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Argumentation in mathematics teachers' beliefs and practices in the context of the Czech Republic

Argumentace v přesvědčeních a praktikách učitelů matematiky v kontextu České republiky

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Prohlašuji, že jsem disertační práci na téma *Argumentation in lower secondary school mathematics in the context of the Czech Republic* vypracovala pod vedením školitelky samostatně za použití v práci uvedených pramenů a literatury. Dále prohlašuji, že tato práce nebyla využita k získání jiného nebo stejného titulu.

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podpis

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ABSTRAKT

Cílem práce bylo popsat, jak na pozadí českého kontextu ovlivňují přesvědčení a praktiky učitelů způsob, kterým je vedena matematická argumentace ve výuce matematiky na druhém stupni základní školy a nižších gymnáziích. V první studii charakterizuji prostřednictví studia dokumentů český kurikulární kontext a aspekty odůvodňování obecných matematických vztahů ve vybraných učebnicích. Druhá studie se na příkladu šesti vybraných učitelů zaměřuje na učitelova přesvědčení a praktiky týkající se argumentace, které byly zjišťovány rozhovory a náslechy na jejich vyučovacích hodinách. Pro poslední studii byli vybráni tři učitelé s odlišnými přístupy k výuce a v jejich výuce byly sledovány konkrétní případy praktik týkajících se argumentace. Praktiky byly posuzovány na pozadí norem, které existují v dané třídě, vzhledem k povaze odůvodňování matematických pravidel a k charakteristikám argumentů. Studie zřetelně poukázala na míru vlivu učebnic a žáků na použité argumenty. Ukázalo se, že učitelův důraz na efektivnost (ve smyslu naplnění školních osnov) na straně jedné a jeho důraz na zprostředkování porozumění žákům na straně druhé vedou k odlišným implementacím kurikula. Analýza kurikulárního kontextu a učitelských praktik naznačuje, že odůvodňování pravidel je ve zkoumaném kontextu obecně chápáno jako důležité, ale postrádá jasný kognitivní záměr pro žáky. V národním kurikulu navíc neexistují jasné pokyny, které by podpořily důraz na zprostředkování porozumění a osvojování si specifických způsobů myšlení.

KLÍČOVÁ SLOVA

matematická argumentace, přesvědčení učitelů, matematické kurikulum, druhý stupeň, odůvodňování, argumenty

ABSTRACT

I aim to describe how teachers' beliefs and practices influence the way mathematical argumentation is conducted in lower secondary mathematics classroom within the Czech curricular context. I present results of two studies: the first one characterises the Czech curricular context, namely, the national curricular document and aspects of justification of mathematical statements in selected series of mathematics textbooks. The second study reports on characteristics of teachers' beliefs and practices as related to argumentation on an example of six purposefully selected teachers, via interviews and observations of their lessons. Finally, I select three teachers with differing approaches to teaching and describe specific observed instances in their practices in relation to classroom norms regarding argumentation, justification of general mathematical truths, and aspects of arguments. I show how teachers' beliefs, a textbook and pupils may influence the observed arguments. The studies show that a teachers' emphasis on efficiency (fulfilling school curriculum demands) on one hand and on sense-making on the other lead to distinct implemented curricula. The curricular context and teachers' practice analysis suggest that justification of general truths is generally seen as important but without clear cognitive aims for pupils. There are no specific guidelines regarding argumentation as an activity to promote sense-making and the learning of particular modes of reasoning.

KEYWORDS

mathematical argumentation, teachers' beliefs, mathematics curriculum, lower secondary education, justification, arguments

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Introduction

Whenever I think about what spurred my interest in justification in mathematics, I trace it down to my first year of teaching mathematics and, in particular, to an episode that occurred during that time. I was teaching algebra at a charter high school in Chicago and my entire formal mathematics education had taken place in the Czech educational system already some substantial span of time before that. In my first mathematics teaching experience, therefore, the challenge was always twofold: a cultural one and a teaching one.

James, a pupil of mine, was solving an equation problem (I do not remember what the context was exactly but it was in an Algebra II class for tenth-graders) and arrived at an apparently wrong answer. While examining his solution, I noticed that one of the steps involved a "fraction" that had zero in the denominator. Glad that I found the culprit of the trouble so quickly, I pointed it out to James. The ensuing conversation went along these lines:

"Oh, look, here, you can't divide by zero like that."

"What do you mean?" the young man replied. His confusion was sincere.

"Well… You cannot divide by zero in math," I tried to clarify.

"Why not? " he asked, incredulously.

Now it was my turn to be surprised: "Well… you never heard of this rule before?"

"No. What kind of rule is that?" I could detect defiance in his voice.

"You must have learned it back in the fourth grade or so? Anyway, you cannot divide by zero. That is the rule."

"That is [nonsense], I never heard that. Why couldn't you divide by zero?" James' voice now conveyed as much distrust as defiance.

"Well …" I honestly could not come up with an actual reason. I realized I did not have one; at least not at the ready. I felt frustrated: I was losing face in front of my pupil, apparently breaking his trust, and I felt angry with myself for not being able to give him a mathematical answer. At the same time, I could not believe that a sixteen-year-old person would not know that division by zero is impossible.

Strongly affected, I shared this episode with a colleague mathematics teacher that very same day. She assured me that this concept is indeed part of the elementary school curricula. Then she added that I should have asked him if he could divide a pizza into zero parts. That made sense. I knew, though, there must be a more high-school mathematics way to explain it. Also, it did not explain why James had forgotten. Or perhaps he was just absent when this was covered? And why was it that I could not justify this simple law of arithmetic? Had I been ever given a justification myself, and did I simply forget it with years? And if not, how come I had never demanded one as a pupil? How much was this a question of chance based on my individual mathematics experience, and how much was this a culture-related phenomenon?

This episode in many ways underlines what I felt throughout my teaching experience in Chicago: the feeling that, somehow, the math was different there. Or rather, that the nature of the teaching, learning, and nature of school mathematics was.

This work, therefore, has been driven, by a strong belief that the perception of mathematics varies from community to community, as well as from individual to individual; and that such perceptions are a result of the schooling experience of mathematics.

The above-sketched personal novice-teacher experience gave me a strong impetus to try to explain the observed and unobserved similarities and differences between my Czech mathematics education and my US mathematic education experience. My first investigative steps were motivated by two questions. On what level and what exactly were the factors underlining such differences? And on what level was this a matter of an individual trajectory of experience that shaped my own knowledge, beliefs, and behaviour in the particular mathematics teaching? In other words, could my novice-teacher view of mathematics, apparently containing the existence of unjustified rules, be the result of the mathematics education I had received? If yes, was I an individual case, either simply lacking or forgetting the mathematical knowledge I should have possessed, or perhaps I was affected by a particular set of my teachers with their own varying professional beliefs and values? Or had this view been intrinsically transmitted by the curriculum across the country?

Unfortunately, it is impossible to go back in time to restore and access my own mathematics learning experience throughout schooling years. An investigation on how my beliefs and knowledge had been formed could not rely on salient enough data. Rather, I focused on a more accessible problem: what role do justification and argumentation play in the secondary mathematics education in the Czech Republic today?

I do not have the ambition to characterize all complexities of the nature of Czech mathematics education in a philosophical sense. However, as the reader can see from the introductory anecdote, the cultural aspect of justification in this particular system has been a strong motivational factor throughout the work. Moreover, one thing that especially became clear in reviewing current literature and research in the field was that the way justification, particular arguments and argumentation are planned, enacted and perceived in a classroom is indeed one of the distinguishing attributes of core philosophical approaches to mathematics teaching and learning. In fact, the same can be said about mathematics as a discipline. “In the opinion of some, the name of the mathematics game is proof; no proof, no mathematics. In the opinion of others, this is nonsense; there are many games in mathematics” (Davis & Hersh, 1981, p. 147). Either way, there is no doubt that mathematical arguments lie at the core of mathematical activity.

To frame my investigation, I am adopting the following theoretical stance, expressed in Ernest (1991):

> Curriculum developments depend greatly on the underlying philosophy of mathematics […], as does the view of mathematics they communicate to learners [...]. In addition to curriculum philosophies, teachers’ personal philosophies also have a powerful impact on the way mathematics is taught […]. (pp xiii-xiv)

My personal experience conforms to both research and theory in the field of mathematics education. A review of literature showed quickly that, indeed, there are more or less fundamentally different approaches to the teaching and learning of mathematics, just as there are different views of mathematics itself. Paul Ernest’s seminal work *The Philosophy of*

Mathematics Education (Ernest, 1991) laid out a theoretical framework of five different philosophies of mathematics education, existing to date in the (Anglo-Saxon) world, illustrating their historical-political context, mostly on the case of the United Kingdom education system.

Such underlying philosophies will be reflected in the classroom practices of teachers as well as in the curricular documents, such as textbooks, available for the use of teachers. A review of literature confirmed that it is meaningful to examine curricular documents, especially textbooks, which would provide the cultural context of the Czech mathematics classroom. Identifying and interpreting teachers' beliefs and their practices has become an important tool for explaining what occurs in the classroom environment.

The context of a particular teacher's classroom (mainly the context of the national and school curriculum, but also the teacher's resources such as textbooks and other materials, along with, of course, the pupils and their own orientations, goals, and resources) are an important factor in studying what happens in the classroom. I endeavour to shed light on the Czech socio-curricular context reflected mostly in available mathematics textbooks and curricular documents in my first investigation, Study 1 (Chapter 2).

Research in mathematics education had already produced a significant amount of literature documenting differences in not only national curricula and content of mathematics education, but especially in the realm of differences in perceptions of the institution (mathematics education) itself. In the 1990's, the "attention of research on mathematics teachers shifted from purely cognitive and mathematical to a domain that allows for sociological and psychological consideration" (Žalská, 2012b, p. 47). It started to make sense to study the personal philosophies of individual players in the field (namely pupils and teachers). One of the main motivations for this particular area of study has been the need to understand why teachers and pupils act the way they do when they are interacting with one another, with mathematics itself and in a specific environment.

There is a reason (many, in fact) why a teacher chooses to expose pupils to the justification of why we do not divide by zero (at least in the school mathematics curriculum), and how they choose to do so. There is also a reason for any kind of deviation from their initial or routine plan when they come across a non-routine situation while carrying the plan out. Schoenfeld (2010) argues that such decisions are a function of the interplay of an individual's orientations (beliefs, values, attitudes, dispositions etc.), their resources (material, intellectual, knowledge, physical, etc.) and their goals. If I want to find out what makes justification and arguments take place in a classroom, I need to look at all three categories closely. Study 2A (Chapter 3) describes six cases of teachers who work within the Czech curricular context, focusing on what mathematical education orientations they profess.

Finally, in Study 2B (Chapter 4), I also look in detail at what happens in the classroom and examine what, in particular, influences the arguments that take place in a classroom, and their qualities. Observing instances (or the lack of them) of mathematical arguments and justifications in a classroom, and looking for their roots through the lenses of teachers' orientations, goals and resources conclude my empirical research report.

Note on the language: I debated for a long time the use of the Czech language as a medium for describing my research endeavours. Finally, I opted for English. The reasons are various: firstly, I would like this work to be readily available to the international expert community as I trust it will contribute to the body of knowledge mathematics education research has built over the years. Secondly, the larger part of literature that inspired and supported the work has been written in English. In building on a body of international research and using its language, I would like to achieve a certain distance and a higher level of objectivity. In other words, I am aware of, and acknowledge, the shortcomings of describing a system "from within" (i.e., while being part and a product of it, at least to a large degree), and I rely on an international perspective to help me lessen these. All non-English sources (including data collected, such as teachers' quotes), if cited, have been translated by me into English and the Czech originals are in some cases (such as textbook pictures) included alongside the translation.

1 Theoretical framework

1.1 Argumentation, justification, warrant and proof in a mathematics classroom

The terms *mathematical justification, argumentation, argument, proof,* and *explanation*, are not only used differently in different areas of human activity, but also within mathematics education research itself. Many a time the concepts overlap. In order to establish the subject of my investigation, I define the way I understand these terms throughout the text below.

Sriraman and Umland (2014) define argumentation in mathematics education as "[the] mathematical arguments that pupils and teachers produce in mathematics classrooms […]" (p. 46) and a mathematical argument in a mathematics classroom as "a line of mathematical reasoning that intend to show or explain why a mathematical result is true. The mathematical result might be a general statement about some class of mathematical objects or it might simply be the solution to a mathematical problem that has been posed." (p. 46).

To appreciate how much has been taken away from the general concept of argumentation by this definition, one can simply imagine everyday classroom situations. Both the teacher and the pupils engage in arguments that neither involve a mathematics statement (e.g., a pupil explaining why they did not bring their notebook) nor do they aim to explain why something is true (e.g., the argument "you need to find the value of *y* in order to plug it in") nor do they involve mathematical reasoning (e.g., the argument "because that's the rule").

The definition of mathematical argumentation includes the provision of arguments towards both general and specific mathematical statements, very often the results of problem-solving activities. Hence, I understand problem-solving activity as my subject of interest as long as such activity involves (by definition) the intention to show why something is true (as opposed to showing why – to what end – it is relevant to do something, such as take certain steps in problem solving).

Further, it is crucial to note the intentionality in the definition of argument here. Surely, this is very relevant to a classroom situation, as there may be attempts to explain or demonstrate why something is true but the argument may lack qualities that would make it acceptable by the intended audience (a class of pupils, particular pupils or the teacher).

I will understand here a (mathematical) argument as in Toulmin's model (Toulmin, 2003), a sequence of statements that is provided with the intention to show that a mathematical claim (specific or general) is true (or not true). Thus, a wrongly constituted, or perceived as such, argument also is a subject of our investigation.

Let us consider now the term mathematical justification as referring to “an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning” (Staples, Bartlo, & Thanheiser, 2012, p. 448).

For a qualitative understanding of arguments, I found it useful to deploy Toulmin’s term *warrant* for one such statement in the argument sequence that directly supports the claim. An argument thus may have more than one warrant and a claim can have more than one argument (we can think of it as a line of reasoning which can be different from another, reaching the same conclusion, or typically in mathematics, the final solution of a problem). Within an episode of argumentation, thus, there can be more than one argument for a single mathematical claim.

The usage of the words *reasoning* and argument in Staples et al.’s (2012) definition is noteworthy. Justification is a type of argument (i.e., there exist arguments that intend to but do not manage to demonstrate or refute the truth of a claim), while reasoning serves as the vehicle or tool employed in carrying it out (i.e., we use reasoning with the objective to justify).

> The socio-mathematical norm aspect of mathematical justification is contained in the word *accept* in the above definition. Clearly, the acceptability (within a community) of an argument plays a key distinctive role. What counts as acceptable mathematical justification, according to our definition, depends on the social factors in the community involved (Brousseau, 1998; Ernest, 1997; Yackel, 2001). The community does not need to be large: Yackel (2001) gives the following example of such acceptability in a second grade classroom in two distinct points of time:
>
> > For a problem such as 5 + 6 = __, pupils initially gave explanations such as, ‘I know that 5 and 5 is 10 and 6 is just one more; it’s just one more on the 5, so the answer is 11; one more than 10’. Later a typical [acceptable] explanation was ‘5 and 5 is 10 so it’s 11’. (p. 16)

Hanna (2000) points out that, in a recent debate on the subject of the validity of proof, mathematicians "agreed that it is imperative to make a clear distinction between a correct proof and a heuristic argument, and that the validity of mathematical results ultimately rests on proof" (p. 12). On the other hand, she states, mathematics educators have been prone to see rigorous mathematical proof, based on a formal apparatus of mathematical logic, as less important (or even undesirable, in more extreme cases), and they stress the importance of the heuristics, intuitive and exploratory nature of mathematics activity[1]. It is evident that the construct of mathematical justification (distinguished from proof) is more relevant to research in the area of mathematics education.

Because this work is also concerned with the representation of mathematics as a discipline, through the apparatus of mathematics education, a parallel should be drawn between the two in terms of mathematical justification. The definition of proof in mathematics classroom found in Stylianides, Stylianides, & Shilling-Traina (2013) fully expresses the socio-mathematical aspect of a mathematics community environment (Stylianides, Bieda, & Morselli, 2016). For my purposes, nonetheless, I will reserve the term proof – unlike Stylianides et al. (2013), but consistently with other authors (Staples et al., 2012; Hanna, 2000; Harel & Sowder, 2005) – as the type of mathematical justification used and accepted in the community of mathematicians, and I will adopt the following definition in understanding the mathematical justification as a kind of proof in a mathematics classroom.

1.1.1 Mathematical justification in this work

Mathematical justification, in this work, is a mathematical argument with these characteristics:

1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;

2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and

[1] It needs to be stated that she provides evidence of this trend using, apart from research literature sources, the curriculum standards for the US and the UK.

3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (Stylianides, 2007, p. 291) (emphasis in original)

In other words, a justification "requires that the set of accepted statements, the modes of argumentation, and the modes of argument representation be readily acceptable by, known to, or within the conceptual reach of the members of a classroom community at a given time" (Stylianides et al., 2013, p. 1466).

To illustrate the use of terminology, let us consider the following two arguments put forth for the claim of "We do not divide by zero".

Argument 1: "We do not divide by zero because it is impossible to divide anything into zero parts." This argument may be a justification in a classroom where pupils accept verbal representation of the physical world and accept that mathematical division represents dividing objects into parts. Pupils also accept as a valid fact (a warrant) that it is impossible to divide anything into zero parts.

Argument 2: "We cannot divide by zero because if we did, we would come to the following contradiction: Consider dividing 1 by this sequence of divisors: $1/0.1 = 10$; $1/0.01 = 100$; $1/0.001$ = 1000; etc. The smaller (closer to zero) the divisor, the higher (closer to infinity) the number. Now, let us consider dividing 1 by a sequence of numbers like this: $1/-0.1 = -10$; $1/-0.01 = -100$; $1/0.001 = -1000$, etc. Here, the closer to zero the divisor, the smaller (closer to negative infinity) the number. Thus, if we divided 1 by zero in both cases, we would reach inconsistent results and the system would not work." This would certainly become a problematic argument in a first year classroom. We could assume that neither the warrants (the individual statements), nor the representation (negative and rational numbers, division with rational numbers) would be readily accepted by them. The modes of reasoning (induction and proof by contradiction) may also not be accessible to such a community. In this sense, Argument 2 would not be a justification in the community of these first-year pupils, although it may be so in the community of pupils in the Year 9. Figure 1.1 illustrates the situation graphically.

ARGUMENT	**We cannot divide by zero because if we did, we would come to the following contradiction: Consider dividing 1 by this sequence of divisors: $\mathbf{1/0.1 = 10}$; $\mathbf{1/0.01 = 100}$; $\mathbf{1/0.001 = 1000}$; etc. The smaller (closer to zero) the divisor, the higher (closer to infinity) the number. Now, let us consider dividing 1 by this sequence of numbers: $\mathbf{1/-0.1 = -10}$; $\mathbf{1/-0.01 = -100}$; $\mathbf{1/0.001 = -1000}$; etc. Here, the closer to zero the divisor, the smaller (closer to negative infinity) the number. Thus, if we divided 1 by zero in both cases, we would reach inconsistent results and the system would not work.**		
		Accepted in Group A (pupils aged 6)	Accepted in Group B (pupils aged 15)
Warrants	All statements, such as $1/0.1 = 10$, etc. The system must "work" and not be inconsistent.	No	Yes
Modes of reasoning	Proof by contradiction; generalizing based on a specific case.	Not likely	Yes
Modes of representation	Rational fractions, negative numbers, infinity	No	Yes
Justification		ARGUMENT does not JUSTIFY CLAIM	ARGUMENT JUSTIFIES CLAIM

Figure 1.1: Breaking down an argument: justification in two different communities.

Finally, there is one more reason I prefer to distinguish between an argument and justification. It is the centrality of justification in a person's or a community's mathematics education viewpoint. Recall that it is what James's need for justification of non-division by zero showcased for me as a mathematics teacher: do all rules need to be justified? Are there any that are agreed conventions? If so, why? So, although justification is, in fact, an argument, whenever I use the term throughout the text, I do so because it is linked with this core overall mathematics education belief and knowledge aspect. In alignment with the above definition by Stylianides et al. (2013), I will consider arguments demonstrating the truth of general mathematical statements, such as mathematical formulas, rules and procedures, the "why things work", through this lens of justification.

Note: for the sake of conciseness and given the social influence on what is accepted as a mathematical argument, justification, warrant etc., I will be using the terms without the adjective *mathematical* unless there is a specific need for it.

1.2 Argumentation in mathematics education

In the following sections, I endeavour to show the different roles that argumentation and justification play in mathematics as a discipline and the communities of mathematics education and classrooms. Educators in mathematics especially point out the need to foster the learning of reasoning skills in classrooms. I discuss further the modes of reasoning in classroom and the importance of learner participation in argumentation.

1.2.1 The roles of argumentation in mathematics education

What is accepted as an argument/warrant/justification is clearly different in different communities. Apart from the difference in modes of representation, reasoning and accepted warrants, what seems to distinguish mathematical justification (proof) as a disciplinary practice of mathematicians (the members of the community of mathematics as a discipline) from mathematical justification as a teaching and learning practice is not only the kind of justification involved, but also the function it primarily fulfils in either community (Bell, 1976; de Villiers, 1990, 1999; Ernest, 1999; Hanna, 1990, 2000; Harel & Sowder, 1998; Knuth, 2002).

Drawing on previous research, Hanna (2000) presents a comprehensive list of the roles that proof plays in the community of mathematicians:

• *verification*, i.e., concerned with the truth of a statement

• *explanation*, i.e., providing insight into why it is true

• *systematization*, i.e., the organization of various results into a deductive system of axioms, major concepts and theorems

• *discovery*, i.e., the discovery or invention of new results

• *communication*, i.e., the transmission of mathematical knowledge

• *construction* of an empirical theory

• *exploration* of the meaning of a definition or the consequences of an assumption

• *incorporation* of a well-known fact into a new framework and thus viewing it from a fresh perspective

This inventory is not all-encompassing: other roles, such as the affective role of intellectual stimulation, or fulfilling a sense of aesthetics, have been noted (Bell, 1976; Burton, 2004; Davis & Hersh, 1981; etc.).

What does this list look like in communities of mathematics education and drawn for the roles of justification? In the community of mathematics education, all the above roles also stand as valid for justification. What they have essentially embedded in their core is the *development of conceptual understanding* and knowledge expansion.

At the same time, the employment of justification, and argumentation in general, in mathematics teaching and learning processes inevitably involves other objectives, fulfilling new didactic and pedagogical roles. From teachers' point of view, for example, the *assessment* role of a pupils' production of an argument is significant: the pupil's "display" (Knuth, 2002, p. 79) of thinking informs the teacher about the pupils' understanding of content or of their ability to justify, or argue, without necessarily affecting the pupils' cognitive process. Other non-cognitive but, still, pedagogical roles have been identified in Staples et al. (2012) as "*Managing Diversity* (offering access to, or reaching, a wider range of pupils)*, Influencing Social Relationships* (shaping pupil-pupil interactions, and moving teacher away from the central authority role)" (p. 453) (emphasis in original). As a typical example of managing diversity in a classroom, a teacher may set a task to justify (prove) a statement, a result etc. to pupils with a higher level of understanding in order to challenge them further where they would otherwise disengage. The same task, on the other hand, can be used in order to encourage the participation of pupils in the teaching and learning process, giving the justifying pupil the role of "the teacher", influencing the social structure of the classroom.

It is educators, rather than mathematicians, who believe that another central role of an argumentation-based activity for a pupil is the development of *advanced thinking skills*, such as reasoning. These skills are, indeed, more often than not, explicitly declared to be major outcomes of mathematics education whilst not, in most cases, referring to purely mathematical thinking. Mathematical argumentation is also understood as an activity focusing on the development of *communication skills*.

One of the educational roles justification plays in a classroom is – whether more or less implicitly – the *(re)presentation* of mathematics as a discipline. In justifying (or the lack of whereof) mathematical truths, the classroom community generates opportunities for establishing personal images of mathematics, i.e., personal sets of beliefs about the nature of mathematics and mathematical activity (Cooney & Wilson, 2002; Ernest, 1991; Goldin, 2002; Handal, 2002; Schoenfeld, 1992; Thompson, 1984, etc.).

For example, consider the (extreme) view that mathematics is a set of given rules and facts established by "someone" (e.g., mathematicians), characterized by memorizing activity, as illustrated by the following claim of a pupil: "In maths you have to remember; in other subjects you can think about it" (Boaler, 2009, p. 35). This conception of mathematics can be a direct result of the individual's experience of mathematics where rules and formulas were not subjected to the "why" question. Such conceptions, to say the least, do not correspond with the strongly logical and creative nature of mathematics as a discipline. A less extreme view can hold that although in mathematics most things can be proved, there are many "conventions" (i.e., the non-division by zero rule) that need to be followed in order for the whole system to work.

Commenting on the use of proof in secondary and post-secondary mathematics, Alibert and Thomas (1991) argue that "mathematics in general, and proof in particular, are presented as a finished product; the pupil is not a partner in the knowledge construction but rather a passive receiver of knowledge" (p. 235, cited in Harel & Sowder, 1998). Thus, justification of general mathematical truths can present both the nature of mathematical knowledge and the nature of mathematical proof. I will look at the various conceptions of nature of mathematics and justification in Section 1.3.

The following list summarises the various possible roles of mathematical argumentation in a classroom. From the socio-mathematical point of view, we could observe that the acceptance of an argument may depend on its role: for example, an argument driven by the teacher's need to assess the pupil's understanding may only have the teacher as audience, and the teachers' knowledge or capability to accept the argument will be detrimental.

- Developing conceptual understanding (ways of understanding, modes of representation)
- Developing mathematical argumentation skills (ways of thinking, modes of reasoning)
- Assessment
- Communication skills
- Presentation of mathematics as a discipline
- Managing diversity
- Influencing social relationship
- Affective aspects (e.g., fulfilling a need of aesthetics or intellectual challenge)

1.2.2 Modes of reasoning in mathematical justification

It follows from Section 1.1.1 that, when enacting or implementing justification, the teacher can have at least two different cognitive objectives:

a) the development of conceptual understanding of particular content (i.e., to reinforce the concept of division as partition, a teacher can explain the zero non-division as: "It is impossible to divide anything into zero parts"), and/or

b) the development of the ability to argue and reason mathematically.

Going back to our definition of justification, the latter objective is closely related to developing modes of reasoning, while the former is associated with warrants (the statements that need no further explanations, i.e., what is understood or known). Modes of representation can be included in either of these objectives: the building of knowledge and understanding as well as arguing using the accepted representations and modes of reasoning.

Harel and Sowder (2005) have developed a conceptual framework[2] with the notion of *ways of understanding* and *ways of thinking*. Ways of understanding involve an individual's ways of understanding particular mathematical concepts, the solutions for particular problems, and the particular evidence and argument used in establishing or refuting a truth. An individual's ways of thinking can be characterized by a person's beliefs (about mathematics), problem-solving approaches, and beliefs about modes of reasoning and representations, i.e., understanding of what constitutes justification. Such beliefs are named *proof schemes* by the authors.

Individuals establish their own understanding of what provides acceptable evidence for a truth, based on their (not only classroom) experience. Let us consider a notorious example reported by Fischbein (1982): university students, after having been shown the proof of a theorem, quite often asked for empirical testing, even though they said they understood the proof. In other words, the classroom community did not find the proof (or its nature) convincing, even if it would have been accepted in a community of mathematicians, or perhaps another group of s.

At school, the social dimension related to the community of mathematicians must be coordinated with the social dimension related to the classroom community: the crucial role of the teacher comes to the forefront, representing contemporaneously the guarantor of the mathematics community and the guarantor of the classroom community. In short, the teacher has to become a cultural mediator and introduce students to the standards of mathematical validation (Mariotti, 2006, p. 188).

Harel and Sowder (2005) argue that teachers often either focus on the general modes of reasoning (e.g., the form of formal proofs, using the apparatus of mathematical logic), without allowing for proper conceptual understanding of mathematics, or they focus on the

[2] The authors frame the definition of Advanced Mathematical Thinking by the "ways of understanding" and "ways of thinking" dualism, as part of their DNR (Duality, Necessity and Repeated Reasoning) framework. Although Harel and Sowder's classification of proof-schemes was of an exploratory nature, it has been adopted by many researchers (e.g., Mariotti, 2006). As already said, the authors study the conceptions of proof, produced by pupils. At the same time, they argue for a strong link between the pupils' mathematical learning experiences and their own proof-schemes, thus, I believe their classification to be a useful one to frame my own exploratory efforts in the context of intended and implemented curricula. Other authors (see, for example, Stacey & Vincent, 2009; Levenson, 2013) have developed less refined or exploratory frameworks.

mathematical concepts without letting pupils develop their own understanding of reasoning modes (e.g., using inductive methods without properly addressing their generalizability; or simply not helping pupils to grasp, evaluate or reflect on the method of justification used).

What proof schemes and modes of reasoning have been identified in classrooms? The least cognitively rigorous of the three main classes of proof scheme system are justifications produced by *external convictions*. These can demonstrate themselves mainly as the dependence on authority: for example, rephrasing a statement into a statement that represents a fact to the justifier, without further understanding; also, consider the case of a mathematically faulty justification accepted by the classroom community solely on the basis because of having been “sanctified” (approved) by the teacher or by another pupil who is considered “clever”.

Empirical proof schemes are believed to be quite common in mathematics classrooms. They involve inductive generalizations based on specific cases, using empirical evidence and involving no other deductive reasoning. For example, a class activity that has pupils cut off the “angles” of various paper-made triangles, and measuring their sum, in order to establish that the sum of all inner angles in a triangle is 180°, is a justification based on empirical evidence. A similar use of one example is used by authors of an Australian textbook where the algorithm of fraction division (“turn over and multiply”) is first demonstrated and verified on simple cases (Figure 1.2), and then stated as a rule (Stacey et al., 2009).

$$3 \div \frac{1}{2} = 6 \text{ and } 3 \times 2 = 6 \qquad \frac{1}{2} \div \frac{1}{4} = 2 \text{ and } \frac{1}{2} \times \frac{4}{1} = 2$$

Figure 1.2: Empirical justification of the algorithm for dividing two fractions. Reprinted from Stacey et al. (2009).

Using *perception* (e.g., visual) in an argument can also lead to a justification: Figure 1.3 gives an example of a (wrong, perception-based) justification of the fact that two congruent triangles can have two different areas.

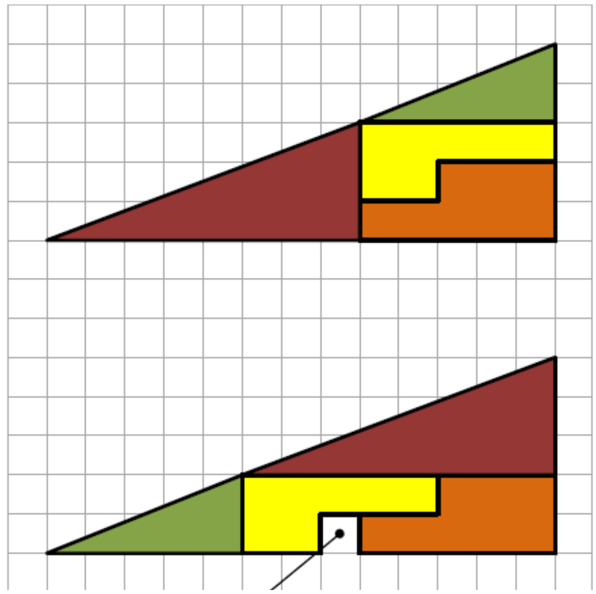


Figure 1.3: Perception-based justification: congruent triangles?[3]

The last case of proof schemes, the *analytical* proof scheme (later designated as *deductive* by the same authors, Harel &Sowder, 2007) is concerned with generalizing, either in a transformational or axiomatic way. Transformational proof scheme generalizes a process rather than separate results, and is also known in literature as justification (or proof) by a *generic example*. For example, the famous story about Gauss's method to add all natural numbers from 1 to 100 by adding first 1 + 100, 2 + 99, 3 + 98, etc. to arrive at the number equal of 50 times 101 is a generic example of justifying why for all natural numbers, the sum of the first $2k$ positive integers is $k(2k + 1)$. "Nobody who could follow Gauss's method in the case $k = 50$ could possibly doubt the general case" (Rowland, 2001). Examples of axiomatic proofs are justifications by counterexample, contradiction, mathematical induction, contraposition, and exhaustion (Stylianides, 2007).

The development of the skill to argue and reason mathematically lies in the notion that "the goal is to help students refine their own conception of what constitutes justification in mathematics:

[3] This puzzle is an entry in Wikipedia, titled "Missing Square Puzzle", referencing the work of Martin Gardner *Mathematics, Magic and Mystery*.

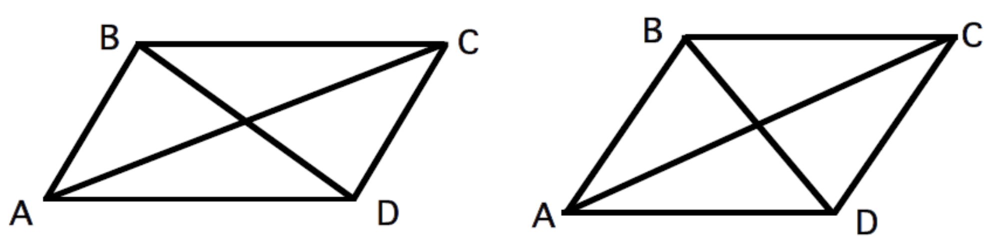
from a conception that is largely dominated by surface perceptions, symbol manipulations and proof rituals, to a conception that is based on intuition, *internal* conviction and necessity” (Harel, & Sowder, 1998, p. 237, emphasis added). As the authors point out, jumping to formal axiomatic proof without exploring other proof schemes leads to epistemological confusion, in terms of what constitutes a mathematically (as a discipline) accepted justification.

1.2.3 Cognitive engagement in justification

Of course, the cognitive engagement and mental investment on the part of an individual needs to be addressed here. The role of pupils’ participation and autonomy in doing mathematics and autonomy comes into the picture and plays further role in an individual’s system of beliefs. A justification by routine, a regurgitated proof (Harel & Sowder, 1992), will hardly have the same effect on pupils’ understanding of a particular mathematics topic, or of what mathematical activity, including proof, as a way of thinking, encompasses.

1. Consider the parallelogram ABCD in Fig. 7a, with diagonals AC and BD. State all the properties of the figure that you are willing to accept. Then, give a complete argument justifying why you believe your assertions to be correct.

2. Suppose you assume in addition that AB = BC, so that the quadrilateral ABCD is a rhombus (Fig. 7b). State all the additional properties of the figure that you are willing to accept. Then, give a complete argument justifying why you believe your additional assertions to be correct.



a. ABCD is a parallelogram. What do you think must be true?

b. ABCD is a rhombus. What else do you think must be true?

Figure 1.4: Two ways of implementing the justification of mathematical truths (Schoenfeld, 1992, p. 87).

As an example, consider the following task: "Prove that the diagonals of a parallelogram bisect each other but are not necessarily mutually perpendicular; prove that diagonals of a rhombus are mutually perpendicular in addition." (Schoenfeld, 1992, p. 87). Now consider an alternative task in Figure 1.4, connected to the justification of the same mathematical truth. The pupils' making own conjectures and having to refute or prove them (note especially the role of acceptability in tasks 1 and 2 in the figure), their discovering and proving new truths engages them in mathematical activity. At the same time, it prevents the phenomenon reported in Harel and Sowder (1992), and Knuth (2002) that pupils view proving as an unnecessary exercise, as they are asked to prove truths long established and proved by generations and generations of pupils like them.

1.3 General Framework for studies – the planned and enacted arguments

What and who determines what arguments, i.e., warrants, forms of reasoning and representations get produced and accepted in a classroom community? What determines the socio-mathematical norms in an individual classroom that guide the production and justification of arguments?

If our goal is to look into argumentation in the planned and enacted curriculum, it is clear that the teacher will be at the centre of our attention. Schoenfeld (2010) demonstrates that the enacted curriculum, the result of a teacher's decision-making within teaching periods, when interacting with pupils and the mathematics content, can be explained by the teacher's orientations (beliefs, values and preferences), their knowledge and resources, and their goals.

Of course, there are other theoretical frameworks in psychology and education research that aim to identify factors that determine one's decision-making. I choose Schoenfeld's here for the following reasons:

a) It is developed and demonstrated in the most relevant context of mathematics teaching, although it is relevant and generalizable for other activities that can be characterized as "knowledge-intensive, goal-directed problem solving" (p. 14), such as cooking or treating a patient.

b) The framework uses constructs that are known, used and studied in the mathematics education research (such as beliefs, knowledge, resources, goals, preferences and values).
c) The constructs are relevant and useful to the subject of my study, i.e., teaching as an activity with Schoenfeld's characteristics (knowledge and goal driven problem solving).

Schoenfeld's framework is mainly concerned with explaining the on-the-spot decisions within a class period. He observes that

> when [a teacher] is on a familiar ground, his activities are structured by his agenda, which is heavily influenced by his orientations. There is a natural goal structure to these activities, structured […] by his agenda and […] by the well practiced routines he selects and implements to achieve the agenda. When something unforeseen happens, [the decision-making] is shaped in fundamental ways by his orientations, and the resources at his disposal. (Schoenfeld, 2010, p. 13)

This resonates with other findings into the connection between teachers' beliefs and their decision-making. When researchers observed behaviour in teacher's practice that was at odds with their core beliefs, it started to make sense to study teacher belief system and practice as sensible systems (Leatham, 2006), assuming a consistency between the two and striving to explain rather than simply point out any outward discrepancies (Žalská, 2012b). Schoenfeld's framework reflects such endeavour well.

