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DOCTORAL THESIS

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Stochastic Differential Equations with Gaussian Noise

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Title: Stochastic Differential Equations with Gaussian Noise

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Abstract: Stochastic partial differential equations of second order with two unknown parameters are studied. The strongly continuous semigroup $(S(t), t \geq 0)$ for the hyperbolic system driven by Brownian motion is found as well as the formula for the covariance operator of the invariant measure $Q_\infty^{(a,b)}$. Based on ergodicity, two suitable families of minimum contrast estimators are introduced and their strong consistency and asymptotic normality are proved. Moreover, another concept of estimation using "observation window" is studied, which leads to more families of strongly consistent estimators. Their properties and special cases are descibed as well as their asymptotic normality. The results are applied to the stochastic wave equation perturbed by Brownian noise and illustrated by several numerical simulations.

Keywords: Stochastic hyperbolic equation, Ornstein–Uhlenbeck process, invariant measure, parameter estimation, strong consistency, asymptotic normality.

Contents

Introduction

Consider the following differential equation

$$\ddot{u} + 2a\dot{u} + bu = 0, \tag{1}$$

where a and b are positive parameters. Since it is a linear differential equation with constant parameters, it is indeed easy to solve it. If we find out the roots of the appropriate characteristic equation

$$\lambda^2 + 2a\lambda + b = 0$$

and if we suppose that $a^2 - b < 0$, we obtain the solution in the form

$$u(t) = C_1 e^{-at} \sin\left(\sqrt{b-a^2}\, t\right) + C_2 e^{-at} \cos\left(\sqrt{b-a^2}\, t\right), \quad t \geq 0.$$

With any given initial condition, it is also possible to determine the coefficients C_1 and C_2 exactly. From the above formula it is easy to see that the solution is a linear combination of sine and cosine functions, which are damped by the exponential multiplicative factor. The equation (1) is a well–known model of oscillator that is suitable for description of damped waves.

In fact, this equation is very important in classical physics, since it may be used for modelling of mechanical waves as well as light waves. The parameter a is called "damping" and the parameter b represents "speed of the wave". (Bigger parameter b means lesser period of the trigonometric functions.) Note that the assumption $a^2 - b < 0$ is needed for the oscillatory behaviour of solutions, since otherwise the solution would be completely different.

There are many ways to generalize the equation (1). One possible choice is to add a noise term, since it is reasonable to assume that we do not detect the perfect information, but some errors may occur in it. That would lead to the equation

$$\ddot{u} + 2a\dot{u} + bu = \eta, \tag{2}$$

where η is some kind of noise. Moreover, that noise need not to be a deterministic one, but rather random with certain probability properties.

Nevertheless, such equation would still be a scalar equation, which is not sufficient for description of (for example) oscillation of solid objects. In order to model the displacements of individual particles in space, they need to be modelled in terms of functions. Hence our state space have to be a linear space of functions, which will transform our equation to an infinite–dimensional one.

Therefore, consider the following stochastic wave equation with strong damping

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t,\xi) &= b\Delta u(t,\xi) - 2a\frac{\partial u}{\partial t}(t,\xi) + \eta(t,\xi), \quad (t,\xi) \in \mathbb{R}_+ \times D, \qquad (3)\\
u(0,\xi) &= u_1(\xi), \quad \xi \in D,\\
\frac{\partial u}{\partial t}(0,\xi) &= u_2(\xi), \quad \xi \in D,\\
u(t,\xi) &= 0, \quad (t,\xi) \in \mathbb{R}_+ \times \partial D,
\end{aligned}$$

where $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, Δ is the Laplace operator and η is a random noise. It may be viewed as a direct generalization of equation (2) to infinite dimension with Dirichlet boundary conditions (i.e., the values of the solution on the boundary ∂D are constant and prescribed). The nature of the solution heavily depends on the structure of the noise. With standard Brownian motion, the solution will be continuous and it will have the (strong) Markov property. If fractional Brownian motion is used, the solution will not possess the Markov property. It will be dependent on the past in some sense instead. And if the driving process is Lévy process, the solution will not be continuous in time, since it inherits the jumps of the Lévy process.

In the Thesis, we use the standard space–time Brownian motion or rather (as will be specified later) the standard cylindrical Brownian motion, since our results are new even in this case. The alternative choices of the noise represent ways for possible future extensions.

It should be said that the equation (3) is an example of a linear stochastic partial differential equation (SPDE)

$$\begin{aligned} dX(t) &= \mathcal{A}X(t)\,dt + \Phi\,dB(t), \\ X(0) &= x_0, \end{aligned} \tag{4}$$

with some linear Hilbert–Schmidt operator Φ and a linear bounded operator $\mathcal{A}$, which is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$.

SPDEs are very frequent and useful generalization of partial differential equations. They are used for modelling not only in physics (see, e.g., [27]), but also in astronomy [31], chemistry [33], biology [29], hydrology [4], climatology [19] and social sciences [2]. SPDEs driven by Lévy process are mostly utilized in finance [25] and option pricing [30].

Now consider the equation (3) with the unknown parameters a and b. The main objective of the Thesis is to provide strongly consistent estimators of the parameters based on the observation of the trajectory of the process $(u(t, \xi)$, $0 \leq t \leq T$, $\xi \in D)$, which is the solution to (3), up to time T. This is the problematics of statistical inference.

Statistical inference for SPDEs driven by standard Brownian motion has been recently extensively studied. This Thesis presents results, which are interesting mostly for two reasons. In the first place, many authors use the maximum likelihood estimator (MLE) for the estimation of unknown parameters in SPDEs (for example [32], where the parameter is linearly built in the drift) or the least squares estimator (for example [6] and [7]), however we are interested in the minimum contrast estimator (MCE). This type of estimator has been studied since 1980's (see the pioneering papers [17] and [16]), but there are also more recent works. For example in [24] and [23], the (MCE) is studied even for the SPDEs driven by fractional Brownian motion.

Secondly, many authors concentrate on stochastic parabolic equations (see [8]), but stochastic hyperbolic equations were not paid too much attention to. We may mention [20] or [21], but, again, only the (MLE) is investigated. Therefore the topic of (MCE) for stochastic hyperbolic equations (such as wave equation or plate equation) is rather new, even if the driving process is ”only” standard Brownian motion.

We will present several families of (MCE) estimators, which use different kind of information, and we will focus on the following themes:

Derivation of the estimators. We will compute the strongly continuous semigroup $(S(t), t \geq 0)$ for the system (3) and show its exponential stability. Thus the solution will have a unique invariant Gaussian measure $\mu_{\infty}^{(a,b)} = N\left(0, Q_{\infty}^{(a,b)}\right)$ and we may follow up the work [23], where (MCE) based on ergodic theorems were derived for analogous parabolic problems. First, the estimators based on norm of the solution are introduced, then we present the estimators based on individual coordinates (or modes) of the solution and their linear combinations. This is the concept of observing the solution through some ”observation window”.

Strong consistency. We would like to have our estimators strongly consistent. It means that with increasing time T, the estimator converges to the true value of the parameter $\mathbb{P}$–almost surely ($\mathbb{P} - a.s.$). Since the estimators were obtained by a suitable ergodic theorem, their strong consistency will be assured for any initial condition x_0.

Asymptotic normality. We will show that each of our estimators is asymptotically normal with certain parameters. It is indeed useful to know that the deviations from the true value of the parameter are normally distributed. In addition, it is the key for construction interval estimators and confidence intervals for statistical hypotheses testing. Although the estimators are asymptotically unbiased (the limiting distribution of deviations has zero mean), the formulae for the variances are rather complicated and they depend on the parameters which are to be estimated. Therefore, it is not easy (or even possible) to construct the interval estimators neither for the parameters themselves, nor for some parametric functions of them.

We will also demonstrate the theoretical results on computer simulations. Although the SPDE, which models oscillations of a rod, is infinite–dimensional, it is quite easy to implement it by the Euler's method (or the finite element method). For that matter, we recommend the book [9], where many numerical methods for stochastic differential equations are described.

Publications related to the Thesis

The Thesis comprises the following two papers:

Josef Janák, *Parameter estimation for stochastic partial differential equations of second order*, Applied Mathematics and Optimization – To appear, (2017). Available at `https://doi.org/10.1007/s00245-018-9506-9` and `https://arxiv.org/abs/1806.04045`.

The paper includes the parameter estimation based on the norm of the solution. The strongly continuous semigroup $(S(t), t \geq 0)$ for the hyperbolic system (3) is derived as well as the formula for the covariance operator $Q_{\infty}^{(a,b)}$ for the limiting measure. Based on ergodicity, strongly consistent family of estimators is established. Moreover, an alternative family of estimators is proposed and comparison of some basic properties shows that this new family of estimators is in some sense better than the "classical" one. The asymptotic normality of estimators is proved and the results are illustrated by several numerical simulations.

Josef Janák, *Parameter estimation for stochastic wave equation based on observation window*, Stochastic Analysis and Applications – Submitted, (2018). Available at `https://arxiv.org/abs/1806.07743`.

This paper extends the results of the first article. The form of the covariance operator $Q_\infty^{(a,b)}$ is reminded and based on observation of the coordiantes of the solution or their linear combination (i.e., "observation window"), further (MCEs) are introduced. Since the resulting formula for the operator $Q_\infty^{(a,b)}$ is rather complicated, only so–called "diagonal case" is assumed. Strong consistency and asymptotic normality are proved and some possible estimation strategies are presented. The simulations are made for the individual coordinates of the solution.

The ideas for the infinite–dimensional case are also valid for the scalar case, which was published across the following three papers:

Josef Janák, *Fractional Brownian motion*, in Proceedings of the 15th Conference on Applied Mathematics, Aplimat, 2016, pp. 557–567.

The main topic of this paper is the concept of scalar fractional Brownian motion (fBm). We recall the definition of stochastic integral of a deterministic function, where the driving process is (fBm). We introduce the solution to the linear stochastic differential equation driven by (fBm) which is called the fractional Ornstein–Uhlenbeck process. In the second part of the paper, some parameter estimation for Ornstein–Uhlenbeck process is discussed. Although the driving process used in this estimation was "only" standard Brownian motion, these were the important initial steps for the whole topic.

Josef Janák, *Asymptotic normality of parameter estimates for stochastic differential equation of second order*, in Proceedings of the 16th Conference on Applied Mathematics, Aplimat, 2017, pp. 704–711.

This short paper contains the proof of the asymptotic normality of estimators, which were derived in the previous article. Since the solution of the wave equation is in the scalar case $\mathbb{R}^2$–valued, the Euclidean norm of the space $\mathbb{R}^2$ is used. The idea of splitting the solution to the individual components and observing their norms separately, has not yet been included.

Josef Janák, *Parameter estimation for scalar stochastic differential equation of second order*, in Proceedings of the 17th Conference on Applied Mathematics, Aplimat, 2018, pp. 487–499.

Finally, the alternative family of estimators was discovered and this paper shows its strong consistency and asymptotic normality for the scalar case. The computed formulae are indeed simplified versions of the more general formulae presented in the Thesis. The final part of the paper is dedicated to implementation of the estimators in R language. The properties of the estimators are discussed on one particular simulation.

Organisation of the Thesis

The Chapter 1 starts with some basic notion on cylindrical Brownian motion and stochastic linear partial differential equations. More specifically, we introduce the concept of the cylindrical Brownian motion $(B(t), t \geq 0)$ on a real separable Hilbert space U together with the extention, which allows the spatial covariance of the process. Such spatial covariance is represented by certain covariance operator Q and the condition, for which the cylindrical Brownian motion takes the values in the Hilbert space U, is that the operator Q needs to be nuclear (see Definition 1.2 and following explanation). Furthermore, we define the stochastic integral for a suitable random operator–valued function $\Phi(\cdot)$ with respect to the cylindrical Brownian motion and we present the linear SPDE with additive noise (cf. (4)). The strong solution $(X^{x_0}(t), t \geq 0)$ to this equation is introduced, however since the assumption $X^{x_0}(t) \in \mathrm{Dom}(\mathcal{A})$ is rather strong, we also introduce so–called mild solution, which is for our purposes satisfactory enough. The mild solution is given by the variation of constants formula and it is unique (see Proposition 1.3). There are many interesting and important theorems on this topic (such as the strong solution is always mild and the contrary is true only with some additional assumptions), however we do not explain such particulars here. For more detail, see [5] or [10], which are the main references for this preface.

We end this preliminary with Proposition 1.4, which states that if the associated semigroup $(S(t), t \geq 0)$ is exponentially stable then the solution has a unique invariant Gaussian measure μ_∞ with covariance operator Q_∞. We do not introduce the concept of Gaussian measures on Hilbert spaces here, but we encourage to read [5, Section 2.3]. The weak convergence of the probability measures to the unique invariant measure holds for each initial condition x_0 and it is essential to our work. Note, however, that the condition of exponential stability of the semigroup $(S(t), t \geq 0)$ may be weakened in the sense of Theorem 11.11 in [5].

In Section 1.2, we provide the rigorous meaning to the equation (3) and its rewriting to the form (4), where the operator $\mathcal{A}$ contains the parameters a and b. We introduce the setup as well as some assumptions which are needed. Then we compute the from of the semigroup $(S(t), t \geq 0)$, which has the infinitesimal generator $\mathcal{A}$. However, since the expression $a^2 - b\alpha_n$ may be negative, positive, or equal to zero (which we do not a priori know, since we do not know the exact values of the parameters a and b), we compute the appropriate semigroup for all three different cases and then we combine the resulting formulae together (see Theorem 1.14). We find the form of the covariance operator $Q_\infty^{(a,b)}$ in a similar way. For each of the three cases, we derive the adjoint operator of $(S(t), t \geq 0)$ and then we use the formula (1.11). The integrals of the operators are computed in a way that the operators are rewritten in the form of Schmidt decomposition and afterwards they are integrated. Note, however, that the eigenvectors of the operators A and Q may differ (we assume the general non–diagonal case here), therefore there is a need to express one basis with respect to the other and the double sums occur. Very interesting result is the fact that although the semigroups in these three cases are totally different, the resulting formulae for the covariance operator $Q_\infty^{(a,b)}$ coincide. Hence they are combined together easily and the resulting formula for the covariance operator is given by Theorem 1.15. This is also the main new result of this chapter.

The main concern of Chapter 2 is the parameter estimation based on norm of the solution. Since the solution $(X^{x_0}(t), t \geq 0)$ to the equation (4) has a unique invariant measure, we use ergodic theorem 2.1 from [23]. The time averages of a certain suitable functional of the solution tend to the space average in the $\mathbb{P}-a.s.$ sense and that is the tool for obtaining some strongly consistent estimators. For the functional $\|\cdot\|_V^2$ the time averages converge to the trace of the covariance operator $Q_\infty^{(a,b)}$. This trace is computed and the family of strongly constistent estimators $(\hat{a}_T, \hat{b}_T)$ is introduced in Theorem 2.3. These estimators have, however, serious disadvantage: In order to compute one estimator (say $\hat{a}_T$), we need to know the true value of the other parameter (in this case b). This is indeed unpleasant limitation, but we discovered a way to overcome it.

Since the solution has two components (the first one is for the actual state $(X_1^{x_0}(t), t \geq 0)$ that belongs to the space $\mathrm{Dom}((-A)^{\frac{1}{2}})$, while the second one is for its time derivative $(X_2^{x_0}(t), t \geq 0)$ that belongs to the space $L^2(D)$), norms of these two may be computed (and observed) separately. Also the trace of $Q_\infty^{(a,b)}$ consists of two terms. Therefore, the use of some appropriate functionals in the ergodic theorem 2.1 provides another family of strongly consistent estimators $(\tilde{a}_T, \tilde{b}_T)$ (see Theorem 2.4). The first comparison of these two families comes naturally: The second family $(\tilde{a}_T, \tilde{b}_T)$ allows to estimate both parameters a and b "at the same time" with no need for any additional information. This is already great improvement, however there is also another one, which will be discussed later on.

Asymptotic normality of the estimators is proved in Section 2.2. In the first part, we show the asymptotic normality of the family $(\hat{a}_T, \hat{b}_T)$. The main strategy of the proof is to find an alternative representation for the process I_T (the process, which the estimators are based on) using Itô's formula for some suitable function. Inspired by [17], we realized that we need to use the function $g(x) = \langle Rx, x\rangle_V, x \in V$, with some self–adjoint linear operator $R : V \to V$ satisfying

$$\langle Rx, \mathcal{A}x\rangle_V = C\|x\|_V^2, \quad \forall x \in \mathrm{Dom}(\mathcal{A}),$$

for some (rather negative) constant C. Such operator R was found and the alternative representation of I_T (see (2.13)) provided a stochastic integral, which was consequently used in central limit theorem for martingales. This is the point, where the asymptotic normality of estimators comes from. See Theorem 2.8 for precise statement.

The asymptotic normality of the family of estimators $(\tilde{a}_T, \tilde{b}_T)$ is formulated in Theorem 2.11. The setup and auxiliary lemmas are similar – there was also a need to find some appropriate operators R_1 and R_2 in order to obtain certain alternative representation for the processes Y_T and H_T.

Although the forms of limiting variances of the estimators are rather complicated (see Remarks 2 and 3), there is a proof that the limiting variances of the family $(\tilde{a}_T, \tilde{b}_T)$ are smaller than the limiting variances of the family $(\hat{a}_T, \hat{b}_T)$ (see Theorem 2.12). This is indeed the other improvement and another reason, why the "new" family of the estimators $(\tilde{a}_T, \tilde{b}_T)$ can be viewed better as the family $(\hat{a}_T, \hat{b}_T)$, which was obtained by the "classical approach".

We end this chapter by Remark 4, where we show that many of the formulae may be considerably simplified if we consider so–called "diagonal case".

In Chapter 3, we study the statistical inference based on "observation window". The idea is that if we able to track the trajectory of the process $(\langle X^{x_0}(t), z\rangle_V,\ t \geq 0)$ for any given $0 \neq z \in V$, we observe the solution "through the window" z. (For example if z is the element of the basis, we observe the appropriate coordinate of the solution.) Based on this information, further strongly consistent estimators of parameters a and b are proposed.

We start Section 3.1 with the ergodic theorem 2.1 again. Using functional $\langle \cdot, z\rangle_V^2$, we obtain the $\mathbb{P}-a.s.$ convergence of the appropriate time averages to the value $\left\langle Q_\infty^{(a,b)} z, z\right\rangle_V$. However, since the formula for the covariance operator $Q_\infty^{(a,b)}$ is rather complicated, we assume only the diagonal case. The explicit form of the estimators $\bar{a}_T$ and $\bar{b}_T$ is given by Theorem 3.1. Nevertheless, the "observation window" z is also rather general and we would prefer some specification. Indeed, for any given concrete z, we may obtain a new formula, which may be used for the estimation. Some of the formulae require the information about the true value of the other parameter, some do not. The formulae may be even combined together, which suggests some certain estimation strategies, which are also described. We will mostly work with the estimators $\bar{a}_{T,z_2}$ (which is obtained by the choice of the window $z = (0, z_2)^\top$) and $\bar{b}_{T,z_1,z_2}$ (which is obtained by using the window $z = (z_1, 0)^\top$ together with the previous estimator $\bar{a}_{T,z_2}$). See Corollary 3.2 for their exact form. These estimators may be specified even further if elements of the bases are used. That leads to the estimators $\bar{a}_{T,k}$ and $\bar{b}_{T,j,k}$, which are used in simulations in Section 4.2.

In Section 3.2, we show the asymptotic normality of these new estimators. The technique of the proof is similar as before, so it may seem that it must be easier. This is indeed true if the "observation window" is just one element of the basis (or sum of two of them), but with general "window" things are not so simple. The proof requires finding of some appropriate operator E, which is used in Itô's formula. (It is similar to the operator R above.) Such operator E is constructed as a linear combination of operators E_k (which would be sufficient for the observing k–th coordinate) and $E_{k,l}$ (which would serve if the solution was observed "through the window" $(0, e_k + e_l)^\top$). All definitions and auxiliary lemmas are provided and the asymptotic normality of the estimator $\bar{a}_{T,z_2}$ is stated in Theorem 3.7. The double summation makes the proof rather technical.

Subsection 3.2.2 starts with an introduction of operators F_k, $F_{k,l}$ and F, which are needed for proving the asymptotic normality of the estimator $\bar{b}_{T,z_1,z_2}$. That is formulated in Theorem 3.11. Moreover, in Subsection 3.2.3, we introduce the estimator $\bar{b}_{T,z_1,a}$, which requires using the "observation window" $(z_1, 0)^\top$ and the true value of the parameter a. This estimator is also strongly consistent and asymptotically normal (see Theorem 3.12).

We end this section by Remark 5, where the limiting variances of the estimators from the previous theorems are simplified (in the case of using the "observational coordinates"). This unveils three interesting facts: First, since the limiting variance of the estimator $\bar{a}_{T,k}$ does not depend on k (it equals to a for any k), we come to the conclusion that it does not matter, which coordinate we observe. All the estimators $\bar{a}_{T,k}$ should behave similarly. Second, the limiting variances of the estimators $\bar{b}_{T,j,k}$, $\bar{b}_{T,j,j}$ and $\bar{b}_{T,f_j,a}$ decrease with increasing j. Therefore, it is better to use the "observation coordinate" with bigger j, if possible. Third fact is probably the most interesting one. Since the limiting variance of the estimator $\bar{b}_{T,j,j}$ is smaller than the limiting variance of the estimator $\bar{b}_{T,f_j,a}$, it is better to use the combination of two "windows" $(0, e_j)^\top$ and $(f_j, 0)^\top$ for estimating the parameter b, instead using the "window" $(f_j, 0)^\top$ with knowing a exactly. These three facts are also justified by the results of simulation in Section 4.2.

We provide two examples of the hyperbolic SPDEs in Section 3.3. Example 1 (the wave equation) is the motivating exemplar for the Thesis, but with our generalization (we use A instead of Δ), we may also present the plate equation in Example 2.

The Chapter 4 includes the implementation and the statistical evidence. We simulate the behaviour of the oscillating rod (see Example 1 with $D = (0, 1)$) for a certain period of time T with given parameters a and b. Then we pretend that these parameters are unknown and we try to estimate them by using previous methods.

Section 4.1 considers the families of estimators $(\hat{a}_T, \hat{b}_T)$ and $(\tilde{a}_T, \tilde{b}_T)$ and starts with the setup. In our programming, we have to restrict ourselves mainly in two areas: First, although the problem is infinite–dimensional, we have to consider some number of finite dimensions N instead. Second, the stochastic processes are in our theory continuous, but we generate "only" discrete versions of them. However, if we use a fine partition of the time scale (i.e., very small Δt), we obtain time series, which approximate the exact solution properly. We have used $N = 10$ (bigger N demands more computer memory and operational time) and $\Delta t = 0.001$ (which was good, however we would not recommend using bigger Δt than this) and the computation run fast and the results are satisfactory. The exact code in R language may be found in Appendix C.

For the terminal time $T = 100$, we show the results from one particular simulated case both numerically and graphically. The time evolution of the estimators (depicted in the presented figures) seems very suitable as well as the results from 100 more simulations. The overall obtained statistics are summarized in Table 4.1. These results show that the estimators have their derived properties and that the family $(\tilde{a}_T, \tilde{b}_T)$ seems better than the family $(\hat{a}_T, \hat{b}_T)$, however with the terminal time $T = 1000$ it is even more demonstrative (see Table 4.2).

The asymptotic normality of the estimators is verified in three ways: By the Wilk–Shapiro tests, by the Q–Q (quantile–quantile) plots and by the comparing the actual variances with the theoretical variances obtained from the respective theorems. Our estimators meet all three criteria.

In Section 4.2, we study the behaviour of the estimators $\bar{a}_{T,k}$, $\bar{b}_{T,j,k}$ and $\bar{b}_{T,j,a}$ for different coordinates j and k. First we show the time evolution of the estimators for one particular trajectory and then we present the results obtained from the larger sample (see the attached pictures and Tables 4.3 and 4.4). The setup for the simulations is the same as in the previous section (except for the fact that $T = 1000$ and $\Delta t = 0.0001$). The results precisely complement the theoretical part of the Thesis.

We conclude with several appendices. The Appendix A recalls the definitions and basic properties of nuclear and Hilbert–Schmidt operators. As the main source, [5, Appendix C] was used. The basic notion on the strongly continuous semigroups can be found in Appendix B. For more information see [3] and [28]. Appendix C provides the computer codes of simulations in R language.

Notation

In the Thesis, we use the following notation for the spaces of linear operators. If U and V are Hilbert spaces, then $\mathcal{L}(U,V)$, $\mathcal{L}_2(U,V)$ and $\mathcal{L}_1(U,V)$ denote the respective spaces of all linear bounded, Hilbert–Schmidt and nuclear operators from U to V. Also $\mathcal{L}(V)$ stands for $\mathcal{L}(V,V)$, etc. For more mathematical symbols and notation see Glossary at the end of the Thesis.

1. Stochastic wave equation

1.1 Preliminaries

All random variables and processes in the Thesis are defined on a given stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Consider the real separable Hilbert space U with norm $\|\cdot\|_U$ and inner product $\langle \cdot, \cdot \rangle_U$.

Definition 1.1. A cylindrical Brownian motion $B = (B(t), t \geq 0)$ on a real separable Hilbert space U is a collection of centered Gaussian scalar random processes $B_x = (B_x(t),\, t \geq 0)$, $x \in U$, such that

$$\mathbb{E}\left(B_x(t)B_y(s)\right) = \min(t,s)\, \langle x, y \rangle_U\,, \quad t, s \geq 0, \tag{1.1}$$

for any $x, y \in U$.

Proposition 1.1. *Let B be a cylindrical Brownian motion on a Hilbert space U.*

1) The mapping $U \to L^2(\Omega)$, $x \longmapsto B_x$ is linear and therefore, for any t_i, $i = 1, \dots, m$, x_j, $j = 1, \dots, n$, the collection of random variables $\{B_{x_j}(t_i), i = 1, \dots, m, j = 1, \dots, n\}$ is a Gaussian system.

2) If u_1, u_2 are the elements of U such that $\|u_1\|_U = \|u_2\|_U = 1$ and $\langle u_1, u_2 \rangle_U = 0$ then the processes $\beta_1 = (\beta_1(t), t \geq 0)$ and $\beta_2 = (\beta_2(t), t \geq 0)$ defined by $\beta_i(t) = B_{u_i}(t)$, $i = 1, 2$, are independent standard Brownian motions.

Proof. 1) By direct computation, for any $a, b \in \mathbb{R}$ and $x, y \in U$, we have

$$\mathbb{E}(B_{ax+by}(t) - aB_x(t) - bB_y(t))^2 = 0, \quad t \geq 0.$$

Cf. [26, Definition 1.1.1].

2) By definition of B, the pair (β_1, β_2) is centered Gaussian process and $\mathbb{E}\beta_i(t)\beta_j(s) = \min(t,s)\, \langle u_i, u_j \rangle_U$, $i, j = 1, 2$, which completes the proof. □

The previous Proposition suggests a representation of $B(t)$ as a series

$$B(t) = \sum_{n=1}^{\infty} \beta_n(t)\, u_n, \quad t \geq 0, \tag{1.2}$$

where $\{u_n, n \in \mathbb{N}\}$ is an orthonormal basis in U and $\beta_n = B_{u_n}$, $n \in \mathbb{N}$, are independent standard Brownian motions. While the series does not converge in U, it is possible to embed U into a bigger Hilbert space, where the series is convergent. See Proposition 1.2.

Definition 1.1 can be extended to allow spatial covariance of the process.

Definition 1.2. Given a non–negative, bounded, self–adjoint operator Q on U, we define the Q–Brownian motion B^Q on U by replacing (1.1) with

$$\mathbb{E}\left(B_x^Q(t)B_y^Q(s)\right) = \min(t,s)\, \langle Qx, y \rangle_U\,, \quad t, s \geq 0. \tag{1.3}$$

If Q has a complete orthonormal system of eigenfunctions $\{u_n, n \in \mathbb{N}\}$ and $Qu_n = \lambda_n u_n$, $\lambda_n > 0$ (i.e., Q is strictly positive) then (1.2) becomes

$$B^Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \beta_n(t) u_n, \quad t \geq 0, \tag{1.4}$$

where $\beta_n = \frac{1}{\sqrt{\lambda_n}} B^Q_{u_n}$, $n \in \mathbb{N}$, are independent standard Brownian motions.

Since

$$\mathbb{E}\left\| \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \beta_n(t) u_n \right\|_U^2 = t \sum_{n=1}^{\infty} \lambda_n, \quad t \geq 0,$$

the sum (1.4) is convergent in $L^2(\Omega)$ if and only if Q is a nuclear operator on U, i.e., $Q \in \mathcal{L}_1(U)$. (For the definition and basic properties of the nuclear and Hilbert–Schmidt operators, see Appendix A.)

There is a correspondence between the concepts of cylindrical Brownian motion and Q–Brownian motion, which is descibed in the following Proposition.

Proposition 1.2. *1) Let B be a cylindrical Brownian motion on a separable Hilbert space U. Let U_1 be a Hilbert space such that U is a dense subset of U_1. Then B is a continuous U_1–valued squared–integrable martingal if and only if the embedding $J : U \longmapsto U_1$ is a Hilbert–Schmidt operator. In this case, B naturally extends to a Q–Brownian motion on U_1 with $Q = JJ^*$ and Q is nuclear operator.*

2) Let B^Q be a Q–Brownian motion on a separable Hilbert space U. Then B^Q is a continuous U–valued squared–integrable martingale if and only if the operator Q is nuclear.

Proof. We provide only an outline of the proof.

1) Let $\{u_n, n \in \mathbb{N}\}$ be an orthonormal basis in U. For any $n \in \mathbb{N}$ we have

$$\|u_n\|_{U_1}^2 = \|J^* u_n\|_U^2 = \langle Qu_n, u_n \rangle_U,$$

where the adjoint operator $J^* : U_1 \longmapsto U$ is defined on the whole space U_1 (cf. [34, Theorem VII.2]).

Since

$$\mathbb{E}\,\|B(t) - B(s)\|_{U_1}^2 = (t-s) \sum_{n=1}^{\infty} \|u_n\|_{U_1}^2, \quad t \geq s \geq 0,$$

the series converges if and only if J is Hilbert–Schmidt. The continuity of B then follows by the Kolmogorov criterion (see [18, Theorem 1.4.1]). Finally, for $x \in U_1$, we set

$$B^Q_x = B_{J^* x}.$$

Part 2) follows from the fact that

$$\mathbb{E}\left\| B^Q(t) - B^Q(s) \right\|_U^2 = (t-s) \sum_{n=1}^{\infty} \langle Qu_n, u_n \rangle_U, \quad t \geq s \geq 0.$$

□

Now we define the stochastic integral for a suitable operator–valued function $\Phi(\cdot)$.

Let V be a real separable Hilbert space with norm $\|\cdot\|_V$ and inner product $\langle \cdot, \cdot \rangle_V$. By $\|\Phi\|_2$ we denote the Hilbert–Schmidt norm of the operator $\Phi \in \mathcal{L}(U, V)$. Moreover, let $(\mathcal{F}_t, t \geq 0)$ be a normal filtration in $\mathcal{F}$ such that

- $B(t)$ is $\mathcal{F}_t$–measurable for all $t \geq 0$,
- $B(t) - B(s)$ is independent of $\mathcal{F}_s$ for all $t \geq s \geq 0$.

Definition 1.3. Let $(B(t), t \geq 0)$ be a cylindrical Brownian motion on a separable Hilbert space U, given by the formal expansion (1.2). Let $\Phi : [0, \infty) \times \Omega \to \mathcal{L}(U, V)$ be $\mathcal{F}_t$–strongly progressively measurable process satisfying

$$\mathbb{E} \int_0^t \|\Phi(s)\|_2^2 \, ds < \infty, \quad t \geq 0. \tag{1.5}$$

The stochastic integral $\int_0^t \Phi(s) \, dB(s)$ is defined as

$$\int_0^t \Phi(s) \, dB(s) := \sum_{n=1}^{\infty} \int_0^t \Phi(s) u_n \, d\beta_n(s), \quad t \geq 0. \tag{1.6}$$

Note that the integral $\int_0^t \Phi(s) u_n \, d\beta_n(s)$ for V–valued simple function $\Phi(\cdot) u_n$ is defined by a standard way. For a general V–valued function, the standard procedure of approximations is used.

Also note that it is elementary to verify that the definition does not depend on the choice of the complete orthonormal basis. Moreover, we have

$$\mathbb{E} \left\| \int_0^t \Phi(s) \, dB(s) \right\|_V^2 = \mathbb{E} \int_0^t \|\Phi(s)\|_2^2 \, ds, \quad t \geq 0.$$

The case of a more general $\Phi(\cdot)$ satisfying

$$\mathbb{P} \left(\int_0^t \|\Phi(s)\|_2^2 \, ds < \infty, \, t \geq 0 \right) = 1$$

instead of condition (1.5), can be obtained by localization (see [5, Section 4.2]).

Given real separable Hilbert spaces $(U, \|\cdot\|_U, \langle \cdot, \cdot \rangle_U)$ and $(V, \|\cdot\|_V, \langle \cdot, \cdot \rangle_V)$, we consider the linear equation with additive noise

$$\begin{aligned} dX(t) &= \mathcal{A} X(t) \, dt + \Phi \, dB(t), \\ X(0) &= x_0, \end{aligned} \tag{1.7}$$

where $(B(t), t \geq 0)$ is a cylindrical Brownian motion on U, $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to V$, $\mathrm{Dom}(\mathcal{A}) \subset V$, $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on V and $\Phi \in \mathcal{L}(U, V)$. The concept of the strongly continuous semigroup as well as its infinitesimal generator is recalled in Appendix B.

We assume that x_0 is $\mathcal{F}_0$–measurable random variable with $\mathbb{E}\|x_0\|_V^2 < \infty$ and that x_0 and $(B(t), t \geq 0)$ are stochastically independent.

We impose the following two conditions:

(A1) $\Phi \in \mathcal{L}_2(U, V)$,

(A2) There exist constants $K > 0$ and $\rho > 0$ such that

$$\|S(t)\|_{\mathcal{L}(V)} \leq K e^{-\rho t}$$

holds for all $t \geq 0$.

The condition (A1) means that the perturbing noise is, in fact, a genuine V–valued Brownian motion and the condition (A2) is the exponential stability of the semigroup generated by $\mathcal{A}$.

Now we introduce two concepts of solution to the equation (1.7).

Definition 1.4. Assume (A1). A V–valued stochastic process $(X^{x_0}(t), t \geq 0)$ is a strong solution of (1.7) if

- $X^{x_0}(t)$ is adapted to the filtration $\mathcal{F}_t$ for all $t \geq 0$,
- $X^{x_0}(t)$ is continuous in t, $\mathbb{P} - a.s.$,
- $X^{x_0}(t) \in \mathrm{Dom}(\mathcal{A})$, almost everywhere on $[0, \infty) \times \Omega$,
- $\int_0^t \|\mathcal{A}X^{x_0}(s)\|_V \, ds < \infty, \quad t \geq 0, \quad \mathbb{P} - a.s.$,
- $X^{x_0}(t) = x_0 + \int_0^t \mathcal{A}X^{x_0}(s) \, ds + \Phi B(t), \quad t \geq 0, \quad \mathbb{P} - a.s.$,

where the process $(\Phi B(t), t \geq 0)$ is understood as the $\Phi\Phi^*$–Brownian motion on V.

In general this concept is rather strong, so we will use a weaker one in later applications.

Definition 1.5. Assume (A1). A V–valued stochastic process $(X^{x_0}(t), t \geq 0)$ is a mild solution of (1.7) if

- $X^{x_0}(t)$ is adapted to the filtration $\mathcal{F}_t$ for all $t \geq 0$,
- $\int_0^t \|X^{x_0}(s)\|_V^2 \, ds < \infty \quad t \geq 0, \quad \mathbb{P} - a.s.$,
- $X^{x_0}(t) = S(t)x_0 + \int_0^t S(t-s)\Phi \, dB(s), \quad t \geq 0, \quad \mathbb{P} - a.s.$

The following two Propositions describe the form of a mild solution to the equation (1.7) and its invariant measure (cf. [5]).

