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PhD Thesis: Extended abstract

The tree property and the continuum function

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1 Introduction

In the thesis, we study the tree property and its interaction with the continuum function.

If $\kappa > \omega$ is a regular uncountable cardinal, we say that κ has *the tree property*, and we denote it by $\mathsf{TP}(\kappa)$, if all κ-trees have a cofinal branch.[1] The tree property of κ is a compactness property which derives its motivation from compactness of the infinitary logic $L_{\kappa,\kappa}$ for an inaccessible κ (see [32] for more details). Indeed, κ is weakly compact if and only if κ has the tree property and it is inaccessible. The notion of the tree property at κ is a priori weaker than weak compactness as it does not require κ to be inaccessible. The existence of κ with the tree property is equiconsistent with the existence of a weakly compact cardinal, and that the tree property can also hold at successor cardinals greater or equal to $\aleph_2$.

A κ-tree T which witnesses the failure of the tree property at κ is called a *κ-Aronszajn tree*, i.e. T is a κ-tree which has no cofinal branches. By results of Aronszajn and Specker ([33] and [49]), GCH ensures the existence of many counterexamples to the tree property:

$$(\forall \kappa \geq \omega)\ (\kappa^{<\kappa} = \kappa \rightarrow \neg \mathsf{TP}(\kappa^+)). \tag{1.1}$$

In particular, the tree property can never hold at $\aleph_1$ (or at the successor of an inaccessible cardinal). In fact, the tree constructed to witness (1.1) can be required to have the additional property that there exists a function $T \rightarrow \kappa$ which is injective on the chains in the tree ordering such trees are called *special Aronszajn trees*. It is consistent that special Aronszajn trees form a strictly smaller family than the Aronszajn trees, and we therefore introduce the notion of the *weak tree property*, and we denote it by $\mathsf{wTP}(\kappa)$: $\mathsf{wTP}(\kappa)$ says that there are no special κ-Aronszajn trees. The existence of κ with the weak tree property is equiconsistent with the existence of a Mahlo cardinal.

The inequality in (1.1) generalises to the weak tree property:

$$(\forall \kappa \geq \omega)\ (\kappa^{<\kappa} = \kappa \rightarrow \neg \mathsf{wTP}(\kappa^+)). \tag{1.2}$$

In fact, the antecedent of the implication in (1.2) can be weakened to the existence of the weak square sequence at κ (denoted $\square^*_\kappa$) (see [4] for more details):

$$(\forall \kappa > \omega)\ (\square^*_\kappa \rightarrow \neg \mathsf{wTP}(\kappa^+)). \tag{1.3}$$

By results of Jensen [30], $\square^*_\kappa$ is actually equivalent to the existence of a special κ^+-Aronszajn tree, and therefore to the failure of the weak tree property.

Recall that the function which maps an infinite cardinal κ to 2^κ is called the *continuum function*. As is well known, the continuum function on regular cardinals can behave very arbitrarily. While large cardinals and the singular strong limit cardinals of uncountable cofinality do reflect the pattern of the continuum function to smaller cardinals – and therefore restrict the freedom of the continuum function –, this limits the arbitrariness of the continuum function on regular cardinals only modulo "large sets" (such as the stationary sets); there is no local control over the continuum function. It is of interest to note that (1.1) does provide such control: for instance $\mathsf{TP}(\aleph_2)$ implies the failure of CH.

The natural question on which we focus in this thesis is the following:

[1] See the full PhD thesis for the definitions and more details for the notions appearing the extended abstract.

(Q) Do the tree property and the weak tree property put more restrictions on the continuum function apart from (1.1) and (1.2)?

This question can in principle be approached either locally or globally, i.e. with the tree property holding at more cardinals at the same time. That is, we may ask how $\mathsf{TP}(\kappa)$ influences the continuum function for a fixed κ, or consider a set of regular cardinals $\{\kappa_i \mid i \in I\}$ (usually an interval) and ask about the influence of $\mathsf{TP}(\kappa_i)$ for all $i \in I$.

