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Report on V. Kovařík's PhD thesis

Absolute and non-absolute $\mathcal{F}$-Borel spaces

The PhD thesis of Vojtěch Kovařík is the union of 3 original research papers, each of them being a chapter of the thesis. In what follows, they will be denoted by [1], [2] and [3]. Paper [1] is a joint work with O. Kalenda, whereas [2] and [3] were written by Kovařík alone. Papers [1] and [2] are already published in very good journals, and [3], which is by far the longest, is submitted and available on `arXiv`.

For brevity, let us agree that the word *space* will always mean "Tychonoff topological space" (*i.e.* Hausdorff and completely regular). The 3 papers are centred around the following vaguely formulated question: what can be said of the descriptive complexity of a space X when considered as a subset of one of its compactifications cX? (This, of course, will depend on the intrinsic properties of X and also of the nature of the compactification.) To be a little more precise, the following notions are of crucial importance in the 3 papers: given a reasonable class $\mathcal{C}$ of sets in topological spaces (for example, $\mathcal{C}$ could be one of the additive or multiplicative Borel classes built from the algebra generated by the open sets), a space X is said to be a $\mathcal{C}$ *space* if it is of class $\mathcal{C}$ in *some* compactification, and *absolutely* $\mathcal{C}$ if it is of classe $\mathcal{C}$ in *every* compactification (equivalently, in every space containing it, if the class $\mathcal{C}$ is reasonable enough). For example the additive and multiplicative Borel classes are absolute, *i.e.* "$\mathcal{C}$ space" $\iff$ "absolutely $\mathcal{C}$" for these classes (this is a well known result due to P. Holický and J. Spurný). The classes mainly considered by Kovařík are the classes $\mathcal{F}_\alpha$, defined for every countable ordinal α as follows : $\mathcal{F}_0$ is the family of closed sets, $\mathcal{F}_1$ is the family of F_σ sets, $\mathcal{F}_2$ is the family of $F_{\sigma\delta}$ sets, and so on (for a limit ordinal α, the convention is that $\mathcal{F}_\alpha$ is multiplicative). The union of all families $\mathcal{F}_\alpha$ is the family of $\mathcal{F}$-*Borel* sets, which is the smallest family containing the closed sets and stable under the operations of countable union and countable intersection. These families of sets are of course well known. Note that "$\mathcal{F}_0$ space" $\iff$ "absolutely $\mathcal{F}_0$" $\iff$ "compact", and that "$\mathcal{F}_1$ space" $\iff$ "absolutely $\mathcal{F}_1$" $\iff$ "K_σ". For the first nontrivial level, *i.e.* $\mathcal{F}_2 = F_{\sigma\delta}$, deep results have been obtained (see below). For arbitray α, the notions of $\mathcal{F}_\alpha$ and absolute $\mathcal{F}_\alpha$ spaces have already been studied by several authors, in a metrizable setting. Kovařík's work seems to be the first attempt to study these notions in full generality.

According to the introduction of the thesis, the work of Kovařík was initially motivated by an old question from Banach space theory: is every WCG Banach space absolutely $F_{\sigma\delta}$ in its weak topology? By a famous 1975 result of M. Talagrand, every such space is $F_{\sigma\delta}$ in its bidual (endowed with the w^{**} topology), and hence an $F_{\sigma\delta}$ space, but the "absoluteness" problem remains open. In the same spirit, S. Argyros, A. Arvanitakis and S. Mercourakis have solved "Talagrand's $K_{\sigma\delta}$ problem" in 2008 by constructing a Banach space which is $\mathcal{K}$-analytic (in its weak topology), but not $F_{\sigma\delta}$ in its bidual. Yet, it may happen that every $\mathcal{K}$-analytic Banach space is an $F_{\sigma\delta}$ space; and this problem remains open as well.