Proposition 1.3. *If (A1) is satisfied then equation* (1.7) *admits a unique mild solution*

$$X^{x_0}(t) = S(t)x_0 + Z(t), \quad t \geq 0, \tag{1.8}$$

where $(Z(t), t \geq 0)$ *is the convolution integral*

$$Z(t) = \int_0^t S(t-s)\Phi \, dB(s), \quad t \geq 0. \tag{1.9}$$

The process $(Z(t), t \geq 0)$ *is a* V*–valued continuous centered Gaussian process with covariance operator given by the formula*

$$Q_t = \int_0^t S(s)\Phi\Phi^* S^*(s) \, ds, \quad t \geq 0. \tag{1.10}$$

Proof. Cf. [5, Theorem 5.2]. $\square$

Proposition 1.4. *If (A1), (A2) are satisfied, then there is a unique invariant measure* $\mu_\infty = N(0, Q_\infty)$ *for the equation* (1.7) *and*

$$w^* - \lim_{t\to\infty} \mu_t^{x_0} = \mu_\infty$$

for each initial condition $x_0 \in V$*, where* $\mu_t^{x_0} = \mathrm{Law}\,(X^{x_0}(t))$ *and* $\mathrm{Law}\,(\cdot)$ *denotes the probability distribution.*

The covariance operator Q_∞ *takes the form*

$$Q_\infty = \int_0^\infty S(t)\Phi\Phi^* S^*(t)\,dt. \tag{1.11}$$

Proof. The exponential stability of the semigroup $(S(t), t \geq 0)$ (i.e., condition (A2)) implies both existence and uniqueness of the invariant measure. See [5, Theorem 11.11]. □

The process $(X^{x_0}(t), t \geq 0)$ is called Ornstein–Uhlenbeck process and the process $(Z(t), t \geq 0)$ stochastic convolution.

1.2 The semigroup and the covariance operator

To interpret stochastic wave equation (3) rigorously, we rewrite it as a first order system in a standard way. Assume that $\{e_n, n \in \mathbb{N}\}$ is an orthonormal basis in $L^2(D)$ and the operator $A : \mathrm{Dom}(A) \subset L^2(D) \to L^2(D)$ is such that

$$\text{(A3)} \begin{cases} Ae_n = -\alpha_n e_n, \\ \forall n \in \mathbb{N} \quad \alpha_n > 0, \\ \alpha_n \to \infty \text{ as } n \to \infty. \end{cases}$$

These assumptions cover the case when the set $D \subset \mathbb{R}^d$ is open, bounded and the boundary ∂D is sufficiently smooth, the operator $A = \Delta|_{\mathrm{Dom}(A)}$ and $\mathrm{Dom}(A) = H^2(D) \cap H_0^1(D)$. ($H^2(D)$ is the standard notation for the Sobolev space $W^{2,2}(D)$ and $H_0^1(D) = W_0^{1,2}(D)$. For more detail, see [1, Chapter 8].)

Next let us assume that Φ_1 is a Hilbert–Schmidt operator on $L^2(D)$ such that $Q = \Phi_1\Phi_1^*$ is strictly positive. Since Q is a symmetric nuclear operator on $L^2(D)$ then there exists an orthonormal basis $\{e_n', n \in \mathbb{N}\}$ of $L^2(D)$ consisting of eigenvectors of Q, that is

$$\text{(A4)} \begin{cases} Qe_n' = \lambda_n e_n', \\ \forall n \in \mathbb{N} \quad \lambda_n > 0, \\ \sum_{n=1}^\infty \lambda_n < \infty. \end{cases}$$

Consider the Hilbert space $V = \mathrm{Dom}((-A)^{\frac{1}{2}}) \times L^2(D)$ endowed with the inner product

$$\begin{aligned} \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_V &= \langle x_1, y_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \langle x_2, y_2 \rangle_{L^2(D)} \\ &= \left\langle (-A)^{\frac{1}{2}} x_1, (-A)^{\frac{1}{2}} y_1 \right\rangle_{L^2(D)} + \langle x_2, y_2 \rangle_{L^2(D)}, \end{aligned}$$

for any $(x_1, x_2)^\top, (y_1, y_2)^\top \in V$.

Also, consider the linear equation

$$\begin{aligned} dX(t) &= \mathcal{A}X(t)\,dt + \Phi\,dB(t), \\ X(0) &= x_0 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \end{aligned} \tag{1.12}$$

where the linear operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) = \mathrm{Dom}(A) \times \mathrm{Dom}((-A)^{\frac{1}{2}}) \to V$ is defined by

$$\mathcal{A}x = \mathcal{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ bA & -2aI \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathrm{Dom}(\mathcal{A}),$$

$a > 0$, $b > 0$ are unknown parameters (which are to be estimated),
$u_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$, $u_2 \in L^2(D)$, $x_0 = (u_1, u_2)^\top \in V$ satisfies $\mathbb{E}\|x_0\|_V^2 < \infty$, where $\|\cdot\|_V := \sqrt{\langle \cdot, \cdot \rangle_V}$, and the linear operator $\Phi : U = V \to V$ is defined by

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_1 \end{pmatrix}.$$

With no loss of generality, we assume that the driving process in (1.12) takes the form $(0, B(t))^\top$, where $(B(t), t \geq 0)$ is a cylindrical Brownian motion on $L^2(D)$.

Note that since the operator Φ_1 is Hilbert–Schmidt on $L^2(D)$, the operator Φ is Hilbert–Schmidt on V. Also note that the orthonormal basis of the space $\mathrm{Dom}((-A)^{\frac{1}{2}})$ is $\{f_n, n \in \mathbb{N}\}$, where $f_n = \frac{1}{\sqrt{\alpha_n}}\, e_n$.

The form of the eigenvalues of the operator $\mathcal{A}$ depends on whether $a^2 - b\alpha_n$ is negative, positive, or equal to zero. Therefore, in order to compute the form of the semigroup $(S(t), t \geq 0)$, we have to consider these three different cases, compute the appropriate semigroups $(S_1(t), t \geq 0)$, $(S_2(t), t \geq 0)$ and $(S_3(t), t \geq 0)$ and then combine them together to obtain the resulting formula (see Theorem 1.14 below).

First, let us divide $\mathbb{N}$ into three (disjoint) sets in this way: $\mathbb{N} = N_1 \cup N_2 \cup N_3$, where

$$\begin{aligned} N_1 &= \left\{ n \in \mathbb{N}, \alpha_n > \frac{a^2}{b} \right\}, \\ N_2 &= \left\{ n \in \mathbb{N}, \alpha_n < \frac{a^2}{b} \right\}, \\ N_3 &= \left\{ n \in \mathbb{N}, \alpha_n = \frac{a^2}{b} \right\}. \end{aligned}$$

Since $\alpha_n \to \infty$ as $n \to \infty$, the sets N_2 and N_3 are finite (or even empty) sets, while the set N_1 is infinite. Let us also write the space V as a direct sum of three closed linear subspaces

$$V = V_1 \oplus V_2 \oplus V_3,$$

where

$$V_i = \mathrm{span}\,\{f_n, n \in N_i\} \times \mathrm{span}\,\{e_n, n \in N_i\}, \quad i = 1, 2, 3.$$

1.2.1 Case $\alpha_n > \frac{a^2}{b}$

In the case $\alpha_n > \frac{a^2}{b}$, the eigenvalues $\{l_n^{1,2}, n \in N_1\}$ of the operator $\mathcal{A}$ are

$$l_n^1 = -a + i\sqrt{b\alpha_n - a^2},$$
$$l_n^2 = -a - i\sqrt{b\alpha_n - a^2}$$

and the operator $\mathcal{A}$ generates a C_0-semigroup on V, which is also exponentially stable (the real parts of the eigenvalues $l_n^{1,2}$ are negative). The form of the semigroup $(S_1(t), t \geq 0)$ is given by Lemma 1.5 below. Define the operator

$$P_1 x = \sum_{n \in N_1} \langle x, e_n \rangle_{L^2(D)} e_n,$$

which is the operator of projection on the span $\{e_n, n \in N_1\}$ (i.e., $P_1 : L^2(D) \to$ span $\{e_n, n \in N_1\}$). Furthermore define the operator $\beta : L^2(D) \to L^2(D)$ by $\beta = (-bA - a^2 I)^{\frac{1}{2}} P_1$, that is

$$\beta x = \sum_{n \in N_1} \sqrt{b\alpha_n - a^2} \langle x, e_n \rangle_{L^2(D)} e_n, \quad \forall x \in \mathrm{Dom}(\beta),$$

where $\mathrm{Dom}(\beta) = \{x \in L^2(D), \sum_{n \in N_1} (b\alpha_n - a^2) \langle x, e_n \rangle^2_{L^2(D)} < \infty\} = \mathrm{Dom}((-A)^{\frac{1}{2}})$.

Similarly, define

$$\beta^{-1} x = \sum_{n \in N_1} \frac{1}{\sqrt{b\alpha_n - a^2}} \langle x, e_n \rangle_{L^2(D)} e_n,$$
$$\sin(\beta t) x = \sum_{n \in N_1} \sin\left(\sqrt{b\alpha_n - a^2}\, t\right) \langle x, e_n \rangle_{L^2(D)} e_n,$$
$$\beta^{-1} \sin(\beta t) x = \sum_{n \in N_1} \frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}} \langle x, e_n \rangle_{L^2(D)} e_n,$$
$$\cos(\beta t) x = \sum_{n \in N_1} \cos\left(\sqrt{b\alpha_n - a^2}\, t\right) \langle x, e_n \rangle_{L^2(D)} e_n,$$

where $x \in L^2(D)$ and $t \geq 0$.

Note that $\beta^{-1} = (-bA - a^2 I)^{-\frac{1}{2}} P_1$, so $\beta^{-1}\beta x = P_1 x$ for any $x \in \mathrm{Dom}(\beta)$ and $\beta^{-1}\beta x = Ix$ for any $x \in \mathrm{Dom}(\beta) \cap$ span $\{e_n, n \in N_1\}$. Also note that the operator $\cos(\beta t)$ evaluated at time $t = 0$ is $\cos(\beta t)|_{t=0}\, x = P_1 x$ for any $x \in L^2(D)$.

The form of the semigroup $(S_1(t), t \geq 0)$, on the subspace defined by the coordinates from the set N_1, is described by the following Lemma.

Lemma 1.5. *For all $x = (x_1, x_2)^\top \in V_1$ we have*

$$S_1(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s_{11}(t) & s_{12}(t) \\ s_{21}(t) & s_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t \geq 0,$$

where

$$s_{11}(t) = e^{-at}\left(\cos(\beta t) + a\beta^{-1}\sin(\beta t)\right),$$
$$s_{12}(t) = e^{-at}\beta^{-1}\sin(\beta t),$$
$$s_{21}(t) = e^{-at}\left(-\beta - a^2\beta^{-1}\right)\sin(\beta t),$$
$$s_{22}(t) = e^{-at}\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right).$$

Proof. It is sufficient to show that

(i)

$$S_1(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

(ii)

$$\frac{d}{dt} S_1(t)x = \mathcal{A} S_1(t)x, \quad \forall x \in \mathrm{Dom}(\mathcal{A}) \cap V_1, \quad \forall t \geq 0.$$

As for (i), it is easy to see that

$$S_1(0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & P_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which is the identity operator for $x_1 \in \mathrm{span}\,\{f_n, n \in N_1\}$, $x_2 \in \mathrm{span}\,\{e_n, n \in N_1\}$.
(ii) may be verified by straightforward computation.

□

The adjoint operator of $(S_1(t), t \geq 0)$ is introduced in Lemma 1.6.

Lemma 1.6. *For all* $x = (x_1, x_2)^\top \in V_1$ *we have*

$$S_1^*(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r_{11}(t) & r_{12}(t) \\ r_{21}(t) & r_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t \geq 0,$$

where

$$\begin{aligned} r_{11}(t) &= e^{-at}(-A)^{-\frac{1}{2}} \left(\cos(\beta t) + a\beta^{-1}\sin(\beta t)\right)(-A)^{\frac{1}{2}}, \\ r_{12}(t) &= e^{-at}(-A)^{-\frac{1}{2}} \left(-\beta - a^2\beta^{-1}\right)\sin(\beta t)(-A)^{-\frac{1}{2}}, \\ r_{21}(t) &= e^{-at}(-A)^{\frac{1}{2}}\beta^{-1}\sin(\beta t)(-A)^{\frac{1}{2}}, \\ r_{22}(t) &= e^{-at}\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right). \end{aligned}$$

Proof. It is easy to verify that

$$\langle S_1(t)x, y\rangle_V = \langle x, S_1^*(t)y\rangle_V, \quad \forall x, y \in V_1, \quad \forall t \geq 0.$$

□

Using Lemma 1.6, it is possible to compute the integrand in (1.11) and consequently to obtain the exact formula for the covariance operator $Q_\infty^{(a,b)}$.

Proposition 1.7. *The covariance operator* $Q_\infty^{(a,b)}$ *takes the form*

$$Q_\infty^{(a,b)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sum_{n \in N_1} \sum_{k \in N_1} \frac{\langle Qe_n, e_k\rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2b(\alpha_n + \alpha_k)} \times \begin{pmatrix} 4a\alpha_n \langle x_1, e_n\rangle_{L^2(D)} e_k + b(\alpha_k - \alpha_n)\langle x_2, e_n\rangle_{L^2(D)} e_k \\ b\alpha_n(\alpha_n - \alpha_k)\langle x_1, e_n\rangle_{L^2(D)} e_k + 2ab(\alpha_n + \alpha_k)\langle x_2, e_n\rangle_{L^2(D)} e_k \end{pmatrix}, \tag{1.13}$$

for any $(x_1, x_2)^\top \in V_1$.

Proof. The integrand in (1.11) can be computed as follows

$$Q_\infty^{(a,b)} = \int_0^\infty S_1(t)\Phi\Phi^* S_1^*(t)\, dt = \int_0^\infty \begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{pmatrix} dt,$$

where

$$q_{11}(t) = e^{-2at}\beta^{-1}\sin(\beta t) Q(-A)^{\frac{1}{2}}\beta^{-1}\sin(\beta t)(-A)^{\frac{1}{2}}, \tag{1.14}$$
$$q_{12}(t) = e^{-2at}\beta^{-1}\sin(\beta t) Q\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right), \tag{1.15}$$
$$q_{21}(t) = e^{-2at}\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right) Q(-A)^{\frac{1}{2}}\beta^{-1}\sin(\beta t)(-A)^{\frac{1}{2}}, \tag{1.16}$$
$$q_{22}(t) = e^{-2at}\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right) Q\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right). \tag{1.17}$$

We need to evaluate the integrals of $q_{11}(t)$, $q_{12}(t)$, $q_{21}(t)$ and $q_{22}(t)$. For any $x = (x_1, x_2)^\top \in V_1$ we have that

$$\begin{aligned} q_{11}(t)x_1 &= e^{-2at}\beta^{-1}\sin(\beta t) Q \sum_{n\in N_1} \alpha_n \frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}} \langle x_1, e_n\rangle_{L^2(D)} e_n \\ &= e^{-2at}\beta^{-1}\sin(\beta t) \sum_{n\in N_1}\sum_{k=1}^\infty \alpha_n \frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}} \\ &\quad \times \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_1, e_n\rangle_{L^2(D)} e_k \\ &= e^{-2at} \sum_{n\in N_1}\sum_{k\in N_1} \alpha_n \frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}} \frac{\sin\left(\sqrt{b\alpha_k - a^2}\, t\right)}{\sqrt{b\alpha_k - a^2}} \\ &\quad \times \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_1, e_n\rangle_{L^2(D)} e_k. \end{aligned}$$

Now we use the fact that

$$\begin{aligned} &\int_0^\infty e^{-2at}\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)\sin\left(\sqrt{b\alpha_k - a^2}\, t\right) dt = \\ &= \frac{4a\sqrt{b\alpha_n - a^2}\sqrt{b\alpha_k - a^2}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)}. \end{aligned}$$

Hence by integrating the formula for $q_{11}(t)x_1$ over t from zero to infinity, we arrive at

$$\left(\int_0^\infty q_{11}(t)\, dt\right) x_1 = \sum_{n\in N_1}\sum_{k\in N_1} \frac{4a\alpha_n \langle Qe_n, e_k\rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle x_1, e_n\rangle_{L^2(D)} e_k.$$

As for $q_{12}(t)$, we have

$$\begin{aligned} q_{12}(t)x_2 &= e^{-2at}\beta^{-1}\sin(\beta t) Q \sum_{n\in N_1} \left(\cos\left(\sqrt{b\alpha_n - a^2}\, t\right) - a\frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}}\right) \\ &\quad \times \langle x_2, e_n\rangle_{L^2(D)} e_n \end{aligned}$$

$$
\begin{aligned}
&= e^{-2at}\beta^{-1}\sin(\beta t)\sum_{n\in N_1}\sum_{k=1}^{\infty}\left(\cos\left(\sqrt{b\alpha_n-a^2}\,t\right)-a\frac{\sin\left(\sqrt{b\alpha_n-a^2}\,t\right)}{\sqrt{b\alpha_n-a^2}}\right)\\
&\quad\times\langle Qe_n,e_k\rangle_{L^2(D)}\langle x_2,e_n\rangle_{L^2(D)}\,e_k\\
&= e^{-2at}\sum_{n\in N_1}\sum_{k\in N_1}\left(\cos\left(\sqrt{b\alpha_n-a^2}\,t\right)-a\frac{\sin\left(\sqrt{b\alpha_n-a^2}\,t\right)}{\sqrt{b\alpha_n-a^2}}\right)\\
&\quad\times\frac{\sin\left(\sqrt{b\alpha_k-a^2}\,t\right)}{\sqrt{b\alpha_k-a^2}}\langle Qe_n,e_k\rangle_{L^2(D)}\langle x_2,e_n\rangle_{L^2(D)}\,e_k.
\end{aligned}
$$

Now we use the fact that

$$
\begin{aligned}
&\int_0^\infty e^{-2at}\sin\left(\sqrt{b\alpha_k-a^2}\,t\right)\left(\cos\left(\sqrt{b\alpha_n-a^2}\,t\right)-a\frac{\sin\left(\sqrt{b\alpha_n-a^2}\,t\right)}{\sqrt{b\alpha_n-a^2}}\right)dt=\\
&=\frac{b(\alpha_k-\alpha_n)\sqrt{b\alpha_k-a^2}}{b^2(\alpha_n-\alpha_k)^2+8a^2b(\alpha_n+\alpha_k)}.
\end{aligned}
$$

Hence by integrating the formula for $q_{12}(t)x_2$ over t from zero to infinity, we obtain

$$
\left(\int_0^\infty q_{12}(t)\,dt\right)x_2=\sum_{n\in N_1}\sum_{k\in N_1}\frac{b(\alpha_k-\alpha_n)\langle Qe_n,e_k\rangle_{L^2(D)}}{b^2(\alpha_n-\alpha_k)^2+8a^2b(\alpha_n+\alpha_k)}\langle x_2,e_n\rangle_{L^2(D)}\,e_k.
$$

The expression for $q_{21}(t)x_1$ is very similar to the previous one,

$$
\begin{aligned}
q_{21}(t)x_1 &= e^{-2at}\left(\cos(\beta t)-a\beta^{-1}\sin(\beta t)\right)Q\sum_{n\in N_1}\alpha_n\frac{\sin\left(\sqrt{b\alpha_n-a^2}\,t\right)}{\sqrt{b\alpha_n-a^2}}\\
&\quad\times\langle x_1,e_n\rangle_{L^2(D)}\,e_n\\
&= e^{-2at}\left(\cos(\beta t)-a\beta^{-1}\sin(\beta t)\right)\sum_{n\in N_1}\sum_{k=1}^{\infty}\alpha_n\frac{\sin\left(\sqrt{b\alpha_n-a^2}\,t\right)}{\sqrt{b\alpha_n-a^2}}\\
&\quad\times\langle Qe_n,e_k\rangle_{L^2(D)}\langle x_1,e_n\rangle_{L^2(D)}\,e_k\\
&= e^{-2at}\sum_{n\in N_1}\sum_{k\in N_1}\alpha_n\left(\cos\left(\sqrt{b\alpha_k-a^2}\,t\right)-a\frac{\sin\left(\sqrt{b\alpha_k-a^2}\,t\right)}{\sqrt{b\alpha_k-a^2}}\right)\\
&\quad\times\frac{\sin\left(\sqrt{b\alpha_n-a^2}\,t\right)}{\sqrt{b\alpha_n-a^2}}\langle Qe_n,e_k\rangle_{L^2(D)}\langle x_2,e_n\rangle_{L^2(D)}\,e_k.
\end{aligned}
$$

Here the integration over t from zero to infinity yields the same result as above with indicies n and k reversed (note that the denominator in the resulting formula remains the same). Hence we obtain that

$$
\left(\int_0^\infty q_{21}(t)\,dt\right)x_1=\sum_{n\in N_1}\sum_{k\in N_1}\frac{b\alpha_n(\alpha_n-\alpha_k)\langle Qe_n,e_k\rangle_{L^2(D)}}{b^2(\alpha_n-\alpha_k)^2+8a^2b(\alpha_n+\alpha_k)}\langle x_1,e_n\rangle_{L^2(D)}\,e_k.
$$

In a similar manner, we have that

$$
\begin{aligned}
q_{22}(t)x_2 = e^{-2at} \sum_{n\in N_1} \sum_{k\in N_1} & \left(\cos\left(\sqrt{b\alpha_k - a^2}\, t\right) - a \frac{\sin\left(\sqrt{b\alpha_k - a^2}\, t\right)}{\sqrt{b\alpha_k - a^2}} \right) \\
& \times \left(\cos\left(\sqrt{b\alpha_n - a^2}\, t\right) - a \frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}} \right) \langle Qe_n, e_k \rangle_{L^2(D)} \\
& \times \langle x_2, e_n \rangle_{L^2(D)} e_k
\end{aligned}
$$

and by evaluating the appropriate integral, we arrive at

$$
\left(\int_0^\infty q_{22}(t)\, dt \right) x_2 = \sum_{n\in N_1} \sum_{k\in N_1} \frac{2ab(\alpha_n + \alpha_k) \langle Qe_n, e_k \rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle x_2, e_n \rangle_{L^2(D)} e_k .
$$

These results may be summarized by the formula (1.13), which completes the proof. □

1.2.2 Case $\alpha_n < \frac{a^2}{b}$

In the case $\alpha_n < \frac{a^2}{b}$, the eigenvalues $\{l_n^{1,2}, n \in N_2\}$ of the operator $\mathcal{A}$ are

$$
\begin{aligned}
l_n^1 &= -a + \sqrt{a^2 - b\alpha_n}, \\
l_n^2 &= -a - \sqrt{a^2 - b\alpha_n}
\end{aligned}
$$

and the operator $\mathcal{A}$ generates a C_0-semigroup on V, which is also exponentially stable (the eigenvalues l_n^1 and l_n^2 are negative). The form of the semigroup $(S_2(t), t \geq 0)$ is given by Lemma 1.8, but let us again introduce some operators, which will be needed further.

First define the operator P_2

$$
P_2 x = \sum_{n\in N_2} \langle x, e_n \rangle_{L^2(D)} e_n,
$$

which is the operator of projection on the span $\{e_n, n \in N_2\}$ (i.e., $P_2 : L^2(D) \to \operatorname{span}\{e_n, n \in N_2\}$). Furthermore define the operator $\gamma : L^2(D) \to L^2(D)$ by $\gamma = (a^2 I + bA)^{\frac{1}{2}} P_2$, that is

$$
\gamma x = \sum_{n\in N_2} \sqrt{a^2 - b\alpha_n} \langle x, e_n \rangle_{L^2(D)} e_n,
$$

where $x \in L^2(D)$. (Since the sum over the set N_2 is finite, it is possible to define the operator γ on the whole space $L^2(D)$.)

Similarly, define

$$
\begin{aligned}
\gamma^{-1} x &= \sum_{n\in N_2} \frac{1}{\sqrt{a^2 - b\alpha_n}} \langle x, e_n \rangle_{L^2(D)} e_n, \\
L_1 x &= (-aP_2 + \gamma)\, x, \\
L_2 x &= (-aP_2 - \gamma)\, x, \\
e^{L_1 t} x &= \sum_{n\in N_2} e^{l_n^1 t} \langle x, e_n \rangle_{L^2(D)} e_n, \\
e^{L_2 t} x &= \sum_{n\in N_2} e^{l_n^2 t} \langle x, e_n \rangle_{L^2(D)} e_n,
\end{aligned}
$$

where $x \in L^2(D)$ and $t \geq 0$.

Note that $\gamma^{-1} = \left(a^2 I + bA\right)^{-\frac{1}{2}} P_2$, so $\gamma^{-1}\gamma x = P_2 x$ for any $x \in L^2(D)$ and $\gamma^{-1}\gamma x = Ix$ for any $x \in \operatorname{span}\{e_n, n \in N_2\}$. Also note that the following properties hold true

$$L_1 - L_2 = 2\gamma, \tag{1.18}$$
$$L_1 L_2 = -bAP_2 \, (= L_2 L_1), \tag{1.19}$$

hence the operators L_1 and L_2 commute. The last remark is that the operator $e^{L_1 t}$ evaluated at time $t = 0$ is $e^{L_1 t}|_{t=0}\, x = P_2 x$ for any $x \in L^2(D)$. (The operator $e^{L_2 t}$ has indeed the same property.)

The form of the semigroup $(S_2(t), t \geq 0)$, on the subspace defined by the coordinates from the set N_2, is described by the following Lemma.

Lemma 1.8. *For all $x = (x_1, x_2)^\top \in V_1$ we have*

$$S_2(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s_{11}(t) & s_{12}(t) \\ s_{21}(t) & s_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t \geq 0,$$

where

$$\begin{aligned} s_{11}(t) &= \frac{1}{2}\gamma^{-1}\left(-L_2 e^{L_1 t} + L_1 e^{L_2 t}\right), \\ s_{12}(t) &= \frac{1}{2}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right), \\ s_{21}(t) &= \frac{1}{2}\gamma^{-1}\left(-L_1 L_2 e^{L_1 t} + L_1 L_2 e^{L_2 t}\right), \\ s_{22}(t) &= \frac{1}{2}\gamma^{-1}\left(L_1 e^{L_1 t} - L_2 e^{L_2 t}\right). \end{aligned}$$

Proof. Analogously to the proof of Lemma 1.5, it is sufficient to show that

(i)
$$S_2(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

(ii)
$$\frac{d}{dt} S_2(t)x = \mathcal{A} S_2(t)x, \quad \forall x \in V_2, \quad \forall t \geq 0.$$

As for (i), it is just matter of evaluating the operators at time $t = 0$ and simplifying. For example the operator $s_{11}(0)$ simplifies as follows

$$s_{11}(0) = \frac{1}{2}\gamma^{-1}\left(L_1 - L_2\right) P_2 = \gamma^{-1}\gamma P_2 = P_2.$$

Consequently we arrive at

$$S_2(0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} P_2 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which is an identity operator for $x_1 \in \operatorname{span}\{f_n, n \in N_2\}$, $x_2 \in \operatorname{span}\{e_n, n \in N_2\}$.

(ii) may be verified by straightforward computation. □

The adjoint operator of $(S_2(t), t \geq 0)$ is introduced in Lemma 1.9.

Lemma 1.9. *For all $x = (x_1, x_2)^\top \in V_2$ we have*

$$S_2^*(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r_{11}(t) & r_{12}(t) \\ r_{21}(t) & r_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t \geq 0,$$

where

$$\begin{aligned}
r_{11}(t) &= \frac{1}{2}\gamma^{-1}\left(-L_2 e^{L_1 t} + L_1 e^{L_2 t}\right), \\
r_{12}(t) &= \frac{1}{2}(-A)^{-\frac{1}{2}}\gamma^{-1}\left(-L_1 L_2 e^{L_1 t} + L_1 L_2 e^{L_2 t}\right)(-A)^{-\frac{1}{2}}, \\
r_{21}(t) &= \frac{1}{2}(-A)^{\frac{1}{2}}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right)(-A)^{\frac{1}{2}}, \\
r_{22}(t) &= \frac{1}{2}\gamma^{-1}\left(L_1 e^{L_1 t} - L_2 e^{L_2 t}\right).
\end{aligned}$$

Proof. It is possible to verify that

$$\langle S_2(t)x, y\rangle_V = \langle x, S_2^*(t)y\rangle_V, \quad \forall x, y \in V_2, \quad \forall t \geq 0.$$

□

Using Lemmas 1.8 and 1.9, it is possible to compute the integrand in (1.11) and to obtain the formula for the covariance operator $Q_\infty^{(a,b)}$ for the case $\alpha_n < \frac{a^2}{b}$.

Proposition 1.10. *The covariance operator $Q_\infty^{(a,b)}$ takes the form*

$$\begin{aligned}
Q_\infty^{(a,b)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \sum_{n \in N_2} \sum_{k \in N_2} \frac{\langle Qe_n, e_k\rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \\
&\times \begin{pmatrix} 4a\alpha_n \langle x_1, e_n\rangle_{L^2(D)} e_k + b(\alpha_k - \alpha_n)\langle x_2, e_n\rangle_{L^2(D)} e_k \\ b\alpha_n(\alpha_n - \alpha_k)\langle x_1, e_n\rangle_{L^2(D)} e_k + 2ab(\alpha_n + \alpha_k)\langle x_2, e_n\rangle_{L^2(D)} e_k \end{pmatrix},
\end{aligned} \tag{1.20}$$

for any $(x_1, x_2)^\top \in V_2$.

Proof. According to (1.11), the covariance operator $Q_\infty^{(a,b)}$ may be expressed as

$$Q_\infty^{(a,b)} = \int_0^\infty S_2(t)\Phi\Phi^* S_2^*(t)\, dt = \int_0^\infty \begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{pmatrix} dt,$$

where

$$\begin{aligned}
q_{11}(t) &= \frac{1}{4}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right) Q(-A)^{\frac{1}{2}}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right)(-A)^{\frac{1}{2}}, \\
q_{12}(t) &= \frac{1}{4}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right) Q\gamma^{-1}\left(L_1 e^{L_1 t} - L_2 e^{L_2 t}\right), \\
q_{21}(t) &= \frac{1}{4}\gamma^{-1}\left(L_1 e^{L_1 t} - L_2 e^{L_2 t}\right) Q(-A)^{\frac{1}{2}}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right)(-A)^{\frac{1}{2}}, \\
q_{22}(t) &= \frac{1}{4}\gamma^{-1}\left(L_1 e^{L_1 t} - L_2 e^{L_2 t}\right) Q\gamma^{-1}\left(L_1 e^{L_1 t} - L_2 e^{L_2 t}\right).
\end{aligned}$$

As in the proof of Proposition 1.7, we need to evaluate the integrals of $q_{11}(t)$, $q_{12}(t)$, $q_{21}(t)$ and $q_{22}(t)$. For any $x = (x_1, x_2)^\top \in V_2$ we have that

$$\begin{aligned}
q_{11}(t)x_1 &= \frac{1}{4}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right) Q \sum_{n\in N_2} \alpha_n \frac{e^{l_n^1 t} - e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \langle x_1, e_n\rangle_{L^2(D)} e_n \\
&= \frac{1}{4}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right) \sum_{n\in N_2}\sum_{k=1}^{\infty} \alpha_n \frac{e^{l_n^1 t} - e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_1, e_n\rangle_{L^2(D)} e_k \\
&= \frac{1}{4}\sum_{n\in N_2}\sum_{k\in N_2} \alpha_n \frac{e^{l_n^1 t} - e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \frac{e^{l_k^1 t} - e^{l_k^2 t}}{\sqrt{a^2 - b\alpha_k}} \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_1, e_n\rangle_{L^2(D)} e_k.
\end{aligned}$$

If we now use the fact that

$$\int_0^\infty \left(e^{l_n^1 t} - e^{l_n^2 t}\right)\left(e^{l_k^1 t} - e^{l_k^2 t}\right) dt = \frac{16a\sqrt{a^2 - b\alpha_n}\sqrt{a^2 - b\alpha_k}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)},$$

we arrive at

$$\left(\int_0^\infty q_{11}(t)\, dt\right) x_1 = \sum_{n\in N_2}\sum_{k\in N_2} \frac{4a\alpha_n \langle Qe_n, e_k\rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle x_1, e_n\rangle_{L^2(D)} e_k.$$

As for the operator $q_{12}(t)$, we have

$$\begin{aligned}
q_{12}(t)x_2 &= \frac{1}{4}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right) Q \sum_{n\in N_2} \frac{l_n^1 e^{l_n^1 t} - l_n^2 e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \langle x_2, e_n\rangle_{L^2(D)} e_n \\
&= \frac{1}{4}\gamma^{-1}\left(e^{L_1 t} - e^{L_2 t}\right) \sum_{n\in N_2}\sum_{k=1}^{\infty} \frac{l_n^1 e^{l_n^1 t} - l_n^2 e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_2, e_n\rangle_{L^2(D)} e_k \\
&= \frac{1}{4}\sum_{n\in N_2}\sum_{k\in N_2} \frac{l_n^1 e^{l_n^1 t} - l_n^2 e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \frac{e^{l_k^1 t} - e^{l_k^2 t}}{\sqrt{a^2 - b\alpha_k}} \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_2, e_n\rangle_{L^2(D)} e_k.
\end{aligned}$$

Now we use the fact that

$$\int_0^\infty \left(l_n^1 e^{l_n^1 t} - l_n^2 e^{l_n^2 t}\right)\left(e^{l_k^1 t} - e^{l_k^2 t}\right) dt = \frac{4b(\alpha_k - \alpha_n)\sqrt{a^2 - b\alpha_n}\sqrt{a^2 - b\alpha_k}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)}.$$

Hence by integrating the formula for $q_{12}(t)x_2$ over t from zero to infinity, we obtain

$$\left(\int_0^\infty q_{12}(t)\, dt\right) x_2 = \sum_{n\in N_2}\sum_{k\in N_2} \frac{b(\alpha_k - \alpha_n) \langle Qe_n, e_k\rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle x_2, e_n\rangle_{L^2(D)} e_k.$$

The expression for $q_{21}(t)x_1$ is similar to the previous one,

$$\begin{aligned}
q_{21}(t)x_1 &= \frac{1}{4}\gamma^{-1}\left(L_1 e^{L_1 t} - L_2 e^{L_2 t}\right) Q \sum_{n\in N_2} \alpha_n \frac{e^{l_n^1 t} - e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \langle x_1, e_n\rangle_{L^2(D)} e_n \\
&= \frac{1}{4}\gamma^{-1}\left(L_1 e^{L_1 t} - L_2 e^{L_2 t}\right) \sum_{n\in N_2}\sum_{k=1}^{\infty} \alpha_n \frac{e^{l_n^1 t} - e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \langle Qe_n, e_k\rangle_{L^2(D)} \\
&\quad \times \langle x_1, e_n\rangle_{L^2(D)} e_k \\
&= \frac{1}{4}\sum_{n\in N_2}\sum_{k\in N_2} \alpha_n \frac{e^{l_n^1 t} - e^{l_n^2 t}}{\sqrt{a^2 - b\alpha_n}} \frac{l_k^1 e^{l_k^1 t} - l_k^2 e^{l_k^2 t}}{\sqrt{a^2 - b\alpha_k}} \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_1, e_n\rangle_{L^2(D)} e_k.
\end{aligned}$$

The integration over t from zero to infinity yields the same result as above with indicies n and k reversed. Hence we obtain that

$$\left(\int_0^\infty q_{21}(t)\,dt\right)x_1 = \sum_{n\in N_2}\sum_{k\in N_2}\frac{b\alpha_n(\alpha_n-\alpha_k)\langle Qe_n,e_k\rangle_{L^2(D)}}{b^2(\alpha_n-\alpha_k)^2+8a^2b(\alpha_n+\alpha_k)}\langle x_1,e_n\rangle_{L^2(D)}e_k.$$

In a similar manner, we have that

$$q_{22}(t)x_2 = \frac{1}{4}\sum_{n\in N_2}\sum_{k\in N_2}\frac{l_n^1e^{l_n^1t}-l_n^2e^{l_n^2t}}{\sqrt{a^2-b\alpha_n}}\frac{l_k^1e^{l_k^1t}-l_k^2e^{l_k^2t}}{\sqrt{a^2-b\alpha_k}}\langle Qe_n,e_k\rangle_{L^2(D)}\langle x_2,e_n\rangle_{L^2(D)}e_k$$

and by evaluating the appropriate integral, we arrive at

$$\left(\int_0^\infty q_{22}(t)\,dt\right)x_2 = \sum_{n\in N_2}\sum_{k\in N_2}\frac{2ab(\alpha_n+\alpha_k)\langle Qe_n,e_k\rangle_{L^2(D)}}{b^2(\alpha_n-\alpha_k)^2+8a^2b(\alpha_n+\alpha_k)}\langle x_2,e_n\rangle_{L^2(D)}e_k.$$

These results may be summarized by the formula (1.20). □

1.2.3 Case $\alpha_n = \frac{a^2}{b}$

In the case $\alpha_n = \frac{a^2}{b}$, the situation is much easier. The double eigenvalue of the operator $\mathcal{A}$ is $-a$, hence the operator $\mathcal{A}$ generates C_0–semigroup on V, which is also exponentially stable.