Let us note in this context that it is highly non-trivial even to obtain a model with a large interval of regular cardinals with the tree property: while an easy modification of the original Mitchell's construction (see [37]; we call the forcing *Mitchell forcing*) yields two successive cardinals with the weak tree property, the existence of two successive cardinals with the tree property requires a major modification of the argument (see [1]). We will not go into details here, but let us mention some crucial problems which make it hard to get long intervals with the tree property: by (1.3), obtaining the weak tree property at the successor of a singular cardinal is hard since it requires the killing of weak square sequences (which exist in core models); and just from (1.2), obtaining the weak tree property at the double successor of a singular strong limit cardinal requires the failure of SCH. Importantly, dealing with these restrictions at more cardinals at the same time complicates the matters even more: it is noteworthy that obtaining $\mathsf{TP}(\aleph_2)$ and $\mathsf{TP}(\aleph_3)$ at the same time requires a much large cardinal strength than $\mathsf{TP}(\aleph_2)$ or $\mathsf{TP}(\aleph_3)$ alone (see [14] for more details).

Returning to our question (Q), we provide three original results which show that the answer to (Q) is negative in some special cases: any behaviour – consistent with (1.1) and (1.2) – of the continuum function on the cardinals considered in our results is consistent with the tree property (locally and globally). Let us say that we expect that the answer to (Q) will be negative even when more cardinals with the tree property are considered. We consider further development and open question in Section 4.

2 The tree property at successor cardinals

The first construction which showed that it is consistent to have the tree property at a successor cardinal is due to Mitchell [37].[2] Starting with regular cardinals $\omega \leq \kappa < \lambda$, Mitchell found a forcing notion which does the following: with GCH, it collapses cardinals in the interval (κ^+, λ) (and no other cardinals), and whenever λ is Mahlo, then the weak tree property holds at $2^\kappa = \lambda = \kappa^{++}$, and whenever λ is weakly compact, then the tree property holds at $2^\kappa = \lambda = \kappa^{++}$. It is important for further development to notice that κ itself is regular, and so Mitchell's construction achieves the tree property at a double successor of a regular cardinal – thus leaving aside successors of singulars, and double successor of singulars. The large cardinal assumptions are optimal in the sense that if the tree property holds at some κ, then κ is weakly compact in L, and if the weak tree property holds at κ, then κ is Mahlo in L. Mitchell [37] gives an argument that the weak tree property can be forced at two successive cardinals, such as $\aleph_2$ and $\aleph_3$, starting with just two Mahlo cardinals. He left it open whether it is consistent to have the tree property at two successive cardinals.

[2] The modern presentation of Mitchell's forcing is due to Abraham [1], and it is the one we use in the thesis.

Abraham [1] solved the question by finding a forcing notion which ensures the tree property at $\aleph_2$ and $\aleph_3$, with $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_3$. Abraham started with a supercompact cardinal and a weakly compact cardinal above it. While the assumption might seem too strong at the first glance,[3] the paper gives an argument (due to Magidor) that two weakly compact cardinals certainly do not suffice since having the tree property at successive cardinals implies the existence of $0^\sharp$. This lower bound was later improved to the level of Woodin cardinals (see [14] for more details).

Another development was the result of Cummings and Foreman [5] who generalised Abraham's construction and obtained a model where the tree property holds at every $\aleph_n$ for $1 < n < \omega$, and $2^{\aleph_m} = \aleph_{m+2}$ for $0 \le m < \omega$ (and GCH elsewhere). They left open whether one can extend the interval of cardinals with the tree property further, in particular to include $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$.