What about the teacher's decisions when it comes to planning a lesson? Schoenfeld stresses the goal-driven character of a teacher's agenda but, surely, the teacher uses their orientations and resources to reach these goals even at the planning stage of the process. Plus, some of the goals are likely to be aligned with external resources such as the national, school or textbook curriculum.

According to Remillard's (2005) review, the textbook curriculum's role is important but the levels of its participation in the intended curriculum vary greatly. The general model (Figure 1.5) holds that a teacher selects tasks from the text, designs their implementation, supplements it with other tasks, and, finally, improvises, based on the pupil contributions (Remillard, 2005).

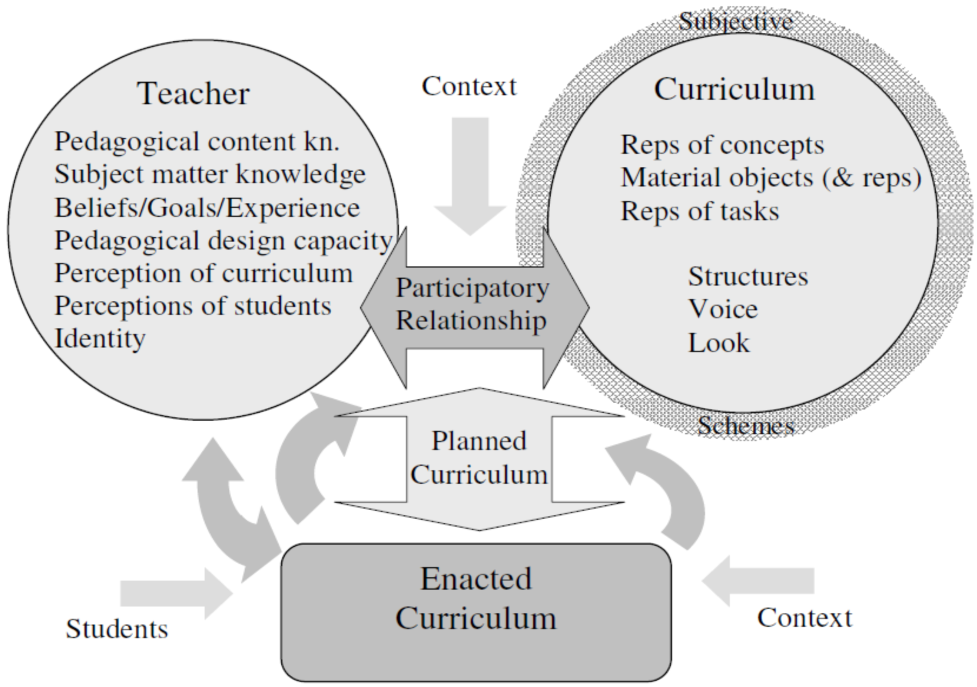


Figure 1.5: Remillard’s model for teacher-curriculum interactions. Reprinted from Remillard (2005).

I adopt Remillard's (2005) model to propose a framework for studying the potential influence of three main participants on argumentation and arguments, or the "enacted" arguments: the teacher, the adopted curriculum (textbook) and the pupils. The following are the ways these participants interact in the planned and enacted curricula.

1) The textbook curriculum provides examples of, requests and opportunities for justification and argumentation (tasks) to be enacted; it may provide pedagogical content guidance for the teacher, being a resource and informing teacher’s knowledge (pedagogical content knowledge or content knowledge, Shulman, 1986). The textbook also reflects the textbook authors’ set of beliefs, resources and goals.

2) Based on their own set of beliefs, resources and goals, the teacher makes choices about planned justifications and their form, evaluates (selects), designs and provides the examples, opportunities and requests for particular arguments as well as anticipates pupils’ reactions. In the enactment stage, the teacher needs to make immediate decisions about justifications

prompted by pupils and pupils' reactions, to adapt the planned justifications, to evaluate arguments put forth by the pupils.

3) The pupils request arguments, ask for clarifications of arguments and may provide their own arguments or claims, which, in turn, are given by their own beliefs, goals and resources.

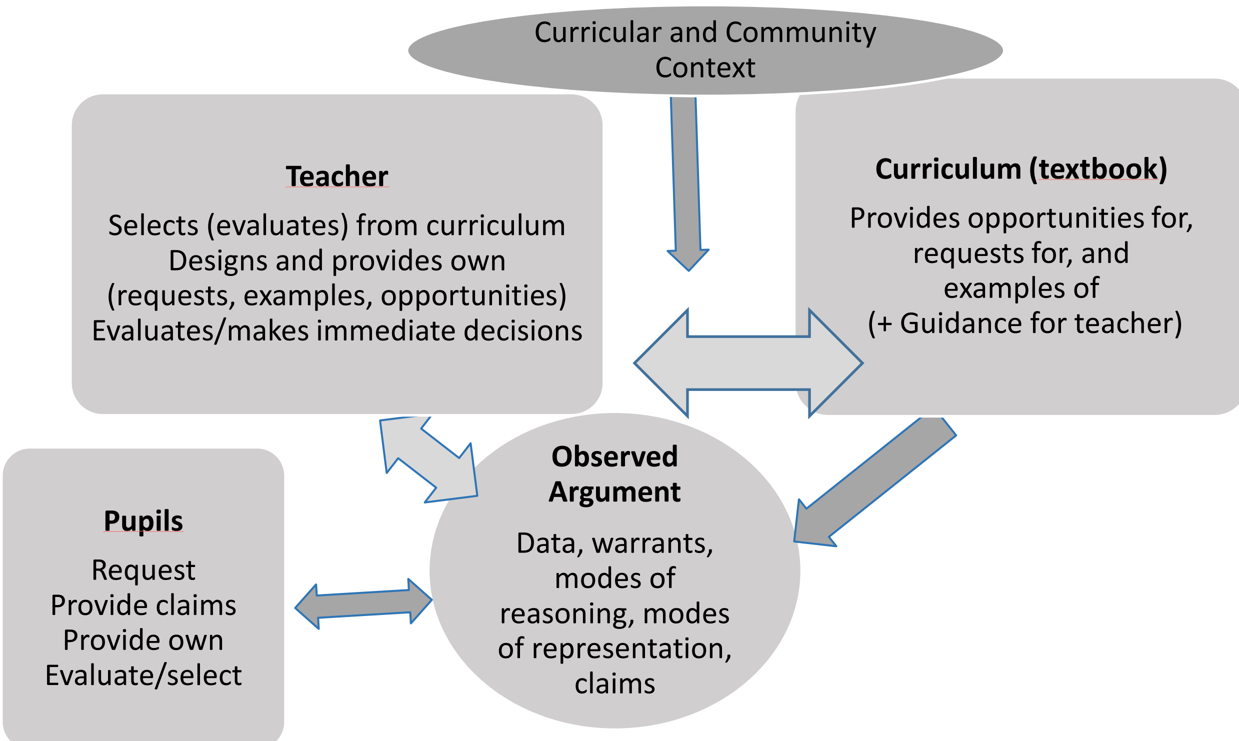


Figure 1.6: Remillard's (2005) participation model adopted for arguments observed in a classroom.

Figure 1.6 illustrates this theoretical model for observing arguments enacted in a classroom. From this point of view, then, it makes sense to explain what happens in the classroom by looking closely both at the teacher as a factor, the used curriculum, and the "participatory relationship" (Remillard, 2005, p. 42) between the teacher and the curriculum. This relationship is characterized by the teacher's nature of interaction with the curricular resource, and involves but is "not limited to, offloading, adapting, improvising, omitting, creating, and replacing." (Remillard, 2005, p. 42). Note that the context of such interaction is also important to take into consideration: above all, the general (national) curriculum likely influences the particular school curriculum; the learning objectives that those documents contain and the strength with which they are adopted and imposed influence strongly the teacher's choice of and interaction with a textbook curriculum when designing lessons.

Apart from the obvious players in the production, or absence, of arguments and justification, and the establishment of socio-mathematical norms pertaining to the acceptance of the former, in the classroom – the teacher and the pupils – there are also other stakeholders who influence the norms: parents and school community members, such as school management, school mathematics department, and other academic bodies involved. Further still, it is the directives and education management decision-makers who influence the curriculum reflecting specific goals of education and mathematics education.

Applying Schoenfeld's theory of decision making to this model, Figure 1.7 illustrates how the "why" of each of the players' actions plays out in a classroom situation.

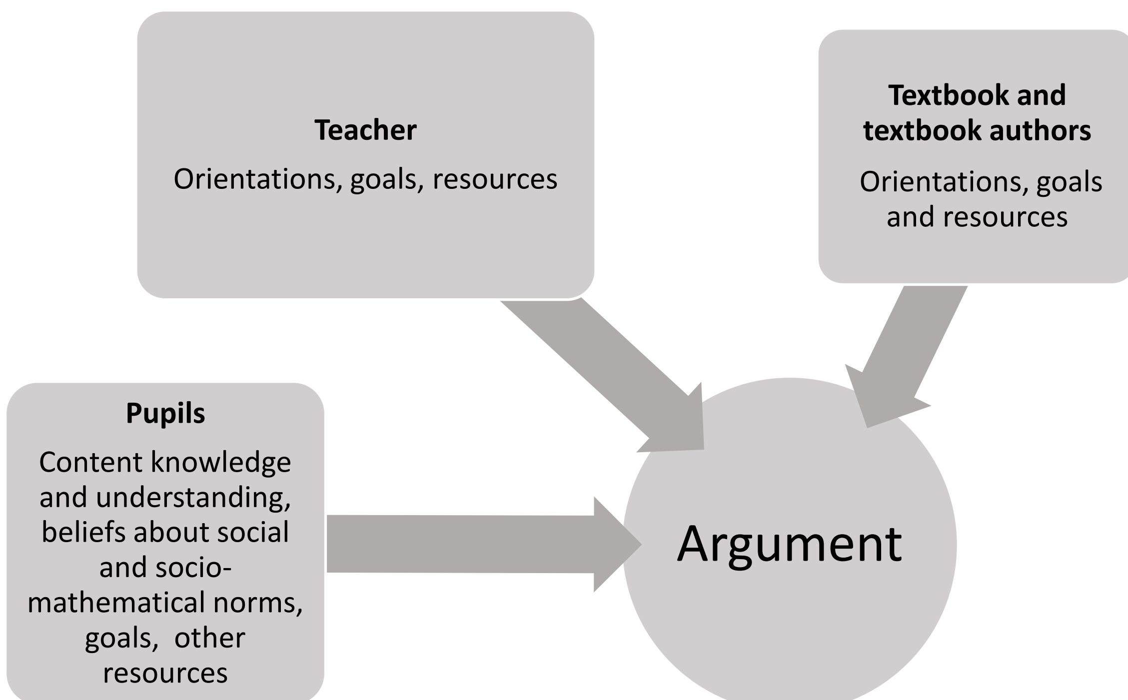


Figure 1.7: Schoenfeld's (2010) factors influencing the actual argument that takes place in a classroom.

1.4 Philosophy of mathematics education and argumentation

The curricular (national) context of argumentation and justification is a starting point if we want to address the particular classroom reality. As I noted through my teaching experience in the USA, the foundations of these contexts vary not only across countries but also within a country (and across time spans). This underpinning philosophy of mathematics education is present in curricular texts and in personal philosophies pertinent to curricular (or textbook) author communities and teachers.

In his seminal work, Paul Ernest (1991) lays out the, historically observed and documented, multiple and multifaceted attitudes towards the nature of mathematics, towards the goals of mathematics education and towards the nature of learning and teaching of mathematics. I include the general descriptors in Appendix C. In the following text, I theorise how argumentation and justification is understood in the five philosophical orientations identified by him. I show how the roles of argumentation, justifications, warrants, modes of reasoning, and modes of representations may vary, in function to the beliefs about the learning and teaching of mathematics, the nature of mathematics and the goals of mathematics education.

1) *The Industrial Trainer*: justification and argumentation is unimportant beyond basic knowledge and skills in mathematics. Knowing how to do things is more important than knowing why things work the way they do. Justification by authority (mathematics, teacher, curriculum) is commonly accepted: The teacher and other authorities are the source and authority in explaining. Argumentation by pupils has primarily an assessment role. School mathematics is represented as hard work and success only comes to those who work hard. There is only one way to represent and that one needs to be practiced, memorised, and/or applied many times. Mathematics is a discipline for the able only, i.e., understanding why is only for the few able.
2) *Technological Pragmatist*: knowing how to solve problems is more important than knowing why things (methods) work. Efficiency in problem solving is paramount and so knowing methods and processes (the how rather than why) and choosing those that are most efficient is more important than reasoning in various representations and individual sense-making. Knowing why is only useful if it helps solving problems more efficiently. Arguments include

steps of problem solving without explicit conceptual connections. Argumentation should have utilitarian, not sense-making, value. Practical, real-life examples are used in order to justify rules, without distinguishing between model and abstract. Mathematics is represented as a real-life problem solving activity. The ultimate goal is to solve a problem, and in the classroom, justification is used for assessment of the use of best methods, of displaying methods to the teacher, for the teacher to evaluate the pupils' ability to apply a method.

3) *Old Humanist*: Old Humanist sustains the image of mathematics as a pure, formal, and beautiful discipline. The aesthetic form of an argument is upheld. Modes of reasoning and representation are as close to those of mathematicians as possible. The stress is on the why, and ways of understanding abstract concepts are central. Modes of reasoning should mirror that of mathematics as a discipline, and the structures of formal proof are important. The Old Humanist justifies mathematical statements through abstract theory rather than practical examples, and values classical argumentation and theory of logic. Pupils should learn to prove things that have already been proved. Old Humanist also perceives mathematics as an "exact science", the ideal world, and embraces correctness and pure mathematical language and form.
4) *Progressive Educator*: argumentation is important on the individual sense-making level. Arguments need to make sense to the individual, and accepted modes of representations can vary from pupil to pupil, and making connections between concepts and representations is important for an individual to learn mathematics. Informal, ability-based arguments are accepted and encouraged. There is no better or worse argument. Progressive Educator believes that there is an affective aspect of argumentation, a child should not be discouraged or evaluated. Pupils should be sheltered from conflict, and although mathematics is seen as absolutist, a pupil's reasoning and arguments should not be openly evaluated as wrong, should that lead to emotional distress.
5) *Public Educator*: the key is both why and how, as long as they are seen as empowering the pupil(s). Public Educator values open discussion of socio-mathematical norms of what is and can be accepted and why. Pupils should be taught to use arguments and ask for justification

to challenge authority effectively, but also to resolve conflicts, and learning ways of thinking is equally empowering as those of understanding. Public Educator encourages multiple arguments (ways of reasoning, warrants and representations), as they represent diversity and help create conflict (also on a cognitive level). Learning to reason and communicate is empowering against populism and demagogy. Public Educator advocates mathematics as a human activity, existing in human mind only and thus, culture-sensitive and loaded with social values.

The above typology is purely theoretical but creates a useful framework at looking at both cultural and personal philosophies of mathematics education. The possible views of the role of and participation on argumentation and the characteristics of arguments in the enacted curriculum give us a sense of a spectrum in which teachers and other community members (textbook authors, parents, school authorities) may find themselves. Research into teachers' beliefs about argumentation (for details see Section 3.1) shows that, indeed, the theoretical philosophies are reflected in the empirical experience.

1.5 Research questions

To understand the above-described features, aspects and mechanisms involved in argumentation, I have undertaken three investigations in the Czech mathematics education context. My guiding research questions are the following:

1) What is the curricular context of justification in Czech school mathematics teaching and learning: is justification present in the textbooks? If so, what arguments, warrants, modes of reasoning count as acceptable/accepted by the mathematics teaching community?
2) What are mathematics teachers' orientations concerning mathematics, its learning and teaching? What orientations do they have towards argumentation and justification?
3) What argumentation actually takes place in the classroom and why? What are the characteristics of arguments? How does textbook, teacher, and pupils influence this?

2 Study 1: Curricular context of justification in Czech lower secondary school mathematics

In this chapter, I present the results of a study into the Czech context of argumentation in secondary school mathematics. I specifically focus on justification of general mathematical statements, such as “in mathematics, we do not divide by zero”.

I first provide a comprehensive overview of the Czech educational system, to match the Czech lower-secondary mathematics classroom with the appropriate age group and the pupils’ national schooling experience.

In order to understand the context of the argumentation and justification in Czech mathematics classrooms, I have examined three types of curricular documents:

a) the nation-wide framework for education, i.e., the Framework Education Program for Basic Education (FEP BE)[4], a document that is binding for all schools on the preschool, primary and secondary level in the Czech Republic,
b) samples of school curricular documents for mathematics, i.e., the relevant section of documents called the School Education Programmes (SEPs), which are required to follow the FEP but provide more detailed, specific, outcomes of each of the content areas, and
c) seven mathematics textbook series for the lower secondary school level.

This chapter first introduces the reader briefly into the structure of the Czech educational system, with focus on the lower secondary level. Next, I present the results of an analysis of the former document (the FEP) with respect to argumentation and justification. To illustrate how closely the school curricular document (SEP) reflects the FEP in terms of argumentation and justification, I chose two particular school documents and looked for relevant instances of outcomes. The comparison is presented in Section 2.2.5.

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[4] Rámcový vzdělávací program pro základní vzdělání (RVP ZV) in Czech. For the sake of conciseness, I will refer to it here as the FEP, leaving out the BE (basic eductiona) part, as the document is valid for all types of schools I have studied.

The chapter's key part follows with a short review of relevant literature about mathematics textbook content, and then presents the results of an analysis of selected Czech lower secondary mathematics textbooks.

2.1 Lower secondary school level in the Czech education system

The Czech schooling system consists of pre-primary schools, 9 years of compulsory first stage (5 years) and second stage (4 years) schools (see Figure 2.1), the post-compulsory general and vocational institutions (upper secondary and post-secondary level also referred to as the third stage, such as vocational schools, general schools, and lyceums), and the tertiary and post-tertiary education institutions.

The lower secondary level (or the second school stage, "druhý stupeň", in the Czech terminology) is the subject of investigation for my thesis. In the Czech school system, pupils attending Years 6 to 9 of compulsory education are typically of ages 11 to 15. There are three different types of schools that provide education at this level. The majority of pupils attend the general basic school, about 11% of pupils attend the multi-year secondary grammar school (the "gymnázium" in Czech)[5], and approximately 0.07% pupils study at conservatoires (Ministry of Education, Youth and Sports of the Czech Republic [MEYS], 2012). In my thesis, I focus on both the first and second school types (basic schools, and the six and eight-year secondary grammar schools). The secondary grammar schools are conceptualised as a general education as opposed to the vocational, professionally specialized schools. At the lower secondary level, they are perceived as the more academically demanding option.

The curriculum in these schools is guided by the FEP, and schools have to prepare their own school educational programmes (SEPs) based on it. The FEP defines nine main educational areas consisting of one or more educational fields designated as Language and language communication, Mathematics and its application, ICT, People and their world, People and society, People and nature, Art and culture, People and their health, People and the world of work. Further, the document defines cross-curricular topics (such as Personal and social

[5] Also refered to in English as the general secondary school.

education or Media Studies) complementary educational fields and key competences for the pupil leaving this school level. The document states the recommended content and expected outcomes of each of these areas (e.g., Mathematics and its applications) for three stages: years 1 to 3, 4 to 5 and 6 to 9. It is up to the SEPs to divide the curriculum into particular years (or other compact parts, e.g., modules) and into subjects. Individual schools define their focus in the SEP.

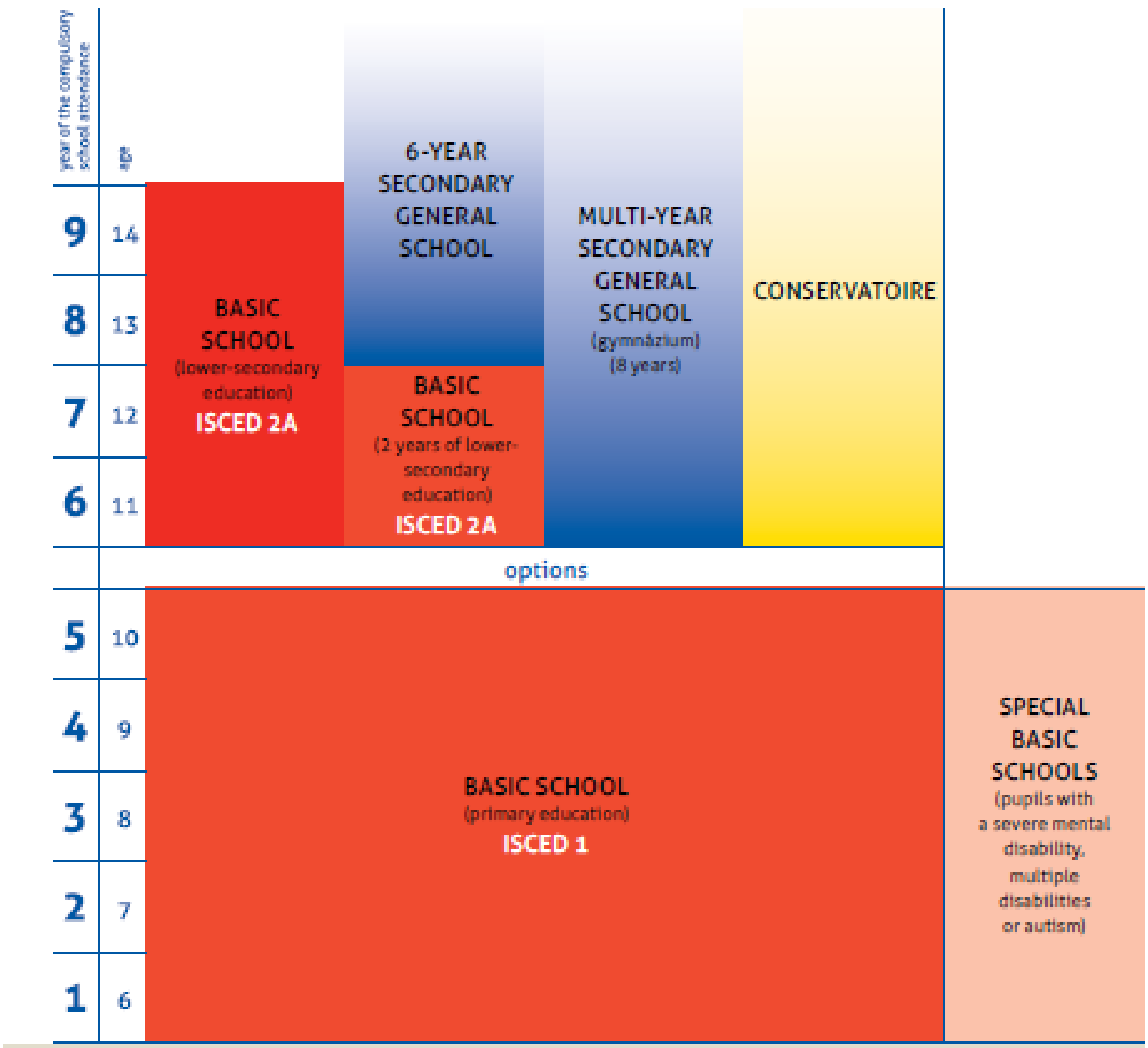

Figure 2.1: The compulsory education years and schools in the Czech Republic. Reproduced from Ministry of Education, Youth and Sports [MEYS] (2011).

2.2 Argumentation in the FEP

What role does the national curricular framework for mathematics education give to mathematical argumentation? What outcomes are expected regarding mathematical argumentation?

2.2.1 Method

For the analysis of the FEP text[6], MEYS (2007), I chose to focus on the presented objectives of mathematics education: a) the general expected education outcomes defined as "key competencies[7]", b) the characteristics and objectives of the mathematics area of education, and c) the expected outcomes related to specific mathematics education content. I selected passages that were relevant to argumentation in general and mathematical argumentation and justification in particular. I then analysed the selected passages from the point of view of the following: the role and intended function of arguments (and argumentation), and the suggested characteristics of arguments, such as modes of reasoning, and modes of representation.

One of the characteristics of the FEP text is that the language regarding ways of reasoning, thinking and argumentation is quite vague: there are several references to "logical thinking" (e.g., p. 28), "basic ways of thinking" (p. 27), "mathematical, logical, and empirical methods" (p. 13), "logical sequences" (p. 13), which I interpret as references to qualities of generally accepted argumentation.

2.2.2 Argumentation in key competencies

Figure B1 in Appendix B lists examples of outcomes related to arguments. Although argumentation is only explicitly mentioned in one of the key competency outcomes (as part of communication skill competencies), others include qualitative elements of argumentation as well. For example, if we consider problem solving an activity that requires the presence of

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[6] I studied the English version from 2007 (Ministry of Education, Youth and Sports [MEYS], (2007). There was an update in 2013 of the Czech document but after inspection, there were no changes relevant to my analysis.

[7] These key competencies are not content-related, apply to the timeframe of Year 1 to 9, and are classified in six areas: learning competencies; problem-solving competencies; communication competencies; social and personal competencies; civil competencies; working competencies.

reasoning (as in mathematical problems) and “methods” to be certain reasoning patterns, then we can infer that the intended problem solving competencies are pertinent to mathematics in that they:

- call for pupils to be able to use arguments as applied “proven methods” (p. 12), i.e., there is an established way of argument, whilst the establishment of the proven may or may not be based on (mathematical) reasoning (i.e., a method can be learned without being understood or justified),
- acknowledge that there exist various arguments, ways of solving a problem, which could apply specifically to mathematical content, and
- acknowledge that reasoning is important when problem-solving.

Likewise, the communication competencies require a pupil to:

- produce arguments in writing and speaking to be acceptable, i.e., following unspecified logical sequences (i.e., accepted modes of argumentation), and
- use “appropriate arguments” (p. 13) to defend a claim.

Finally, the description of the desired learning competencies also includes elements of reasoning abilities, specifically, the ability to draw conclusions from experiments and observations.

2.2.3 Argumentation in general characteristics and objectives of school mathematics

In terms of the general mathematical content characteristics and objectives, Figure B2 in Appendix B displays the relevant passages of the FEP text and offers further insight into the roles and characteristics it assigns to argumentation.

It seems that the text draws out the following features of mathematical argumentation:

- Solving problems is the tool for “developing combinatory and logical thinking, critical judgment and comprehensible and factual argumentation” (p. 28), as well as for conceptual understanding.
- There are multiple arguments for a solution of the same problem, multiple ways to solve one problem.

- There are multiple representations in arguments, e.g., multiple models represent one situation.
- The language of arguments should be precise and efficient (“succinct”, p. 28).
- It is important to learn to check a problem solution, i.e., follow or formulate an argument that will verify the solution.
- Arguments are also meant to be used for verification or rejection of a hypothesis that a pupil forms and for verifying it or rejecting it by a counterexample.
- Verifying and problem solving also plays an affective role and reinforces pupils’ confidence.
- The solving of non-standard problems is separated from the content-related problems, and is meant to encourage logical thinking and may even serve to “encourage pupils who are less apt at mathematics” (p. 28).
- Solving problems in collaboration with other pupils gives arguments a social, communicative role.

2.2.4 Arguments in the FEP

To respect the document’s rhetoric and my own definition of mathematical argumentation, the content-specific outcomes related to problem solving are likely to involve or aim to involve mathematical arguments.

The two areas where reasoning is mentioned specifically in the descriptions of the outcomes are areas of geometry. Pupils are expected to justify[8] “and apply the positional and metric properties of basic two-dimensional figures when solving tasks and simple practical problems” (p. 31) and “apply theorems on congruent and similar triangles when making argumentations and calculations” (p. 32). The general description of the subject of school mathematics includes the expectation to understand the rationale behind numerical operations (p. 27).

The framework also identifies a specific content area, “non-standard application exercises and problems” (p. 32), where the specific outcomes finally include the ability to reason logically, and use combinatory deduction.

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[8] The English text says “reason“ but I believe “justify“ is more accurate for “zdůvodňuje”.

The text defines the ability to find various solutions to one problem as an expected outcome. In the terminology of this dissertation, this means that the modes of representations, possibly modes of reasoning, and warrants in arguments that form the solutions should be varied/different. Note that this is only required in this one area of the curriculum (non-standard problems), which are recommended to be taught interspersed throughout and in between specific mathematics content areas (p. 27).

Overall, it appears that mathematical argumentation is presented by the FEP as a skill inherent in problem solving across various mathematical areas and topics, and, in addition, a certain set of non-standard problems is meant to develop logical reasoning skills and the ability to solve problems using various representations and modes of reasoning. The roles assigned to argumentation are cognitive roles that are complemented by the role of developing communication skills, presentation of mathematics as a discipline, as well as social and affective roles.

Understanding the justification (the why) of mathematical content is stated as an outcome for the areas of

1) numerical operations and their justification, i.e., understanding why algorithms are used the way they are when performing operations on numbers is included in the content description, and
2) positional and metric properties of geometrical objects in a plane.

2.2.5 School mathematics education programmes and argumentation

To illustrate the relationship between the FEP and SEP when it comes to the aspects (e.g., roles and use) of argumentation and justification, I examined two specific SEPs[9], using the same method as in the FEP analysis.

[9] There are as many School Educational Plans (SEPs) in the Czech Republic as there are schools. These specimens were both lower secondary mathematics curricular documents that I selected because they come from schools where my investigation took place in Study 2. In order to preserve participants' anonymity, I will refer to the Schools as School A and School B and to the documents as SEP A and SEP B, respectively.

I found that, in the SEPs, argumentation is addressed with slightly more detail and relevance to the key competencies, and that schools include some teaching and learning strategies in the description of these desired competency outcomes. These echo the FEP's language about problem solving and multiple solutions, about non-standard problems helping develop logical thinking, and about the need for verification. In addition to that, the documents went into more specifics in some features: for example, they included the notions of particular representations (e.g., tables), and examples of employing argumentation and its role. Figure B3 in Appendix B compares the two school texts, giving a sense of the differences found across school's language in formulating outcomes and strategies that relate to arguments and argumentation.

Apart from the general language of these goals[10], School A strategies involve a few particular teaching goals, such as working with error, and the need to look for the number of possible solutions. In the SEP for School B, problem solving drives the activity and quality of argumentation, applying a particular "method" to more problems (as if creating an argument template). Further, the text asserts that the formulation of ideas itself, perhaps the warrants or arguments themselves serve as a connection-making and learning tool.

The specific outcomes of particular mathematical topics, however, do not significantly develop in the schools' documents from those outlined in the FEP and the particular measurable output does not involve argumentation specifically, except as part of problem solving. Only several topics (and only in SEP A) specifically reflect the general strategies and goals related to argumentation. Table B2 in Appendix B shows some examples of specific outcomes from SEP A.

2.3 Justification in textbooks: international research

Mathematics textbooks have been the subject of analysis in many studies, in various geographical regions, and on various age group. Undoubtedly, the literature accessible internationally, and in English, does not include all work published within individual countries[11]. There are various

[10] Note for example School A's emphasis on the teachers' responsibility to "teach" and the existence of social setting, in contrast with the focus on the individual pupil "solving" relevant problems in SEP from School B. Also notable is the lack of defining the audience for argumentation (Table B1 in Appendix B).

[11] Chang and Silalahi (2017) identify 44 studies conducted between 1953 and 2015.

approaches and foci of such investigations and before delving into my own, I introduce a few of those that most align with mathematical justification of general patterns and procedures.

Current research literature addresses the subject of mathematical justification either directly (as, for example, in Davis (2012), Newton and Newton (2007), Silverman and Even (2015), Stacey and Vincent (2009), Stylianides (2009), Thompson, Sharon, and Johnson (2012)) or as part of broader analysis (for example, in Charalambous, Delaney, Hsu, and Mesa (2010), Haggarty and Pepin (2002), Howson (1995), Son and Senk (2010)).

2.3.1 Presence and quality of justifications in textbooks

Newton and Newton (2007) lament the lack of support UK elementary teachers get from mathematics textbooks regarding the reasons for patterns and procedures. They dissect eighteen textbooks and find instances of purpose (functional justification) and causal justification. From the number of categories they identify, only one is concerned with mathematical justification in our sense of the term, namely, providing "reasons underpinning mathematical assertions for the child" (p. 78). Such reasons comprised only 9% of the data set (on average, with range from 0% to 31%). The authors warn that the overall message the books gave was that of mathematics as a domain of "computational skill development through routines, algorithms and practice" (p. 69).

Stacey and Vincent's (2009) [12] investigation into Australian 8th grade textbooks (nine textbook series) identifies the presence and quality of mathematical justification in presenting seven mathematical topics: dividing fractions, the area of a trapezium, the area of a circle, the angle sum of a triangle, the distributive law, multiplication of powers and multiplication of two negative integers. Their findings show the use of different modes of reasoning across the topics. Importantly, only four of the nine series use any sort of deductive or empirical reasoning to support the presentation of all seven topics. The differences were not only present across the textbooks but also across the topics: unsurprisingly, the area of a trapezoid was explained using

[12] One of the contributions of Stacey and Vincent's (2009) paper is a finer categorization of modes of reasoning in textbooks, which gives a useful framework for further researchers, including my work (see Section 2.4.3).

a deductive proof scheme in all nine books, while the division of fractions turned out to be the most problematic to explain deductively for the textbook authors (only one book used deductive reasoning here).

Similarly, Thompson et al. (2012) analyse twenty US high school textbooks and their use of proof-related reasoning in units dealing with the topics of exponents, logarithms, and polynomials. Again, the extent of proof-related reasoning varied by topic and textbook: about 50% of the identified properties in the 3 topic areas were mathematically justified, of which roughly 60% were justifications with a general argument and about 40% empirical justifications (generalizing from a specific case).

In Israeli 7th grade textbooks, Silverman and Even (2015) find that in eight textbook series and across ten different topics (geometric, arithmetic and algebraic) only one textbook series did not justify one topic (all others were justified in all books) , and that 98% of the explanations were either of exploratory or deductive character. They also report differences between topics in the quality of explanation, and that two textbooks that were intended for low achievement pupils made prevalent use of empirical explanations, rather than deductive ones.

2.3.2 Reader involvement in justification

Studies that focus on the nature of the tasks textbooks provide for pupils to perform give an insight on the intended pupil involvement in the justification (discovery, deduction) of some general mathematical statements. Davis (2005) and Stylianides (2009) both look at opportunities for pupils to engage in reasoning and proving. Davis (2005) examines one topic of instruction (the polynomial functions) in three high school textbooks that he depicts as "reform-oriented, conventional, and hybrid" (p. 467). As empirical approaches are generally one of the tenets of reform-oriented teaching, he finds that the reform-oriented text tends to foster the empirical proof scheme building, while the other two were likely to promote authoritative proof schemes. At the same time, none of the books made explicit demands on pupils to conjecture or test a conjecture, and proofs were given directly.

Stylianides (2008) takes a look at one particular "reform-based" (p. 194) series of US middle school textbooks and looks for the presence of "proof tasks" (p. 195) (tasks requiring pupils to make a mathematically valid argument corresponding to my definition of mathematical argument). Out of the 4.578 tasks provided by the series, only about 5% were proof tasks. Importantly, the study also looks at the support teachers get from the authors in the accompanying teacher's books. He finds that for 90% of the "proof tasks", teachers were not provided didactic guidance (in addition to a possible solution) either in terms of providing insight into the importance of or potential difficulties with the proof task, or into the knowledge of mathematical proof per se. Figure 2.2 gives an example of both a proof task and the commentary in the textbook's teacher's edition.

Even and Dolev (2015) report that in 7th grade Israeli textbooks, the algebraic topic they investigated (linear equations) had considerably smaller percentages of tasks where pupils were meant to justify their answer than in the case of the geometric topic (triangle properties).

Problem (p. 38): Rectangle A is similar to rectangle B and also similar to rectangle C. Can you conclude that rectangle B is similar to rectangle C? Explain your answer. Use drawings and examples to illustrate your answer.

Commentary in the Teacher's Edition about this Problem (p. 40): Yes, rectangles B and C are similar.

Possible explanation: Since rectangle A is similar to rectangle B, the ratio of the short side of rectangle A to the long side of rectangle A is the same as the ratio of the short side of rectangle B to the long side of rectangle B. But since rectangle B is similar to rectangle C, the short side of rectangle C to the long side of rectangle C must equal this same ratio. This means the ratio between sides in rectangle C equals the ratio between sides in rectangle A, making rectangles C and A similar.

Figure 2.2: Guidance for teachers in justifying mathematical statements. Reproduced from Stylianides (2009).

2.3.3 Justification in textbook-based comparative studies

A considerable amount of studies has been devoted to the comparison of textbooks used in different countries. I focus here on those that involve mathematical justification.

