivations of the (split-)octonions.

3.1 Definition. G_2 (resp. $\tilde{G}_2$) is the group of *algebra*-automorphisms of the octonions $\mathbb{O}$ (resp. split-octonions $\tilde{\mathbb{O}}$).

$$G_2 := \mathrm{Aut}_{\mathrm{alg}}(\mathbb{O})$$
$$\tilde{G}_2 := \mathrm{Aut}_{\mathrm{alg}}(\tilde{\mathbb{O}})$$

As a subgroup of $GL(8)$, G_2 ($\tilde{G}_2$) inherits the topology.

3.2 Lemma. G_2 ($\tilde{G}_2$) is closed in $GL(8)$. Thus it inherits the smooth structure making it a Lie group.

Proof. Let a sequence $\{X_n\} \subset G_2$ has a limit $X \in GL(8)$. We have $X(ab) = \lim_{n\to\infty} X_n(ab) = \lim_{n\to\infty} X_n(a)X_n(b) = X(a)X(b)$ since the octonionic multiplication is linear and therefore continuous. The final assertion is a well known theorem - see for example [2], page 17. □

3.3 Theorem. Lie algebra $\mathfrak{g}_2$ of G_2 is the Lie algebra of derivations of $\mathbb{O}$, i.e.

$$\mathfrak{g}_2 = \mathrm{Der}(\mathbb{O}) := \{Y \in \mathrm{End}(\mathbb{O}) : Y(ab) = Y(a)b + aY(b)\}$$

with Lie bracket the usual commutator. Similarly[5]

$$\tilde{\mathfrak{g}}_2 = \mathrm{Der}(\tilde{\mathbb{O}}) := \{Y \in \mathrm{End}(\tilde{\mathbb{O}}) : Y(ab) = Y(a)b + aY(b)\}$$

Proof. Let $t \mapsto X_t$ be a smooth curve in G_2 such that $X_0 = \mathrm{id}$ and $\dot{X}_0 = \lim_{t\to 0} \frac{X_t - X_0}{t} =: Y \in T_eG_2 = \mathfrak{g}_2$.

$$\begin{aligned}(\lim_{t\to 0} \frac{X_t - X_0}{t})(ab) &= \lim_{t\to 0} \frac{X_t(a)X_t(b) - ab}{t} = \\ &= \lim_{t\to 0} \frac{(X_t(a) - a)(X_t(b) - b) + (X_t(a) - a)b + a(X_t(b) - b)}{t} = \\ &= Y(a)b + aY(b)\end{aligned}$$

One easily verifies that all the operations are correct. As easily seen, any algebra is closed under the commutator, therefore forms a Lie algebra. The result for $\tilde{G}_2$ is completely analogous. □

[5] $\mathrm{End}(\mathbb{O})$ denotes the vector space endomorphisms, not algebra-endomorphisms.

3.4 Lemma. Every $X \in G_2$ (resp. $\tilde{G}_2$) preserves the splitting

$$\mathbb{O} = \mathbb{R}1 \oplus \operatorname{Im} \mathbb{O}$$

of the vector space. Consequently

$$X(a)^* = X(a^*)$$

Proof. We have $X(1) = 1$ by the definition of algebra-homomorphism. Now we proceed to verify $X(a) \in \operatorname{Im} \mathbb{O}$ for every $a \in \operatorname{Im} \mathbb{O}$. Set $R := \operatorname{Re}(X(a))$ and $I := \operatorname{Im}(X(a))$. Recall 2.7.

$$X(a)^2 = X(a^2) = -X(\|a\|^2) = -\|a\|^2$$

On the other hand

$$X(a)^2 = (R + I)^2 = R^2 + I^2 + 2RI = R^2 - \|I\|^2 + 2RI$$

and we obtain

$$-2RI = R^2 - \|I\|^2 + \|a\|^2$$

$I \neq 0$ for otherwise $X(a) = R = X(R)$ would contradict the injectivity of X. Hence $R = 0$ and consequently $X(a) \in \operatorname{Im} \mathbb{O}$. □

3.5 Lemma. $G_2 \subset O(7)$, $\tilde{G}_2 \subset O(3, 4)$

Proof.

- Let $X \in G_2$, then

$$\begin{aligned}\langle X(a), X(b)\rangle &= \frac{1}{2}(X(a)X(b)^* + X(b)X(a)^*) = \frac{1}{2}(X(a)X(b^*) + X(b)X(a^*)) = \\ &= \frac{1}{2}(X(ab^* + ba^*)) = \frac{1}{2}(ab^* + ba^*) = \langle a, b\rangle\end{aligned}$$

  where we used 3.4.

- The lemma 3.4 implies that X, viewed as a matrix, has the block decomposition

$$X = \begin{pmatrix} 1 & 0 \\ 0 & M(X) \end{pmatrix}$$

  for some matrix $M(X) \in GL(7, \mathbb{R})$. Then $M(X) \in O(7)$ since $X \in O(8)$. Now the map

$$\begin{aligned} G_2 &\overset{\varphi}{\hookrightarrow} O(7) \\ X &\mapsto M(X) \end{aligned}$$

  is obviously an injective smooth group-homomorphism.

Analogous proof works for $\tilde{G}_2$. □

From now on we will consider G_2 as subgroup of $O(7)$ or $O(8)$ as will be convenient at the moment.

3.2 Associated 3-forms

We define certain 3-forms associated to the algebra of (split-)octonions and show that these forms are of type 5 and 8 as defined in [4]. In 3.6 and 3.8 we compute the stabilizer of these forms under the $GL(7)$-action.

$GL(7, \mathbb{R})$ acts via the matrix multiplication on $V := \mathbb{R}^7$. This induces a natural action on all tensor powers of V. Namely for $\omega \in \otimes^3 V^*$ and $g \in GL(7)$ we have

$$(g \cdot \omega)(a, b, c) = \omega(ga, gb, gc)$$

3.6 Theorem. Consider a trilinear map $\omega : \mathrm{Im}(\mathbb{O})^3 \to \mathbb{R}$ (i.e. $\omega \in \otimes^3 \mathrm{Im}(\mathbb{O})^*$) given by

$$\omega(a, b, c) := \langle ab, c \rangle \quad \text{for } a, b, c \in \mathrm{Im}(\mathbb{O})$$

where $\langle , \rangle$ is the inner product (1) on $\mathbb{O}$. Then ω is skew-symmetric and $\mathrm{Stab}_{GL(7,\mathbb{R})}(\omega) = G_2$. In basis $\{e^1, \ldots, e^7\}$ of $\mathrm{Im}(\mathbb{O})$, we have

$$\omega = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356} \tag{7}$$

Proof. First we prove the skew-symmetry. $\langle ab, c \rangle = \langle -ba - 2\langle a, b \rangle, c \rangle$ is by 2.7. Then $\langle \langle a, b \rangle, c \rangle = 0$ immediately implies

$$\langle ab, c \rangle = -\langle ba, c \rangle$$

Next

$$\langle ac, b \rangle = \langle a, bc^* \rangle = -\langle a, bc \rangle = \langle a, cb \rangle = \langle ab^*, c \rangle = -\langle ab, c \rangle$$

where the first and the fourth equality comes from 2.11. Finally

$$\langle cb, a \rangle = -\langle bc, a \rangle = \langle ba, c \rangle = -\langle ab, c \rangle$$

by the previous computations.

Let $X \in G_2$ and $a, b, c \in \mathrm{Im}(\mathbb{O})$. We have

$$(X \cdot \omega)(a, b, c) = \langle X(a)X(b), X(c) \rangle = \langle X(ab), X(c) \rangle = \langle ab, c \rangle = \omega(a, b, c)$$

because $X \in G_2 \subset O(8)$ by 3.5. Thus G_2 stabilizes ω.

On the other hand let $X \in GL(7, \mathbb{R})$ in the sense that X is a vector space automorphism of $\mathrm{Im}(\mathbb{O})$. We can extend X to a vector space automorphism of the whole $\mathbb{O}$ by setting $X(e_0) = e_0$. Suppose X stabilizes ω, i.e. $\langle X(a)X(b), X(c) \rangle = \langle ab, c \rangle$. We want to show that $X \in G_2$, i.e. prove that X is an algebra automorphism of $\mathbb{O}$. Since $X(e_0) = e_0$ by definition, only $X(ab) = X(a)X(b)$ for $a, b \in \mathrm{Im}(\mathbb{O})$ remains to be proved. It suffices to verify this for a basis. Clearly $\{X(e_1), \ldots, X(e_7)\}$ is a basis of $\mathrm{Im}(\mathbb{O})$.

$$X(e_ie_j) = X(\sum_{k=0}^{7} \langle e_ie_j, e_k\rangle e_k) = \sum_{k=0}^{7} \langle e_ie_j, e_k\rangle X(e_k) =$$
$$= \sum_{k=0}^{7} \langle X(e_i)X(e_j), X(e_k)\rangle X(e_k) = X(e_i)X(e_j)$$

The coordinate formula is obtained by a simple computation. □

3.7 Remark. The skew-symmetry was proved by an elegant coordinatefree computation. We could have avoided using the formula 2.11 and proceeded by straightforward twisted $\mathbb{Z}_2^3$ computations in basis :

Let $i, j, k \in \mathbb{Z}_2^3 - \{000\}$. $\langle e_ie_j, e_k\rangle = \alpha(i, j)\delta_{i+j,k}$ so it suffices to restrict to the case $i + j = k$. Now $\langle e_ie_k, e_j\rangle = \alpha(i, i + j)$ and $\langle e_ke_j, e_i\rangle = \alpha(i + j, j)$. So all we have to verify is

$$\alpha(i, j) = -\alpha(i, i + j) \quad \text{for } i, j \in \mathbb{Z}_2^3 - \{000\}, i \neq j$$

This is just a simple though unpleasant verification of certain symmetries in the twisting table for $\mathbb{O}$.