We will leave $\aleph_{\omega+1}$ aside for a moment and focus on $\aleph_{\omega+2}$. Since $\aleph_\omega$ is strong limit in the model in [5] and GCH holds at $\aleph_\omega$, the tree property necessarily fails at $\aleph_{\omega+2}$. In the second part of [5] (attributed to Foreman), they give an argument which shows how to get the tree property at κ^{++} for a strong limit singular κ with countable cofinality (starting with a supercompact κ and a weakly compact above it). They also claim that their construction generalises to collapse κ to $\aleph_\omega$, and ensure the tree property at $\aleph_{\omega+2}$. However, they provided no argument, and in hindsight it does not seem that an easy modification of their argument for κ^{++} with the tree property generalises to $\aleph_{\omega+2}$: The problem is that Prikry forcing with collapses prevents the use of the type of forcing they used in [5].[4] Today, there are four different arguments available for the tree property at $\aleph_{\omega+2}$ (to our knowledge): the first one is the construction of Friedman and Halilović [15], followed by Gitik's construction in [21], the construction in [10] due to Cummings and others, and our present construction (see Theorem 3.4). The construction in [15] is completely different from the argument of Cummings and Foreman in [5]: first, it uses just a weakly compact strong cardinal, and second it uses Sacks iteration at κ of length λ, followed by Prikry forcing with collapses, to achieve the desired goal.[5] The constructions in [10] and our construction in Theorem 3.4 are similar, but differ in important aspects. They both use Mitchell forcing followed by Prikry forcing with collapses. However Theorem 3.4 uses only a strong cardinal of a suitable degree, while [10] uses a supercompact cardinal.[6] Furthermore, the construction in Theorem 3.4 achieves any desired finite gap at $\aleph_\omega$. Regarding the Gitik's construction, it proceeds from a sequence of short extenders and it is optimal with respect to the large cardinal assumptions (however, it is not known how to generalise it to achieve a larger gap than 2 at $\aleph_\omega$).

Remark 2.1. The optimal lage cardinal assumption for the tree property at $\aleph_{\omega+2}$ is close to a weakly compact strong cardinal.[7] We will not give too many details, but let us say

[3] It is used only once to lift an embedding using a master condition argument.

[4] See Footnote 8 for more details.

[5] The use of Sacks forcing enforces a direct method of proof: there is no "product-style" analysis used with the Mitchell forcing. A common restriction related to an iteration with support κ applies: it is possible to achieve only gap 2 at $\aleph_\omega$, i.e. $2^{\aleph_\omega} = \aleph_{\omega+2}$. In retrospect, the use of Sacks forcing probably makes the argument more complicated than the methods for the Mitchell-like forcings (unless we want to achieve some sort definability result together with the tree property – in this setting an iteration is the primary option; see Section 4.4 for more details).

[6] The paper [48] by Sinapova and Unger contains an argument for the tree property at κ^{++} for a large strong limit κ of countable cofinality, with gap 3.

[7] Note that $\aleph_\omega$ violates SCH, so lower bounds for the failure of SCH apply.

that it is not so important that we start with a weakly compact strong embedding – a tall embedding $j : V \to M$ which sends κ above a weakly compact cardinal λ in M would also suffice – the issue is whether we need to assume the existence of one big extender, or a sequence of short extenders would suffice. As it turns out, the optimal large cardinal strength is indeed formulated with a sequence of short extenders as we mentioned above (see [21]).

Let us return to the case of the tree property at $\aleph_{\omega+1}$. Let us first note that in all the models we discussed, with the tree property at $\aleph_{\omega+2}$, the tree property at $\aleph_{\omega+1}$ fails. However, by itself the tree property at $\aleph_{\omega+1}$ is achievable as shown by Magidor and Shelah in [36] (the key ingredient of the construction is a theorem in ZFC, proved in [36], which says that if λ is a singular limit of strongly compact cardinals, then the tree property holds at λ^+). However, the methods in [36] force SCH at $\aleph_\omega$, so the tree property at $\aleph_{\omega+2}$ fails in the model in [36].

