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Report on doctoral thesis by Mr. Vít Musil
"Classical operators of harmonic analysis in Orlicz spaces "

1 General information

Mr Vít Musil has completed his PHD thesis under the supervision of Professor Lubos Pick. The thesis are concerned with problems of boundedness of the classical operators in harmonic analysis, as well as Sobolev type embeddings, between rearrangement invariant spaces in the Euclidean setting, with the particular emphasis on Orlicz spaces.

The dissertation is taking into account four papers (numeration as in the Bibliography in the PHD Thesis):

[27] (with A. Cianchi) Optimal domain spaces in Orlicz-Sobolev embeddings, accepted in Indiana Univ. Math. J., 2018;

[36] (with D.E. Edmunds, Z. Mihula and L. Pick) Boundedness of the classical operators on rearrangement invariant spaces, preprint;

[56] Fractional maximal operator in Orlicz spaces, preprint;

[57] Optimal Orlicz domains in Sobolev embeddings into Marcintiewicz spaces, J. Funct. Anal. **270**(7) (2016), 2653-2690.

in which Vít Musil was the author or coauthor.
The presentation is almost self-contained as most of the related facts are presented together with their proofs, often in the modernized form.

It is worth to mention, that papers [27] and [28] are written together with the very experineced and recognized mathematicians (A. Cianchi, D.E. Edmunds, L. Pick) . On the other hand, solid and long paper [57] is written by the author alone and is already published in the very prestigious journal which is Journal of Functional Analysis. Also, paper [56] is the author's individual achievement.

The motivations to undertake such a problem are clear. Since many years the problem was undertaken by mathematicians building the theory of harmonic analysis, such as for example Calderón, O'Neel, Riesz, Sharpley, Stein, Trudinger, as well as distinguishing cooperators of Vít Musil - Lubos Pick and Andrea Cianchi. Such problems are related to existence and regularity in PDE's. For example, when we deal with the A-harmonic elliptic equations and use the elliptic regularity theory, we need the embeddings in the Orlicz setting. Despite the fact that a big effort has been done so far to understand boundedness of classical and fractional maximal functions, Hardy, Riesz or Laplace transforms, there are still many difficult problems open, such as sharpness results. This is where Vít Musil wanted to contribute, closing and deepening the already existing discussion.

The methods are based often on the techniques from papers by Andrea Cianchi and Lubos Pick. That techniques require very precise and difficult computations, solid knowledge about function spaces and recent literature.

The final effect of this dissertation: the self-contained presentation ending on deep, sharp and new results has been obtained with success.

2 Description of the thesis

The dissertation is 139 pages long and includes 73 positions in its Bibliography. It consists of 8 Chapters, where after Introduction and the preliminary chapter the author discusses boundedness for: Hardy, fractional maximal, Hardy Littlewood maximal - operators, Riesz potential, Laplace transform as well as he contributes to Sobolev type embeddings. The rearrangememt invariant type spaces are mostly Orlicz, Marcinkiewicz and Lorentz, and they build the related Sobolev type spaces.

After the introductory and preliminary chapters, in *Chapter 3* the author deals with boundedness for the general Hardy operator with the special attention to sharpness. For optimality, there is taken into account the domain and target space. Firstly, it is fucussed on boundedness between Orlicz spaces and results decsribed relay on research done by Strömberg, A. Cianchi and A. Cianchi with L. Pick. The individual achievement of the author is the analysis within weak Orlicz spaces, based on research in papers [57] and [27]. Some parts were also present in papers by A. Cianchi and L. Pick. The important tool here is usage of Boyd indexes to simplify pointwise inequalities in terms of Young functions and characterization of bounedness of Hardy operators in terms of pointwise inequalities for Young functions. Similar schemma will be used in further parts of the thesis.

Chapter 4 is devoted to the analysis of embeddings from m-th order Sobolev spaces defined on domains, to Orlicz spaces on domains, possibly with respect to the so-called Frostman measure. Such measures, if nontivial, are naturally lower dimensional, which means that they can be supported on the set of lower dimension. The example measure is d-dimensional Hausdorff measure on d-dimensional submanifold, so $n-1$ dimensional Hausdorff measure defined on the boundary of the sufficiently regular domain embedded in $\mathbf{R}^n$, can be also available. However, another example could be as well the fractal subset of the closure of the domain. The author has presented the comprehensive approach to obtaining sharp Sobolev embeddings, unifying the known results for Sobolev type embeddings and trace type Sobolev embeddings. What is important, the embeddings are equivalent to the embeddings for Hardy transforms, so this chapter is the natural continuation of previous one. The individual contribution of the author here is the analysis of the embeddings involving Marcinkiewicz type spaces, based on papers [57] and [27]. Moreover, the author contributes to the very recent results, for example he applies results from preprint of A. Cianchi L. Pick and L. Slavikova. The analysis is supported by several convincing examples.