3.8 Theorem. Consider $\omega \in \otimes^3 \mathrm{Im}(\tilde{\mathbb{O}})^*$ given by

$$\omega(a, b, c) := \langle ab, c\rangle \quad \text{for } a, b, c \in \mathrm{Im}(\tilde{\mathbb{O}})$$

Then ω is skew-symmetric and $\mathrm{Stab}_{GL(7,\mathbb{R})}(\omega) = \tilde{G}_2$. In basis $\{e^1, \ldots, e^7\}$ of $\mathrm{Im}(\tilde{\mathbb{O}})$, we have

$$\omega = -e^{123} - e^{145} + e^{167} + e^{246} - e^{257} - e^{347} + e^{356} \tag{8}$$

Proof. Copy the proof of 3.6, change $\mathbb{O}$ to $\tilde{\mathbb{O}}$, G_2 to $\tilde{G}_2$ and $O(8)$ to $O(4, 4)$.□

3.9 Lemma. Let $u, v \in \mathrm{Im}(\mathbb{O})$ (resp. $\mathrm{Im}(\tilde{\mathbb{O}})$) and $\omega \in \wedge^3 \mathrm{Im}(\mathbb{O})$ (resp. $\wedge^3 \mathrm{Im}(\tilde{\mathbb{O}})$) as defined in 3.6 (resp. 3.8). Then

$$\omega \wedge (\iota_u\omega) \wedge (\iota_v\omega) = -2\langle u, v\rangle e^{1234567}$$

Proof. Obviously there is a bilinear form $B : \mathrm{Im}(\mathbb{O})^2 \to \mathrm{Im}(\mathbb{O})$ such that $\omega \wedge (\iota_u \omega) \wedge (\iota_v \omega) = B(u,v) e^{1234567}$. We have

$$\begin{aligned} B(e_i, e_j) &= (\omega \wedge (\iota_{e_i}\omega) \wedge (\iota_{e_j}\omega))(e_1, \ldots, e_7) = \\ &= \frac{1}{3!2!2!} \sum_{\sigma \in S_7} \mathrm{sgn}(\sigma)(\omega \otimes (\iota_{e_i}\omega) \otimes (\iota_{e_j}\omega))(e_{\sigma(1)}, \ldots, e_{\sigma(7)}) = \\ &= \frac{1}{24} \sum_{\sigma \in S_7} \mathrm{sgn}(\sigma) \langle e_{\sigma(1)} e_{\sigma(2)}, e_{\sigma(3)} \rangle \langle e_i e_{\sigma(4)}, e_{\sigma(5)} \rangle \langle e_j e_{\sigma(6)}, e_{\sigma(7)} \rangle \end{aligned}$$

Consider the indexes of the standard basis vectors as elements of $\mathbb{Z}_2^3$, recall the twisted computations. By properties of the scalar product, the permutations $\sigma \in S_7$ for which the summand is nonzero are those satisfying

$$\begin{aligned} \sigma(1) + \sigma(2) + \sigma(3) &= 0 \\ i + \sigma(4) + \sigma(5) &= 0 \\ j + \sigma(6) + \sigma(7) &= 0 \end{aligned} \tag{9}$$

Summing all these equations we obtain $i = j$ since obviously $\sum_{g \in \mathbb{Z}_2^3 - \{000\}} g = 000$. This proves

$$B(e_i, e_j) = 0 \quad \text{for } i \neq j$$

We proceed to the case $i = j$. It is easy to convince yourself that for any $i \in \mathbb{Z}_2^3 - \{000\}$ there is a $\sigma \in S_7$ satisfying the system (9). Moreover $i = \sigma(1)$ without loss of generality. Swapping $\sigma(2)$ with $\sigma(3)$ provides a new solution and similarly for the pairs $\sigma(4), \sigma(5)$ and $\sigma(6), \sigma(7)$. Yet another solutions arise from swapping these whole pairs. This is altogether $2^3 3! = 48$ solutions and one easily sees that these are exactly all the solutions. Inspecting the signs, one verifies that all the 48 summands are equal ($\mathrm{sgn}(\sigma)$ and skew-symmetry of the scalar product cooperate). It remains to decide whether all the summands are equal to $+1$ or -1. By the above reasoning, it suffices to compute the triple scalar product for one of the 48 permutations only for each i. We eventually find that the result for the octonions is indeed -1 for each i. This yields finally $B(e_i, e_i) = \frac{1}{24} 48(-1) = -2$. For the split octonions we have the corresponding result. □

3.3 Basic properties

Now we obtain basic topological properties of G_2 and $\tilde{G}_2$. Namely we show how these group sit in the corresponding special orthogonal group (3.10) and then we construct certain principal bundles (3.15,3.16 and 3.18). To prove these theorems we introduce the useful notion of basic triples and we also prove a nice interpretation of the 3-forms associated to the groups. For G_2 we later exploit the bundle structures to compute its first 5 homotopy groups.

3.10 Theorem. There are canonical inclusions

$$G_2 \subset SO(7)$$
$$\tilde{G}_2 \subset SO(3,4)$$

Proof. We want to prove that $X \in G_2 \subset O(7)$ preserves a volume form. Choose $u \in \mathrm{Im}(\mathbb{O})$ such that $\|u\| = 1$. Then by 3.9 we have $e^{1234567} = -\frac{1}{2}\omega \wedge (\iota_u \omega) \wedge (\iota_u \omega)$.

$$\begin{aligned} X \cdot e^{1234567} &= -\frac{1}{2} X\omega \wedge (\iota_{X^{-1}(u)} X\omega) \wedge (\iota_{X^{-1}(u)} X\omega) = \\ &= -\frac{1}{2}\omega \wedge (\iota_{X^{-1}(u)}\omega) \wedge (\iota_{X^{-1}(u)}\omega) = \|X^{-1}(u)\|^2 e^{1234567} = \\ &= \|u\|^2 e^{1234567} = e^{1234567} \end{aligned}$$

since $X \cdot (\iota_u \omega) = \iota_{X^{-1}(u)} X \cdot \omega$ and we also used 3.5. Thus

$$G_2 \overset{\varphi}{\hookrightarrow} SO(7)$$

where φ is as in 3.5. The proof for $\tilde{G}_2$ is once again completely analogous.□

3.11 Claim. G_2 is not normal in $SO(7)$.

Proof. It is well known that $SO(7)$ contains no normal subgroup. □

3.12 Definition. A triple $a, b, c \in \mathrm{Im}(\mathbb{O})$ of imaginary octonions is called **basic triple** iff a, b, c are pairwise orthogonal of norm 1 and $\langle ab, c\rangle = 0$.

A triple $a, b, c \in \mathrm{Im}(\tilde{\mathbb{O}})$ of imaginary split-octonions is called **basic triple** iff a, b, c are pairwise orthogonal, $\|a\|^2 = -1$, $\|b\|^2 = \|c\|^2 = 1$ and $\langle ab, c\rangle = 0$.

3.13 Lemma. Fix a basic triple $a, b, c \in \mathrm{Im}(\mathbb{O})$ (resp. $\mathrm{Im}(\tilde{\mathbb{O}})$). Any $X \in G_2$ (resp. $\tilde{G}_2$) maps a, b, c to a basic triple. Conversly given a mapping f of a, b, c to a basic triple $f(a), f(b), f(c)$, there is exactly one $X \in G_2$ ($\tilde{G}_2$) such that $X|_{\{a,b,c\}} = f$. Thus G_2 acts transitively on the sphere $S^6 = \{v \in \mathrm{Im}(\mathbb{O}) : \|v\| = 1\}$ and on the Stiefel manifold $V_{2,7}$. $\tilde{G}_2$ acts transitively on the hyperboloid $H^{3,4} = \{v \in \mathrm{Im}(\tilde{\mathbb{O}}) : \|v\|^2 = 1\}$.

Proof. Let $X \in G_2$. Then $X(a), X(b), X(c)$ are pairwise orthogonal because X preserves the scalar product. Finally $\langle X(a)X(b), X(c)\rangle = \langle X(ab), X(c)\rangle = \langle ab, c\rangle = 0$, so $X(a), X(b), X(c)$ is a basic triple.

Now we claim that elements of $B = \{a, b, ab, c, ac, bc, (ab)c\}$ are pairwise orthogonal, thus form a basis of $\mathrm{Im}(\mathbb{O})$. The verification involves a long but straightforward computation using 2.7, 2.11 and the alternativity of the octonions. For example

$$\langle b, (ab)c\rangle = \langle bc^*, ab\rangle = -\langle bc, ab\rangle = \langle cb, ab\rangle = \langle c, a\rangle = 0$$

Suppose we are given f as in the theorem and we are to construct X. The claim implies that X is given uniquely by values on B. We have $X(a) = f(a), X(b) = f(b), X(c) = f(c)$. In order to have $X \in G_2$ we must define $X(ab) = X(a)X(b) = f(a)f(b)$ and similarly for the other elements of B. It remains to verify $X \in G_2$. Because $X(a), X(b), X(c)$ is a basic triple we see that $X \in GL(7)$. Finally $X(kl) = X(k)X(l)$ holds for the basis vectors by definition of X and thus holds for all of $\mathrm{Im}(\mathbb{O})$.

For $\tilde{G}_2$ we just mention the following general fact: Let V be a vector space of signature $(p+, q-, 0)$, $S \subset V$ any linearly independent subset. Let $u \in V$ with $\|u\|^2 \neq 0$. Then $\langle u, S\rangle = 0$ implies that $u \cup S$ is a linearly independent set. This justifies proving the linear independence via orthogonality. Otherwise the proof is formally the same. □

3.14 Remark. We have even proved that G_2 acts transitively on basic triples with null stabilizer. Space of basic triples is a topological subspace of $V_{3,7}$. This establishes the inclusion $G_2 \subset V_{3,7}$.