Haggarty and Pepin (2002) make a comparison, studying the use of textbooks in classroom (in France, UK and Germany) and what mathematics is available in the textbook, on the topic of measuring angles, looking at explanations, opportunities to make generalizations, and making connections. The authors conclude that, indeed, the mathematics presented to pupils in each country is different. The interpretation of results is guided by Ernest’s (1991) framework of conceptions of mathematics, and the authors observe the following: in Germany, the conception of mathematics is rather Platonist: mathematics is perceived as a pre-discovered body of knowledge and the focus of mathematics education of a pupil is “to understand mathematical concepts through exposure to theoretical ideas in mathematics, and engagement in exercises” (p. 586). The French perceptions of mathematics appeared more varied: traditional formal view giving way to more dynamic – and, especially, novel – programs that provide space for discovery, while, at the same time, there co-existed a “utilitarian view recognizable in the French mathematics books where mathematics had to be useful, as well as exciting, and accessible for all“ (p. 586). Finally, the authors found English textbooks and their use in classrooms to be presenting mathematics as a utilitarian set of rules, with “superficial attempts” (p. 586) at developing processing skills, such as investigations. The overall picture from this part of the study was that learners in English classrooms were generally engaged in completing routine practice mathematics tasks, and understood mathematics as something “to be done” (p. 587).

More generally, Howson’s (1995) monograph presents the TIMSS 1995 analysis of eight 8th grade mathematical textbooks (USA, Japan, UK, Spain, the Netherlands, Norway, France, and Switzerland) in a philosophical and pedagogical comparison that also looks at the functional justification (i.e., to what end, rather than why) of mathematics, and the representation of mathematics as a discipline. Among his findings, Howson also reports that two of the eight countries use textbooks that provide formal proofs in the 8th grade.

2.3.4 Justification and argumentation in Czech textbooks

In the context of Czech mathematics textbooks, Břehovský (2011) looks for the use of heuristic methods (either inductive or deductive based activities, also in alignment with the construct of mathematical justification) in both introducing new content to the reader and in pupil practice tasks. After analysing two main upper secondary series of books, he finds out that only 7% of the content is introduced through inductive tasks (problem solving). His analysis does not include the mathematical justification component in general (i.e., the content can well be justified but may not involve the reader, or it might not be justified at all). In the task section, the authors of the two upper secondary textbook series require pupils to use heuristics in certain content areas more than others. For example, the section about complex numbers (in one series) would involve pupils in heuristics problem-solving in 19% of the total of the tasks, while the analytical geometry section would do so in only 2% of the practice tasks. The importance of the topic area in the tasks involving pupils seems to be a phenomenon corresponding with Stacey and Vincent's (2009) findings about content introductions in middle school (lower secondary) textbooks.

Another example from the Czech context is in Nováková (2013), who explores the topic of solving linear equations and, among others, analyses its presentation in ten Czech middle-school textbook series. It is clear that although authors use various ways of reasoning and different models for the concept of linear equation, they all justify the solution procedures. Nováková also looks at the nature[13] of the tasks offered to pupils, albeit not with special details. It is interesting to see the variety across books. All of them include the verification element of "proof" or "check" but only four include tasks potentially leading to justification, such as "justify", "explain", or "solve by reasoning".

The literature that I reviewed suggests several conclusions: the presence and quality of justification in textbooks varies across textbooks, across countries, and across topics (especially geometric and algebraic or arithmetic topics). The following section introduces the results of an analysis of the mathematical textbook context in the Czech Republic.

[13] based on the verb used in the text of task instructions

2.4 Justification in Czech lower secondary textbooks

2.4.1 Textbook series

Deciding on what textbook series to analyse was not a difficult task, as all textbook series in the Czech Republic need to be given authorization by the Czech Ministry of Education, Youth and Sports. At the time of the analysis[14], there were eight major textbook series available on the market (see also Žalská, 2012a). In 2014, a survey conducted by a research project[15] of the Czech Science Foundation among Czech lower-secondary teachers further confirmed that six of these were among the most frequently cited as a mathematics teaching/learning resource by lower-secondary mathematics teacher participants teaching at the general track school. The seventh textbook was the only series authorized for teaching at the academic track (gymnázium) school. For clarity, the individual textbook series are given a letter code (see Table C1 in Appendix C). Because only two of the series at that time included an accompanying teacher's manual, the analysis involved just student books.

2.4.2 Choice of mathematical topics

I chose the topics based on several considerations: the first was my own interest in the topic of zero and general statements involving operations with it; secondly, I chose topics based of the review of internationally published research literature (such as Stacey and Vincent, 2009); and lastly, it was the FEP's treatment of the subject of mathematical justification that confirmed the selection.

The six topics selected are:

a) Non-division by zero: why we do not divide by zero.
b) The zero[th] power: why for all real numbers x: $x^0 = 1$ (in case of lower secondary schools, whole numbers).

---

[14] Some textbooks have been updated between that analysis and the time this chapter was written, and the topic of the area of a circle was added to the list of topics analyzed for justification. When relevant, I make a distinction between the different editions. For a whole list of textbooks analyzed, see Appendix D.
[15] GA ČR P407/11/1740.

c) Square root of zero and negative real numbers.

d) Multiplication of whole numbers, especially why for any natural numbers n, m: $(-n) \cdot (-m) = n \cdot m$.

e) Division of fractions: namely, why for all whole numbers n, m, r, s: $\frac{n}{m} \div \frac{r}{s} = \frac{n}{m} \cdot \frac{s}{r}$.

f) Area of a circle: why the area of a circle is the product of the square of its radius and π.

2.4.3 Framework for analysis

I identified the selected topics in all seven series and analysed the passages that introduced the concepts or rules. I conducted the analysis in accordance with a framework based on the underlying theoretical framework introduced in Chapter 1, and the framework developed by Stacey and Vincent (2009). I focused on answering questions in these four areas:

a) The presence of mathematical justification: is justification given for the general statement?

b) Participation: how do authors present the argument, does the justification activity involve the reader/pupils or is it presented and shown directly by the authors?

c) Modes of representation and number of arguments: what representations or representation do authors choose in their warrants and justifications, and are there various models involved? Are there alternative justifications provided within a text, and across the textbook series?

d) Modes of reasoning[16]: what modes of reasoning are used? Here, I adopted the framework developed by Stacey and Vincent (2009), namely, I looked for these modes of reasoning:

- Appeal to authority: null explanation or reliance on an external source of authority.[17]
- Qualitative analogy: reliance on a surface similarity to non-mathematical situations.

---

[16] Modes of reasoning and modes of representation were only analyzed in detail for three topics: Non-division by zero, division of fractions, and the area of a circle, as examples of the two FEP mentioned areas, i.e., numerical operations and properties of geometric shapes.

[17] Stacey and Vincent (2009) include appeal to authority as mathematical justification. However, I will treat statements such as „in mathematics, we do not divide by zero“ as mathematically unjustified. This is to reflect the belief that in appealing to authority, the nature of mathematics is portrayed as authoritative, e.g., as a set of rules that authorities (mathematicians, society, textbook authors, etc.) decided on.

- Experimental demonstration: identifying a pattern after checking selected examples.
- Concordance of a rule with a model: comparing specific results of a rule and a model.
- Deduction using a model: a model that serves to illustrate a mathematical structure.
- Deduction using a specific case: an inference process conducted using a special case.
- Deduction using a general case: an inference process conducted using a general case.

2.4.4 The findings

Presence of mathematical justification of selected topics

Table 2.1 summarises the presence of mathematical justification[18] across the selected topics and textbook series. We can see that the presence of justification varies both across the textbooks and across the topics. For example, I found instances of the lack of justification of a general mathematical statement when dealing with the topic of the zeroth power (textbook A), square root of zero or a negative number (textbook D) and the multiplication of two negative numbers (textbooks C and G)."[19]

All analysed textbook series provided some justification for the procedure of dividing two fractions, and the calculation of the area of a circle. Three of the seven series provided justification for all six topics. One of the textbook series (A) avoided the concept of zero, including when dealing with powers or square roots of numbers.

[19] The non-division by zero is perhaps a special case in that the authors explicitly refer to the fact that "we already know that we cannot divide by zero" (series D and G), referring to a previously established fact. Authors presented the topic of non-division by zero in either the revision of operations with natural number (in the Year 6 textbooks) or when the topic of fractions was introduced (in the Year 7 textbooks). In one case the topic was completely absent (in series A).

Table 2.1

Mathematical justification in Czech lower-secondary mathematics textbooks.
Legend: J (justified), U (unjustified or authoritative justification provided), NA (no mention).

TEXT SERIES/ TOPICS	A	B	C	D	E	F	G
Non-division by zero	NA	J	J	U	J	J	U
The zeroth power	U	J	J	J	J	J	J
Square root of zero/negative	NA	J	U/NA	U	J	J	J
Multiplication of whole numbers	J	J	U	J	J	J	U
Division of fractions	J	J	J	J	J	J	J
Area of circle	J	J	J	J	J	J	J

Pupil participation in justifying

When investigating how authors present justifications, I found three types of such presentation, differing in the nature of the task for potential readers (pupils). I coded the one with the least pupil involvement as Telling (T), where authors simply present the justification without posing a task to the reader. Another way of presenting a justification were tasks where the reader is asked to observe a pattern, or an example, possibly to arrive at a conclusion that manifests the targeted concept (for example, see Figure 2.3). I coded this as Observing (O).

16 Počítejte a pozorujte.

$24:8=$ $\quad 60:12=$ $\quad 36:30=$ $\quad 6{,}75:2{,}5=$ $\quad 3:6=$ $\quad 3:6=$

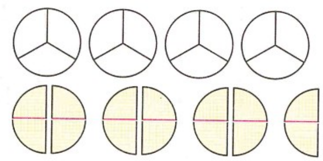
$24:3=$ $\quad 60:5=$ $\quad 36:1{,}2=$ $\quad 6{,}75:2{,}7=$ $\quad 3:0{,}5=$ $\quad 3:\frac{1}{2}=$

17 Kolik půldolarů dostaneme za *10* dolarů? Kolik třetin koláče nakrájíme ze *2* koláčů?

Příklad 17: a) $4:12=\frac{1}{3}$

$4:\frac{1}{3}=12 \qquad 4:\boxed{\frac{1}{3}}=12=4\cdot 3=4\cdot\boxed{\frac{3}{1}}$

b) $\frac{7}{2}:\frac{1}{4}=14 \qquad \frac{7}{2}:\boxed{\frac{1}{4}}=14=\frac{7}{2}\cdot\boxed{\frac{4}{1}}$



Dělit zlomkem znamená násobit k němu převráceným číslem.

English translation:

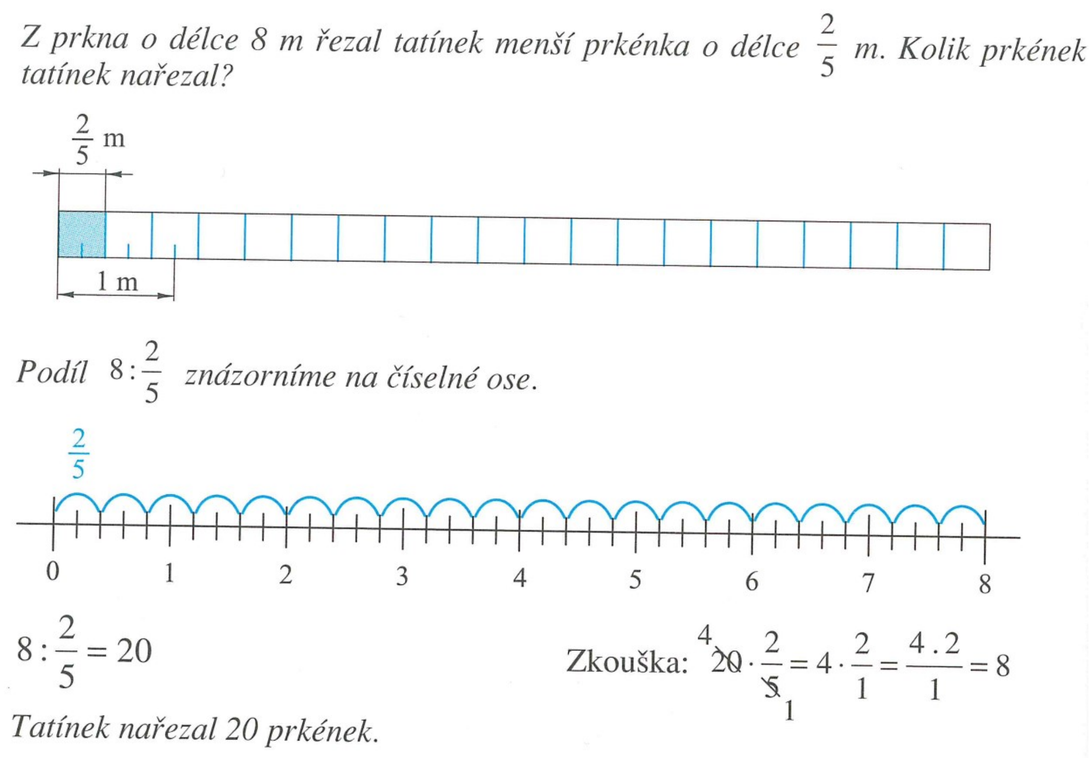
Task 16: Calculate and observe.

Task 17: How many half-dollars do you get for 10 dollars? How many thirds of a pie can we make out of 2 pies?

Yellow text: To divide by a fraction means to multiply by its inverse number.

Figure 2.3: Mathematical justification of fraction division through an observation task (textbook series D). Reprinted and translated from Molnár, Emanovský, Lepík, Lišková, and Slouka (1999).

Finally, posing a problem to the reader was also a common way to justify the mathematical statements. Sometimes, this would be a simple question (such as “Can you divide a pie into zero pieces?”), sometimes a more elaborate problem, or a series of problems. Figure 2.4 has an example of a problem used as the first of two that establish that dividing a natural number by a fraction is the same as multiplying it by its inverse. I designated such justification tasks as Problem (P).



Z prkna o délce 8 m řezal tatínek menší prkénka o délce $\frac{2}{5}$ *m. Kolik prkének tatínek nařezal?*

$\frac{2}{5}$ m

1 m

Podíl $8 : \frac{2}{5}$ *znázorníme na číselné ose.*

$\frac{2}{5}$

0 1 2 3 4 5 6 7 8

$8 : \frac{2}{5} = 20$

Zkouška: $\overset{4}{\cancel{20}} \cdot \frac{2}{\cancel{5}_{1}} = 4 \cdot \frac{2}{1} = \frac{4 \,.\, 2}{1} = 8$

Tatínek nařezal 20 prkének.

English translation:

Dad was cutting a plank of the length of 8 m into smaller pieces that were $\frac{2}{5}$ m long. How many smaller pieces did he cut?

Let us mark the quotient $8 : \frac{2}{5}$ on the number line.

Dad cut out 20 pieces of wood.

Figure 2.4: Posing a problem to provide warrants to justify the procedure of dividing by fraction (textbook series A). Reprinted and translated from Coufalová (2007).

Note that in nearly all[20] instances of reader involvement, the authors provide solutions and their own conclusions, i.e., the text always "tells" or summarizes what the reader should have learned completing the observation or problem-solving tasks (or series of tasks).

[20] With the exception of a task in series B, where pupils are asked to predict what a division by zero would yield and then check their prediction using the inverse operation in an equation.

The participation codes across topics and series are shown in Table 2.2. We can see that there are also differences both across topics and across textbooks. In the case of topics, I noted that some of the topics seem to lend themselves more to justification by simply telling, while others are more often justified through a problem-solving or observation activity. For example, the value of the zeroth power of a number was justified through telling in the majority of textbooks (in all five identified justifications), while in the case of multiplication of whole numbers and division of fraction, the tasks of problem solving or observation was given to the reader in order to establish an understanding behind a general statement. The nature of the text itself does not seem to carry much weight in the sample of topics that I chose: with the exception of textbook series F, all textbooks use a mixture of telling, as well as engaging the reader in observation and/or problem solving when asserting a general mathematical statement. In some cases (for example, in the case of the area of a circle in series A, B, and G), multiple approaches and multiple justifications are shown.

Table 2.2

Engaging textbook readers in justification of general statements. U (no justification or authoritative justification provided), NA (no mention of concept/topic), T (Telling), O (Observe) task), P (Problem).

TEXT SERIES/ TOPICS	A	B	C	D	E	F	G
Non-division by zero	NA	P	T	U	T	P	U
The zeroth power	U	T	T	T	T	NA	T
Square root of zero/negative	NA	T	U	U	T	O	T
Multiplication of whole numbers	P	P	U	P	O	O	U
Division of fractions	P	P	O	O, P	O	P	P
Area of circle	O, T	P, O	T	T	T	P	O, T

Modes of representation and number of arguments

To get an idea of the qualitative character of the justifications in textbooks, I identified the modes of representations used in these justifications. The number of justifications per topic in one book is also a relevant indicator of how authors present a mathematical truth, and the nature of mathematical reasoning. Further, the presence of alternative justifications may signify a didactic strategy of making explanations accessible to more pupils; varied modes of representations also can be a didactic choice in that they help connection-making between representations and concepts.

My sample showed only few instances of multiple justifications of one topic in a particular textbook (see Table 2.3). One textbook (E) provided two justifications for non-division by zero. Only one textbook (A) provided alternative justifications for both the multiplication of two negative numbers, and for the area of a circle. The procedure for dividing by fractions was justified in two different ways in three different series (E, F and G).

Table 2.3

Alternative justifications in textbooks. 0 – no justification, 1 – one justification, 2 – two distinct justifications.

TEXT SERIES/ TOPICS	A	B	C	D	E	F	G
Division by zero	0	1	1	0	2	1	0
The zeroth power	0	1	1	1	1	0	1
Square root of zero/negative	0	1	0	0	1	1	1
Multiplication of whole numbers	2	1	0	1	1	1	0
Division of fractions	1	1	1	1	2	2	2
Area of circle	2	1	1	1	1	1	1

Justifying non-division by zero at the Year 7 level appeared with two different modes of representation and models: a) using the concept of partition: we cannot divide anything into zero parts, and b) using inverse operations and introducing this problem: If $a \div 0 = b$, for any $a, b \neq 0$, then $a \cdot b = 0$ and we come to a contradiction, because this will not hold true for any $a, b \neq 0$. For more detail, see Žalská (2012c).

The zeroth power was justified across the textbooks by using the definition of power (multiple multiplication) and the division of powers. Similarly, the justification of the square root of zero or the non-existence (within real numbers) of the square root of a negative number was explained using the inverse operation, i.e., the power (e.g., $0 \cdot 0 = 0 \text{ or } 0^2 = 0$).

In justifying the multiplication of two negative numbers resulting in a positive number, authors used a) the analogy of changing direction (taking steps and changing directions analogy), b) the establishment of the warrant of $-a \cdot b = -(a \cdot b)$ for positive real a and b, in combination with a multiplication sequence (see Figure 2.5), or c) the establishment of the warrant of $(-a) \cdot b = -a \cdot b$ for positive real a and b and then using the warrant that multiplying by (-1) yields a number with the opposite polarity and then deducing the rule on a specific example that e.g., $-3 \cdot (-7) = -1 \cdot 3 \cdot (-7) = -1 \cdot (-21) = 21$.

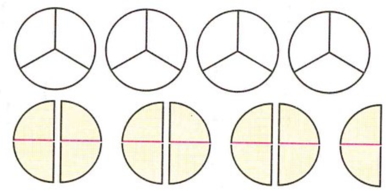
Arguments for division by fractions varied by their mode of reasoning as well as the models it used for division. Authors used warrants based on division as partition, some on division as quotition. One textbook (F) uses the equivalent of expanded division[21] in combination with warrants built on the idea of dividing a fraction by a whole number. Two textbooks (C and D) use explicitly the idea of inverse operations, i.e., when we divide by a number, we are looking for a number to multiply the dividend with, in order to obtain the divisor (Figure 2.5). Interestingly, the combination of warrants (as with modes of reasoning) varied across all textbooks.

[21] Using proportion or equivalency of expanded fractions "to divide two thirds among three people is the same as to divide 2 wholes among 6 people" (Herman, Chrápavá, Jančovičová, & Šimša, 2004, p. 83, translated).

Příklad 17: a) $4 : 12 = \frac{1}{3}$

$4 : \frac{1}{3} = 12 \qquad 4 : \boxed{\frac{1}{3}} = 12 = 4 \cdot 3 = 4 \cdot \boxed{\frac{3}{1}}$

b) $\frac{7}{2} : \frac{1}{4} = 14 \qquad \frac{7}{2} : \boxed{\frac{1}{4}} = 14 = \frac{7}{2} \cdot \boxed{\frac{4}{1}}$



Z obecné školy už víte, že dělení je početní výkon „opačný" k násobení: Určit podíl $a : b$ dvou čísel a, b ($b \neq 0$) znamená vlastně najít takové číslo x, které splňuje rovnost $b \cdot x = a$. Prozkoumejme, zda náš způsob dělení zlomku zlomkem má také tuto vlastnost. Tak například podíl $\frac{2}{3} : \frac{3}{4}$ by měl být roven takovému číslu x, pro které platí

$$\frac{3}{4} \cdot x = \frac{2}{3}.$$

Najděme tedy takové číslo, jehož tři čtvrtiny jsou rovny zlomku $\frac{2}{3}$. Pokud tři čtvrtiny čísla x jsou rovny $\frac{2}{3}$, je jedna čtvrtina čísla x rovna $\frac{2}{3} : 3 = \frac{2}{9}$, číslo x je pak čtyřnásobkem čísla $\frac{2}{9}$ a platí

$$x = 4 \cdot \frac{2}{9} = \frac{8}{9}.$$

English translation:

You already know from primary school that division is an action „opposite" to multiplication: To calculate the quotient $a : b$ of two numbers a, b $(b \neq 0)$ means to find a number x so that it satisfies the equality $b \cdot x = a$. Let us investigate whether our way of dividing a fraction by another fraction also has this characteristic. For example, the quotient $\frac{2}{3} \div \frac{3}{4}$ should equal a number x for which

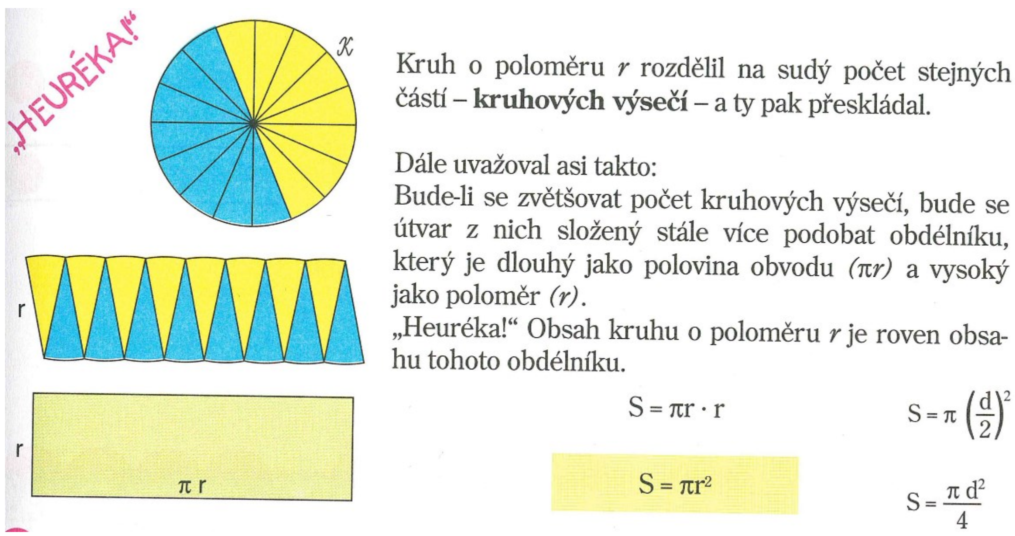
$\frac{3}{4} \cdot x = \frac{2}{3}$.

Let us find then such a number of which three quarters are equal to the fraction $\frac{2}{3}$. If three quarters of number x equal $\frac{2}{3}$, one quarter of number x equals $\frac{2}{3} \div 3 = \frac{2}{9}$, number x is then the quadruple of the number $\frac{2}{9}$ and it is true that

$x = 4 \cdot \frac{2}{9} = \frac{8}{9}$.

Figure 2.5: Using the concept of inverse operations as warrants in justifying the procedure for dividing by fractions (textbook series D and F). Reprinted from Molnár et al. (1999) and translated from Herman et al. (2004).

In determining the area of a circle, all authors utilize the idea of rearranging a circle into a polygon (a rectangle or a parallelogram – see Figure 2.6). Some textbooks asked pupils to cut up their own models of a circle from paper and all of the representations were pictorial. As I found out, even when all authors use the same idea for their justification, the limit idea (of dividing the circle into smaller and smaller sectors to approximate a parallelogram) is only explicitly stated in 5 of the 7 textbooks: in textbook series C and E, the authors state that the area of the circle is equal to the "almost-rectangle" (series F) or "is nearing the area of the parallelogram" (series C). In four series (A, B, D, and E), exploration tasks precede the justification of rearranging the circle.



English translation:

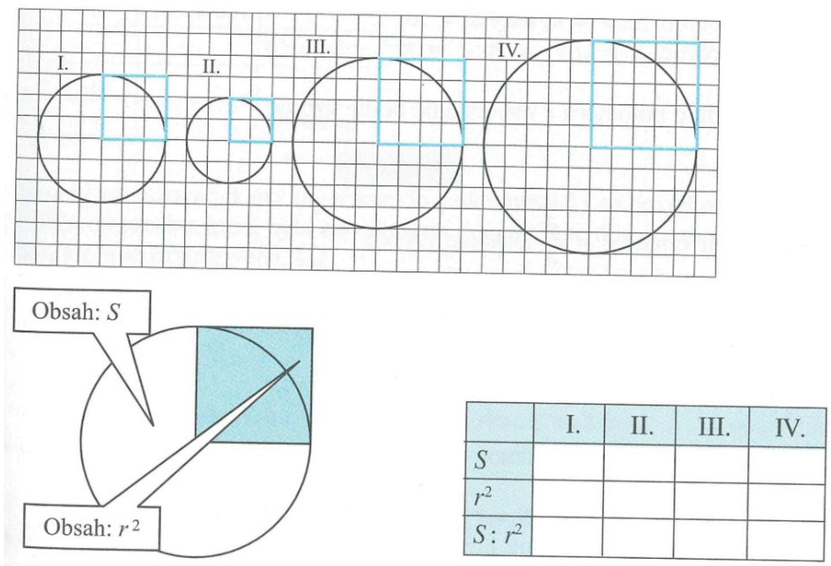
He divided the circle with radius *r* into an even number of equal parts – circle sectors – and then he rearranged them.

Then his reasoning went like this: If the number of sectors increases, the shape that we get from them will more and more resemble a rectangle that is as long as a half of the circumference (πr) and as wide as the radius (r).

„Eureka!" The area of a circle with the radius *r* equals the area of this rectangle.

Figure 2.9: Determining the area of a circle using rearrangement (series D). Reprinted and translated from Molnár et al. (2000).

Alternative arguments included estimating the area of an inscribed and a circumscribed square, or by using a square grid to estimate the areas of inscribed and circumscribed polygons. Textbook series A was the only one that draws conclusion based on estimated data in a table (Figure 2.7). One textbook (C) suggests the use of software (for example, Cabri), showing a picture of the program's screen where a dodecagon is inscribed in a circle and the areas of both shapes are shown as calculated by the software. The use of software is possibly intending to show the approximation of a circle by inscribing regular polygons with shorter and shorter edges, although it is not clear from the text.



	I.	II.	III.	IV.
S				
r^2				
$S : r^2$				

Podle tabulky se zdá, že pro obsah S kruhu platí

$$S : r^2 = \frac{S}{r^2} = \pi$$

English translation:

From the table it seems that for the area S of a circle the following is true:

Figure 2.7: Justifying the area of a circle using estimation (textbook A). Reprinted and translated from Coufalová (2007).

Modes of reasoning

What kind of reasoning do authors use when justifying the selected mathematical statements? Are their explanations based on deductive reasoning or do they fall into the category of empirical proof schemes? Overall, it seems that in treating the topics selected, authors are careful to use deductive reasoning (except for the cases where authors appeal to authority). Recall that two of the topics (fraction division and the area of the circle) are justified in all of the examined textbooks, and yield an insight into a finer-grained analysis. I applied Stacey and Vincent's (2009) framework, and found cases of all three deductive modes of reasoning: deduction using a specific case (GSC), deduction using a general case (DGC), and deduction using a model (DM). Further, I was able to identify examples of experimental demonstrations (ED), in my case, those in determining the area of a circle, as well as instances of concordance with a rule (CRM). Table 2.4 shows the particular distribution across textbooks and the two topics.

Table 2.4

Modes of reasoning in textbook justification of two topics. CRM (concordance of a model with a rule), DM (deduction using a model), DGM (deduction using a general case), DSC (deduction using a specific case), and ED (experimental demonstration).

<table>
<tr><th>TEXT SERIES/ TOPICS</th><th>A</th><th>B</th><th>C</th><th>D</th><th>E</th><th>F</th><th>G</th></tr>
<tr><td rowspan="2">Division of fractions</td><td rowspan="2">CRM</td><td rowspan="2">DM</td><td rowspan="2">DM</td><td rowspan="2">CRM</td><td>DM + DCS</td><td>DSC</td><td>DSC</td></tr>
<tr><td>CRM</td><td>CRM</td><td>CRM</td></tr>
<tr><td rowspan="2">Area of circle</td><td>ED + CRM</td><td>ED</td><td rowspan="2">DGC</td><td rowspan="2">DGC</td><td rowspan="2">DGC</td><td rowspan="2">DGC</td><td rowspan="2">DGC</td></tr>
<tr><td>DGC</td><td>DGC</td></tr>
</table>

From the results, it is apparent that determining the area of a circle, a geometric concept, was deduced using a general example in all books, even if there were exploratory tasks that involved experimental demonstration (cutting out and cutting up of one specific circle by the pupils). In the case of fraction division, two textbooks chose to provide solely the non-deductive mode of reasoning (CRM) when they simply established that one way of calculating the division by a fraction happened to give an equal answer to the multiply-by-the-inverse procedure. There was no warrant that explained this phenomenon, i.e., why this is so, and why it works for all fractions.

2.5 Conclusions and discussion about argumentation and the Czech curricular context

The examination of the Czech national curricular document for lower secondary schools (the FEP) showed that aspects of argumentation (in general) are woven into the conceptions of key communication, problem-solving, and learning competencies. Mathematical argumentation is not particularly distinguished from problem-solving activities in the document. The FEP draws out the appreciation for and existence of multiple arguments, multiple representations, as well as the preciseness and efficiency of language and argument. In terms of modes of reasoning, the document refers to logical reasoning, but does not specify what modes of reasoning are acceptable. It also acknowledges that verifying or rejecting own solutions or hypotheses is important for pupils' confidence, and sets common-sense problem solving skills and arguments apart from the content-related ones, suggesting that reasoning "logically", i.e., not using mathematical apparatus, and reasoning using the mathematical content are cognitively distinct activities. Understanding the real-life meaning of numerical operations and the justification of algorithms commonly used for these operations is expected, as is the understanding of metric properties of shapes in a plane. The document does not give enough detail for either modes of reasoning, or different kinds of arguments, nor modes of representations.

One feature that stands out regarding argumentation is the vagueness of the language used in the document. Specifically, terminology such as *logical thinking* seems to be prevalent but undefined. It may be a good moment now to take a short excursion into the nature of the term logical thinking as it is understood in relevant literature. First of all, let me note that the term commonly used in literature written in English is reasoning. However, the Czech translation of the term reasoning can take on these two forms: *uvažování*, which reflects the element of thinking (*úvaha* = a thought, meditation on something), and *odůvodňování*, which brings along the element of justification, i.e., providing a reason (*důvod* = a reason).

This linguistic dichotomy between thinking and argumentation can also be observed in other contexts. For example, the Encyclopaedia of Mathematics Education's entry about mathematical competency frameworks (Kilpatrick, 2014) refers to competencies identified by a Danish project, where there is a distinction (Niss, 2003) between *mathematical thinking* or the "mastery of mathematical modes of thought" (p. 122) and *mathematical reasoning*. The study lists the following modes for mathematical reasoning: posing questions characteristic of mathematics, understanding limitations of a concept, extending the scope and abstracting some of its properties, generalizing results, distinguishing between different kinds of mathematical statements. For mathematical thinking, the authors distinguish: following and assessing a chain of arguments, knowing what a mathematical proof is or is not and distinguishing it from other mathematical reasoning, constructing formal and informal mathematical arguments.

In the case of the Czech curricular documents, it seems that reasoning and thinking may be used interchangeably. Let us now consider the terms *reasoning* and *logical reasoning*. Reasoning is the mental process of moving from one related thought to another. According to Hanna (2014), reasoning is "the common human ability to make inferences, deductive or otherwise" (p. 405). Logical reasoning is the process of doing so following certain rules. Logic is a discipline that provides these rules. At the same time, as Anderson (1990) points out, human reasoning often involves the generalization and evaluation of logical arguments (involving deduction from conditional statements and statements using quantifiers, and inductive hypothesis testing) within a wider context than the rules of formal logic.

Looking further at literature in education and psychology, Piaget's cognitive developmental stages are perhaps the most relevant to the understanding of what logical thinking may encompass in the context of learning mathematics. In Piaget's works (e.g., Inhelder & Piaget, 1958), mental processes called *operations* are logical in a concrete stage when a child can operate mentally using concrete physical objects, these operations are based on the logic of classes and the logic of relations (so that a child can decide on a class inclusion of an object or perform serial-ordering). On the lower-secondary level, children enter an abstract operation stage when they start operating with propositions and even if they do not separate these operations from content, as a logician would, they are able to operate abstractly when they start reasoning using propositions, especially in generating hypotheses.

The above-described mental processes are thus closely connected to the notion of logical thinking. To bring about a comparative perspective for the language of curricular documents, I can contrast the language and content regarding argumentation of the Czech FEP analysis with that of the national curricular guiding documents referenced across schools in the United States, such as the National Council of Teachers of Mathematics (NCTM) Standards and Principles or the Common Core State Standards.[22] The former document lists standards that highlight content related outcomes but, more pertinently to my research, five of the standards are aimed at not content but processes: Problem Solving, Reasoning and Proof, Communication, Connections, and Representation. While problem solving and communication match the Czech key competence categories, it is apparent that the defining of standards for reasoning and proof gives educators a much clearer picture of the expectations in what the Czech context calls, for example, "logical thinking".

A similar contrast between curricular documents can be found at the content-specific outcome level. For example, in the Common Core State Standards for Mathematics[23], under the topic of fraction division, the standard outcome is stated as "use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.)" (p. 37). Not only is the explanation required as an outcome but the justification by a specific example of an argument itself is exemplified.

The relative vagueness of terminology and sparsity of content specific outcomes regarding argumentation and justification in the Czech national curricular document is likely to be perpetuated in the schools' curricular documents. As we saw in the two examined examples, schools are given freedom to formulate their own approaches and priorities and express them in a school document but it appears that the general tone of the former document does not inspire further elaboration and focus of outcomes in the latter. This may mean that textbook curricula and documents such as teaching manuals might be the influential guide teachers have for the

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22 I am aware that the historical, cultural, political and economic differences in the reasons for and the ways in both documents have come into existence are significant. However, I believe the comparison helps to set the Czech curricular context within a more global context.

[23] National Governors Association Center for Best Practices, Council of Chief State School Officers (2010). Common Core State Standards for Mathematics. Retrieved from www.corestandards.org/assets/CCSSI_Math%20Standards.pdf on March 31, 2018.

actual interpretation and application of argumentation in school mathematics in the Czech Republic.

My survey of justification of specific numerical operations and one geometrical property in textbooks suggests an alignment with the national curricular framework in that it shows that authors use multiple models or arguments to justify, and that they tend to explain the rules for arithmetic operations (such as multiplying two negative numbers and dividing by fractions) as well as the metric properties (at least in the case of the area of a circle).

Further, Czech textbook authors take the provision of mathematical justification of general mathematical statements, such as formulas and procedures, relatively seriously. Across the topics I investigated, most were justified (or assumed to have been established/justified in earlier years[24], as in the case of non-division by zero), and it appears that the authors feel an obligation to justify, even when they render a rigorous argument unavailable. For example, in the case of justifications using the concordance of a model with a rule in fraction division, or the lack of the mention of a limit case when justifying the formula for the area of a circle. This may be interpreted as a feature of the school mathematics context: some justification or explanation is expected of the authors (generally viewed as an authority on school mathematics). Further, the rule or general mathematics statement was finally stated in all textbooks across the topics (with the exception of the case mentioned on p. 56), which highlights the need of a method or procedure, or a proven mathematically succinct language/representation.

As in similar studies conducted in other cultural contexts, the justification modes of reasoning, modes of representation and the number of supportive arguments vary across topics, and across texts. However, access to comparable results in other national contexts also helps us identify nuances in the approach to justifications in the Czech contexts. For example, in comparison with Stacey and Vincent (2009), the justification of division of fraction is present in all analysed Czech textbooks, while in Australia, two out of six books appealed to authority (Stacey & Vincent, 2009).

[24] This assumption might be problematic, as within the same publishing house, the primary and lower secondary textbooks were written by different authors (at time of the analysis in Study 1).

On the other hand, two out of seven Czech textbooks gave the procedure for multiplying two negative numbers without explaining why the result is a positive number, while seven out of seven Australian texts provided at least some sort of explanation, albeit mathematically less rigorous (such as qualitative analogy, e.g., the analogy drawn between the multiplication of directed numbers and observed back and forth movements across a film screen).

Silverman and Even (2015) report that across ten topics (algebraic and geometric) and eight textbooks, there was only one case (the area of a circle) where there was no attempt to justify other than by appealing to authority. Multiplication by two negative numbers as well as non-division by zero were justified. The number of arguments used towards justification is also different in Israeli and Czech books. It seems that multiple explanations are commonly presented in the case of Israel, while Czech authors tend to present one single justification for the studied topics. At the same time, it is true that Czech authors use different models (like the model of expanding fractions in one textbook) to form arguments or to explore the topic, such as in the case of the area of a circle.

