In the Light of Intuitionism: Two Investigations in Proof Theory

This dissertation focuses on two specific interconnections between the classical and the intuitionistic proof theory. In the first part, we will propose a formalization for Gödel's informal reading of the BHK interpretation, using the usual classical arithmetical proofs. His provability interpretation of the propositional intuitionistic logic, first appeared in [1], in which he introduced the modal system, **S4**, as a formalization of the intuitive concept of provability and then translated **IPC** to **S4** in a sound and complete manner. His work suggested the search for a concrete provability interpretation for the modal logic **S4** which itself leads to a concrete provability interpretation for the intutionistic logic. In the first chapter of this work, we will try to solve this problem. For this purpose, we will generalize Solovay's provability interpretation of the modal logic **GL** to capture other modal logics such as **K4**, **KD4** and **S4**. Then, using the mentioned Gödel's translation, we will propose a formalization for the BHK interpretation via classical proofs. As a consequence, it will be shown that the BHK interpretation is powerful enough to admit many different formalizations that surprisingly capture different propositional logics, including intuitionistic logic, minimal logic and Visser-Ruitenburg's basic logic. We will also present some negative results to show that there is no provability interpretation for any extension of the system **KD45** and as we expected there is no BHK interpretation for the classical propositional logic.

In the second half of the dissertation, we change our focus to the other direction of the interconnection to investigate the applications of the intuitionistic viewpoint in the realm of classical proof theory. For this purpose, we will develop a complexity sensitive version of the classical Dialectica interpretation to deal with the bounded theories of arithmetic. More precisely, we will define a notion called the computational flow which is a pair consisting of a sequence of computational problems of a certain sort and a sequence of computational reductions among them. We will develop a theory for these flows to provide a sound and complete interpretation for bounded theories of arithmetic. This property helps us to transform a first order arithmetical proof to a sequence of computational reductions by which we can extract the computational content of low complexity statements in some bounded theories of arithmetic including $I\Delta_0$, T^k_n, $I\Delta_0(\exp)$ and PRA. Then, in the last section, by generalizing term-length flows to ordinal-length flows, we will extend our investigations from bounded theories to strong unbounded systems such as PA and PA+TI(α) to capture their total NP search problems.

Keywords: Provability Interpretation, BHK Interpretation, Proof Mining,

References

[1] K. Gödel, Eine Interpretation des Intuitionistichen Aussagenkalküls, Ergebnisse Math Colloq. Vol. 4 (1933), pp. 39-40.