Consider the basic triple $e^1, e^2, e^4 \in \mathrm{Im}(\mathbb{O})$. In the formula (7) for the G_2-form we notice the exceptional position of e_4. This motivates the following : We can define a complex structure on the vector space V with basis $e_1, e_2, e_3, e_5, e_6, e_7$ by $J(x) = xe_4$. Indeed $J^2(x) = (xe_4)e_4 = -x$. Obviously $J(e_k) = e_{k+4}$. Then denote $z^k := e^k + ie^{k+4}$ and $\overline{z}^k := e^k - ie^{k+4}$ for $k = 1, 2, 3$ the complex coordinates. We have the holomorphic volume form

$$\Omega_3 := z^{123} = (e^1 + ie^5) \wedge (e^2 + ie^6) \wedge (e^3 + ie^7)$$

and Kähler form

$$\omega_3 := \frac{i}{2}(z^1\overline{z}^1 + z^2\overline{z}^2 + z^3\overline{z}^3) = e^{15} + e^{26} + e^{37}$$

The point is that

$$\omega = \mathrm{Re}(\Omega_3) - e^4 \wedge \omega_3$$

Recall that the complex determinant of $X \in GL(V, \mathbb{C})$ satisfies

$$\det_{\mathbb{C}} X = \Omega_3(X(e_1), X(e_2), X(e_3))$$

This is clear by the basis representation of X as an element of $GL(V, \mathbb{R})$ and by recalling the action of J on e_k's. Finally recall that V is equipped with a canonical hermitean form $\langle\rangle_{\mathbb{C}}$ defined by

$$\langle x, y\rangle_{\mathbb{C}} := \langle x, y\rangle + i\langle x, Jy\rangle$$

Now we are prepared to prove the following :

3.15 Theorem. There is a diffeomorphism

$$G_2/SU(3) \cong S^6$$

thus a principal $SU(3)$-bundle

$$SU(3) \hookrightarrow G_2 \twoheadrightarrow S^6$$

Proof. We already know - 3.13 - that G_2 acts transitively on S^6. Let $u \in S^6 = S^7 \cap \mathrm{Im}(\mathbb{O})$ and we want to see what is $\mathrm{Stab}_{G_2} u$. Without any loss of generality assume $u = e_4$. As in the introduction, consider $V := \mathrm{Im}(\mathbb{O}) \cap (\mathbb{R}u)^{\perp} = \mathbb{R}\{e_1, e_2, e_3, e_5, e_6, e_7\}$ with the complex structure $J(a) := au$. Let $X \in \mathrm{Stab}_{G_2}(u)$.

$$JX(a) = X(a)u = X(a)X(u) = X(au) = XJ(a)$$

thus $X \in GL(V, \mathbb{C})$.

$$\begin{aligned}\langle X(a), X(b)\rangle_{\mathbb{C}} &= \langle X(a), X(b)\rangle + i\langle X(a), JX(b)\rangle = \langle a, b\rangle + i\langle X(a), XJ(b)\rangle = \\ &= \langle a, b\rangle_{\mathbb{C}}\end{aligned}$$

thus $X \in U(3)$.

$$\begin{aligned}\mathrm{Re}(\det_{\mathbb{C}} X) &= \mathrm{Re}(\Omega_3(X(e_1), X(e_2), X(e_3))) = \\ &= (\omega + e^4 \wedge \omega_3)(X(e_1), X(e_2), X(e_3)) = \\ &= \omega(X(e_1), X(e_2), X(e_3)) + (e^{415} + e^{426} + e^{437})(X(e_1), X(e_2), X(e_3))\end{aligned}$$

Now use invariance of ω under the G_2-action and $\langle X(e_1), e_4\rangle = \langle X(e_1), X(e_4)\rangle = \langle e_1, e_4\rangle = 0$ and analogous $\langle X(e_2), e_4\rangle = \langle X(e_3), e_4\rangle = 0$. Thus

$$\mathrm{Re}(\det_{\mathbb{C}} X) = \omega(e_1, e_2, e_3) = 1$$

Because $X \in U(3)$ implies $|\det_{\mathbb{C}} X| = 1$, we get $\det_{\mathbb{C}} X = 1$ and consequently $\mathrm{Stab}_{G_2}(u) \subset SU(3)$.

Now we are going to prove the converse inclusion $SU(3) \subset \mathrm{Stab}_{G_2}(u)$. Let $X' \in SU(3)$ where we view $SU(3)$ as acting on $V = \mathbb{R}\{e_1, e_2, e_3, e_5, e_6, e_7\}$ with a complex structure $J(a) = ae_4$. Extend X' to $X \in O(\mathrm{Im}(\mathbb{O}))$ by setting $X(e_4) = e_4$ and $X(e_i) = X'(e_i)$ for $i \neq 4$. We easily see that all we have to verify in order to prove $X \in G_2$ is $X(e_1)X(e_2) = X(e_3)$. Indeed look how X maps the basis $\{e_1, e_2, e_3 = e_1e_2, e_4, e_5 = e_1e_4, e_6 = e_2e_4, e_7 = (e_1e_2)e_4\}$: $e_5 = e_1e_4$ is mapped onto $XJ(e_1) = JX(e_1) = X(e_1)e_4 = X(e_1)X(e_4)$ which is OK and similarly e_6, e_7, the only unclear case being $X(e_1)X(e_2) = X(e_3)$. But we have

$$1 = \mathrm{Re}(\det_{\mathbb{C}} X) = \omega(X(e_1), X(e_2), X(e_3)) = \langle X(e_1)X(e_2), X(e_3)\rangle$$

the second equality being proved as in the previous case. But $X(e_1)X(e_2)$ and $X(e_3)$ are of norm 1, thus the desired equality. □

3.16 Theorem. There is a diffeomorphism

$$G_2/SU(2) \cong V_{2,7}$$

Proof. This is analogous to the previous theorem. The determinant condition is managable by a direct basis computation. Since we don't use this theorem any further, we ommit details. □

3.17 Theorem. G_2 is compact, connected 14-dimensional Lie group.

Proof. Connectedness, dimension and compactness follows immediately from the principal bundle structure on G_2. □

Now we mimic the previous procedure for $\tilde{G}_2$. We have an interpretation of the $\tilde{G}_2$ form ω (see (3.8)) similar as before : Denote

$$\Omega_3 := (e^1 + ie^5) \wedge (e^2 + ie^6) \wedge (e^3 + ie^7)$$
$$\omega_3' := -e^{15} + e^{26} - e^{37}$$

Then

$$\omega = -\mathrm{Re}(\Omega_3) - e^4 \wedge \omega_3'$$

Let's define

$$SU(1, 2)$$

to be the group of complex 3×3 matrices of complex determinant 1 which preserve a fixed hermitean form h given by

$$h(x, y) := +x_1\overline{y_1} - x_2\overline{y_2} - x_3\overline{y_3} \quad \text{for all } x, y \in \mathbb{C}^3$$

In our case this group $SU(1,2)$ will arise from a 6-dimensional vector space V with a metric $\langle\rangle$ of signature $(2+,4-)$ with a complex structure J satisfying $\|J(v)\|^2 = \|v\|^2$. Then the familiar formula

$$\langle x, y\rangle_{\mathbb{C}} := \langle x, y\rangle + i\langle x, Jy\rangle$$

defines the desired hermitean metric.

3.18 Theorem. There is a diffeomorphism

$$\tilde{G}_2/SU(1,2) \cong H^{3,4}$$

Proof. This is analogous to the lemma 3.15 for G_2. In fact only the very last argument concerning the equality $X(e_1)X(e_2) = X(e_3)$ requires a different reasoning due to the indefinite signature : $\|X(e_1)X(e_2)\|^2 = -1$ and we consider its projections to the basis $X(e_1), \ldots, X(e_7)$. We show that $\langle X(e_1)X(e_2), X(e_i)\rangle = 0$ for $i = 2, 4, 6$, so $X(e_1)X(e_2)$ lies in a subspace of negative signature. By $1 = \langle X(e_1)X(e_2), X(e_3)\rangle$ the conclusion follows. So for $i = 2$ the equality $\langle X(e_1)X(e_2), X(e_1)\rangle = 0$ is obvious. For $i = 4$ we have

$$\begin{aligned}\langle X(e_1)X(e_2), X(e_4)\rangle &= \langle X(e_1)X(e_2), e_4\rangle = -\langle X(e_2), X(e_1)e_4\rangle = \\ &= -\langle X(e_2), JX(e_1)\rangle = -\langle X(e_2), X(e_5)\rangle = 0\end{aligned}$$

since X preserves the metric.

$$\begin{aligned}\langle X(e_1)X(e_2), X(e_6)\rangle &= \langle X(e_1)X(e_2), X(e_2)X(e_4)\rangle = \\ &= -\langle X(e_1)X(e_2), X(e_4)X(e_2)\rangle = \\ &= -\langle X(e_1), X(e_4)\rangle = \langle X(e_1)X(e_4), 1\rangle = \\ &= \langle X(e_1)e_4, 1\rangle = \langle JX(e_1), 1\rangle = \langle X(e_5), 1\rangle = 0\end{aligned}$$

3.19 Theorem. $\tilde{G}_2$ is noncompact connected 14-dimensional Lie group.

Proof. $H^{3,4}$ is obviously noncompact and connected, so the claim follows from 3.18. □

3.4 More on octonions - Triality

We obtain some deeper results on (split-)octonions. Namely we prove the Moufang identities 3.21 which are useful substitutes for the missing associativity. Finally we obtain the triality principle 3.25. The exposition in this section is adapted from [5].

3.20 Lemma. Let A be a good alternative[6] algebra and $a, b, c, d \in A$. Then

$$\langle ab, cd\rangle + \langle ad, cb\rangle = 2\langle a, c\rangle\langle b, d\rangle$$

Proof.

$$\begin{aligned}\langle ab, cd\rangle + \langle ad, cb\rangle &= \langle (ab)d^*, c\rangle + \langle (ad)b^*, c\rangle = \\ &= \langle a(bd^*) + (a, b, d^*) + a(db^*) + (a, d, b^*), c\rangle = \dots\end{aligned}$$

Now use 2.8 and the alternativity to get $(a, b, d^*) + (a, d, b^*) = 0$. Thus

$$\dots = \langle a(bd^* + db^*), c\rangle = \langle a2\langle b, d\rangle, c\rangle = 2\langle a, c\rangle\langle b, d\rangle$$

3.21 Theorem (Moufang identities). Let A be a nondegenerate alternative algebra. For all $x, y, z \in A$ we have

1. $x(yz)x = (xy)(zx)$
2. $x(yz) = (xyx)(x^{-1}z)$
3. $(xy)z = (xz^{-1})(zyz)$

Proof. We use 2.11 and 3.20.