However, the main inspiration for most of Kovařík's work is a 1985 paper by Talagrand, which will be denoted by [Tal] in what follows. In this paper, Talagrand constructs an $F_{\sigma\delta}$ space $\mathbf{T}$ which is not absolutely $F_{\sigma\delta}$, and in fact not even absolutely $\mathcal{F}$-Borel.

The paper [1] is short, but very interesting. It is devoted to $F_{\sigma\delta}$ spaces. The main resut is that any *hereditarily Lindelöf* space is absolutely $F_{\sigma\delta}$; which implies in particular that any separable Banach space is absolutely $F_{\sigma\delta}$ in its weak topology. This result is in fact obtained in 2 steps. It is first shown that a space X is absolutely $F_{\sigma\delta}$ as soon as it admits a complete sequence of countable disjoint F_σ covers. This is a nice analogue of a classical characterization of $F_{\sigma\delta}$ spaces (not necessarily absolute) due to Z. Frolik. Then, it is shown that hereditarily Lindelöf spaces do admit such complete sequences of covers. I find both results quite interesting and potentially very useful.

Another result from [1] which I would like to mention is a criterion for a space X which happens to to $F_{\sigma\delta}$ is some compactification cX, to be also $F_{\sigma\delta}$ in some smaller compactification dX. Recall that dX is smaller than cX if there is a (uniquely determined) continuous map $\varphi : cX \to dX$ which is the identity on X (considering X as a subset of both cX and dX). Note that if X is of class $\mathcal{F}_\alpha$ in some compactification dX (for any α), it is also of class $\mathcal{F}_\alpha$ in any larger compactification cX; but the converse is not true since any space X has a largest compactification (namely βX) and there exists $F_{\sigma\delta}$ spaces which are not absolutely $F_{\sigma\delta}$. It follows from the criterion proved in [1] that if there are only countably many φ-fibers $\varphi^{-1}(x)$ not reduced to a single point and if X is $F_{\sigma\delta}$ in the "large" compactification cX, then it is also $F_{\sigma\delta}$ in the "small" compactification dX.

Before describing the contents of [2] and [3], I have to say that technically, what I'm going to write is sometimes not completely correct. This is so because there is a conflict of notation between [2] and [3] for the classe $\mathcal{F}_\alpha$ and the translation from one notation to the other cannot be made perfectly accurate. Nevertheless, I think it is clearer to write things the way I did, so that in particular one can fully appreciate why the results of [3] greatly improve those of [2].