Define the operator P_3 in a similar fashion as P_1 and P_2 above

$$P_3x = \sum_{n\in N_3}\langle x,e_n\rangle_{L^2(D)}e_n.$$

That is the operator of projection on the span $\{e_n, n\in N_3\}$ (i.e., $P_3 : L^2(D)\to$ span $\{e_n, n\in N_3\}$). The form of semigroup $(S_3(t), t\geq 0)$ is given by the following Lemma.

Lemma 1.11. *For all $x=(x_1,x_2)^\top\in V_3$ we have*

$$S_3(t)\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}(1+at)e^{-at}P_3 & te^{-at}P_3\\ -a^2te^{-at}P_3 & (1-at)e^{-at}P_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},\quad t\geq 0.$$

Proof. If we evaluate the above operator $S_3(t)$ at time $t=0$, we obtain

$$S_3(0)\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}P_3 & 0\\ 0 & P_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},$$

which is an identity operator for $x_1\in$ span $\{f_n, n\in N_3\}$, $x_2\in$ span $\{e_n, n\in N_3\}$.

The property

$$\frac{d}{dt}S_3(t)x = \mathcal{A}S_3(t)x,\quad \forall x\in V_3,\quad \forall t\geq 0,$$

may also be verified by straightforward computation. □

The adjoint operator of $(S_3(t), t\geq 0)$ is introduced in the following Lemma.

Lemma 1.12. *For all $x = (x_1, x_2)^\top \in V_3$ we have*

$$S_3^*(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1+at)e^{-at}P_3 & -bte^{-at}P_3 \\ \frac{a^2}{b}te^{-at}P_3 & (1-at)e^{-at}P_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t \geq 0.$$

Proof. It is possible to verify that

$$\langle S_3(t)x, y \rangle_V = \langle x, S_3^*(t)y \rangle_V, \quad \forall x, y \in V_3, \quad \forall t \geq 0.$$

□

As in the two previous cases, we may use Lemmas 1.11 and 1.12 to compute the covariance operator $Q_\infty^{(a,b)}$.

Proposition 1.13. *The covariance operator $Q_\infty^{(a,b)}$ takes the form*

$$Q_\infty^{(a,b)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4ab}P_3Q & 0 \\ 0 & \frac{1}{4a}P_3Q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1.21}$$

for any $(x_1, x_2)^\top \in V_3$.

Proof. According to (1.11), the operator $Q_\infty^{(a,b)}$ may be expressed as

$$\begin{aligned} Q_\infty^{(a,b)} &= \int_0^\infty S_3(t)\Phi\Phi^* S_3^*(t)\, dt \\ &= \int_0^\infty \begin{pmatrix} \frac{a^2}{b}t^2e^{-2at}P_3QP_3 & (1-at)te^{-2at}P_3QP_3 \\ \frac{a^2}{b}t(1-at)e^{-2at}P_3QP_3 & (1-at)^2e^{-2at}P_3QP_3 \end{pmatrix} dt \end{aligned}$$

and the result is just straightforward integration. Since we consinder only $(x_1, x_2)^\top$ from the space V_3, the first (right–hand side) projection P_3 may be omitted. □

The formula (1.21) may also be written in the form like (1.13) in Proposition 1.7 or (1.20) in Proposition 1.10, i.e.,

$$Q_\infty^{(a,b)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sum_{n \in N_3} \sum_{k \in N_3} \langle Qe_n, e_k \rangle_{L^2(D)} \begin{pmatrix} \frac{1}{4ab} \langle x_1, e_n \rangle_{L^2(D)} e_k \\ \frac{1}{4a} \langle x_2, e_n \rangle_{L^2(D)} e_k \end{pmatrix},$$

for any $(x_1, x_2)^\top \in V_3$, which is in fact the same formula as (1.13) (or (1.20)), where $\alpha_n = \frac{a^2}{b} = \alpha_k$ and summations over the set N_3 are used. It is indeed some kind of consistency of these formulae (1.13), (1.20), (1.21).

1.2.4 Summary

We have computed the semigroups $(S_1(t), t \geq 0)$, $(S_2(t), t \geq 0)$ and $(S_3(t), t \geq 0)$ for the coordinates from the sets N_1, N_2 and N_3. The semigroup $(S(t), t \geq 0)$ (with the infinitesimal generator $\mathcal{A}$) is in fact their combination and its form is stated in the following Theorem.

Theorem 1.14. *The operator $\mathcal{A}$ is an infinitesimal generator of the strongly continuous semigroup $(S(t), t \geq 0)$ on V, which takes the following form*

$$S(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = S_1(t)\begin{pmatrix} P_1 & 0 \\ 0 & P_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + S_2(t)\begin{pmatrix} P_2 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + S_3(t)\begin{pmatrix} P_3 & 0 \\ 0 & P_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{1.22}$$

for any $x = (x_1, x_2)^\top \in V$ and $t \geq 0$.

Moreover, the semigroup $(S(t), t \geq 0)$ is exponentially stable.

Proof. For any $x \in V$, its projections to the space V_i, $i = 1, 2, 3$, are taken and then the appropriate semigroup to the appropriate coordinates is applied. From the proofs of Lemmas 1.5, 1.8 and 1.11 it is also clear that

(i)
$$S(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

(ii)
$$\frac{d}{dt}S(t)x = \mathcal{A}S(t)x, \quad \forall x \in \mathrm{Dom}(\mathcal{A}), \quad \forall t \geq 0,$$

which means that this is the form of semigroup $(S(t), t \geq 0)$ with the infinitesimal generator $\mathcal{A}$. Exponential stability is implied by the exponential stability of semigroups $(S_1(t), t \geq 0)$, $(S_2(t), t \geq 0)$ and $(S_3(t), t \geq 0)$. □

The covariance operator $Q_\infty^{(a,b)}$ is in fact combined in the same way (we could have used marking $Q_{\infty,1}^{(a,b)}, Q_{\infty,2}^{(a,b)}$ and $Q_{\infty,3}^{(a,b)}$ in the previous cases), but since (1.13), (1.20) and (1.21) coincide, the resulting formula is rather simple and is given by the following Theorem.

Theorem 1.15. *If (A1) – (A4) are satisfied then there is a unique invariant measure $\mu_\infty^{(a,b)} = N\left(0, Q_\infty^{(a,b)}\right)$ for the equation* (1.12) *and*

$$w^* - \lim_{t\to\infty} \mu_t^{x_0} = \mu_\infty^{(a,b)}$$

for each initial condition $x_0 \in V$. The covariance operator $Q_\infty^{(a,b)}$ takes the form

$$Q_\infty^{(a,b)}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{\langle Qe_n, e_k\rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2b(\alpha_n + \alpha_k)} \times \begin{pmatrix} 4a\alpha_n \langle x_1, e_n\rangle_{L^2(D)} e_k + b(\alpha_k - \alpha_n)\langle x_2, e_n\rangle_{L^2(D)} e_k \\ b\alpha_n(\alpha_n - \alpha_k)\langle x_1, e_n\rangle_{L^2(D)} e_k + 2ab(\alpha_n + \alpha_k)\langle x_2, e_n\rangle_{L^2(D)} e_k \end{pmatrix}, \tag{1.23}$$

for any $(x_1, x_2)^\top \in V$.

Proof. The existence of invariant measure $\mu_\infty^{(a,b)}$ is given by Proposition 1.4. The formula for the covariance operator $Q_\infty^{(a,b)}$ follows from Propositions 1.7, 1.10 and 1.13. □

2. Statistical inference based on norm

2.1 Parameter estimation

Consider the stochastic differential equation (1.12) with unknown parameters $a > 0$, $b > 0$. Our aim is to propose strongly consistent estimators of these parameters based on observation of the trajectory of the process $(X^{x_0}(t), 0 \leq t \leq T)$ up to time T.

Since the linear differential equation (1.12) has a unique invariant measure $\mu_\infty^{(a,b)}$ (by Theorem 1.15), we may use the following ergodic theorem for arbitrary solution (see [23, Theorem 4.9]).

Theorem 2.1. *Let $(X^{x_0}(t), t \geq 0)$ be a solution to (1.12) with $\Phi \in \mathcal{L}_2(U, V)$. Let $\varrho : V \to \mathbb{R}$ be a functional satisfying the following local Lipschitz condition: Let there exist real constants $K > 0$ and $m \geq 0$ such that*

$$|\varrho(x) - \varrho(y)| \leq K \|x - y\|_V \left(1 + \|x\|_V^m + \|y\|_V^m\right) \tag{2.1}$$

holds for all $x, y \in V$. Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho\left(X^{x_0}(t)\right)\, dt = \int_V \varrho(y)\, \mu_\infty(dy), \quad \mathbb{P} - a.s. \tag{2.2}$$

for all $x_0 \in V$.

We will be specifically interested in a functional $\varrho : V \to \mathbb{R}$, $\varrho(y) = \|y\|_V^2$, $y \in V$. Then all the conditions of above Theorem are satisfied with $m = 1$ and

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho\left(X^{x_0}(t)\right)\, dt &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \|X^{x_0}(t)\|_V^2\, dt \\ &= \int_V \|y\|_V^2\, \mu_\infty^{(a,b)}(dy) \\ &= \operatorname{Tr} Q_\infty^{(a,b)}, \quad \mathbb{P} - a.s., \end{aligned} \tag{2.3}$$

where $\operatorname{Tr}(\cdot)$ denotes the trace of the (nuclear) operator. Hence we first introduce the trace of the operator $Q_\infty^{(a,b)}$.

Lemma 2.2. *Trace of the nuclear operator $Q_\infty^{(a,b)}$ takes the form*

$$\operatorname{Tr} Q_\infty^{(a,b)} = \frac{1}{4ab} \sum_{n=1}^\infty \lambda_n + \frac{1}{4a} \sum_{n=1}^\infty \lambda_n \tag{2.4}$$

$$= \frac{b+1}{4ab} \operatorname{Tr} Q. \tag{2.5}$$

Proof. According to the definition of the trace

$$\operatorname{Tr} Q_\infty^{(a,b)} = \sum_{j=1}^\infty \left\langle Q_\infty^{(a,b)} \begin{pmatrix} f_j \\ 0 \end{pmatrix}, \begin{pmatrix} f_j \\ 0 \end{pmatrix} \right\rangle_V + \sum_{j=1}^\infty \left\langle Q_\infty^{(a,b)} \begin{pmatrix} 0 \\ e_j \end{pmatrix}, \begin{pmatrix} 0 \\ e_j \end{pmatrix} \right\rangle_V.$$

With (1.23) in mind, we start with the summand of the first sum

$$\left\langle Q_\infty^{(a,b)} \begin{pmatrix} f_j \\ 0 \end{pmatrix}, \begin{pmatrix} f_j \\ 0 \end{pmatrix} \right\rangle_V =$$
$$= \left\langle \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\langle Qe_n, e_k \rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \begin{pmatrix} 4a\alpha_n \langle f_j, e_n \rangle_{L^2(D)} e_k \\ b\alpha_n(\alpha_n - \alpha_k) \langle f_j, e_n \rangle_{L^2(D)} e_k \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} f_j \\ 0 \end{pmatrix} \right\rangle_V$$
$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{4a\alpha_n \langle Qe_n, e_k \rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle f_j, e_n \rangle_{L^2(D)} \langle f_j, e_k \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}.$$

Since

$$\langle f_j, e_n \rangle_{L^2(D)} = \frac{1}{\sqrt{\alpha_j}} \delta_{j,n},$$
$$\langle f_j, e_k \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} = \sqrt{\alpha_k} \delta_{j,k},$$

where δ stands for the Kronecker delta, there is only one nonzero summand, which corresponds to $n = k = j$, so we arrive at

$$\frac{1}{4ab} \langle Qe_j, e_j \rangle_{L^2(D)}.$$

If we sum up these terms over j, we will obtain the first term on the right–hand side of (2.4), that is

$$\frac{1}{4ab} \sum_{j=1}^{\infty} \langle Qe_j, e_j \rangle_{L^2(D)} = \frac{1}{4ab} \sum_{j=1}^{\infty} \lambda_j.$$

Note that

$$\operatorname{Tr} Q = \sum_{j=1}^{\infty} \left\langle Qe'_j, e'_j \right\rangle_{L^2(D)} = \sum_{j=1}^{\infty} \lambda_j = \sum_{j=1}^{\infty} \langle Qe_j, e_j \rangle_{L^2(D)},$$

where the last equality follows from the fact that the definition of the trace does not depend on the choice of orthonormal basis of $L^2(D)$.

In a similar fashion, we compute the summand of the second sum

$$\left\langle Q_\infty^{(a,b)} \begin{pmatrix} 0 \\ e_j \end{pmatrix}, \begin{pmatrix} 0 \\ e_j \end{pmatrix} \right\rangle_V =$$
$$= \left\langle \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\langle Qe_n, e_k \rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \begin{pmatrix} b(\alpha_k - \alpha_n) \langle e_j, e_n \rangle_{L^2(D)} e_k \\ 2ab(\alpha_n + \alpha_k) \langle e_j, e_n \rangle_{L^2(D)} e_k \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ e_j \end{pmatrix} \right\rangle_V$$
$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2ab(\alpha_n + \alpha_k) \langle Qe_n, e_k \rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle e_j, e_n \rangle_{L^2(D)} \langle e_j, e_k \rangle_{L^2(D)}$$
$$= \frac{1}{4a} \langle Qe_j, e_j \rangle_{L^2(D)}.$$

If we sum up these terms over j, we will obtain the second term on the right–hand side of (2.4), that is

$$\frac{1}{4a}\sum_{j=1}^{\infty}\langle Qe_j, e_j\rangle_{L^2(D)} = \frac{1}{4a}\sum_{j=1}^{\infty}\lambda_j.$$

□

Based on above Lemma and Theorem 2.1, strongly consistent estimators of parameters a and b may be proposed.

Theorem 2.3. *If we set*

$$I_T = \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_V^2\, dt, \tag{2.6}$$

then the processes

$$\hat{a}_T = \frac{b+1}{4bI_T}\operatorname{Tr} Q, \tag{2.7}$$

$$\hat{b}_T = \frac{\operatorname{Tr} Q}{4aI_T - \operatorname{Tr} Q} \tag{2.8}$$

are strongly constistent estimators of the parameters a and b, respectively, i.e., $\hat{a}_T \to a$, $\hat{b}_T \to b$, $\mathbb{P} - a.s.$ as $T \to \infty$.

Proof. From (2.3) and (2.5) it follows that

$$\lim_{T\to\infty} I_T = \frac{b+1}{4ab}\operatorname{Tr} Q, \quad \mathbb{P} - a.s.$$

Hence we obtain the desired limits $\hat{a}_T \to a$, $\hat{b}_T \to b$, $\mathbb{P} - a.s.$ as $T \to \infty$. □

Remark 1. The estimators $\hat{a}_T$ and $\hat{b}_T$ may be easily implemented, but they have one major disadvantage: We need to know the true value of the other parameter. In order to compute the estimator $\hat{a}_T$, we need to know not only the quantity I_T (which can be computed from the observation of the trajectory of the process $(X^{x_0}(t), 0 \le t \le T)$), the trace of the operator Q (which is supposed to be given by the model), but we also need to know the true value of the parameter b. (And similarly for the estimator $\hat{b}_T$.)

However another family of estimators $(\tilde{a}_T, \tilde{b}_T)$ is proposed now, which does not possess this disadvantage. Since

$$\|x\|_V^2 = \|x_1\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \|x_2\|^2_{L^2(D)},$$

for any $x = (x_1, x_2)^\top \in V$, the integral in (2.6) may be split into two parts

$$\begin{aligned} I_T &= \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_V^2\, dt \\ &= \frac{1}{T}\int_0^T \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt + \frac{1}{T}\int_0^T \|X_2^{x_0}(t)\|^2_{L^2(D)}\, dt \\ &=: Y_T + H_T, \end{aligned} \tag{2.9}$$

where $X^{x_0}(t) = (X_1^{x_0}(t), X_2^{x_0}(t))^\top \in V$ is the solution to the equation (1.12).

From the proof of Lemma 2.2 (and also from the formula (2.4)) it is easy to see that the trace of $Q_\infty^{(a,b)}$ may also be split into two parts. In the following Theorem, we show that these parts converge individually to their corresponding limits and based on this convergence, we may introduce new family of estimators $\tilde{a}_T$ and $\tilde{b}_T$.

Theorem 2.4. *The estimators*

$$\tilde{a}_T = \frac{\operatorname{Tr} Q}{4H_T}, \tag{2.10}$$

$$\tilde{b}_T = \frac{H_T}{Y_T} \tag{2.11}$$

are strongly constistent estimators of the parameters a and b, respectively.

Proof. Consider the functional $\varrho_1 : V \to \mathbb{R}$, $\varrho_1(y) = \|y_1\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}$, $y = (y_1, y_2)^\top \in V$. Then all the conditions of Theorem 2.1 are satisfied with $m = 1$ and

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho_1\left(X^{x_0}(t)\right) dt &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \, dt \\ &= \int_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \|y_1\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \, \mu_{\infty,1}^{(a,b)}(dy_1) \\ &= \frac{1}{4ab} \operatorname{Tr} Q, \quad \mathbb{P} - a.s.,\end{aligned}$$

where $\mu_{\infty,1}^{(a,b)}$ is the Gaussian measure with zero mean and covariance operator given by

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{4a\alpha_n \langle Qe_n, e_k \rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle x, e_n \rangle_{L^2(D)} e_k, \quad \forall x \in \mathrm{Dom}((-A)^{\frac{1}{2}}),$$

i.e., $\mu_{\infty,1}^{(a,b)}$ is "the first marginal" of the measure $\mu_\infty^{(a,b)}$.

Hence $Y_T \to \frac{1}{4ab} \operatorname{Tr} Q$, $\mathbb{P} - a.s.$ as $T \to \infty$.

Similarly, consider the functional $\varrho_2 : V \to \mathbb{R}$, $\varrho_2(y) = \|y_2\|^2_{L^2(D)}$, $y = (y_1, y_2)^\top \in V$. Then all the conditions of Theorem 2.1 are satisfied with $m = 1$ and

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho_2\left(X^{x_0}(t)\right) dt &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \|X_2^{x_0}(t)\|^2_{L^2(D)} \, dt \\ &= \int_{L^2(D)} \|y_2\|^2_{L^2(D)} \, \mu_{\infty,2}^{(a,b)}(dy_2) \\ &= \frac{1}{4a} \operatorname{Tr} Q, \quad \mathbb{P} - a.s.,\end{aligned}$$

where $\mu_{\infty,2}^{(a,b)}$ is the Gaussian measure with zero mean and covariance operator given by

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2ab(\alpha_n + \alpha_k) \langle Qe_n, e_k \rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle x, e_n \rangle_{L^2(D)} e_k, \quad \forall x \in L^2(D),$$

i.e., it is "the second marginal" of the measure $\mu_\infty^{(a,b)}$.

Hence $H_T \to \frac{1}{4a}\operatorname{Tr} Q$, $\mathbb{P}-a.s.$ as $T\to\infty$ and the convergence of $\tilde a_T$ to the true value of parameter a follows. Similarly

$$\tilde b_T = \frac{H_T}{Y_T} \to \frac{\frac{\operatorname{Tr} Q}{4a}}{\frac{\operatorname{Tr} Q}{4ab}} = b, \quad T\to\infty, \quad \mathbb{P}-a.s.$$

□

2.2 Asymptotic normality of the estimators

2.2.1 Asymptotic normality of the estimators $(\hat a_T, \hat b_T)$

In this part, we will show the asymptotic normality of estimators (2.7) and (2.8), i.e., the weak convergences of $\operatorname{Law}\left(\sqrt{T}\,(\hat a_T - a)\right)$ and $\operatorname{Law}\left(\sqrt{T}\left(\hat b_T - b\right)\right)$ to Gaussian distributions. To this aim, define an operator $R: V\to V$ by

$$Rx = R\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} bI - \frac{4a^2}{b+1}A^{-1} & -\frac{2a}{b+1}A^{-1} \\ \frac{2a}{b+1}I & I \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V.$$

The properties of R needed in the sequel are summarized in the following Lemma.

Lemma 2.5. *The operator R is a self–adjoint linear isomorphism of V. Moreover,*

$$\langle Rx, \mathcal{A}x\rangle_V = -\frac{2ab}{b+1}\|x\|_V^2, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \operatorname{Dom}(\mathcal{A}). \tag{2.12}$$

Proof. It is evident that $R\in\mathcal{L}(V)$ and for any $x=(x_1,x_2)^\top \in V$ and $y = (y_1,y_2)^\top \in V$ we have

$$\begin{aligned}
\langle Rx, y\rangle_V &= \left\langle \begin{pmatrix} bx_1 - \frac{4a^2}{b+1}A^{-1}x_1 - \frac{2a}{b+1}A^{-1}x_2 \\ \frac{2a}{b+1}x_1 + x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_V \\
&= b\left\langle (-A)^{\frac12}x_1, (-A)^{\frac12}y_1\right\rangle_{L^2(D)} - \frac{4a^2}{b+1}\left\langle (-A)^{\frac12}A^{-1}x_1, (-A)^{\frac12}y_1\right\rangle_{L^2(D)} \\
&\quad - \frac{2a}{b+1}\left\langle (-A)^{\frac12}A^{-1}x_2, (-A)^{\frac12}y_1\right\rangle_{L^2(D)} + \frac{2a}{b+1}\langle x_1, y_2\rangle_{L^2(D)} \\
&\quad + \langle x_2, y_2\rangle_{L^2(D)} \\
&= \langle x, Ry\rangle_V,
\end{aligned}$$

hence $R=R^*$. The equation (2.12) can be derived by similar computation. Indeed, for any $x=(x_1,x_2)^\top\in\operatorname{Dom}(\mathcal{A})$ we have

$$\langle Rx, \mathcal{A}x\rangle_V = \left\langle \begin{pmatrix} bx_1 - \frac{4a^2}{b+1}A^{-1}x_1 - \frac{2a}{b+1}A^{-1}x_2 \\ \frac{2a}{b+1}x_1 + x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ bAx_1 - 2ax_2 \end{pmatrix} \right\rangle_V$$

$$
\begin{aligned}
&= b\left\langle(-A)^{\frac{1}{2}}x_1,(-A)^{\frac{1}{2}}x_2\right\rangle_{L^2(D)} - \frac{4a^2}{b+1}\left\langle(-A)^{\frac{1}{2}}A^{-1}x_1,(-A)^{\frac{1}{2}}x_2\right\rangle_{L^2(D)}\\
&\quad - \frac{2a}{b+1}\left\langle(-A)^{\frac{1}{2}}A^{-1}x_2,(-A)^{\frac{1}{2}}x_2\right\rangle_{L^2(D)} + \frac{2ab}{b+1}\left\langle x_1,Ax_1\right\rangle_{L^2(D)}\\
&\quad - \frac{4a^2}{b+1}\left\langle x_1,x_2\right\rangle_{L^2(D)} + b\left\langle x_2,Ax_1\right\rangle_{L^2(D)} - 2a\left\langle x_2,x_2\right\rangle_{L^2(D)}\\
&= -\frac{2ab}{b+1}\left\langle(-A)^{\frac{1}{2}}x_1,(-A)^{\frac{1}{2}}x_1\right\rangle_{L^2(D)} - \frac{2ab}{b+1}\left\langle x_2,x_2\right\rangle_{L^2(D)}\\
&= -\frac{2ab}{b+1}\|x\|_V^2.
\end{aligned}
$$

□

In the proof of Theorem 2.8, we will also need an alternative formula for the process I_T, which was defined by (2.6).

Proposition 2.6. *The process I_T admits the following representation*

$$
\begin{aligned}
I_T &= \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_V^2\,dt\\
&= -\frac{b+1}{4abT}\left(\langle RX^{x_0}(T),X^{x_0}(T)\rangle_V - \langle Rx_0,x_0\rangle_V\right)\\
&\quad + \frac{b+1}{2abT}\int_0^T \langle RX^{x_0}(t),\Phi\,dB(t)\rangle_V + \frac{b+1}{4ab}\operatorname{Tr}Q.
\end{aligned}
\tag{2.13}
$$

Proof. Define the function $g : V \to \mathbb{R}$ by

$$
g(x) = \langle Rx,x\rangle_V\,,\quad \forall x\in V. \tag{2.14}
$$

The Itô's formula (see e.g. [5, Theorem 4.17]) is not applicable to the process $g(X^{x_0}(t))$ directly, because $(X^{x_0}(t), t\geq 0)$ is not a strong solution to the equation (1.12). We apply it to suitable finite–dimensional projections.

Let $\{h_n, n\in\mathbb{N}\}$ be an orthonormal basis in V consisting of elements from $\mathrm{Dom}(\mathcal{A})$ and let P_N be the operator of projection on the span $\{h_n, n=1,\dots N\}$, that is

$$
P_N x = \sum_{n=1}^{N}\langle x,h_n\rangle_V\,h_n,\quad \forall x\in V,\quad \forall N\in\mathbb{N}.
$$

Choose $N\in\mathbb{N}$ and set

$$
X^{x_0,N}(t) := P_N X^{x_0}(t),\quad t\geq 0.
$$

The expansion for the $X^{x_0,N}(t)$ is finite, so $X_1^{x_0,N}(t)\in\mathrm{Dom}(A)$, $X_2^{x_0,N}(t)\in\mathrm{Dom}((-A)^{\frac{1}{2}})$ and consequently $X^{x_0,N}(t)\in\mathrm{Dom}(\mathcal{A})$ for all $t\geq 0$. Now we may apply Itô's formula to the function $g(X^{x_0,N}(t))$, which yields

$$
dg(X^{x_0,N}(t)) = 2\left\langle RX^{x_0,N}(t),dX^{x_0,N}(t)\right\rangle_V + \frac{1}{2}\operatorname{Tr}\left(2R\Phi\Phi^*\right)\,dt. \tag{2.15}
$$

The second term may be simplified via following calculation

$$
\frac{1}{2}\operatorname{Tr}\left(2R\Phi\Phi^*\right) = \operatorname{Tr}\begin{pmatrix} 0 & -\frac{2a}{b+1}A^{-1}Q\\ 0 & Q\end{pmatrix} = \operatorname{Tr}Q.
$$

Using that fact and Lemma 2.5, the expression (2.15) implies

$$\begin{aligned}
&dg(X^{x_0,N}(t)) = \\
&= 2\left\langle RX^{x_0,N}(t), \mathcal{A}X^{x_0,N}(t)\right\rangle_V dt + 2\left\langle RX^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V + \operatorname{Tr} Q\, dt \\
&= -\frac{4ab}{b+1}\left\|X^{x_0,N}(t)\right\|_V^2 dt + 2\left\langle RX^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V + \operatorname{Tr} Q\, dt.
\end{aligned}$$

After integrating previous formula over the interval $(0,T)$, we arrive at

$$\begin{aligned}
\frac{1}{T}\int_0^T \left\|X^{x_0,N}(t)\right\|_V^2 dt = &-\frac{b+1}{4abT}\left(\left\langle RX^{x_0,N}(T), X^{x_0,N}(T)\right\rangle_V - \left\langle Rx_0^N, x_0^N\right\rangle_V\right) \\
&+ \frac{b+1}{2abT}\int_0^T \left\langle RX^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V + \frac{b+1}{4ab}\operatorname{Tr} Q. \quad (2.16)
\end{aligned}$$

Since

$$\left\|X^{x_0,N}(t)\right\|_V \le \|X^{x_0}(t)\|_V, \quad \forall t \ge 0, \quad \forall N \in \mathbb{N},$$

we may use the random variable $\|X^{x_0}(t)\|_V^2$ as an integrable majorant for the integral on the left–hand side. Also,

$$\int_0^T \left\langle RX^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V \to \int_0^T \langle RX^{x_0}(t), \Phi\, dB(t)\rangle_V, \quad N \to \infty \text{ in } L^2(\Omega),$$

because, for some positive constant $C > 0$, we have that

$$\mathbb{E}\left|\int_0^T \left\langle R\left(X^{x_0,N}(t) - X^{x_0}(t)\right), \Phi\, dB(t)\right\rangle_V\right|^2 \le C\int_0^T \mathbb{E}\left\|X^{x_0,N}(t) - X^{x_0}(t)\right\|_V^2 dt,$$

which tends to 0 as $N \to \infty$, since

$$X^{x_0,N}(t) \to X^{x_0}(t), \quad \forall t \ge 0, \quad N \to \infty \text{ in } L^2(\Omega; V).$$

Hence we obtain (2.13) by passing N to infinity in (2.16). □

We will also need the following Lemma.

Lemma 2.7. *Let $(X^{x_0}(t), t \ge 0)$ be a solution to the linear equation* (1.12) *and $R \in \mathcal{L}(V)$. Then*

$$\frac{1}{\sqrt{t}}\langle RX^{x_0}(t), X^{x_0}(t)\rangle_V \to 0$$

in $L^1(\Omega)$ as $t \to \infty$.

Proof. There exist some positive constants $C > 0$ and $C_1 > 0$, so that

$$\begin{aligned}
\mathbb{E}\left|\frac{\langle RX^{x_0}(t), X^{x_0}(t)\rangle_V}{\sqrt{t}}\right| &\le \frac{C}{\sqrt{t}}\mathbb{E}\|X^{x_0}(t)\|_V^2 \\
&\le \frac{2C}{\sqrt{t}}\mathbb{E}\|S(t)x_0\|_V^2 + \frac{2C}{\sqrt{t}}\mathbb{E}\|Z(t)\|_V^2 \\
&\le \frac{C_1}{\sqrt{t}}e^{-2\rho t}\mathbb{E}\|x_0\|_V^2 + \frac{2C}{\sqrt{t}}\operatorname{Tr} Q_t.
\end{aligned}$$

Since

$$\sup_{t \ge 0}\operatorname{Tr} Q_t < \infty,$$

(which is equivalent to the existence of an invariant measure, see [5, Theorem 11.7]), both terms tend to 0 as $t \to \infty$. □

Finally, define the operator $\tilde{R} : V \to L^2(D)$ by

$$\tilde{R}x = \begin{pmatrix} \frac{2a}{b+1}I & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{2a}{b+1}x_1 + x_2, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V. \tag{2.17}$$

Note that the adjoint operator of $\tilde{R}$ has the following form

$$\tilde{R}^* : L^2(D) \to V, \quad \tilde{R}^* x = \begin{pmatrix} -\frac{2a}{b+1}A^{-1} \\ I \end{pmatrix} x = \begin{pmatrix} -\frac{2a}{b+1}A^{-1}x \\ x \end{pmatrix}, \tag{2.18}$$

for any $x \in L^2(D)$.

Asymptotic normality of the estimators $\hat{a}_T$ and $\hat{b}_T$ is formulated in the following Theorem.

Theorem 2.8. *1) The estimator $\hat{a}_T$ is asymptotically normal, i.e., $\mathrm{Law}\left(\sqrt{T}\,(\hat{a}_T - a)\right)$ converges weakly to the centered Gaussian distribution with variance $\frac{4a^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q\tilde{R}Q_\infty^{(a,b)}\tilde{R}^*\right)$, i.e.,*

$$\mathrm{Law}\left(\sqrt{T}\,(\hat{a}_T - a)\right) \xrightarrow{w^*} N\left(0, \frac{4a^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q\tilde{R}Q_\infty^{(a,b)}\tilde{R}^*\right)\right), \quad T \to \infty. \tag{2.19}$$

2) The estimator $\hat{b}_T$ is asymptotically normal, i.e.,

$$\mathrm{Law}\left(\sqrt{T}\left(\hat{b}_T - b\right)\right) \xrightarrow{w^*} N\left(0, \frac{4b^2(b+1)^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q\tilde{R}Q_\infty^{(a,b)}\tilde{R}^*\right)\right), \quad T \to \infty. \tag{2.20}$$

Proof. 1) Using formula (2.7) for the estimator $\hat{a}_T$ and Proposition 2.6 for the representation of I_T, it is possible to compute the following

$$\begin{aligned} \sqrt{T}\,(\hat{a}_T - a) &= \sqrt{T}\left(\frac{b+1}{4bI_T}\operatorname{Tr} Q - a\right) = \sqrt{T}\frac{(b+1)\operatorname{Tr} Q - 4abI_T}{4bI_T} \\ &= \frac{\sqrt{T}}{4bI_T}\left(\frac{b+1}{T}\left(\langle RX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Rx_0, x_0\rangle_V\right)\right. \\ &\quad \left. - \frac{2(b+1)}{T}\int_0^T \langle RX^{x_0}(t), \Phi\, dB(t)\rangle_V\right) \\ &= \frac{b+1}{4bI_T}\frac{1}{\sqrt{T}}\left(\langle RX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Rx_0, x_0\rangle_V\right) \\ &\quad - \frac{b+1}{2bI_T}\frac{1}{\sqrt{T}}\int_0^T \langle RX^{x_0}(t), \Phi\, dB(t)\rangle_V. \end{aligned} \tag{2.21}$$

The first term on the right–hand side converges to zero in probability as $T \to \infty$, since

$$\lim_{T\to\infty} \frac{b+1}{4bI_T} = \frac{a}{\operatorname{Tr} Q}, \quad \mathbb{P} - a.s.$$

by Theorem 2.3 and

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}}\left(\langle RX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Rx_0, x_0\rangle_V\right) = 0, \quad \text{in } L^1(\Omega)$$

by Lemma 2.7. Define

$$\begin{aligned} v(T) &= \frac{1}{\sqrt{T}} \int_0^T \langle RX^{x_0}(t), \Phi\, dB(t)\rangle_V \\ &= \frac{1}{\sqrt{T}} \int_0^T \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left\langle \tilde{R}X^{x_0}(t), e_n' \right\rangle_{L^2(D)} d\beta_n(t), \end{aligned}$$

where we have used the representation of V–valued Brownian motion $B(t)$. For any $n \in \mathbb{N}$ $\beta_n(t) = \langle B(t), e_n\rangle_V$ are mutually independent scalar Browinan motions (see (1.2)).

By the central limit theorem for martingales (see e.g. [22]), Law $(v(T))$ converges weakly to the Gaussian distribution with a zero mean and variance given by

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n \left\langle \tilde{R}X^{x_0}(t), e_n' \right\rangle_{L^2(D)}^2 dt = \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \left\langle Q^{\frac{1}{2}} \tilde{R}X^{x_0}(t), e_n' \right\rangle_{L^2(D)}^2 dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \|Q^{\frac{1}{2}} \tilde{R}X^{x_0}(t)\|_{L^2(D)}^2 \, dt \\ &= \mathbb{E}\|Q^{\frac{1}{2}} \tilde{R}X(\infty)\|_{L^2(D)}^2 \\ &= \operatorname{Tr}\left(Q\tilde{R}Q_\infty^{(a,b)}\tilde{R}^*\right), \quad \mathbb{P}-a.s., \end{aligned}$$

where $X(\infty)$ is a V–valued Gaussian random variable with zero mean and covariance operator $Q_\infty^{(a,b)}$ (i.e., Law $(X(\infty)) = \mu_\infty^{(a,b)}$).

