

Report on the PhD thesis

Rich Families of Projections and Retractions

by Jacopo Somaglia

The thesis deals with the concept of a retractional skeleton in compact topological spaces, as well as its counterpart in the category of Banach spaces. The dissertation consists of three parts containing the main results, available also in the form of two published articles and one preprint. The author starts with a nice introduction to the subject, explaining the background, history, principal notions, and finally the main problems addressed during his PhD project. He also sketches his main achievements, described in detail in the next sections, which in turn are copies of the manuscripts forming the PhD dissertation. One needs to notice that all these works are without co-authors. This fact should be taken very positively, as most real collaborations arise during postdoctoral positions and later, when the researcher becomes more and more independent. In my opinion, the most important achievements of Somaglia are in the last two sections, dealing with compact trees. The first section (work) contains non-trivial results, too, parallel to the commutative case obtained several years ago by his supervisor Ondřej Kalenda.

It turns out that projectional and retractional skeletons are particular cases of a general category-theoretic concept which (according to my knowledge) has not been thoroughly studied besides the categories of compact spaces and Banach spaces. From this general point of view, Somaglia's thesis contains new and significant insights around this notion in the two categories mentioned above. His results may provide some good inspirations for further research, possibly going into the general category-theoretic direction.

Below I comment on Somaglia's works, quoting the titles from his PhD's table of contents.

Weakly non-commutative Valdivia compacta. This concerns the following published article:

J. Somaglia, *On the class of continuous images of non-commutative Valdivia compacta*, *Topology and its Applications* 210 (2016) 147–167

The author studies continuous images of spaces with retractional skeletons, proving several non-trivial stability properties. It seems that the main notion is a *weakly non-commutative Corson countably compact space*. It apparently turned out to be useful in detecting several stability properties for larger classes of (countably) compact spaces. By the way, the definition of a retractional skeleton asserting that the values of retractions are compact and metrizable is very natural and leads to reasonable results beyond the realm of compact spaces. The last part of this work contains interesting results in Banach space theory, in particular, concerning a new class of spaces whose dual balls are weakly non-commutative Corson. In my opinion, the stability result on ordinal sums (in the dissertation called $[0, \eta]$ -sums) is highly non-trivial, it requires a really good knowledge of the subject. Also, the stability result concerning Alexandrov duplicates is quite involved. The author could perhaps look at a more general version, where a fixed subset of the underlying space is duplicated. I have only one critical remark here. Namely, the author writes on page 15:

In [2] it is proved that a compact space is Corson if and only if it has a full retractional skeleton.

Actually, this has been proved much earlier in [W. Kubiś, *Compact spaces generated by retractions*, Topology Appl. 153 (2006) 3383–3396]. Specifically, in this paper I proved that a countably tight space in a rather large class of compacta (involving retractions and continuous images) is Corson; this class clearly contains spaces with retractional skeletons. In any case, this is only a bibliographic remark and I find the study of non-commutative Corson/Valdivia (countably) compact spaces an interesting further direction of research.

Retractional skeletons on trees. This concerns the following published work:

J. Somaglia, *New examples of non-commutative Valdivia compact spaces*, Fundamenta Mathematicae 243 (2018) 143–154¹

The author characterizes compact trees (with the coarse wedge topology) that have a retractional skeleton. Taking advantage of his characterization, he finds a new example of a non-Valdivia compact space with a retractional skeleton and of character ω_1 , therefore with no homeomorphic copy of the ordinal interval $\omega_2 + 1 = [0, \omega_2]$. Previously, the only known examples of so-called non-commutative Valdivia compacta were based on the ordinal $\omega_2 + 1$, therefore this new example answers in the negative a

¹In the dissertation, the author classifies this work as “to appear”. As it happens, the manuscript has already been officially published (see DOI: 10.4064/fm520-2-2018).

natural question² asking whether a non-commutative Valdivia compact must necessarily contain a copy of $\omega_2 + 1$. Summarizing, compact trees turned out to be a useful source of examples in the theory of retractional skeletons. One has to admit that the above-mentioned characterization (Theorem 3.3.1 in the dissertation), yet intuitively clear, cannot be called trivial at all. The proof that Example 3.4.3 is not Valdivia compact requires an elegant Baire category-theoretic argument. This clearly shows that the author is capable of doing a high level research, discovering new and sometimes unexpected relations and arguments.

Compact trees. This concerns the following published work:

J. Somaglia, *On compact trees with the coarse wedge topology*, preprint (arXiv:1803.11107)

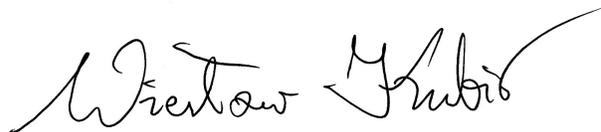
Here, the author continues his study of compact trees with the coarse wedge topology, in the context of properties related to retractional skeletons. He obtained a nice (although not extremely surprising) result saying that every probability measure on a tree has a metrizable support. The proof is rather non-trivial, using transfinite induction. Let us note that compact trees with the coarse wedge topology can contain quite complicated topological spaces (simply consider the full binary tree of a large height). In particular, they are typically far from being scattered, therefore the result on measures does not seem to follow from any general theory capturing compact trees. The paper mentioned in this section contains also a short “almost” characterization of Valdivia compacta among trees. A very interesting problem is left: how to characterize Valdivia compacta among trees of height precisely ω_2 ?

Style of the dissertation. The remarks below do not decrease the value of Somaglia’s PhD thesis. This is rather a suggestion for some changes concerning the final forms of PhD dissertations consisting of published works. Namely, in my opinion a dissertation should be either a more or less self-contained small research monograph or an enlarged abstract containing descriptions of results that have already been published, with detailed bibliographic references. The PhD thesis of Jacopo Somaglia is not a readable mathematical text, as it contains 4 bibliographies and many repetitions. On the other hand, the author indicates that the contents of his thesis consists of 3 articles (two of them published) and I do not see any sense to copy these articles, it would be much better to write a survey text instead, including all the main results.

²In fact, I personally asked this question few times during various conferences, unfortunately it seems that it has not been officially written in any published text.

Moreover, two articles of Somaglia have been published in rather prestigious journals (Fundamenta Mathematicae, Topology and its Applications) therefore one should assume that the results are correct, checked by professional referees. It makes sense to include the full contents of the third article (which, as I understand, is under refereeing), since its correctness has not been fully checked yet. Finally, preparing the dissertation as a long survey article would actually make it a nice source for inspirations for younger researchers, while in its current form this is just a text that will be used in the refereeing process and which will very soon end up in the university archive. I am pretty sure that nobody will be interested in reading it, simply because the published works with identical content are (or there soon will be) available in the internet and in good mathematical libraries.

Conclusion. The submitted thesis of Jacopo Somaglia contains several original results that solve some open problems in the area of Banach space theory and related compact topological spaces. I am fully convinced that the thesis fulfills requirements for the PhD degree and I strongly recommend its acceptance.

A handwritten signature in black ink, reading "Wiesław Kubiś". The signature is written in a cursive style with a long, sweeping flourish extending from the end of the name.

Wiesław Kubiś

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