Most of [2] is devoted to a detailed analysis of [Tal], which leads to refinements of Talagrand's result mentioned above. Let us first describe very roughly some ideas of [Tal]. One starts with a family $\mathcal{E}$ of subsets of ω^ω containing only countable sets. To this family $\mathcal{E}$ is associated a space $\mathbf{T}_\mathcal{E}$ defined as follows. As a set, $\mathbf{T}_\mathcal{E} = \omega^\omega \cup \{\infty\}$, *i.e.* one adds just one point ∞ to ω^ω. As for the topology, each $\sigma \in \omega^\omega$ is declared to be isolated, and a neighbourhood subbasis of ∞ consists all sets of the form $\{\infty\} \cup (\omega^\omega \setminus E)$, where $E \in \mathcal{E}$. It is shown in [Tal] that if each set $E \in \mathcal{E}$ is closed and discrete in ω^ω (with respect to the usual product topology!), then $\mathbf{T}_\mathcal{E}$ is an $F_{\sigma\delta}$ space. On the other hand, a transfinite hierarchy of families $\mathcal{E}_\alpha$, $\alpha < \omega_1$ is defined in [Tal], and it is shown that, given $\alpha < \omega_1$, there is an ordinal $\widetilde{\alpha}$ related to α in a precise way, such that $\mathbf{T}_\alpha := \mathbf{T}_{\mathcal{E}_{\widetilde{\alpha}}}$ is not an absolute $\mathcal{F}_\xi$ space for any $\xi < \alpha$. In the opposite direction, the main contribution of [2] is to show that, for any even $\alpha \geq 4$, the space $\mathbf{T}_\alpha$ is in fact absolutely $\mathcal{F}_\alpha$. To prove this result, Kovařík proceeds as follows. First, he defines a notion of *admissible mapping* $\varphi : T \to \omega^{<\omega}$, where T is a tree on ω. (This notion turns out to be crucial, in [2] as well as in [3]). An admissible mapping is an increasing map $\varphi : T \to \omega^{<\omega}$ such that for any $t = (t_0, \ldots, t_r) \in T$, the length of $\varphi(t)$ is equal to $t_0 + \cdots + t_r$. Now, let X be any space of the form $\mathbf{T}_\mathcal{E}$, and let cX be any compactification of X. Given a tree T on ω, one can define a set $Y_T \subseteq cX$ as $\{\infty\}$ plus the set of all $x \in cX$ such that $x \in \bigcap_{t \in T} \overline{\mathcal{N}(\varphi(t))}^{cX}$, for some admissible mapping $\varphi : T \to \omega^{<\omega}$ (depending on x). Here, $\mathcal{N}(s)$ is the canonical open subset of ω^ω defined by the sequence $s \in \omega^{<\omega}$. It is almost obvious that Y_T always contains $X = \omega^\omega \cup \{\infty\}$. Moreover, it is shown in [2] that if T is equal to some canonical tree T_α of height α, then the complexity of $Y_\alpha := Y_{T_\alpha}$ in cX can be precisely estimated: if $\alpha = \lambda + m$ where $m \in \omega$ and λ is a limit ordinal (or 0), then $Y_\alpha \in \mathcal{F}_{\lambda+2m}(cX)$. The final piece of the argument is then essentially to show (for even $\alpha \geq 4$) that if $\mathcal{E} = \mathcal{E}_{\widetilde{\alpha}}$, so that X is the space $\mathbf{T}_\alpha$, then in fact $X = Y_{\widetilde{\alpha}}$ for any compactification cX; which gives the required complexity for X because of the way in which $\widetilde{\alpha}$ is related to α.

To summarize : the main result of [2] says that for any even $\alpha \geq 4$, there is an $\mathcal{F}_2$ space $\mathbf{T}_\alpha$ which is absolutely $\mathcal{F}_\alpha$ but not better.

Apart from this, it is shown in [2] that "the complexity of metrizable separable spaces is absolute": if a metrizable separable space X happens to be an $\mathcal{F}_\alpha$ space, then it is in fact absolutely $\mathcal{F}_\alpha$. The subtlety here is that one has to look at *all* the compactifications of X, not only the metrizable ones. The proof is short, but rather tricky.

Using these two results and building appropriate *topological sums*, one can go a little bit further: for any $2 \leq \gamma \leq \alpha$ with α even, there exists a space X_γ^α which is an $\mathcal{F}_\gamma$ space but not better, and also an absolute $\mathcal{F}_\alpha$ space but not better.

Finally, it is also shown in [2] that "(absolute) complexity is hereditary with respect to closed subspaces": any closed subspace of an (absolutely) $\mathcal{F}_\alpha$ space is (absolutely) $\mathcal{F}_\alpha$. Again, the proof is short but tricky.

As already mentioned, paper [3] is by far the longest part of the thesis. It contains lots of interesting results and ideas. Due to my own limitations, I do not feel able to do full justice to this very nice piece of work, but I'll try to give a sketch of the picture.