Since the multiplicative factor $-\frac{b+1}{2bI_T}$ of $v(T)$ on the right–hand side of (2.21) converges to $-\frac{2a}{\operatorname{Tr} Q}$, $\mathbb{P}-a.s.$ as $T \to \infty$, we arrive at

$$\operatorname{Law}(v(T)) \xrightarrow{w^*} N\left(0,\, \operatorname{Tr}\left(Q\tilde{R}Q_\infty^{(a,b)}\tilde{R}^*\right)\right), \quad T \to \infty, \tag{2.22}$$

$$\operatorname{Law}\left(\sqrt{T}\,(\hat{a}_T - a)\right) \xrightarrow{w^*} N\left(0,\, \frac{4a^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q\tilde{R}Q_\infty^{(a,b)}\tilde{R}^*\right)\right), \quad T \to \infty.$$

2) In a similar fashion, using formula (2.8) for the estimator $\hat{b}_T$ and Proposition 2.6, it is possible to compute the following

$$\begin{aligned} \sqrt{T}\left(\hat{b}_T - b\right) &= \sqrt{T}\left(\frac{\operatorname{Tr} Q}{4aI_T - \operatorname{Tr} Q} - b\right) = \frac{\sqrt{T}}{4aI_T - \operatorname{Tr} Q} \left(\operatorname{Tr} Q - 4abI_T + b \operatorname{Tr} Q\right) \\ &= \frac{\sqrt{T}}{4aI_T - \operatorname{Tr} Q} \left(\frac{b+1}{T} \left(\langle RX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Rx_0, x_0\rangle_V\right)\right. \\ &\quad \left. - \frac{2(b+1)}{T} \int_0^T \langle RX^{x_0}(t), \Phi\, dB(t)\rangle_V \right) \\ &= \frac{b+1}{4aI_T - \operatorname{Tr} Q} \frac{1}{\sqrt{T}} \left(\langle RX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Rx_0, x_0\rangle_V\right) \\ &\quad - \frac{2(b+1)}{4aI_T - \operatorname{Tr} Q} \frac{1}{\sqrt{T}} \int_0^T \langle RX^{x_0}(t), \Phi\, dB(t)\rangle_V . \end{aligned} \tag{2.23}$$

Similarly as above, the first term on the right–hand side converges to zero in probability as $T \to \infty$ and the multiplicative factor $-\frac{2(b+1)}{4aI_T - \operatorname{Tr} Q}$ of $v(T)$ on the right–hand side of (2.23) converges to $-\frac{2b(b+1)}{\operatorname{Tr} Q}$, $\mathbb{P} - a.s.$ as $T \to \infty$. Hence we obtain the result

$$\operatorname{Law}\left(\sqrt{T}\left(\hat{b}_T - b\right)\right) \xrightarrow{w^*} N\left(0, \frac{4b^2(b+1)^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q\tilde{R}Q_\infty^{(a,b)}\tilde{R}^*\right)\right), \quad T \to \infty.$$

$\square$

Remark 2. We may specify the variance of the limiting Gaussian distribution in (2.22). By Theorem 1.15, we obtain

$$\operatorname{Tr}\left(Q\tilde{R}Q_\infty^{(a,b)}\tilde{R}^*\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{16a^3 + 2ab(b+1)^2(\alpha_n + \alpha_k)}{b^2(\alpha_n - \alpha_k)^2 + 8a^2b(\alpha_n + \alpha_k)} \frac{1}{(b+1)^2} \langle Qe_n, e_k\rangle_{L^2(D)}^2 . \tag{2.24}$$

2.2.2 Asymptotic normality of the estimators $(\tilde{a}_T, \tilde{b}_T)$

The family of estimators $(\tilde{a}_T, \tilde{b}_T)$ is also asymptotically normal, which will be shown in Theorem 2.11. The proof uses the same technique as the proof of Theorem 2.8, so the setup and auxiliary Lemmas will be similar to those in previous subsection.

We start with the definition of the operators $R_1 : V \to V$ and $R_2 : V \to V$,

$$R_1 x = R_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} bI & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V,$$

$$R_2 x = R_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} bI - 4a^2A^{-1} & -2aA^{-1} \\ 2aI & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V.$$

The properties of these two operators are summarized in the following Lemma.

Lemma 2.9. *The operators R_1 and R_2 are self–adjoint linear isomorphisms of V. Moreover,*

$$\langle R_1 x, \mathcal{A}x\rangle_V = -2a\|x_2\|^2_{L^2(D)}, \qquad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \operatorname{Dom}(\mathcal{A}), \tag{2.25}$$

$$\langle R_2 x, \mathcal{A}x\rangle_V = -2ab\|x_1\|^2_{\operatorname{Dom}((-A)^{\frac{1}{2}})}, \qquad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \operatorname{Dom}(\mathcal{A}). \tag{2.26}$$

Proof. It is evident that $R_1, R_2 \in \mathcal{L}(V)$ and for any $x = (x_1, x_2)^\top \in V$ and $y = (y_1, y_2)^\top \in V$ we have

$$\begin{aligned}\langle R_1 x, y\rangle_V &= \left\langle \begin{pmatrix} bx_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_V \\ &= b\left\langle (-A)^{\frac{1}{2}}x_1, (-A)^{\frac{1}{2}}y_1\right\rangle_{L^2(D)} + \langle x_2, y_2\rangle_{L^2(D)} \\ &= \langle x, R_1 y\rangle_V\end{aligned}$$

and

$$\begin{aligned}
\langle R_2 x, y\rangle_V &= \left\langle \begin{pmatrix} bx_1 - 4a^2A^{-1}x_1 - 2aA^{-1}x_2 \\ 2ax_1 + x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_V \\
&= b\left\langle (-A)^{\frac{1}{2}}x_1, (-A)^{\frac{1}{2}}y_1 \right\rangle_{L^2(D)} - 4a^2\left\langle (-A)^{\frac{1}{2}}A^{-1}x_1, (-A)^{\frac{1}{2}}y_1 \right\rangle_{L^2(D)} \\
&\quad - 2a\left\langle (-A)^{\frac{1}{2}}A^{-1}x_2, (-A)^{\frac{1}{2}}y_1 \right\rangle_{L^2(D)} + 2a\left\langle x_1, y_2\right\rangle_{L^2(D)} + \left\langle x_2, y_2\right\rangle_{L^2(D)} \\
&= \langle x, R_2 y\rangle_V ,
\end{aligned}$$

hence $R_1 = R_1^*$ and $R_2 = R_2^*$. The equation (2.25) can be derived by a simple computation. For any $x = (x_1, x_2)^\top \in \mathrm{Dom}(\mathcal{A})$ we have

$$\begin{aligned}
\langle R_1 x, \mathcal{A}x\rangle_V &= \left\langle \begin{pmatrix} bx_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ bAx_1 - 2ax_2 \end{pmatrix} \right\rangle_V \\
&= b\left\langle (-A)^{\frac{1}{2}}x_1, (-A)^{\frac{1}{2}}x_2 \right\rangle_{L^2(D)} + b\left\langle x_2, Ax_1\right\rangle_{L^2(D)} - 2a\left\langle x_2, x_2\right\rangle_{L^2(D)} \\
&= -2a\|x_2\|^2_{L^2(D)}.
\end{aligned}$$

Similar computation yields (2.26):

$$\begin{aligned}
\langle R_2 x, \mathcal{A}x\rangle_V &= \left\langle \begin{pmatrix} bx_1 - 4a^2A^{-1}x_1 - 2aA^{-1}x_2 \\ 2ax_1 + x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ bAx_1 - 2ax_2 \end{pmatrix} \right\rangle_V \\
&= b\left\langle (-A)^{\frac{1}{2}}x_1, (-A)^{\frac{1}{2}}x_2 \right\rangle_{L^2(D)} - 4a^2\left\langle (-A)^{\frac{1}{2}}A^{-1}x_1, (-A)^{\frac{1}{2}}x_2 \right\rangle_{L^2(D)} \\
&\quad - 2a\left\langle (-A)^{\frac{1}{2}}A^{-1}x_2, (-A)^{\frac{1}{2}}x_2 \right\rangle_{L^2(D)} + 2ab\left\langle x_1, Ax_1\right\rangle_{L^2(D)} \\
&\quad - 4a^2\left\langle x_1, x_2\right\rangle_{L^2(D)} + b\left\langle x_2, Ax_1\right\rangle_{L^2(D)} - 2a\left\langle x_2, x_2\right\rangle_{L^2(D)} \\
&= -2ab\|x_1\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}.
\end{aligned}$$

□

We will also need the alternative formulae for the processes Y_T and H_T, which were defined by (2.9).

Proposition 2.10. *1) The process Y_T admits the following representation*

$$\begin{aligned}
Y_T &= \frac{1}{T}\int_0^T \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt \\
&= -\frac{1}{4abT}\left(\langle R_2 X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle R_2 x_0, x_0\rangle_V\right) \\
&\quad + \frac{1}{2abT}\int_0^T \langle R_2 X^{x_0}(t), \Phi\, dB(t)\rangle_V + \frac{1}{4ab}\operatorname{Tr} Q.
\end{aligned} \tag{2.27}$$

2) The process H_T admits the following representation

$$\begin{aligned}
H_T &= \frac{1}{T}\int_0^T \|X_2^{x_0}(t)\|^2_{L^2(D)}\,dt \\
&= -\frac{1}{4aT}\left(\langle R_1 X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle R_1 x_0, x_0\rangle_V\right) \\
&\quad + \frac{1}{2aT}\int_0^T \langle R_1 X^{x_0}(t), \Phi\, dB(t)\rangle_V + \frac{1}{4a}\operatorname{Tr} Q.
\end{aligned} \tag{2.28}$$

Proof. 1) Define the function $g_1 : V \to \mathbb{R}$ by

$$g_1(x) = \langle R_1 x, x\rangle_V , \quad \forall x \in V. \tag{2.29}$$

The application of Itô's formula to the function $g_1(X^{x_0,N}(t))$ (we also have to use suitable projections, see the proof of Proposition 2.6), yields

$$dg_1(X^{x_0,N}(t)) = 2\left\langle R_1 X^{x_0,N}(t), dX^{x_0,N}(t)\right\rangle_V + \frac{1}{2}\,\mathrm{Tr}\,(2R_1\Phi\Phi^*)\; dt. \tag{2.30}$$

Since the second term equals to

$$\frac{1}{2}\mathrm{Tr}\,(2R_1\Phi\Phi^*) = \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} = \mathrm{Tr}\,Q,$$

the expression (2.30) and Lemma 2.9 imply

$$\begin{aligned} & dg_1(X^{x_0,N}(t)) = \\ & = 2\left\langle R_1 X^{x_0,N}(t), \mathcal{A}X^{x_0,N}(t)\right\rangle_V dt + 2\left\langle R_1 X^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V + \mathrm{Tr}\,Q\,dt \\ & = -4a\left\|X_2^{x_0,N}(t)\right\|^2_{L^2(D)} dt + 2\left\langle R_1 X^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V + \mathrm{Tr}\,Q\,dt. \end{aligned}$$

After integrating previous formula over the interval $(0, T)$ and passing N to infinity, we arrive at (2.28).

2) Similarly, if we define the function $g_2 : V \to \mathbb{R}$ by

$$g_2(x) = \langle R_2 x, x\rangle_V , \quad \forall x \in V, \tag{2.31}$$

and apply Itô's formula to the function $g_2(X^{x_0,N}(t))$, we will obtain

$$dg_2(X^{x_0,N}(t)) = 2\left\langle R_2 X^{x_0,N}(t), dX^{x_0,N}(t)\right\rangle_V + \frac{1}{2}\,\mathrm{Tr}\,(2R_2\Phi\Phi^*)\; dt. \tag{2.32}$$

Since the second term equals to

$$\frac{1}{2}\mathrm{Tr}\,(2R_2\Phi\Phi^*) = \mathrm{Tr}\begin{pmatrix} 0 & -2aA^{-1}Q \\ 0 & Q \end{pmatrix} = \mathrm{Tr}\,Q,$$

the expression (2.32) and Lemma 2.9 imply

$$\begin{aligned} & dg_2(X^{x_0,N}(t)) \\ & = 2\left\langle R_2 X^{x_0,N}(t), \mathcal{A}X^{x_0,N}(t)\right\rangle_V dt + 2\left\langle R_2 X^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V + \mathrm{Tr}\,Q\,dt \\ & = -4ab\left\|X_1^{x_0,N}(t)\right\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} dt + 2\left\langle R_2 X^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V + \mathrm{Tr}\,Q\,dt. \end{aligned}$$

(2.27) follows from the integrating of above formula over the interval $(0, T)$ and passing N to infinity. □

Also define the operator $\tilde{R}_1 : V \to L^2(D)$ by

$$\tilde{R}_1 x = \begin{pmatrix} 0 & I \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V,$$

and the operator $\tilde{R}_2 : V \to L^2(D)$ by

$$\tilde{R}_2 x = \begin{pmatrix} 2aI & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2ax_1, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V.$$

Note that their adjoint operators equal to

$$\tilde{R}_1^* : L^2(D) \to V,\ \tilde{R}^* x = \begin{pmatrix} 0 \\ I \end{pmatrix} x = \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad \forall x \in L^2(D),$$

$$\tilde{R}_2^* : L^2(D) \to V,\ \tilde{R}^* x = \begin{pmatrix} -2aA^{-1} \\ 0 \end{pmatrix} x = \begin{pmatrix} -2aA^{-1}x \\ 0 \end{pmatrix}, \quad \forall x \in L^2(D).$$

Asymptotic normality of the estimators $\tilde{a}_T$ and $\tilde{b}_T$ is formulated in the following Theorem.

Theorem 2.11. *1) The estimator $\tilde{a}_T$ is asymptotically normal, i.e.,*

$$\operatorname{Law}\left(\sqrt{T}\,(\tilde{a}_T - a)\right) \xrightarrow{w^*} N\left(0, \frac{4a^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q\tilde{R}_1 Q_\infty^{(a,b)} \tilde{R}_1^*\right)\right), \quad T \to \infty. \tag{2.33}$$

2) The estimator $\tilde{b}_T$ is asymptotically normal, i.e.,

$$\operatorname{Law}\left(\sqrt{T}\left(\tilde{b}_T - b\right)\right) \xrightarrow{w^*} N\left(0, \frac{4b^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q\tilde{R}_2 Q_\infty^{(a,b)} \tilde{R}_2^*\right)\right), \quad T \to \infty. \tag{2.34}$$

Proof. 1) If we use formula (2.10) for the estimator $\tilde{a}_T$ and representation (2.28) for H_T from Proposition 2.10, we obtain

$$\begin{aligned}
\sqrt{T}\,(\tilde{a}_T - a) &= \sqrt{T}\left(\frac{1}{4H_T}\operatorname{Tr} Q - a\right) = \frac{\sqrt{T}}{4H_T}\left(\operatorname{Tr} Q - 4aH_T\right) \\
&= \frac{\sqrt{T}}{4H_T}\left(\frac{1}{T}\left(\langle R_1 X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle R_1 x_0, x_0\rangle_V\right)\right. \\
&\quad \left. - \frac{2}{T}\int_0^T \langle R_1 X^{x_0}(t), \Phi\, dB(t)\rangle_V\right) \\
&= \frac{1}{4H_T}\frac{1}{\sqrt{T}}\left(\langle R_1 X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle R_1 x_0, x_0\rangle_V\right) \\
&\quad - \frac{1}{2H_T}\frac{1}{\sqrt{T}}\int_0^T \langle R_1 X^{x_0}(t), \Phi\, dB(t)\rangle_V.
\end{aligned} \tag{2.35}$$

The first term on the right–hand side converges to zero in probability as $T \to \infty$, since

$$\lim_{T\to\infty} \frac{1}{4H_T} = \frac{a}{\operatorname{Tr} Q}, \quad \mathbb{P} - a.s.$$

by Theorem 2.4 and

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}}\left(\langle R_1 X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle R_1 x_0, x_0\rangle_V\right) = 0, \quad \text{in } L^1(\Omega)$$

by Lemma 2.7. Also define

$$
\begin{aligned}
v_1(T) &= \frac{1}{\sqrt{T}} \int_0^T \langle R_1 X^{x_0}(t), \Phi\, dB(t)\rangle_V \\
&= \frac{1}{\sqrt{T}} \int_0^T \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left\langle \tilde{R}_1 X^{x_0}(t), e_n' \right\rangle_{L^2(D)} d\beta_n(t),
\end{aligned}
$$

where we have used the representation of V–valued Brownian motion $B(t)$.

By the central limit theorem for martingales, Law$(v_1(T))$ converges weakly to the Gaussian distribution with a zero mean and variance given by

$$
\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n \left\langle \tilde{R}_1 X^{x_0}(t), e_n' \right\rangle_{L^2(D)}^2 dt &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \| Q^{\frac{1}{2}} \tilde{R}_1 X^{x_0}(t)\|_{L^2(D)}^2\, dt \\
&= \operatorname{Tr}\left(Q \tilde{R}_1 Q_\infty^{(a,b)} \tilde{R}_1^*\right), \quad \mathbb{P} - a.s.
\end{aligned}
$$

Since the multiplicative factor $-\frac{1}{2H_T}$ of $v_1(T)$ on the right–hand side of (2.35) converges to $-\frac{2a}{\operatorname{Tr} Q}$, $\mathbb{P} - a.s.$ as $T \to \infty$, we arrive at

$$
\begin{aligned}
\operatorname{Law}(v_1(T)) &\xrightarrow{w^*} N\left(0,\ \operatorname{Tr}\left(Q \tilde{R}_1 Q_\infty^{(a,b)} \tilde{R}_1^*\right)\right), \quad T \to \infty, \\
\operatorname{Law}\left(\sqrt{T}\,(\tilde{a}_T - a)\right) &\xrightarrow{w^*} N\left(0,\ \frac{4a^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q \tilde{R}_1 Q_\infty^{(a,b)} \tilde{R}_1^*\right)\right), \quad T \to \infty.
\end{aligned}
$$

2) Similarly, using formula (2.11) for the estimator $\tilde{b}_T$ and Proposition 2.10 for representation of Y_T and H_T, we may compute the following

$$
\begin{aligned}
\sqrt{T}\left(\tilde{b}_T - b\right) &= \sqrt{T}\left(\frac{H_T}{Y_T} - b\right) = \frac{\sqrt{T}}{Y_T}\left(H_T - bY_T\right) \\
&= \frac{\sqrt{T}}{Y_T}\Bigg(-\frac{1}{4aT}\left(\langle R_1 X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle R_1 x_0, x_0\rangle_V\right) \\
&\quad + \frac{1}{2aT}\int_0^T \langle R_1 X^{x_0}(t), \Phi\, dB(t)\rangle_V \\
&\quad + \frac{1}{4aT}\left(\langle R_2 X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle R_2 x_0, x_0\rangle_V\right) \\
&\quad - \frac{1}{2aT}\int_0^T \langle R_2 X^{x_0}(t), \Phi\, dB(t)\rangle_V \Bigg) \\
&= \frac{1}{4aY_T}\frac{1}{\sqrt{T}}\left(\langle (R_2 - R_1) X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle (R_2 - R_1) x_0, x_0\rangle_V\right) \\
&\quad - \frac{1}{2aY_T}\frac{1}{\sqrt{T}}\int_0^T \langle (R_2 - R_1) X^{x_0}(t), \Phi\, dB(t)\rangle_V . \qquad (2.36)
\end{aligned}
$$

Analogously as above the first term on the right–hand side converges to zero in probability, since

$$
\lim_{T\to\infty} \frac{1}{4aY_T} = \frac{\operatorname{Tr} Q}{b}, \quad \mathbb{P} - a.s.
$$

by Theorem 2.4 and

$$
\lim_{T\to\infty} \frac{1}{\sqrt{T}}\left(\langle (R_2 - R_1) X^{x_0}(T), X^{x_0}(T)\rangle_V - \langle (R_2 - R_1) x_0, x_0\rangle_V\right) = 0, \quad \text{in } L^1(\Omega)
$$

by Lemma 2.7. If we denote

$$
\begin{aligned}
v_2(T) &= \frac{1}{\sqrt{T}} \int_0^T \langle (R_2 - R_1) X^{x_0}(t), \Phi\, dB(t) \rangle_V \\
&= \frac{1}{\sqrt{T}} \int_0^T \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left\langle \tilde{R}_2 X^{x_0}(t), e_n' \right\rangle_{L^2(D)} d\beta_n(t),
\end{aligned}
$$

then Law $(v_2(T))$ converges weakly to the Gaussian distribution with a zero mean and variance given by $\operatorname{Tr}\left(Q\tilde{R}_2 Q_\infty^{(a,b)} \tilde{R}_2^*\right)$. Since the multiplicative factor $-\frac{1}{2aY_T}$ of $v_2(T)$ on the right–hand side of (2.36) converges to $-\frac{2b}{\operatorname{Tr} Q}$, $\mathbb{P}-a.s.$ as $T \to \infty$, we obtain the result

$$
\operatorname{Law}\left(\sqrt{T}\left(\tilde{b}_T - b\right)\right) \xrightarrow{w^*} N\left(0, \frac{4b^2}{(\operatorname{Tr} Q)^2} \operatorname{Tr}\left(Q\tilde{R}_2 Q_\infty^{(a,b)} \tilde{R}_2^*\right)\right), \quad T \to \infty.
$$

□

Remark 3. It is also possible to specify the variance of the limiting Gaussian distribution of $v_1(T)$ and $v_2(T)$ as

$$
\operatorname{Tr}\left(Q\tilde{R}_1 Q_\infty^{(a,b)} \tilde{R}_1^*\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{2ab(\alpha_n + \alpha_k)}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle Qe_n, e_k \rangle_{L^2(D)}^2, \tag{2.37}
$$

$$
\operatorname{Tr}\left(Q\tilde{R}_2 Q_\infty^{(a,b)} \tilde{R}_2^*\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{16a^3}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle Qe_n, e_k \rangle_{L^2(D)}^2. \tag{2.38}
$$

The family of estimators $(\tilde{a}_T, \tilde{b}_T)$ may be viewed as better than the family of estimators $(\hat{a}_T, \hat{b}_T)$, because their respective limiting variances are smaller, which is stated in the following Theorem.

Theorem 2.12. *1) The limiting variance of $\sqrt{T}\,(\tilde{a}_T - a)$ is smaller than the limiting variance of $\sqrt{T}\,(\hat{a}_T - a)$, i.e.,*

$$
\frac{4a^2}{\operatorname{Tr} Q^2} \operatorname{Tr}\left(Q\tilde{R}_1 Q_\infty^{(a,b)} \tilde{R}_1^*\right) < \frac{4a^2}{\operatorname{Tr} Q^2} \operatorname{Tr}\left(Q\tilde{R} Q_\infty^{(a,b)} \tilde{R}^*\right). \tag{2.39}
$$

2) The limiting variance of $\sqrt{T}\left(\tilde{b}_T - b\right)$ is smaller than the limiting variance of $\sqrt{T}\left(\hat{b}_T - b\right)$, i.e.,

$$
\frac{4b^2}{\operatorname{Tr} Q^2} \operatorname{Tr}\left(Q\tilde{R}_2 Q_\infty^{(a,b)} \tilde{R}_2^*\right) < \frac{4b^2(b+1)^2}{\operatorname{Tr} Q^2} \operatorname{Tr}\left(Q\tilde{R} Q_\infty^{(a,b)} \tilde{R}^*\right). \tag{2.40}
$$

Proof. By Remarks 2 and 3, $\operatorname{Tr}\left(Q\tilde{R} Q_\infty^{(a,b)} \tilde{R}^*\right)$ equals to

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{16a^3 + 2ab(b+1)^2(\alpha_n + \alpha_k)}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \frac{1}{(b+1)^2} \langle Qe_n, e_k \rangle_{L^2(D)}^2 \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{16a^3}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \frac{1}{(b+1)^2} \langle Qe_n, e_k \rangle_{L^2(D)}^2 \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{2ab(\alpha_n + \alpha_k)}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \langle Qe_n, e_k \rangle_{L^2(D)}^2 \\
&= \frac{1}{(b+1)^2} \operatorname{Tr}\left(Q\tilde{R}_2 Q_\infty^{(a,b)} \tilde{R}_2^*\right) + \operatorname{Tr}\left(Q\tilde{R}_1 Q_\infty^{(a,b)} \tilde{R}_1^*\right).
\end{aligned}
$$

Since both above terms are positive, (2.39) and (2.40) follow. □

Remark 4. If we consider so–called "diagonal case", i.e, $Qe_n = \lambda_n e_n$ for orthonormal basis $\{e_n, n \in \mathbb{N}\}$ in $L^2(D)$, many of the previous formulae may be considerably simplified. The covariance operator $Q_\infty^{(a,b)}$ from Theorem 1.15 will take the form

$$Q_\infty^{(a,b)} = \begin{pmatrix} \frac{1}{4ab} Q & 0 \\ 0 & \frac{1}{4a} Q \end{pmatrix}, \tag{2.41}$$

with the same trace given by Lemma 2.2,

$$\operatorname{Tr} Q_\infty^{(a,b)} = \frac{1}{4ab} \operatorname{Tr} Q + \frac{1}{4a} \operatorname{Tr} Q = \frac{b+1}{4ab} \operatorname{Tr} Q.$$

Also the limiting variances of Gaussian distributions in Theorems 2.8 and 2.11 may be further specified as

$$\operatorname{Law}\left(\sqrt{T}\,(\hat{a}_T - a)\right) \xrightarrow{w^*} N\left(0,\ \frac{1}{(\operatorname{Tr} Q)^2}\left(\frac{4a^3}{b(b+1)^2} \operatorname{Tr}\left(Q^2(-A)^{-1}\right) + a \operatorname{Tr} Q^2\right)\right),$$

$$\operatorname{Law}\left(\sqrt{T}\left(\hat{b}_T - b\right)\right) \xrightarrow{w^*}$$

$$N\left(0,\ \frac{1}{(\operatorname{Tr} Q)^2}\left(4ab \operatorname{Tr}\left(Q^2(-A)^{-1}\right) + \frac{b^2(b+1)^2}{a} \operatorname{Tr} Q^2\right)\right),$$

$$\operatorname{Law}\left(\sqrt{T}\,(\tilde{a}_T - a)\right) \xrightarrow{w^*} N\left(0,\ a\,\frac{\operatorname{Tr} Q^2}{(\operatorname{Tr} Q)^2}\right),$$

$$\operatorname{Law}\left(\sqrt{T}\left(\tilde{b}_T - b\right)\right) \xrightarrow{w^*} N\left(0,\ 4ab\,\frac{\operatorname{Tr}(Q^2(-A)^{-1})}{(\operatorname{Tr} Q)^2}\right),$$

as $T \to \infty$.

These results are illustrated by simulations in Section 4.1.

3. Statistical inference based on observation window

3.1 Parameter estimation

Consider the stochastic differential equation (1.12) with unknown parameters $a > 0$, $b > 0$. Unlike in the previous chapter, where the estimators were dependent on the information provided by the norm of the solution (or the norms of the individual components), our new aim is to propose strongly consistent estimators of parameters a and b based on observation of the trajectory through some "observation window" $z \in V$.

More specifically, let $0 \neq z \in V$ be arbitrary and suppose that we are able to track the trajectory of the process $(\langle X^{x_0}(t), z\rangle_V, 0 \leq t \leq T)$ up to time T.

Since the linear differential equation (1.12) has a unique invariant measure $\mu_\infty^{(a,b)}$, we may also use Theorem 2.1 in an appropriate setup.

Let $z \in V$ be arbitrary. Using a functional $\varrho : V \to \mathbb{R}$, $\varrho(y) = \langle y, z\rangle_V^2$, $y \in V$, all the conditions of Theorem 2.1 are satisfied with $m = 1$ and

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} \int_0^T \varrho\left(X^{x_0}(t)\right) dt &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle X^{x_0}(t), z\rangle_V^2 \; dt \\
&= \int_V \langle y, z\rangle_V^2 \; \mu_\infty^{(a,b)}(dy) \\
&= \left\langle Q_\infty^{(a,b)} z, z\right\rangle_V
\end{aligned} \tag{3.1}$$

holds $\mathbb{P} - a.s.$

Since the formula (1.23) for the covariance operator $Q_\infty^{(a,b)}$ from Theorem 1.15 does not allow to derive the formulae for unknown parameters explicitly, we will restrict ourselves only to diagonal case. In the rest of this chapter, we assume that the covariance operator $Q_\infty^{(a,b)}$ takes the form from Remark 4, i.e.,

$$Q_\infty^{(a,b)} = \begin{pmatrix} \frac{1}{4ab} Q & 0 \\ 0 & \frac{1}{4a} Q \end{pmatrix}.$$

Using this simpler formula and the convergence above, some strongly consistent estimators of parameters a and b may be proposed.

Theorem 3.1. *Let $z \in V$ be arbitrary, $z = (z_1, z_2)^\top$, where $z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$, $z_2 \in L^2(D)$. Define*

$$J_T = \frac{1}{T} \int_0^T \langle X^{x_0}(t), z\rangle_V^2 \; dt. \tag{3.2}$$

1) If $z \neq 0$ then the process

$$\bar{a}_T = \frac{1}{4bJ_T} \langle Qz_1, z_1\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \frac{1}{4J_T} \langle Qz_2, z_2\rangle_{L^2(D)} \tag{3.3}$$

is strongly consistent estimator of the parameter a.

2) If $z_1 \neq 0$ then the process

$$\bar{b}_T = \frac{\langle Qz_1, z_1\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}}{4aJ_T - \langle Qz_2, z_2\rangle_{L^2(D)}} \tag{3.4}$$

is strongly consistent estimator of the parameter b.

Proof. From (3.1) and (2.41) it follows that

$$\lim_{T\to\infty} J_T = \left\langle Q_\infty^{(a,b)} z, z \right\rangle_V = \left\langle \begin{pmatrix} \frac{1}{4ab}Q & 0 \\ 0 & \frac{1}{4a}Q \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle_V$$
$$= \frac{1}{4ab} \left\langle Qz_1, z_1 \right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \frac{1}{4a} \left\langle Qz_2, z_2 \right\rangle_{L^2(D)}, \quad \mathbb{P} - a.s.$$

Hence we obtain the desired convergence $\bar{a}_T \to a$, $\mathbb{P} - a.s.$ as $T \to \infty$ unless $z = 0$. Similarly, if $z_1 \neq 0$ then we obtain the convergence $\bar{b}_T \to b$, $\mathbb{P} - a.s.$ as $T \to \infty$. ☐

The estimators $\bar{a}_T$ and $\bar{b}_T$ have similar disadvantage as the family of estimators $(\hat{a}_T, \hat{b}_T)$ introduced in Chapter 2: In order to compute the estimator $\bar{a}_T$, we have to know the true value of the other parameter b (and vice versa for the estimator $\bar{b}_T$). This problem may be overcome by a more specific choice of the "observation window". Therefore, consider the following special cases or estimation strategies:

1. If $z = (0, z_2)^\top$, i.e., $z_1 = 0$, $z_2 \neq 0$, then

$$\bar{a}_T = \frac{\left\langle Qz_2, z_2 \right\rangle_{L^2(D)}}{\frac{4}{T}\int_0^T \left\langle X_2^{x_0}(t), z_2 \right\rangle_{L^2(D)}^2 dt},$$

where $X^{x_0}(t) = (X_1^{x_0}(t), X_2^{x_0}(t))^\top \in V$ is the solution to the equation (1.12). In order to make such an estimator, only the observation of the second component of the solution is needed.

2. If $z = (z_1, 0)^\top$, i.e., $z_2 = 0$, $z_1 \neq 0$, then we have that

$$\lim_{T\to\infty} J_T = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle X_1^{x_0}(t), z_1 \right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}^2 dt = \frac{1}{4ab} \left\langle Qz_1, z_1 \right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}$$

and it is possible to estimate either the product ab, or one of the parameters if the true value of the other one is known. (In this case the formulae (3.3) and (3.4) actually coincide.)

3. It is possible to combine the two previous strategies together. First, using the "window" $z = (0, z_2)^\top$, $z_2 \neq 0$, we get an estimator of a, that is

$$\bar{a}_T = \frac{\left\langle Qz_2, z_2 \right\rangle_{L^2(D)}}{\frac{4}{T}\int_0^T \left\langle X_2^{x_0}(t), z_2 \right\rangle_{L^2(D)}^2 dt}$$

and then, using the "window" $z = (z_1, 0)^\top$, $z_1 \neq 0$, we get an estimator of b, that is

$$\bar{b}_T = \frac{\left\langle Qz_1, z_1 \right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}}{\left\langle Qz_2, z_2 \right\rangle_{L^2(D)}} \cdot \frac{\frac{1}{T}\int_0^T \left\langle X_2^{x_0}(t), z_2 \right\rangle_{L^2(D)}^2 dt}{\frac{1}{T}\int_0^T \left\langle X_1^{x_0}(t), z_1 \right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}^2 dt}.$$

4. It is also possible to generalize the previous procedure further. First, using the "window" $z = (0, z_2)^\top$, $z_2 \neq 0$, we get an estimator of a, that is

$$\bar{a}_T = \frac{\langle Q z_2, z_2 \rangle_{L^2(D)}}{\frac{4}{T} \int_0^T \langle X_2^{x_0}(t), z_2 \rangle_{L^2(D)}^2 \, dt}$$

and then, using any "window" $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)^\top$, $\tilde{z}_1 \neq 0$, we get an estimator of b, that is

$$\begin{aligned} \bar{b}_T &= \frac{\langle Q \tilde{z}_1, \tilde{z}_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}}{4 \bar{a}_T \frac{1}{T} \int_0^T \langle X^{x_0}(t), \tilde{z} \rangle_V^2 \, dt - \langle Q \tilde{z}_2, \tilde{z}_2 \rangle_{L^2(D)}} \\ &= \frac{\langle Q \tilde{z}_1, \tilde{z}_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}}{\langle Q z_2, z_2 \rangle_{L^2(D)} \frac{\frac{1}{T} \int_0^T \langle X^{x_0}(t), \tilde{z} \rangle_V^2 \, dt}{\frac{1}{T} \int_0^T \langle X_2^{x_0}(t), z_2 \rangle_{L^2(D)}^2 \, dt} - \langle Q \tilde{z}_2, \tilde{z}_2 \rangle_{L^2(D)}} \\ &= \frac{\langle Q \tilde{z}_1, \tilde{z}_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \int_0^T \langle X_2^{x_0}(t), z_2 \rangle_{L^2(D)}^2 \, dt}{\langle Q z_2, z_2 \rangle_{L^2(D)} \int_0^T \langle X^{x_0}(t), \tilde{z} \rangle_V^2 \, dt - \langle Q \tilde{z}_2, \tilde{z}_2 \rangle_{L^2(D)} \int_0^T \langle X_2^{x_0}(t), z_2 \rangle_{L^2(D)}^2 \, dt}. \end{aligned}$$

For the practical reasons there is an incentive to observe the solution to the equation (1.12) through the "observation window" componentwise (i.e., we observe the processes $(\langle X_1^{x_0}, z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}, 0 \leq t \leq T)$ and $(\langle X_2^{x_0}, z_2 \rangle_{L^2(D)}, 0 \leq t \leq T)$ for any given $0 \neq z_1 \in V$, $0 \neq z_2 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ separately), so we will prefer the strategy 3. Let us introduce these estimators once again with the new notation.

Corollary 3.2. *1) Let $0 \neq z_2 \in L^2(D)$ be arbitrary. The process*

$$\bar{a}_{T,z_2} = \frac{\langle Q z_2, z_2 \rangle_{L^2(D)}}{\frac{4}{T} \int_0^T \langle X_2^{x_0}(t), z_2 \rangle_{L^2(D)}^2 \, dt} \tag{3.5}$$

is strongly consistent estimator of the parameter a.

2) Moreover, let $0 \neq z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ be arbitrary. The process

$$\bar{b}_{T,z_1,z_2} = \frac{\langle Q z_1, z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}}{\langle Q z_2, z_2 \rangle_{L^2(D)}} \cdot \frac{\frac{1}{T} \int_0^T \langle X_2^{x_0}(t), z_2 \rangle_{L^2(D)}^2 \, dt}{\frac{1}{T} \int_0^T \langle X_1^{x_0}(t), z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}^2 \, dt} \tag{3.6}$$

is strongly consistent estimator of the parameter b.

Proof. It is the direct consequence of Theorem 3.1. □

The previous estimators may be specified even further if the "observation window" is the element of the orthonormal basis.