Chapter 5 deals with fractional maximal operator M_γ related to an index $0 < \gamma < n$ and is based on author's result [56]. Earlier related results in this direction were obtained by A. Cianchi (1999) and E. Harboure, O. Salinas and B. Viviani (2002), but the author's result seems to be most closing. Main achievements there are two representative statements: Theorem 5.3.1 and 5.3.5, which give complete description of optimal target and domain space for continuity of M_γ, respectively, when it is defined on domain Ω is of finite measure. Similar statements are also formulated on $\mathbf{R}^n$. The presentation follows previous schema: derivation of the reduction principle, which characterizes the inequality in question in terms of equivalent inequalities for Hardy type operators. Hardy inequalities reduce to the pointwise inequalities due to analysis from Chapter 3. Finally, the optimality conditions are expressed in terms of certain inequalities for Boys indexes. The analysis provided there is deep. One has an impression that the discussion has been completely closed.

In *Chapter 6* the author discusses the Hardy - Littlewood maximal operator M. Questions posed there were first investigated by Kita, in papers starting from 1997. Independently, they were also investigated by A. Cianchi. Such results contribute to the celebrated Stein result which states boundedness of M between $LLogL$ space and L^1. The methods are similar to the one of Kita and relay on Cavalieri principles and the classical pointiwise inequalities between rearrangements of maximal functions and the maximal functions of the rearrangements. At first one finds the statements contributing to the inequalities between Orlicz spaces with optimal Orlicz domains. As an interesting new point in that discussion, based on results from [36], I found the proof that such optimal Orlicz domain spaces are also optimal as domains within all rearrangement invariant spaces where the target space is some fixed Orlicz space. Among results there one finds the characterization of optimal domain space for the embeddings where the domain space can be an arbitrary r.i. space while the target space is the given Orlicz one. It appears that the symmetric question about optimal target space within r.i. spaces, when the domain space is the given Orlicz one, is open. Here in one point I was lost, as unfortunately I could not understand the notation in Example 6.4.1 - the essential point in discussion, that is the notation of $L(LogL)^{\bar{A}+1}$ where $\bar{A}$ is an interval. Despite that, I found the remaining discussion very interesting, as well as reduction principle dealing with weak Orlicz space as domain, the characterization of optimal target space within Marcinkiewicz spaces (Proposition 6.4.5), theorems about the equivalence of the embeddings with Orlicz and Lorentz type domain space on $\mathbf{R}^n$, and embeddings with the endpoint cases, involving Marcinkiewicz spaces and Lorentz spaces.

In *Chapter 7* there is considered boundedness problem for Riesz potential acting on $\mathbf{R}^n$. Due to certain pointwise estimates which involve the conjugate Hardy transform of the Hardy Littlewood maximal function of the rearrangement, the problem reduces to the considerations from Chapter 3 and so, the embeddings, together with the optimality results are natural consequence of the proposed approach from the beginning part of the Thesis. Such problem in Orlicz setting was already solved by A. Cianchi in 1999. It is mentioned here, that in paper [36] one can find further analysis: of boundedness property in the larger class of rearrangement invariant spaces.

In last chapter, *Chapter 8* the author considers boudedness and optimality problem for the Laplace transform $\mathcal{L}$. He proposes the unified approach, consequent with the presentation in previous chapters, which leads to sharpness. Here the author is focusing on pointwise estimates from recent paper of E. Buriánková, D.E. Edmunds and L. Pick (2017), which give the handy tool for the reduction principle, while the problem of boundedness in the Orlicz setting, reminds open. Therefore the author posed the question about the estimates in weak type spaces. One of the example results there is Proposition 8.2.3, where sharpness condition for boudedness of $\mathcal{L}$ from Orlicz space to the Marcinkiewicz one, is derived in terms of pointwise estimates for Orlicz functions involved. There is also proposed an approach to the analysis when $\mathcal{L}$ is bounded as acting between Orlicz spaces. For this the author obtains the sufficient condition and uses interpolation type argument. Then, when focusing on L^p and L^q spaces as domain and target, he derives precise boudedness and sharpness result. I found very interesting there the analysis done on page 130, where appears weighted Lorentz space and then, in Lemma 8.3.5 one finds the sufficient condtion for the embedding of Orlicz space into such a Lorentz space. That arguments involve Stepanov techniques for weighted Lorentz spaces from 1993. It is worth to mention, that the very recent papers are involved in the presented analysis, namely, the paper by A. Alberico, A. Cianchi, L. Pick and L. Slavikova from this year, the paper by E. Buriánková, D.E. Edmunds and L. Pick from 2017 and the paper by O. Galdames Bravo from 2017.

3 Conclusion

Vít Musil proposes in his Thesis the unified approach to several problems considered previously as separate. All the problems dealing with sharpness conditions are considered difficult and the results obtained are deep, strong and very precise. The presented analysis requires very solid computations and the presentation is almost self-contained. It is also visible, that Vít Musil contributes actively to the very recent research on the field of harmonic analysis.

Taking into account the provided discussion I confirm that the presented thesis meets all customary and statutory requirements imposed for doctoral dissertations and prove Mr. Vít Musil ability to the creative scientific work in the future. Moreover, if the defense of doctoral thesis would be in my country, according to our roles, I would vote for the status of the outstanding doctoral thesis.

Yours sincerely,

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