A lot of recent research has been focused on combining the above-mentioned results and obtaining the tree property at all regular cardinals in the interval $[\aleph_2, \aleph_{\omega+2}]$. There has been an important progress, but the main question is still unanswered (see for instance [38, 46, 52, 48, 53, 47] for more details). With the natural goal being to force the tree property at all regular cardinals, it is also important to consider singular cardinals with uncountable cofinality; there has been some important progress here as well (see Sinapova [45] for $\aleph_{\omega_1+1}$ and Golshani and Mohammadpour [25] for κ^{++}, κ singular with uncountable cofinality for more details).

Remark 2.2. The results reviewed so far work with $\aleph_\omega$ being strong limit. If we relax this requirement, then it is consistent that both $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$ have the tree property by a result of Fontanella and Friedman [12]. It is an intriguing question whether $\aleph_\omega$ can at all be strong limit with the tree property holding at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$, especially because the tree property can consistently hold at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ with $\aleph_{\omega^2}$ strong limit (see Sinapova and Unger [48]). It may very well be that $\aleph_\omega$ is a special case, whose properties are governed by theorems provable in ZFC (as is the bound on $2^{\aleph_\omega}$ identified by Shelah).

Stepping back to the weak tree property (or equivalently to the failure of the weak square principle), it turns out that killing all special Aronszajn trees is much easier than killing all Aronszajn trees. As we already said, Mitchell [37] gave a proof of the tree weak property holding, for instance, at $\aleph_2$ and $\aleph_3$. This construction was generalised by Unger [51] to all cardinals at the interval $[\aleph_2, \aleph_\omega)$, starting with infinitely many Mahlo cardinals.

Moving on to our thesis and research, it is important to state that all the results reviewed so far did not specifically control the continuum function, and therefore achieve the least possible gap at the relevant cardinal: if the tree property holds at κ^{++}, then $2^\kappa = \kappa^{++}$. It is therefore natural to ask whether one can control the continuum function on regular cardinals in the presence of the tree property as freely as in the case of the usual Easton theorem.

3 Original results of the thesis

Let us briefly introduce the results in the thesis and discuss how they relate to existing results.

The results in Section 5 of the full thesis, joint with Radek Honzik, were submitted as [27] and deal with the tree property and the weak tree property at cardinals $\aleph_n$, $1 < n < \omega$. We show that the tree property and the weak tree property at these cardinals do not put any restrictions on the continuum function below $\aleph_\omega$ apart from the trivial implication that $\mathsf{wTP}(\aleph_{n+2})$ implies $2^{\aleph_n} > \aleph_{n+1}$ for $0 \leq n < \omega$.

A succinct statement of the theorems is as follows:

Theorem 3.1. (GCH) *Assume there are infinitely many weakly compact cardinals. Let f be a function from ω to ω which satisfies*

(i) For all $m, n < \omega$, $m < n \rightarrow f(m) \leq f(n)$.
(ii) $f(2n) \geq 2n + 2$ for all $n < \omega$.

Then there is a model where the tree property holds at every $\aleph_{2n}$, $0 < n < \omega$, and the continuum function below $\aleph_\omega$ obeys f: i.e. $2^{\aleph_n} = \aleph_{f(n)}$ for all $n < \omega$.

Theorem 3.2. (GCH) *Assume there are infinitely many Mahlo cardinals. Let f be a function from ω to ω which satisfies*

(i) For all $m, n < \omega$, $m < n \rightarrow f(m) \leq f(n)$,
(ii) $f(n) > n + 1$ for all $n < \omega$.

Then there is a model where the weak tree property holds at every $\aleph_n$, $1 < n < \omega$, and the continuum function below $\aleph_\omega$ obeys f: i.e. $2^{\aleph_n} = \aleph_{f(n)}$ for all $n < \omega$.