Indeed, if $z_2 = e_k \in L^2(D)$ for any $k \in \mathbb{N}$ then $\bar{a}_{T,z_2}$ takes the form

$$\bar{a}_{T,k} := \bar{a}_{T,e_k} = \frac{\lambda_k}{\frac{4}{T} \int_0^T \langle X_2^{x_0}(t), e_k \rangle_{L^2(D)}^2 \, dt}. \tag{3.7}$$

Moreover, if $z_1 = f_j \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ for any $j \in \mathbb{N}$ then $\bar{b}_{T,z_1,z_2}$ takes the form

$$\bar{b}_{T,j,k} := \bar{b}_{T,f_j,e_k} = \frac{\lambda_j}{\lambda_k} \cdot \frac{\frac{1}{T}\int_0^T \langle X_2^{x_0}(t), e_k\rangle^2_{L^2(D)}\, dt}{\frac{1}{T}\, \langle X_1^{x_0}(t), f_j\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt}. \tag{3.8}$$

These estimators are using only observation of some given modes in the expansion of the solution.

3.2 Asymptotic normality of the estimators

3.2.1 Asymptotic normality of the estimator $\bar{a}_{T,z_2}$

In this part we show asymptotic normality of the estimator (3.5), i.e., the weak convergence of $\sqrt{T}(\bar{a}_{T,z_2} - a)$ to a Gaussian distribution.

Let $k \in \mathbb{N}$ be arbitrary and define the operator $E_k : V \to V$ by

$$E_k x = E_k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} E_{k,1} & E_{k,2} \\ E_{k,3} & E_{k,4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V,$$

where $E_{k,2} = 0$, $E_{k,3} = 0$ and

$$\begin{aligned} E_{k,1} &: x_1 \longmapsto b\,\langle x_1, e_k\rangle_{L^2(D)}\, e_k, \\ E_{k,4} &: x_2 \longmapsto \langle x_2, e_k\rangle_{L^2(D)}\, e_k, \end{aligned}$$

for any $x_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ and $x_2 \in L^2(D)$. Hence the operator E_k evaluates as

$$E_k x = \begin{pmatrix} b\,\langle x_1, e_k\rangle_{L^2(D)}\, e_k \\ \langle x_2, e_k\rangle_{L^2(D)}\, e_k \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V.$$

Let $k, l \in \mathbb{N}$ be arbitrary and define the operator $E_{k,l} : V \to V$ by

$$E_{k,l} x = E_{k,l} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{D_{k,l}} \begin{pmatrix} E_{k,l,1} & E_{k,l,2} \\ E_{k,l,3} & E_{k,l,4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V,$$

where

$$\begin{aligned} E_{k,l,1} &: x_1 \longmapsto 16a^2 b\alpha_l\,\langle x_1, e_l\rangle_{L^2(D)}\, e_k + 16a^2 b\alpha_k\,\langle x_1, e_k\rangle_{L^2(D)}\, e_l, \\ E_{k,l,2} &: x_2 \longmapsto 4ab(\alpha_k - \alpha_l)\,\langle x_2, e_k\rangle_{L^2(D)}\, e_l + 4ab(\alpha_l - \alpha_k)\,\langle x_2, e_l\rangle_{L^2(D)}\, e_k, \\ E_{k,l,3} &: x_1 \longmapsto 4ab\alpha_l(\alpha_k - \alpha_l)\,\langle x_1, e_l\rangle_{L^2(D)}\, e_k + 4ab\alpha_k(\alpha_l - \alpha_k)\,\langle x_1, e_k\rangle_{L^2(D)}\, e_l, \\ E_{k,l,4} &: x_2 \longmapsto 8a^2(\alpha_k + \alpha_l)\,\langle x_2, e_k\rangle_{L^2(D)}\, e_l + 8a^2(\alpha_k + \alpha_l)\,\langle x_2, e_l\rangle_{L^2(D)}\, e_k, \end{aligned}$$

for any $x_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$, $x_2 \in L^2(D)$ and $D_{k,l}$ defined by

$$D_{k,l} = b(\alpha_k - \alpha_l)^2 + 8a^2(\alpha_k + \alpha_l).$$

Note that $D_{k,l}$ is the denominator from the formula (1.23) divided by b and that $D_{k,l} = D_{l,k}$. Also note that $D_{k,k} = 16a^2\alpha_k$.

The properties of the operators E_k and $E_{k,l}$ are summarized in the following Lemma.

Lemma 3.3. *1) The operator $E_k \in \mathcal{L}(V)$ is self–adjoint for any given $k \in \mathbb{N}$. Moreover,*

$$\langle E_k x, \mathcal{A}x \rangle_V = -2a \langle x_2, e_k \rangle^2_{L^2(D)}, \tag{3.9}$$

for any $x = (x_1, x_2)^\top \in \mathrm{Dom}(\mathcal{A})$.

2) The operator $E_{k,l} \in \mathcal{L}(V)$ is self–adjoint for any given $k, l \in \mathbb{N}$. Moreover,

$$\langle E_{k,l} x, \mathcal{A}x \rangle_V = -4a \langle x_2, e_k \rangle_{L^2(D)} \langle x_2, e_l \rangle_{L^2(D)}, \tag{3.10}$$

for any $x = (x_1, x_2)^\top \in \mathrm{Dom}(\mathcal{A})$.

Proof. 1) Let $k \in \mathbb{N}$ be arbitrary. It is evident that $E_k \in \mathcal{L}(V)$ and for any $x = (x_1, x_2)^\top \in V$ and $y = (y_1, y_2)^\top \in V$ we have

$$\begin{aligned}
\langle E_k x, y \rangle_V &= \left\langle \begin{pmatrix} b \langle x_1, e_k \rangle_{L^2(D)} e_k \\ \langle x_2, e_k \rangle_{L^2(D)} e_k \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_V \\
&= b \langle x_1, e_k \rangle_{L^2(D)} \langle y_1, e_k \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \langle x_2, e_k \rangle_{L^2(D)} \langle y_2, e_k \rangle_{L^2(D)} \\
&= b\alpha_k \langle x_1, e_k \rangle_{L^2(D)} \langle y_1, e_k \rangle_{L^2(D)} + \langle x_2, e_k \rangle_{L^2(D)} \langle y_2, e_k \rangle_{L^2(D)} \\
&= \langle x, E_k y \rangle_V,
\end{aligned}$$

hence $E_k = E_k^*$. Moreover, for any $x = (x_1, x_2)^\top \in \mathrm{Dom}(\mathcal{A})$ we have

$$\begin{aligned}
\langle E_k x, \mathcal{A}x \rangle_V &= \left\langle \begin{pmatrix} b \langle x_1, e_k \rangle_{L^2(D)} e_k \\ \langle x_2, e_k \rangle_{L^2(D)} e_k \end{pmatrix}, \begin{pmatrix} x_2 \\ bAx_1 - 2ax_2 \end{pmatrix} \right\rangle_V \\
&= b \langle x_1, e_k \rangle_{L^2(D)} \left\langle (-A)^{\frac{1}{2}} x_2, (-A)^{\frac{1}{2}} e_k \right\rangle_{L^2(D)} \\
&\quad + b \langle x_2, e_k \rangle_{L^2(D)} \langle Ax_1, e_k \rangle_{L^2(D)} - 2a \langle x_2, e_k \rangle^2_{L^2(D)} \\
&= -2a \langle x_2, e_k \rangle^2_{L^2(D)}.
\end{aligned}$$

2) Let $k, l \in \mathbb{N}$ be arbitrary. It is evident that $E_{k,l} \in \mathcal{L}(V)$ and similarly as above it is possible to verify that $E_{k,l} = E_{k,l}^*$ and that (3.10) holds true for any $x \in \mathrm{Dom}(\mathcal{A})$. ☐

Choose $0 \neq z_2 \in L^2(D)$ taking the form

$$z_2 = \sum_{k=1}^{\infty} \langle z_2, e_k \rangle_{L^2(D)} e_k = \sum_{k=1}^{\infty} z_{2,k} e_k,$$

i.e., $\{z_{2,k}, k \in \mathbb{N}\}$ is the set of coordinates of the element z_2 with respect to the orthonormal basis in $L^2(D)$. Finally, define the operator $E : V \to V$ by

$$E = \sum_{k=1}^{\infty} z_{2,k}^2 E_k + \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} z_{2,k} z_{2,l} E_{k,l}. \tag{3.11}$$

The properties of the operator E needed in the sequel are summarized in the following Lemma.

Lemma 3.4. *The operator $E \in \mathcal{L}(V)$. Moreover, it is self–adjoint and*

$$\langle Ex, \mathcal{A}x \rangle_V = -2a \langle x_2, z_2 \rangle^2_{L^2(D)}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathrm{Dom}(\mathcal{A}). \tag{3.12}$$

Proof. There exists a positive constant $C > 0$ (which does not depend on k) such that $\|E_k\|_{\mathcal{L}(V)} \le C$ for any $k \in \mathbb{N}$. Hence

$$\left\|\sum_{k=1}^{\infty} z_{2,k}^2 E_k\right\|_{\mathcal{L}(V)} \le \sum_{k=1}^{\infty} z_{2,k}^2 \|E_k\|_{\mathcal{L}(V)} \le C \sum_{k=1}^{\infty} z_{2,k}^2 = C\|z_2\|^2_{L^2(D)} < \infty$$

and the operator defined by the first sum in (3.11) belongs to the space $\mathcal{L}(V)$.

The convergence of the double series is fulfilled by the denominator $D_{k,l}$. For any $k, l \in \mathbb{N}$ we have

$$\frac{\alpha_k + \alpha_l}{D_{k,l}} \le \frac{\alpha_k + \alpha_l}{8a^2(\alpha_k + \alpha_l)} = \frac{1}{8a},$$

$$\begin{aligned}
\frac{\sqrt{\alpha_k}|\alpha_k - \alpha_l|}{D_{k,l}} &= \sqrt{\frac{\alpha_k}{D_{k,l}}}\frac{|\alpha_k - \alpha_l|}{\sqrt{D_{k,l}}} < \frac{1}{\sqrt{8a^2}} \frac{|\alpha_k - \alpha_l|}{\sqrt{b(\alpha_k - \alpha_l)^2 + 8a^2(\alpha_k + \alpha_l)}} \\
&< \frac{1}{\sqrt{8a^2}} \frac{|\alpha_k - \alpha_l|}{\sqrt{b}|\alpha_k - \alpha_l|} = \frac{1}{8a^2 b},
\end{aligned}$$

$$\frac{\sqrt{\alpha_k\alpha_l}}{D_{k,l}} < \frac{2\sqrt{\alpha_k\alpha_l}}{D_{k,l}} \le \frac{2\sqrt{\alpha_k\alpha_l}}{8a^2(\alpha_k + \alpha_l)} \le \frac{1}{8a^2}.$$

The desired convergence is then accomplished by the convergence of the series

$$\sum_{k=1}^{\infty} z_{2,k}^2 < \infty, \quad \sum_{k=1}^{\infty} \langle x_2, e_k\rangle^2_{L^2(D)} < \infty, \quad \sum_{k=1}^{\infty} \alpha_k \langle x_1, e_k\rangle^2_{L^2(D)} < \infty.$$

The linear combination of the self–adjoint operators is also the self–adjoint operator, hence $E = E^*$.

The property (3.12) follows by (3.9), (3.10) and the following computation

$$\begin{aligned}
&\langle Ex, \mathcal{A}x\rangle_V = \\
&= \left\langle \left(\sum_{k=1}^{\infty} z_{2,k}^2 E_k + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} z_{2,k}z_{2,l}E_{k,l}\right) x, \mathcal{A}x\right\rangle_V \\
&= \sum_{k=1}^{\infty} z_{2,k}^2 \langle E_k x, \mathcal{A}x\rangle_V + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} z_{2,k}z_{2,l} \langle E_{k,l}x, \mathcal{A}x\rangle_V \\
&= -2a\left(\sum_{k=1}^{\infty} z_{2,k}^2 \langle x_2, e_k\rangle^2_{L^2(D)} + 2\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} z_{2,k}z_{2,l} \langle x_2, e_k\rangle_{L^2(D)} \langle x_2, e_l\rangle_{L^2(D)}\right) \\
&= -2a\left(\sum_{k=1}^{\infty} z_{2,k} \langle x_2, e_k\rangle_{L^2(D)}\right)^2 \\
&= -2a\left\langle x_2, \sum_{k=1}^{\infty} z_{2,k}e_k\right\rangle^2_{L^2(D)} \\
&= -2a \langle x_2, z_2\rangle^2_{L^2(D)}.
\end{aligned}$$

□

We will need an alternative representation for the process $\frac{1}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle^2_{L^2(D)}\, dt$.

Proposition 3.5. *The process* $\frac{1}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle^2_{L^2(D)}\, dt$ *admits the following representation*

$$\begin{aligned}
&\frac{1}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle^2_{L^2(D)}\, dt = \\
&= -\frac{1}{4aT}\left(\langle EX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Ex_0, x_0\rangle_V\right) \\
&\quad + \frac{1}{2aT}\int_0^T \langle EX^{x_0}(t), \Phi\, dB(t)\rangle_V + \frac{1}{4a}\langle Qz_2, z_2\rangle_{L^2(D)}\,. \qquad (3.13)
\end{aligned}$$

Proof. Define the function $g_3 : V \to \mathbb{R}$ by

$$g_3(x) = \langle Ex, x\rangle_V\,, \quad \forall x \in V. \qquad (3.14)$$

The application of Itô's formula to the function $g_3(X^{x_0,N}(t))$ (we also have to use suitable projections, see the proof of Proposition 2.6), yields

$$dg_3(X^{x_0,N}(t)) = 2\left\langle EX^{x_0,N}(t), dX^{x_0,N}(t)\right\rangle_V + \frac{1}{2}\operatorname{Tr}(2E\Phi\Phi^*)\, dt. \qquad (3.15)$$

First, we simplify the second term by the following calculation

$$\begin{aligned}
\operatorname{Tr}(E\Phi\Phi^*) &= \operatorname{Tr}\left(\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & Q\end{pmatrix}\right) = \operatorname{Tr}\begin{pmatrix} 0 & E_2Q \\ 0 & E_4Q\end{pmatrix} = \operatorname{Tr}(E_4Q) \\
&= \operatorname{Tr}\left(\sum_{k=1}^{\infty} z_{2,k}^2 E_{k,4}Q + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\frac{z_{2,k}z_{2,l}}{D_{k,l}}E_{k,l,4}Q\right) \\
&= \sum_{k=1}^{\infty} z_{2,k}^2\operatorname{Tr}(E_{k,4}Q) + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty}\frac{z_{2,k}z_{2,l}}{D_{k,l}}\operatorname{Tr}(E_{k,l,4}Q).
\end{aligned}$$

Now we compute the partial traces $\operatorname{Tr}(E_{k,4}Q)$ and $\operatorname{Tr}(E_{k,l,4}Q)$. According to the definition of the trace

$$\begin{aligned}
\operatorname{Tr}(E_{k,4}Q) &= \sum_{j=1}^{\infty}\langle E_{k,4}Qe_j, e_j\rangle_{L^2(D)} = \sum_{j=1}^{\infty}\lambda_j\langle E_{k,4}e_j, e_j\rangle_{L^2(D)} \\
&= \sum_{j=1}^{\infty}\lambda_j\langle e_j, e_k\rangle^2_{L^2(D)} = \sum_{j=1}^{\infty}\lambda_j\delta_{j,k} = \lambda_k.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\operatorname{Tr}(E_{k,l,4}Q) &= \sum_{j=1}^{\infty}\langle E_{k,l,4}Qe_j, e_j\rangle_{L^2(D)} \\
&= \sum_{j=1}^{\infty}\lambda_j\Big(8a^2(\alpha_k+\alpha_l)\langle e_j, e_k\rangle_{L^2(D)}\langle e_l, e_j\rangle_{L^2(D)} \\
&\quad +8a^2(\alpha_k+\alpha_l)\langle e_j, e_l\rangle_{L^2(D)}\langle e_k, e_j\rangle_{L^2(D)}\Big) \\
&= \sum_{j=1}^{\infty}16a^2\lambda_j(\alpha_k+\alpha_l)\delta_{j,k}\delta_{j,l} = 0,
\end{aligned}$$

since $k \neq l$ in (3.11). Therefore

$$\operatorname{Tr}(E\Phi\Phi^*) = \sum_{k=1}^{\infty}\lambda_k z_{2,k}^2 = \langle Qz_2, z_2\rangle_{L^2(D)}\,.$$

Using this formulae and Lemma 3.4, the expression (3.15) implies

$$\begin{aligned}
dg_3(X^{x_0,N}(t)) &= 2\left\langle EX^{x_0,N}(t), \mathcal{A}X^{x_0,N}(t)\right\rangle_V \, dt + 2\left\langle EX^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V \\
&\quad + \langle Qz_2, z_2\rangle_{L^2(D)} \, dt \\
&= -4a\left\langle X_2^{x_0,N}(t), z_2\right\rangle_{L^2(D)}^2 \, dt + 2\left\langle EX^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V \\
&\quad + \langle Qz_2, z_2\rangle_{L^2(D)} \, dt.
\end{aligned}$$

Integrating previous formula over the interval $(0,T)$, we arrive at

$$\begin{aligned}
&\frac{1}{T}\int_0^T \left\langle X_2^{x_0,N}(t), z_2\right\rangle_{L^2(D)}^2 \, dt = \\
&= -\frac{1}{4aT}\left(\left\langle EX^{x_0,N}(T), X^{x_0,N}(T)\right\rangle_V - \left\langle Ex_0^N, x_0^N\right\rangle_V\right) \\
&\quad + \frac{1}{2aT}\int_0^T \left\langle EX^{x_0,N}(t), \Phi\, dB(t)\right\rangle_V + \frac{1}{4a}\langle Qz_2, z_2\rangle_{L^2(D)} . \qquad (3.16)
\end{aligned}$$

Since

$$\left|\left\langle X_2^{x_0,N}(t), z_2\right\rangle_{L^2(D)}\right| \le \|X_2^{x_0}(t)\|_{L^2(D)} \|z_2\|_{L^2(D)}, \quad \forall t \ge 0, \quad \forall N \in \mathbb{N},$$

we may use the random variable $\|X_2^{x_0}(t)\|_{L^2(D)}^2$ as an integrable majorant for the integral on the left–hand side. For the convergence of the stochastic integral on the right–hand side, see the proof of Lemma 2.6. (3.13) is obtained by passing N to infinity in (3.16). □

We will also need the following Lemma for convergence of some cross terms to zero.

Lemma 3.6. *1) Let $z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ and $z_2 \in L^2(D)$ be arbitrary. Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \langle X_2^{x_0}(t), z_2\rangle_{L^2(D)} \, dt = 0, \quad \mathbb{P}-a.s.$$

2) Let $f_k, f_l \in \mathrm{Dom}((-A)^{\frac{1}{2}})$, $k \neq l$, be arbitrary. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle X_1^{x_0}(t), f_k\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \langle X_1^{x_0}(t), f_l\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \, dt = 0, \quad \mathbb{P}-a.s.$$

3) Let $e_k, e_l \in L^2(D)$, $k \neq l$, be arbitrary. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle X_2^{x_0}(t), e_k\rangle_{L^2(D)} \langle X_2^{x_0}(t), e_l\rangle_{L^2(D)} \, dt = 0, \quad \mathbb{P}-a.s.$$

Proof. 1) Let $z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ and $z_2 \in L^2(D)$. Using Theorem 2.1 for a functional $\varrho : V \to \mathbb{R}$, $\varrho(y) = \left\langle y, \begin{pmatrix} z_1 \\ 0 \end{pmatrix}\right\rangle_V \left\langle y, \begin{pmatrix} 0 \\ z_2 \end{pmatrix}\right\rangle_V$, $y \in V$ and $m = 1$, we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \langle X_2^{x_0}(t), z_2\rangle_{L^2(D)} \, dt =$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle X^{x_0}(t), \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \right\rangle_V \left\langle X^{x_0}(t), \begin{pmatrix} 0 \\ z_2 \end{pmatrix} \right\rangle_V dt \\
&= \int_V \left\langle y, \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \right\rangle_V \left\langle y, \begin{pmatrix} 0 \\ z_2 \end{pmatrix} \right\rangle_V \mu_\infty^{(a,b)}(dy) \\
&= \left\langle Q_\infty^{(a,b)} \begin{pmatrix} z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2 \end{pmatrix} \right\rangle_V \\
&= \left\langle \begin{pmatrix} \frac{1}{4ab}Q & 0 \\ 0 & \frac{1}{4a}Q \end{pmatrix} \begin{pmatrix} z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2 \end{pmatrix} \right\rangle_V = 0, \quad \mathbb{P}-a.s.
\end{aligned}
$$

2) Let $f_k, f_l \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ with $k \neq l$. Using Theorem 2.1 for a functional $\varrho : V \to \mathbb{R}$, $\varrho(y) = \left\langle y, \begin{pmatrix} f_k \\ 0 \end{pmatrix} \right\rangle_V \left\langle y, \begin{pmatrix} f_l \\ 0 \end{pmatrix} \right\rangle_V$, $y \in V$ and $m = 1$, we get

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle X_1^{x_0}(t), f_k\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \langle X_1^{x_0}(t), f_l\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt = \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle X^{x_0}(t), \begin{pmatrix} f_k \\ 0 \end{pmatrix} \right\rangle_V \left\langle X^{x_0}(t), \begin{pmatrix} f_l \\ 0 \end{pmatrix} \right\rangle_V dt \\
&= \left\langle \begin{pmatrix} \frac{1}{4ab}Q & 0 \\ 0 & \frac{1}{4a}Q \end{pmatrix} \begin{pmatrix} f_k \\ 0 \end{pmatrix}, \begin{pmatrix} f_l \\ 0 \end{pmatrix} \right\rangle_V \\
&= \frac{1}{4ab} \langle Q f_k, f_l \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \\
&= \frac{\lambda_k}{4ab}\delta_{k,l} = 0, \quad \mathbb{P}-a.s.
\end{aligned}
$$

3) Let $e_k, e_l \in L^2(D)$ with $k \neq l$. The proof is analogous to the previous one with functional $\varrho : V \to \mathbb{R}$, $\varrho(y) = \left\langle y, \begin{pmatrix} 0 \\ e_k \end{pmatrix} \right\rangle_V \left\langle y, \begin{pmatrix} 0 \\ e_l \end{pmatrix} \right\rangle_V$, $y \in V$. □

Asymptotic normality of the estimator $\bar{a}_{T,z_2}$ is formulated in the following Theorem.

Theorem 3.7. *Let $0 \neq z_2 \in L^2(D)$ be arbitrary. The estimator $\bar{a}_{T,z_2}$ is asymptotically normal, i.e.,*

$$
\begin{aligned}
&\mathrm{Law}\left(\sqrt{T}\,(\bar{a}_{T,z_2} - a)\right) \xrightarrow{w^*} \\
&N\left(0, \frac{8a^3}{\langle Qz_2, z_2\rangle^2_{L^2(D)}} \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{\lambda_k \lambda_n (\alpha_k + \alpha_n) z_{2,k}^2 z_{2,n}^2}{b(\alpha_k - \alpha_n)^2 + 8a^2(\alpha_k + \alpha_n)}\right), \quad T \to \infty.
\end{aligned}
$$

Proof. Set $0 \neq z_2 \in L^2(D)$. If we use formula (3.5) for the estimator $\bar{a}_{T,z_2}$ and Proposition 3.5, we obtain

$$
\begin{aligned}
&\sqrt{T}\,(\bar{a}_{T,z_2} - a) = \\
&= \frac{\sqrt{T}}{\frac{4}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle^2_{L^2(D)}\, dt} \left(\langle Qz_2, z_2\rangle_{L^2(D)} - 4a\frac{1}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle^2_{L^2(D)}\, dt \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{T}}{\frac{4}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle_{L^2(D)}^2 \, dt} \Bigg(\frac{1}{T} \left(\langle EX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Ex_0, x_0\rangle_V\right) \\
&\quad - \frac{2}{T}\int_0^T \langle EX^{x_0}(t), \Phi\, dB(t)\rangle_V \Bigg) \\
&= \frac{1}{\frac{4}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle_{L^2(D)}^2 \, dt} \frac{1}{\sqrt{T}} \left(\langle EX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Ex_0, x_0\rangle_V\right) \\
&\quad - \frac{1}{\frac{2}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle_{L^2(D)}^2 \, dt} \frac{1}{\sqrt{T}} \int_0^T \langle EX^{x_0}(t), \Phi\, dB(t)\rangle_V . \qquad (3.17)
\end{aligned}
$$

The first term on the right–hand side converges to zero in probability as $T \to \infty$, since

$$
\lim_{T\to\infty} \frac{1}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle_{L^2(D)}^2 \, dt = \frac{1}{4a} \langle Qz_2, z_2\rangle_{L^2(D)}, \quad \mathbb{P} - a.s.
$$

by Theorem 3.1 and

$$
\lim_{T\to\infty} \frac{1}{\sqrt{T}} \left(\langle EX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Ex_0, x_0\rangle_V\right) = 0, \quad \text{in } L^1(\Omega)
$$

by Lemma 2.7. Define

$$
\begin{aligned}
w(T) &= \frac{1}{\sqrt{T}}\int_0^T \langle EX^{x_0}(t), \Phi\, dB(t)\rangle_V \\
&= \frac{1}{\sqrt{T}}\int_0^T \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left\langle EX^{x_0}(t), \begin{pmatrix} 0 \\ e_n \end{pmatrix} \right\rangle_V d\beta_n(t).
\end{aligned}
$$

First, let us express scalar product in the above series

$$
\left\langle EX^{x_0}(t), \begin{pmatrix} 0 \\ e_n \end{pmatrix} \right\rangle_V = \langle E_3 X_1^{x_0}(t) + E_4 X_2^{x_0}(t), e_n\rangle_{L^2(D)} .
$$

Next we have

$$
\begin{aligned}
\langle E_3 X_1^{x_0}(t), e_n\rangle_{L^2(D)} &= \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{2,k} z_{2,l}}{D_{k,l}} \langle E_{k,l,3} X_1^{x_0}(t), e_n\rangle_{L^2(D)} \\
&= \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{2,k} z_{2,l}}{D_{k,l}} \Big(4ab\alpha_l(\alpha_k - \alpha_l) \langle X_1^{x_0}(t), e_l\rangle_{L^2(D)} \delta_{k,n} \\
&\quad + 4ab\alpha_k(\alpha_l - \alpha_k) \langle X_1^{x_0}(t), e_k\rangle_{L^2(D)} \delta_{n,l} \Big) \\
&= \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{2,k} z_{2,l}}{D_{k,l}} 4ab\alpha_l(\alpha_k - \alpha_l) \langle X_1^{x_0}(t), e_l\rangle_{L^2(D)} \delta_{k,n} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{2,k} z_{2,l}}{D_{k,l}} 4ab\alpha_k(\alpha_l - \alpha_k) \langle X_1^{x_0}(t), e_k\rangle_{L^2(D)} \delta_{n,l} \\
&= (I) + (II),
\end{aligned}
$$

where we have used the fact that $E_{k,3}=0$. Furthermore, compute

$$
\begin{aligned}
&\langle E_4 X_2^{x_0}(t), e_n\rangle_{L^2(D)} = \\
&= \left\langle \sum_{k=1}^{\infty} z_{2,k}^2 E_{k,4} X_2^{x_0}(t) + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{2,k}z_{2,l}}{D_{k,l}} E_{k,l,4} X_2^{x_0}(t), e_n \right\rangle_{L^2(D)} \\
&= \sum_{k=1}^{\infty} z_{2,k}^2 \left\langle X_2^{x_0}(t), e_k\right\rangle_{L^2(D)} \delta_{k,n} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{2,k}z_{2,l}}{D_{k,l}} 8a^2(\alpha_k+\alpha_l) \left\langle X_2^{x_0}(t), e_k\right\rangle_{L^2(D)} \delta_{n,l} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{2,k}z_{2,l}}{D_{k,l}} 8a^2(\alpha_k+\alpha_l) \left\langle X_2^{x_0}(t), e_l\right\rangle_{L^2(D)} \delta_{k,n} \\
&= (III) + (IV) + (V).
\end{aligned}
$$

By the central limit theorem for martingales, Law $(w(T))$ converges weakly to a Gaussian distribution with a zero mean and variance given by the limit in probability

$$
\lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n \left((I)+\ldots+(V)\right)^2 \, dt.
$$

However, all the limits in the rest of the proof will be considered in the $\mathbb{P}-a.s.$ sense.

The limits of the cross terms are zero by Lemma 3.6. For example

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n (I)(II)\, dt = \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n \left(\sum_{l=n+1}^{\infty} \frac{z_{2,n}z_{2,l}}{D_{n,l}} 4ab\alpha_l(\alpha_n-\alpha_l) \left\langle X_1^{x_0}(t), e_l\right\rangle_{L^2(D)} \right) \\
&\quad \times \left(\sum_{k=1}^{n-1} \frac{z_{2,k}z_{2,n}}{D_{k,n}} 4ab\alpha_k(\alpha_n-\alpha_k) \left\langle X_1^{x_0}(t), e_k\right\rangle_{L^2(D)} \right) dt \\
&= \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty}\sum_{k=1}^{n-1} \lambda_n \frac{z_{2,n}^2 z_{2,l} z_{2,k}}{D_{n,l}D_{k,n}} 16a^2b^2\alpha_l\alpha_k(\alpha_n-\alpha_l)(\alpha_n-\alpha_k) \\
&\quad \times \left(\lim_{T\to\infty} \frac{1}{T}\int_0^T \left\langle X_1^{x_0}(t), e_l\right\rangle_{L^2(D)} \left\langle X_1^{x_0}(t), e_k\right\rangle_{L^2(D)} \, dt \right) = 0,
\end{aligned}
$$

because $k \neq l$, $\left\langle X_1^{x_0}(t), e_k\right\rangle_{L^2(D)} = \frac{1}{\sqrt{\alpha_k}} \left\langle X_1^{x_0}(t), f_k\right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}$ (and similarly with the index l) and we may use Lemma 3.6, part 2).

Also

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n (I)(III)\, dt = \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n \left(\sum_{l=n+1}^{\infty} \frac{z_{2,n}z_{2,l}}{D_{n,l}} 4ab\alpha_l(\alpha_n-\alpha_l) \left\langle X_1^{x_0}(t), e_l\right\rangle_{L^2(D)} \right) \\
&\quad \times \left(z_{2,n}^2 \left\langle X_2^{x_0}(t), e_n\right\rangle_{L^2(D)} \right) dt
\end{aligned}
$$

$$= \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty}\lambda_n\frac{z_{2,n}^3 z_{2,l}}{D_{n,l}}4ab\alpha_l(\alpha_n-\alpha_l)$$
$$\times\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle X_1^{x_0}(t),e_l\rangle_{L^2(D)}\langle X_2^{x_0}(t),e_n\rangle_{L^2(D)}\,dt\right)=0,$$

since $n\neq l$ and the result of the limit follows from Lemma 3.6, part 1). The remaining limits of the cross terms are handled similarly.

Now we compute the limits of the "diagonal" terms.

$$\begin{aligned}(A) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(I)^2\,dt\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n\left(\sum_{l=n+1}^{\infty}\frac{z_{2,n}z_{2,l}}{D_{n,l}}4ab\alpha_l(\alpha_n-\alpha_l)\langle X_1^{x_0}(t),e_l\rangle_{L^2(D)}\right)^2 dt\\ &\overset{(*)}{=} \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty}\lambda_n\frac{z_{2,n}^2 z_{2,l}^2}{D_{n,l}^2}16a^2b^2\alpha_l^2(\alpha_n-\alpha_l)^2\langle X_1^{x_0}(t),e_l\rangle_{L^2(D)}^2\,dt\\ &= \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty}\frac{1}{D_{n,l}^2}4ab\lambda_n\lambda_l\alpha_l(\alpha_n-\alpha_l)^2 z_{2,n}^2 z_{2,l}^2,\end{aligned}$$

since

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle X_1^{x_0}(t),e_l\rangle_{L^2(D)}^2\,dt=\frac{\lambda_l}{4ab\alpha_l}$$

and in the equality $(*)$ we have also used Lemma 3.6 for the cross summands.

In a similar manner, we have

$$\begin{aligned}(B) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(II)^2\,dt\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=1}^{n-1}\frac{z_{2,k}z_{2,n}}{D_{k,n}}4ab\alpha_k(\alpha_n-\alpha_k)\langle X_1^{x_0}(t),e_k\rangle_{L^2(D)}\right)^2 dt\\ &\overset{(*)}{=} \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\lambda_n\frac{z_{2,k}^2 z_{2,n}^2}{D_{k,n}^2}16a^2b^2\alpha_k^2(\alpha_n-\alpha_k)^2\langle X_1^{x_0}(t),e_k\rangle_{L^2(D)}^2\,dt\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{1}{D_{k,n}^2}4ab\lambda_n\lambda_k\alpha_k(\alpha_n-\alpha_k)^2 z_{2,k}^2 z_{2,n}^2,\end{aligned}$$

$$\begin{aligned}(C) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(III)^2\,dt\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n z_{2,n}^4\langle X_2^{x_0}(t),e_n\rangle_{L^2(D)}^2\,dt\\ &= \frac{1}{4a}\sum_{n=1}^{\infty}\lambda_n^2 z_{2,n}^4,\end{aligned}$$

since

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle X_2^{x_0}(t),e_n\rangle_{L^2(D)}^2\,dt=\frac{\lambda_n}{4a}.$$

Next, we have

$$
\begin{aligned}
(D) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n (IV)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n-1} \frac{z_{2,k} z_{2,n}}{D_{k,n}} 8a^2 (\alpha_k + \alpha_n) \left\langle X_2^{x_0}(t), e_k \right\rangle_{L^2(D)} \right)^2 dt \\
&\overset{(*)}{=} \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty}\sum_{k=1}^{n-1} \lambda_n \frac{z_{2,k}^2 z_{2,n}^2}{D_{k,n}^2} 64a^4 (\alpha_k + \alpha_n)^2 \left\langle X_2^{x_0}(t), e_k \right\rangle_{L^2(D)}^2 dt \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{n-1} \frac{1}{D_{k,n}^2} 16a^3 \lambda_n \lambda_k (\alpha_k + \alpha_n)^2 z_{2,k}^2 z_{2,n}^2,
\end{aligned}
$$

$$
\begin{aligned}
(E) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n (V)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n \left(\sum_{l=n+1}^{\infty} \frac{z_{2,n} z_{2,l}}{D_{n,l}} 8a^2 (\alpha_n + \alpha_l) \left\langle X_2^{x_0}(t), e_l \right\rangle_{L^2(D)} \right)^2 dt \\
&\overset{(*)}{=} \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty} \lambda_n \frac{z_{2,n}^2 z_{2,l}^2}{D_{n,l}^2} 64a^4 (\alpha_n + \alpha_l)^2 \left\langle X_2^{x_0}(t), e_l \right\rangle_{L^2(D)}^2 dt \\
&= \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty} \frac{1}{D_{n,l}^2} 16a^3 \lambda_n \lambda_l (\alpha_n + \alpha_l)^2 z_{2,n}^2 z_{2,l}^2.
\end{aligned}
$$

The resulting formula for the limiting variance of $w(T)$ is the sum of the five above terms, however it may be further simplified. Since

$$
\sum_{n=1}^{\infty}\sum_{k=1}^{n-1} a_{n,k} = \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} a_{n,k}, \quad \text{if } \forall k \in \mathbb{N}\, \forall n \in \mathbb{N} \quad a_{n,k} \geq 0,
$$

we may switch the sums in the term (B) and by changing indices $n \mapsto k$, $l \mapsto n$ in the term (A), we arrive at

$$
(A) + (B) = \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} \frac{1}{D_{k,n}^2} 4ab \lambda_k \lambda_n (\alpha_k + \alpha_n)(\alpha_n - \alpha_k)^2 z_{2,k}^2 z_{2,n}^2.
$$

Similary, if we switch the sums in the term (D) and change indices $n \mapsto k$, $l \mapsto n$ in the term (E), we find out that $(D) = (E)$, so

$$
(D) + (E) = \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} \frac{1}{D_{k,n}^2} 32a^3 \lambda_k \lambda_n (\alpha_k + \alpha_n)^2 z_{2,k}^2 z_{2,n}^2
$$

and consequently

$$
\begin{aligned}
&(A) + (B) + (D) + (E) = \\
&= \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} \frac{1}{D_{k,n}^2} 4a \lambda_k \lambda_n (\alpha_k + \alpha_n) z_{2,k}^2 z_{2,n}^2 \left(b(\alpha_n - \alpha_k)^2 + 8a^2(\alpha_k + \alpha_n) \right) \\
&= \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} \frac{1}{D_{k,n}} 4a \lambda_k \lambda_n (\alpha_k + \alpha_n) z_{2,k}^2 z_{2,n}^2. \qquad (3.18)
\end{aligned}
$$

Since

$$\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} a_{n,k} = \frac{1}{2}\sum_{k=1}^{\infty}\sum_{n=1,n\neq k}^{\infty} a_{n,k}, \quad \text{if } \forall k \in \mathbb{N}\,\forall n \in \mathbb{N} \quad a_{k,n} = a_{n,k},$$

the sum on the right–hand side of (3.18) equals to

$$2a\sum_{k=1}^{\infty}\sum_{n=1,n\neq k}^{\infty} \frac{1}{D_{k,n}}\lambda_k\lambda_n(\alpha_k+\alpha_n)z_{2,k}^2 z_{2,n}^2. \tag{3.19}$$

The summand (C) is the corresponding sum to (3.19), where $n = k$, so we may add it and we end up with the formula for the limiting variance of $w(T)$, that is

$$\operatorname{Var}(w(T)) = 2a\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{D_{k,n}}\lambda_k\lambda_n(\alpha_k+\alpha_n)z_{2,k}^2 z_{2,n}^2, \quad T\to\infty.$$

Since the multiplicative factor $-\frac{1}{\frac{2}{T}\int_0^T \left\langle X_2^{x_0}(t), z_2\right\rangle_{L^2(D)}^2 dt}$ of $w(T)$ on the right–hand side of (3.17) converges to $\frac{-2a}{\langle Qz_2,z_2\rangle_{L^2(D)}}$ as $T\to\infty$, we arrive at

$$\operatorname{Law}\left(\sqrt{T}\left(\bar{a}_{T,z_2}-a\right)\right) \xrightarrow{w^*}$$
$$N\left(0, \frac{8a^3}{\langle Qz_2,z_2\rangle_{L^2(D)}^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\lambda_k\lambda_n(\alpha_k+\alpha_n)z_{2,k}^2 z_{2,n}^2}{b(\alpha_k-\alpha_n)^2+8a^2(\alpha_k+\alpha_n)}\right), \quad T\to\infty.$$

□

3.2.2 Asymptotic normality of the estimator $\bar{b}_{T,z_1,z_2}$

The estimator $\bar{b}_{T,z_1,z_2}$ (defined by (3.6)) is also asymptotically normal. The proof uses similiar technique as the proof of Theorem 3.7, so the setup and auxiliary Lemmas will be analogous to those in previous subsection.