1. Let $t \in A$ be arbitrary.

$$\begin{aligned}\langle (xy)(zx), t\rangle &= \langle xy, t(x^*z^*)\rangle = -\langle x(x^*z^*), ty\rangle + 2\langle x, t\rangle\langle y, x^*z^*\rangle = \\ &= -\|x\|^2\langle z^*, ty\rangle + 2\langle x, t\rangle\langle xy, z^*\rangle = \\ &= -\|x\|^2\langle z^*y^*, t\rangle + 2\langle x, t\rangle\langle x, z^*y^*\rangle = \\ &= -\|x\|^2\langle (yz)^*, t\rangle + 2\langle x, t\rangle\langle x, (yz)^*\rangle\end{aligned}$$

This proves

$$(xy)(zx) = -\|x\|^2(yz)^* + 2\langle x, (yz)^*\rangle x \tag{10}$$

We see that $(xy)(zx)$ depends only on x and on the product yz. So changing the pair y, z to another one with the same product leaves the result unchanged. Replace y by yz and z by 1 to get

$$(xy)(zx) = (x(yz))(1x)$$

[6] In fact we need only skew-symmetry of the associator in the second and third variable.

2. Let $t \in A$ be arbitrary.

$$\langle (xyx)(x^{-1}z), t\rangle = \langle x(yx), t(z^*(x^{-1})^*)\rangle = \dots$$

Observe that $(x^{-1})^* = \frac{x}{\|x\|^2}$.

$$\begin{aligned} \dots &= -\frac{1}{\|x\|^2}\langle x(z^*x), t(yx)\rangle + \frac{2}{\|x\|^2}\langle x, t\rangle\langle yx, z^*x\rangle = \\ &= -\frac{1}{\|x\|^2}\langle z^*, x^*(t(yx))x^*\rangle + 2\langle x, t\rangle\langle y, z^*\rangle = \dots \end{aligned}$$

Now we apply the Moufang identity we have already proved to get $x^*(t(yx))x^* = (x^*t)(yxx^*) = \|x\|^2(x^*t)y$:

$$\dots = -\langle z^*, (x^*t)y\rangle + 2\langle x, t\rangle\langle yz, 1\rangle = -\langle (yz)^*, x^*t\rangle + 2\langle x, t\rangle\langle yz, 1\rangle$$

Thus

$$(xyx)(x^{-1}z) = -x(yz)^* + 2(\mathrm{Re}(yz))x = -x(yz)^* + x(yz + (yz)^*) = x(yz)$$

3. This is similar to the previous case. □

3.22 Lemma. If

$$(xy)z = x(yz)$$

holds for all $x, z \in \mathbb{O}$ (resp. $\tilde{\mathbb{O}}$), then $y \in \mathbb{R}$.

Proof. Let $a, b, c \in \mathrm{Im}(\mathbb{O})$ be such that $\|a\|^2 = \|c\|^2 = 1$ and

$$(ab)c = -a(bc) \tag{11}$$

Then

$$\langle y, b\rangle = \langle (ay)c, (ab)c\rangle$$

Now use the assumption of the lemma to swap brackets in the first slot and use (11) to swap brackets in the second slot :

$$\langle y, b\rangle = \langle a(yc), -a(bc)\rangle = -\langle yc, bc\rangle = -\langle y, b\rangle$$

Thus (11) implies $\langle y, b\rangle = 0$. For every $j = 1, \dots, 7$ we have $\langle y, e_j\rangle = 0$ since for every e_j we can find e_i, e_k satisfying (11). For example for e_1 the triple is e_2, e_1, e_4 since $(e_2e_1)e_4 = -e_2(e_1e_4)$.

For the split octonions the proof is analogous, we also permit $\|a\|^2 = \|b\|^2 = -1$ and then find the triples. □

3.23 Lemma. Let $a \in \mathbb{O}$ (resp. $\tilde{\mathbb{O}}$). Denote $r(a)$ the reflection along a hyperplane orthogonal[7] to a. Then there is $a \in \mathbb{O}$ ($\tilde{\mathbb{O}}$) such that

$$\begin{aligned} r(a)r(1)(x) &= \frac{1}{\|a\|^2} axa \\ r(1)r(a)(x) &= \frac{1}{\|a\|^2} a^{-1}xa^{-1} \end{aligned}$$

for all $x \in \mathbb{O}$ ($\tilde{\mathbb{O}}$).

Proof. We have $r(1)(x) = x - 2\langle x, 1\rangle = -x^*$ and $r(a)(-x^*) = -x^* - 2\frac{\langle -x^*, a\rangle}{\|a\|^2} a$. Thus

$$\|a\|^2 r(a)r(1)(x) = -\|a\|^2 x^* + 2\langle x^*, a\rangle a$$

and by the equality (10) we have

$$\|a\|^2 r(a)r(1)(x) = axa$$

The second equality is analogous. □

3.24 Lemma. Let $A, B, C \in SO(8)$ (resp. $SO(4,4)$). Then $A(xy) = B(x)C(y)$ for all $x, y \in \mathbb{O}$ (resp. $\tilde{\mathbb{O}}$) iff there are $b, c \in \mathbb{O}$ ($\tilde{\mathbb{O}}$) such that $A(xy) = (A(x)b)(cA(x))$. The (split-)octonions b, c are called the **companions** of A. The companions uniquely determine and are uniquely determined by B, C via the following relations :

$$\begin{aligned} B(x) &= A(x)b, \quad b = C(1)^{-1} \\ C(x) &= cA(x), \quad c = B(1)^{-1} \end{aligned}$$

Proof. Put $y = 1$ to obtain $A(x) = B(x)C(1)$ and set $b := C(1)^{-1}$. Put $x = 1$ to obtain $A(y) = B(1)C(y)$ and set $c := B(1)^{-1}$. The converse is obvious.

For split octonions we have to check whether the inverses are defined, i.e. whether $\|B(1)\|^2$ and $\|C(1)\|^2$ are nonzero. But we have

$$\langle A(x), A(y)\rangle = \langle B(x)C(1), B(y)C(1)\rangle = \|C(1)\|^2\langle B(x), B(y)\rangle = \|C(1)\|^2\langle x, y\rangle$$

hence $\|C(1)\|^2 = 1$ and similarly for $B(1)$. □

[7]Orthogonality is meant with respect to the scalar product $\langle\rangle$. For the split octonions this lacks the usual geometric sense.

3.25 Theorem (Triality). For every $A \in SO(8)$ (resp. $SO(4,4)$) there are $B, C \in SO(8)$ ($SO(4,4)$) such that

$$A(xy) = B(x)C(y) \quad \text{for all } x, y \in \mathbb{O}\ (\tilde{\mathbb{O}})$$

The only other pair of elements of $SO(8)$ ($SO(4,4)$) with this property is $-B, -C$. Equivalently, $A \in SO(8)$ ($SO(4,4)$) has companions $b, c \in S^7$ (resp. $\in H^{4,4}$, the hyperboloid) and the only other pair of companions of A is $-b, -c$.

Proof.

1. Existence of B, C :
By the Cartan-Dieudonné theorem ([8]) there is even number of reflections $r(a_i)$ ($i = 1, \ldots, 2n$) such that $A = r(a_1) \cdots r(a_{2n})$. We can suppose $\|a_i\| = 1$. Write

$$A = r(a_1)r(1)r(1)r(a_2) \cdots r(a_{2n-1})r(1)r(1)r(a_{2n})$$

By lemma 3.23 we have $r(a_i)r(1)(xy) = (a_i x)(y a_i)$ and $r(1)r(a_i)(xy) = (a_i^{-1}x)(y a_i^{-1})$ because of the Moufang identity.

$$A(xy) = (a_1(a_2^{-1}(\cdots(a_{2n-1}(a_{2n}^{-1}x)\cdots)(\cdots(y a_{2n}^{-1})a_{2n-1})\cdots a_2^{-1})a_1)$$

Thus

$$A(xy) = (L_{a_1}L_{a_2^{-1}} \cdots L_{a_{2n-1}}L_{a_{2n}^{-1}})(x)(R_{a_1}R_{a_2^{-1}} \cdots R_{a_{2n-1}}R_{a_{2n}^{-1}})(y)$$

$B(x) := (L_{a_1}L_{a_2^{-1}} \cdots L_{a_{2n-1}}L_{a_{2n}^{-1}})(x)$ is in $SO(8)$ since a_i's are of norm 1. And for $C(y)$ similarly.

2. Uniqueness property :
Let $A \in SO(8)$ and let both r, s and r', s' are its companions. We have

$$(A(x)r)(sA(y)) = (A(x)r')(s'A(y))$$

Choose $t \in \mathbb{O}$ such that $r' = rt$. Choose $x \in \mathbb{O}$ such that $A(x) = r^{-1}$ (we are going to abbreviate this by saying "let $A(x) = r^{-1}$") and let $A(y) = 1$ to obtain

$$s = ts', \text{ i.e. } s' = t^{-1}s$$

So we have

$$(A(x)r)(sA(y)) = (A(x)(rt))((t^{-1}s)A(y))$$

Let $A(x) = (rt)^{-1}$ so that we get

$$t^{-1}(sA(y)) = (t^{-1}s)A(y)$$

Let $A(y) = (t^{-1}s)^{-1}$ so that we get

$$(A(x)r)t = A(x)(rt)$$

This yields

$$(A(x)r)(sA(y)) = ((A(x)r)t)(t^{-1}(sA(y)))$$

Finally let $X = A(x)r$ and let $tY = sA(y)$ to get

$$X(tY) = (Xt)Y$$

Since X, Y can be arbitrary elements of $\mathbb{O}$, we can apply lemma 3.22 to get $r \in \mathbb{R}$.