Theorem 3.1 is based on the construction in [17, Section 5] – which just ensures $2^{\aleph_m} = \aleph_{m+2}$, $0 \leq m < \omega$, and the tree property at $\aleph_{2n}$, $0 < n < \omega$ –, and adds extra forcings to control the continuum function. Similarly, Theorem 3.2 builds on the proof in Unger [51] and adds extra forcings to control the continuum function while ensuring the weak tree property at every $\aleph_n$, $1 < n < \omega$. In both cases we used the product of Cohen forcings at relevant cardinals, and computed that their presence will not destroy the tree property ensured by the rest of the forcing.

The results in Section 6 of the full thesis, joint with Sy-David Friedman and Radek Honzik, were submitted as [18] and focus on the tree property at the double successor of a singular strong limit cardinal κ with countable cofinality.

A succinct statement of the theorem is as follows:

Theorem 3.3. *Assume* GCH *and let κ be a Laver-indestructible supercompact cardinal, λ a weakly compact cardinal and μ a cardinal of cofinality greater than κ such that $\kappa < \lambda < \mu$. Then there is a forcing notion $\mathbb{R}$ such that the following hold:*

(i) $\mathbb{R}$ preserves cardinals $\leq \kappa^+$ and $\geq \lambda$.
(ii) $V[\mathbb{R}] \models (\kappa^{++} = \lambda \ \& \ 2^\kappa = \mu \ \& \ \mathrm{cf}(\kappa) = \omega \ \& \ \kappa$ is strong limit).
(iii) $V[\mathbb{R}] \models \mathsf{TP}(\lambda)$.

Theorem 3.3 generalises the construction in [5] in which Cummings and Foreman obtained a singular strong limit cardinal with countable cofinality with $2^\kappa = \kappa^{++}$ and $\mathsf{TP}(\kappa^{++})$, starting with a Laver-indestructible supercompact κ. We modify their original forcing – which integrates Prikry forcing with Mitchell forcing – by adding more Cohen subsets of κ to control the continuum function at κ so that $2^\kappa = \mu$ for any $\mu \geq \lambda$ of cofinality greater than κ. This modification required substantial changes in the argument built as it is on reflecting Prikry forcing defined after adding λ-many Cohen subsets of κ: we add μ-many subsets of κ, with $\mu > \lambda$, and therefore the reflection is more complicated.

In Section 7 of the full thesis, submitted as [19] and joint with Sy-David Friedman and Radek Honzik, we bring the cardinal κ in Theorem 3.3 down to $\aleph_\omega$. The method of the proof is different from [5] and [18]: we do not integrate Prikry forcing with collapses into Mitchell forcing, but force with Prikry forcing after Mitchell forcing.[8] Also, as the value of $2^{\aleph_\omega}$ cannot be arbitrarily high, we only achieve an arbitrary finite gap. It is an open question whether we can achieve an infinite gap.

A succinct statement of the theorem is as follows:

Theorem 3.4. *Suppose* GCH *holds in the universe. Assume* n *is a natural number,* $2 \leq n < \omega$, $\kappa < \lambda$ *are cardinals such that* λ *is the least weakly compact cardinal above* κ, *and* κ *is* $H(\lambda^{+n-2})$*-strong. Then there is a forcing extension where the following hold:*

(i) $\kappa = \aleph_\omega$ *is strong limit;*
(ii) $2^{\aleph_\omega} = \aleph_{\omega+n}$*;*
(iii) $\mathsf{TP}(\aleph_{\omega+2})$.

4 Further progress and open questions

By way of conclusion, we discuss topics for future research and mention some open questions.

Let us start by introducing some other principles which are similar to the tree property in the sense that they postulate a variant of compactness at a successor cardinal. We will then formulate open questions and problems in this more general framework.

Let κ be an uncountable cardinal in what follows (unless said otherwise).