Further, the general statements in Czech textbooks are presented as the preferred, efficient, methods or procedures – e.g., invert and multiply for the division of fractions, while, for example, Son and Senk (2010) report the use of "common denominator"[25] and other methods in US and Korean textbooks. This again aligns with the Czech national curricular context: the efficient use of proven methods forms part of the curricular outcomes, as we saw in Section 2.2.2.

If we agree with researchers and educators who assert that connection making when explaining a new topic/concept/procedure is crucial for learner's understanding, or that diverse representations are important for the learning of learners with diverse backgrounds, we may conclude that Czech textbooks, individually, do not offer enough support to teachers in this sense. This suggests that the idea of justification (showing that something is true) is taken seriously while the explanation of the meaning of a statement (what it means) is secondary. Still, the analysis reveals a certain variety of approaches in explaining. Even though there may be

[25]For example, $4 \div 4/5 = 20/5 \div 4/5 = 20 \div 4 = 5$.

dominant arguments for some topics (e.g., the area of a circle or the zeroth power), others offer alternative representations that complement an argument or alternative arguments (warrants) altogether. Even if multiple explanations are not provided in most books and most topics, the variety across texts seems to suggest that teachers can have access to multiple explanations and approaches if they use more texts as reference. With current development of electronically (and internationally) shared materials and resources, the pool of possible ways to explain and activities to engage pupils in discovery or argumentation is likely to expand as well.

The context presented in this chapter is illustrative as the general backdrop to teachers' beliefs and practices; we cannot say, at this point, to what extent this reflects actual teachers' practices. The ministry-approved textbooks and curricular documents tell us about officially accepted roles of argumentation, and what is officially accepted as arguments (i.e., provided in approved textbooks). With this context in mind, I next investigated teachers' beliefs about mathematics, school mathematics and argumentation in particular.

3 Study 2: Argumentation in teachers' beliefs and practices

In the previous study, I identified some characteristics of the national curriculum for lower secondary school mathematics, and the textbook curricula that are available for teachers to work with. Even though the national curriculum is a binding document for schools' curricula, we could see that its general nature allows schools and teachers freedom in enacting it. In alignment with my theoretical framework, in this chapter, I focus on teachers working within this curricular context. I present the results of a study of mathematics education beliefs held by six lower secondary teachers and I focus in detail on their beliefs about mathematical argumentation. Investigation into the actual enactment of arguments in the classroom is the subject of Chapter 4.

3.1 Research on argumentation in teachers' beliefs and practices

Teachers' beliefs about the nature of mathematics and about the teaching and learning of mathematics have been studied intensively for the past 25 years (for a review of the literature, see Žalská, 2012b)[26]. Analogically to analyses that reveal diversity in how mathematics is presented in textbooks, empirical research on mathematics teacher's beliefs has reported on disparities not only between cultures (such as individual countries) but also within one curricular community. Both quantitative[27] and qualitative[28] studies tend to describe types or categories of teachers' orientations, and some point out the range teachers cover between them[29].

A similar endeavour has so far not been carried out in the context of Czech lower secondary mathematics education, although there are several studies that show some variety in orientations or teaching practices in the Czech Republic (for example, Jirotková, 2012; Rendl et al., 2013).

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[26] In Schoenfeld's theoretical model, the term orientations includes the concepts of beliefs, values and preferences, while the term resources includes knowledge. Values and preferences are often tacitly included in such research, or at least not clearly excluded. That is why, when reviewing current results of empirical research on mathematics teachers' orientations (as understood by Schoenfeld), I sought to consult those studies concerning teachers' beliefs.

[27] For example, Andrews and Hatch (1999), Barkatsas and Malone (2005) or Hoz and Weizman (2006).

[28] For example, Cross (2009), Eichler (2006) or Stipek, Givvin, Salmon, and MacGyvers (2001).

[29] For example, Askew (1997).

Skott, Mosvold, and Sakonidis (2017) assert that teachers' beliefs are increasingly studied with connection to classroom practice:

> in the case of beliefs, research on teachers and teaching has increasingly moved towards a concern for the complex, dynamic and emerging character of classroom practice [and research needs to find ways to] acknowledge the significance of the multiple micro and macro factors that may influence how learning and lives in classrooms unfold. (p. 12, draft)

The following sections summarise findings from empirical research endeavours within the intersection of the two areas: teachers' beliefs about justification and argumentation and teachers' practices. Naturally, the investigations vary in methodology and aims: some studies focus on practice only[30], and put forward speculations about the underlying reasons (including teachers' orientations and beliefs). Others[31] intentionally investigate the connection of teachers' practices and beliefs.

In the Czech context, Rendl et al. (2013) give us some insight into the beliefs about argumentation and justification in mathematics classroom in studies that qualitatively analysed interviews with Czech primary and lower secondary mathematics teachers about pupils' difficulties with learning school mathematics, and their experience with teaching practices.

3.1.1 Argumentation as observed in teachers' practices

The first example of a study that reports on teachers' practices relevant to argumentation, Jacobs, Hiebert, Givvin, Hollingsworth, and Wearne (2006), is unique in its quantitative scale. The authors found that in the 50 video recordings (and their transcripts) of US 8th grade mathematics lessons, collected as random and representative data in the TIMSS 1995 and TIMSS 1999 studies, not one contained problems that would involve justification according to the following description: "[the] teacher or students verified or demonstrated that the result must be true by reasoning from the given conditions to the result using a logically connected sequence of steps"

[30] Such as Bergqvist and Lithner (2012), Connor (2017) or Drageset (2015).

[31] Such as Bieda (2010) or Staples et al. (2012)

(Jacobs, 2006, p. 21). The same was true for other forms of justification, such as explaining or motivating a mathematical assertion or procedure, generalizing or finding counterexamples for refuting a conjecture. Although the study's descriptive nature does not give us information about the underlying factors, it is nonetheless a confirmation of the fact that argumentation may not be an inherent element of the mathematics classroom.

Looking at the reasons behind such statistics, Bieda (2010) investigated what goes on in classrooms when problems involving justification are actually part of the intended curriculum. To that end, she studied the communication patterns in episodes involving opportunities of mathematical justification in seven middle-school classrooms. Her findings showed that teachers' priorities and lack of time marked decidedly the implementation of justification: under time restrictions, the teachers chose not to work on problems requiring justification, or not to discuss pupils' answers after problem-solving group-work had concluded. Apparently, in teacher's minds, the practice of classroom-enacted justification and argumentation was secondary to other practices or activities. When pupils did justify, the author reports a remarkable lack of feedback that was given to them: no feedback was given in 30% of instances and only one event that led to a pupil's own revision of his example-based justification was identified.

Drageset (2015) set out to study the opposite instance, i.e., how teachers respond when a pupil gives out an answer (e.g., to a mathematical problem or a question) without giving an explanation (either why or how they arrived at the answer). The data from classroom observations of five experienced teachers showed that teachers were more likely to engage in explaining why and how when they respond to an unexplained answer given by a pupil than in the rest of the classroom discourse situations. A finer-grained analysis ascertained that when such answer was correct, teachers typically requested pupils to explain the how, what and why and in response to incorrect answers, teachers typically responded with correcting questions.

Regarding the quality of teachers' interventions involving argumentation, Conner (2017) illustrates contrasting practices of posing questions in a classroom. One of the teachers Connor observed used her questioning mostly to promote argumentation in our sense of the term (why a statement is true), the other to mostly promote functional argumentation (arguments explaining to what end something is done).

Bergqvist and Lithner (2012) studied modes of reasoning present in teachers' explanations in a classroom. Specifically, they focused on the difference between the so-called *imitative reasoning* (algorithmic reasoning and memory reasoning) and *creative reasoning* (arguments that justify a statement in an epistemological way and using mathematical foundations). They looked for the presence of aspects that characterise both of these ways of problem solving as observed in a teacher's presentation of task solutions (i.e., also enactments of explanations and justifications in a classroom). They found that in the 23 instances of the observable reasoning situations, the majority involved algorithmic reasoning.

Discussing the impact of the practices observed, Conner (2017) reasons that arguments which support both elements (arguments about why something is true, and to what end something is done) are conductive to pupils' autonomy and understanding. This argument is supported also by Bergqvist and Lithner (2012) who contend that conducting an algorithm without comments or only with descriptive comments is likely to lead to pupils' rota imitative reasoning (algorithmic reasoning and memory reasoning). Presenting an algorithm with functional arguments (to what end something is done) is more likely to lead to pupils' not only performing an algorithm but also understanding why, and is more similar to creative reasoning with mathematical foundation.

What beliefs, then, can we identify behind the practice of implementing (or not) argumentation in a classroom?

Firstly, the role teachers assign to the practice of justification, i.e., what they believe argumentation and justification should be used *for* in a classroom, is one of the factors that determine the quality and quantity of arguments that take place.

3.1.2 Beliefs about the role of argumentation and justification in a classroom

Recall that Staples et al. (2012) identified numerous ways teachers perceive the role of argumentation and justification in a classroom (see Section 1.1.1). The following studies report on some of these in more detail.

Ayalon and Hershkowitz (2018) found out that when teachers were asked to choose potential tasks for argumentation (the intended argumentation) and to describe the rationale for their choice, they valued three aspects of these activities: the mathematics in which the argumentation is embedded, the socio-cultural aspects related to argumentation, and pupils' ways of mathematical thinking and conceptual development. The teachers always attended to the mathematics immediately connected with the task, and the pupils' conceptual understanding, less so to broader skills and meta-skills, or social aspects of argumentation.

Similarly, Rendl et al. (2013) point out that teachers in the Czech context embrace the explanatory quality of justification (i.e., the role related to the conceptual grasp of mathematical content). However, when gauging the extent to which Czech teachers perceive the role of argumentation and justification as a practice for the development of reasoning skills, the authors find their discourse rather confounding: on the one hand, teachers express the belief that mathematics develops logical thinking. On the other, when speaking about the learning processes in mathematics, they do not speak about logical thinking as a *result of*, but rather the *condition for* learning mathematics. Note that this is resonant of the ambiguity of the language used in the Czech national curricular document (the FEP), as described in Chapter 2.

Investigating teachers' beliefs about the roles of proof in school mathematics, Knuth (2002) interviewed 17 secondary teachers and presented them with arguments that varied in terms of their validity as proofs as well as the degree to which they played an explanatory role. He reports that teachers did not see the explanatory role in proof (or justification), but did value the fact that it verifies, i.e., that it shows that something is true. Staples et al. (2012), on the other hand, report that teachers mostly left this central role of justification – verification – out as well as the justification's role to allow for axiomatic systematization, and discovery. These seemingly contradictory findings can be the result of the context within the groups of teachers under each investigation. Staples et al.'s (2012) participants were middle-school teachers taking part in a project that explicitly promoted the use of justification and argumentation in a classroom, while Knuth's (2002) data came from middle and high-school teachers who were "committed to reform in mathematics education" (p. 67) but participating in less strictly determined professional development programs. Also, and perhaps more importantly, Staples' study focuses on teachers who understood justification as a pupil practice, which was promoted in their program, while Knuth's participants were speaking about the concept of proof in classrooms in general, i.e., not only as performed by pupils. Finally, Knuth's (2002) justifications were given the label "proof", and were considered hypothetically, without being seen in the context of a particular classroom situation.

Finally, Rendl et al. (2013) also report that Czech teachers valued the role of justification in classroom as representing (their view of) the nature of mathematics as a discipline. They expressed the belief that in mathematics things do not appear “out of nowhere”, and that it is important to show this to their pupils.

3.1.3 Teachers’ beliefs and practices: argumentation and its characteristics

According to Rendl et al.’s (2013) findings, representations seem to take up a considerable space in Czech teachers’ discourse. The teachers put a strong emphasis on the way concepts are represented when introduced to pupils in connection with their conceptual understanding. For example, they expressed that they use representations of “apples and pears” when teaching operations on algebraic expressions, or the thermometer analogy when introducing negative numbers. They also complained about the impossibility or difficulty to represent the operations of multiplication and division of negative numbers, fractions or algebraic expressions in a meaningful way (for them, this should be, for example, an analogy with something the pupils already know, i.e., real, concrete, but also already familiar mathematical objects and relationships). The teachers seemed implicitly convinced about the effectiveness of such an analogy, and this conviction was likely grounded in their own teaching experience.

Shedding more light on representations and the use of analogies, Diamond (2018) finds that some teachers believe in the use of *associations* in problem solving. The paper reports the results of a study of teachers’ beliefs about *transfer*[32] in problem solving when they select tasks and a lesson plan. In observing how teachers understand pupils’ learning (transfer, in particular) when planning lessons on slope, the author found that teachers held different beliefs about being able to apply experiences from one slope problem to another. Some teachers believed that associations (and procedures) were most important. That means, in our terminology, that at least

[32] The author characterises transfer as the phenomenon of one’s generalization and application or other use of what one knows to a novel situation. Specifically in this study, it is the situation of problem solving (and hence arguments).

one of the warrants refers to a situation in a previously solved problem, e.g., “walking towards something”, without referring to the underlying mathematical concepts, e.g., slope as a ratio. Another group of teachers, on the other hand, believed that transfer is based on the “ways in which pupils interpret the mathematical activity and the meaning [mathematically valid interpretation] they develop for mathematical topics” (pp. 14 – 15, online). This result shows that the use of analogy (including the representations described in Rendl et al., 2013) above is not automatically connected to the belief that conceptual understanding is necessary for making an argument when solving a problem.

Nardi, Biza, and Zachariades (2011) probe for the reasons behind teachers’ decisions to accept (or not) an argument laid out by a pupil. They show that the teacher’s reasoning *and* resolution behind acceptance of an answer[33] vary. For example, they show cases where one teacher is very clear what he would accept in class as a warrant (the visual warrant was meant to only inform and support class discussion and hypothesis that will need to be proved by the employment of definition and algebraic solution) while another teacher is not (they would accept some images as warrants but not others). A further examination of the former case showed that the decision-making was much more complex: the teacher considers what is acceptable as proof in mathematics, but also what is acceptable at the school level of mathematics, and finally, he internally applauds the intuition and originality of the visual argument.

Apart from the belief- and practice-related phenomena described above, there are two other aspects about argumentation that are recurring in either research findings or the interpretations of them: the beliefs related to argumentation in function to pupils’ characteristics, and the resources and knowledge teachers have access to.

[33] In this case, it was specifically the acceptance of a visual/graphical vs. algebraic solution of a upper secondary school tangent function problem.

3.1.4 Teachers’ beliefs about pupil dispositions

One belief category that seems to be explicitly or implicitly present in teachers’ beliefs in research, or the interpretation of their actions, is that of how teachers perceive argumentation and justification in relation to the diversity of pupils they teach.

Firstly, Bieda (2010) suggests that teachers’ beliefs about the goals of instruction and about their pupils’ abilities played a crucial role in their implementation of justification-related tasks. She documents teachers’ “uncertainty about whether students are capable of understanding conceptually the mathematics in the investigation, let alone justifying their thinking. Justification is seen as something ‘especially impressive’ or something for students who are developmentally ready” (p. 380).

The fact that teachers’ beliefs about pupils’ abilities affect their expectations for argumentation have also been confirmed by Raudenbush, Rowan, and Cheong (1993) and Zohar, Degani, and Vaaknin (2001) cited in Staples (2012): when it comes to high-order thinking practices, such as justification, teachers tend to engage only those pupils who they perceive as more able.

Bergqvist and Lithner (2012), speculate that the teachers’ implementation of arguments and modes of reasoning in problem solving in front of the class also depends on their view of how *useful* creative reasoning (argumentation) skills are for low- and high-performing pupils.

Planas and Gorgorió (2004) show that in addition to the pupils’ ability, ethnicity can be a distinguishing factor for teachers’ decision-making: the expectations for different (groups of) pupils are implicit in the classroom discourse patterns. Their classroom observations in a multi-ethnic mathematics classroom showed that local and immigrant pupils were “not expected to behave in the same way, nor [treated] in the same way” (pp. 35–36). The authors report that immigrant pupils were expected to use real-life context to explain/solve a problem, while local pupils were expected to use the academic context of mathematics only.

Hříbková and Páchová (2013) further confirm that Czech teachers distinguish between pupils mostly based on the pupils’ cognitive abilities, thus speaking about pupils who are smart or gifted on the one hand, and weak and below average on the other. At the same time, the authors also notice another differentiation among pupils in teachers’ discourse – effort. In the study, the teachers seemed to especially speak about the group of pupils designated as “gifted slackers“[34] (p. 246), i.e., pupils who are smart but do not like to make much effort, and “diligent dummies“[35] (p. 246), i.e., pupils who are not so smart but make an effort. At the same time, the authors report that in the teachers’ discourse, the existence of these (strongly defined) types of pupils has no effect on the teachers’ approach to teaching (or practices). Still on the subject of perceiving pupils’ abilities in combination with justification practices, Rendl et al. (2013) also note that some teachers believe that memorizing a rule is an option for a pupil in cases where they feel that the underlying content is too complex for certain pupils (for example, the pupils perceived as “diligent dummies”).

3.1.5 Teachers’ pedagogical content knowledge and resources

Rendl and Páchová (2013) argue that the looseness of the curricular documents, i.e., the absence of specific didactic guidelines, leads to a situation when teachers, left to their own devices, tap into their creativity, and value the originality of their own representations over a carefully chosen systematic set of models and representations, choosing analogies ad hoc[36]. My own analysis of representations in Czech textbooks does not have enough evidence in terms of confirming or disputing the presence of a systematic approach to using models and representations. However, there is a chance that at least some of the textbooks do not provide analogies that are meaningful to all pupils (which is observed by teachers in Rendl and Páchová (2013), who mention the impossibility to represent the multiplication of negative numbers, fractions, etc.). This is an example of how pedagogical content knowledge interacts with the resources (such as textbooks) content knowledge, and beliefs.

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[34] A gifted slacker is “chytrý flink“ in Czech. The authors of the analysis used gender-biased grammatical feminine noun to further underline that teachers also made differences between the two genders.

[35] A dilligent dummy is “hloupá snaživka“ in Czech. The authors of the analysis used gender-biased grammatical masculine noun to further underline that teachers also made differences between the two genders.

[36] And the authors go on to argue that this, in turn, makes some pupils (as teachers report) prefer learning by rota, where such analogical connection is not (or in their experience has not been) effective for them, for whatever reason.

Bergqvist and Lithner (2012) discuss the lack of creative mathematical reasoning in teachers' presentation of a task solution in their study and suggest a variety of reasons. These range from the general public's view of the purpose of mathematics education to the curriculum and assessment restrictions[37] and, finally, to teachers' beliefs about these aspects.

Diamond (2018) points out that the reason teachers believe in the effectiveness (for pupils' understanding, or ability to solve problems, in this particular case) of one representation or mode of reasoning over another appears dependent on the topic and the teacher's pedagogical content knowledge.

Another phenomenon that has been reported is that some teachers see opportunities for argumentation in textbook tasks that are not phrased as justification or argumentation tasks (Ayalon & Hershkowitz, 2018). On the other hand, as we saw in Bieda (2010), they are also likely to drop the intended justification task or the argumentation activity if they feel restricted by time.

From the above research, it is clear that in studying how and why argumentation happens in a classroom, we need to take into account two kinds of beliefs:

a) a teacher's beliefs about argumentation and justification that are global – for example, about their view of role of argumentation in respect to their beliefs about the aims of mathematical education, or in respect to its significance for pupils of different levels of achievement and other characteristics, etc., as well as

b) those that are local in relation to particular mathematics content, such as the beliefs exemplified by the teachers in Nardi et al. (2011) or Diamond (2018).

In Study 2A I aim to identify individual teachers' global orientations (beliefs, preferences and values) as pertaining to argumentation and justification: its roles and forms, its roles and forms as connected to their orientations about mathematics, mathematics teaching and learning and the outcomes of mathematical education, their beliefs about the curricular framework, and,

[37] Swedish teachers tend to align their practice with textbooks, which are not under official control, rather than with national curricula and assessments (Lithner, 2008).

specifically, their beliefs about pupils' abilities in connection with argumentation and justification.

In addition to beliefs, it is apparent that teachers' knowledge (in the case of experienced teachers, especially pedagogical content knowledge) plays an important role in the way arguments occur, get accepted, etc. in a classroom, as do other resources such as textbooks and curricular documents.

In Study 2b, I endeavour to recognize individual teachers' local beliefs, investigate how they interplay with the teachers' local resources (such as the textbook curriculum available to the teacher, and pedagogical content knowledge), their global orientations and the pupils' actions, and to explain how and why argumentation takes place.

3.2 Methods, data and participants

3.2.1 Methods and data

The method design I chose for the exploration of Czech lower-secondary teachers is a multiple-case study (Yin, 1994).

The study of beliefs is, methodologically, a difficult task: in qualitative studies, inference plays the most important part in identifying teachers' beliefs (see review Žalská, 2012b): interviews, prompted (written) reflections/journals, and observations are the most commonly used methods for data collection.

In a pilot study[38], I was able to confirm that the use of specific classroom occurrences as prompts for the teacher's reflection helped to strengthen the initial interview data reliability and, especially, provided opportunities to elicit more detailed insight into beliefs about argumentation (as well as for the collection of data for Study 2b, the study of practices). Thus, I designed a sequence of interviews, woven around a sequence of lesson observations.

[38] I conducted a pilot study with three teachers and focused on finding out what the participants perceived as problematic for Czech pupils when learning mathematics. I collected the data through interviews that included various prompts, such as textbook materials and classroom events observed in their lessons.

Figure 3.1 describes the design in more detail: the tools I chose are semi-structured interview and non-participant observation. The aim of the initial interview was to generate data for identification (inferential) of a teacher's beliefs and practices. The topics I targeted in these interviews were, for example: the teacher's own experience with school mathematics and their teaching career, why they teach mathematics, their view of current curricula, their use of textbooks, the importance of school mathematics for pupils, of mathematics in general, a successful pupil in mathematics, a personal and professional experience with school mathematics and mathematics, their description of the group of pupils that were going to be part of the observations, etc.

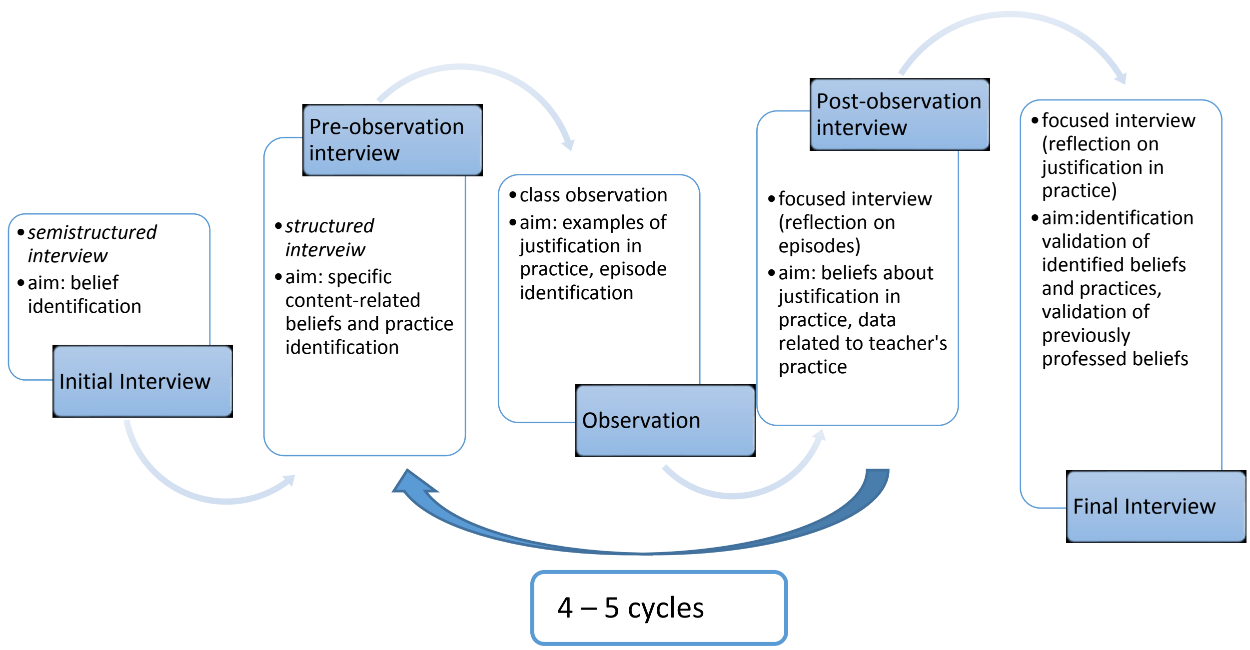

Figure 3.1: Chronological outline of data collection and primary aims.

I planned to complement the lesson observations by pre- and post-lesson interviews and to carry out these interviews in four or five cycles, within one topic unit of teaching (e.g., five lessons about percentage in a Year 7 class). As statements about the intended curriculum and lesson plan have been successfully employed in research on teachers' beliefs in the past (see Žalská, 2012b), I focused on the teacher's aims for the lesson, a rough outline of what they expected to do in the lesson, as well as any concerns regarding pupils, or individual pupils in each pre-observation interview.

Observations were non-participant, and primarily non-selective. In each session, though, I, as the sole observer, tried to identify main episodes and/or phenomena related to the use of justification. I then used these as prompt material for the post-observation focused interviews.

Finally, a preliminary analysis of all the data gathered in the above described steps (including the initial interview data) informed me about the structure of the final interview which included specific questions about justification-related beliefs and practices as well as reformulated some of the belief-oriented questions from prior interviews (to further increase validity). I designed the repeated observations along with the focused and prompt-based multiple interviews to secure the reliability and validity of gathered data.

Up to the final interview, I made no explicit reference to the investigation's focus on justification, as such information could have an effect on the teacher's behaviour. To preserve investigation ethics[39], this objective was revealed to the teacher in the final interview.

I followed this initial design whenever possible. However, as is often the case in educational research, time restrictions allowed for less ideal data: namely, it proved impossible to carry out some of the planned pre- and post-observation interviews. One of these short interviews had to be conducted during a lesson, while pupils were individually completing a task, and some post-lesson interviews had to be combined into one. Also, it was not always possible to observe the same group of pupils in five cycles, so in some cases the observations of the same teacher were conducted in two different classes (see Table 3.1). However, I believe that the interviews still yielded data that are relevant and valid in the investigation of teachers' beliefs about mathematics education and argumentation in particular.

[39] A consensus was obtained from each teacher to use an audio-recording device during all stages of the cycle, as well as to take photographs of any relevant class material (excluding any visual material in which a particular pupil could be identified, unless their parents' consensus was previously obtained).

Table 3.1

Participants and the observed lessons.

Teacher	Class 1: Year Topic # of lessons	Class 2: Year Topic # of lessons	School type	School location	Years of experience
Karen	Year 7 Percent 5	--	Basic	capital	16
Barbara	Year 7 Fractions 4	--	Basic	capital	9
Zachary	Year 11 Solid Geometry 4	--	Gymnázium	capital	27
Jenny	Year 8 Circle 4	Year 10/Logic/2	Gymnázium	non-capital	25
Victor	Year 9 Solid Geometry 2	Year 12 Analytical Geometry 2	Gymnázium	non-capital	15
Charles	Year 7 Circle 4*	Year 10 Algebra 1	Gymnázium	non-capital	20

As it proved impossible to obtain all participants' consent for video-recordings, interviews and lessons were audio-recorded with two devices (as a back-up, and for clarity, in case of the lesson observations), and photographs were taken of artefacts relevant to the lesson and/or interviews (mainly of the blackboard, pupils' work, and task-related materials). All recordings of interviews and lessons were filed and transcribed, including field notes (such as the number of pupils, the pupils' names and general comments about the run of the lesson) and the pertaining photographs.

3.2.2 Participants

The six participants chosen for the exploration study of beliefs were teachers teaching at the secondary level (either at the basic school, or at the more academic multi-year secondary grammar school track, where they also taught classes on the upper-secondary level). All of them were, at the time of the study, also teaching classes in lower secondary school – Years 6 to 9 (see Table 3.1).

I made the initial selection of the participants on the basis of exploration of richness of relevant features of the phenomena under study, i.e., in the hope to find both literally and theoretically replicable cases (Yin, 1994). With the aim to capture a wide range of beliefs, I selected the teachers because of my partial knowledge of their different teaching styles (from previous research activities or upon recommendation). All six teachers were recommended as (or had, within their school community, the reputation of) successful professionals. All of them also agreed to share their personal views and practices with me, for research purposes, upon initial contact – something that indicated a certain level of self-confidence and openness. Other criteria for selection included the length of experience (all teachers had had at least 10 years of experience in teaching mathematics). Three of the teachers were teaching in schools in the capital, three in other regions of the country). The topics and classes observed were subject to teachers' convenience and availability (see also Section 3.2.1). The names of the teachers have been changed to preserve anonymity.

3.3 Study 2A – Data analysis

I first analysed the transcripts qualitatively, based on Paul Ernest's (1991) typology (see Appendix A, Table A1) of philosophical views of mathematics education. As expected, the declared beliefs of any one teacher did not fit the columns of the ideal prototypes. To provide a salient picture of the individual participants, I identified the beliefs as expressed by each participant about the individual categories (e.g., View of Nature of Mathematics) and noted their alignment with the characteristics of the theoretical (Ernest's) types. For example, if a teacher's espoused beliefs aligned most with the statement that the aim of mathematics education is to grow a human being, to develop creativity and self-learning through learning experience, to foster an autonomous inquirer, and to strengthen individual's self-efficacy, then this teacher's set of beliefs was assigned the type Progressive Educator under the category Aims of Mathematics Education. I then understood the teachers' orientations as individual compositions of the various types and categories.

Still, this categorical composition of characteristics leaves behind some qualitatively important information (e.g., which category is more central and which more peripheral in the teacher's overall orientation). To add a more holistic element to the story, I complemented the analysis according to Ernest's framework with an overall narrative of each teachers' expressed orientations.

Finally, I looked specifically at teachers' beliefs about argumentation and about its place in mathematics education. Informed by the cited empirical findings (Section 3.1), I paid specific attention to teachers' views about the roles and forms of justification in the teaching and learning of mathematics, and how these fit into their orientations towards mathematics, mathematical teaching and learning and the outcomes of mathematical education. I further inspected the data for connections between the beliefs about justification and the teachers' beliefs about the curricular framework as well as their beliefs about pupils' abilities in connection with argumentation and justification.

3.4 Study 2A – Results

3.4.1 Espoused mathematics education beliefs

Figure 3.2 outlines the individual participants' expressed beliefs as I interpreted them to match the general orientations of the theoretical types. The numbers in each box account for the number of categories that matched the certain theoretical orientation (type). We can see the spread of orientations spanning all categories – from Charles with mostly Public Educator beliefs to Karen, who seemed to hold mostly beliefs fitting the Technological Pragmatist and Industrial Trainer orientations. Note that the orientation type is independent of the kind of school (basic school or gymnázium) where the teachers teach. Also, teachers held beliefs about one category that matched various orientation types (for example, a teacher believed that pupils learn through problem solving, questioning, and decision-making but also that hard work, effort, and practice put into the intellectual activity of mathematics is paramount to learning it). To give the reader a better picture of the participants' beliefs, I briefly characterize each teacher to depict their unique professional beliefs, values and preferences, in addition to showing individual teachers' composition of categories and types.

Type / Teacher	Industrial Trainer	Technological Pragmatist	Old Humanist	Progressive Educator	Public Educator	School Type
Charles	1	1	3	9	10	G
Barbara				9	7	BS
Victor	2	4	7		3	G
Jenny	2	5	6	1	1	G
Zack	2	10	6		1	G
Karen	6	12				BS

Figure 3.2: Mathematics education orientations of participants and their school types. BS – basic school, G – gymnázium.

Charles – the Discovery Realist

Charles's orientations are strongly leaning towards that of Ernest's Public Educator, and he also espouses Progressive Educator and Old Humanist views (see Figure E1 in Appendix E). What characterizes him is distinguishing between the "general, average" thinking, teaching, or "mastering" and himself, his views and his own teaching. Charles's ability to reflect on practice and to verbalize beliefs is outstanding among the participants.

Charles's biggest concern is with independent critical thinking, which a) takes effort and which, in his opinion and experience, b) is seriously lacking in the existing (Czech) education system. This is especially true in all the other subjects, where memorization and fact recitation is valued, rather than connection-making and relating ideas and information. He does not think that critical thinking is the sole virtue of mathematics. Thus, the goal of education is to grow individually independent thinkers: thinking for yourself is important, rather than being a diligent and obedient "hard-worker" and/or a consumer of technology tools or given procedures. Ideally, pupils learn how to think about what and why they are doing it, challenging themselves as well as challenging established routines or means for problem solving.

In Charles’s opinion, in order to learn something, it is important to figure things out for yourself and work hard on this – not to be given something (e.g., a result, a formula) from the teacher or other resources 'for free', i.e., without making a mental effort:

> If children don't work actively themselves [on a problem] or don't at least partially discover it for themselves, it's not really worth it. […] I never approached [teaching] mathematics believing that I'm going to give a lecture on something and everyone will learn it and know it like I do. I never [believed that], I know that is nonsense.

At the same time, Charles realizes that although discovery supports retention it does not guarantee it. Given the current school mathematics curriculum, which contains a wide range of separated concepts and areas, he believes that it is essential to interconnect topics, so that things do not get forgotten. However, he expresses his doubt whether it is attainable to do so with all the currently prescribed topics in the curriculum. Retention of core concepts and procedures happens when these are embedded in discovery and problem solving in other areas of mathematics, for example, algebraic expressions. He believes that skills (e.g., "manual" calculation skills) are learned throughout problem solving and should be its by-product. At the same time, the deeper you investigate an area or a topic in mathematics, the more there is to "remember" and readily use in problem solving:

> The longer we spend on a given [topic], the worse it gets because they need to kind of remember more. If [all topics] were all mixed together, that would be absolutely in … in … inaccessible, unmanageable for most of them.

The amount of mental energy and the willingness to expend it seem to be key to learning. Such willingness is mostly the product of social factors and personal experience, and the aim of (not only mathematics) education is to cultivate it.

According to Charles, the role of the teacher is to provide materials and tasks that lead to investigation and discovery, and to facilitate class discussion. It is important for the teacher to also attend to pupils' individual misconceptions and struggles, to problem-solve on a pedagogical level, and to make sure that his pupils are engaged in problem solving at the level they are able to do so. That also means telling pupils when they make an error and leading them to self-correction.

At the same time, Charles admits that the discovery method he uses might not be the best way of teaching because it creates dependency on the teacher's preparation of the discovery problems. It may be, paradoxically, possible that someone who is taught in a traditional way needs to make much more effort to figure things out for themselves, and therefore become a more independent problem-solver.

Charles’s beliefs about argumentation and justification

In alignment with his beliefs about learning and teaching, it is the pupils in Charles’s classes that are actively engaged in problem solving that is designed to discover mathematical relationships and it is they who produce justification and arguments. Further, he values that pupils explain and produce arguments in individual consultations with him or spontaneously in pairs, rather than in all-class discussions (which would take up too much class time). Unlike other teachers in this study, he does not think that there are topics in the curriculum that need to be presented without mathematical justification.

As we saw in the previous section, Charles primarily sees pupil-centred activity as essential for learning and for conceptual understanding. For example, he expressed his dislike of a textbook series that gives too many rules *“*they are the kind of rules that if the pupils don’t even know what the letters [in algebraic expressions] mean, they do not know what the rule actually says*”*. He confirms his conviction in the need for pupils to work content out on their own, describing another textbook series: “It is well explained and there are some good problems but the fundamental approach is wrong [i.e., it explains and does not let the reader do the mental work].”

Even so, he expresses his worry about the pupils’ dependence on Charles’s ways of explaining, and considers training them in reading other texts with understanding to avoid procedural mastery without understanding:

> They are not used to studying a text with … understanding, in general, actually. […] so they read the text to only be able to repeat the example methods/procedures that are

> there. They really just look at the examples [of solved problems]. I should use the textbook more often so that they can learn to read but…

As critical thinking (or the lack of it) is at the forefront of his mind, Charles also believes that verification is of utmost importance in all areas of human activity, and in mathematics in particular. He stated that not accepting a statement just because someone (a teacher, authority, media) claims it is something he finds essential, and he lamented the fact that most pupils are not used to the idea that everything needs to be “built on some foundations”.

> Anything you tell them is a fact, they are too trusting, they forget or just cannot evaluate how important or not important something is. […] If I gave them a wrong formula, they would not protest.

Charles also demonstrated a strong awareness of the lack of reasoning skills in terms of proof and understanding of valid modes of reasoning, which he does see as an outcome of mathematics education. This is something that his pupils in particular are only beginning to develop, and he admitted that although they can justify a simple (mathematical) statement, to “prove something really from the beginning” is a different story. In his experience, they still have no grasp of the “start and the destination” of a justification (this was prompted during a conversation after a lesson where pupils were working on justifying Thales theorem – more about this in Section 4.3.5). For this reason, it is important to discuss the validity of modes of reasoning (e.g., an equivalence statement needs to be shown as two implications in both ways) even if not everyone would grasp the subject, yet.

