Supervisor's evaluation of the PhD thesis

Jacopo Somaglia: *Rich Families of Projections and Retractions*

The presented thesis is devoted to investigation of compact spaces with a retractional skeleton and Banach spaces with a projectional skeleton. This is an important topic in the context of nonseparable Banach spaces – it provides a method of decomposing large spaces into smaller pieces and to transfer certain properties of separable spaces to a large class of nonseparable ones.

Compact spaces with a retractional skeleton generalize Valdivia compact spaces and are sometimes called 'noncommutative Valdivia compact spaces'. The reason is that by a result of Kubiś and Michalewski a compact space is Valdivia if and only if it admits a commutative retractional skeleton. So, the study of compact spaces admitting a retractional skeleton (and related Banach spaces) naturally splits into three basic directions. The first one consists in generalizing results on Valdivia compact spaces to the noncommutative context. The second one is the search of the differences between the commutative and noncommutative cases and of witnessing examples. Finally, the last one is concerned with characterization of Valdivia compact spaces within non-commutative ones.

The thesis consists of four chapters – and introductory one and three research papers (one published, one accepted for publication and one submitted). Let me comment the three papers.

The first one (contained in Chapter 2) is devoted to the study of the class of continuous images of non-commutative Valdivia compact spaces. The starting point is my paper from 2003 where similar things were studied for continuous images of Valdivia compact space. Hence, this paper follows the first direction of research mentioned above. Some results generalize trivially, but it should be pointed out that it is far from being a general rule. The reason is that Valdivia compact spaces are defined using a certain type of embedding, while non-commutative ones are defined using a family of retractions. Therefore, generalizing certain results on Valdivia compact spaces to the noncommutative context often requires finding a new proof for the commutative case. It should be also mentioned that the applicant pointed out a mistake in the above-mentioned paper which was due to an incorrect definition of one-point countably compact modifications and presented a correction here.

The second paper, contained in Chapter 3, follows the second direction of research. As mentioned in its introduction, it was originally motivated by the question whether a noncommutative Valdivia compact space containing no copy of $[0, \omega_2]$ is necessarily Valdivia. I conjectured that the answer may be positive – it was motivated by the study of certain class of Banach spaces ('almost WLD' ones). However, the applicant provided a negative answer. He found in the literature the coarse wedge topology on trees and realized that it is closely related to retractional skeletons. As a result, a new wide class of compact spaces with retractional skeleton has been discovered, showing that non-commutative Valdivia compact spaces form a richer class than expected.

In the third paper (Chapter 4) the class of compact trees with coarse wedge topology is investigated in more detail. Firstly, it is proved that any such tree has property (M), i.e., any Radon measure on it has metrizable support. This is a general result, not depending on retractional skeletons. Secondly, the ideas from Chapter 3 are elaborated to a partial characterization of Valdivia trees. This is not a complete characterization, as there is a necessary condition and a sufficient condititon which are not the same (although they seem to be near to each other). However, the proof of the sufficiency is rather involved. Finally, the relationship of the properties of a tree T and the space $C(T)$ of continuous functions on T is studied. It is known that $C(K)$ has a 1-projectional skeleton whenever K is a compact space with a retractional skeleton (and similarly in the commuttative case). Here the author proves that for trees the converse hold – this is not very deep, but it is interesting,

because the converse is known to be true if K has a dense set of G_δ points but it fails in general. (Note that trees need not contain G_δ points.) But what is even more interesting, it is shown that $C(T)$ has a commutative projectional skeleton (i.e., it is Plichko, the upper bound of norms is computed) whenever T has small height, regardless of the existence of a retractional skeleton on T.

All the papers contain nontrivial original results. The papers are mutually related, the third one is a direct continuation of the second one. The topic is a part of an active field as witnessed by the bibliography. The applicant had to learn a lot of material, combine approaches from various areas and come up with new ideas. I am convinced that the thesis demonstrates that the applicant is able of an independent scientific work.

Therefore I have no reservations to strongly recommend to accept the submitted thesis as a PhD thesis.

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Prof. RNDr. Ondřej Kalenda, Ph.D., DSc.
Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University