Now since $B, C \in SO(8)$, the norm of the companions $r = C(1)^{-1}, s = B(1)^{-1}$ must be equal to 1. Therefore the $t \in \mathbb{R}$ from the previous discussion must be equal either $+1$ or -1.

3. For split-octonions the proof is formally same, we just ocassionally check correctness of inverses. □

3.5 Homogenenous spaces $SO(7)/G_2$ and $SO(3,4)/\tilde{G}_2$

Now we exploit the results of the previous section to obtain identifications of the homogeneous spaces mentioned.

3.26 Lemma ([5]). If a, b are the companions of A and c, d are the companions of B, then $(ab)(A(c)a), (bA(d))(ab)$ are the companions of AB.

Proof.

$$\begin{aligned} AB(xy) = A((B(x)c)(dB(y))) = (A(B(x)c)a)(bA(dB(y))) = \\ = [((AB(x)a)(bA(c)))a][b((A(d)a)(bAB(y)))] = \dots \end{aligned}$$

by the definition of the companions. Further by the Moufang identities 3.21 we obtain

$$\begin{aligned} \dots = [((AB(x)a)a^{-1})(a(bA(c))a)][(b(A(d)a)b)(b^{-1}(bAB(y)))] = \\ = [AB(x)((ab)(A(c)a))][((bA(d))(ab))AB(y)] \end{aligned}$$

3.27 Example. The companions of L_a are a, a^{-2}. This is a direct consequence of the Moufang law 3.21, 2. and uniqueness of the companions. Analogously, the companions of R_a are a^{-2}, a. By the previous lemma 3.26, we get the companions for $L_a R_{a^{-1}}$, namely a^3, a^{-3}. Thus

$$a(xy)a^{-1} = (axa^3)(a^{-3}xa^{-1}) \quad \text{for all } x, y \in \mathbb{O}\ (\tilde{\mathbb{O}}) \tag{12}$$

3.28 Lemma. Let $A \in SO(8) = SO(\mathbb{O})$ be such that $A(1) = 1$, so $A \in SO(7)$. Then there is $r(A) \in S^7 \in \mathbb{O}$, determined uniquely up to sign, satisfying

$$A(xy) = (A(x)r(A))(r(A)^{-1}A(y)) \quad \text{for all } x, y \in \mathbb{O}$$

The map

$$SO(7) \xrightarrow{r} \mathbb{R}P^7$$

is well defined, surjective and $\mathrm{Ker}(r) := \{A \in SO(7) : r(A) = \pm 1 \in \mathbb{R}P^7\} = G_2$.

Proof. By the triality principle 3.25 we have some $r, s \in S^7$ such that $A(xy) = (A(x)r)(sA(y))$ for all $x, y \in \mathbb{O}$. Put $x = y = 1$ to get $1 = rs$. Thus A has companions r, r^{-1}. $\|r\| = 1$ because $A \in SO(8)$. Uniqueness also follows from the triality. $\mathrm{Ker}(r) = G_2$ is obvious. To prove surjectivity let $\pm z \in \mathbb{R}P^7$. There is $w \in S^7$ such that $w^3 = z$. Indeed, we can choose an unit octonion orthogonal to z, then the algebra generated by this element and by z is obviously isomorphic to the complex numbers. There we are able to solve the equation $w^3 = z$. Surely $\|w\| = 1$. Set

$$A(x) := wxw^{-1} = L_w R_{w^{-1}}(x)$$

We are to check $w(xy)w^{-1} = ((wx)w^3)(w^{-3}(yw^{-1}))$, but this is (12). $\square$

3.29 Theorem. There is a homeomorphism

$$SO(7)/G_2 \cong \mathbb{R}P^7$$

Proof. Use the map r from the previous lemma to define

$$\begin{aligned} SO(7)/G_2 &\xrightarrow{R} \mathbb{R}P^7 \\ [A] &\mapsto r(A) \end{aligned}$$

where $[A]$ denotes a class modulo G_2. First we observe that R is well defined : let $[A] = [B]$, i.e. $B = AG$ for some $G \in G_2$. By lemma 3.26 we have $r(A) = r(AG)$. Obviousy R is surjective and is easily seen to be even injective.

The continuity of R depends on the choice of the reflections in the proof of the triality principle 3.25. But this refers back to the Cartan-Dieudonné theorem, so we would have to analyze the proof of this theorem. We don't give any formal proof and we just rely on reader's geometric intuition. Taking the continuity of R granted, the continuity of the inverse follows from compactness. □

Rewriting the proof of the previous 2 claims for $\tilde{\mathbb{O}}$ we obtain :

3.30 Theorem. There is a homeomorphism

$$SO(3,4)/\tilde{G}_2 \cong PH^{4,4}$$

where $PH^{4,4}$ is the projectivization of the hyperboloid $H^{4,4}$. □

There are very useful tables of homotopy groups in [6]. We use these without mentioning throughout the paper. Namely we consider known low homotopy groups of the following spaces (at least for small dimensions) : spheres, $SU(n)$, $Spin(n)$ and real Stiefel manifolds.

3.31 Theorem. The first five homotopy groups of G_2 are as follows :

i	1	2	3	4	5
$\pi_i(G_2)$	0	0	$\mathbb{Z}$	0	0

Proof. Use 3.15 and homotopy long exact sequence (HLES) to compute $\pi_1(G_2) = 0$. Then use 3.29 and HLES to obtain the rest. □

4 G_2-structures on manifolds

4.1 Basic properties

In this section we introduce the well known notion of G_2-manifold. We give 4 definitions of G_2 manifolds : via reduction of the frame bundle, via existence of the 3-form associated to G_2, via cross product and finally via octonionic structure which formalizes the intuitive idea of identifying the tangent bundle with the imaginary octonions. In 4.9 and 4.10 we prove equivalence of all the definitions.

Through this text, M^d denotes a smooth manifold[8] of dimension d.

4.1 Definition. Let G be a Lie subgroup of $GL(d, \mathbb{R})$. A G-structure on a manifold M^d is a principal G-subbundle of the frame bundle of M.

[8]We consider hausdorff and second countable manifolds only.

Equivalently, a G-structure is a reduction of the structure group of the tangent bundle TM to the group G. It's well known that there is a G-structure on the manifold M iff M admits an oper cover trivializing TM such that its transition functions take their values in $G \subset GL(d)$.

4.2 Theorem. A 7-manifold M admits a G_2-structure iff there is a global form $\omega \in \wedge^3 T^*M$ and an open cover $\mathcal{U}$ of M trivializing the tangent bundle TM such that ω is given by

$$\omega = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356} \tag{13}$$

on arbitrary $U \in \mathcal{U}$.

Proof.

- Suppose there is an open cover $\mathcal{U}$ with G_2-valued transition functions of the tangent bundle. On every $U \in \mathcal{U}$ define ω_U by the local formula (7). Choose any two $U, V \in \mathcal{U}$ be such that $U \cap V \neq \emptyset$. Let ψ_{VU} be the transition function of the tangent bundle, we have $\psi_{VU} \in G_2$. So ω transforms as $\omega_V(a, b, c) = \omega_U(\psi_{VU}a, \psi_{VU}b, \psi_{VU}c) = (\psi_{VU} \cdot \omega_U)(a, b, c)$. Since G_2 is the stabilizer of ω (by 3.6) and U, V are arbitrary, the form ω is defined globally on M.
- Conversely, if there is an open cover such that (13) defines a global form ω, then the differentials of the transition functions of the manifold take their values in the stabilizer of ω, i.e. in G_2. □

4.3 Theorem. A 7-manifold M admits a $\tilde{G}_2$-structure iff there is a global form $\omega \in \wedge^3 T^*M$ and an open cover $\mathcal{U}$ of M trivializing the tangent bundle TM such that ω is given by

$$\omega = -e^{123} - e^{145} + e^{167} + e^{246} - e^{257} - e^{347} + e^{356}$$

on arbitrary $U \in \mathcal{U}$.

Proof. This is analogous to the previous statement - use 3.8. □

4.4 Theorem. Let M admit a G_2 ($\tilde{G}_2$) structure given by $\omega \in \wedge^3 T^*M$. Then

1. M is orientable.
2. Let $\Omega \in \wedge^7 T^*M$ be the volume corresponding to a choice of orientation. There is a metric g on M canonically associated to ω and Ω by

$$\omega \wedge \iota_u \omega \wedge \iota_v \omega = -2g(u, v)\Omega$$

Proof. 1. $G_2 \subset SO(7)$ by 3.10, so the transition functions of TM with the G_2-reduction preserve the orientation. Thus we can define a global 7-form on M. The same proof goes for $\tilde{G}_2$.

2. This is essentially 3.9. □

4.5 Definition. Let (M, g) be a 7-dimensional (pseudo-)riemannian manifold. Let

$$\mu \in T^*M \otimes T^*M \otimes TM$$

For every $m \in M$ we define a $*$-algebra O_m as follows :

$$O_m := \mathbb{R}1 \oplus T_mM$$

as a vector space. The multiplication on O_m is given by

1. 1 is the unit of O_m
2. For $x, y \in T_mM$ define

$$xy := \mu_m(x, y) - g_m(x, y)1$$

The conjugation is defined by

$$(r1 + x)^* := r1 - x \quad \text{for } r \in \mathbb{R}, x \in T_mM$$

$\mu \in T^*M \otimes T^*M \otimes TM$ is called the **(split-)octonionic structure** iff for every $m \in M$ the $*$-algebra O_m is isomorphic to the (split-)octonions $\mathbb{O}$ (resp. $\tilde{\mathbb{O}}$).

Therefore $\mu_m(x, y)$ corresponds to $\mathrm{Im}(xy)$ for $x, y \in \mathrm{Im}(\mathbb{O})$ (resp. $\mathrm{Im}(\tilde{\mathbb{O}})$). Observe that in fact $\mu \in \wedge^2(T^*M) \otimes TM$.