Before going into any detail, let us introduce the following definition (taken from [3]). For any $\mathcal{K}$-analytic space X, we will denote by $\mathrm{Comp}(X)$ the set of all $\alpha \leq \omega_1$ such that, for some compactification cX, we have $X \in \mathcal{F}_\alpha(cX)$ but not better. In other words, $\mathrm{Comp}(X)$ is the set of attainable complexities for X. (Note that by definition, $\mathcal{F}_{\omega_1}$ is the family of Souslin-$\mathcal{F}$ sets, $i.e.$ the sets built from the closed sets using the Souslin operation. By a result of Hansell, any $\mathcal{K}$-analytic space is Souslin-$\mathcal{F}$ in any space containing it, so the definition of $\mathrm{Comp}(X)$ does make sense.) For example, Talagrand's space $\mathbf{T}$ mentioned above satisfies $\{2, \omega_1\} \subseteq \mathrm{Comp}(\mathbf{T}) \subseteq [2, \omega_1]$. Also, for $2 \leq \alpha$ even, the space X_2^α from [2] satisfies $\{2, \alpha\} \subseteq \mathrm{Comp}(X_2^\alpha) \subseteq [2, \alpha]$.

Formally, the main *results* of [3] may be considered to be the following.

(a) For any closed interval $I \subseteq [2, \omega_1]$, there is a space X_I such that $\mathrm{Comp}(X) = I$.

(b) For an interval of the form $I = [2, \alpha]$, one can take as X_I the Talagrand broom space $\mathbf{T}_\alpha$.

(c) The complexity of hereditarily Lindelöf spaces is absolute: if a hereditarily Lindelöf space X happens to be an $\mathcal{F}_\alpha$ space, then it is in fact absolutely $\mathcal{F}_\alpha$. (This generalizes the main result of [1].)

In my opinion, these are beautiful results. However, I agree with the author that the methods developed to prove them are perhaps even more iportant than the results themselves.

To give an idea of the proof of (c), we need some notation. For any class $\mathcal{C}$ of subsets of some set Y, define the families $\mathcal{C}_\alpha$, $\alpha < \omega_1$ in exactly the same way as the classes $\mathcal{F}_\alpha$, starting from $\mathcal{C}_0 := \mathcal{C}$. Also, for any well-founded tree T on ω (whose set of leaves is denoted by $l(T)$) and any family of sets $\mathbf{C} = (C(t))_{t \in l(T)}$ in Y, define $C(s)$ for every $s \in T$ by alternating the operations of union and intersection going down the tree T, starting with union; and set $\widehat{\mathbf{C}} := \mathbf{C}(\emptyset)$. The key point for proving (c) is to show that (for any $\mathcal{C}$) a set $X \subseteq Y$ belongs to $\mathcal{C}_\alpha$ if and only if it admits a *simple representation* by sets from $\mathcal{C}$, which means that it has the form $X = \widehat{\mathbf{C}}$ for some family of $\mathcal{C}$-sets $\mathbf{C}$ indexed by a tree T of height $\leq \alpha$. This is applied with a space X, a compactification $Y = cX$ and taking as $\mathcal{C}$ the family $\mathcal{F}(cX) \wedge \mathcal{G}(cX)$ (intersections of an open set and a closed set in cX). Using the absoluteness of the classes $(\mathcal{F} \wedge \mathcal{G})_\alpha$ (which is due to Holický and Spurný) and an idea from [1], the result follows.

For the proof of (a), Kovařík uses as a "blackbox" the fact that for any $2 \leq \gamma \leq \alpha$ there exists a space X_γ^α such that $\{\gamma, \alpha\} \subseteq \mathrm{Comp}(X_\gamma^\alpha) \subseteq [\gamma, \alpha]$. For even α, this follows essentially from [2]; but for arbitrary α, this is in fact a key step in the

