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To whom it may concern

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Referee's report on the PhD thesis submitted by Vít Musil
to the Faculty of Mathematics and Physics, Charles University, Prague

Title: **Classical operators of harmonic analysis in Orlicz spaces**

This thesis is concerned with Orlicz spaces and a number of very famous, classical operators acting between them. This includes, in particular, the Hardy-Littlewood maximal operator, the Riesz potential, the Laplace transform, the maximal operator of fractional order, the embedding operator and the Hardy-type integral operator. In all these cases questions of their boundedness are studied, as well as the optimality or sharpness of the corresponding setting. Though partial results in this direction have been obtained before already, this presentation appears to be the first nearly complete and systematic treatment in my opinion. For the convenience of the reader the author recalls and collects known results (with proper reference given) and complements and extends them by his new findings.

In other words, the main object is to study the question of boundedness of

$$T : L^A(\Omega) \to L^B(\Omega),$$

where $L^A(\Omega)$, $L^B(\Omega)$ are Orlicz spaces with Young functions $A, B : [0,\infty) \to [0,\infty]$, $\Omega \subset \mathbb{R}^n$, and T stands for one of the classical operators mentioned above. Furthermore, optimality means in this context the quest for the *best possible* (in terms to be made precise later) Orlicz space as a target space $L^B(\Omega)$ (or source space $L^A(\Omega)$, respectively), assuming that the source space $L^A(\Omega)$, (or target space $L^B(\Omega)$, respectively,) is fixed. Such highly non-trivial questions have been studied before in similar settings to some extent by Andrea Cianchi, Luboš Pick (the PhD supervisor), and further co-authors. Their papers attracted a lot of attention, so I expect similar interest in the systematic approach contained in the present work.

The PhD thesis consists of 139 pages, sub-divided into 8 chapters and further sections. After a nice and short introductory chapter in which the scene is set, Chapter 2 deals with the general background, explaining the main notions (function spaces, operators) used in this work. Beginning with Chapter 3 the above described program is followed, starting with the study of the Hardy operator, more precisely,

$$H^R_{\alpha,\beta} : L^A(0,R) \to L^B(0,R^{1/\beta}), \quad f \mapsto \int_{t^\beta}^R f(s)s^{\alpha-1}\mathrm{d}s, \quad 0 < t < R^{1/\beta},$$

where $R \in (0,\infty]$ is arbitrary, but only the cases $R = 1$ and $R = \infty$ are of particular interest. The parameters α, β satisfy $\alpha \in (0,1)$, $\beta > 0$ with $\alpha + 1/\beta \geq 1$. The first main tool is the reduction principle for Hardy operators presented in Theorem 3.3.2, which is used to find optimal Orlicz target spaces, whereas the second reduction principle, dealt with in Section 3.5, leads to the corresponding assertions for the optimal Orlicz domain space. In both cases the outcome depends also on additional assumptions on the other Orlicz function in that sense, that if some condition is not satisfied, there is no optimal target or source space within the scale of Orlicz spaces, see Theorems 3.4.1 and 3.6.1 ($R = \infty$), as well as Theorems 3.4.3 and 3.6.3 ($R = 1$). This phenomenon can be found in all subsequent considerations, too. Chapter 4 concentrates on the embedding operator, hence embedding results of Sobolev type. Thus Sobolev spaces built upon Orlicz spaces, $W^{m,A}(\Omega)$, have to be introduced first. Here $m \in \mathbb{N}$ denotes the order of smoothness. The argument follows a similar structure as in the previous chapter, however an important role is now played by the quality of the domain and associated boundary conditions. The calculations become even more involved and tricky. Of course one might ask for more general scales of smoothness function spaces, including fractional order of smoothness, but this would certainly require completely different tools and are likely to cause other phenomena.

Chapters 5 and 6 are devoted to maximal operators of Hardy-type,

$$M_\gamma f(x) = \sup_{Q \ni x} |Q|^{\frac{\gamma}{n}-1} \int_Q |f(y)|\mathrm{d}y, \quad x \in \Omega, \quad 0 \leq \gamma < n,$$

where $Q \subseteq \Omega$ ranges over all cubes with sides parallel to the coordinate axes, containing $x \in \Omega$, with the very classical setting $M_0 = M$.The Riesz potential

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\mathrm{d}y, \quad x \in \mathbb{R}^n, \quad 0 < \gamma < n,$$

is treated in the short Chapter 7, while the presentation ends with some short digression to the Laplace transform in Chapter 8.

The text is very carefully written in a well-structured and self-contained way.

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All the necessary background material is either recalled or referred to properly; I have found almost no misprints. The proofs are given explicitly. Though the reduction principles are used mainly as elegant tool to prove optimality results, they are of their own interest and will surely find further applications in the future. In particular I also like the few examples, mainly in the realm of Zygmund spaces, which link the new and more general descriptions to well-known classical ones.

I have a very positive view of the thesis. It deals with an interesting problem and demonstrates not only considerable technical ability but also insight of a high order into what is possible. The results obtained are very good: it is not surprising that some have already appeared in first-rate journals. Moreover, the thesis is written in a remarkably clear, accurate and efficient way.

I conclude that the submitted work 'Classical operators of harmonic analysis in Orlicz spaces' completely fulfils the requirement for the PhD thesis and strongly recommend it to be accepted as doctoral work of Vít Musil . It is a fine piece of mathematics.

Dorothee Haroske

Rostock, August 23, 2018

(Dorothee D. Haroske)