Let $k\in\mathbb{N}$ be arbitrary and define the operator $F_k : V \to V$ by

$$F_k x = F_k\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_{k,1} & F_{k,2} \\ F_{k,3} & F_{k,4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V,$$

where

$$\begin{aligned}
F_{k,1} &: x_1 \longmapsto \left(b+\frac{4a^2}{\alpha_k}\right)\langle x_1, e_k\rangle_{L^2(D)} e_k,\\
F_{k,2} &: x_2 \longmapsto \frac{2a}{\alpha_k}\langle x_2, e_k\rangle_{L^2(D)} e_k,\\
F_{k,3} &: x_1 \longmapsto 2a\langle x_1, e_k\rangle_{L^2(D)} e_k,\\
F_{k,4} &: x_2 \longmapsto \langle x_2, e_k\rangle_{L^2(D)} e_k,
\end{aligned}$$

for any $x_1 \in \operatorname{Dom}((-A)^{\frac{1}{2}})$ and $x_2 \in L^2(D)$.

Let $k,l\in\mathbb{N}$ be arbitrary and define the operator $F_{k,l} : V \to V$ by

$$F_{k,l}x = F_k\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{D_{k,l}}\begin{pmatrix} F_{k,l,1} & F_{k,l,2} \\ F_{k,l,3} & F_{k,l,4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V,$$

where

$$
\begin{aligned}
F_{k,l,1} : x_1 \longmapsto\ & 8a^2\sqrt{\frac{\alpha_l}{\alpha_k}}\Big(8a^2 + b(\alpha_k + \alpha_l)\Big)\left\langle x_1, e_l\right\rangle_{L^2(D)} e_k \\
&+ 8a^2\sqrt{\frac{\alpha_k}{\alpha_l}}\Big(8a^2 + b(\alpha_k + \alpha_l)\Big)\left\langle x_1, e_k\right\rangle_{L^2(D)} e_l, \\
F_{k,l,2} : x_2 \longmapsto\ & 4a\sqrt{\frac{\alpha_k}{\alpha_l}}\Big(8a^2 + b(\alpha_k - \alpha_l)\Big)\left\langle x_2, e_k\right\rangle_{L^2(D)} e_l \\
&+ 4a\sqrt{\frac{\alpha_l}{\alpha_k}}\Big(8a^2 + b(\alpha_l - \alpha_k)\Big)\left\langle x_2, e_l\right\rangle_{L^2(D)} e_k, \\
F_{k,l,3} : x_1 \longmapsto\ & 4a\sqrt{\alpha_k\alpha_l}\Big(8a^2 + b(\alpha_k - \alpha_l)\Big)\left\langle x_1, e_l\right\rangle_{L^2(D)} e_k \\
&+ 4a\sqrt{\alpha_k\alpha_l}\Big(8a^2 + b(\alpha_l - \alpha_k)\Big)\left\langle x_1, e_k\right\rangle_{L^2(D)} e_l, \\
F_{k,l,4} : x_2 \longmapsto\ & 16a^2\sqrt{\alpha_k\alpha_l}\left\langle x_2, e_k\right\rangle_{L^2(D)} e_l + 16a^2\sqrt{\alpha_k\alpha_l}\left\langle x_2, e_l\right\rangle_{L^2(D)} e_k,
\end{aligned}
$$

for any $x_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ and $x_2 \in L^2(D)$ with $D_{k,l}$ defined above.

The properties of the operators F_k and $F_{k,l}$ are summarized in the following Lemma.

Lemma 3.8. *1) The operator $F_k \in \mathcal{L}(V)$ is self–adjoint for any given $k \in \mathbb{N}$. Moreover,*

$$
\left\langle F_k x, \mathcal{A}x\right\rangle_V = -2ab\left\langle x_1, f_k\right\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}, \tag{3.20}
$$

for any $x = (x_1, x_2)^\top \in \mathrm{Dom}(\mathcal{A})$.

2) The operator $F_{k,l} \in \mathcal{L}(V)$ is self–adjoint for any given $k, l \in \mathbb{N}$. Moreover,

$$
\left\langle F_{k,l} x, \mathcal{A}x\right\rangle_V = -4ab\left\langle x_1, f_k\right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\left\langle x_1, f_l\right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}, \tag{3.21}
$$

for any $x = (x_1, x_2)^\top \in \mathrm{Dom}(\mathcal{A})$.

Proof. 1) Let $k \in \mathbb{N}$ be arbitrary. Obviously $F_k \in \mathcal{L}(V)$ and for any $x = (x_1, x_2)^\top \in V$ and $y = (y_1, y_2)^\top \in V$ we have

$$
\begin{aligned}
\left\langle F_k x, y\right\rangle_V &= \left\langle \begin{pmatrix} F_{k,1}x_1 + F_{k,2}x_2 \\ F_{k,3}x_1 + F_{k,4}x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_V \\
&= \left(b + \frac{4a^2}{\alpha_k}\right)\left\langle x_1, e_k\right\rangle_{L^2(D)}\left\langle y_1, e_k\right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \\
&\quad + \frac{2a}{\alpha_k}\left\langle x_2, e_k\right\rangle_{L^2(D)}\left\langle y_1, e_k\right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \\
&\quad + 2a\left\langle x_1, e_k\right\rangle_{L^2(D)}\left\langle y_2, e_k\right\rangle_{L^2(D)} + \left\langle x_2, e_k\right\rangle_{L^2(D)}\left\langle y_2, e_k\right\rangle_{L^2(D)} \\
&= (b\alpha_k + 4a^2)\left\langle x_1, e_k\right\rangle_{L^2(D)}\left\langle y_1, e_k\right\rangle_{L^2(D)} + 2a\left\langle x_2, e_k\right\rangle_{L^2(D)}\left\langle y_1, e_k\right\rangle_{L^2(D)} \\
&\quad + 2a\left\langle x_1, e_k\right\rangle_{L^2(D)}\left\langle y_2, e_k\right\rangle_{L^2(D)} + \left\langle x_2, e_k\right\rangle_{L^2(D)}\left\langle y_2, e_k\right\rangle_{L^2(D)} \\
&= \left\langle x, F_k y\right\rangle_V,
\end{aligned}
$$

hence $F_k = F_k^*$. Moreover, for any $x = (x_1, x_2)^\top \in \mathrm{Dom}(\mathcal{A})$ we have

$$\begin{aligned}
\langle F_k x, \mathcal{A}x\rangle_V &= \left\langle \begin{pmatrix} F_{k,1}x_1 + F_{k,2}x_2 \\ F_{k,3}x_1 + F_{k,4}x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ bAx_1 - 2ax_2 \end{pmatrix} \right\rangle_V \\
&= \left(b + \frac{4a^2}{\alpha_k}\right) \langle x_1, e_k\rangle_{L^2(D)} \left\langle (-A)^{\frac{1}{2}} x_2, (-A)^{\frac{1}{2}} e_k \right\rangle_{L^2(D)} \\
&\quad + \frac{2a}{\alpha_k} \langle x_2, e_k\rangle_{L^2(D)} \left\langle (-A)^{\frac{1}{2}} x_2, (-A)^{\frac{1}{2}} e_k \right\rangle_{L^2(D)} \\
&\quad + 2ab \langle x_1, e_k\rangle_{L^2(D)} \langle Ax_1, e_k\rangle_{L^2(D)} - 4a^2 \langle x_1, e_k\rangle_{L^2(D)} \langle x_2, e_k\rangle_{L^2(D)} \\
&\quad + b \langle x_2, e_k\rangle_{L^2(D)} \langle Ax_1, e_k\rangle_{L^2(D)} - 2a \langle x_2, e_k\rangle^2_{L^2(D)} \\
&= 2ab \langle x_1, e_k\rangle_{L^2(D)} \langle Ax_1, e_k\rangle_{L^2(D)} \\
&= -2ab \langle x_1, f_k\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} .
\end{aligned}$$

2) Let $k, l \in \mathbb{N}$ be arbitrary. It is clear that $F_{k,l} \in \mathcal{L}(V)$ and similarly as above it is possible to verify that $F_{k,l} = F_{k,l}^*$ and that (3.21) holds true for any $x \in \mathrm{Dom}(\mathcal{A})$. $\square$

Choose $0 \neq z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ taking the form

$$z_1 = \sum_{k=1}^{\infty} \langle z_1, f_k\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} f_k = \sum_{k=1}^{\infty} z_{1,k} f_k,$$

i.e., $\{z_{1,k}, k \in \mathbb{N}\}$ is the set of coordinates of the element z_1 with respect to the orthonormal basis in $\mathrm{Dom}((-A)^{\frac{1}{2}})$. Finally, define the operator $F : V \to V$ by

$$F = \sum_{k=1}^{\infty} z_{1,k}^2 F_k + \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} z_{1,k} z_{1,l} F_{k,l}. \tag{3.22}$$

The properties of the operator F are summarized in the following Lemma.

Lemma 3.9. *The operator $F \in \mathcal{L}(V)$. Moreover, it is self–adjoint and*

$$\langle Fx, \mathcal{A}x\rangle_V = -2ab \langle x_1, z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathrm{Dom}(\mathcal{A}). \tag{3.23}$$

Proof. Using inequalities in the proof of Lemma 3.4, it is possible to verify that $F \in \mathcal{L}(V)$. Moreover, the linear combination of the self–adjoint operators is also the self–adjoint operator, hence $F = F^*$.

The property (3.23) comes from (3.20) and (3.21). The proof is analogous to the proof of Lemma 3.4. $\square$

We will also need an alternative representation for the process $\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt$.

Proposition 3.10. *The process $\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt$ admits the following representation*

$$\begin{aligned}
&\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt = \\
&= -\frac{1}{4abT} \left(\langle FX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Fx_0, x_0\rangle_V \right) \\
&\quad + \frac{1}{2abT} \int_0^T \langle FX^{x_0}(t), \Phi\, dB(t)\rangle_V + \frac{1}{4ab} \langle Qz_1, z_1\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} .
\end{aligned} \tag{3.24}$$

Proof. Define the function $g_4 : V \to \mathbb{R}$ by

$$g_4(x) = \langle Fx, x \rangle_V \,, \quad \forall x \in V.$$

The application of Itô's formula to the function $g_4(X^{x_0,N}(t))$ (we also have to use suitable projections, see the proof of Proposition 2.6), yields

$$dg_4(X^{x_0,N}(t)) = 2 \left\langle FX^{x_0,N}(t), dX^{x_0,N}(t) \right\rangle_V + \frac{1}{2} \operatorname{Tr}\left(2F\Phi\Phi^*\right) \, dt. \tag{3.25}$$

We also start with simplification of the second term

$$\begin{aligned}
\operatorname{Tr}\left(F\Phi\Phi^*\right) &= \operatorname{Tr}\left(F_4 Q\right) \\
&= \sum_{k=1}^{\infty} z_{1,k}^2 \operatorname{Tr}\left(F_{k,4} Q\right) + \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \frac{z_{1,k} z_{1,l}}{D_{k,l}} \operatorname{Tr}\left(F_{k,l,4} Q\right) \\
&= \langle Q z_1, z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \,,
\end{aligned}$$

since $\operatorname{Tr}\left(F_{k,4}Q\right) = \lambda_k$, $\operatorname{Tr}\left(F_{k,l,4}Q\right) = 0$, because $k \neq l$, and

$$\sum_{k=1}^{\infty} \lambda_k z_{1,k}^2 = \langle Q z_1, z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \,.$$

Lemma 3.9 and formula (3.25) imply

$$\begin{aligned}
dg_4(X^{x_0,N}(t)) &= 2 \left\langle FX^{x_0,N}(t), \mathcal{A}X^{x_0,N}(t) \right\rangle_V dt + 2 \left\langle FX^{x_0,N}(t), \Phi \, dB(t) \right\rangle_V \\
&\quad + \langle Q z_1, z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \, dt \\
&= -4ab \left\langle X_1^{x_0,N}(t), z_1 \right\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} dt + 2 \left\langle FX^{x_0,N}(t), \Phi \, dB(t) \right\rangle_V \\
&\quad + \langle Q z_1, z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \, dt.
\end{aligned}$$

By integrating the above expression over the interval $(0, T)$ and passing N to infinity, we arrive at (3.24). $\square$

Denote

$$Q_1 := \langle Q z_1, z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})} \,, \quad Q_2 := \langle Q z_2, z_2 \rangle_{L^2(D)} \,.$$

Asymptotic normality of the estimator $\bar{b}_{T,z_1,z_2}$ is formulated in the following Theorem.

Theorem 3.11. *Let* $0 \neq z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$, $0 \neq z_2 \in L^2(D)$ *be arbitrary. The estimator* $\bar{b}_{T,z_1,z_2}$ *is asymptotically normal, i.e.,* $\mathrm{Law}\left(\sqrt{T}\left(\bar{b}_{T,z_1,z_2} - b\right)\right)$ *converges weakly to a centered Gaussian distribution with variance given by*

$$\begin{aligned}
&\frac{64a^3 b}{Q_1^2} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_k \lambda_n z_{1,k}^2 z_{1,n}^2}{b(\alpha_k - \alpha_n)^2 + 8a^2(\alpha_k + \alpha_n)} \\
&+ \frac{8ab^2}{Q_1^2 Q_2^2} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_k \lambda_n}{b(\alpha_k - \alpha_n)^2 + 8a^2(\alpha_k + \alpha_n)} \Big((Q_1 z_{2,k} z_{2,n} \sqrt{\alpha_k} - Q_2 z_{1,k} z_{1,n} \sqrt{\alpha_n})^2 \\
&+ (Q_1 z_{2,k} z_{2,n} \sqrt{\alpha_n} - Q_2 z_{1,k} z_{1,n} \sqrt{\alpha_k})^2 \Big) \,.
\end{aligned} \tag{3.26}$$

Proof. Set $0 \neq z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$ and $0 \neq z_2 \in L^2(D)$. Using formula (3.6) for the estimator $\bar{b}_{T,z_1,z_2}$ and Propositions 3.5 and 3.10, we obtain

$$\sqrt{T}\left(\bar{b}_{T,z_1,z_2} - b\right) = \tag{3.27}$$

$$= \frac{\sqrt{T}}{Q_2 \frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt}\left(\frac{Q_1}{T}\int_0^T \langle X_2^{x_0}(t), z_2\rangle^2_{L^2(D)}\, dt\right.$$
$$\left. -\frac{bQ_2}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt\right)$$
$$= \frac{\sqrt{T}}{Q_2 \frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt}\left(-\frac{Q_1}{4aT}\left(\langle EX^{x_0}(T), X^{x_0}(T)\rangle_V\right.\right.$$
$$\left. - \langle Ex_0, x_0\rangle_V\right) + \frac{Q_1}{2aT}\int_0^T \langle EX^{x_0}(t), \Phi\, dB(t)\rangle_V +$$
$$\left. + \frac{Q_2}{4aT}\left(\langle FX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Fx_0, x_0\rangle_V\right) - \frac{Q_2}{2aT}\int_0^T \langle FX^{x_0}(t), \Phi\, dB(t)\rangle_V\right)$$
$$= -\frac{Q_1}{4aQ_2 \frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt}\frac{1}{\sqrt{T}}\left(\langle EX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Ex_0, x_0\rangle_V\right)$$
$$+ \frac{1}{4a\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt}\frac{1}{\sqrt{T}}\left(\langle FX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Fx_0, x_0\rangle_V\right)$$
$$+ \frac{1}{2aQ_2 \frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt}\frac{1}{\sqrt{T}}\int_0^T \langle (Q_1E - Q_2F)X^{x_0}(t), \Phi\, dB(t)\rangle_V$$

The first two terms on the right–hand side converge to zero in probability as $T \to \infty$, since

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt = \frac{Q_1}{4ab}, \quad \mathbb{P} - a.s.$$

by Theorem 3.1 and

$$\lim_{T\to\infty}\frac{1}{\sqrt{T}}\left(\langle EX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Ex_0, x_0\rangle_V\right) = 0, \quad \text{in } L^1(\Omega),$$
$$\lim_{T\to\infty}\frac{1}{\sqrt{T}}\left(\langle FX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Fx_0, x_0\rangle_V\right) = 0, \quad \text{in } L^1(\Omega),$$

by Lemma 2.7. Define

$$\begin{aligned} w_1(T) &= \frac{1}{\sqrt{T}}\int_0^T \langle (Q_1E - Q_2F)X^{x_0}(t), \Phi\, dB(t)\rangle_V \\ &= \frac{1}{\sqrt{T}}\int_0^T \sum_{n=1}^{\infty}\sqrt{\lambda_n}\left\langle (Q_1E - Q_2F)X^{x_0}(t), \begin{pmatrix} 0 \\ e_n \end{pmatrix}\right\rangle_V d\beta_n(t). \end{aligned}$$

Furthermore, we compute the scalar product in the above series

$$\left\langle (Q_1E - Q_2F)X^{x_0}(t), \begin{pmatrix} 0 \\ e_n \end{pmatrix}\right\rangle_V =$$
$$= \langle (Q_1E_3 - Q_2F_3)X_1^{x_0}(t), e_n\rangle_{L^2(D)} + \langle (Q_1E_4 - Q_2F_4)X_2^{x_0}(t), e_n\rangle_{L^2(D)}.$$

By the definition of the operators E_3, F_3, E_4, F_4 we have

$$
\begin{aligned}
&\langle (Q_1 E_3 - Q_2 F_3) X_1^{x_0}(t), e_n\rangle_{L^2(D)} = \\
&= \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{4a}{D_{k,l}} \Big(Q_1 z_{2,k} z_{2,l} b\alpha_l(\alpha_k - \alpha_l) - Q_2 z_{1,k} z_{1,l}\sqrt{\alpha_k\alpha_l}\left(8a^2 + b(\alpha_k - \alpha_l)\right)\Big) \\
&\quad \times \langle X_1^{x_0}(t), e_l\rangle_{L^2(D)}\,\delta_{k,n} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{4a}{D_{k,l}} \Big(Q_1 z_{2,k} z_{2,l} b\alpha_k(\alpha_l - \alpha_k) - Q_2 z_{1,k} z_{1,l}\sqrt{\alpha_k\alpha_l}\left(8a^2 + b(\alpha_l - \alpha_k)\right)\Big) \\
&\quad \times \langle X_1^{x_0}(t), e_k\rangle_{L^2(D)}\,\delta_{n,l} \\
&\quad - \sum_{k=1}^{\infty} 2aQ_2 z_{1,k}^2 \langle X_1^{x_0}(t), e_k\rangle_{L^2(D)}\,\delta_{k,n} \\
&= (I) + (II) + (III),
\end{aligned}
$$

$$
\begin{aligned}
&\langle (Q_1 E_4 - Q_2 F_4) X_2^{x_0}(t), e_n\rangle_{L^2(D)} = \\
&= \sum_{k=1}^{\infty} \left(Q_1 z_{2,k}^2 - Q_2 z_{1,k}^2\right) \langle X_2^{x_0}(t), e_k\rangle_{L^2(D)}\,\delta_{k,n} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{8a^2}{D_{k,l}} \left(Q_1 z_{2,k} z_{2,l}(\alpha_k + \alpha_l) - 2Q_2 z_{1,k} z_{1,l}\sqrt{\alpha_k\alpha_l}\right) \langle X_2^{x_0}(t), e_k\rangle_{L^2(D)}\,\delta_{n,l} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{8a^2}{D_{k,l}} \left(Q_1 z_{2,k} z_{2,l}(\alpha_k + \alpha_l) - 2Q_2 z_{1,k} z_{1,l}\sqrt{\alpha_k\alpha_l}\right) \langle X_2^{x_0}(t), e_l\rangle_{L^2(D)}\,\delta_{k,n} \\
&= (IV) + (V) + (VI).
\end{aligned}
$$

By the central limit theorem for martingales, Law $(w_1(T))$ converges weakly to a Gaussian distribution with a zero mean and variance given by the limit in probability

$$
\lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n \left((I) + \ldots + (VI)\right)^2\, dt.
$$

The limits of the cross terms are zero due to Lemma 3.6 (see the proof of Theorem 3.7). We compute the limits of the "diagonal" terms in the $\mathbb{P} - a.s.$ sense.

$$
\begin{aligned}
(A) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n (I)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty} \lambda_n \left(\sum_{l=n+1}^{\infty} \frac{4a}{D_{n,l}} (\cdots) \langle X_1^{x_0}(t), e_l\rangle_{L^2(D)}\right)^2 dt \\
&\overset{(*)}{=} \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty} \lambda_n \frac{16a^2}{D_{n,l}^2} (\cdots)^2 \langle X_1^{x_0}(t), e_l\rangle_{L^2(D)}^2\, dt \\
&= \sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty} \frac{1}{D_{n,l}^2} \frac{4a\lambda_n\lambda_l}{b\alpha_l} \\
&\quad \times \Big(Q_1 z_{2,n} z_{2,l} b\alpha_l(\alpha_n - \alpha_l) - Q_2 z_{1,n} z_{1,l}\sqrt{\alpha_n\alpha_l}\left(8a^2 + b(\alpha_n - \alpha_l)\right)\Big)^2,
\end{aligned}
$$

since

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle X_1^{x_0}(t), e_l\rangle^2_{L^2(D)}\, dt = \frac{\lambda_l}{4ab\alpha_l}$$

and in the equality $(*)$ we have also used Lemma 3.6 for the cross summands. $(\cdots)$ stands for the bracket from the definition of (I).

Similarly, we have

$$\begin{aligned}
(B) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n (II)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n-1} \frac{4a}{D_{k,n}} (\cdots) \langle X_1^{x_0}(t), e_k\rangle_{L^2(D)} \right)^2 dt \\
&\overset{(*)}{=} \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \lambda_n \frac{16a^2}{D_{k,n}^2} (\cdots)^2 \langle X_1^{x_0}(t), e_k\rangle^2_{L^2(D)}\, dt \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{1}{D_{k,n}^2} \frac{4a\lambda_n\lambda_k}{b\alpha_k} \\
&\quad \times \left(Q_1 z_{2,k} z_{2,n} b\alpha_k(\alpha_n - \alpha_k) - Q_2 z_{1,k} z_{1,n} \sqrt{\alpha_k\alpha_n} \left(8a^2 + b(\alpha_n - \alpha_k)\right) \right)^2, \\
(C) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n (III)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} 4a^2 Q_2^2 \lambda_n z_{1,n}^4 \langle X_1^{x_0}(t), e_n\rangle^2_{L^2(D)}\, dt \\
&= \frac{aQ_2^2}{b} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{\alpha_n} z_{1,n}^4, \\
(D) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n (IV)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n \left(Q_1 z_{2,n}^2 - Q_2 z_{1,n}^2 \right)^2 \langle X_2^{x_0}(t), e_n\rangle^2_{L^2(D)}\, dt \\
&= \frac{1}{4a} \sum_{n=1}^{\infty} \lambda_n^2 \left(Q_1 z_{2,n}^2 - Q_2 z_{1,n}^2 \right)^2,
\end{aligned}$$

since

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle X_2^{x_0}(t), e_n\rangle^2_{L^2(D)}\, dt = \frac{\lambda_n}{4a}.$$

Next, we have

$$\begin{aligned}
(E) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n (V)^2\, dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n-1} \frac{8a^2}{D_{k,n}} (\cdots) \langle X_2^{x_0}(t), e_k\rangle_{L^2(D)} \right)^2 dt \\
&\overset{(*)}{=} \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \lambda_n \frac{64a^4}{D_{k,n}^2} (\cdots)^2 \langle X_2^{x_0}(t), e_k\rangle^2_{L^2(D)}\, dt \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{1}{D_{k,n}^2} 16a^3 \lambda_n\lambda_k \left(Q_1 z_{2,k} z_{2,n}(\alpha_k + \alpha_n) - 2Q_2 z_{1,k} z_{1,n} \sqrt{\alpha_k\alpha_n} \right)^2,
\end{aligned}$$

$$
\begin{aligned}
(F) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n (VI)^2 \, dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \lambda_n \left(\sum_{l=n+1}^{\infty} \frac{8a^2}{D_{n,l}} (\cdots) \left\langle X_2^{x_0}(t), e_l \right\rangle_{L^2(D)} \right)^2 \, dt \\
&\overset{(*)}{=} \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{n=1}^{\infty} \sum_{l=n+1}^{\infty} \lambda_n \frac{64a^4}{D_{n,l}^2} (\cdots)^2 \left\langle X_2^{x_0}(t), e_l \right\rangle_{L^2(D)}^2 \, dt \\
&= \sum_{n=1}^{\infty} \sum_{l=n+1}^{\infty} \frac{1}{D_{n,l}^2} 16a^3 \lambda_n \lambda_l \left(Q_1 z_{2,n} z_{2,l} (\alpha_n + \alpha_l) - 2Q_2 z_{1,n} z_{1,l} \sqrt{\alpha_n \alpha_l} \right)^2 .
\end{aligned}
$$

The limiting variance of $w_1(T)$ is the sum of the six above terms $(A)+\ldots+(F)$, which can be simplified (analogously as in the proof of Theorem 3.7) to

$$
\begin{aligned}
&\frac{16a^3 Q_2^2}{b} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_k \lambda_n z_{1,k}^2 z_{1,n}^2}{D_{k,n}} \\
&+ 2a \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_k \lambda_n}{D_{k,n}} \Big(\left(Q_1 z_{2,k} z_{2,n} \sqrt{\alpha_k} - Q_2 z_{1,k} z_{1,n} \sqrt{\alpha_n} \right)^2 \\
&+ \left(Q_1 z_{2,k} z_{2,n} \sqrt{\alpha_n} - Q_2 z_{1,k} z_{1,n} \sqrt{\alpha_k} \right)^2 \Big) .
\end{aligned}
$$

Since the multiplicative factor

$$
\frac{1}{2aQ_2 \frac{1}{T} \int_0^T \left\langle X_1^{x_0}(t), z_1 \right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}^2 \, dt}
$$

of $w_1(T)$ on the right–hand side of (3.27) converges to $\frac{2b}{Q_1 Q_2}$ as $T \to \infty$, we arrive at (3.26). □

3.2.3 Asymptotic normality of the estimator $\bar{b}_{T,z_1,a}$

Based on the estimation strategy 2., another estimator of the parameter b may be introduced. Using "observation window" $(z_1, 0)^\top$, $z_1 \neq 0$, with the parameter a known, we may define

$$
\bar{b}_{T,z_1,a} = \frac{\left\langle Qz_1, z_1 \right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}}{4a \frac{1}{T} \int_0^T \left\langle X_1^{x_0}(t), z_1 \right\rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}^2 \, dt}. \tag{3.28}
$$

This estimator $\bar{b}_{T,z_1,a}$ is also strongly consistent as $T \to \infty$ by Theorem 3.1 and we show its asymptotic normality.

Theorem 3.12. *Let $0 \neq z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$, $a > 0$ be arbitrary. The estimator $\bar{b}_{T,z_1,a}$ is asymptotically normal, i.e.,* $\mathrm{Law}\left(\sqrt{T}\left(\bar{b}_{T,z_1,a} - b\right)\right)$ *converges weakly to a centered Gaussian distribution with variance given by*

$$
\begin{aligned}
&\frac{64a^3 b}{Q_1^2} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_k \lambda_n z_{1,k}^2 z_{1,n}^2}{b(\alpha_k - \alpha_n)^2 + 8a^2(\alpha_k + \alpha_n)} \\
&\quad + \frac{8ab^2}{Q_1^2} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_k \lambda_n (\alpha_k + \alpha_n) z_{1,k}^2 z_{1,n}^2}{b(\alpha_k - \alpha_n)^2 + 8a^2(\alpha_k + \alpha_n)}.
\end{aligned} \tag{3.29}
$$

Proof. Set $0 \neq z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$. Using formula (3.28) for the estimator $\bar{b}_{T,z_1,a}$ and Proposition 3.10, we obtain

$$\begin{aligned}
&\sqrt{T}\left(\bar{b}_{T,z_1,a} - b\right) = \\
&= \frac{\sqrt{T}}{4a\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt}\left(Q_1 - \frac{4ab}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt\right) \\
&= \frac{1}{4a\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt}\frac{1}{\sqrt{T}}\left(\langle FX^{x_0}(T), X^{x_0}(T)\rangle_V - \langle Fx_0, x_0\rangle_V\right) \\
&\quad - \frac{1}{2a\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), z_1\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt}\frac{1}{\sqrt{T}}\int_0^T \langle FX^{x_0}(t), \Phi\, dB(t)\rangle_V . \qquad (3.30)
\end{aligned}$$

The first term on the right–hand side converges to zero in probability as $T \to \infty$ (see the proof of Theorem 3.11) and define

$$\begin{aligned}
w_2(T) &= \frac{1}{\sqrt{T}}\int_0^T \langle FX^{x_0}(t), \Phi\, dB(t)\rangle_V \\
&= \frac{1}{\sqrt{T}}\int_0^T \sum_{n=1}^{\infty} \sqrt{\lambda_n}\left\langle FX^{x_0}(t), \begin{pmatrix} 0 \\ e_n \end{pmatrix}\right\rangle_V d\beta_n(t).
\end{aligned}$$

Since the stochastic integral $w_2(T)$ is a special case of the stochastic integral $w_1(T)$ from the proof of Theorem 3.11 (use $E = 0$, $Q_2 = 1$ and omit the minus sign), we will handle it much easier than above. The required scalar products equal to

$$\begin{aligned}
&\langle F_3X_1^{x_0}(t), e_n\rangle_{L^2(D)} = \\
&= \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{1,k}z_{1,l}}{D_{k,l}} 4a\sqrt{\alpha_k\alpha_l}\left(8a^2 + b(\alpha_k - \alpha_l)\right)\langle X_1^{x_0}(t), e_l\rangle_{L^2(D)}\,\delta_{k,n} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{1,k}z_{1,l}}{D_{k,l}} 4a\sqrt{\alpha_k\alpha_l}\left(8a^2 + b(\alpha_l - \alpha_k)\right)\langle X_1^{x_0}(t), e_k\rangle_{L^2(D)}\,\delta_{n,l} \\
&\quad + \sum_{k=1}^{\infty} 2az_{1,k}^2\langle X_1^{x_0}(t), e_k\rangle_{L^2(D)}\,\delta_{k,n} \\
&= (I) + (II) + (III),
\end{aligned}$$

and

$$\begin{aligned}
&\langle F_4X_2^{x_0}(t), e_n\rangle_{L^2(D)} = \\
&= \sum_{k=1}^{\infty} z_{1,k}^2\langle X_2^{x_0}(t), e_k\rangle_{L^2(D)}\,\delta_{k,n} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{1,k}z_{1,l}}{D_{k,l}} 16a^2\sqrt{\alpha_k\alpha_l}\langle X_2^{x_0}(t), e_k\rangle_{L^2(D)}\,\delta_{n,l} \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \frac{z_{1,k}z_{1,l}}{D_{k,l}} 16a^2\sqrt{\alpha_k\alpha_l}\langle X_2^{x_0}(t), e_l\rangle_{L^2(D)}\,\delta_{k,n} \\
&= (IV) + (V) + (VI).
\end{aligned}$$

The appropriate $\mathbb{P}-a.s.$ limits of the "diagonal" terms equal to

$$(A)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(I)^2\,dt=\sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty}\frac{z_{1,n}^2z_{1,l}^2}{D_{n,l}^2}\frac{4a\lambda_n\lambda_l\alpha_n}{b}\left(8a^2+b(\alpha_n-\alpha_l)\right)^2,$$

$$(B)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(II)^2\,dt=\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{z_{1,k}^2z_{1,n}^2}{D_{k,n}^2}\frac{4a\lambda_n\lambda_k\alpha_n}{b}\left(8a^2+b(\alpha_n-\alpha_k)\right)^2,$$

$$(C)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(III)^2\,dt=\frac{a}{b}\sum_{n=1}^{\infty}\frac{\lambda_n^2}{\alpha_n}z_{1,n}^4,$$

$$(D)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(IV)^2\,dt=\frac{1}{4a}\sum_{n=1}^{\infty}\lambda_n^2z_{1,n}^4,$$

$$(E)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(V)^2\,dt=\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{z_{1,k}^2z_{1,n}^2}{D_{k,n}^2}64a^3\lambda_n\lambda_k\alpha_n\alpha_k,$$

$$(F)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\lambda_n(VI)^2\,dt=\sum_{n=1}^{\infty}\sum_{l=n+1}^{\infty}\frac{z_{1,n}^2z_{1,l}^2}{D_{n,l}^2}64a^3\lambda_n\lambda_l\alpha_n\alpha_l.$$

Analogously as above, Law $(w_2(T))$ converges weakly to a Gaussian distribution with a zero mean and variance given by $(A)+\ldots+(F)$, which can be simplified to

$$\frac{16a^3}{b}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\lambda_k\lambda_nz_{1,k}^2z_{1,n}^2}{D_{k,n}}+2a\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\lambda_k\lambda_n(\alpha_k+\alpha_n)z_{1,k}^2z_{1,n}^2}{D_{k,n}}.$$

Since the multiplicative factor of $w_2(T)$ on the right–hand side of (3.30) converges to $-\frac{2b}{Q_1}$, we arrive at (3.29). □

Remark 5. If we consider some special cases of "observation windows", the formulae for the limiting variances from Theorems 3.7, 3.11 and 3.12 may be considerably simplified.