4.6 Remark. There would be a more natural definition of the octonionic structure for 8-dimensional manifolds. However it turns out that in this case the 1-dimensional trivial bundle comming from $1 \in \mathbb{O}$ splits off the tangent bundle.

4.7 Definition. Let (M, g) be a pseudoriemannian manifold. r**-fold cross product structure** on (M, g) is a TM-valued skew-symmetric global r-form P on M, i.e.

$$P \in \wedge^r T^*M \otimes TM$$

subject to axioms

1. $\|P(v_1,\ldots,v_r)\| = \|v_1 \wedge \ldots \wedge v_r\| = \det(g(v_i,v_j))_{i,j}$

2. $g(P(v_1,\ldots,v_r),v_j) = 0$ for $1 \le j \le r$.

See [7] for more details.

4.8 Lemma. For $a, b \in \mathrm{Im}(\mathbb{O})$ we have

$$\langle \mathrm{Im}(ab), \mathrm{Im}(ab)\rangle = \|a\|^2\|b\|^2 - \langle a,b\rangle^2$$

Proof. Use 2.7 and 2.11 :

$$\begin{aligned}\langle \mathrm{Im}(ab), \mathrm{Im}(ab)\rangle &= \frac{1}{4}\langle ab - ba, ab - ba\rangle = \\ &= \frac{1}{4}\langle 2ab + 2\langle a,b\rangle, 2ab + 2\langle a,b\rangle\rangle = \\ &= \langle ab, ab\rangle + 2\langle a,b\rangle\langle 1, ab\rangle + \langle a,b\rangle^2 = \\ &= \|a\|^2\|b\|^2 + 2\langle a,b\rangle\langle b^*, a\rangle + \langle a,b\rangle^2 = \\ &= \|a\|^2\|b\|^2 - \langle a,b\rangle^2\end{aligned}$$

4.9 Theorem. The following conditions are equivalent for a 7-manifold M:

1. M admits a G_2-structure.

2. There is a global form $\omega \in \wedge^3 T^*M$ and an open cover $\mathcal{U}$ of M trivializing the tangent bundle TM such that ω is given by

$$\omega = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}$$

on arbitrary $U \in \mathcal{U}$.

3. There is a riemannian metric g on M and an octonionic structure μ associated to this metric.

4. There is a riemannian metric g on M and a cross product structure P associated to this metric.

Proof.

- "(1) $\Leftrightarrow$ (2)" is 4.2.

- "(2) $\Rightarrow$ (3)" There is the riemannian metric g by 4.4. Define $\mu \in T^*M \otimes T^*M \otimes TM$ by

$$g(\mu(x,y),z) := \omega(x,y,z) \tag{14}$$

We immediately see that μ is skew-symmetric. We want to verify that the $*$-algebra O (subscript m ommited) is isomorphic to $\mathbb{O}$. By the Hurewicz theorem 2.36, it suffices to show O is positive definite (which is immediate from the definition of O and the signature of g) and satisfies $\|ab\|^2 = \|a\|^2\|b\|^2$ for $a, b \in O$ (with $\|a\|^2 = aa^*$). Denote the real part of $a \in O$ by a_r and the imaginary part by a_i. Abbreviate $g(x) := g(x, x)$. We have

$$\|a\|^2 = a_r^2 - a_i^2 = a_r^2 - \mu(a_i, a_i) + g(a_i) = a_r^2 + g(a_i)$$
$$ab = a_r b_r + a_r b_i + a_i b_r + \mu(a_i, b_i) - g(a_i, b_i)$$

Using these formulas and definition of μ one easily computes

$$\|ab\|^2 = (a_r b_r - g(a_i, b_i))^2 + a_r^2 g(b_i) + 2a_r b_r g(a_i, b_i) + b_r^2 g(a_i) + g(\mu(a_i, b_i))$$
$$\|a\|^2\|b\|^2 = (a_r b_r)^2 + a_r^2 g(b_i) + b_r^2 g(a_i) + g(a_i)g(b_i) \tag{15}$$

So we are to verify

$$g(\mu(a_i, b_i)) = g(a_i)g(b_i) - g(a_i, b_i)^2 \tag{16}$$

It suffices to prove this in the octonionic formalism, since $\mu(a_i, b_i)$ has the same definition (14) as the imaginary part of the octonionic multiplication of two imaginary octonions. But it has been done in advance in the previous lemma.

- "(3) $\Rightarrow$ (2)" Define

  $$\omega(x, y, z) := g(\mu(x, y), z)$$

  By the definition of $O \cong \mathbb{O}$ we have

  $$\mu(x, y) = \mathrm{Im}(xy) \text{ and } g(x, y) = \langle x, y\rangle$$

  Therefore $\omega(x, y, z) = \langle \mathrm{Im}(xy), z\rangle = \langle xy, z\rangle$ and use 3.6.

- "(3) $\Rightarrow$ (4)" Define the cross product $P \in \wedge^2(T^*M) \otimes TM$ by

  $$P(x, y) := \mu(x, y)$$

  Let's verify the axioms of the cross product :

  1. $g(P(x, y)) = g(x)g(y) - g(x, y)^2$? Switch to octonions by the definition of μ and then this is again the previous lemma.

2. $g(P(x,y),y) = g(P(x,y),y) = 0$? We have $\langle \mathrm{Im}(xy), y\rangle = \langle xy, y\rangle = \|y\|^2\langle x, 1\rangle = 0$ and the other equality is analogous.

- "(4) $\Rightarrow$ (3)" Define the octonionic structure μ by

$$\mu(x,y) := P(x,y)$$

The idea is very similar to that of "(2) $\Rightarrow$ (3)" - we use the Hurewicz theorem to prove $O \cong \mathbb{O}$. All the computations go the same way (vanishing of certain terms of (15) is because of the second axiom of the scalar product) until we end up with verifying (16) again. This time it holds thanks to the first axiom of the cross product. □

4.10 Theorem. The following conditions are equivalent for a 7-manifold M:

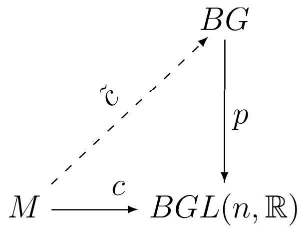
1. M admits a $\tilde{G}_2$-structure.
2. There is a global form $\omega \in \wedge^3 T^*M$ and an open cover $\mathcal{U}$ of M trivializing the tangent bundle TM such that ω is given by (8) on arbitrary $U \in \mathcal{U}$.
3. There is a pseudoriemannian metric g of signature $(3+, 4-)$ on M and an split-octonionic structure μ associated to this metric.
4. There is a pseudoriemannian metric g of signature $(3+, 4-)$ on M and a cross product structure associated to this metric.

Proof. It is analogous to 4.9 using the corresponding results, namely 2.37 instead of the Hurewicz theorem. □

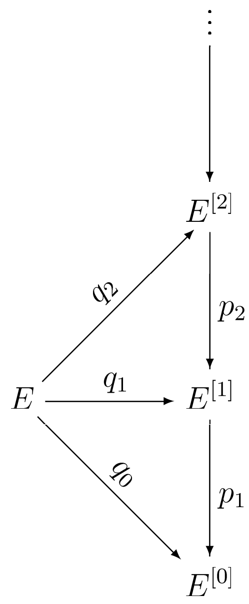
4.2 Topological toolbox

The purpose of this section is mainly to set the notation. We collect some well known topological facts. We ommit any proofs here.

4.11 Theorem. Let $G \subset GL(n,\mathbb{R})$ and M^n be a manifold with the frame bundle $F(M)$ considered as a principal $GL(n,\mathbb{R})$-bundle. Denote c the classifying map $M \xrightarrow{c} BGL(n)$ of the frame bundle. Then there is a G-structure on M iff there is a lift $\tilde{c}$ of c from $BGL(n)$ to BG such that the following diagram commutes :



4.12 Theorem (Moore-Postnikov decomposition). Let $E \xrightarrow{p} B$ be a fibration with B connected. Then there are fibrations $E^{[n+1]} \xrightarrow{p_n} E^{[n]}$ and maps $E \xrightarrow{q_n} E^{[n]}$ for $n \geq 0$ such that the following diagram commutes.



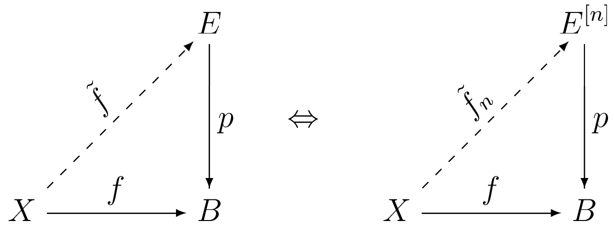
Moreover the following holds :

$$F^{[n]} \to E \xrightarrow{q_n} E^{[n]}$$
$$K(\pi_{n+1}(F^{[n]}), n+1) \to E^{[n+1]} \xrightarrow{p_{n+1}} E^{[n]}$$
$$E^{[0]} := B, \quad F^{[0]} := F$$

and

$$\pi_k(E) \underset{q_{n\#}}{\xrightarrow{\cong}} \pi_k(E^{[n]}) \text{ for } k \leq n$$
$$\pi_{n+1}(E) \underset{q_{(n+1)\#}}{\twoheadrightarrow} \pi_{n+1}(E^{[n]})$$
$$\pi_k(F^{[n]}) \cong \begin{cases} 0 \text{ for } k \leq n \\ \pi_k(F) \text{ for } k > n \end{cases}$$

4.13 Theorem. Let X be a CW-complex with $\dim(X) \leq n$ and $E \xrightarrow{p} B$ a fibration. Then the lift $\tilde{f}$ exists iff $\tilde{f}_n$ exists :



4.14 Theorem ([6]). Let R be a ring. Denote $R[x_1, \dots, x_n]$ a polynomial algebra over R generated by elements $x_1, \dots, x_n$ with degrees specified.