We say that κ^+ satisfies *the stationary reflection*, and we write it as $\mathsf{SR}(\kappa^+)$, if every stationary subset of $\kappa^+ \cap \mathrm{cof}(<\kappa)$ reflects at a point of cofinality κ, i.e. for every stationary $S \subseteq \kappa^+ \cap \mathrm{cof}(<\kappa)$ there is $\gamma < \kappa^+$ of cofinality κ such that $S \cap \gamma$ is stationary in γ. Stationary reflection has been extensively studied in literature, see for instance [2, 7, 29, 42, 6, 8, 9].

If we require that the stationary subsets reflect simultaneously, we get stronger principles introduced in Magidor [35]: We say that κ^+ satisfies *the simultaneous stationary reflection*, and we write it as $\mathsf{SSR}(\kappa^+)$, if every two stationary subsets of $\kappa^+ \cap \mathrm{cof}(<\kappa)$ reflect at a common point of cofinality κ. An even stronger principle is the following: We say that κ^+ satisfies *the club stationary reflection*, and we write it as $\mathsf{CSR}(\kappa^+)$, if every stationary subset of $\kappa^+ \cap \mathrm{cof}(<\kappa)$ reflects on a κ-club subset of κ^+ (an unbounded subset of κ^+ closed at limit stages of cofinality κ).

Fact 4.1. *Let* κ *be an uncountable cardinal.* $\mathsf{CSR}(\kappa^+) \to \mathsf{SSR}(\kappa^+) \to \mathsf{SR}(\kappa^+)$.

Recall the definition of *the approachability ideal* $I[\kappa^+]$. Let $\langle a_\alpha \mid \alpha < \kappa^+ \rangle$ be some sequence of bounded subsets of κ^+. We say that a limit ordinal $\gamma < \kappa^+$ is approachable with respect to the sequence if there is an unbounded subset A of γ of ordertype $\mathrm{cf}(\gamma)$ such

[8] This modification is necessary: the original method does not work since λ (the weakly compact cardinal above κ) must be first collapsed to κ^{++}, and only then Prikry forcing with collapses can be used.

that $\{A \cap \beta \mid \beta < \gamma\} \subseteq \{a_\beta \mid \beta < \gamma\}$. We define $I[\kappa^+]$ as the collection of all $S \subseteq \kappa^+$ for which there is a sequence $\langle a_\alpha \mid \alpha < \kappa^+\rangle$ as above and a club subset C of κ such that every $\gamma \in S \cap C$ is approachable with respect to the sequence.

The ideal $I[\kappa^+]$ has proved to be closely connected with many topics in combinatorial set theory, for example PCF theory in Shelah's [43], saturated ideals in Foreman's and Magidor's [13], and the extent of diamond in Rinot's [40] (see also [22] and [24]).

We say that κ^+ has the *approachability property* if $\kappa^+ \in I[\kappa^+]$, and we write it as $\mathsf{AP}(\kappa^+)$. $\mathsf{AP}(\kappa^+)$ is a weak form of the square principle on κ, and therefore we consider $\neg\mathsf{AP}(\kappa^+)$ as a compactness property of κ^+.

We list some fact related to these notions (for more details see [4]).

Fact 4.2. *Let κ be an uncountable cardinal.*

(i) $\square_\kappa \to \neg\mathsf{SR}(\kappa^+)$.
(ii) $\square^*_\kappa \to \mathsf{AP}(\kappa^+)$.
(iii) A Mahlo cardinal suffices to get $\mathsf{SR}(\kappa^+)$ *(and it is necessary). See Harrington and Shelah [26].*
(iv) A weakly compact cardinal suffices to get $\mathsf{SSR}(\kappa^+)$ *and* $\mathsf{CSR}(\kappa^+)$ *(and it is necessary). See Magidor [35].*
(v) A Mahlo cardinal suffices to get $\neg\mathsf{AP}(\kappa^+)$ *(and it is necessary). See Cummings and others [10].*

Let us know mention some open problems and areas of future research.

4.1 The continuum function

Let us first consider direct generalisations of the problems we studied in this thesis.