- The estimator $\bar{a}_{T,k}$ (defined by (3.7)) has limiting variance a for any $k\in\mathbb{N}$, i.e.,
$$\text{Law}\left(\sqrt{T}\,(\bar{a}_{T,k}-a)\right)\xrightarrow{w^*}N(0,a),\quad T\to\infty.$$
Since the result does not depend on k it does not matter which coordinate of the second component we observe. All the estimators $\bar{a}_{T,k}$ behave similarly.

- The estimator $\bar{b}_{T,j,k}$ (defined by (3.8)) satisfies
$$\text{Law}\left(\sqrt{T}\left(\bar{b}_{T,j,k}-b\right)\right)\xrightarrow{w^*}N\left(0,\frac{4ab}{\alpha_j}+\frac{2b^2}{a}\right),\quad T\to\infty,$$
for any $j,k\in\mathbb{N}$, $j\neq k$.

- However, if $j=k$ then the estimator $\bar{b}_{T,j,j}$ satisfies
$$\text{Law}\left(\sqrt{T}\left(\bar{b}_{T,j,j}-b\right)\right)\xrightarrow{w^*}N\left(0,\frac{4ab}{\alpha_j}\right),\quad T\to\infty.$$

- The estimator $\bar{b}_{T,f_j,a}$ (defined by (3.28)) satisfies

$$\mathrm{Law}\left(\sqrt{T}\left(\bar{b}_{T,f_j,a} - b\right)\right) \xrightarrow{w^*} N\left(0, \frac{4ab}{\alpha_j} + \frac{b^2}{a}\right), \quad T \to \infty,$$

for any $j \in \mathbb{N}$.

By comparing the last three points, we obtain two following observations, which are illustrated by simulations in Section 4.2:

1. Since the limiting variance of the estimator $\bar{b}_{T,j,k}$ is greater than the limiting variance of the estimator $\bar{b}_{T,f_j,a}$, it seems that it is better to know the true value of the parameter a exactly, instead of estimating it. However, if $j = k$ then the limiting variance of the estimator $\bar{b}_{T,j,j}$ is even smaller. So estimating the parameter a by the "window" $(0, e_j)^\top$ and then using the "window" $(f_j, 0)^\top$ to estimate the parameter b should be even better than knowing a exactly.

2. Since $\alpha_j \to \infty$ as $j \to \infty$, the limiting variance $\frac{4ab}{\alpha_j}$ gets smaller with bigger j. Hence it is better to use the "observation coordinate" with bigger j rather than using the smaller one.

3.3 Examples

Example 1. Consider the stochastic wave equation with Dirichlet boundary conditions

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,\xi) &= b\Delta u(t,\xi) - 2a\frac{\partial u}{\partial t}(t,\xi) + \eta(t,\xi), \quad (t,\xi) \in \mathbb{R}_+ \times D, \\ u(0,\xi) &= u_1(\xi), \quad \xi \in D, \\ \frac{\partial u}{\partial t}(0,\xi) &= u_2(\xi), \quad \xi \in D, \\ u(t,\xi) &= 0, \quad (t,\xi) \in \mathbb{R}_+ \times \partial D, \end{aligned} \tag{3.31}$$

where $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, η is a noise process that is the formal time derivative of a space dependent Brownian motion and $a > 0$, $b > 0$ are unknown parameters.

We rewrite the hyperbolic system (3.31) as an infinite dimensional stochastic differential equation (1.12)

$$\begin{aligned} dX(t) &= \mathcal{A}X(t)\,dt + \Phi\,dB(t), \\ X(0) &= x_0 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned}$$

for $t \geq 0$, setting $A = \Delta|_{\mathrm{Dom}(A)}$, $\mathrm{Dom}(A) = H^2(D) \cap H_0^1(D)$, $\mathrm{Dom}(\mathcal{A}) = \mathrm{Dom}(A) \times \mathrm{Dom}((-A)^{\frac{1}{2}})$ and

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ bA & -2aI \end{pmatrix}.$$

The operator $\mathcal{A}$ generates strongly continuous semigroup in the space $V = \mathrm{Dom}((-A)^{\frac{1}{2}}) \times L^2(D)$. The driving process may take a form $B(t) = (0, \tilde{B}(t))^\top$, where $(\tilde{B}(t), t \geq 0)$ is a cylindrical Brownian motion on $L^2(D)$. The noise η is modelled as the formal derivative $\Phi_1 \frac{d\tilde{B}(t)}{dt}$, $\Phi_1 \in \mathcal{L}_2(L^2(D))$ and $\Phi \in \mathcal{L}_2(V)$ is given by

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_1 \end{pmatrix}.$$

With this setup, the assumptions (A1) – (A4) are fulfilled, so Theorems 2.3, 2.4 and 3.1 may be used for estimation of parameters. Theorems 2.8, 2.11, 3.7, 3.11 and 3.12, which show asymptotic normality of these estimators, may be applied as well.

The operator $Q = \Phi_1 \Phi_1^*$ which appears in the formulae for estimators established in these Theorems may be interpreted as the "covariance in space" of the driving process $(\Phi_1 \tilde{B}(t), t \geq 0)$, i.e.,

$$\mathbb{E} \left\langle \Phi_1 \tilde{B}(t, \cdot), x \right\rangle_{L^2(D)} \left\langle \Phi_1 \tilde{B}(s, \cdot), y \right\rangle_{L^2(D)} = \min(t, s) \langle Qx, y \rangle_{L^2(D)}, \quad t, s \geq 0,$$

for any $x, y \in L^2(D)$ (see Definition 1.2).

Example 2. Consider the stochastic plate equation with Dirichlet boundary conditions

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t,\xi) &= -b\Delta^2 u(t,\xi) - 2a\frac{\partial u}{\partial t}(t,\xi) + \eta(t,\xi), \quad (t,\xi) \in \mathbb{R}_+ \times D, \qquad (3.32)\\
u(0,\xi) &= u_1(\xi), \quad \xi \in D,\\
\frac{\partial u}{\partial t}(0,\xi) &= u_2(\xi), \quad \xi \in D,\\
u(t,\xi) &= 0, \quad (t,\xi) \in \mathbb{R}_+ \times \partial D,
\end{aligned}$$

where D, η, a and b satisfy the conditions in Example 1.

We rewrite the hyperbolic system (3.32) as an infinite dimensional stochastic differential equation (1.12), setting $A = \Delta|_{\mathrm{Dom}(A)}$, $\mathrm{Dom}(A) = H^2(D) \cap H_0^1(D)$, $\mathrm{Dom}(\mathcal{A}) = \mathrm{Dom}(A^2) \times \mathrm{Dom}(A)$ and

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -bA^2 & -2aI \end{pmatrix}.$$

The operator $\mathcal{A}$ generates strongly continuous semigroup in the space $V = \mathrm{Dom}(A) \times L^2(D)$. The driving process may take a form $B(t) = (0, \tilde{B}(t))^\top$, where $(\tilde{B}(t), t \geq 0)$ is a cylindrical Brownian motion on $L^2(D)$. The noise η is modelled as the formal derivative $\Phi_1 \frac{d\tilde{B}(t)}{dt}$, $\Phi_1 \in \mathcal{L}_2(L^2(D))$ and $\Phi \in \mathcal{L}_2(V)$ is given by

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_1 \end{pmatrix}.$$

The interpretation of the noise term is the same as in Example 1.

In this case, the assumptions (A1) – (A4) are also satisfied.

4. Simulations

4.1 Statistical evidence for $(\hat{a}_T, \hat{b}_T)$ and $(\tilde{a}_T, \tilde{b}_T)$

We have generated a trajectory of the solution to the stochastic differential equation (3.31) from Example 1 in the program R by the Euler's method (see [9, Chapter 2] and Appendix C). The setup of Example 1 is specified as follows:

- $D = (0,1)$ – We consider the wave equation for the oscillating rod modelled as a function from the space $L^2((0,1))$.
- The choice of the orthonormal basis of the space $L^2((0,1))$ is
$$\{e_n(\xi) = \sqrt{2}\sin(n\pi\xi), n = 1, \dots, N\},$$
whose elements satisfy the boundary condition $u(t,0) = 0 = u(t,1)$, for any $t > 0$.
- $N = 10$ – Due to possible memory limitations, we have restricted the expansion of the previous basis only to $N = 10$ functions. The accuracy of our results may suffer due to this limitation, nevertheless we will show that our results are sufficiently satisfactory.
- $T = 100$ – The length of the time interval.
- $\Delta t = 0.001$ – The mesh of the partition of the time interval $[0, T]$.
- The intial functions u_1 and u_2 have the following form
$$u_1(\xi) = \sqrt{2}\sum_{n=1}^{N}\sin(n\pi\xi) = u_2(\xi).$$
This means that $\langle u_1, e_n\rangle_{L^2(D)} = 1 = \langle u_2, e_n\rangle_{L^2(D)}$ for any $n = 1, \dots, N$, so the initial conditions are the same in all N dimensions.
- $a = 1$, $b = 0.2$ – The values of the parameters that are to be estimated.
- $-\alpha_n = -n^2\pi^2$ – The eigenvalues of the operator A. With this setup the operator A is the Laplacian operator $A = \Delta|_{\mathrm{Dom}(A)}$ with $\mathrm{Dom}(A) = H^2((0,1)) \cap H_0^1((0,1))$.
- $\lambda_n = \frac{1000}{n^2}$ – The eigenvalues of the operator Q. (The eigenvalues of the operator Φ_1 equal to $\sqrt{\lambda_n}$ for any $n = 1, \dots, N$.) The eigenvalues are chosen in the way that the sum $\sum_{n=1}^{\infty}\lambda_n$ is convergent. The multiplication factor is chosen in order to increase the values of the λ_n. Otherwise the noise would be in "higher" dimensions so small that it would be practically vanishing.
- We consider the "diagonal case", i.e., the eigenvectors of the operators A and Q coincide and form the basis $\{e_n(\cdot), n = 1, \dots, N\}$.

From the generated trajectory we obtained the following results: The value of the statistic I_T (on which the estimators $\hat{a}_T$ and $\hat{b}_T$ are based on (see Theorem 2.3)) is $I_T = 2740.959$, while the trace of the operator $Q_\infty^{(a,b)}$ equals to $\operatorname{Tr} Q_\infty^{(a,b)} = \frac{b+1}{4ab}\sum_{n=1}^N \lambda_n = 2324.652$ (since we have restricted ourselves to just $N = 10$ dimensions, we use only the sum of the N eigenvalues to compute $\operatorname{Tr} Q$). The estimators of a and b are $\hat{a}_T = 0.8481$ and $\hat{b}_T = 0.1646$ and their time evolution is shown in Figure 4.1.

Let us compute the estimators $\tilde{a}_T$ and $\tilde{b}_T$ from Theorem 2.4. The results are the following

$$\begin{aligned} Y_T &= 2330.218, & \frac{1}{4ab}\sum_{n=1}^N \lambda_n &= 1937.210, \\ H_T &= 410.741, & \frac{1}{4a}\sum_{n=1}^N \lambda_n &= 387.442, \\ \tilde{a}_T &= 0.9433, & \tilde{b}_T &= 0.1763. \end{aligned}$$

Time evolution of the estimators $\tilde{a}_t$ and $\tilde{b}_t$ is shown in the Figure 4.2.

From the figures and from the results it seems that the family of estimators $(\tilde{a}_T, \tilde{b}_T)$ was better than the family $(\hat{a}_T, \hat{b}_T)$, nevertheless we have made 100 more simulations in a similar manner. The values of the estimators $\hat{a}_T$ and $\hat{b}_T$ are depicted in Figure 4.3 and the values of the estimators $\tilde{a}_T$ and $\tilde{b}_T$ are depicted in Figure 4.4. The overall statistics can be found in Table 4.1.

The row "Var" stands for the variance of $\sqrt{T}(\hat{a}_T - a)$ (and its analogues in the following columns). The actual variances of the estimators are 100 times smaller. The theoretical values of the limiting variances (see formulae in Remark 4) can be found in the row "Var – Theoretical".

Since the absolute errors of the estimators can be viewed in Figures 4.3 and 4.4, we mention only relative errors: maximal (which is the relative error of the worst estimator) and typical (that is the level below which 75 % of the errors belong).

The p–values of the Wilk–Shapiro test of normality can be found in the last row. Since they are greater than 0.05, we do not reject the hypothesis of normality on 5%–significance level. The Q–Q plots of the centered and rescaled estimators are shown in Figures 4.5 and 4.6.

From the previous simulations the main three observations follow:

	$\hat{a}_T$	$\hat{b}_T$	$\tilde{a}_T$	$\tilde{b}_T$
Mean	0.9994	0.2003	0.9948	0.2013
Var	0.9473	0.0545	0.4218	0.0298
Var – Theoretical	1.0466	0.0776	0.4505	0.0343
Relative error – Maximal	26 %	30 %	20 %	24 %
Relative error – Typical	$\leq$ 10 %	$\leq$ 10 %	$\leq$ 5 %	$\leq$ 7 %
p–value	0.2746	0.2728	0.3790	0.5800

Table 4.1: The results of the simulations

- The family of the estimators $(\tilde{a}_T, \tilde{b}_T)$ has similar mean as the family $(\hat{a}_T, \hat{b}_T)$, but in addition it has smaller variances and smaller relative errors. That behaviour is the consequence of Theorem 2.12.
- From the comparing of the rows "Var" and "Var – Theoretical" it seems that the limiting variances from Remark 4 are accurate.
- From Figures 4.5, 4.6 and from the results of the Wilk–Shapiro tests it seems that the estimators are asymptotically normally distributed as prescribed.

Although these results for time $T = 100$ are satisfactory enough, we have also made simulations for time $T = 1000$. The results from one particular trajectory are the following

$$
\begin{aligned}
I_T &= 2360.458, & \frac{b+1}{4ab}\sum_{n=1}^{N}\lambda_n &= 2324.652,\\
Y_T &= 1975.777, & \frac{1}{4ab}\sum_{n=1}^{N}\lambda_n &= 1937.210,\\
H_T &= 384.681, & \frac{1}{4a}\sum_{n=1}^{N}\lambda_n &= 387.442,\\
\hat{a}_T &= 0.9848, & \hat{b}_T &= 0.1964,\\
\tilde{a}_T &= 1.0072, & \tilde{b}_T &= 0.1947.
\end{aligned}
$$

Time evolution of the estimators $(\hat{a}_T, \hat{b}_T)$ is shown in Figure 4.7 and time evolution of the estimators $(\tilde{a}_T, \tilde{b}_T)$ can be seen in Figure 4.8.

From this one particular trajectory it seems that the families $(\hat{a}_T, \hat{b}_T)$ and $(\tilde{a}_T, \tilde{b}_T)$ do not differ much, but let us take a closer look at the results of 100 simulations. Figures 4.9 and 4.10 show values of all obtained estimators with corresponding Q–Q plots depicted in Figures 4.11 and 4.12. The overall statistics can be found in Table 4.2 with the same meaning as above.

The conclusions of these simulations are similar as above: The family of estimators $(\tilde{a}_T, \tilde{b}_T)$ can be viewed better as the family $(\hat{a}_T, \hat{b}_T)$ since it has smaller variances and smaller relative errors. Moreover, we can compare the results from Tables 4.1 and 4.2:

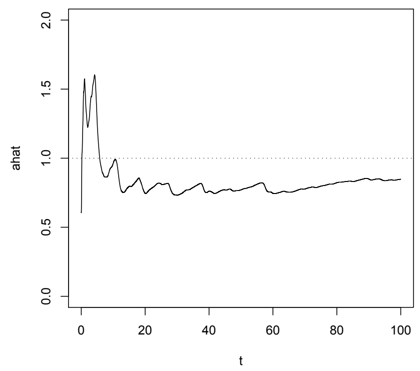
- The estimators for the time $T = 1000$ have 10 times lesser variances than those for the time $T = 100$. (The actual variances of the estimators for the

	$\hat{a}_T$	$\hat{b}_T$	$\tilde{a}_T$	$\tilde{b}_T$
Mean	0.9921	0.1982	0.9916	0.2001
Var	1.1285	0.0648	0.5186	0.0280
Var – Theoretical	1.0466	0.0776	0.4505	0.0343
Relative error – Maximal	9 %	12 %	6 %	6 %
Relative error – Typical	≤ 4 %	≤ 5 %	≤ 3 %	≤ 3 %
p–value	0.8690	0.7913	0.7093	0.4192

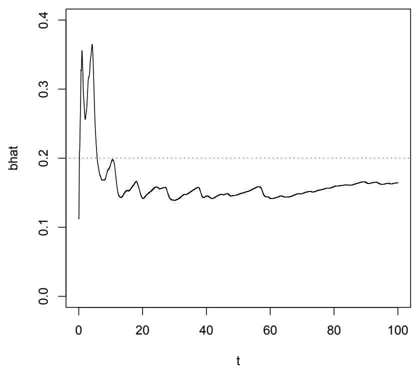
Table 4.2: The results of the simulations for time $T = 1000$

time $T = 1000$ are 1000 times smaller than the numbers in the raw "Var" in Table 4.2.)

- The estimators for the time $T = 1000$ have about two times smaller relative errors than those for the time $T = 100$.
- From the Q–Q plots and from the results of the Wilk–Shapiro tests it seems that the asymptotic normality of estimators is better for greater time T as expected.

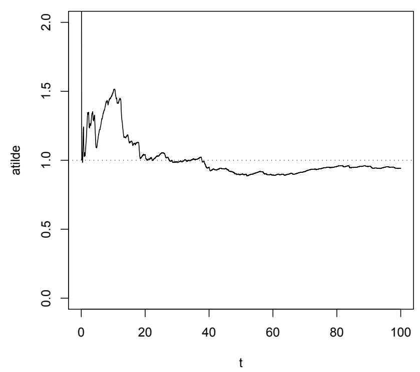


(a) The estimator $\hat{a}_t$

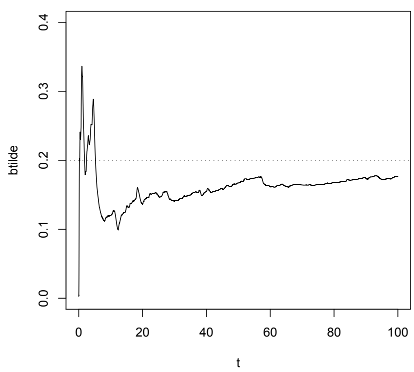


(b) The estimator $\hat{b}_t$

Figure 4.1: The time evolution of the estimators $\hat{a}_t$ and $\hat{b}_t$

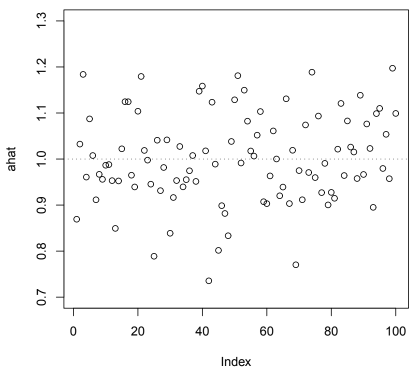


(a) The estimator $\tilde{a}_t$

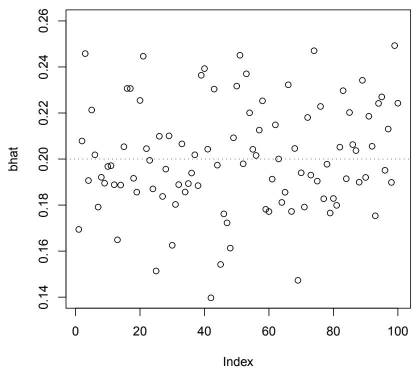


(b) The estimator $\tilde{b}_t$

Figure 4.2: The time evolution of the estimators $\tilde{a}_t$ and $\tilde{b}_t$

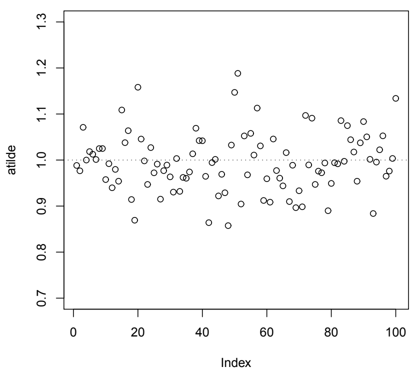


(a) The values of $\hat{a}_T$

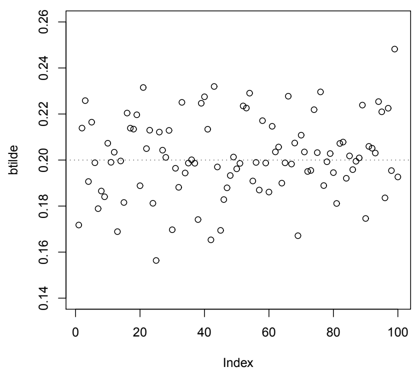


(b) The values of $\hat{b}_T$

Figure 4.3: The estimators $\hat{a}_T$ and $\hat{b}_T$ – Overall

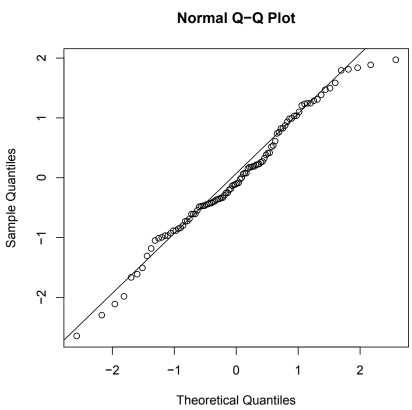


(a) The values of $\tilde{a}_T$

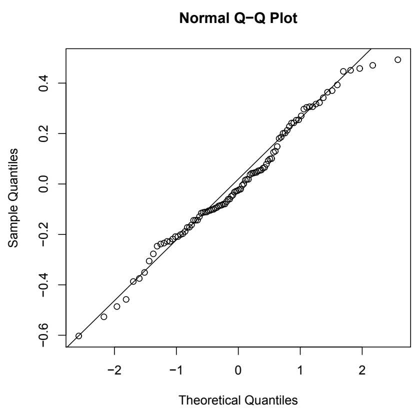


(b) The values of $\tilde{b}_T$

Figure 4.4: The estimators $\tilde{a}_T$ and $\tilde{b}_T$ – Overall

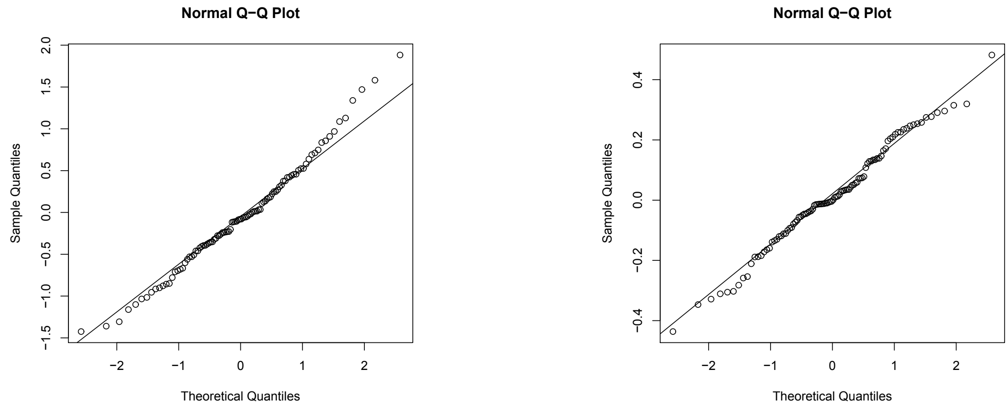


(a) Q–Q plot of $\sqrt{T}(\hat{a}_T - a)$



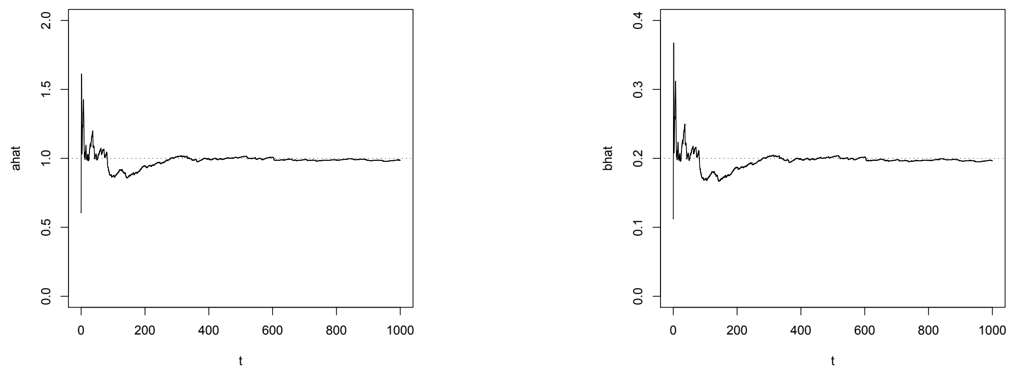
(b) Q–Q plot of $\sqrt{T}(\hat{b}_T - b)$

Figure 4.5: Asymptotic normality of $\hat{a}_T$ and $\hat{b}_T$



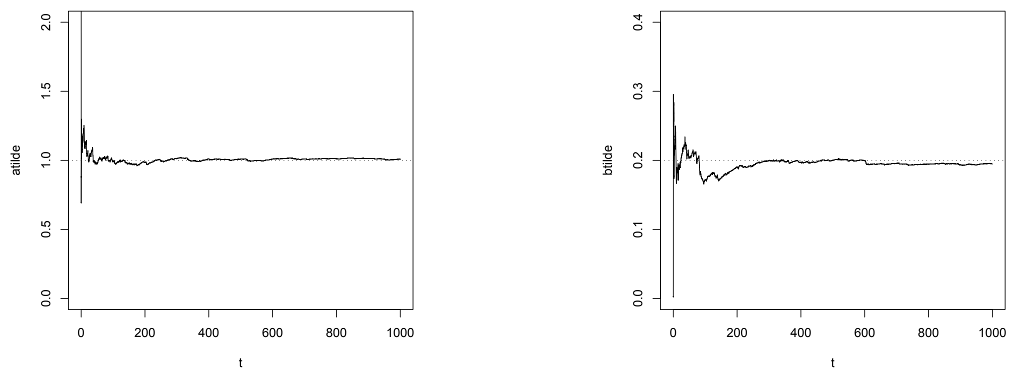
(a) Q–Q plot of $\sqrt{T}(\tilde{a}_T - a)$ (b) Q–Q plot of $\sqrt{T}(\tilde{b}_T - b)$

Figure 4.6: Asymptotic normality of $\tilde{a}_T$ and $\tilde{b}_T$



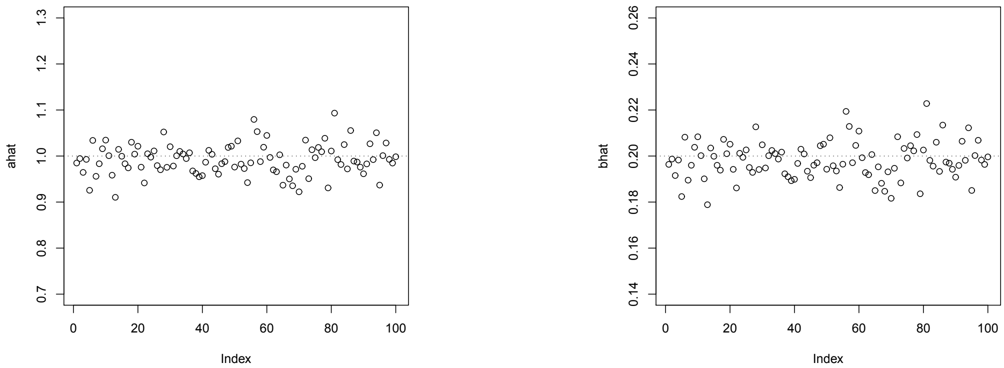
(a) The estimator $\hat{a}_t$ (b) The estimator $\hat{b}_t$

Figure 4.7: The time evolution of the estimators $\hat{a}_t$ and $\hat{b}_t$, $T = 1000$



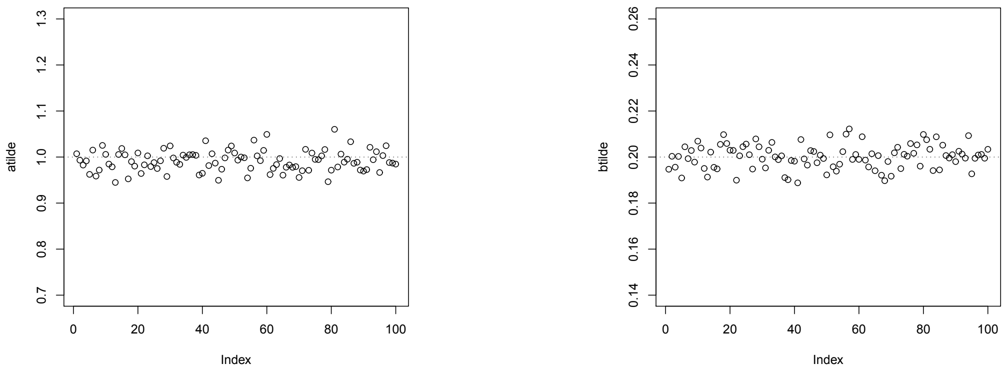
(a) The estimator $\tilde{a}_t$ (b) The estimator $\tilde{b}_t$

Figure 4.8: The time evolution of the estimators $\tilde{a}_t$ and $\tilde{b}_t$, $T = 1000$



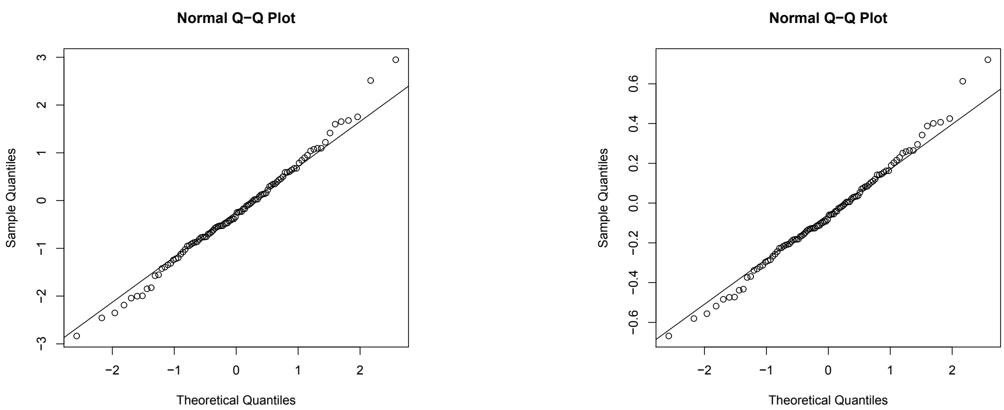
(a) The values of $\hat{a}_T$ (b) The values of $\hat{b}_T$

Figure 4.9: The estimators $\hat{a}_T$ and $\hat{b}_T$, $T = 1000$ – Overall



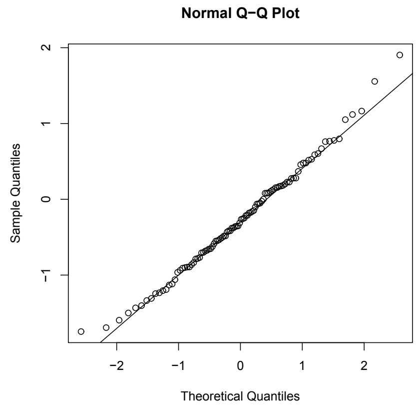
(a) The values of $\tilde{a}_T$ (b) The values of $\tilde{b}_T$

Figure 4.10: The estimators $\tilde{a}_T$ and $\tilde{b}_T$, $T = 1000$ – Overall

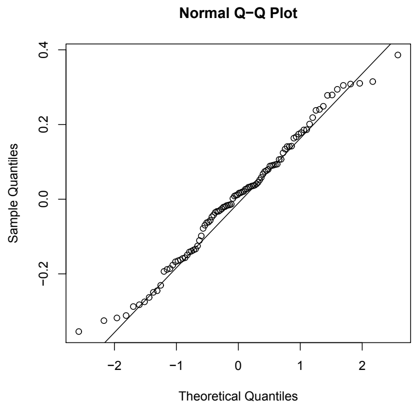


(a) Q–Q plot of $\sqrt{T}(\hat{a}_T - a)$ (b) Q–Q plot of $\sqrt{T}(\hat{b}_T - b)$

Figure 4.11: Asymptotic normality of $\hat{a}_T$ and $\hat{b}_T$, $T = 1000$



(a) Q–Q plot of $\sqrt{T}(\tilde{a}_T - a)$



(b) Q–Q plot of $\sqrt{T}(\tilde{b}_T - b)$

Figure 4.12: Asymptotic normality of $\tilde{a}_T$ and $\tilde{b}_T$, $T = 1000$

4.2 Statistical evidence for $\bar{a}_{T,k}, \bar{b}_{T,j,k}$ and $\bar{b}_{T,j,a}$

In this section we will focus on the parameters established in Chapter 3. We will use the same setup of Example 1 as above, except the fact that $T = 1000$ and $\Delta t = 0.0001$. Since there is no need to compute the norm of the solution (observation of some coordinates of the solution is satisfactory enough), simulations do not require that much computer memory as earlier, so we may afford to extend the time interval $[0, T]$ as well as to refine the mesh of the partition Δt.

We will observe the estimation of the parameter a using the first and tenth coordinate of the second component, i.e., $\bar{a}_{T,k=1}$ and $\bar{a}_{T,k=10}$ (see (3.7)), and the estimation of the parameter b using the estimators $\bar{a}_{T,k=1}$ and $\bar{a}_{T,k=10}$ of a together with the first and tenth coordinate of the second component, respectively, i.e., $\bar{b}_{T,j=1,k=1}$ and $\bar{b}_{T,j=10,k=10}$ (see (3.8)). We will also study the estimators $\bar{b}_{T,j=1,a=1}$ and $\bar{b}_{T,j=10,a=1}$, which depend only on the "observation window" $(f_1, 0)^\top$ (or $(f_{10}, 0)^\top$) with the parameter a supposed to be known (see (3.28)).

Using the generated trajectory, we obtained the following results:

$$
\begin{aligned}
\bar{a}_{T,k=1} &= 0.9994, & \bar{a}_{T,k=10} &= 0.9978, \\
\bar{b}_{T,j=1,k=1} &= 0.1902, & \bar{b}_{T,j=10,k=10} &= 0.2001, \\
\bar{b}_{T,j=1,a=1} &= 0.1901, & \bar{b}_{T,j=10,a=1} &= 0.1997.
\end{aligned}
$$

Time evolution of these estimators is depicted in Figures 4.13, 4.14 and 4.15.

Although the results seems satisfactory (especially for the estimator $\bar{b}_{T,j=10,k=10}$), we have made 100 more simulations in a similar manner. The values of the estimators $\bar{a}_{T,k=1}$ and $\bar{a}_{T,k=10}$ are depicted in Figure 4.16 and the values of the estimators $\bar{b}_{T,j=1,k=1}$ and $\bar{b}_{T,j=10,k=10}$ are depicted in Figure 4.17. Moreover, the values of the estimators $\bar{b}_{T,j=1,a=1}$ and $\bar{b}_{T,j=10,a=1}$ are shown in Figure 4.18. The overall statistics can be found in Tables 4.3 and 4.4.

The row "Var" stands for the variance of $\sqrt{T}(\bar{a}_{T,k} - a)$ (and its analogues in the following columns). The actual variances of the estimators are 1000 times smaller. The theoretical values of the limiting variances (see formulae in Remark 5) can be found in the row "Var – Theoretical". The following rows have the same meaning as in previous section. We also do not reject the hypothesis of normality

	$\bar{a}_{T,k=1}$	$\bar{a}_{T,k=10}$
Mean	0.9995	0.9673
Var	1.1039	1.0036
Var – Theoretical	1.0000	1.0000
Relative error – Maximal	10 %	10 %
Relative error – Typical	≤ 4 %	≤ 6 %
p–value	0.9161	0.6340

Table 4.3: The results of the simulations – Window – Part I

	$\bar{b}_{T,j=1,k=1}$	$\bar{b}_{T,j=10,k=10}$	$\bar{b}_{T,j=1,a=1}$	$\bar{b}_{T,j=10,a=1}$
Mean	0.1989	0.2000	0.1988	0.1935
Var	0.0616	0.0008	0.1033	0.0402
Var – Theoretical	0.0811	0.0008	0.1211	0.0408
Relative error – Maximal	9 %	1 %	12 %	10 %
Relative error – Typical	≤ 5 %	≤ 0.6 %	≤ 6 %	≤ 6 %
p–value	0.7904	0.2986	0.2825	0.6668

Table 4.4: The results of the simulations – Window – Part II

on 5%–significance level for any estimator. The Q–Q plots of the centered and rescaled estimators are shown in Figures 4.19, 4.20 and 4.21.