Charles is particularly concerned with an apparent lack of the ability to generalize in pupils and, again, he explains this through the social environment they grow up in, e.g., the fact that they are taught and used to concretise in other school subjects. He believes that “reasoning and thinking” is what will distinguish his pupils from the rest of people but these activities are not enforced in other environments (especially other school subjects) and that is what makes some pupils give up, too. According to Charles, thinking and reasoning need to be an expectation, the standard, but it is not, and society is now based on learning procedures, not systems.

Charles also feels that the school environment (although it is not clear whether he refers to the mathematics curriculum, including his lessons, or again a general perception of the schooling system) is what brings formality and dwelling on exact definitions usage, etc. in problem solving and argumentation. In his experience, school expectations are more pedantic than mathematicians doing their work are.

Finally, Charles has expressed that in his experience, there is a difference between male and female pupils' approach to problem solving. This was prompted by the events in his two lessons (a female group and a male group separately working on the same problems) where pupils were proving Thales theorem. Charles distinguishes between the straightforward solution (including justification) professed in his experience mostly by female pupils, and the more creative one by male pupils. Male pupils, in his experience, are likely to produce a more creative solution/proof (as in the example of Thales theorem), female pupils are more precise and likely to come up with only the "straightforward", obvious, way.

Barbara – the Connectivist Educator

Figure E2 in Appendix E shows that Barbara espouses important Progressive Educator beliefs but also embraces some of Public Educator's values. Her professional beliefs mirror the approach to mathematics teaching and learning of the Hejný method[40], an approach based on the principles of social constructivism, and she defines her teaching strongly against the image of traditional mathematics teaching, especially against her own schooling experiences.

Collective activity and awareness is very important for Barbara. In her view, a teacher needs to create an environment (i.e., to create classroom management rules and a need for pupils to communicate effectively), in which pupils can work on mathematics collectively and learn from each other and from their own mistakes. Fostering oral communication and argumentation is important. She also stresses that the more general principles of social justice and democratic values cannot be separated from mathematics education: pupils should be brought up in an

[40] For more information on this method, see, for example, http://www.h-mat.cz/en.

environment of partnership, not authority, and should be encouraged to demand and participate in the justification of a norm, a mark, a rule, etc. At the same time, awareness of social justice, collective experience, and ownership should be fostered. For example, the class should be asked to compete as a team, not only on the individual level.

In Barbara's classroom, as she says, the values of respect and partnership are essential. She herself disagrees with imposed authority (with no proper justification), for example, standardized testing. She loves meaningful, problem-solving activity in her job, and dislikes more routine and standardized tasks, like writing formal reports or assessment. She distances herself and her teaching from any form of blind obedience and doing things just because the teacher says so. This is what she tries to establish in the relationships within her class community. Likewise, an individual's need or rationale for doing mathematics is the basis for everything, and Barbara sees the essence of her job in creating such need (internal motivation) by providing the rationale and in choosing problems that are meaningful, challenging and varied and that connect various concepts and allow for various ways of solving. She knows she needs to be flexible and prepared mathematically, to analyse and assess a situation they may have not prepared for; this requires a lot of preparation but it pays off.

Barbara strongly believes that mathematics needs to be constructed in each pupil's mind at their own pace, and that this process takes place at different rates for each child, and with the help of the social environment of the classroom. A recurring theme in her discourse about mathematics learning is “readiness”. She uses it as a point of reference on both the individual and the collective level. Thus, some pupils are more ready than others to, for example, start using a more general algorithm, a "shortcut". For example, some pupils will add two fractions in their heads while others will still need to draw their pie representations. Barbara believes that mathematical maturity (rather than ability) is a function of the social environment. For example, she sees correlation between the progress pupils who joined her class more recently have made and the time they have been taking part in the class. On the collective level, the class discovers and clarifies things together and there are members of the class that will "pull the class along", coming up with original answers and/or communicating them to the others. Time is of little relevance:

the topics intertwine in the curriculum and there are built-in opportunities for individuals and for the class to return and expand concepts in a sort of spiral way.

According to Barbara, school mathematics is both socially and individually constructed in class. She believes that children learn best by engaging in trying to solve problems individually and confronting the ideas and solutions with peers. These problems as well as the open discussion of them are the source of each child's concept-building. Barbara further believes that a teacher's role is that of a facilitator of class discussion, and, the teacher should not resort to being a source of mathematical knowledge, e.g., by showing and/or telling pupils what the correct or appropriate solution is. Like Charles, she believes that if a teacher explains, it does not help the pupils' learning. A teacher cannot "give" rules, facts, nor can they generalize for the pupils.

Barbara sees mathematics as interconnected: she believes in the connections and consistency within mathematics, and also that topics in (school) mathematics cannot be separated from each other, they all interconnect; e.g., geometry cannot be separated from arithmetic, and it makes no sense to teach them separately. Unlike Charles, she believes that this is definitely possible, she has no qualms about curricular obstacles (such as too many separated topics).

Barbara described mathematics as a kind of worldview, or lenses through which to look at the world. The purpose of school mathematics should be to bring up critical problem-solvers, help pupils develop “a way of thinking that nobody will take away from them”. The knowledge of mathematical topics and concepts themselves is not as important as the problem solving: “The fact that they are solving some kind of problem, they are creating connections, and they are thinking.” It is more important to be solving a problem than to do division. Algorithms are secondary, although important, as if a by-product: they help with efficiency but if you have not internalized their mathematical nature, they are not useful and are a source of misconceptions, in the long run:

> And really, [pupils] have to mature into [algorithms]. […] they will also get the routine skills, but not until they really try to get to the bottom of it, when they know why it is done and they are already familiar with the mechanisms.

Mathematics, in Barbara's view, has a unique position in that it can be found in, and connected to, all school subjects, including music, art, the language, manual projects, as well as to all areas of human activity. In this sense, school mathematics is useful for everyone in their everyday problem-solving activities (for example, as she says, a carpenter needs to solve a lot of geometry problems, or her pupils need mathematics to figure out how much money they need to collect for the school's annual Christmas performance). Similarly, mathematics is necessary for problem solving at a social and professional level; it gives a useful theoretical foundation:

> And my image of a mathematician is someone who knows how to solve these [social and scientific] problems mathematically**.** Let's say physics describes it, medicine knows that something works in a certain way, but mathematics can give it a scientific foundation.

Barbara's beliefs about argumentation and justification

As apparent from the text above, Barbara believes that mathematical activity needs to be done by pupils, not by the teacher. Mathematical argumentation is no exception. Moreover, according to Barbara, in her classroom, there are no general mathematical truths or algorithms that are dictated by the community of mathematicians or mathematics educators, in the formal sense. For example, she says her pupils do not take notes, in fact, Barbara feels that there is nothing important for her pupils to write down in their notebooks, that is, unless they personally have the need to do so.

As we can see, Barbara's foremost concern is that of motivation and the pupils' need to explain, argue or justify. For argumentation to take place, there must be a need to explain or justify. In Barbara's view, this need can only come through the understanding of the following aspects of argumentation: a) the collectively accepted norm that a mathematical claim can only be true if it can be justified (in an accepted way), and the accepted personal responsibility for a claim made; b) the need to be understood by others.

There are two other important functions argumentation overtly fulfils in Barbara's class: self-correction (when a pupil realizes his or her own erroneous warrant in the process of voicing out an argument) and learning (the building, regulation or refinement of an image or concept).

Finally, Barbara may ask for a justification to get the necessary feedback from weaker pupils:

> So, this ‘convince me’ … of course they all nod ‘yes’ but on the other hand we, teachers, don’t know if someone’s nodding yes but has no idea, so also that is a reason [to ask why]. And I know who the kids… I’m not sure about … I say: ‘Come to the board and calculate it for us', and in essence, they all participated in the solution, first they solved it individually, then in a group (collectively).

According to Barbara, her pupils accept and understand all of the above rationales for argumentation.

In our interviews, Barbara distinguished clearly between the ability to solve a problem and communicating an argument or an explanation. Explaining (to others) is something that all of her pupils need to learn, each on some level, in order to work in the social learning environment of her class. This kind of explanation needs to be valid within the classroom community.

Apart from that, Barbara acknowledges that especially the stronger pupils will need to learn and develop another skill: that of presenting a more formal argument, in order to communicate with the world outside the classroom, for example, when participating in a mathematics competition. She notes that her pupils in Year 7 might need to start working on learning how to explain a problem solution in writing and offers a parallel with her own dislike of formally presenting the solution of a solved problem (in this case, a report on a project) itself:

> So, here, I will need to try to break this [norm/skill] in a bit. […] but I must say I’m that way, too, carrying out a six-month-long project, I do it, take pictures and all but then writing a report on it, well, you know…

As all of the argumentation is done in the class informally and by pupils, who are forming their knowledge individually, the form of an argument depends on the individual pupil's preference, and Barbara believes that one of the aims of mathematical instructions is to give pupils enough opportunities to try out different forms (e.g., modes of representations). The variety of representations in arguments on the collective level is necessary for making connections and understanding concepts on the individual level.

Notably, reasoning can take the form of references to a collective experience with analogical problems or even moments when these problems were solved. For example, the warrant "That’s just like when I cut off Kuba’s finger." may refer to a moment when a class was making an argument about a particular cube net because "when they were making it in the Year 3 [he cut the other boy’s finger, there was a lot of blood]" may be accepted as a warrant by the class.

Victor – the Conflicted Realist

Victor’s orientation profile (see Figure E3 in Appendix 3) is a mixture of all Ernest’s types, distancing himself only from the Progressive Educator’s values. One of Victor's main themes is the tension between the ideal teaching conditions (such as enough time for all the mathematics content, less administrative demands from the system, pupils with a certain level of knowledge and willingness to make an effort, and being able to let the low achievers find a different academic track when they do not apply themselves sufficiently) and the reality, which makes it impossible to apply methods that he thinks are ideal and match his beliefs about the aims of mathematics education. For example, he believes that discovering something on your own will help retention of the information, but he himself is not given (by the system) enough class time with his pupils to use discovery methods, and prefers to explain to the whole class, give example problems, and then let pupils practice on their own.

Victor strictly distances himself from the figure of a teacher-nurturer. Life is not fair and it is not about play and a constant (feeling of) success. School should simulate a certain level of tension, and teach about personal/individual responsibility for one's learning and success, which, in Victor's view, translates as working hard and studying on your own. This also means that the class as a whole should not be slowed down by re-explaining previously learned concepts and basic skills that some individual pupils might lack. These pupils should work on catching up in their own time.

For Victor, mathematics in general is a tool for functioning in the established system: it develops many real-life skills, especially systematic work, being able to read and work with symbolic language and representation (e.g., tables and graphs) in practical ways, such as dealing with administrative tasks like filing your taxes, being able to reason beyond given information. He believes that mathematics education, ultimately, should be producing a population of critical thinkers that would be able to keep the politicians in power in check. At the same time, he seems to uphold the elitist view that mathematics beyond a certain (lower secondary) level does not need to be compulsory. Different bodies of mathematical knowledge and skills should be the result of mathematics education at different levels (and different curricula) and should be standard within these levels. Naturally, the gymnázium mathematics is also useful for many further fields of study and professional application.

Victor's elitist view manifests itself also in the beliefs he expressed about the primary level mathematics, its learning and teaching: anybody can learn mathematics at the primary (and lower secondary) school, it is enough to work and practice algorithms. Teaching methods are important at that level to keep a certain level of motivation and interest, not to discourage or bore pupils too early. Word problems are the most challenging and most important part of elementary school mathematics because

> they involve exactly what mathematics should teach: you get a text, you need to find what you need in it, […] do something with it using maths, and […] interpret the result. […] Maths is the means of communication between the person who poses the problem and the person who solves it and delivers the solution.

At the higher level, a mathematics pupil needs both inner curiosity as motivation and work discipline to succeed and experience satisfaction. If these qualities are not present, the teacher (and their teaching methods) can do very little to help at this level. The best the teacher can do is not to discourage and/or demotivate pupils further.

According to Victor, mathematics is not about memorizing, it is about thinking and deducing. To understand mathematics means to be able to solve problems but also to be able to pass mathematics on, explain, communicate it, to see it in a global perspective, to be able to answer questions (for example, peers'). However, he also admits the existence of "math minds" that do not communicate their ideas well, but still can "see" the concepts, relationships, etc., they are typically weaker in language(s) and communication in general. For Victor, understanding in mathematics is also being able to correct yourself, that is, examine, verify and review your solution critically.

Victor does not differ much in practice in general from Charles – in fact, Victor has expressed admiration for Charles's materials and work – but Charles makes pupils think on their own even if the discovery in the end happens as a class discovery.

Victor's beliefs about argumentation and justification

In Victor's classroom, the teacher presents and justifies general mathematical statements in front of the class, typically using the technique of Socratic dialogue[41], involving the class as a whole by asking guiding questions.

Like other teachers in the study, Victor believes that not all general mathematical statements need to be justified (although he tries to whenever it is possible). There are times he needs to say: "Here is the formula. Deducing it is too complicated or too time-consuming so take it [as it is]." Notwithstanding, he believes that mathematics in general is about learning to reason, to "deduce a conclusion from something, not just 'he said it, so I will start saying it, too'" and he sees the role of school mathematics as a training for empowered citizenship. It is not clear, though, how learning this takes place, other than in the following situations.

Victor asks his pupils to justify to the whole class to make sure everyone knows what is going on: "Oftentimes someone working at the board [combines two steps into one], and I ask, also for the benefit of the whole class, because not everyone can see the two steps." So he wants to ensure that the whole class can follow and, for example "understand an algorithm".

Victor understands a pupils' explanation (justification) as a proof that they really understand the mathematics in a problem. For example, as a didactic tool he says he uses a specific prompt to justify (e.g., "Really?") to give them an opportunity to display understanding. Some will panic and delete the solution even if it is correct; he perceives this as a lack of confidence and the common perception that the teacher is an authority (but also a readily available resource) to tell them what is right and what is wrong. Argumentation, therefore, is important for Victor as a display of

[41] By the term Socratic dialogue in this text, I understand a dialogue that teacher leads with a pupil or the whole class by posing questions and eliciting answers that lead to discovery of a particular truth, e.g., a solution to a problem. In Schoenfeld's (2010) terminology, the technique corresponds to the term *interactive lecture*.

understanding and ability to solve problems, whether to the assessment-maker, the teacher or the pupils themselves.

Further, he believes that following one's own argument in order to check where in the problem-solving process they may have made a mistake is also an important skill and it shows that someone really understands. It is not clear to what extent these solutions involve procedural or conceptual argumentation, though.

Victor believes that strong problem-solvers should also be able to write and explain the argument but in Victor's experience, either they are "lazy", or the justification is beyond their communicative and language abilities.

When presenting a new topic or relationship, Victor says that he prefers to use a (physical) model or representation, if he perceives it as leading to pupils' longer retention of the concept or relationship. He chooses the mode of representation, in accordance with his belief in what is memorable, and he perceives physical models as more effective that way. At the same time, the availability of the model depends on the mathematical topic and Victor's resources (including content and pedagogical content knowledge).

It seems that for Victor, creative solutions, justification and argumentation are meant for the more able. He stated that in some of his classes with weaker pupils, he just needed to be content when *"they master typical problems, I don't give them any others on the tests anymore, and still the average mark is a three[42]"*.

Jenny – the Nurturing Pragmatist

Jenny is a teacher who holds beliefs that are predominantly a combination of old humanist and technical pragmatist types (see Figure E4 in Appendix E). She had been teaching mathematics and biology for 26 years at both the lower and upper secondary level of the gymnázium. She perceives school mathematics as being different (and serving different aims) at the lower and upper secondary school levels.

[42] In the Czech school system, the marks are traditionally 1, 2, 3, 4 and 5, with 1 being the best mark.

While it is important to foster and tap into the younger pupils' natural curiosity and nurture their motivation, encouraging them through a variety of form of assessment, she believes that the nature of higher secondary school mathematics and, especially, the quantity of topics in the curriculum are constraining the learning approach to rota learning, drill and cramming. Her perception of the upper secondary (gymnázium) mathematics is that of a much more formal body of knowledge, as expected by the curricula and, according to Jenny, by the universities. Her mission as a teacher at the upper secondary level is to prepare her pupils for the leaving exam and university entrance exams[43] and because of the quantitative demands of external authorities, teaching and learning mathematics at the higher level is about a teacher explaining, followed by pupils drilling and cramming, and there is "no time to play".

Complementing this perception, Jenny also distinguished consistently between the "younger" (the lower secondary years) and "older" (the higher secondary years) pupils throughout the interviews. She expressed on various occasions that younger pupils are motivated by play (such as mathematical games) and competition. They are more curious, likely to be enthusiastic about solving common-sense and logic problems. On the other hand, she adds that "unfortunately, as they get older, and then perhaps in puberty … I wonder if it might be also because of us [the teachers], that this activeness stops and the interest peters away".

Apart from the utility of mathematics in further talons of the educational system, Jenny sees school mathematics as a tool for training one's brain in a certain way of thinking (she says "synthesis, analysis, and deduction") and "even" memory. Mathematics as a discipline is very theoretical but it came out of people's need to solve practical problems, and applied mathematics is important (e.g., IT, space science or biology). School mathematics shows how mathematics as a discipline came about (from solving practical problems). It is very much part of the human culture and civilization, and the basis of technological progress: "Mathematics has a certain kind

[43] In the Czech Republic, the leaving exam has both a standardized national part as well as an elective part prepared by individual schools. The entrance exams are conducted independently based on each university's regulations. Recall that the gymnázium is supposed to prepare pupils for both the leaving exam and university entrance exams.

of logic, and even when it is not exactly about something practical, [realizing] that a human is able to think this abstractly. There is no other being that is able to do that, right?"

Mathematics, in her view, is key for a productive (non-consumerist) life, for contributing to society and human progress. It has a personal utilitarian value (immediate application of calculating real-life mathematical problems). Above all, though, it is the way of thinking that will help people with other areas of their lives (e.g., languages). "[Pupils] learn various methods/procedures/processes[44] and they train their brain so that they can use these methods in, let's say, even in language or other subjects." We can see that although Jenny contrasts the thinking in mathematics with memorizing in other subjects (such as biology or history), she, at the same time, sees mathematics activity as advancing pupils' memory skills.

Another theme prevailing in Jenny's discourse was that of a teacher as a pedagogue (i.e., not just a methodologist/didactician) and a personality with a natural authority and respect, and her perception of herself as lacking in that regard.

> It's not about what you motivate them with in mathematics […] but the personality itself. Because there are teachers who enter the classroom and have a natural respect [of the pupils and] they deliver it in such a way that they inspire the pupils. Even though I try, I do not belong to that category of teachers.

For Jenny, it is important to motivate and support pupils in their learning, and she works hard to achieve that (because she does not believe she has the natural gift as some teachers). Jenny loves to "play" herself, i.e., to invent activities, problems and ways to explain to make the content more interesting and accessible to her pupils. She identifies herself as a didactic problem-solver, i.e., she enjoys figuring out the best way to present a topic or to plan a lesson (she prepares a different lesson plan for each of her classes, even when they are the same year working on the same topic).

At the same time, she emphasizes that play and drill need to be balanced because

[44] In Czech, "postupy".

> children get distracted when you play too much. So [just like it is said that] when the teacher… just the blackboard and chalk, and calculate, calculate, calculate, … is bad methodology etcetera, then also when the methods vary too much, and you do a lot of riddles and games, it isn't good because drill is necessary. […] nothing should be overdone.

The logical thinking and understanding of mathematics, according to Jenny, needs to be accompanied by retention of concepts, methods, and procedures. The belief that pupils need to revise mathematical topics and processes in order to be successful also guides Jenny's lesson planning and she says she always revises (through warmers, revision sessions of chapters, etc.) because otherwise pupils forget. When she speaks about the need to include problems that have a real-world context (but this can also mean fictional, the real-life context is understood as distinct from the mathematical, abstract, symbolic, etc. world), it is in the spirit of helping pupils remember (a way to solve) rather than of developing conceptual understanding.

Jenny also makes a distinction between "gifted" pupils (who understand but are lazy to learn routines and use conventions, such as common mathematics symbols or a way to display a solution that is accepted and understood by the whole classroom) and "average" pupils (who succeed through hard work and drill). In her experience, she says that those who try hard end up with better academic results. Hard work is important and diligence is more important than gift but everybody can be successful. This mirrors her belief in the innate ability to be an inspiring and respected teacher. Embracing a nurturing attitude towards her pupils, Jenny considers pupils' affect, too. For example, she believes that it is important to have varied forms of assessment, so that pupils do not experience disappointment or embarrassment.

Jenny's beliefs about argumentation and justification

When justifying general truths in lessons, Jenny said that she got inspiration from various resources and from her interview, it is clear that the justification is done as a whole class.

Jenny also stated that she did not deduce (justify) everything in all classes. She says that it is impossible to do so, or even to justify everything to any class, because there is "really no time to justify each of the formulas, rather, you show them example problems, where you can use it and how it can help". It is true that the examples she gives (e.g., goniometric identities) are from the upper secondary curriculum. When observing Jenny's class in which she introduced the topic of circumference in a lower secondary class, I noticed that she elicited the formula from the pupils (because she knew some of them had known it from their physics class, as she stated afterwards), as well as the approximate value of pi and then verified experimentally that it worked[45].

It seems that the roles Jenny assigns justification and argumentation are of cognitive and communicative character: she does not expect pupils to be able to justify general formulas as an outcome of any topic unit. Jenny feels that to demand such justification would be outside the goals of school mathematics education (she perceives this sort of justification as something that would be "knowledge" that can be memorized). On the other hand, in Jenny's view, it is important that pupils write down some record of their problem solving. This is more important than the correct calculation. What is more, it is important to keep this record in a language and form that is understandable to all (the teacher, the pupils in class). This form of an argument is important for communication, thus, there is a commonly accepted way of showing how one proceeded in solving problem, and it seems that Jenny is the authority on this form.

Recording the solution (and perhaps the written argument or its representation) in a certain way is important because some pupils process longer, and they need to be able to get back to the record at home and understand what they wrote down. It is ok to copy without understanding in the moment in the lesson, but time at home and concentration should lead to understanding eventually.

Finally, Jenny asks pupils (in class) how they solved a problem especially at the initial stage after a topic has been introduced, so that through this repetition, the pupils retain it better. As an

[45] They discussed the approximate value and looked at some history of determining its approximate values. Then she let the class determine the (approximate) value of pi from measuring various round objects and using the formula. They verified that the "pi value" for each measurement was close to the actual pi. Jenny concluded that this is "how the ancient Egyptians came up with a number that can be used to determine [circumference]".

example of prompts, she provides a prompt, such as "Why is it so, what does it mean?". She also remarked that if she kept asking repeatedly, pupils "eventually answer", meaning that she believes that the procedure or the argument will finally sink in after multiple repetitions.

Zack – the Academic Pragmatist

Zack's expressed views of mathematics education are mostly a mixture of the platonic-like orientations of Old Humanist and Technological Pragmatist. The themes permeating the interviews were: effort, his own experience with learning and studying mathematics, the systematic (i.e., driven by the education system) utility of mathematics and the differences between mathematics and mathematics teaching and learning between school levels (Zack had taught at the basic school prior to his current gymnázium teacher post).

Zack often referred to his own experience with learning and teaching mathematics and it seems to have influenced his orientation and practice. For example, he stated that at the gymnázium, the "homogeneity" (referring to the fact that to attend a gymnázium type of school, pupils usually have a certain achievement level across all subjects, including mathematics) of the pupils means that content becomes more important than the methods for delivering this content. This, he feels, is in contrast with the basic school where the range of pupils (in terms of achievement) is wider and teaching evolves around choosing appropriate methods to reach or involve most pupils. Consequently, he feels that the show-and-tell method is suitable for his gymnázium pupils, who are higher achievers, more disciplined and likely to understand mathematics when taught in this way.

Apart from pupil characteristics, Zack also finds that the mathematics itself has a different nature at individual levels. For example, at the lower secondary level, mathematics is practically applicable and at the upper secondary level, more theoretical and self-serving (i.e., for mathematics itself). In his own experience, at the higher – university – level, it is a strictly built structure and perfectly logical but not accessible, not practical, formal. With this, he contrasted the "intuitiveness" of mathematics at the secondary school (e.g., talking about space in 3D geometry is intuitive at that level, as is also the proof of the Pythagorean Theorem) and its design as being that of tools and methods for problem solving (e.g., problem analysis).

He connected this to his view of the general aim of doing mathematics:

> [School mathematics is] brain gymnastics/exercise. It teaches one a certain preciseness, ensuring that a task is stated with exactness and to also exactly finding out what needs to be done and what needs to be found and then some methods – analysing a problem, selecting a relevant tool for its solution and completion.

Zack feels that it is important to include all the curricular content because one never knows which area or topic will be required by which pupils' career choice (or future choices), i.e., he sees mathematics education as valuable for finding a place on the job market.

Clearly, Zack espouses beliefs about school mathematics as valuable for developing certain meta-cognitive skills, and, in alignment with this, a teacher should present content and problem solving efficiently and this efficiency should be pointed out to pupils. For example, his presentations of a problem solution included language such as "it seems efficient to" or "we use […] because …" to draw attention to the functionality of a method or the use of symbols. This confirms Zack's need to explain why (to what end) things are done the way they are, in mathematics.

When it comes to the learning of mathematics, Zack is convinced that hard work and diligence are paramount in order to be successful. This is true in mathematics, in school as well as in one's professional life. Learning is directly proportionate to pupils' willingness to learn, to receive knowledge and develop skills, and the willingness to make the effort. This effort can be partly replaced by talent, the (innate) ability to think mathematically. Still, one can gain understanding through practice and hard work, as some of his pupils do, and Zack did himself at the university level: "At first, I just memorized it and then the longer I used it, the better I saw how it worked."

I also noted that, as the other teachers in this study, Zach distinguishes between pupil types, and believes that innate abilities and dispositions influence the way his pupils master and do mathematics, and the way individual pupils learn. To memorize shown procedures and steps and not understand (at first) is also a way of success in the educational system. On the other hand, pupils with innate mathematical curiosity and talent as well as innate individual characteristic (such as 3D vision/imagination) do not need as much structure in problem solving (methods, procedures, steps) in order to be successful.

Zack's beliefs about argumentation and justification

Zack believes that it is the teacher's responsibility to explain and justify to the class, at least during class time. The teacher is also the authority on the subject matter and Zack feels the necessity to exemplify exactness and rigour in mathematics. Thus, justification is important to him as a way of preserving the nature of mathematics as he perceives it (its objectivity, logic, and exact structure). He himself feels the need to present with exact language and terminology when explaining.

At the same time, he admitted that his own explanation was not always effective for all pupils. For example, he commented on an episode where a pupil approached him with an erroneous solution:

> […] and I was trying to convince her that this [has to be true], the two sides are only adjacent through this edge and that I can see it there […] I told her 'Look, I don't think I can give you a better argument, but try to cut the net out'.

Notably, Zack does not see any merit in helping pupils develop reasoning skills through their own engagement in argumentation in class. Rather, he lets "weaker" pupils explain in class in order to increase their self-efficacy or as a tool for classroom management, e.g., letting pupils with a need to socialize or communicate explain or justify in order to keep discipline in class under control. Further, he said that he does not let the able pupils explain, because "no one would understand that, anyway". Apparently, a talented pupil does not have to be able to explain or justify to everyone, i.e., they are not expected to communicate their ideas so that everyone understands. This further highlights his belief that rigorous mathematics (and justification) is not meant for everyone and seems to indicate that in Zack's view, being able to communicate one's reasoning to wider audience is not something that a strong mathematics pupil needs to profess.

Zack agrees that the modes of reasoning for justification as well as the absence or presence of it are subject dependent, as well as pupil-characteristic dependent. Throughout our interviews, Zack distinguished between the ideal mathematically correct justification and less formal ways of reasoning or models. When speaking about justification of general mathematics statements, he admitted freely that some things cannot be justified because the warrants were out of the pupils' realm of knowledge. This can be due to the curriculum:

> Today we came across a lot of things, e.g., [we said] that the lines are parallel but we still cannot prove that because the chapter about parallel objects comes in a bit later and the fact that there need to be some pairs of parallel lines, well, I cannot use that, yet.

In addition, the pupils' achievement level in mathematics is also a factor in justifying general truths.

Therefore, justification is not necessary (or possible) for everything – for example, he says he does not algebraically deduce the formula for finding the roots of a quadratic equation. Instead, he verifies that it works through experimenting with specific examples. He justified this choice by saying that the pupils would not be engaged (or would not enjoy it), and would not be able to understand the algebraic proof. Clearly, he understood the justification to be valid only in a proof-like form and needed to express the inferiority of the proof by specific examples (or rather "compliance with a model" mode of reasoning we saw in the textbook analysis in Section 2.4.3), and the intuitiveness and experimental nature of school mathematics, in the context of the interview. This could be the effect of the status he assigned to the interviewer as a representative of the higher education (i.e., in his view the formal mathematics) institution. Even if he is aware of the faulty reasoning and lack of justification, he sees them as valid in his practice, for pupils, apparently, as a sort of verification that the tool (e.g., the formula) works and thus can be used to problem-solve.

In sum, Zack assigns argumentation the role of verification. Being able to verify/check that one's problem solving is correct is perhaps the most important skill he wants his pupils to develop when problem solving. In that sense, he sees heuristic warrants and modes of representations as useful in their explanatory and verifying power (and he uses them himself in his own mathematics activities, for example, cutting up a paper model net of a cube, to verify). However, he does not feel they are valid in mathematics. This is in alignment with his distinction between formal mathematics and secondary school mathematics, which he sees as intuitive, without theoretical foundations.

Overall, Zack sees the role of justification and argumentation in the following: verifying the problem's solution, increasing weaker pupils' self-efficacy, classroom management, means to show efficient and effective problem solving methods, means to explain concepts.

Karen – the Efficient Pragmatist

In Ernest's terminology, Karen seems to adhere to many views compliant with the Technological Pragmatist type. The prevailing themes across her thoughts about the teaching and learning of mathematics are her concerns with time - and consequently with efficiency – and with effort. Everyone can be successful in school mathematics as well as in life if they only try and put in the effort. The success in the learning of school mathematics is an inevitable means to success, achievable by determination and effort, albeit devoid of pleasure. At the same time, this success can be often undermined by circumstances (e.g., financial restrictions) or "the system", which Karen sees as sometimes unfair, inefficient, etc. but whose existence is inevitable and cannot be directly influenced by an individual.

Karen views mathematics as a highly utile area of human activity and has a strong sense of school mathematics content being an important tool for real-life situations. When speaking about school mathematics, she expressed a concern that not all mathematics (even middle school mathematics) can be used in real-life contexts. She believes that the middle school mathematics curriculum should be restricted to content that is useful in solving practical problems of an individual in everyday life (for example, excluding operations on more complex algebraic expressions like polynomial fractions). At the same time, success in mathematics is important for moving on to higher levels of the education system and in getting qualified in a field that will increase the chances of employment on the job market. It also seems that Karen does not view mathematics activity as something to be enjoyed, stressing the fact that it is ultimately a matter of hard work and effort: "[Pupils] must think, put things together, it is not that demanding, if they think a bit, they can be successful at least."

In Karen's experience, children inherently avoid effort, especially the effort to think. They need structure and guidance from parents and teachers. They also need to be told what is right/correct and wrong/not correct. Parents are responsible for their children's choice (i.e., signing up for extracurricular practice) and time-management, i.e., the time a child spends studying, practicing.

They should help their child in understanding the importance of work ethics and studying. Children are brainwashed by media, adopt the same values from their parents. Meaningfulness or usefulness of mathematics is no motivation for small children – they will not appreciate it; what motivates them is a mark.

At the same time, Karen expressed her belief that "weaker" children should be protected from feelings of failure by experiencing success (in something). The weaker pupils need nurturing and extra time, all, of course bound by the condition that they make the effort; work ethic is paramount. The young child finds pleasure in activities (play, experience, etc.) but older children respond to fewer, or only some kind of, stimuli and expect to be entertained or [the teacher] has to know them well to know "what works for them".

When speaking about the teaching and learning of mathematics, Karen is convinced that the best way to learn is by observing how things are done and then trying for yourself, getting a lot of practice on your own. The teacher's role is to show how things are done in the best (i.e., the most efficient) way, provide pupils with an example (or a few) and work it out with them. To avoid giving facts, the teacher deduces, derives, explains, and justifies with the class. The teacher's job is to ask effective questions and logically sequence topics. Using a procedure precedes the understanding of its use, sometimes conceptual understanding of a procedure/formula does not have to come at all (or at least not for some pupils). Still, everyone can be successful in school mathematics. It is easier for the majority to use a given (shown) procedure.

> If you have 20 children, 5 will understand it in the first lesson, you use new problems in the next lesson and 5 more will catch on, and so on. That is reinforcement based on practice. In the end, 2 to 3 children are so weak that they are not able to understand it and they will end up with a bad mark.

Consequently, Karen values feedback, i.e., discussing an error, as very important (unlike in the upper secondary school where, in Karen's words, "the teacher dictates and pupils write things down, there is no communication"), and in alignment with her view of the nature of a child, Karen feels that she needs to be the confident authority and have all the answers. For example, in the case of 3D geometry, which is a subject matter about which she feels least confident, she chooses to involve pupils who will do a better job explaining, in order to conceal her perceived weakness: “I am better at math, they can see things better. We complement each other."

In concordance with Karen's utilitarian view of mathematics and its consequent need for rigorous work ethic, she feels that using a real-life context (and pictures) will increase pupils’ ability to understand but not their effort. An introductory problem should be from the real world, e.g., about how much a lollipop costs, or how much gas a car uses. Pictures (or drawings) are equally important when introducing a topic, so pupils can imagine it better. This connection to the real world also shows pupils where mathematics came from. After that, it is all about drill.

In addition, Karen feels that this drill or practice should be done mostly individually, which corresponds with her view of children as inherently prone to distraction and to circumventing effort: "Group work only leads to one pupil doing all the work." Explaining, therefore, should happen mostly on the all-group level, orchestrated and modelled by the teacher.

Mathematics as a discipline, in Karen’s view, has a hierarchical structure: all is built atop of everything else and if you do not acquire the basic skills, you cannot go on. Mathematics is a set of rules and procedures that are logically tied together, and its purpose is to solve practical problems efficiently, even if understanding their origin may be beyond one's knowledge.

Karen’s beliefs about argumentation and justification

Karen understands justification of mathematics concepts, procedures and rules as an important part of showing pupils a) the usefulness and b) the logical, deductive nature of the mechanisms in problem solving.

The teacher is the foremost actor in justification and explanation because time is of utmost significance. Of course, the act of deduction should be done with the pupils, i.e., with the class as a whole:

> We always deduce, derive, explain, justify together. I have to ask effective questions. Pupils solve a [motivational/introductory] problem their way, then I show them a different way [more effective one], using a new concept [e.g., ratios], that way they see [how/that] it *works*.

It is not time-efficient, or even possible, for pupils to come up with general rules on their own; the teacher needs to show or guide them. Time spent over practice is more valuable than time spent explaining why the practiced procedure works, and it may not be efficient to explain everything to everyone, for some pupils such explanations are a waste of time, intimidating or even confusing. For example, when talking about operations on fractions, Karen said: “I have some examples and pictures which I use to explain [the mechanisms] so that they know that the procedure did not just come out of nowhere. But after that, it’s a matter of practice.”

Consequently, when an explanation of a rule/procedure is once given and explained by the teacher, this rule or procedure can be used as a warrant or referred to as a fact. When the curriculum content requires the class to get back to it after some time, it is important to recall or refresh such a fact/procedure (e.g., the fact that you cannot divide by zero needs to be recalled when working with algebraic expressions).

Argumentation done by pupils serves several purposes in Karen's belief system:

a) most of all, for the teacher to check if a pupil’s reasoning is right, if they “got it” (this feature appeared as the most significant),
b) correction of a pupil’s error, an opportunity to reinforce understanding, especially if they are strong and wrong: “Also, learning with mistakes. When they make a mistake, you return it to them, and they discuss it and realize for themselves that it can’t be right.”
c) improving pupils' image of self-efficacy, especially if they are perceived as a weaker pupil and provide a correct solution,
d) taking off teacher's workload, or providing more quality explanation where the teacher is lacking a certain ability (as in the instance of 3D geometry for Karen).

Note that Karen seems to see no merit in justifying for the pupil him/herself, unless their reasoning is incorrect, or unless they are a weak pupil and it makes them feel successful. When asked directly, Karen agreed that arguing/presenting an answer may be also an important skill for pupils’ lives but in school mathematics, it is the correct answer that matters.

Karen also seems to believe that argumentation as a self-correcting tool is exclusively for stronger pupils, while the weaker ones will need the teacher to step in and explain again. In observing the interaction of Karen’s classroom, an error was usually a clear signal for the teacher to intervene and take over the argumentation.

Overall, it seems that Karen understands argumentation as a sort of by-product of problem solving, an inherent part of mathematics activity that does not necessarily need to be developed as a skill itself. In her classroom, the predominant function of a pupil's argument is to display and gauge the correctness of their reasoning to the teacher, fulfilling the role of assessment and developing (or correcting) understanding.

3.4.2 Teachers’ beliefs – synthesis and discussion

In this study, I used Ernest's (1991) typology framework of philosophical views of mathematics education and was able to confirm that even in a small sample of Czech teachers, the fundamental orientations ranged widely within a community guided by the one mandated national curriculum. As expected, the declared beliefs of no one teacher fit the columns of the ideal prototypes. I noted some context-specific aspects of the educational system in the interview analyses. For example, the subject of race and ethnic differences was not broached in the interviews, as it was not relevant to the participants' pupils. Further, participants’ mathematics education beliefs seemed to be also dependent on the teaching experience the teachers had with teaching at schools with different age-level and school-type curricula: i.e., the three different age levels (primary, lower secondary and the upper secondary gymnázium track), which is specific to the Czech educational system (e.g., the existence of an academic track curriculum at the gymnázium).