Q1. Is it possible to show that $\mathsf{TP}(\aleph_n)$ holds for each $1 < n < \omega$ while the continuum function is arbitrary (subject to the condition that GCH must fail below $\aleph_\omega$)?

It seems natural to start with the model constructed in [5] by Cummings and Foreman and use Cohen forcings to control the continuum function.

Q2. Let $\aleph_n$ for $1 < n < \omega$ be fixed. Is it possible show that $\neg\mathsf{AP}(\aleph_n)$ and $\mathsf{SR}(\aleph_n)$ pose no restriction on the continuum function (except for the restriction exerted by $\neg\mathsf{AP}(\aleph_n)$ as we discussed above)?

A challenging extension of this problem adds the requirement to control the value of $2^{\aleph_\omega}$ with $\aleph_\omega$ strong limit (see [23] for more details) while having some compactness principles below $\aleph_\omega$.

Q3. The above two questions can also be studied on $\aleph_{\omega+2}$. In particular, is it possible to show that the compactness principles at $\aleph_{\omega+2}$ are consistent with an arbitrary finite gap at $\aleph_\omega$, i.e. with $2^{\aleph_\omega} = \aleph_{\omega+n}$ for any $2 \le n < \omega$?

Notice that the results in this thesis show that this is possible for the tree property.

Q4. The previous question can be formulated with an infinite gap, i.e. with $\aleph_\omega$ strong limit and $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for some countable α. More specifically, one can ask whether we can get (to start modestly) the tree property at $\aleph_{\omega+2}$ with $2^{\aleph_\omega} = \aleph_{\omega+\omega+1}$.

Note that an infinite gap was first shown by Magidor in [34] (for $\alpha = \omega$) and generalised by Shelah in [41]. These methods use supercompact cardinals and collapse cardinals above the large cardinal which gets collapsed to $\aleph_\omega$ so we presume that the proofs would be quite different from those in this thesis.

4.2 Mixing the compactness principles

Let us now mention questions which study the interactions between the various compactness principles.

By the results of [10] by Cummings and others, it is possible to force any Boolean combination of truth and falsity of the principles $\mathsf{TP}(\kappa^+)$, $\mathsf{SR}(\kappa^+)$ and $\mathsf{AP}(\kappa^+)$ for a fixed cardinal κ^+ in the set $\{\aleph_n \mid 2 \leq n < \omega\} \cup \{\aleph_{\omega+2}\}$.[9]

With $\aleph_\omega$ strong limit, we can ask the following:

Q5. Is it possible to generalise the results of [10] by Cummings and others to include the variants of the principles $\mathsf{TP}(\kappa^+)$ and $\mathsf{SR}(\kappa^+)$ which we introduced above? As a test case, is it possible to achieve $\mathsf{TP}(\aleph_n) + \mathsf{CSR}(\aleph_n) + \neg\mathsf{AP}(\aleph_n)$ for some $1 < n < \omega$?

Note that the known method to obtain $\mathsf{CSR}(\kappa^+)$ (see Magidor [35]) requires an additional forcing over a model of $\mathsf{SSR}(\kappa^+)$, hence it is not obvious how this combines with the methods to obtain for instance $\mathsf{TP}(\kappa^+)$.

Q6. One might ask whether a Mahlo cardinal is sufficient to obtain certain configurations of the compactness principles at $\aleph_n$ for some $1 < n < \omega$. In particular, is it possible to force $\neg\mathsf{AP}(\aleph_n) + \mathsf{SR}(\aleph_n)$ from a Mahlo cardinal? The latter configuration was achieved in [10] by Cummings and others using a weakly compact cardinal (both with $\neg\mathsf{TP}(\aleph_n)$ and with $\mathsf{TP}(\aleph_n)$).

Note that the fact that $\neg\mathsf{AP}(\aleph_n)$ and $\mathsf{SR}(\aleph_n)$ require by themselves just a Mahlo cardinal does not necessarily imply that their combinations do. However, we conjecture it is the case and that a Mahlo cardinal should be sufficient.