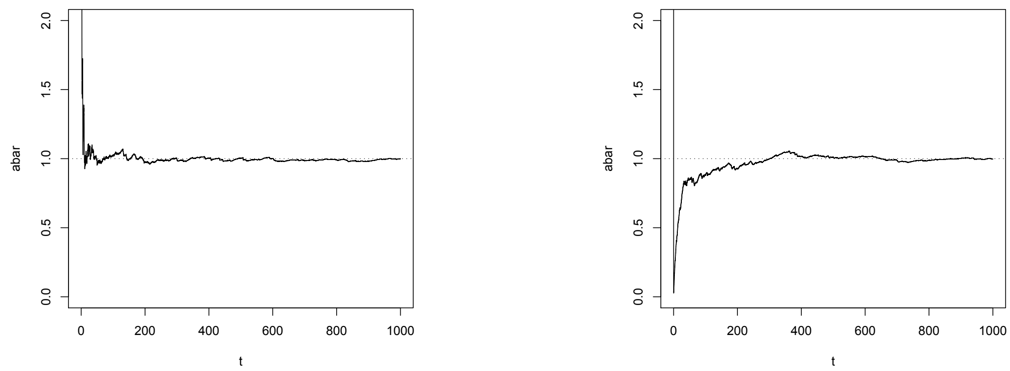
From the previous simulations the main three observations follow:

- The estimators $\bar{a}_{T,k=1}$ and $\bar{a}_{T,k=10}$ behave similarly (the estimator $\bar{a}_{T,k=10}$ would require some bigger time T, though), however there is a big difference between the estimators of the parameter b. Not only that the estimator $\bar{b}_{T,j=10,k=10}$ behaves better than the estimator $\bar{b}_{T,j=1,k=1}$ (it has better mean and lesser variance and relative errors), but also by comparing the estimator $\bar{b}_{T,j=1,k=1}$ with $\bar{b}_{T,j=1,a=1}$ (and $\bar{b}_{T,j=10,k=10}$ with $\bar{b}_{T,j=10,a=1}$), it seems that it is better to work with the paramater a unknown. (See Remark 5.)

- From the comparing of the rows ”Var” and ”Var–Theoretical” it seems that the computed limiting variances from Remark 5 are accurate.

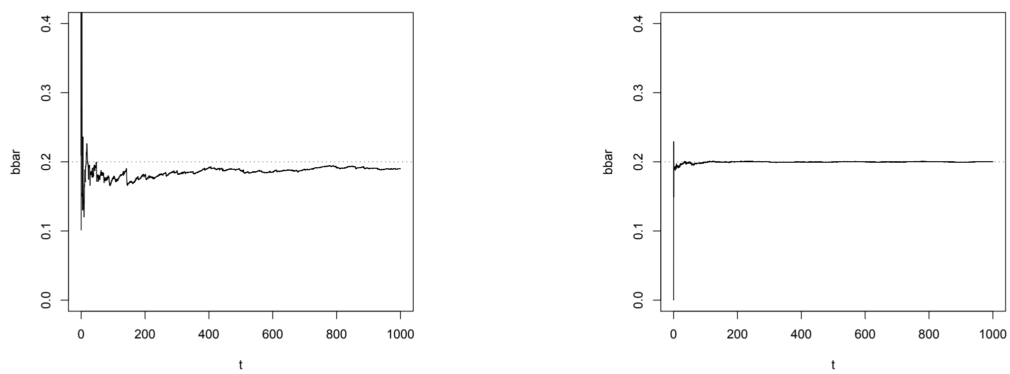
- From the Figures 4.19, 4.20 and 4.21 and from the results of Wilk–Shapiro tests it seems that the estimators are asymptotically normal as prescribed by Theorems 3.7, 3.11 and 3.12.

These simulations precisely match the results obtained in the theoretical part of the Thesis.



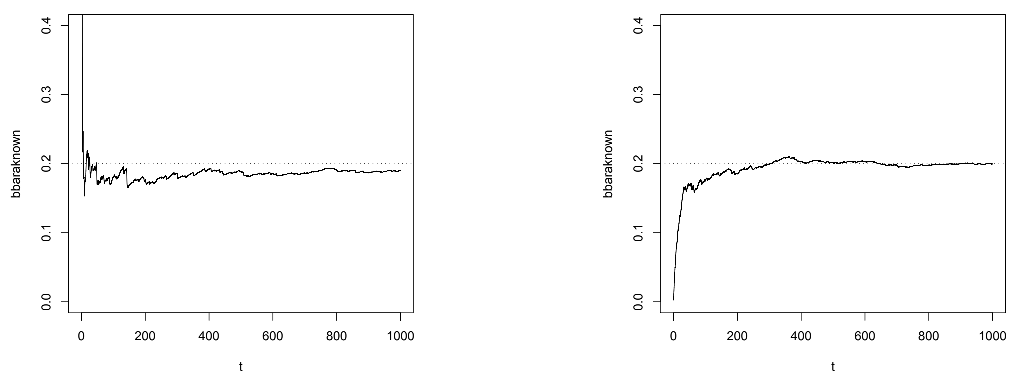
(a) The estimator $\bar{a}_{t,k=1}$ (b) The estimator $\bar{a}_{t,k=10}$

Figure 4.13: The time evolution of the estimators $\bar{a}_{t,k}$ for $k = 1$ and $k = 10$



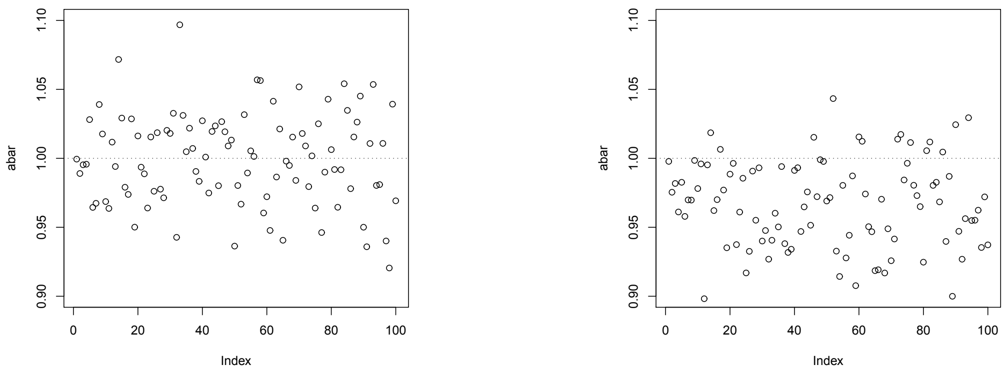
(a) The estimator $\bar{b}_{t,j=1,k=1}$ (b) The estimator $\bar{b}_{t,j=10,k=10}$

Figure 4.14: The time evolution of the estimators $\bar{b}_{t,j,k}$ for $j = k = 1$ and $j = k = 10$



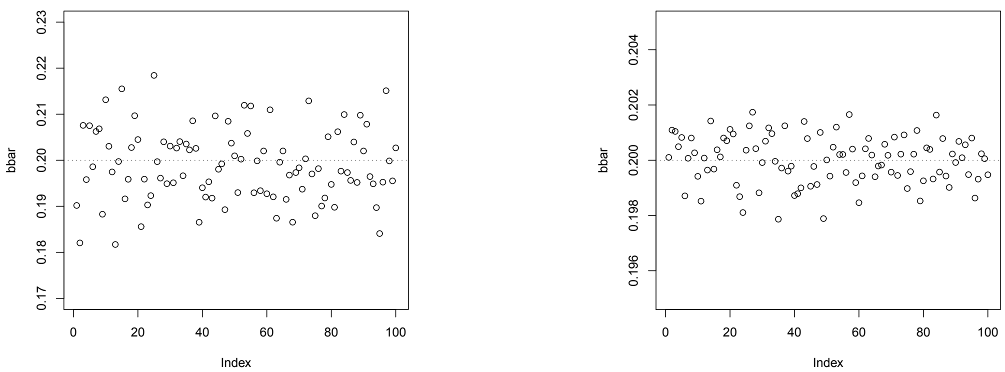
(a) The estimator $\bar{b}_{t,j=1,a=1}$ (b) The estimator $\bar{b}_{t,j=10,a=1}$

Figure 4.15: The time evolution of the estimators $\bar{b}_{t,j,a}$ for $j = 1$, $j = 10$ and $a = 1$



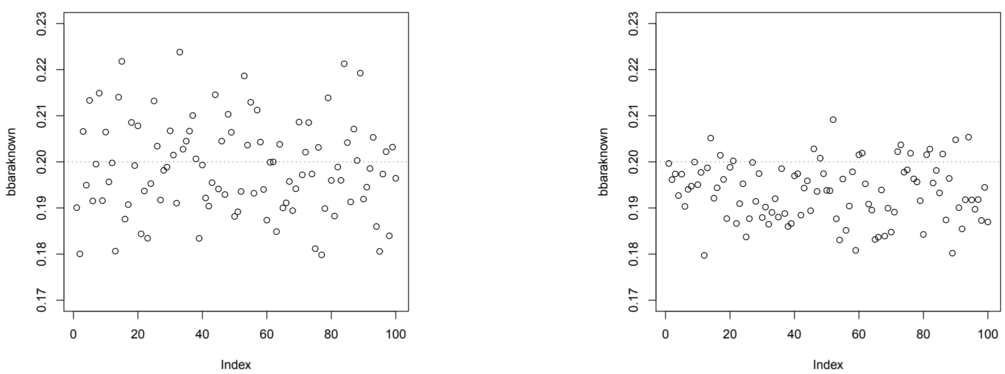
(a) The values of $\bar{a}_{T,k=1}$ (b) The values of $\bar{a}_{T,k=10}$

Figure 4.16: The estimators $\bar{a}_{t,k}$ for $k = 1$ and $k = 10$ – Overall



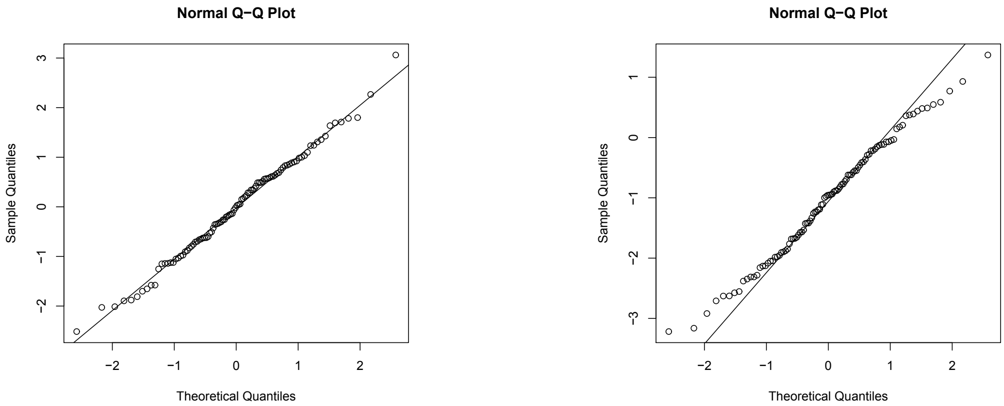
(a) The values of $\bar{b}_{T,j=1,k=1}$ (b) The values of $\bar{b}_{T,j=10,k=10}$

Figure 4.17: The estimators $\bar{b}_{t,j,k}$ for $j = k = 1$ and $j = k = 10$ – Overall



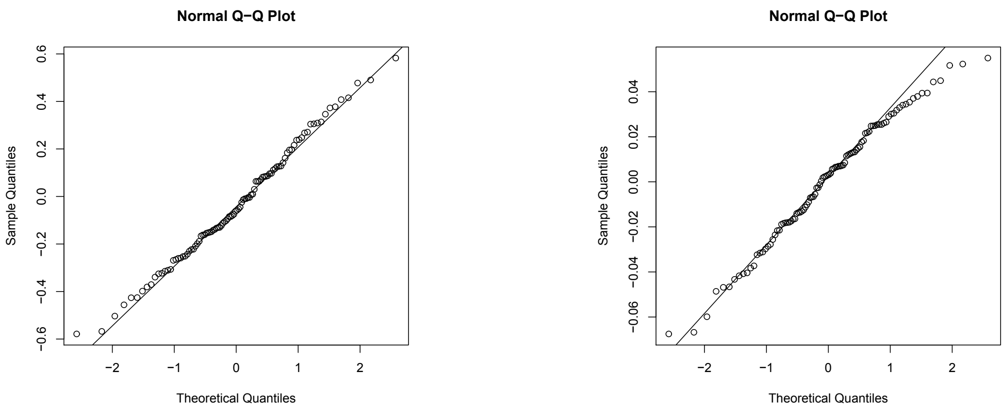
(a) The values of $\bar{b}_{T,j=1,a=1}$ (b) The values of $\bar{b}_{T,j=10,a=1}$

Figure 4.18: The estimators $\bar{b}_{t,j,a}$ for $j = 1$, $j = 10$ and $a = 1$ – Overall



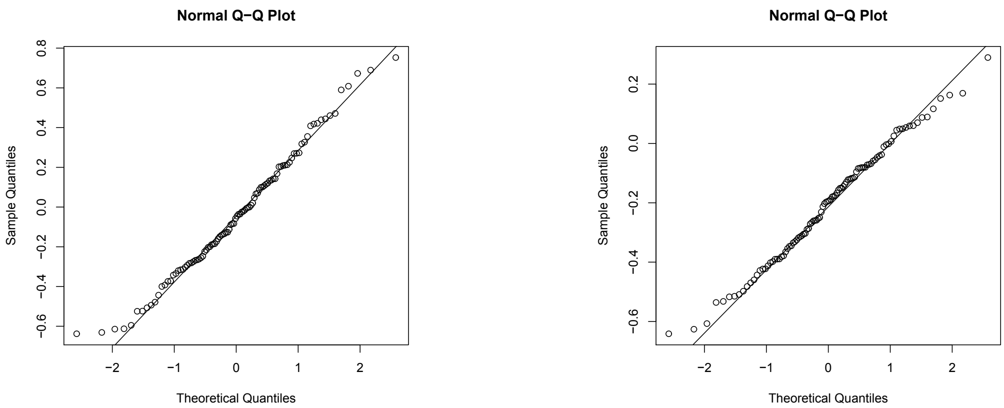
(a) Q–Q plot of $\sqrt{T}(\bar{a}_{T,k=1} - a)$ (b) Q–Q plot of $\sqrt{T}(\bar{a}_{T,k=10} - a)$

Figure 4.19: Asymptotic normality of $\bar{a}_{T,k}$ for $k = 1$ and $k = 10$



(a) Q–Q plot of $\sqrt{T}(\bar{b}_{T,j=1,k=1} - b)$ (b) Q–Q plot of $\sqrt{T}(\bar{b}_{T,j=10,k=10} - b)$

Figure 4.20: Asymptotic normality of $\bar{b}_{T,j,k}$ for $j = k = 1$ and $j = k = 10$



(a) Q–Q plot of $\sqrt{T}(\bar{b}_{T,j=1,a=1} - b)$ (b) Q–Q plot of $\sqrt{T}(\bar{b}_{T,j=10,a=1} - b)$

Figure 4.21: Asymptotic normality of $\bar{b}_{T,j,a}$ for $j = 1$, $j = 10$ and $a = 1$

Conclusion

The Thesis enriches the subject of statistical inference for stochastic hyperbolic partial differential equations. In particular, minimum constrast estimators are considered, which is not a frequent tool. The main and the most interesting results of the Thesis include the following:

The form of the covariance operator $Q_\infty^{(a,b)}$ of the invariant measure $\mu_\infty^{(a,b)}$ for the hyperbolic system is new, even if the driving process is the cylindrical Brownian motion. The exact formula for the non–diagonal case is stated in Theorem 1.15. The diagonal case is mentioned in Remark 4.

The derivation of the strongly consistent family of minimum constrast estimators $(\hat{a}_T, \hat{b}_T)$ by the ergodic theorem is surpassed by finding the family $(\tilde{a}_T, \tilde{b}_T)$, which has better properties. Strong consistency of the estimators is shown in Theorems 2.3 and 2.4 and their asymptotic normality is proved in Theorems 2.8 and 2.11.

Based on certain "observation windows", several new estimators are proposed, which are accomplished by various estimation strategies. The strong consistency of these estimators is described by Theorem 3.1 and Corollary 3.2. Their asymptotic normality is the main topic of Section 3.2 with some interesting conclusions in Remark 5.

All these results were verified by numerical simulations in Chapter 4 with very satisfying outcome.

Appendices

A. Nuclear and Hilbert–Schmidt operators

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be real Banach spaces and let $\mathcal{L}(E,F)$ be the Banach space of all linear bounded operators from E to F endowed with the usual supremum norm. We denote by E^* and F^* the dual spaces of E and F, respectively.

Definition A.1. An operator $T \in \mathcal{L}(E,F)$ is called nuclear (or trace–class) if there exist two sequences $\{a_j, j \in \mathbb{N}\} \subset F$ and $\{\varphi_j, j \in \mathbb{N}\} \subset E^*$ such that $\sum_{j=1}^{\infty} \|a_j\|_F \|\varphi_j\|_{E^*} < \infty$ and T has the representation

$$Tx = \sum_{j=1}^{\infty} a_j \varphi_j(x), \quad x \in E. \tag{A.1}$$

The space of all nuclear operators from E to F, endowed with the norm

$$\|T\|_1 = \inf \left\{ \sum_{j=1}^{\infty} \|a_j\|_F \|\varphi_j\|_{E^*},\, Tx = \sum_{j=1}^{\infty} a_j \varphi_j(x) \right\},$$

is a Banach space and we denote it by $\mathcal{L}_1(E,F)$. Also $\mathcal{L}_1(E)$ stands for $\mathcal{L}_1(E,E)$.

Definition A.2. Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ be a real separable Hilbert space and $\{e_j, j \in \mathbb{N}\}$ be a complete orthonormal basis in H. For $T \in \mathcal{L}_1(H)$, we define the trace of the operator T by

$$\operatorname{Tr} T = \sum_{j=1}^{\infty} \langle Te_j, e_j \rangle_H . \tag{A.2}$$

Proposition A.1. *If $T \in \mathcal{L}_1(H)$ then $\operatorname{Tr} T$ is well–defined number independent on the choice of the orthonormal basis $\{e_j, j \in \mathbb{N}\}$.*

Proof. See [5, Proposition C.1]. □

Also note that

$$|\operatorname{Tr} T| \le \|T\|_1, \quad T \in \mathcal{L}_1(H).$$

Proposition A.2. *If $T \in \mathcal{L}_1(H)$ and $S \in \mathcal{L}(H)$ then $TS, ST \in \mathcal{L}_1(H)$ and*

$$\operatorname{Tr} TS = \operatorname{Tr} ST \le \|T\|_1 \|S\|.$$

Proof. See [5, Corollary C.2]. □

Proposition A.3. *A non–negative operator $T \in \mathcal{L}(H)$ is nuclear if and only if $\operatorname{Tr} T < \infty$. Moreover, in this case $\operatorname{Tr} T = \|T\|_1$.*

Proof. See [5, Proposition C.3]. □

Definition A.3. Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ and $(G, \|\cdot\|_G, \langle\cdot,\cdot\rangle_G)$ be real separable Hilbert spaces and $\{e_j, j \in \mathbb{N}\}$ be a complete orthonormal basis in H. An operator $T \in \mathcal{L}(H,G)$ is called Hilbert–Schmidt if

$$\sum_{j=1}^{\infty} \|Te_j\|_G^2 < \infty. \tag{A.3}$$

Similarly as above, the number

$$\|T\|_2 = \left(\sum_{j=1}^{\infty} \|Te_j\|_G^2 \right)^{\frac{1}{2}}$$

is independent on the choice of the orthonormal basis $\{e_j, j \in \mathbb{N}\}$ and defines the norm on the space $\mathcal{L}_2(H, G)$. That is the space of all Hilbert–Schmidt operators from H to G, which is also a separable Hilbert space equipped with the scalar product

$$\langle T, S \rangle_2 = \sum_{j=1}^{\infty} \langle Te_j, Se_j \rangle_G, \quad T, S \in \mathcal{L}_2(H, G).$$

We also denote $\mathcal{L}_2(H, H)$ by $\mathcal{L}_2(H)$.

Proposition A.4. *Let $(H, \|\cdot\|_H, \langle \cdot, \cdot \rangle_H)$, $(G, \|\cdot\|_G, \langle \cdot, \cdot \rangle_G)$ and $(K, \|\cdot\|_K, \langle \cdot, \cdot \rangle_K)$ be real separable Hilbert spaces. If $T \in \mathcal{L}_2(H, G)$ and $S \in \mathcal{L}_2(G, K)$ then $TS \in \mathcal{L}_1(H, K)$ and*

$$\|TS\|_1 \leq \|T\|_2 \|S\|_2.$$

Proof. See [5, Proposition C.4]. □

B. Strongly continuous semigroups

We recall the basic notion on the strongly continuous semigroups on real separable Hilbert spaces and on their infinitesimal generators. As the main reference, we have used [3, Chapter 2], but see also [28].

Definition B.1. Let $(V, \|\cdot\|_V, \langle\cdot,\cdot\rangle_V)$ be a real separable Hilbert space. A strongly continuous semigroup (or C_0–semigroup) is an operator–valued function $S(t)$ from $[0,\infty)$ to $\mathcal{L}(V)$ that satisfies the following properties

- $S(t+s) = S(t)S(s)$ for $t, s \geq 0$,
- $S(0) = I$, (the identity operator on V),
- $\|S(t)v - v\|_V \to 0$ as $t \to 0_+$ for all $v \in V$.

Theorem B.1. *A strongly continuous semigroup $(S(t), t \geq 0)$ on a Hilbert space V has the following properties*

1) $\|S(t)\|_{\mathcal{L}(V)}$ *is bounded on every finite subinterval of* $[0,\infty)$,
2) $S(t)$ *is strongly continuous for all* $t \geq 0$,
3) *For all* $v \in V$ *we have that* $\frac{1}{t}\int_0^t S(s)v\,ds \to v$ *as* $t \to 0_+$,
4) *If* $\omega_0 = \inf_{t>0}\left(\frac{1}{t}\log\|S(t)\|_{\mathcal{L}(V)}\right)$ *then* $\omega_0 = \lim_{t\to\infty}\left(\frac{1}{t}\log\|S(t)\|_{\mathcal{L}(V)}\right) < \infty$,
5) *For all* $\omega > \omega_0$ *there exists a constant* M_ω *such that* $\|S(t)\|_{\mathcal{L}(V)} \leq M_\omega e^{\omega t}$ *holds for all* $t \geq 0$. *The constant* ω_0 *is called the growth bound of the semigroup.*

Proof. See [3, Theorem 2.1.6]. □

Since it is only assumed that $S(t)v$ is continuous, it is generally not possible to differentiate it. However, since the main aim is to relate $S(t)v$ to the solution of an abstract (stochastic) differential equation, we introduce the concept of an infinitesimal generator A.

Definition B.2. The infinitesimal generator A of a strongly continuous semigroup on a Hilbert space V is defined by

$$Av = \lim_{t\to 0_+}\frac{1}{t}\left(S(t) - I\right)v, \tag{B.1}$$

whenever the limit exists. The domain of A, $\mathrm{Dom}(A)$, being the set of elements in V for which the limit exists.

Theorem B.2. *Let $(S(t), t \geq 0)$ be a strongly continuous semigroup on a Hilbert space V with the infinitesimal generator A. Then the following results hold*

1) *For* $v_0 \in \mathrm{Dom}(A), S(t)v_0 \in \mathrm{Dom}(A), \quad \forall t \geq 0$,

2) $\frac{d}{dt}(S(t)v_0) = AS(t)v_0 = S(t)Av_0$ *for* $v_0 \in \mathrm{Dom}(A)$, $t > 0$,

3) $\frac{d^n}{dt^n}(S(t)v_0) = A^n S(t)v_0 = S(t)A^n v_0$ *for* $v_0 \in \mathrm{Dom}(A^n)$, $t > 0$,

4) $S(t)v_0 - v_0 = \int_0^t S(s)Av_0\, ds$ *for* $v_0 \in \mathrm{Dom}(A)$,

5) $\int_0^t S(s)v\, ds \in \mathrm{Dom}(A)$ *and* $A \int_0^t S(s)v\, ds = S(t)v - v$ *for all* $v \in V$. *Moreover,* $\mathrm{Dom}(A)$ *is dense in* V,

6) A *is a closed linear operator,*

7) $\bigcap_{n=1}^{\infty} \mathrm{Dom}(A^n)$ *is dense in* V.

Proof. See [3, Theorem 2.1.10]. □

The previous Theorem shows that for a theory of linear, infinite–dimensional systems of the form

$$\begin{aligned} \dot{v}(t) &= Av(t), \quad t \geq 0, \\ v(0) &= v_0, \end{aligned}$$

we require A to be the infinitesimal generator of a strongly continuous semigroup. Consequently, the following Theorem on the characterization of infinitesimal generators is very important.

Theorem B.3 (Hille–Yoshida)**.** *A necessary and sufficient condition for a closed, densely defined, linear operator* A *on a Hilbert space* V *to be the infinitesimal generator of a strongly continuous semigroup is that there exist real numbers* M *and* ω *such that for all real* $\alpha > \omega, \alpha \in \rho(A)$, *the resolvent set of* A, *and*

$$\|R(\alpha, A)^r\|_{\mathcal{L}(V)} \leq \frac{M}{(\alpha - \omega)^r}, \quad \forall r \geq 1,$$

where $R(\alpha, A) = (\alpha I - A)^{-1}$ *is the resolvent operator.*

In this case

$$\|S(t)\|_{\mathcal{L}(V)} \leq M e^{\omega t}, \quad \forall t \geq 0.$$

Proof. See [3, Theorem 2.1.12]. □

C. Implementation in R

This appendix provides the codes, which were used for simulations in Chapter 4. The R language is rather intuitive, therefore we believe that the implementation is understandable. For more detail, see [9] or the brief R manual ”An Introduction to R” that comes with every installed version of R.

For generating the trajectory of the solution to the equation (3.31) from Example 1, we have used the following code. The declaration of variables follows the setup from Section 4.1 with some changes in notation. The variable K stands for the depth of the expansion (instead of N), while the variable N determines the number of iterations. The first component of the solution (i.e., the process $(X_1^{x_0}(t), t \geq 0)$) is recorded in matrix X and the second component of the solution (i.e., the process $(X_2^{x_0}(t), t \geq 0)$) is recorded in matrix Y.

```
> T <- 100
> N <- 100000
> Delta <- T/N
> K <- 10
> t <- seq(0, T, length = N+1)
> X <- matrix(0, K, N+1)
> Y <- matrix(0, K, N+1)
> X0 <- c(rep(1,K))
> Y0 <- c(rep(1,K))
> a <- 1
> b <- 0.2
> sequence <- 1:K
> alpha <- sequence^2*pi^2
> lambda <- 1000/sequence^2
> Sq <- sqrt(lambda)
> S <- sum(lambda)
> X[,1] <- X0
> Y[,1] <- Y0
> set.seed(222)
> for (i in 2:(N+1)){
+ X[,i] <- X[,i-1] + Y[,i-1]*Delta
+ Y[,i] <- Y[,i-1] - (b*alpha*X[,i-1] + 2*a*Y[,i-1])*Delta +
+ Sq*rnorm(K)*sqrt(Delta)
+ }
```

The implementation of the family of estimators $(\hat{a}_T, \hat{b}_T)$ as well as the computation of the necessary statistics I_T (see Theorem 2.3) is as follows.

```
> I <- numeric(N+1)
> ahat <- numeric(N+1)
> bhat <- numeric(N+1)
> I[1] <- sum(alpha*X[,1]^2) + sum(Y[,1]^2)
> ahat[1] <- S*(b+1)/(4*b*I[1])
> bhat[1] <- S/(4*a*I[1] - S)
> for (i in 2:(N+1)){
```

```
+ I[i] <- (I[i-1]*(i-1) + sum(alpha*X[,i]^2) + sum(Y[,i]^2))/i
+ ahat[i] <- S*(b+1)/(4*b*I[i])
+ bhat[i] <- S/(4*a*I[i] - S)
+ }
> Tr <- S*(b+1)/(4*a*b)
> Tr
> I[N+1]
> ahat[N+1]
> bhat[N+1]
> plot(t, I, type = "l")
> abline(h = Tr, lty = 3)
> plot(t, ahat, ylim = c(0,2), type = "l")
> abline(h = a, lty = 3)
> plot(t, bhat, ylim = c(0,0.4), type = "l")
> abline(h = b, lty = 3)
```

The following piece of code shows the implementation of the family of estimators $(\tilde{a}_T, \tilde{b}_T)$ established in Theorem 2.4. Note that the code uses IX for the record of Y_T and IY for the record of H_T.

```
> IX <- numeric(N+1)
> IY <- numeric(N+1)
> atilde <- numeric(N+1)
> btilde <- numeric(N+1)
> IX[1] <- sum(alpha*X[,1]^2)
> IY[1] <- sum(Y[,1]^2)
> atilde[1] <- S/(4*IY[1])
> btilde[1] <- IY[1]/IX[1]
> for (i in 2:(N+1)){
+ IX[i] <- (IX[i-1]*(i-1) + sum(alpha*X[,i]^2))/i
+ IY[i] <- (IY[i-1]*(i-1) + sum(Y[,i]^2))/i
+ atilde[i] <- S/(4*IY[i])
+ btilde[i] <- IY[i]/IX[i]
+ }
> S/(4*a*b)
> S/(4*a)
> IX[N+1]
> IY[N+1]
> atilde[N+1]
> btilde[N+1]
> plot(t, IX, type = "l")
> abline(h = S/(4*a*b), lty = 3)
> plot(IY, type = "l")
> abline(h = S/(4*a), lty = 3)
> plot(t, atilde, ylim = c(0,2), type = "l")
> abline(h = a, lty = 3)
> plot(t, btilde, ylim = c(0,0.4), type = "l")
> abline(h = b, lty = 3)
```

The last part of code was used for computing the estimators $\bar{a}_{T,k}$, $\bar{b}_{T,k,k}$ and $\bar{b}_{T,k,a}$ (see formulae (3.7), (3.8), (3.28) and Section 4.2).

```
> k <- 1 ### choose k - the number of the coordinate
> IX <- numeric(N+1)
> IY <- numeric(N+1)
> abar <- numeric(N+1)
> bbar <- numeric(N+1)
> bbaraknown <- numeric(N+1)
> IX[1] <- alpha[k]*X[k,1]^2
> IY[1] <- Y[k,1]^2
> abar[1] <- lambda[k]/(4*IY[1])
> bbar[1] <- IY[1]/IX[1]
> bbaraknown[1] <- lambda[k]/(4*a*IX[1])
> for (i in 2:(N+1)){
+ IX[i] <- (IX[i-1]*(i-1) + alpha[k]*X[k,i]^2)/i
+ IY[i] <- (IY[i-1]*(i-1) + Y[k,i]^2)/i
+ abar[i] <- lambda[k]/(4*IY[i])
+ bbar[i] <- IY[i]/IX[i]
+ bbaraknown[i] <- lambda[k]/(4*a*IX[i])
+ }
> lambda[k]/(4*a*b)
> lambda[k]/(4*a)
> IX[N+1]
> IY[N+1]
> abar[N+1]
> bbar[N+1]
> bbaraknown[N+1]
> plot(IX, type = "l")
> plot(IY, type = "l")
> plot(t, abar, type = "l")
> plot(t, bbar, type = "l")
> plot(t, bbaraknown, type = "l")
```

After running many simulations (also with different parameters a, b, N, T, Δt, u_1, u_2, λ_n – see the setup in Section 4.1), we claim that all estimators have their derived properties and that our implementation is correct and fully functional.

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Glossary

Spaces of operators and functions

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$	stochastic basis
$H^2(D)$	Sobolev space $W^{2,2}(D)$
$H_0^1(D)$	Sobolev space $W_0^{1,2}(D)$
$\mathcal{L}(U, V)$	space of linear bounded operators from U to V
$\mathcal{L}_1(U, V)$	space of nuclear operators from U to V
$\mathcal{L}_2(U, V)$	space of Hilbert–Schmidt operators from U to V
$\mathcal{L}(V)$	space of linear bounded operators from V to V
$L^p(D)$	Lebesgue space of functions defined on D
$L^p(\Omega)$	Lebesgue space of real random variables
$L^p(\Omega; V)$	Lebesgue space of $V-$valued random variables

Operators

$\mathbb{E}$	expected value operator
Var	variance operator
Δ	Laplace operator
I	identity operator
$\frac{d}{dt}$	derivative with respect to t
$\frac{\partial}{\partial t}$	partial derivative with respect to t
$\frac{\partial^2}{\partial t^2}$	second partial derivative with respect to t

Miscellaneous symbols

∂D	boundary of $D \subset \mathbb{R}^d$
δ	Kronecker delta
$\Delta\|_{\mathrm{Dom}(A)}$	restriction of Δ to $\mathrm{Dom}(A)$
$V_1 \oplus V_2 \oplus V_3$	direct sum of linear spaces V_1, V_2 and V_3
$\mathrm{Dom}\,(\cdot)$	domain of the operator
$\mathrm{Law}\,(\cdot)$	probability distribution
$N(\mu, \sigma^2)$	Gaussian distribution with mean μ and variance σ^2
$N(m, Q)$	Gaussian distribution on Hilbert space with mean m and covariance operator Q
$\mathbb{P} - a.s.$	almost surely under the measure $\mathbb{P}$
$\mathbb{R}$	interval $(-\infty, \infty)$
$\mathbb{R}_+$	interval $[0, \infty)$
$\mathrm{span}\{\cdots\}$	closed linear span
$\mathrm{Tr}\,(\cdot)$	trace of the operator
$w^* - \lim$	weak limit of probability measures; also $\xrightarrow{w^*}$

List of Publications

[1] J. Janák, *Fractional Ornstein–Uhlenbeck bridge*, in Proceedings of the 19th Annual Conference of Doctoral Students, WDS, 2010, pp. 201–206.

[2] J. Janák, *Fractional Brownian motion*, in Proceedings of the 15th Conference on Applied Mathematics, Aplimat, 2016, pp. 557–567.

[3] J. Janák, *Asymptotic normality of parameter estimates for stochastic differential equation of second order*, in Proceedings of the 16th Conference on Applied Mathematics, Aplimat, 2017, pp. 704–711.

[4] J. Janák, *Parameter estimation for stochastic partial differential equations of second order*, Applied Mathematics and Optimization – To appear, (2017). Available at `https://doi.org/10.1007/s00245-018-9506-9` and `https://arxiv.org/abs/1806.04045`.

[5] J. Janák, *Parameter estimation for scalar stochastic differential equation of second order*, in Proceedings of the 17th Conference on Applied Mathematics, Aplimat, 2018, pp. 487–499.

[6] J. Janák, *Parameter estimation for stochastic wave equation based on observation window*, Stochastic Analysis and Applications – Submitted, (2018). Available at `https://arxiv.org/abs/1806.07743`.

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[12] ——, *Asymptotic normality of parameter estimates for stochastic differential equation of second order*, in Proceeding of the 16th Conference on Applied Mathematics, Aplimat, 2017, pp. 704–711.

[13] ——, *Parameter estimation for stochastic partial differential equations of second order*, Applied Mathematics and Optimization – To appear, (2017). Available at `https://doi.org/10.1007/s00245-018-9506-9` and `https://arxiv.org/abs/1806.04045`.

[14] ——, *Parameter estimation for scalar stochastic differential equation of second order*, in Proceeding of the 17th Conference on Applied Mathematics, Aplimat, 2018, pp. 487–499.

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