Except for Barbara, all teachers had had experience teaching in both the lower and the upper secondary school (the gymnázium) and they made a clear distinction between school mathematics (and the teaching and learning of it) at the two levels. The lower secondary mathematics (especially at the basic school) seemed to be viewed from the perspective of Technological Pragmatist, or even Industrial trainer, while at the gymnázium level there was as a mix of Technological Pragmatist, Old Humanist, and Public Educator beliefs.

All participants expressed that the aim of mathematics education is to teach and learn to “think”, or to think in a certain way. Recall from Section 2.5 that the meaning of the Czech word for thinking, “myšlení”, also encompasses the act of reasoning (the private act of reasoning, without the necessity to express thoughts publicly). Teachers connected this “thinking” directly to skills involved in problem solving: analysing a problem, choosing a method or tools to solve it (Zack), logical thinking, problem analysis, synthesis (Jenny), and generalization (Charles).

Simultaneously, all participants conceived mathematics education at least on some level as something that fosters critical thinking (challenging the routine, not accepting every piece of information they are given as a fact and using own judgement to evaluate the trueness of it). They viewed both of these ideal outcomes of mathematics education as unique to school mathematics at various degrees. For example, we saw that Victor seems to give mathematics the unique status of being the source of fostering critical thinking, problem solving, reasoning, etc. while Charles believes they should be the fundamental goals for any school subjects (e.g., history) although they are currently not. Barbara believes the goals are common for all subjects within the school curriculum through interconnecting concepts and human activity in general.

While all participants understood the benefits of mathematics education as teaching reasoning on the private thinking level, their perception of the role of justifying or arguing was less unified, less clear and less determined.

First of all, there were some conflicting statements about the distinction between the ability to solve a problem or reason (“think”) and the ability to justify (for example, a problem solution) to others. Let us take Karen as an example: she spoke about calling on weaker pupils to explain easier problems in order to boost their confidence on problems they could solve, as if the ability to justify (or explain) came along with being able to solve a problem and was not a separate skill. It is not clear, though, what such easy explanation would involve. Victor says directly that being able to explain (e.g., a problem solution) to someone is equivalent to understanding the mathematics involved. Yet, in the case of “math minds” (gifted pupils), he agreed that many lack the ability to communicate their solution in a comprehensible way, i.e., justify it to others. In other words, if you can explain, you understand but understanding is not a sufficient condition for being able to explain. The phrase “they just see it” came up in interviews with Barbara, Charles, Victor, and Zack. Karen also expressed she generally has a hard time convincing her stronger pupils to write down the record of their solution in steps she would understand.

It seems then that understanding, insight and problem-solving proficiency in mathematics are understood as distinct, or even disjointed, from the communicative part of justification. Note Victor's observation: "The quicker they were in problem solving, seeing the solution, the harder it was for them to write it down or they were too lazy to record it and they were weaker in subjects like Czech or foreign languages." Teachers are aware of this phenomenon and some try to encourage their (mathematically strong) pupils to provide arguments by explaining why this skill is important. Even in Barbara's class, where the class norm is to explain to others, she notes that stronger pupils sometimes have a hard time making themselves understood, and she clearly states that that skill is also something they need to learn, especially in class time.

Zack does not let the gifted pupils or the pupils "who can just see it" explain to class because they would not be able to give a justification comprehensible to others. Both Victor and Karen note that the stronger pupils do not provide proper justification (or a display of a solution), either because they do not know how or because they "are lazy", and this latter notion is especially true about written records of problem solving. Charles and Victor highlighted the fact that even the more able and gifted pupils lack the ability to display their reasoning and justify the solutions, specifically, in mathematics competitions, where it is required in writing and in a systematic way, comprehensible to someone outside the usual mathematics classroom community.

This highlights teacher's awareness of the different audiences to which pupils might need to justify. What is fine among peers in class or at the blackboard, might not be accepted as a justification at a math competition. Although Barbara knows her pupils are used to being called upon to justify their ideas in class, she also feels that to justify something to an external audience (especially in writing) is something the pupils will eventually need to learn.

Of course, all the teachers acknowledge the importance of pupils' display of a problem solution in order to comprehend it themselves, as teachers, to evaluate the pupils' understanding or ability to solve a problem, especially in written assignments and tests. Karen and Jenny believe that showing pupils a particular way of organizing this record is also helpful for their ability to solve particular problems, i.e., they see that this specific way will be a tool to be used by the pupils in their problem solving. It is not clear to what extent this may mean that pupils are learning to use imitative rather than creative reasoning (as in Bergqvist & Lithner, 2012).

Does everything in mathematics need to be justified? All teachers agreed that it is important for pupils to see where the mathematics they are learning came from, to establish a non-arbitrary nature of mathematics. They also agreed the justification can be useful for the understanding and problem solving, although this did not apply to all topics or mathematical truths. Except for Barbara and Charles (the two teachers that do not believe in the use of the whole-class presentation or Socratic dialogue with the whole class), participants felt that it was more desirable not to justify in the following instances:

a) When they felt it would take too much time to do in the class.
b) When the justification they knew involved concepts not available to pupils at that level.
c) When they felt the justification was beyond the particular group's level of mastery (for example, Karen justifies the area of a cone).
d) When they felt it was not going to help pupils' ability to use the concept/statement in problem solving.
e) When any combination of the above was true.

It seems that all teachers feel the need to justify a general statement, even though they do not believe in the benefit it would have for all pupils. Note, for example, Karen's observation about a class that she describes as less mathematically able:

> [We were doing the surface area of a cone] and there were two or three children who understood what it was all about. The rest of the children just, well, copies it. [In the justification] you use [the concept of] similarity but only two or three children are aware of it; the rest just takes it as a fact.

Taking a formula as a fact and understanding what each of the symbols mean is what can make the pupils successful, in the participants' views.

This is different from Charles and Barbara. Charles believes that all topics are justifiable and should be justified, or, rather, worked out through carefully selected problems. Barbara does not see the benefit of stating any general statements that need justification per se, as all understanding all concepts and relationships should be the result of an individual's engagement (through problems) with them and the collective pupils' discussion of this activity.

These perceptions of the necessity to justify are directly linked to the teachers' beliefs about mathematics learning and teaching practices. Both Barbara and Charles believe that it is the pupil (or a group of pupils) that need to work out a problem, including the justification of a general truth, themselves, in order to learn. As much as the others may agree that such practice is helpful for learning, they feel that:

a) involving pupils in active justification of a general mathematical truth is impossible because of the time restrictions they work under, sometimes in combination with their pupils' knowledge and abilities and
b) it is their responsibility to providing pupils with problems to solve, but they feel it is each pupil's responsibility to do work on their own, and that working out or checking the problem solution as a class solution is the best teaching practice, even if it just means copying the solution and later "getting it" at home.

In Charles's view, this has to be done individually; in Barbara's, this effort is owned by the whole group community.

None of the teachers felt that justification of general truths should be part of mathematics assessment.

In alignment with the findings in Rendl et al. (2013), some teachers expressed their intention to introduce a topic through representations that were using real-world models. They felt that this was especially necessary at the lower secondary level (and not at the upper secondary one), either to motivate pupils with something non-mathematical and/or to help them connect the mathematics to something concrete they knew. I observed that, as for example in Jenny's case, this connection to the real world is understood as something to help pupils remember a certain mathematical procedure or a method to solve a problem. However, it is not clear whether Jenny means a connection on the level of an association[46] or the more fundamental connection between a real-world model and mathematical objects and relationships.

When discussing justification and argumentation, the teachers did not refer to particular modes of reasoning other than the general notions of logical and analytical thinking and common sense already described above. One exception was Charles, who, in his discussion of pupils' ways to prove Thales theorem, expressed awareness of the difficulty pupils had with conducting a proof, i.e., hypothesising and working towards something given (such as the hypothesis) rather than solving a problem based on given data. It seems that this type of justifying was not present in our conversations with the other teachers.

As is already apparent from the above paragraphs, the perceived differences between types of pupils reported in Rendl et al. (2013) also come through in speaking about justification with the participants in my study. The characteristics of successful learners as being hard-working or clever were a unified theme among all teachers.

All teachers (even Barbara) also distinguished between weaker and stronger pupils but they displayed varied beliefs about practices used in connection to ability and justification. Charles also added a gender distinction in the way pupils are likely to solve a problem (e.g., to hypothesize and prove Thales theorem). Male pupils, in Charles experience, are likely to come up with more original and creative arguments and solutions, while female pupils look for a straightforward one, and are much better at recording and communicating the solution (argument) on paper.

The teachers distinguish between weaker and stronger groups of pupils, and (apart from Charles and Barbara) believe that practices like justifying general mathematical truths or allowing pupils to discover or justify themselves is less efficient and not effective. Further, if pupils do not grasp, it is because they do not apply themselves enough (e.g., they do not work at home, neither revise

[46] In the sense in which Diamond (2018) uses it: "Association refers to the notion that a student links a specific word, phrase, or image to a particular mathematical response. For example, when confronted with a set of three graphs and asked which one represents 'walking toward an object,' a student circles the graph showing a line slanted down (as one looks from left to right), because of class activities in which the student has been told that the term 'toward' is associated with the image representing negative slope." (p. 11)

on their own, etc.). Everybody can master mathematics if they spend enough time practicing, studying, and thinking about the mathematics or problems – they eventually "get it”.

Finally, there is also the distinction between older and younger pupils, which some teachers perceive as crucial in choosing real-life models in explaining topics, or taking advantage of pupils’ still existent willingness (as a group) to think and reason on their own and want to work and discover at the lower secondary level.

Thus, all the gymnázium teachers acknowledge the need for other methods than the watch – process – understand – practice method especially at the higher years, but (except for Charles) they feel that applying this method is the most effective for various reasons:

a) They can afford it – their students are at a similar academic level, trained and capable of grasping. (Zack)
b) Unpreparedness from lower years disqualifies pupils from discovering/problem-solving on their own, and remedial content is again possible only to a small degree, albeit necessary. (Victor)
c) Other methods take too much time and the curriculum is mercilessly crammed with concepts pupils need to master. (Jenny)

Even when teachers complain about the lack of time to teach everything (especially at the upper secondary level – even Charles asks himself whether it is possible to understand and demonstrate the understanding of the quantity of concepts within the secondary school curriculum), no teacher challenges the national curriculum itself. Apparently, the document gives enough freedom to schools and teachers to implement different approaches to teaching but the declared quantity of topics makes teachers reach for the proven method of Socratic dialogue with a class or even the “show and tell how” in classes with a lower achievement level.

4 Study 2B: Argumentation in teachers' classrooms[47]

Study 2A has established that justification and argumentation are viewed differently in teachers' individual orientations, even within one curricular context, in the Czech education. In the second part of the study, I undertook an investigation into how argumentation takes place in classrooms: what are the enacted practices involving argumentation? What arguments are put forward and accepted by the teacher and the classroom community, what warrants, modes of reasoning and modes of representations are used in these arguments? Ultimately, I aimed to look for the roots of these phenomena: how do teachers' orientations, goals, and resources influence this? What specific beliefs are central to the teachers' decision-making? What role does the textbook curriculum play? How do pupils affect the arguments?

Recall from Chapter 1 that the main players in the classroom event, such as the occurrence of an argument, are the teacher, the textbook, and the pupils. In the general model, a teacher selects tasks from the text, designs their implementation, supplements it with other tasks, and, finally, improvises, based on the pupil contributions (Remillard, 2005). The levels of the textbook participation in the intended curriculum, and therefore its role in influencing arguments in the classroom, vary greatly (Remillard, 2005).

Schoenfeld's model (2010) explains the actions of an individual by their orientations, goals, and resources (including knowledge) and argues that this is especially true for a goal-driven professional action, such as teaching.

Further, research (such as Planas and Gorgorió (2004) or Levenson, Tirosh, and Tsamir (2006)) shows that the individual pupils' mathematical knowledge and their perceptions of what is expected of them in producing an argument can differ from a teacher's expectations. The pupils contribute, request or choose arguments based on their own knowledge, beliefs, and goals and they "have their weight in the negotiation of socio-mathematical norms regarding mathematical arguments and explanations" (Žalská, 2017, p. 292).

[47] Some results presented in this chapter have been presented at CERME10 conference and published in the proceedings as Žalská (2017).

To study the way argumentation takes place in a classroom and the influence of teachers' orientations and goals, their resources and the classroom environment (including pupils), I chose three of the teachers from Study 2A, based on their declared orientations. The three teachers (Karen, Barbara and Charles) demonstrated very distinct beliefs about effective teaching practices, and their approaches to teaching were strongly rooted in their orientations.

Given the context of the Czech curricular documents discussed in Study 1, it is not surprising that all three teachers had a choice in selecting the actual detailed curriculum, i.e., the textbook[48]. It became apparent during Study 2A that these choices strongly aligned with the participants' beliefs. The three teachers also had a different relationship with the curriculum they chose. Karen was using a standard textbook. Barbara was using a curriculum that had been co-written by her and developed using the theory of generic models (Hejný, 2012), designed to support teachers in teaching with the scheme-oriented approach, which she had adopted as her own approach to teaching. Finally, Charles was using a curriculum text that he had been writing and editing entirely by himself, based on a discovery approach, which he refers to as "realistic".

4.1 Study 2B – Participants and data

Karen is an experienced mathematics teacher who I identified in Study 2A as a teacher with strong utilitarian beliefs about mathematics education. She is a lower secondary mathematics and geography teacher at a public school in the capital. Her teaching experience amounted to about 20 years of teaching mathematics, mostly at that particular school. She had been working with her class for almost two years prior to the data collection.

During the observations, there were 12 to 15 pupils, about a half of them boys, present in the class. Karen frequently referred to this group of pupils as "good", or "clever". The pupils' formal marks ranged from 1 to 3 (on the traditional Czech 1 to 5 scale, 1 being the best evaluation). The pupils all had both the student book and workbook at their disposal. The class used a mainstream textbook series, one of the most popular ones in the country at that time (textbook E in Appendix

[48] I will refer to these texts as textbooks although two of the texts take on a form different from a typical textbook but they are the texts that are available to pupils to work with, consult, etc.

B). Karen was among the teachers who approved the choice of the textbook in her school. The topic taught was percent and percentage.

Barbara is an experienced mathematics teacher who professes a strong orientation toward non-traditional beliefs about the teaching and learning of mathematics. In Study 2A, we saw that she espoused beliefs that are mostly a combination of Progressive and Public Educator. She is a primary school teacher who was one of the lead figures in promoting and practicing the Hejný method, a non-traditional way of teaching, based on a scheme-oriented approach. She had been teaching mathematics for 9 years at the time of data collection.

The observations took place in her Year 7 classroom with 16 to 18 pupils present in the observed classes. There was an even mix of boys and girls. Barbara characterized the class as "average" in terms of skills and abilities. She had been teaching this particular group of children since their third year: she was their class teacher up to Year 5; after that, she was allowed (upon her own request) to continue teaching them mathematics in the lower secondary school. The class was using a textbook that was in the piloting stages[49]. The analysed text is what the pupils had available in and out of class. There were no teacher manuals or workbooks. The text consisted of problems and tasks with no explanation or arguments provided. The topic explored in the observed lessons was identified by Barbara as operations with fractions.

Charles is an experienced mathematics and physics gymnázium teacher who had had 20 years of experience teaching at the upper secondary levels at the time this research took place. His orientation in terms of mathematics education had mostly Public Educator characteristics, and he publicly characterises his approach to teaching as “realistic”, distancing himself from what he perceived as “extremes” of both constructivism and traditional transmissive teaching.

The lower secondary group (Year 7) that I observed in this investigation was the first one Charles was teaching at that level and he had been teaching them for almost two years. There were 29

[49] The book had been unpublished at that time, I obtained a copy of the pdf document from the teacher, with a permission to use it for analysis. I will refer to this document as Barbara (pilot).

pupils (15 females) in the first two lessons. The third and fourth observations were done in lessons when the class was separated in two: the third observation took place in a lesson with the male pupils and the fourth observation was the same lesson plan carried out with a group of female pupils[50]. Charles characterised this group as a bit unruly, with several stronger and several weaker pupils. The topic was circle and disk[51]. Charles has been writing his own teaching materials and publishing them online for both his pupils, other mathematics and physics teachers and the general public since the year 2010. The materials form a cohesive text that covers the topics defined in the national curriculum across upper secondary and, at the time of data collection, Year 6 and 7 of the lower secondary school[52]. They contain tasks and their solutions and the text also includes some Charles's pedagogical comments intended for teachers' use.

The data I used in this study for each of the selected teachers, as well as the way I collected it, are described in Section 3.2.1. Recall that in the interviews with teachers I also collected data about the teachers' perception of pupils as "stronger" and "weaker". Further, I selected those sections in textbooks used by the three teachers that contained the topic (and material) the teachers taught in the observed lessons to analyse the text's influence on the subject of my investigation.

4.2 Study 2B – Data analysis

Let us first re-visit some terminology (Section 1.1.1). A (mathematical) *argument* denotes a sequence of statements (including written statements) that is provided with the intention to show that a mathematical *claim* (specific or general) is true (or not). In my thesis, arguments include the explaining of an answer to a problem, as well as the working out of the answer to a

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[50] This separation was determined by the fact that physical education lessons were separated by gender and the schedule matched this with the math lessons.

[51] Recall that the Czech language and the mathematics curriculum distinguishe between the two concepts: Circle is a closed plane curve consisting of all points at a given distance from a point within it called center. Disk is defined as the portion of a plane bounded by such a curve.

[52] For this study, I used the freely available pdf documents from the website, downloaded in 2014. I also took photographs of the projected text during observed lessons. To preserve the participants' anonymity, I refer to these texts as Charles (online).

problem. A *warrant* is one such statement that directly supports the claim. In the context of a classroom, it is a statement that is accepted as true and does not require further explanation. Two arguments will be considered to be different if they contain different warrants, different modes of representations or different modes of reasoning.

The three textbooks used by the teachers were analysed in accordance with the theoretical framework, i.e., the provided arguments were analysed in terms of warrants, modes of representations, and modes of reasoning. The textbook tasks were analysed for requests and opportunities for arguments (e.g., towards a claim that contains a problem's solution).

The transcripts of observations of lessons were first analysed for episodes of argumentation to establish specific context for argumentation and social norms in the classroom. In doing so, the kind of pupil and teacher participation on the argument was taken into account, to separate the cases of arguments provided by the teacher (including when a teacher elicited an argument step by step and pupils only provided the final part of a requested warrant) from those constructed by pupils themselves. I identified the social norms of the practices including argumentation (especially regarding pupil participation on arguments).

Next, the identified episodes of argumentation were divided into individual arguments and identified warrants, modes of reasoning, and modes of representation, to determine differences between arguments.

Further, I compared the observed arguments to the examples of arguments (if present at all[53]) in the textbook curricula, comparing warrants, modes of reasoning and modes of representation[54]. Finally, I considered the teachers' own comments about particular arguments, warrants, reasoning or representations in class and during interviews to gain insight into the beliefs behind their decisions.

[53] Barbara's textbook did not contain any problem solutions, explanations or examples of arguments.

[54] For Karen's textbook, I also included the relevant part of teacher's manual in the analysis and looked for commentaries and any additional rationale given to a particular argument to get insight into the beliefs of the textbook authors.

4.3 Study 2B – Results

Given the three participants' diverse approaches to teaching and classroom norms, the data I collected showed the influence of both the textbook and pupils in varied levels of detail. For example, given the nature of Barbara's classroom social norms, I captured richer data about pupil-supplied arguments, as arguments were happening more often publicly. In Charles's classroom, where pupils were expected to work individually on problems most of the lesson, the publicly displayed arguments were fewer because Charles had individual conversations with pupils during initial stages of the problem solving activity (also captured in recordings but not the main subject of the investigation). In the following text, I describe phenomena observed in each classroom and pertaining to argumentation: the classroom social norms regarding argumentation, how general statements were justified, and the three characteristics of enacted arguments. Finally, I describe the particular influence of the teacher, pupils and the used curriculum (text) on the phenomena.

4.3.1 Pupil participation – the social norms in the classrooms

The social norms affecting the pupils' participation on arguments in each of the three classrooms were in general alignment with the respective teachers' beliefs about mathematics education.

Karen's classroom

The first noticeable aspects of Karen's lessons were the quick pace and a high level of apparent pupil involvement. Pupils appeared to be listening to or actively participating in a collective (whole-class) dialogue, or working individually on given tasks and problems throughout the lesson. Most of the activity took on the form of either teacher-led whole class (explanation, problem solving, and answer checking), or individually solved tasks. The pupils were expected to follow the teacher's instructions (including problem solving) and answer her questions, they were expected to provide or try to provide arguments upon the teacher's request.

It was obviously the teacher's job to ask for arguments and thus decide when arguments were needed (this follows from Karen's beliefs about the role of argumentation discussed in Section 3.4.1). Pupils rarely offered their own arguments without a prompt from Karen.

Karen was also the person responsible for providing examples of problems and arguments that underlie *methods* of pupils' problem solving, especially steps that are needed to solve particular problems. Pupils were expected to take active part in this process, which usually took on the form of a Socratic dialogue.

In written work (i.e., tests, homework, individual problem solving on paper or blackboard), pupils were expected to always show (the teacher) how they solved the problem. Furthermore, there was an expectation about the specific form this explanation should take for word problems. Karen did not use the word "explanation" but "record", which consists of organizing the given information in a specific way.[55]

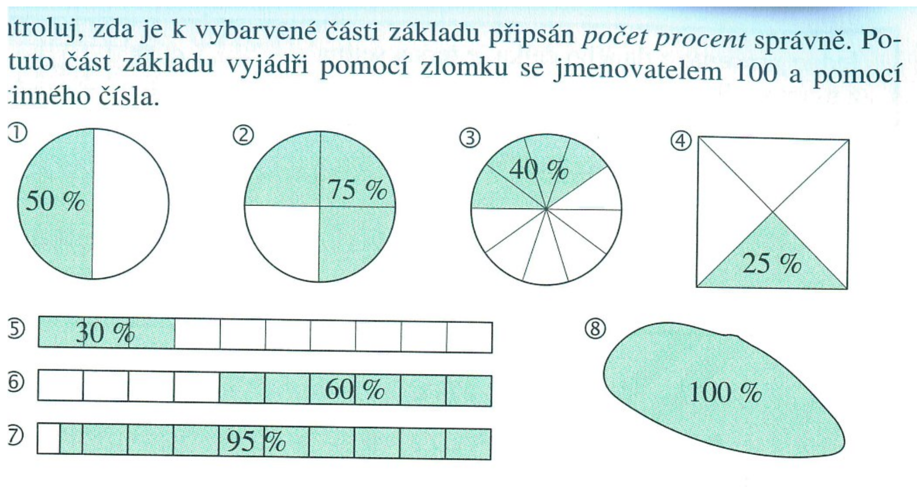
Finally, Karen was also the person who gave the ultimate backing for arguments, i.e., decided what is correct or incorrect. In addition, she made qualitative judgments (such as what counts as a more efficient method) and expressed them in class.

Karen was the one responsible for finding and correcting erroneous arguments and solutions. When she detected an error, she tended to take over and elicit the correct answer, from the pupil or from the class, through a series of questions. Alternatively, she provided the correct argument herself. In the following situation, Klara is asked what fraction is represented by a stick model of 30% (see Figure 4.1).

Klara:	This is ... one third (uncertainly).
Karen:	Wait a minute. 30% as a fraction. How many parts do you divide the whole into, if you have 30%? (waiting)
Klara:	Into ten?
Karen:	Into ten. And how many parts are coloured?
Klara:	Three.
Karen:	So when you have ten parts, one part is what percent part?
Klara:	One tenth.
Karen:	One tenth or also …

[55] This form of record is commonly encouraged in Czech mathematics classroom across the country. When, in the last observed lesson, I asked the class to solve a word problem individually and to "explain their solutions", the teacher rephrased this as "provide a record" ("udělejte zápis" in Czech) and repeated this phrase several times while the students were working on the task. See Figure 4.19 for an example of such a record.

Klara: (unintelligible)
Karen: And when you have three parts coloured?
Klara: That is three tenths.
Karen: Well. Three tenths or also thirty percent.



English translation:

Check whether the coloured parts of the whole have been assigned *the percentage* correctly. Then express this part of a whole as a fraction with the denominator 100 and as a decimal.

Figure 4.1: Klara's problem (Problem #5). Reprinted from Odvárko and Kadleček (2004).

It was acceptable for pupils in the class to contribute with a different idea or to ask a question, Karen usually responded positively to these. As a rule, she did not dismiss other correct arguments and solutions but she would, whether explicitly or not, demonstrate the need to promote the one that is, in her view, more appropriate. In fact, Karen felt responsible for pointing out the most efficient, universal, or common method. I describe this phenomenon in detail in Sections 4.3.3 and 4.4.1.

Barbara's classroom

Throughout the lessons, Barbara's pupils were expected to work – individually, in pairs, or in groups – on problems presented by Barbara, to listen and contribute to class discussion concerning these problems. The teacher, acting as a facilitator, decided what problems would be presented to be solved in the lessons, when individual (or group) work on a problem was stopped or when discussion started or whether a problem would need to be abandoned and left for later. These decisions seemed to be based on her perception of the pupils' understanding, ability to concentrate, etc.

There were clearly two practices in place that prompted argumentation: a) the selection of the problems themselves and b) a mechanism for correcting errors. The first one showed in the nature of the problems: they did not always have one correct solution only and could be stated informally, or vaguely, so that solutions could include various interpretations of the data and there could be more than one correct solution. One such example was the following problem: "A gardener bought a support stick for a plant: he wrapped tape around a half of the stick, and also some wire around two thirds of it. What part of the stick is covered by both tape and wire?"

Pupils worked individually first and then discussed their various solutions as a class, they agreed that there were two correct ones and noticed that there might be others. Barbara encouraged pupils to "try finding one more correct solution and convince us that it is correct". This problem was revisited in the next lesson where the class worked on finding all possible solutions (an interval). The nature of the problem and the pupils' different interpretations prompted the pupils to argue. In the following example, a discussion took place after Barbara put up three different answers, which she had noted earlier amongst the pupils' individual solutions (see Figure 4.2). After the pupils evaluated their correctness, they began to produce their arguments.

Barbara:	Well, those are the solutions you have come up with. Does anyone have anything else? Well, so, now what? (sounds puzzled)
Bara:	It's not a third.
	(other suggestions, unintelligible)
Pupil 1:	I'd say it will be three (emphatically) sixths.
Barbara:	Bara says it is not a third. (pupils are interacting: [it must be a half, a sixth]) Matyas? Don't tell me (emphatically), I know the answer.
Pupil 2:	[It must be a half.]
Pupil 3:	A sixth also not, a fifth either, so it has to be a half.
Pupil 4:	It has to be a half because the wire stops in the middle and that's where the tape starts...

And a little further into this discussion, Adela offers an argument for someone else's solution:

Adela: Well, one sixth, someone maybe thought that if the half (unintelligible) and then the thirds (unintelligible) then one sixth is left over.

$\frac{1}{3}$ $\frac{1}{6}$ $a\ half$ $\frac{1}{2}$ $\frac{1}{5}$

Figure 4.2: Barbara creates a need for pupils to argue: displaying various solutions.

The latter practice (also visible in the above episode) can be described as a certain way of working with error: Barbara did not tell pupils what solution is correct or incorrect but, rather, displayed the solution or all solutions for pupils to evaluate and argue why something was correct and, especially, why something was *not* correct. For example, in the problem "Decide which of these is the biggest: a third of a fourth, a half of a half of a half, a quarter of a third and a fifth of a half", one of the pupils (Lukas) marked "a fifth of a half" as the largest. When pupils evaluated each expression as a whole class (1/12, 1/8, 1/12 and 1/10, respectively), the teacher asked Lukas:

Barbara: So, is an eighth definitely bigger than a tenth, Lukas?
Lukas: Yes.
Barbara: Did they convince you?
Lukas: I think so.
Barbara: And where did you make a mistake, do you know?
Lukas: Well, because, well, […] I did not realize that …, and I took the fifth of the whole, sort of, hmm…

Although Barbara accepted all correct arguments, she had a need to clarify some of them, summarizing and/or consolidating the argument that the class put forward for "the whole class". For example, in the following situation the class were solving a textbook problem about Egyptian fractions (see Figure 4.3) establishing whether it is Jakub's or Suyen's solution that is done according to the Egyptian way of dividing. The teacher felt she had to clarify by putting forth her own argument.

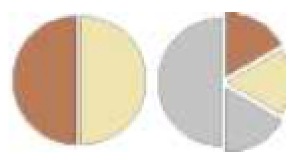
		
Suyen's solution: "I'd divide each loaf into thirds and then each of us would take a third from each loaf."	Honza: "I'd cut out a third from each loaf, and maybe Jakub could take these cut out thirds."	Jakub: "No way! We'll cut each loaf in half, each of us will take a half. Then the remaining half into thirds and each of us will take one of those."

Figure 4.3: Egyptian fraction problem prompt in Barbara's textbook. Reproduced and translated from Barbara (pilot, p. 21).

Barbara: So, is [your group] right or is it Lukas? Why do you think it is Jakub? Marcela?
Marcela: Because … over a half, it changes[56]? And then for the third it… (unintelligible)
Barbara: Well, and how many cuts were there? How many cuts did Jakub make?

[56] The meaning of this warrant is not clear to me as the observer.

Pupil:	Four.
Barbara:	Four. How many [did Suyen make]? How many cuts, Lukas?
Kristof:	Six?
Barbara:	You are Kristof. How many cuts [did Honza make]? (pupil: Four) Also four. So who then… (pupils suggest, unintelligible), well?
Pupil:	(probably pointing at Honza's solution) So that everyone has the same number of parts.
Barbara:	And that's why Jakub is upset. Because he would get pieces (unintelligible) because this way one person would get two thirds and the other two thirds, but he would get two thirds in two halves, while the others would get the whole [piece].

Barbara also sometimes takes over to hurry things up (when other pupils are acting up), or when she perceives it as easy, i.e., established practice, that only is the means to a more complex idea. For example, the following dialogue took place after she had asked several pupils to mark five different fractions on a number line, in the interval [0, 1] separated into 24 parts, in order to help them make a new connection (see also Section 4.3.4). She is guiding a weaker pupil (Kiki) to complete the task.

Barbara:	You're supposed to do one sixth, where it is. (Pupils are making noise in the background. Kiki is looking. It is taking him time.) One twelfth, how many is that? Squares?
Pupil:	Two.
Barbara:	So a sixth?
Pupil:	Four.
Barbara:	(towards Kiki) Kiki, four.

On several other occasions, Barbara elicits arguments step by step to establish a particular connection or warrant. I will show in Section 4.3.3 that it is not always accepted by pupils.

Charles's classroom

In Charles's classroom, pupils were expected to work individually on problems from the provided text (a sequence of problems) assigned one by one or in a sequence by the teacher. Typically, Charles would have pupils work on a problem, walk around, evaluate and discuss their work individually. Pupils were expected to solve a problem, ask for help, and if told by Charles that the solution was incorrect, to review the solution. He would reconvene the class when he noticed that too many pupils were stuck, to elicit or provide warrants or to point out common misreading of the problem, mistakes etc., or when he felt it was time to consolidate the pupils' individual work.

For example, the following conversations (individual and whole-class) took place in the first observed lesson. The task (Task 4, see Fig. 4.4) had been put up on the board and pupils had started working on it individually.

> **Task 4:** Draw (do not construct) two circles: *k* (*A*; 5 km), *l* (*B*; 3 km), $|AB| = 3$ km. In the picture, draw a point:
>
> a) *C*; $|CA| = |CB| = 2$ km,
> b) *D*; $|DA| = 5$ km; $|DB| = 3$ km,
> c) *E*; $|EA| < 5$ km; $|EB| > 3$ km,
> d) *F*; $|FA| \geq 5$ km; $|FB| < 3$ km.
>
> Think about all the possible places where we can place each of the points *C, D, E,* and *F*.
> For better clarity, draw a new picture for each part of the problem.

Figure 4.4: Task 4 in Charles's Lesson 1 on disk and circle.

Charles: (to Pupil 1) I am not quite sure that the picture reflects what the text says. In fact, I am really sure that it does not. (to Pupil 2) Well, I don't think this is right.

Pupil 2: How so?

Charles: Well, because you have (are given) the radii and how far the points *A* and *B* are supposed to be apart. And when you lay [draw] this out, it certainly cannot be this way.

Pupil 2: Yes, I know, because this here would be longer than… but…

Charles: So, you have to, sort of, cross this out and start over. This picture does not correspond to what you should have in it [the data from the problem].

[…]

Charles: (to Pupil 3) This is not right. Well, how much is this (probably pointing at distances in the pupil's drawing)? And how much is this? So how can it look this way?

We can see that Charles was pointing back to the data to show pupils that their drawing was incorrect. After a while, he called the class together to consolidate on the board.

Charles: (to the whole class) So, let' shave a look at what you got […] I am going to draw the first circle (drawing). So, this is …?

Pupils:	k.
Charles:	k. So. Where do I need to draw point *B* now?
Pupil:	So that it should touch. … so that the circle touches, like, the *A* point. So that it goes through *A*.
Charles:	Well, I ...
Pupils:	(many voices) …point *B* must be simply 3 kilometres from *A*.
Charles:	So that means … so should I make another, sort of … circle? Like this? (drawing on the board)
Pupil:	No, no, no. The circle must run through *SA*.
Charles:	Well … I do not know, actually … what you want me to do at the moment. I really don't know. […] I do not know what Petr wants me to draw. I would have said something completely different, you see, if I was asking about where to draw point *B*. So, […] come draw it on the board, and all will be clear. […]

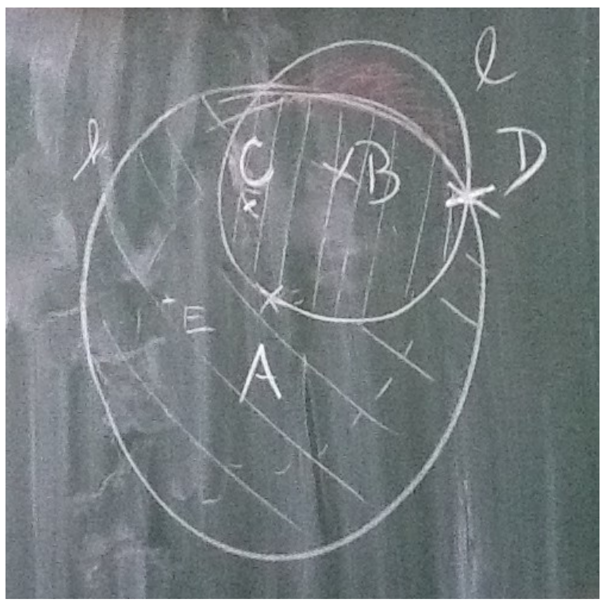


Figure 4.5: Task 4 – solutions on the board.

Petr apparently drew a circle around point *A* (dashed in Figure 4.5) and chose a point *B* on it. Charles consolidated, pointed out the most common error that he had observed and referred back to the problem's data to explain why that solution was incorrect:

Charles:	Great. […] So, now I see what you wanted me to do. […] So everyone agrees with this, right? Yes? The most common error was that the points *B* appeared

here (points probably at the circle) but that's not possible, right? This must be three and this must be five (pointing). So it needs to look something like this.

4.3.2 Justification of general truths

Karen – sometimes it is impossible[57]

The below example of a dialogue gives us a sense of how Karen's beliefs about the need to provide mathematical justification for methods and general mathematical statements manifested themselves when the class discussed the percent – decimal relationship.

Karen: So, if we have 18% (writing on the board), how do we get a decimal?
Pupils: Eighteen divided by 100.
Karen: We divide by 100. Why? Because 18% is 18 hundredths (writing 18% = 18/100 = 0.18 on the board), to divide by a 100 means 18 hundredths.

Karen expressed her belief in having the responsibility to provide pupils with justification of mathematical statements. This responsibility was felt even in the one moment in the observed lessons when Karen acknowledged that she did not know how to provide a mathematical argument for the procedure, and stated that pupils just "have to remember". The problem Karen posed to class was: "From a class of 22 pupils, six participated in a math competition. What percent of the class was that?" Karen went on to exemplify two methods for solving the argument.

Karen: The first one is the 1% method. Again, I think that this method is more convenient and easier… ok, what's the base in this problem?
Pupils: (suggest ideas)
Karen: Yes, base or 100% is 22 pupils. There are 22 pupils. (writing a record of the solution on the board, writes "1% =") Now, we'll calculate. Ada?
Ada: 1% will be 0.22. (Karen writes this on the board)
Karen: Now you just have to remember that the percent, […] I don't know how to help you remember … you need to remember. You can calculate the percent this way […] we divide the percent part we want to express in percent by one percent.

[57] The results in this section come from Žalská (2017).

The argument that she was reluctant to share with her pupils is in fact the ratio argument used in the rule-of-three method: firstly, that the *percent part : percent* ratio is a constant, and for all non-zero real numbers a, b, c, and d, if $a : b = c : d$ then $a = c \cdot b / d.$ Clearly, this presented a conflict of beliefs for her, and she chose not to present the argument, because it was too complicated in her opinion. Some pupils would not grasp it with their current knowledge.