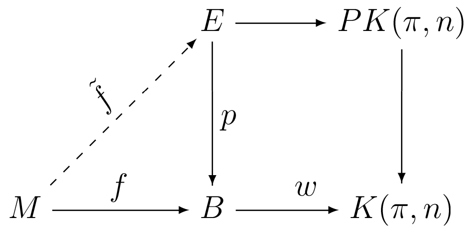
1. Stiefel-Whitney classes

$$H^*(BO(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \dots, w_n] \text{ with } \deg w_i = i$$
$$H^*(BSO(n), \mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_3, \dots, w_n] \text{ with } w_i \text{ as above}$$

2. Chern classes

$$H^*(BU(n), \mathbb{Z}) = \mathbb{Z}[c_1, c_2 \dots, c_n] \text{ with } \deg c_i = 2i$$

4.15 Lemma. Let $E \xrightarrow{p} B$ be a pullback of a path fibration $PK(\pi, n) \to K(\pi, n)$ via a map $w : B \to K(\pi, n)$.



Denote $\iota \in H^n(K(\pi, n), \pi)$ the fundamental class. Then the lift $\tilde{f}$ of f exists iff $(wf)^*(\iota) = 0$, where $(wf)^*$ is the map induced by the composite wf on cohomology.

4.3 Existence

We finally solve the problem of the existence of the G_2-structure on a 7-manifold. We rely on some knowledge of BG_2 and $SO(7)/G_2$ from the previous sections. We briefly discuss the difficulties we face if we try to use the same method for $\tilde{G}_2$.

4.16 Theorem. M is orientable iff the first Stiefel-Whitney class $w_1(M)$ vanishes.

Proof. Denote f the classifying map of the frame bundle of M. Orientability is equivalent to the existence of a lift of f from $BO(n)$ to $BSO(n)$. There is a fibration

$$\mathbb{Z}_2 \cong O(n)/SO(n) \to BSO(n) \xrightarrow{p} BO(n)$$

Let $BO(n) \xrightarrow{w} K(\mathbb{Z}_2, 1)$ be a map to be specified later. Let $E_1 \xrightarrow{p_1} K(\mathbb{Z}_2, 1)$ be a pullback of the path fibration $PK(\mathbb{Z}_2, 1) \to K(\mathbb{Z}_2, 1)$ via w. Consider the following diagram, the map q is to be defined later :

$$\begin{array}{ccccc}
 & & \mathbb{Z}_2 & & \mathbb{Z}_2 \\
 & & \downarrow & & \downarrow \\
BSO(n) & \overset{q}{\dashrightarrow} & E_1 & \longrightarrow & PK(\mathbb{Z}_2, 1) \\
\downarrow p & & \downarrow p_1 & & \downarrow \\
BO(n) & \xrightarrow{=} & BO(n) & \xrightarrow{w} & K(\mathbb{Z}_2, 1)
\end{array}$$

From the homotopy exact sequence we see that $p_{1\#}$ induces isomorphism $\pi_k E_1 \cong \pi_k BO(n)$ for all $k \geq 2$. We want to choose w so that p_1 induces isomorphism even on π_1 and moreover q exists and makes the diagram commutative. For π_1 we have the following commutative diagram :

$$\begin{array}{ccc}
\pi_1 E_1 & \longrightarrow & \pi_1 PK(\mathbb{Z}_2, 1) = 0 \\
\big\downarrow p_{1\#} & & \big\downarrow \\
\mathbb{Z}_2 = \pi_1 BO(n) & \xrightarrow{w_\#} & \pi_1 K(\mathbb{Z}_2, 1) = \mathbb{Z}_2
\end{array}$$

$p_{1\#}$ is mono so we get $\pi_1 E_1 = 0$ iff $w_\# \neq 0$. By the Hurewicz isomorphism we have $w_\# \neq 0$ iff $H^1(K(\mathbb{Z}_2, 1), \mathbb{Z}_2) \xrightarrow{w^*} H^1(BO(n), \mathbb{Z}_2)$ is nonzero. So the desired w is the unique one satisfying $w^*(\iota) = w_1$, the first Stiefel-Whitney class.

Since w_1 pulls back to 0 via p^*, by 4.15 we get the existence of q lifting p. Applying homotopy exact sequences to the left square we find that $q_\#$ induces isomorphism on π_k's for $k \geq 1$. The case $k = 1$ is trivial since $\pi_1 BSO(n) = \pi_1 E_1 = 0$. So $q_\#$ is a weak equivalence, hence $[X, BSO(n)] \xrightarrow{q_\#} [X, E_1]$ is bijection. Thus f lifts to $BSO(n)$. $\square$

4.17 Example. Using the long exact sequence for the fibration

$$\mathbb{Z}_2 \to S^n \to \mathbb{R}P^n$$

for $n \geq 2$ we find that

$$\pi_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}_2 \text{ for } k = 1 \\ \pi_k(S^n) \text{ for } k \neq 1 \end{cases}$$

In particular, the only nontrivial homotopy group of $\mathbb{R}P^7$ up to the sixth is $\pi_1(\mathbb{R}P^7) = \mathbb{Z}_2$.

4.18 Theorem. M admits a G_2-structure iff the first two Stiefel-Whitney classes $w_1(M), w_2(M)$ vanish.

Proof. By 4.4, the necessary condition for the existence of G_2-structure is the orientability of M. By 4.16 this implies $w_1(M) = 0$. Thus it suffices to restrict to orientable manifolds and consider the lifting problem from $BSO(7)$ to BG_2 only. As seen in 3.29, the fibration

$$SO(7)/G_2 \to BG_2 \to BSO(7)$$

has in fact fiber $\mathbb{R}P^7$ and by the computation of the homotopy groups of $\mathbb{R}P^7$ the only obstruction lies in $H^2(BSO(7), \mathbb{Z}_2) = \mathbb{Z}_2 w_2$, w_2 being the second Stiefel-Whitney class.

$$\begin{array}{ccccccc}
F_1' & \longrightarrow & SO(7)/G_2 & \xrightarrow{\iota} & K(\mathbb{Z}_2,1) & & \\
\downarrow \simeq & & \downarrow & & \downarrow & & \\
F_1 & \longrightarrow & BG_2 & \longrightarrow & E_1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & BSO(7) & \xrightarrow{=} & BSO(7) & \xrightarrow{\tau(\iota)} & K(\mathbb{Z}_2,2)
\end{array}$$

By the Serre exact sequence we have

$$H^1(BG_2, \mathbb{Z}_2) \to H^1(\mathbb{R}P^7, \mathbb{Z}_2) \xrightarrow{\tau} H^2(BSO(7), \mathbb{Z}_2) \to H^2(BG_2, \mathbb{Z}_2)$$

By 3.31 we have $H^1(BG_2, \mathbb{Z}_2) = H^2(BG_2, \mathbb{Z}_2) = 0$. Therefore τ is an isomorphism and $\tau(\iota) = w_2$ for the fundamental class ι, the generator of $H^1(\mathbb{R}P^7, \mathbb{Z}_2)$. Thus the lift exists iff $w_2(M) = 0$. $\square$

For $\tilde{G}_2$ the analogous attack is far more complicated. We need some information on $H^*(B\tilde{G}_2)$ and $H^*(BSO(3,4))$, but we don't have any. Further we need to know the first six homotopy groups of $SO(3,4)/\tilde{G}_2$. Even though we have the identification of this homogeneous space with the projectivized hyperboloid $PH^{3,4}$ (which deformation retracts to $\mathbb{R}P^2$), we are not able to compute a single homotopy group using the homotopy exact sequence without further knowledge of the homomorphisms.

5 One more 3-form

5.1 Formulation

In this short section we show that the problem of the existence of 3-form ω of type 3 on a 7-manifold M is, under some simplifying conditions, equivalent to the existence of an almost complex structure on certain 6-dimensional subbundle of TM.

The multisymplectic 3-form of type 3 (see [4]) is the one which has local coordinates expression

$$\omega = e^1 \wedge (e^{27} - e^{36} + e^{45}) \tag{17}$$

which means that there is an open cover $\mathcal{U}$ trivializing TM such that ω is given by (17) on each $U \in \mathcal{U}$.

5.1 Lemma. Let M be a 7-manifold admitting a 3-form ω of type 3. Denote

$$\begin{aligned} Z_\omega &:= \{\alpha \in TM^* : \alpha \wedge \omega = 0\} \\ D_\omega &:= \{v \in TM : \alpha(v) = 0 \text{ for all } \alpha \in Z_\omega\} \end{aligned}$$

Then D_ω is 6-dimensional vector subbundle of TM, thus

$$TM = D_\omega \oplus C$$

for an 1-dimensional complement bundle C to D_ω.

Proof. The dimension of D_ω is easily seen from the local formula (17). We just remark that in the local coordinates Z_ω is spanned by e^1 and D_ω is spanned by $e_2, e_3, \ldots, e_7$. □

5.2 Lemma. If C of 5.1 is trivial, i.e. there is a nowhere vanishing vector field v on M spanning C, then $\iota_v \omega$ is a nondegenerate 2-form on the bundle D_ω.

Proof. Again, this follows directly from (17). □

5.3 Theorem. There exists a nondegenerate 2-form on a vector bundle ξ iff there exists an almost complex structure on ξ.

Proof. We perform the constructions locally on a vector space V.

- Assume there is a complex structure J on V. Take a metric $\tilde{g}$ on V. Define
$$g(x,y) := \tilde{g}(x,y) + \tilde{g}(Jx,Jy)$$
g is indeed a metric and we easily see that $g(Jx,Jy) = g(x,y)$. Define
$$\omega(x,y) := g(Jx,y)$$
Then $\omega(x,y) = g(Jx,y) = g(J^2x,Jx) = -g(x,Jy) = -g(Jy,x) = -\omega(y,x)$ shows the skew-symmetry of ω. The nondegeneracy follows from regularity of J.