proof of (b). Taking this as granted, (a) is proved by making extensive use of a nice topological tool that Kovařík calls the *zoom space* construction. Zoom spaces are defined as follows. Let Y be a space. Denote by $I(Y)$ the set of all isolated points of Y, and and let $\mathcal{X} = (X_i)_{i \in I(Y)}$ a family of spaces. The zoom space $Z(Y, \mathcal{X})$ is the disjoint union $(Y \setminus I(Y)) \cup \bigcup \mathcal{X}$ equipped with the topology generated by all open subsets of X_i, $i \in I(Y)$ and all sets of the form $(U \setminus I(Y)) \cup \bigcup_{i \in U \cap I(Y)} X_i$, where U is open in Y. For example, if $Y = I$ is a discrete space, then $Z(I, \mathcal{X})$ is the topological direct sum $\bigoplus_{i \in I} X_i$. As shown by Kovařík, zoom spaces behaves nicely under many natural operations; for example, for any compactifications cY and cX_i, $i \in I(Y)$, the zoom space $Z(cY, c\mathcal{X})$ is a compactification of $Z(Y, \mathcal{X})$. Moreover, one can compute the complexity of a zoom space $Z(Y, \mathcal{X})$ inside $Z(cY, c\mathcal{X})$: it is exactly equal to $\max\big(\mathrm{Comp}(Y, cY), \sup_{i \in I(Y)} \mathrm{Comp}(X_i, cX_i)\big)$ (the meaning of $\mathrm{Comp}(E, cE)$ for any space E should be clear). Using these ideas, Kovařík is able to give a rather soft proof of (a).

The proof of (b) is the most difficult part of [3]. One key ingredient is a general theorem saying that a subset X of some space Y belongs to $\mathcal{F}_\alpha(Y)$ is and only if it admits what one may call (following Kovařík) a *regular representation of complexity at most α*. The definition is as follows. If $\mathcal{C} = (C(t))_{t \in \omega^{<\omega}}$ is Souslin scheme of subsets of Y and if T is tree on ω, define $\mathbf{R}_T(\mathcal{C})$ to be the set of all $x \in C(\emptyset)$ such that $x \in \bigcap_{t \in T} C(\varphi(t))$ for some admissible mapping $\varphi : T \to \omega^{<\omega}$. (Recall that admissible mappings were introduced in [2].) A set $X \subseteq Y$ is said to admit a regular representation of complexity at most α if $X = \mathbf{R}_{T_\alpha}(\mathcal{C})$ for some closed Souslin scheme $\mathcal{C}$ and some tree T_α canonically associated with α. I strongly believe that the general theorem about regular representations obtained by Kovařík is likely to become a "standard" tool for the study of $\mathcal{F}_\alpha$ spaces. Another key ingredient in the proof of (b) is a topological tool that Kovaárík calls *amalgamation spaces*. I will not give the definition here. In some sense, these amalgamation spaces are close in spirit to zoom spaces, yet this is a rather different notion. However, they also have nice formal properties, especially with respect to compactifications, so that one can use them to construct various compactifications of Talagrand's broom spaces $\mathbf{T}_\alpha$. Finally, the third main ingredient of the proof of (b) is, of course, a very careful analysis of the arguments in Talagrand's paper [Tal], which is in the spirit of [2] but (I think) of a much deeper nature.

There are other interesting results and remarks in [3]; but since this report is already long enough, it does not seem necessary to give any further detail.

In my opinion, the 3 papers presented by V. Kovařík are of great mathematical value. There are lots of interesting ideas in them, and they show without any doubt the mathematical creativity of Kovařík, his outsanding ability to analyze and to explain difficult ideas, and its strong technical skills. Moreover, I have no doubt that some of the tools presented by Kovařík (zoom spaces, regular representations, amalgamation spaces) could be extremely useful in various situations. Finally, I

found the papers very well written, with sometimes a very delicate touch of humour (am I wrong?).

To temper a little bit my enthusiasm, I do have one criticism: I would have liked this PhD thesis even more if it had been a real "thesis" rather than the union of 3 papers. This would have avoided some redundancies (for example, the definitions given several times, or the overlap between some results in [2] and their improvements in [3]). Also, the notational conflict between [2] and [3] would have disappeared. However, this is rather inessential in view of the mathematical value of the work.

Altogether, it is quite clear to me that V. Kovařík's work can be qualified as a high quality PhD thesis.

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