In Karen's textbook, the authors let the reader observe the first warrant through a series of examples, and then simply refer to the rule-of-three as practice established in the previous unit on proportion. However, in the teacher's book they also admit that the equivalence of the two equations is, as yet, in the curriculum, inaccessible to pupils.

Barbara – no general truths to justify?

In alignment with Barbara's beliefs about teaching and learning mathematics, I observed no mathematical statements that were generalizing a relationship, a pattern, or a procedure. In fact, it seems that when Barbara does decide to lead the class to discover a common general pattern, this is not met with understanding from her pupils. The following episode from the third lesson I observed illustrates a moment where Barbara wants to justify the general idea of reducing fractions, using a particular warrant.

The following took place after a problem-solving activity where the answers were expected to be simplified fractions. Barbara asked pupils to clarify why they worked with "the smallest [fractions]". They replied that it was "for better clarity". The teacher proceeded to write the fraction 128/256 on the blackboard, asking how much it was. The pupils quickly figured out that it was 1/2. She acted surprised at the quickness and asked (rhetorically) which number was easier to draw. Next, she seemed to make a quick decision to go further and work on the justification of this statement. She asked the pupils to factorise the fraction, or rather, the numerator and the denominator (in Czech usage, the literal translation of the verb "to factorise" is "to decompose into a product of primes").

We can see from the following transcript that the pupils found Barbara's request confusing, perhaps unnecessary and irrelevant. They had shown her that they were able to tell her why (to what end) they simplify and also that they were able to simplify the fraction at hand. The class seemed to be protesting. After all, they already solved the "how much is it?" and the teacher was now asking them for another proof, without telling them why. Also, Barbara uncharacteristically chooses to use a formal wording "factorize", which in Czech is a very formal expression that translates as “decompose into a product of primes". After a while, the pupils settled down and started working individually.

Barbara:	You know what, factorise 128 and factorise 256 and write it in a fraction. (pupils protesting) Come on, two times two times two…
Pupils:	Ah, ok, like this (discussions and protests).
Barbara:	Decompose into a product of primes, if this means anything to you, I wonder.

After a while, Barbara asked for the answer, and from the pupils' reactions it seems they were not sure about what such a task involved. For example, Adela clearly thought that she was just supposed to justify the answer 1/2. She got frustrated when she was not sure "how to say it". In other words, it is likely that she was expressing her lack of understanding about what Barbara wants to hear from her.

Barbara:	Adela?
Adela:	Well, 256, a half is 128 and I just copy that.
Barbara:	So, 128 will be decomposed how? Adela?
Adela:	128? Well...
Barbara:	Two times what?
Adela:	64.
Barbara:	Two times 64, 64 is not a prime ... what is the decomposition into primes you have, just dictate it to me. (pupils offering some suggestions)
Adela:	But I don’t know how to say it. How am I supposed to say it? (pupils suggesting)
Pupil:	I have the same thing.
Barbara:	128 (writing on the board) Lenka is going to tell me.
Lenka:	128 by two is 64.
Barbara:	Well? ... Marcela? ...
Marcela:	128 by two is 64 by two is 32 by two is 16 ...
Barbara:	And where is the product? ... Tell me.

After a while, a pupil at last offers the factorised numerator, and Barbara writes it on the board. Then she asks about the denominator.

Barbara: And 256? Shortly put?

Pupil: 128 times two.

Barbara: Well, that's true but 128 is not a prime number. But if 128 is this (pointing at the factorised numerator), I can just repeat it (writing out the product into the denominator, aligning it with the expression in the numerator, and adding a two at the bottom end of the line; see Figure 4.6).

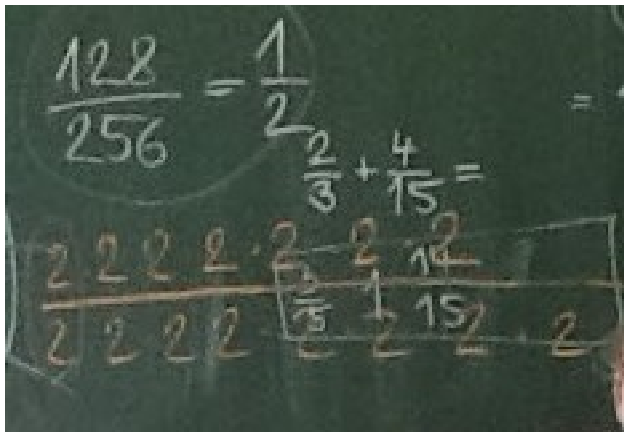


Figure 4.6: "Why do we simplify a fraction?" – a justification unrealised in Barbara's lesson.

To mark the importance of the point she wants to make, Barbara asks the class to "concentrate like never". She is trying to elicit the answer "number one" but the pupils do not seem to grasp her idea and start responding with irrelevant formal terminology instead. She makes a decision to leave the problem unresolved and moves on to the next activity.

Barbara: OK (approvingly), now concentrate like never, […] What is this number? (circling the first two in both the numerator and denominator) What is this?

Pupils: (uncertainly) The same.

Barbara: Come on, quickly, [give me] something. What is this?

Pupils: [a common multiple, the largest common divisor, the smallest common divisor]

Barbara: Look (resignation), let's go to the back [of the classroom].

Charles – working with geometrical properties

In his lessons, Charles's class explored the property known as Thales theorem. Pupils first made a hypothesis based on observation and measurement of a specific situation. The teacher led the class through a series of partial proofs, providing some warrants. I will revisit this in more detail in Section 4.3.5.

There was one other problem that involved a general method – the construction of a centre of a circle. Pupils were able to show the steps but it is not clear whether the underlying method was justified. For more detail, I refer the reader to Section 4.3.4. Rather, the pupils were using their understanding of other concepts and properties in problem solving. Charles also hit upon the intuitive aspect of conviction "he just sees it" that he is willing to accept at this point because it shows that the relevant concepts were understood. At the same time, he has a need to acknowledge that this is not a general proof.

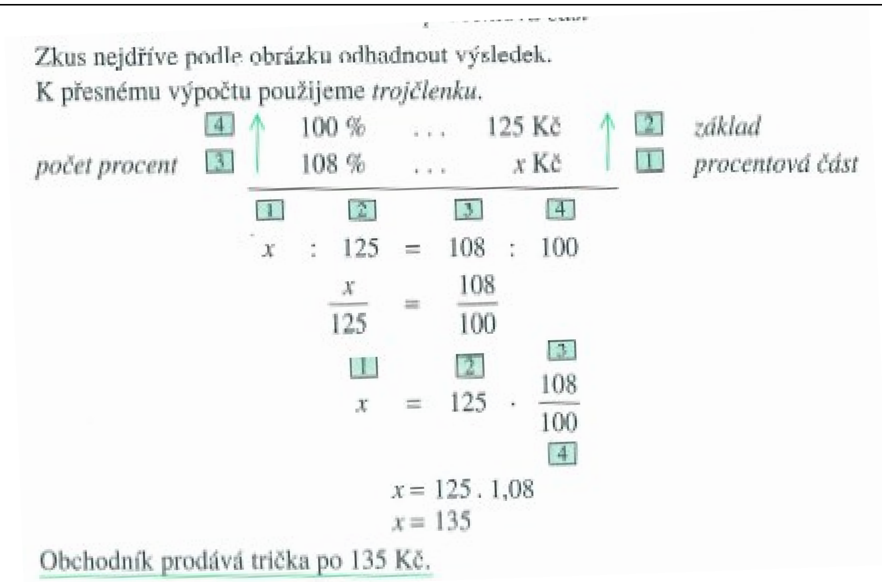
4.3.3 Characteristics of arguments: warrants

Karen and the textbook: the rule of three

The textbook used by Karen's class introduces one method for solving word problems with percent. The authors base the arguments on the concept of direct proportion, in particular, on the fact that the percent part changes in the same ratio as the percent. This idea is then used as a warrant in the method of the ratio-based rule of three (see Figure 4.7), which is explained and practiced in an earlier chapter in the book, in the unit on ratios.

In contrast, Karen did not use this method at any moment in her classes. The arguments that she did show pupils were given names ("one percent", "with a decimal", and "ratio") and referred to as "methods". The majority of warrants for methods were based on the multiplicative relationship of the percent part and the base, and on the definition of one percent, as corresponding to one hundredth, either as a fraction or decimal (recall the percent method from Section 4.3.2).

In the teacher's manual, the authors assign the use of the ratio warrant the prominent role of helping pupils get an insight into the problem. This belief about a need to understand the problem through the use of a particular method or warrant seemed to collide with Karen's beliefs about what is important for her pupils. Rather, she values efficiency and straightforwardness in problem solving. Hence, she introduced neither the rectangular representation nor the rule-of-three arguments when solving word problems in her teaching. In fact, she discouraged her pupils from using it (albeit acknowledging its existence and its effectiveness):


Zkus nejdříve podle obrázku odhadnout výsledek.
K přesnému výpočtu použijeme *trojčlenku*.

	$\boxed{4}$	100 %	. . .	125 Kč	$\boxed{2}$ *základ*
počet procent	$\boxed{3}$	108 %	. . .	x Kč	$\boxed{1}$ *procentová část*

$$\underset{\boxed{1}}{x} : \underset{\boxed{2}}{125} = \underset{\boxed{3}}{108} : \underset{\boxed{4}}{100}$$

$$\frac{x}{125} = \frac{108}{100}$$

$$\underset{\boxed{1}}{x} = \underset{\boxed{2}}{125} \cdot \frac{\overset{\boxed{3}}{108}}{\underset{\boxed{4}}{100}}$$

$x = 125 . 1{,}08$
$x = 135$

Obchodník prodává trička po 135 Kč.

English translation:

Try to first estimate the answer.
We will use the *rule-of-three* for the exact calculation.

percent (left of diagram, next to box “3”)
base (next to box “2”)
percent part (next to box “1”)

The salesman sells T-shirts for 135 CZK each.

Figure 4.7: The rule-of-three method in Karen’s textbook. Reproduced from Odvárko and Kadleček (2004).

Karen: Someone mentioned a third method, in case you study from your textbook, [I don't recommend it, only if someone gets] really lost and needs a crutch […] but in the time you write it all out (referring to the method), you might as well have finished other three problems [using the other methods].

Karen and pupils’ warrants

The following passages will show examples when different arguments were provided by pupils. The exchanges took place at the beginning of the second lesson, pupils were converting a series of fractions into percent. They had just converted 4/5 by expanding to tenths and then hundredths. Now Sam tried to convert 3/8 in the same way:

Sam:	I'll multiply the fraction by twelve and a half.
Karen:	Why twelve and a half?
Sam:	Because if I multiplied 8 times 125 (unintelligible)
Karen:	So by 125, right?
Sam:	But that will be a thousand, so …
Karen:	Doesn't matter. But (writing on the board) 8 times 125 is 1 000. What is 3 times 125?
Pupil:	375.

Sam was trying to expand the fraction to hundredths (realizing that expanding by 125 and simplifying to hundredths is the same as expanding by 12.5) but the teacher felt that this was not straightforward and accessible to all pupils, so she took over and broke the argument down. After a few more simple problems, where pupils did not need to calculate, they were asked to convert the fraction 9/40. At first, Will suggested to reduce by two and expand by five. Then he added:

Will:	Or multiply (sic) by two and a half.
Karen:	Excellent, two and a half. Do you [all] agree?
Kim:	And couldn't you expand to thousands?
Karen:	Also. And if you were to do that, by what number would you expand?
Kim:	So, that would be times … (thinking) … two hundr …
Karen:	Twenty-five. Either, as Will said, we expand by two and a half, <u>which is not very common</u>, (turning to the board and writing) if we want hundredths in the denominator we expand by two and a half (she writes this on the board), do you agree? Forty times two and a half is one hundred, right? And the numerator … 18 and 4 and a half […] 22 and a half. So what percent is 9/40?
Pupils:	Twenty-two and a half.
Karen:	Or, as Kim said, expand by 25 (writing on the board), the numerator (sic) is 1 000, do you agree? And the denominator (sic) is …
Pupils:	225.
Karen:	And <u>we got the same thing,</u> 22.5 %.

At this point, Karen allowed a pupil (Will) to carry out an argument that is (like Sam's) based on expanding by decimals, but this time the pupil broke it down into two warrants first, and Karen praised it. Will felt encouraged to suggest expanding by a decimal. Finally, another pupil supplied an argument based on the expansion to thousands (which had been shown by Karen before, see the transcript above). Both methods were now endorsed by the teacher, publicly, as valid arguments, and demonstrated on the board. When Karen summarized these approaches, however, she qualified Will's solution as "not very common".

There is another moment in the lessons that exemplifies Karen's decision-making when it comes to unexpected arguments. Pupils had successfully solved the following problem: "There are 36 pupils in a class, 75% are learning German, 50% are learning English. How many pupils are learning English? How many pupils are learning German?" Karen added another question to this: "How many are learning both languages?" The class was able to argue towards the answer "a quarter, 12 pupils". Then a pupil suggested a different possible solution. Karen's first impulse was to dismiss this warrant (as incorrect) but then realized what the pupil meant and admitted that the problem is ambiguous and automatically restated the pupils' data interpretation ("There are pupils who do not study either language.").

Pupil:	And what if those who were learning English were also learning German?
Karen:	No, that is the 75% … and German (ponderingly) … though … the problem is stated so that we don't know if there might be somebody who is not studying any language or what the situation actually is, right.

Again, we can see that Karen acknowledges the validity of her pupil's warrant (the data interpretation) but does not take it up with the whole class, and seems to have the need to move on to the next activity. Obviously, problems that are subject to interpretations or problems with multiple solutions are not part of Karen's classroom desired curriculum.

Barbara's and pupils' warrants

In Section 4.3.2, I have already illustrated that the warrants Barbara wants pupils to find were not always accessible to or discovered by them. The following is another illustration of this fact, and we can observe that Barbara persisted with eliciting the notion in various lessons and various problems, deeming it important.

During solving problems (from the book) involving Egyptian fractions, I observed that Barbara's particular argument did not seem to be fully accepted by the class. Barbara used these problems as an opportunity to create the pupils' need for adding fractions. When pupils add up the Egyptian fractions (the bread pieces allocated to one person) for each problem, the result should be the fraction that represents the number of bread loaves and the number of people. Barbara does not make that specific warrant explicit, instead, she is expecting pupils to notice it. However, the transcripts give no straightforward evidence of whether they did or did not.

The Egyptian fraction problems appeared in three of the four lessons. In the first lesson, the class worked in groups with cut out paper models of circular bread loaves to first divide 3 loaves among 4 people, then 2 loaves among 5 people. They presented their results on the smartboard (see Figure 4.8) and Barbara tried to draw their attention to the correspondence of "2 loaves among 5 people" and the resulting $2/7$ fraction.

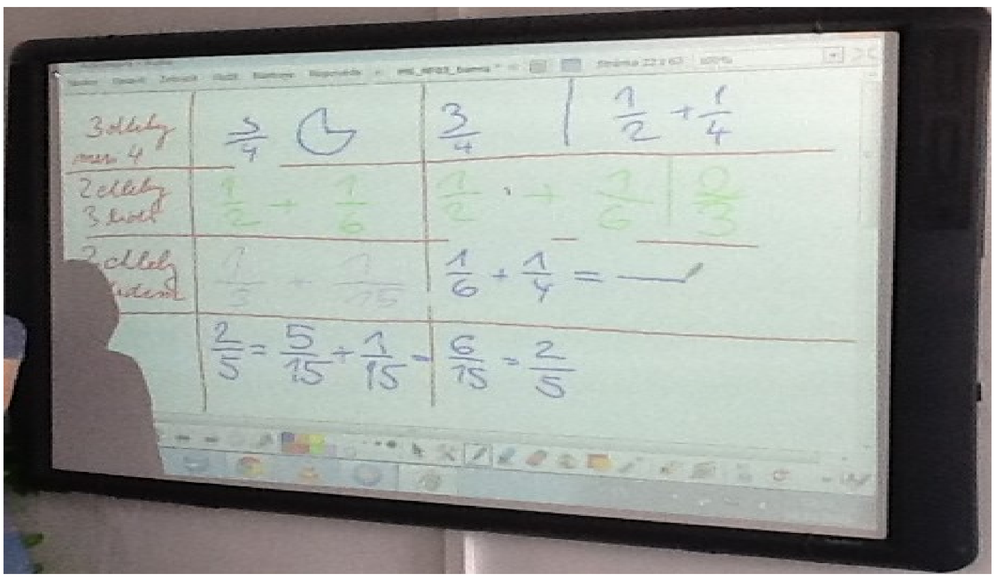


Figure 4.8: Adding up Egyptian fractions (Barbara's Lesson 1).

She then went on to elicit the addition $(1/3 + 1/15)$ and simplifying to 2/5. When they reached that number, she recorded it on the board and pointed out the equality, without actually stating it, as if it were a noteworthy "coincidence". Then Barbara turned attention to another (this time incorrect) solution, which had been presented by Lucas, 1/6 + 1/6 (see Figure 4.8), and asked "What did Lucas divide? How many loaves among how many people?", and then proceeded immediately to elicit the addition, arriving at 5/12 (the picture was taken before this was noted on the board).

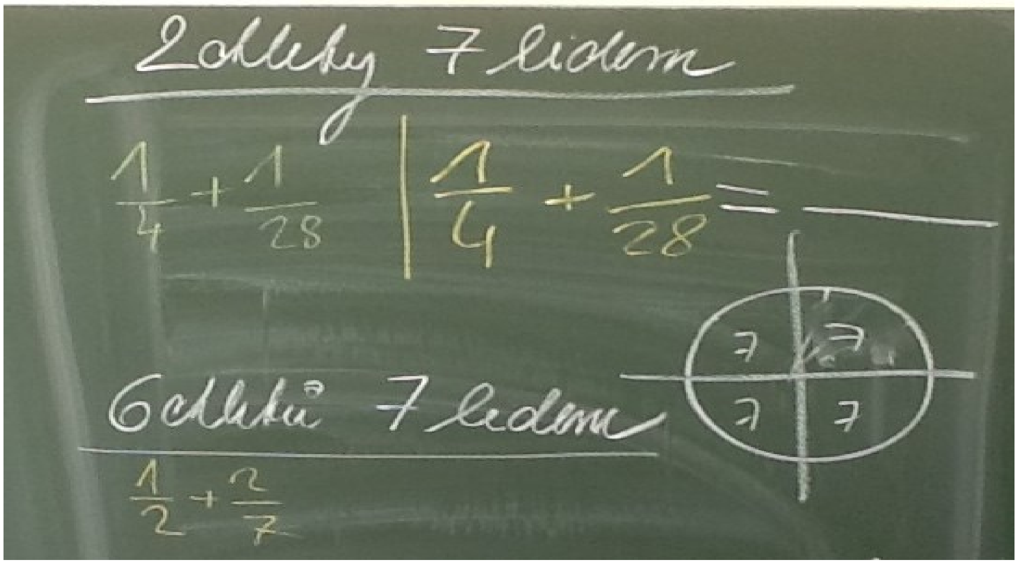
Barbara:	Five twelfths. So what did Lukas do?
Pupil:	Added..
Barbara:	Divided… how many loaves… he was supposed to divide two loaves among five people … not just him, the whole group … and they divided five loaves among how many people?
Pupil:	[Twelve?]

We can see from the pupils' reactions that they had probably not grasped Barbara's intended argument. When they gave her the expected answer, the inflection in the voices indicates a question – they might have simply been filling in the gap in Barbara's sentence by stating the only other number involved (i.e., 12 in the fraction 5/12).

In the second lesson, I noticed a similar dissonance. The pupils had been working individually on dividing 2 loaves among 7 people and had written one solution on the board (see Figure 4.9). Barbara then gave them a new problem, emphasizing that it is a challenging one (six loaves among seven people).

After a while, Barbara realized that she needed to bring the lesson to a close, even if the pupils had not come up with an answer. She drew their attention to the strategy of verifying, i.e., she tried to bring up the point that the sum of the fractions should reflect the number of loaves as well as the number of people. The reactions from the pupils, though, indicate that the pupils understood (also correctly) that the verification would consist in adding all the divided parts and getting the original number of loaves.

Barbara:	So, stop for now… [try to solve this problem before tomorrow’s lesson] but … how can you be absolutely sure that you got the right answer?
Pupil:	Add it all together and it has to be six.
Barbara:	Ah, add it all together and it has to match. So, here (pointing at the first problem of dividing two loaves among seven people) we must get what?
Pupil:	Two loaves.
Barbara:	Two loaves. Among how many [people]? (pupils suggesting seven) So, what must be the result here (pointing at the addition of two fractions)? (silence) Two among seven. ... Marcela? What must we get? […] Well, Kiki? (silence) Ok, let’s see what we get then. What equal parts do we need to divide a quarter and a twenty-eighth? … How many is this twenty-eighths? A quarter – how many is that twenty-eighths?



English translation:

2 loaves among 7 people

6 loaves among 7 people

Figure 4.9: Adding up Egyptian fractions (Barbara's Lesson 2).

Barbara then tried to elicit the sum of the two fractions to make the point but when the class did not seem to be able to come up with the answer to her question, she abandoned the problems for the day.

Pupil:	A seventh.
Barbara:	How many twenty-eighths is a quarter?
Pupil:	Teacher, what is a seventh of three quarters of a loaf?
Barbara:	A seventh of three quarters of a loaf? (pupils working.) Stop for now, don't try to solve the six loaves now, you'll have all afternoon, but how will you check [the result]? … A quarter – how many is that twenty-eighths? (silence) A quarter – how many is that eighths?
Pupil:	Two.
Barbara:	Two eighths. You know that. How many sixteenths is that?
Pupils:	(after a while) Four.
Barbara:	Four sixteenths. How many is that thirty-seconds?
Pupils:	(after a while) Eight.

Barbara:	Eight. How many is that twenty-eighths?
Pupils:	(discussing)
Barbara:	Well? How many? (silence, pupils working) Well, how many is it? [...] You are tired (pupils agreeing), I can see […] so, the last one, how many is it? Nikol? [No correct answer.] Ok, let’s finish, until tomorrow.

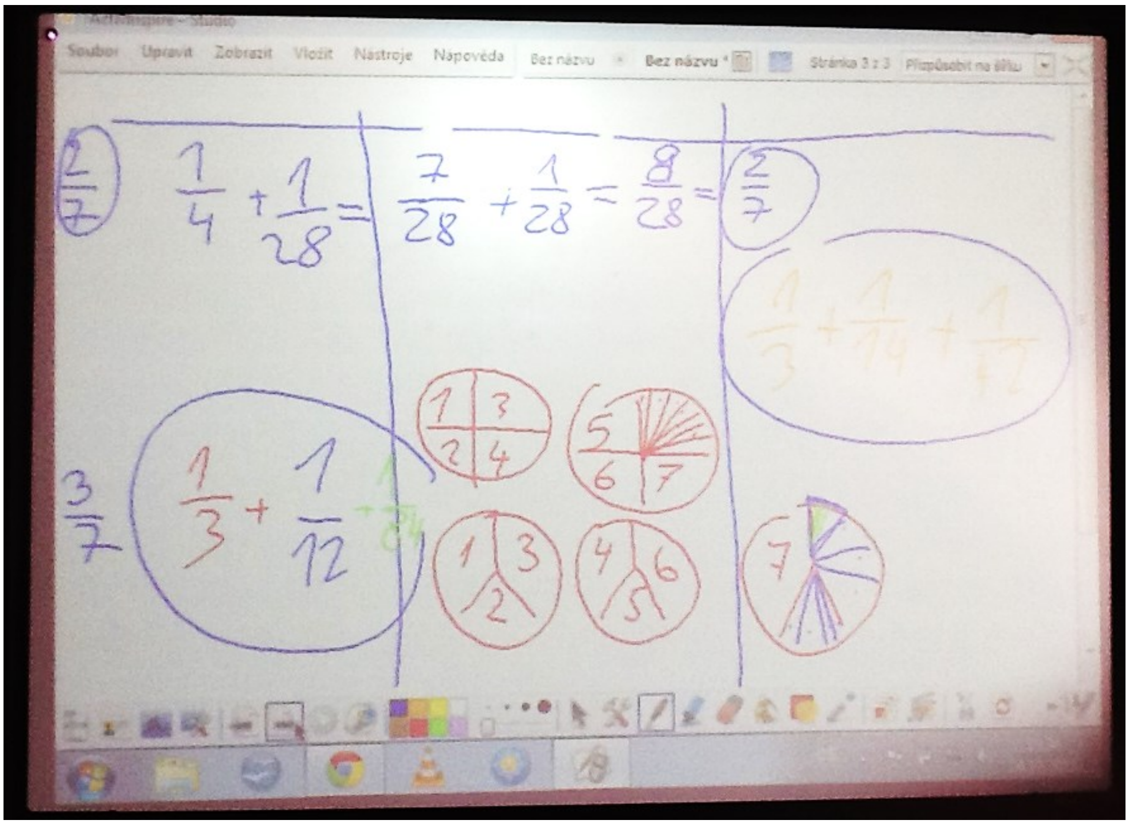


Figure 4.10: Adding up Egyptian fractions (Barbara’s Lesson 4).

The class revisited the two problems in the last observed lesson. Pupils as a class revised dividing 2 loaves among 7 people and then in groups worked on dividing 3 loaves among 7 people. Note that this time Barbara stated the problem as a fraction on the left-hand side of the smartboard (Figure 4.10). The Egyptian fraction was recorded on the smartboard as $1/4 + 1/28$. Again, Barbara elicited the idea of verifying by adding the fractions, and went on to elicit the sum, this time they arrived at 8/28 and simplified to 2/7 without problems.

Barbara: Ok, so only five minutes left, ok? How can we decide that it is correct, not just because all three [groups] had that but how can we decide that it is correct?

Pupil: We'll add.

Barbara: Add. Ok, I'll put it here (writing) and we'll write the solution here (pointing at the board). How much is that (pointing at the first problem)? How much is one fourth plus one twenty-eighth? (eliciting and recording the sum on the smartboard)

Barbara: So, two sevenths. Is that two loaves among seven people? (pupils agreeing.) So, we could decide in a similar way here (pointing to the problem with $3/7$. Pupils worked in groups and got these results: $1/3 + 1/14 + 1/42, 1/3 + 1/12 + +1/84$ – see Figure 4.10). We won't erase this, I will write over it but let's leave it for now (transitioning to a next activity – pupils react with disagreement).

This time it seems that pupils made the connection, although we do not have a clear evidence of that; it was Barbara again who generalized it onto the other problem (3 loaves among 7 people). It is hard to speculate how much of the general statement was actually accepted. The argument "dividing n units between m entities is represented by the fraction n/m, and the corresponding Egyptian fraction is equivalent to it" was never stated openly. Rather, there were several (four) specific instances shown by the teacher with the intention that the pupils would notice the pattern. It seems that this was only accepted by the pupils – if at all – in the last lesson.

Charles and pupils' warrants

From analysing Charles's textbook, it became apparent that the task sequences are designed to allow pupils to use warrants from previous problems.

For example, when pupils were solving Task 2 in Lesson 2 (looking for a centre of a circle), Charles expected that pupils would use the idea of constructing axes of any two chords[58]. The previous problem prepared some warrants for them – finding a distance of a chord to the centre of a circle involved the idea of the diameter lying on the axis of symmetry of a chord. After the lesson,

[58] In the textbook lesson (which I did not observe enacted in the classroom for logistical reasons) immediately preceding this observed lesson, I found this task: "Construct a circle and any one of its chords. Construct an axis of this chord. What interesting property does the constructed axis have? Try to justify why any axis of any chord has this same property." (Charles, online) In the pedagogical commentary to the problem in the observed lesson, the author states that the solution can make use of a rule about chords.

Charles commented that this was the most common solution. Here is how it was presented in front of the class (Figure 4.11, on the left) as a whole:

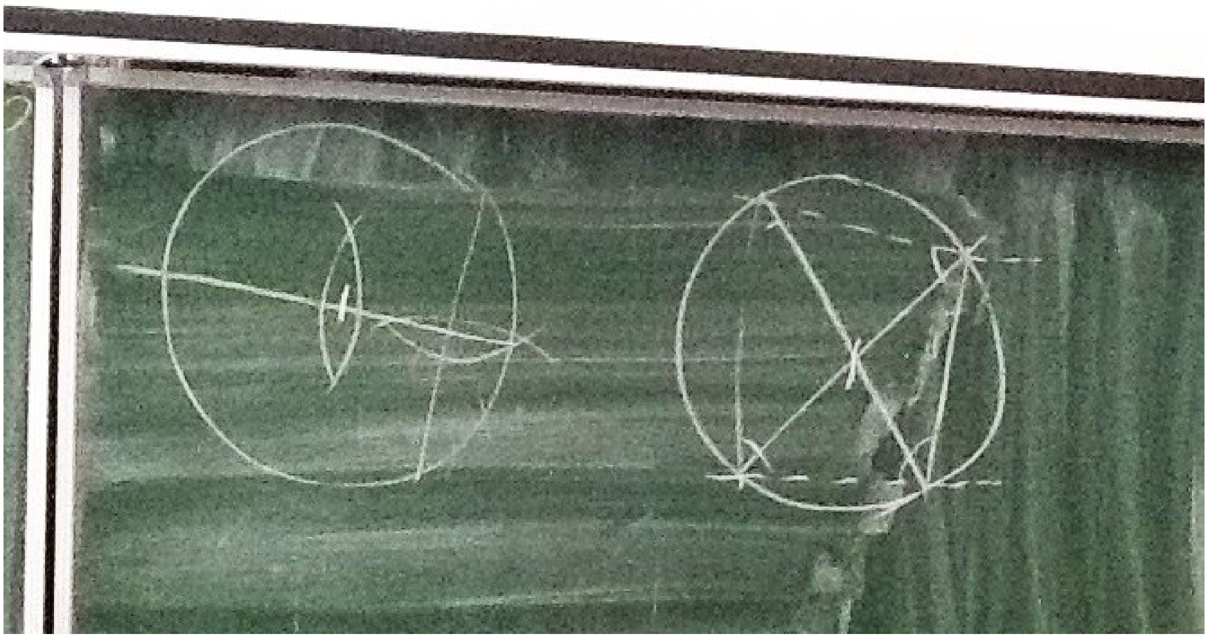


Figure 4.11: Finding a centre of a circle in Charles's lesson.

Julie:	I would construct a chord.
Charles:	What chord?
Julie:	Any. It actually doesn't ….
Charles:	So we make a chord, good.
Julie:	So now, I would construct an axis of the chord.
Charles:	OK, I'll make an axis of the chord (drawing on the board). OK.
Julie:	Now I would construct an axis of the axis of the chord. (laughter in class)
Charles:	OK.
Julie:	And where (unintelligible), that's where the centre is.
Charles:	Here is the centre. Is that clear? (discussion among pupils)
Charles:	OK, do you understand this?
Pupil:	I think so.
Charles:	It's actually like the problem we had a minute ago (referring to the first task).
Pupil:	Aha.

However, the pupils came up with two other solutions that Charles deemed invalid, as in case of Mirka, who wanted to find the centre by drawing a circumscribed square.

Mirka:	[...] I can make a square around it, not sure if I did it right. And so I would actually make a square around it, and so it's going to always, like, the line with the ruler is always going to touch up there in one point. And the point ... they are actually two points mostly, and the two points is the longest line segment that could be created there.
Charles:	And how … the only thing I am going to ask is … how do you make the square?
Mirka:	Well, that I don't know (laughing).
Charles:	You see …
Mirka:	I will make an inscribed circle.
Charles:	Well, you see, when you don't know how to make that circle...
Mirka:	Well, here it is in a square (meaning the cut out square with a circle drawn in the middle).
Charles:	But, well, that is a square, which is all lopsided and cut with my scissors, that I would not rely on at all, if you had seen how I cut it, I would not [rely on] any precision.
Mirka:	Well, I fortunately did not.
Charles:	Look, you see, if that square had been constructed, then this would have probably worked.
Mirka:	Hmm, right.

We can see that the pupil was arguing using a concrete physical model as a warrant. Charles is making the distinction between construction and the physical model, or between the theoretical geometric entities and their graphical representations (Laborde, 2005). It is not clear whether Mirka sees his point.

Another solution came from Franta. Again, we see that the pupils are struggling to see the difference between the validity of a warrant that uses construction (the theoretical space) and experimental measurement (the representation) [59]:

Franta:	So, we used the compass to take the radius of the circle.
Pupils:	Exactly…
Franta:	And measured it.

[59] This is a common problem of distiguishing between the theoretical geometric properties and the properties of its grafical representation also reported in the Czech context by Vondrová (2015).

Charles:	And how did you know …?
Franta:	A half. A half of the ...
Charles:	But wait and how did you know, well, you do not know, you do not know the radius of the circle.
Franta:	Well, I measured it.
Charles:	And how would you measure it?
Franta:	Well, with a compass...
Charles:	But you see, just using, just putting the compass there and trying to ... that is not construction. You see? Construction is when you have some method, and (unintelligible) will show you, not when you play around with trying to set the compass.
Pupil:	Well, it turned out to be wrong, anyway.

Finally, Adam came up with a method to correctly construct a centre of a circle (see Figure 4.11, on the right). Charles upholds this solution but the justification (why that works) does not seem to take place.

Adam:	Well, we have the circle.
Charles:	OK.
Adam:	And then (unintelligible, explaining how he used a chord and constructed a rectangle inside the circle, and constructed its diagonals).
Charles:	(drawing the situation on the board). So when I make the diagonals, I get the centre. Is that clear? He actually constructed some rectangle, that's what it is about, right? Not that we have exactly proved that this will [always] work, but it works for you.

Charles acknowledges that the justification is not present and, instead, agrees that it worked in this particular case and that is enough, accepting the solution.

In the case of incorrect solutions, Charles took care to point out the mathematically invalid warrants. He did not accept an argument that involved a faulty warrant or a way of reasoning that was based on it. It is unclear to me as an observer whether pupils accepted this cognitively, or whether they saw Charles as the authority to decide on the validity.

At the same time, even when Charles knew Adam's solution was right, he chose to leave it without a justification. He did acknowledge in front of the class, instead, that they had not provided a proper justification but said that the method worked in that particular case and that was enough. It was uncharacteristic of Charles. I asked him for some comments after the class.

He praised the solution for its originality and commented that "some people just don't need a proof, they see it right away".

He also confirmed that Julie's solution was the most frequented in pupils' individual solutions. Adam is one of the stronger pupils in class. Perhaps in Charles's view, this problem was about how to construct a centre of a circle, and as long as pupils used the underlying concepts for its construction, he did not feel it to be important to revisit the concepts in a justification, as long as the construction was correct.

4.3.4 Characteristics of arguments: modes of representation

Karen and the textbook: pictorial representation of percent

To solve problems involving percent, Karen's textbook introduces the rectangular representation (see Figure 4.12) of the problem as part of the problem-solving process; the authors sketch out the known and unknown quantities.

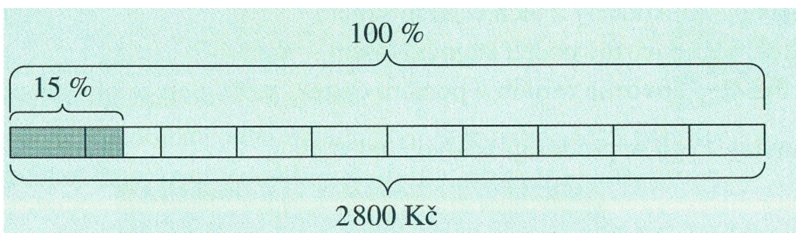


Figure 4.12: A rectangular representation of a 15% percent discount. Reproduced from Odvárko and Kadleček (2004).

Karen does let pupils work on a few problems using pictorial representations in the introductory part of the unit, when pupils match various pictorial representations (see Fig 4.3.1) to the percent and decimal representations. However, these representations (including the rectangular one) are not used by Karen at any moment in her classes when solving a problem. Her preferred representation, built into the methods she uses, are decimals, percent, and fractions (all expressed arithmetically).

Barbara – modes of representations in arguments about fractions

The representations of fractions in Barbara's classroom were varied and, apart from arithmetical symbols, included notably pictorial representations (as an example, see Figure 4.13, the representation of the stick problem described in Section 4.3.1) of the same concept, also that of a clock face, and a circle with angular measure. Pupils also worked on a few Egyptian fractions problem with a manipulative paper model of a pie. Finally, Barbara led the class to solve one problem using decimal number representation.

Figure 4.13: The gardener stick problem representations in Barbara's first lesson.

Obviously, pupils were familiar with these representations and used them in arguments or problem solutions.

All of these representations are present in the textbook material, where the concept of (operations on) fractions is embedded in various contexts. The contexts establish connections between fractions and minutes and hours on the clock dial, degree angle measurements in a circular sector, the act of dividing m circular objects into n equal parts, part of a line segment, rational numbers, ratio and (later in the book) percent. These contexts and representations influenced the solutions and the arguments produced were based on the representations relevant to the concepts.

Typically, a problem would make a connection between a particular model (such as the pie representation of bread loaves) and fractions, thus the mode of representation was determined by the task itself. Still, there were situations when Barbara suggested or requested particular modes of representations by supplying her own problems.