Q7. We may consider compactness principles holding at successive cardinals, or on an interval of regular cardinals.

We reviewed the existing results for the tree property and the weak tree property in Section 2. Stationary reflection on multiple cardinals was studied in papers by Jech and Shelah [29] and Shelah [42]. Recently, Unger [54] considers successive failures of AP ($\aleph_\omega$ is not strong limit in his model). It is natural to study this question for other compactness principles and their combinations. In particular, is it consistent to combine the tree property on cardinals below $\aleph_\omega$ with $\mathsf{SR}(\aleph_n)$, $\mathsf{SSR}(\aleph_n)$ and $\mathsf{CSR}(\aleph_n)$ for $1 < n < \omega$, and if so, under which large cardinal assumptions?

This question can be extended to the context where SCH fails at $\aleph_\omega$, a first step to obtaining compactness principles at $\aleph_{\omega+2}$ (which imply the failure of SCH). A paper by Unger [52] shows that this is possible for the tree property; it is worth asking this question for SR and $\neg\mathsf{AP}$.

[9] In fact, they showed it for any κ^+ such that κ is a successor cardinal; we apply their result here in the context of the cardinals close to $\aleph_\omega$ on which we focus.

With $\aleph_{\omega_1}$ strong limit, we may analogously ask:

Q8. Is it possible to force compactness principles at $\aleph_{\omega_1+2}$?

We think it is possible, using a suitable version of the Radin forcing.

4.3 Generalised cardinal invariants

The cardinal invariants of the continuum provide as a finer classification of the properties relevant for the real numbers (identified with $2^{\aleph_0}$). The invariants are an interesting topic of study if CH fails (if CH holds they are typically equal to $2^{\aleph_0}$). Since both wTP and ¬AP imply the negation of CH it is natural to ask what cardinal invariants patterns can be realised in models where they hold at $\aleph_2$.

There are more forcings available to force $\mathsf{TP}(\aleph_2)$ in addition to Mitchell forcing: for instance Sacks forcing (see [31] or [17]) and its variants (see [28] or [50]) and forcings with side conditions such as [39]. It is also known by the result of Friedman and Torres [20] that MA can hold with $\mathsf{TP}(\aleph_2)$, starting just from a weakly compact cardinal.[10]

Q9. What is the structure of the cardinal invariants in the models with compactness principles at $\aleph_2$, where $2^{\aleph_0} = \aleph_2$?

Cardinal invariants generalise to larger cardinals (see [11], [44] or [3]).

Q10. What is the structure of the generalised cardinal invariants in the models with compactness principles at $\aleph_n$, for $1 < n < \omega$?

4.4 Definability

Finally let us consider the question of definability. It is known that SCH can fail definable at $\aleph_\omega$ in the sense that there is a lightface definable wellorder of the subsets of $\aleph_\omega$ in $H(\aleph_{\omega+1})$ with $\aleph_\omega$ strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$ (see [16]).

It is natural to ask whether the definability can be combined with the tree property at $\aleph_{\omega+2}$.

Q11. Is it possible for SCH to fail definably at $\aleph_\omega$ (in the above sense) with the tree property at $\aleph_{\omega+2}$?

Notice that in this context Mitchell forcing will probably not work as the coding of the wellorder usually requires an iteration. One may conjecture that the method of the proof in [16] might be modified to yield the desired result since it is based on a coding using a variant of Sacks forcing (which is known to force the tree property, see [31] and [17] for more details).

[10] In the model constructed in [20] the continuum has size $\aleph_2$. It is open whether the size of the continuum can be larger with MA and $\mathsf{TP}(\aleph_2)$ (note in this context that PFA implies $\mathsf{MA} + \mathsf{TP}(\aleph_2)$ but also $2^{\aleph_0} = \aleph_2$).

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