- Assume there is a nondegenerate 2-form ω on V. Let g be an arbitrary metric on V. By nondegeneracy of ω there is unique automorphism A of V such that
$$g(Ax,y) = \omega(x,y)$$
A is skew-symmetric : $g(Ax,y) = \omega(x,y) = -\omega(y,x) = -g(Ay,x) = -g(x,Ay)$. Hence for $x \neq 0$ we have $g(-A^2x,x) = g(Ax,Ax) > 0$, i.e. $-A^2$ is positive. By the well known spectral theorem we find a positive B such that $-A^2 = B^2$. We claim that
$$J := B^{-1}A$$
is a complex structure on V. To see this we have to show $AB = BA$ because then
$$(B^{-1}A)^2 = B^{-1}AB^{-1}A = B^{-2}A^2 = -1$$
First we observe that ABA^{-1} is positive : Let v be an eigenvector of B with eigenvalue $\lambda > 0$. Then $ABA^{-1}(Av) = AB(v) = \lambda A(v)$ hence B and ABA^{-1} have the same eigenvalues - all positive.

  Now recall that for any positive automorphism there is its uniquely determined positive square root. We find two positive square roots for the positive automorphism $-A^2$: We have $B^2 = -A^2$ and also $(ABA^{-1})^2 = AB^2A^{-1} = A(-A^2)A^{-1} = -A^2$. Hence
$$B = ABA^{-1}$$
as required.

The standard arguments allow us to do everything pointwise and continuously. We just remark that A and B above are determined uniquely. □

5.2 Existence

We prove the theorem 5.5 on the existence of an almost complex structure on a 6-dimensional vector bundle over an oriented 7-manifold and use it to partially solve the problem of the previous section.

5.4 Lemma. The first six homotopy groups of $SO(6)/U(3)$ are as follows :

i	1	2	3	4	5	6
$\pi_i(SO(6)/U(3))$	0	$\mathbb{Z}$	0	0	0	0

Proof. For $k = 1, \dots, 5$ this is taken from [9]. For $k = 6$ we apply the homotopy exact sequence :

$$\pi_6(SO(6)) \to \pi_6(SO(6)/U(3)) \xrightarrow{\alpha} \pi_5(U(3)) \xrightarrow{\beta} \pi_5(SO(6)) \to \pi_5(SO(6)/U(3))$$

so

$$0 \to \pi_6(SO(6)/U(3)) \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \to 0$$

We see that β is epi and thus iso and this implies $\alpha = 0$, thus the result. □

5.5 Theorem. There is an almost complex structure on an orientable 6-dimensional vector bundle ξ over a 7-manifold M iff $\beta w_2(\xi) = 0$, β being the Bockenstein homomorphism and w_2 the second Stiefel-Whitney class.

Proof. The existence of an almost complex structure on ξ is equivalent to the lifting of the classifying map from $BSO(6)$ to $BU(3)$. In the previous lemma we have obtained the homotopy groups of the fiber of the fibration

$$SO(6)/U(3) \to BU(3) \to BSO(6)$$

up to the sixth. This implies that there is only one obstruction.

$$\begin{array}{ccccc}
F_1' & \longrightarrow & SO(6)/U(3) & \xrightarrow{\iota} & K(\mathbb{Z},2) & & \\
\downarrow \simeq & & \downarrow & & \downarrow & & \\
F_1 & \longrightarrow & BU(3) & \longrightarrow & E_1 & & \\
& & \downarrow & & \downarrow & & \\
& & BSO(6) & \xrightarrow{=} & BSO(6) & \xrightarrow{\tau(\iota)} & K(\mathbb{Z},3)
\end{array}$$

At first we compute the first three integral cohomology groups of $BSO(6)$. We have

$$Z_2 = \pi_1 SO(6) \cong \pi_2 BSO(6) \cong H_2(BSO(6), \mathbb{Z})$$
$$0 = \pi_2 SO(6) \cong \pi_3 BSO(6) \twoheadrightarrow H_3(BSO(6), \mathbb{Z})$$

by the Hurewicz theorem. By UCT we get

$$0 \to \mathrm{Ext}(H_1(BSO(6), \mathbb{Z}), \mathbb{Z}) \to H^2(BSO(6), \mathbb{Z}) \to \mathrm{Hom}(H_2(BSO(6), \mathbb{Z}), \mathbb{Z}) \to 0$$

hence

$$H^2(BSO(6)) \cong \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}) = 0$$

Similarly

$$H^3(BSO(6)) \cong \mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$$

Consider the coefficient exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$$

and the associated cohomology exact sequence applied to $BSO(6)$:

$$0 = H^2(BSO(6), \mathbb{Z}) \to H^2(BSO(6), \mathbb{Z}_2) \xrightarrow{\beta} H^3(BSO(6), \mathbb{Z}) = \mathbb{Z}_2$$

So β is mono and therefore it maps $0 \neq w_2 \in H^2(BSO(6), \mathbb{Z}_2)$ to a nonzero element of $H^3(BSO(6), \mathbb{Z})$, hence β is iso. Concluding, $H^3(BSO(6))$ is generated by $\beta(w_2)$ of order 2.

Serre sequence for the fibration $BU(3) \xrightarrow{p} BSO(6)$ gives

$$H^2(K(\mathbb{Z}_2)) \xrightarrow{\tau} H^3(BSO(6)) \to H^3(BU(3)) = 0$$

Hence τ is surjective and therefore $\tau(\iota) = \beta(w_2)$.

Thus the lift exists iff $\beta(w_2(\xi)) = 0$. □

5.6 Theorem. There is a nowhere vanishing vector field on an orientable manifold M iff the Euler class $e(M)$ vanishes.

Proof ([11]). Let M be n-dimensional. The existence problem is equivalent to lifting the classifying map of TM from $BSO(n)$ to $BSO(n-1)$. The fibration

$$S^{n-1} \cong SO(n)/SO(n-1) \to BSO(n-1) \to BSO(n)$$

has fiber S^{n-1} so the only obstruction for the lifting is in dimension n.

$$\begin{array}{ccccccc}
F_1' & \longrightarrow & S^{n-1} & \xrightarrow{\iota} & K(\mathbb{Z}, n-1) & & \\
\downarrow \simeq & & \downarrow & & \downarrow & & \\
F_1 & \longrightarrow & BSO(n-1) & \longrightarrow & E_1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & BSO(n) & \xrightarrow{=} & BSO(n) & \xrightarrow{\tau(\iota)} & K(\mathbb{Z}, n)
\end{array}$$

By the Serre sequence we get

$$H^{n-1}(S^{n-1}) \xrightarrow{\tau} H^n(BSO(n)) \xrightarrow{p^*} H^n(BSO(n-1))$$

We compute $\operatorname{Im} \tau = \operatorname{Ker} p^*$ using the Gysin sequence :

$$\mathbb{Z}1 = H^0(BSO(n)) \xrightarrow{e \cup \cdot} H^n(BSO(n)) \xrightarrow{p^*} H^n(BSO(n-1))$$

We see $\operatorname{Ker} p^* = \mathbb{Z}e$, where e is the Euler class. Recall that e is a nontorsion element vanishing for n odd.

The generator $\iota \in H^{n-1}(S^{n-1})$ maps onto a generator of $\operatorname{Im} \tau$, i.e. $\tau(\iota) = \pm e$. Thus the lift exists iff $e(M) = 0$. □

5.7 Theorem. Let M be an orientable 7-manifold. If $\beta(w_2(M)) = 0$, then TM admits a 3-form of type 3. Conversly if M admits a 3-form of type 3 such that C of 5.1 *is trivial*, then $\beta(w_2(M)) = 0$.

Proof.
- Let M admit ω and C be trivial, i.e. spanned by nowhere vanishing vector field v. By 5.2 there is a nondegenerate 2-form on D_ω. Since TM is orientable so is D_ω. By 5.5 we obtain $\beta(w_2(D_\omega)) = 0$. But $w_2(M) \equiv w_2(TM) = w_2(D_\omega \oplus C) = w_2(D_\omega)$ since C is trivial.

- Let $\beta(w_2(M)) = 0$. Because $e(M) = 0$ for odd dimensional manifolds, by 5.6 we have a nowhere vanishing vector field v on TM. Denote C the trivial line bundle spanned by v. We have

$$TM = C \oplus D$$

for some 6-dimensional vector bundle D over M. As before we have $\beta(w_2(D)) = 0$ and by 5.5 and 5.3 there is a nondegenerate 2-form $\tilde{\omega}$ on D. Choose $v^* \in T^*M$ dual to v and define

$$\omega := v^* \wedge \tilde{\omega}$$

ω is indeed a multisymplectic form of type 3 as seen by using the classification of [4] - $r(\omega) = 0$. □

References

[1] Adams J. F. (1996) : *Lectures on Exceptional Lie Groups*. The University of Chicago Press

[2] Adams J. F. (1969) : *Lectures on Lie Groups*. W. A. Benjamin, Inc.

[3] Baez J. C. (2001) : *The Octonions*

[4] Bureš J., Vanžura J. (2003) : *Multisymplectic forms of degree three in dimension seven*. The Proceedings of the 22^{th} Winter School Geometry and Physics, Suppl. Rend. Circ. Math. Palermo, Serie II, **71** (2003), pp. 73-91

[5] Conway J. H., Smith D. A. (2003) : *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*. A K Peters

[6] Dodson C. T. J., Parker P. E. (1997) : *A User's Guide to Algebraic Topology*. Kluwer Academic Publishers

[7] Gray A. (1969) : *Vector Cross Products on Manifolds*. Trans. Amer. Math. Soc. **141** (1969), pp. 465–504

[8] Harvey F. R. (1990) : *Spinors and calibrations*. Academic Press, Inc.

[9] Massey W. S. (1961) : *Obstructions to the existence of almost complex structure*. Bull. Amer. Math. Soc. **67** (1961), 559-564

[10] Milnor J. W., Stasheff J. D. (1974) : *Characteristic classes*. Princeton University Press and University of Tokyo Press

[11] Thomas E. (1966) : *Seminar on Fiber Spaces*. Lecture Notes in Mathematics, Springer-Verlag

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