For example, before letting pupils work on a series of problems in a book that asked pupils to find the fractions of a clock face given by a certain number of minutes, Barbara let pupils work on the problem $\frac{1}{3}+\frac{5}{6}+\frac{2}{12}$. She then drew the pie model with twelfths on the board to elicit the solution. She did this in order to prepare pupils for working on expressing minutes on a clock face as fractions, as is apparent from this utterance:

Barbara: [Stop working now. Nikol will come to the board and mark one third]. We're going to work with a clock in a minute, so I made a clock. Mark one third.

Figure 4.14 shows then the representations drawn by one of the pupils who was working on the clock task, finding the fraction of a clock face given by minutes (notice the clock face drawing with numbers 1 to 12 and cut into six pieces). The only argument expressed publicly during this activity was the one that justified the fact that three minutes represented one twentieth of a clock face by "three times 20 is 60".

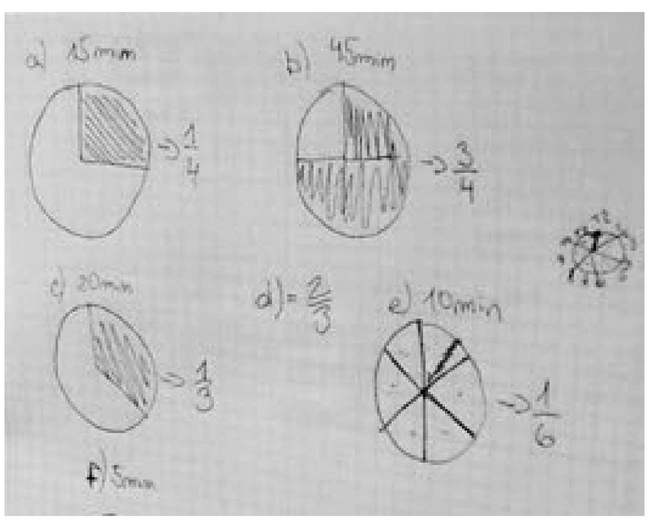


Figure 4.14: Clock face and pie representations in Barbara's pupils' solutions.

Barbara chose another representation when the class were working on a problem where they needed to show that the parts (a half, a quarter and an eighth) of a herd of eight camels were not equal to the same parts of a herd of seven camels. This time the idea of a fraction of a certain number was represented by decimals, and presented in a table on the board (see Figure 4.15).

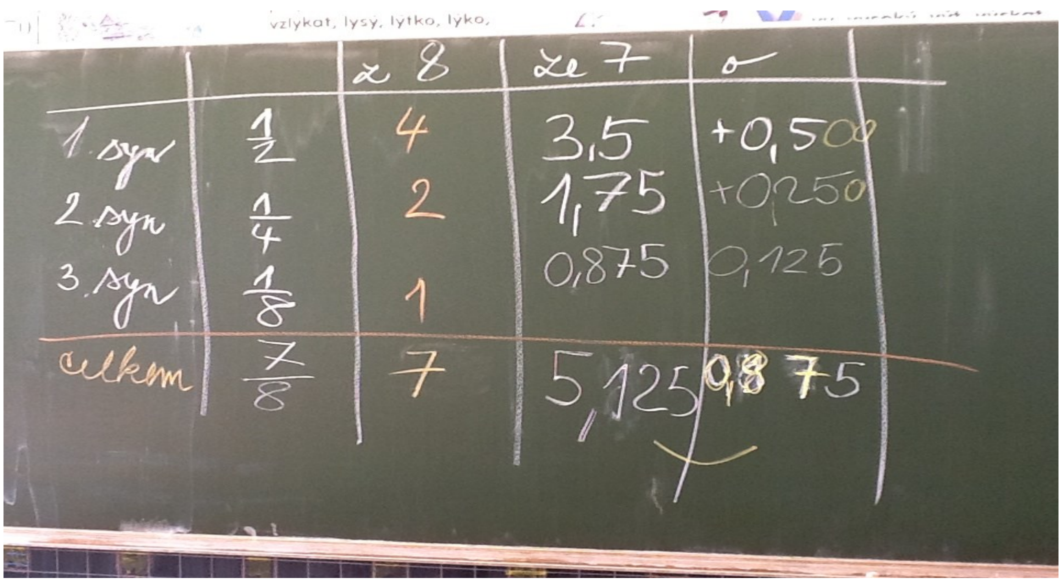


Figure 4.15: Warrants based on decimals (the camel problem).

It was not clear why Barbara chose this particular representation but there were no protests from pupils regarding this choice, suggesting that they were used to representing fractions by decimals, too.

Barbara – modes of representation and pupils' understanding

At the beginning of the first lesson, Barbara introduced the stick problem (see Section 4.3.1). The problem became the source of rich discussion that took up about fifteen minutes of the first lesson and the bigger part of the following lesson.

First, the pupils discovered that there was more than one solution because it transpired that it was possible to interpret the situation in ways that would lead to different answers. They argued for two solutions (one half or one sixth of the stick is covered by both materials), clarifying the problem's conditions, i.e., whether the gardener started winding from the top or the bottom of the stick with either material, and decide that it could have been from "anywhere". Upon Barbara's prompting, they offered one more solution (one quarter of the stick), which corresponds to the alternative problem (i.e., the wire had been cut in half and then wound from both the top and the bottom each part one third of the stick's length). Figure 4.16 captures the three visual representations used in the arguments, the first two were chosen by the pupils themselves, the representation in a one-by-twelve-square shape in a grid was prompted by Barbara (on the right, the bottom three squares represent the fourth that is wrapped in both materials).

In the next lesson, the class revisited the two original solutions drawn in the grid and when Barbara asked to find other solutions, she restricted the conditions: the material cannot be cut in more pieces. She reminded pupils that it is OK to start winding from any point. Pupils quickly came up with three more solutions by drawing them in the grid: $1/3, 1/4$ and $5/12$ (see Figure 4.16). In other words, they find all solutions in the (discrete) set of twelfths. The mode of representation itself seems to have hindered pupils from finding more solutions.

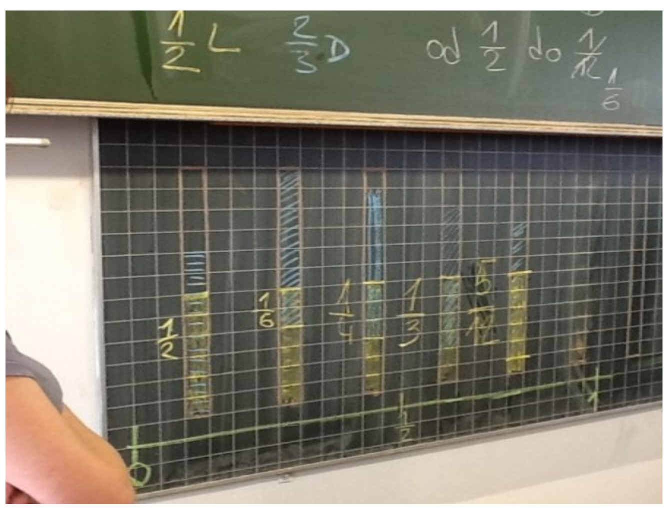


Figure 4.16: Numerical, geometrical and number line representations in the stick problem.

At that point, Lukas suggested a hypothesis: the solutions would be between $1/12$ and $1/2$. Barbara let the pupils discard $1/12$ as a solution (the warrant was that it was impossible to draw the situation in the stick model), corrected it to $1/6$, and upheld the idea of an interval, saying that it would be "from something to something". Then Barbara decided to use a new model to test the hypothesis – this time a number line, an interval $[0, 1]$ drawn horizontally and containing 24 subintervals (marked by grid lines on the board, see Figure 4.16, at the bottom of the board).

Pupils marked the found solutions on the number line one by one, and Barbara asked again if they saw another solution, pointing to the rectangular stick model solutions above. Pupils started offering unorthodox ideas (like cutting the stick, winding in one spot) but none of them realized what Barbara had in mind, i.e., that the twelfths in the stick model can be halved and further divided, in other words, that we don't need to be restricted by the pre-marked twelfths and can start winding from *any*where. Naturally, the idea of infinite divisibility and thus continuity was likely still too abstract for them at this stage. Again, the representation of the divided number line was not effective.

Barbara finally decided to move back to the non-grid board and drew another thick vertical line as a stick to represent another solution, starting to "wind" at an unmarked point just below the top of the "stick" (see Figure 4.17 on the left side of the board).

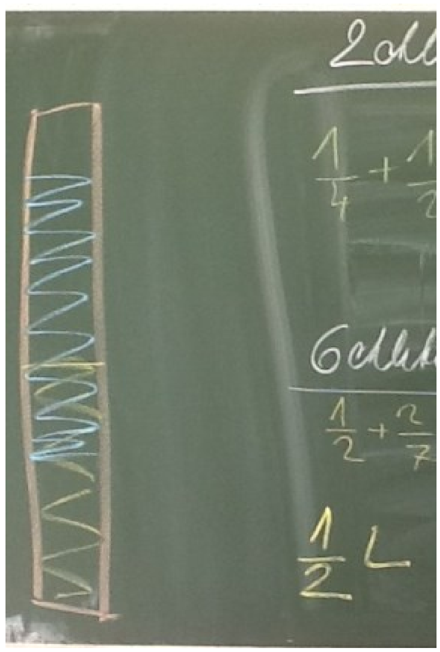


Figure 4.17: Barbara's non-grid representation of a stick (on the left).

She asked the pupils "Where did I start winding from?" – pointing at the previous solutions on the grid board. Finally, someone offered "from a half of the small square." They clarified what the solution was (someone offered $1/8$ at first), using the following sequence of warrants: it is three and a half squares, one half-square is a twenty-fourth, 7 half-squares is $7/24$. Barbara told pupils to mark $7/24$ on the number line below and asked" "Have we managed to fit within (referring to the original hypothesis of the interval)?" She got no response and specified: "Into the orange part here (pointing at the number line model). Is the hypothesis true, then?" and again got no response. It seems that pupils may not have established a straightforward connection between their stick solutions and the number line, i.e., that the mode of representation offered by the teacher was not accepted, although they (at least some) seem to have accepted the stick model warrants. When Barbara asked if they saw more possible solutions, they still referred to the squares rather than to fractions or numbers on the number line. This is apparent from the suggestions in the following discussion. Note that even Pavel, who was called upon to summarise for the class, did not use the number line data for warrants, and offered his own representation of the solution instead. The teacher then summarised his solution in her own words. Note that this does not seem to be for the whole class, as she addressed only Pavel.

Barbara: But we found another solution. Did anyone find one more? I will not want it in numbers, just whether there exist some other solutions, what you think. [Pupils suggest.] Nela? (Nela unintelligible) Hmm, and do I need to start in the middle (half) of the square?

Pupils: No (more suggestions).

Pupil: So I can start at one twenty-fourth, for example.

Barbara: Then I can … so how many solutions is there going to be?

Pupils: Lots.

Pupil: Also in one thousandth.

Pavel: From two to six.

Barbara: From two to six? From one half to one sixth, included. Simply, all numbers that I could find here? (pupils discussing) Pavel will conclude this.

Pavel: I calculated this in numbers, that it will overlap, so it will be like from 2 to 6 any number.

Barbara: You reduced the problem to superposing six squares over eight squares, is that so? … And you can move it any way you want, even by thousandths, and it will change accordingly. […] Great.

Charles – geometrical modes of representation

In Charles's case, the modes of representations were given by the text/tasks he was using: geometrical representations (either precise construction or sketch), or algebraic and numerical representations, e.g., in the case of Lesson 2, Task 1 (Figure 4.18). These were used based on the task instruction (chosen by the teacher when writing the text) or his instruction during class. For example, in the same Task 1 in Lesson 2 he instructed class to draw the situation first. Figure 4.18 shows four such problems that pupils worked on during the observed lessons.

We already saw in Section 4.3.3 that pupils used constructed or sketched representations of geometrical objects as modes of representations. There is an interesting moment when in the first task in Lesson 2, Charles's formulation of the problem includes a verification of a numerical solution using construction (i.e., a physical measurement). In Lesson 3, construction and a physical measurement is supposed to be used to state a hypothesis (involving Thales theorem).

Task 2.1 Points *A*, *B* lie on a circle $k(S;\ 5\,\text{cm})$, $AB = 8$ cm. Calculate the distance of the chord *AB* from the centre of the circle. Use construction to check the correctness of your solution.
Task 2.2 There is a circle printed on the handout. Use construction to find its centre.
Task 3.1 Draw a circle *k* (*S*; 5 cm) and its diameter *AB*. On the circle, draw a point *C*, different from points *A*, *B,* on the circle (choose a point that is different from your neighbour's). Draw triangle *ABC*. Does it have any special property? Measure it.
Task 3.2: Prove the property that we discovered in the previous task.

Figure 4.18: Problems requiring multiple representations in Charles's lessons.

In the observations, Charles leads pupils to use sketched geometrical representations as a base for warrants that justify the use of the Pythagoras theorem in the former task and then again in the general proof of Thales theorem in the following lessons. He is deliberate about the use of construction and sketches.

4.3.5 Characteristics of arguments: modes of reasoning

Karen – generalizing with a specific example

In section 4.3.2, I gave a sense of how Karen's justification of general procedures was carried out as generalization using a specific example. In the following procedure, she explains the procedures for expressing percent as a decimal, and vice versa.

Karen:	So, if we have 18% (writing on the board), how do we get a decimal?
Pupils:	Eighteen divided by 100.
Karen:	We divide by 100. Why? Because 18% is 18 hundredths (writing $18\% = 18/100 = 0.18$ on the board), to divide by a 100 means 18 hundredths. […] When we have a decimal (writing 0.278), this will be very important in word-problems.
Pupils:	27.8.
Karen:	Yes. Why? Because we can express this as a fraction with the denominator … [pupils offer: a hundred] a hundred (writing $0.278 = 27.8/100$). Is that right? And in a fraction with a denominator one hundred, the numerator tells you the number of percent. Is that clear?

Karen – problem solving methods as arguments

The methods that Karen introduced to pupils were mostly used for problem solving and Karen demands the record from pupils when they solve problems. As an illustration, the example problem was the following: "Only 4% of the pupils at our school did not participate in the Athletics Day. We also know that there were 16 pupils who did not participate. What's the total of pupils in our school?" Figure 4.19 shows a transcribed and translated version of the record on the blackboard, the first method (the "one percent method") is recorded on the left hand side. It consists in finding the quantity corresponding to one percent and then determining the quantity corresponding to the desired percent part (100% in the example here).

Karen carried out the argument for the solution (using the one percent method, see Figure 4.19, on the left) in the following way:

Calculating the base

Prob[60]: 4% of school pupils not part. in sport compet. 16 pupils not part.

How many pupils are there in the school?

Using 1%:	Using dec. no.
$4\% = 16\,p.$	
$1\% = 16 \div 4 = 4p.$	$4\%\ of\ bas. = 16$
$100\% = 400\,p.$	$0{,}04 \cdot bas. = 16$
	$bas. = 16 \div 0{,}04 \quad /\cdot 100$[61]
	$1600 \div 4 = 400$

Figure 4.19: Blackboard record of two arguments for finding the base in a word problem. Transcribed and translated from a photograph.

Pupil: 4% pupils is 16.

Karen: Yes, 4% of all pupils of the school corresponds to 16 children. And again, the number of all pupils is the … base, that will be the whole. And here I recommend and I think that it will be most efficient to calculate using …

Pupils: 1%

Karen: 1% (writing on board „Using 1%“, she writes out the record, see Figure 4.19). Using one percent is…, Klara?

Klara: ...

Karen: One percent is how many times less?

Klara: Four times.

Karen: Yes, that is four pupils and the base or the whole is …

Pupil: 100%

Karen: 100%. How many is 100%?

Pupil: 400.

Karen: So we can see the answer right away and that is …?

Pupil: The school has 400 pupils.

Karen: Excellent, and we are going to write the answer out.

[60] Karen shortened words in this manner on the board.

[61] This denotes expansion by 100 and was aparenty an established practice in the classroom.

Karen – using estimation to evaluate correctness

Karen emphasised the method of estimation when she was letting pupils check their answers. As with other methods, she exemplified this method on various occasions and then used it as a prompt for pupil self-correction.

For example, in the following situation pupils were working out the original price of a blouse that was twice discounted and now cost 320 Czech crowns.

Pupil:	Teacher, I got two thousand.
Karen:	So if a blouse cost two thousand (emphasises), they put a 20% discount on it, that's a fifth, and then again another fifth, do you think we would get to 320?
Pupil:	Well, maybe [not] …
Karen:	Try to do this in reverse, if it works out. Tom, how much did you get?
Tom:	3 000.
Teacher:	3 000.
	(pupils suggesting other answers)
Karen:	So, if you discounted by one fifth, that is how much, a fifth is…
Tom:	A fifth is …
Karen:	… is 600? Right? So you have 2 400, then you discount again a fifth, which is less than 600. Are you going to get to 320?

In the textbook, the authors recommend estimation of an expected answer to a problem: the problem posed in the book is accompanied in the teacher's book by the suggestion that pupils should estimate before they try to solve and then verify their estimation by calculation/solving. This is persistent across topics and the authors' pedagogical rationale is the need for a "preliminary analysis[62] of the problem and forming a general idea of what the results may be." (Odvárko & Kadleček, 1999, p. 8). The authors put emphasis on estimation as a way of gaining insight into the given problem. On the other hand, Karen shows pupils how to use this technique *a posteriori*, to be able to detect a faulty answer independently. Again, we see that in Karen's utilitarian perception, the goal is to get a correct answer, which includes being able to reflect and detect an answer that does not make sense.

[62] In Czech, "Zamyšlení se".

Barbara – generalizing with many specific examples

As I discussed in Section 4.3.2, generalisation did not happen formally in Barbara's class. At the same time, Barbara is convinced that by solving the specific sequences of problems, pupils are creating their ways of understanding the underlying concepts and are able to solve problems involving operations with fractions and provide justifications.

Charles – making and proving a hypothesis, reverse implication, and proof by contradiction

As we saw in Section 4.3.4, Charles let his pupils discover and prove Thales theorem. He first made them experimentally hypothesise (Problem 3.1 in Figure 4.18) that angles subtended by the diameter from any point of the circle (except for the endpoints of the diameter) are right angles, and then prove it (Task 3.2 in Figure 4.18), any way they wanted to. Finally, he used a task (Task 3.3 in Figure 4.20) to show that the reversed implication is also true.

Task 3.3 Draw a circle *k* (*S*;6 cm) and a line segment *AB* which is its diameter. Outside the circle, find a point *D* so that the triangle *ABD* is right-angled, with the right angle at vertex *D*.

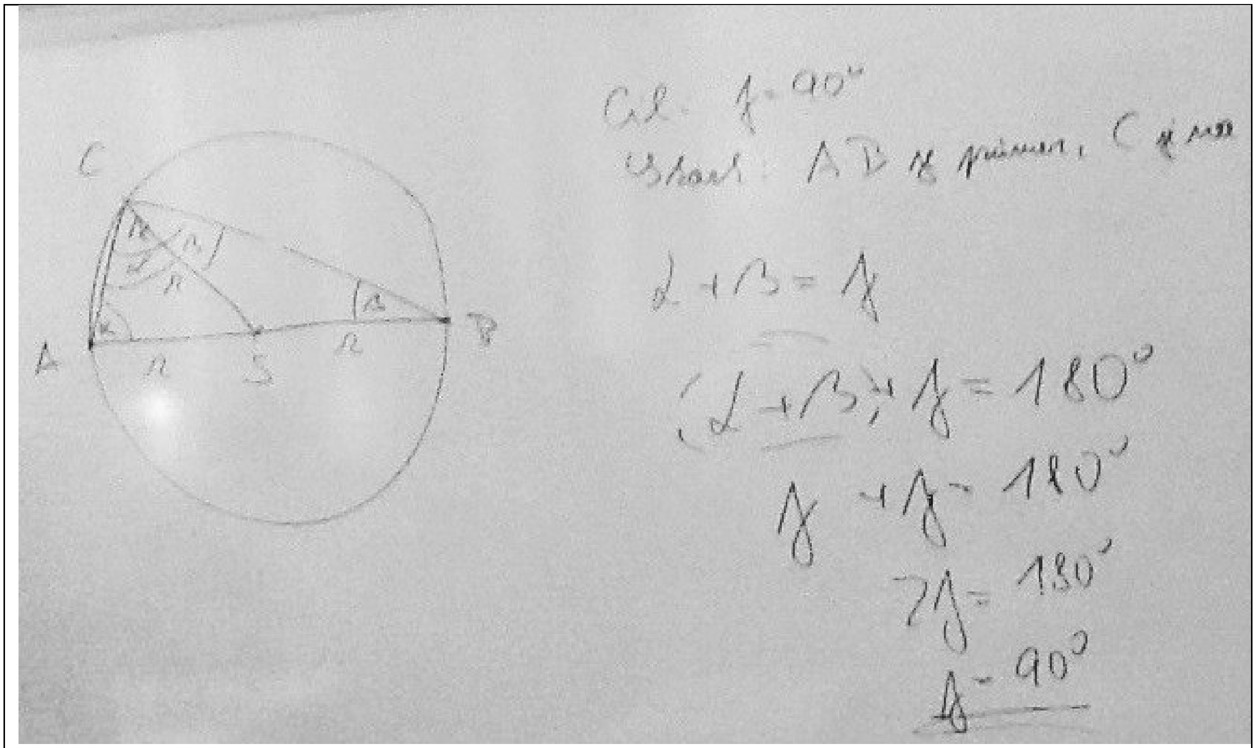
Figure 4.20: Problem 3.3 in Charles's lesson about Thales theorem.

This lesson was conducted twice for two parts of the class separately (the female group first, the male group afterwards). In both lessons, Charles guided the proof at the board (Figure 4.21), eliciting ideas and stopping after completing the sketch (Figure 4.21 on the left) to let pupils finish the proof individually. In both lessons (male and female groups), the pupils struggled with the idea of not using the hypothesis (the claim) as a warrant in their proof, i.e., they were struggling with this particular mode of reasoning (proving a hypothesis). The following two conversations between a pupil (first female, then male) and Charles illustrates this struggle.

Charles: Well, so, this does not have to be 45 and 45 [degrees] (referring to angles alpha and beta). That is definitely not true. It doesn't even look like it is 45 and 45 here. That is not true. You see? The only thing we know for sure is that the two angels together give you gamma.

Pupil 1: Ah, ok.

Charles: Now either. We don't even know that gamma is 90, that's something we need to get to, you cannot use that [as a fact] along the way.



Top right text translation: “Goal: $\gamma = 90^{o}$; Start: *AB* is a diameter, *C* is on the circle”

Figure 4.21: Proving Thales theorem in Charles’s lessons.

Pupil 2:	These are the same, these are the same (referring to the pairs of congruent angles in the triangles), and this together (referring to angle gamma) is 90.
Charles:	You don’t know that, you need to prove that.

Charles emphasised the structure of the reasoning: “I must start with something. I must end with something. I need to get the most information from the starting data.”

Charles used the task following the first proof (Task 3.3 in Figure 4.20) to show that the reversed implication is also true. First, he let them find out that there is no solution to Task 3.3. Again, he made them place the solution in their drawings and they kept finding that the point they are looking for is located on the circle. All of this was experimental. Then Charles asked: “Do you think it is a coincidence?” and set a task to prove that there are no such points, no other points in the plane that have the quality of forming a right angle with the end points of a diameter. Here he felt the need to clarify the difference between the two statements:

Charles: We said earlier that if this is the diameter, and a point on the circle, then it has to be 90 degrees. But that does not mean that it could not be somewhere else, right? Do you see? We did not exclude for it to be 90 degrees here (pointing probably outside the circle), do you understand me? That is a different thing. Do you feel that it is not the same when we said … we said if it is here [probably point at the circle], it will be 90 degrees. And now I asked you to make 90 degrees somewhere else and you are telling me it is not possible. But we do not know that, yet, why it is so, right?

Again, Charles shows first the case of a point being outside the circle (starting to perform a proof by contradiction – Figure 4.22).

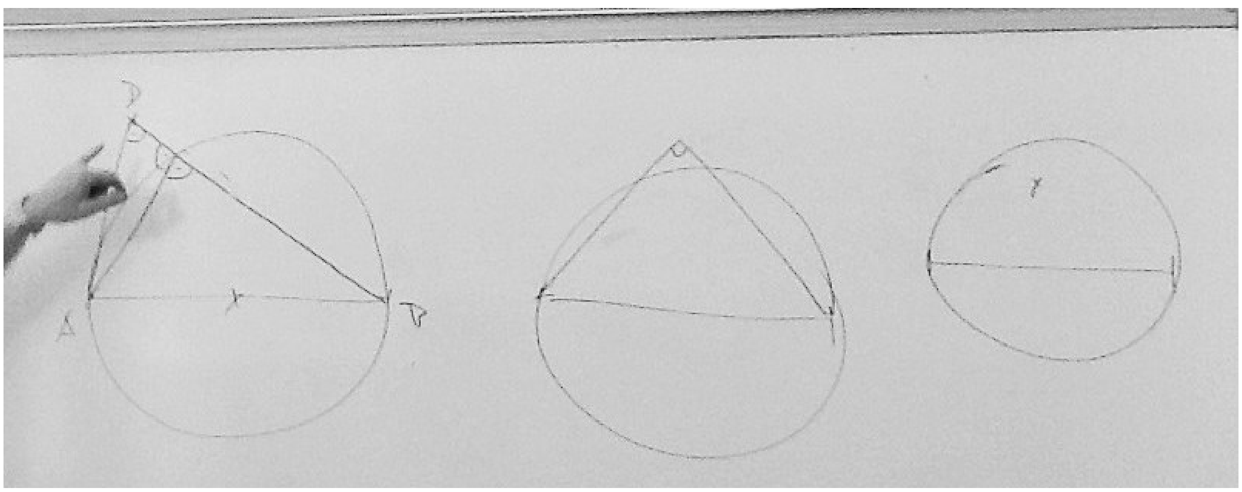


Figure 4.22: Proof by contradiction in Charles's class.

He asks the class to prove the same for a point inside the circle. He lets them work on this individually and then consolidates at the board:

Charles: So, not only if the point *C* is on the circle the angle is right, but now we also just found out what? What have we just shown? A moment ago, at the beginning of the lesson, we showed that if the *C* is on the circle, the angle there is a right angle. And now we proved what?

Pupils: That the right angle cannot be anywhere else.

Charles: That the right angle cannot be anywhere else but the circle. Is that so? Yes? And the whole thing is called Thales theorem, right? You will need to probably write this down in your notebooks. I have it written here (on the projector), so let's see if you can recognize it there.

In the interview after the lessons, Charles admitted that he had expected the struggle the pupils went through, and that this mode of reasoning – proving a general claim, was something that pupils were not familiar with and that this was the second time (or third) they experienced this type of reasoning (proving a hypothesis). Recall also from Study 2A (Section 3.4.1) that Charles distinguished between justifying a "simple statement" and "proving something from the beginning". Charles believes it is important to show pupils these ways of thinking, and although he does not expect them to know how to perform this proof (which would become likely a memorised procedure, something he is opposed to in mathematics).

4.4 Discussion of Study 2 results – the why of enacted arguments

For Study 2, I chose three teachers with different orientations. Having identified the teachers' beliefs and orientations in the preceding study (Study 2A), I looked at what practices they conducted and how arguments were put forth in their classrooms. I was able to observe both repeated practices as well as episodes when teachers dealt with the "unforeseen" (Schoenfeld, 2010, p. 13) characteristics of arguments. Ultimately, my goal was to find out what exactly might have determined the occurrence of these arguments and, especially, the decision teachers made about them. I observed the following routine practices or social norms in each of the classrooms.

Karen provided pupils with example arguments, general methods for problem solving, that she justified (except for one method). Pupils then used these methods in their problem solving and argued or explained their solutions using the steps of the methods. These methods were also used when correcting pupils' errors.

Barbara provided pupils with problems she believed they could solve individually or in groups. She also offered clues (warrants) when pupils as a class struggled. She elicited arguments (or they were offered spontaneously by pupils) especially after the class reconvened and both correct and incorrect solutions were displayed. Problems sometimes allowed for multiple solutions or interpretations and this also prompted arguments. Majority of arguments or warrants were supplied by the pupils. Barbara resorted to consolidating herself (or through a Socratic dialogue) on very few occasions, when she seemed to have the need to clarify a pupils' correct solution to others.

Charles provided pupils with problems and tasks he believed they would be able to solve individually, and let them try. He monitored and offered clues (warrants) when pupils struggled, sometimes reconvening as a class, but he expected them to finish the problem individually. He would let pupils display their solutions if he noticed they varied, or consolidate the solution himself for the whole class.

Because of the distinct participatory relationship the teachers had with their textbook material (driven, again, by the teachers' professional orientations), I was able to see varying degrees of adaptations of the text itself.

Because of the social norms established by teachers' orientations about mathematics education and its goals, I observed a varying extent of influence pupils had on the observed arguments in each of the three classrooms. Looking for episodes involving what Schoenfeld calls "unplanned" occurrences, I identified the following phenomena that had the potential to affect arguments in the classrooms.

4.4.1 Pupils' original (unexpected) solutions

Firstly, I noticed how individual pupils' arguments or solutions that were original or unexpected were accepted by the teachers, and in what way were they further utilized, if at all.

Karen allowed pupils to provide arguments that she had not intended to take place and accepted them as long as they were mathematically correct. At the same time, she manipulated such publicly expressed arguments according to her perception of accessibility to all pupils and made frequent evaluative comments about the methods and arguments, labelling them as efficient, common practice, convenient, easier, or universal.

This qualitative evaluation springs from her beliefs about her pupils' mathematical ability and what it means to be good in mathematics: in her view, some pupils are better at understanding the problem, and innately capable of finding and choosing the most efficient, original, or convenient method, an attribute she also gives mathematicians in general. For the others, she needs to show simply which method to use, and they need to learn it by solving many similar problems, i.e., for some pupils drilling is the only path to succeeding in mathematics. The episodes seemed to confirm that this belief corresponded with the pupils' contributions: the weaker pupils would rely on arguments promoted by Karen, while pupils who felt confident in their own warrants, could keep using their own.

Ambiguity and different (valid) interpretation of data or problems with multiple correct answers, is also something Karen wants to avoid, as we saw in the episode with problem about learning languages (Section 4.3.3). Again, we can assign this decision-making to Karen's perception of the class as a whole, who she believes might find such ambiguity confusing. After all, in her view, the goal is to get the answer right, which is something they had, as a whole class, done. The other aspect is the value Karen places on time, which leaves no resources (class time) for unplanned explorations. This particular episode is in stark contrast with what happened with a mathematically analogical problem (the farmer's stick problem) in Barbara's classroom.

In Barbara's classroom, I witnessed that pupils' arguments were not only the source of an immediate discussion, but they would become a vehicle for future activities and problem solving. So, in her class, a pupils' argument about their initial solution of the farmer's stick problem made Barbara revisit the problem, exploring the suggested interpretation, in the following lesson. Thanks to the pupil's contribution, Barbara saw that the problem in this interpretation would provide her class with more opportunities for sense making and discussion, something that she expressed as her core teaching values.

In a different scenario, Barbara publicly restated an argument that was original or used an original representation, such as Pavel's argument about the set of solutions of the farmer's stick problem, acknowledging its validity. Even though she restated it, she did not offer its further explanation or discussion, respecting the social norms of the class. She reformulated the argument as another pupil's idea, not a method that she needed pupils to learn, and did not make any qualitative remarks about it in respect to the other's arguments. This seems to be in alignment with Barbara's belief in pupils' construction of their own knowledge and their ownership of such.

I also observed that Barbara made a decision to offer pupils more problems with Egyptian fractions when they were not making the connection she wanted them to make, and gave them thus more opportunities to notice the desired warrant.

In Charles's class, pupils' arguments and solutions during the individual problem solving activity would inform the teacher about any need for supplying additional warrants to the whole class. When he did so, he followed the class norms – he would supply some but not all of the warrants, true to his belief that pupils cannot be "given" or "served" mathematics.

There is also an interesting effect that pupils' arguments have had on Charles's beliefs. This became especially salient in the lesson on Thales theorem that was taught in two different groups, first with female pupils and then with male pupils. In the male group, several pupils started offering solutions to finding a proof immediately after the property of the triangle was hypothesized and Charles clarified the starting point and end-point of the proof. There were a few solutions with erroneous modes of reasoning (as we saw in Section 4.3.5) and one argument was not complete but in terms of attempted warrants, the male group definitely confirmed Charles's expressed belief about male and female pupils as problem-solvers. Nonetheless, the two lessons themselves were conducted following the same general outline, including the same warrants in the proofs and opportunities for pupils' individual work on them. I could observe no difference in the routine practice regarding argumentation.

I observed two arguments that were original and publicly validated by Charles. In both cases, the teacher acknowledged the solutions, or the partial argument. In Adam's solution (see Section 4.3.3), he sanctified the solution as correct and valid, although he recognized the shortcomings of the incomplete argument (why the method would always work). What was important for him was that it worked for this particular pupil, i.e., Charles believed that this pupil had an understanding of the underlying concepts. This faith likely sprang from Charles's perception of Adam as a stronger pupil. Adam's internal conviction was, thus, enough. There is a connecting theme with Charles speaking about his pupils who participate in math competitions. The competitions require pupils to present their reasoning or explain their solutions and Charles knows that his pupils in particular have a difficulty with this part of the process. One possible interpretation of this situation is that Charles's own respect of the (strong) pupil's internal conviction as displayed in the observed class and commented on afterwards (that "they just see it"), helps to cultivate this phenomenon.

The other argument that appeared as original (i.e., not following the same line of reasoning as the whole class) was an alternative proof of Thales theorem that was recorded during the individual solving stage. The teacher acknowledged that the direction the pupil was headed was promising but asked the pupil to abandon it for the moment and join the class in working on the proof Charles had prepared. From Charles's apologetic tone, it was clear that he was making a difficult decision, and he indicated that time was a factor. Still, he offered to return to the pupils' idea "if there is time" later (but there was no time). Obviously, given his approach to teaching practice, Charles has to balance carefully individual and whole class time, and needs to make decisions that he deems best for the class as a whole.

The responsibility for the class as a whole seems to be an important factor that limits teachers' willingness to display and explore the path further. Looking at this from Schoenfeld's (2010) perspective, the teachers' goals (for the lesson, for the whole class, etc.) are what determines how they use the time.

When a pupil was right and correct in their original argument or warrant, all teachers validated it in some way, recognizing the belief that pupils have their own way of seeing mathematical objects and relationships. It seems, however, that Charles and Karen assign this quality only to the pupils they perceive as strong. Both Barbara and Charles value such originality above other qualities, while Karen acknowledges it as a natural phenomenon and feels her responsibility towards the whole class, which she believes will succeed by conforming to the way mathematics is done in the curriculum intended by her (and ultimately by standard assessment). In the case of Charles and Barbara, we saw that their intended curriculum changed based on some pupils' unexpected arguments. It seems that we can link this decision to the two teachers' belief that in order to understand mathematics, pupils need to be given opportunities for (individual or collective) sense making.

4.4.2 Pupils' errors

A different situation when pupils' contributions had influence the arguments in all three classrooms was that of providing an invalid or incorrect answer, or a faulty argument. The three teachers had different way of reacting to these.

Recall that one of the functions Karen assigns argumentation is getting feedback from pupils about their understanding of the problem and then being able to guide them to see "for themselves that it can't be right." This happened in her lessons with various levels of participation from the pupils. Typically, Karen took over and let the pupil or the class discover the correct answer, leading them through the correct argument (as we saw in Section 4.3.1) or she prompted the method of estimation (Section 4.3.5), which she had previously exemplified. The "why not" arguments were either following the prescribed problem-solving procedure or arguments using the (prescribed) estimation method (working backward from an answer). In alignment with her beliefs and teaching practice, her error correction practice exposed pupils to simply more examples of the methods for problem solving (or verification) she had shown, preserving the same representations and warrants.

In Barbara's classroom, incorrect answers formed a basis for a discussion among pupils and pupils were asked to evaluate their correctness. On various occasions, pupils explained the "why not", spontaneously or prompted by the teacher. On all such occasions, pupils identified the faulty warrant (or data they started solving from). Of course, it was ultimately Barbara who decided on the correction but the social norm of the classroom expected pupils to find their (or their classmates') erroneous answers or reasoning. It is Barbara's responsibility to facilitate this process. Recall that Barbara believes that each pupil gradually forms their own world of mathematical concepts and representations, even as this formation takes place in a community of 15 other pupils and a teacher. The "why not" arguments have both the effect of practicing self-reflection and for Barbara (and the class) to access information about this private world.

Finally, I observed that Charles monitored pupils during their individual problem-solving activity and told them when something was not correct. He provided the arguments "why not" himself – referring to the data in the problem (see, e.g., Section 4.3.1), or arguing about an incorrect warrant (as in Section 4.3.3), or about the mode of reasoning (as in Section 4.3.5). In his pedagogical notes in the textbook, he provides common erroneous solutions or misconceptions. For example, when pupils are supposed to prove that there is no point outside the circle that forms a right triangle with a diameter as its hypotenuse (Task 3.3 in Figure 4.20), Charles writes this pedagogical note in the textbook document:

> Pupils start protesting that to find a point outside is not possible because that is what we proved in the previous problem. I point out to them that this is not true, that we proved that if the point *C* is on the circle, there is a right angle, but that by far does not mean that a point outside the circle could not have a right angle (I give them an example with the class: the fact that all pupils in the classroom are seventh-graders does not mean that all pupils outside the classroom are not (there is always someone absent)). If they think that point *D* does not exist, they need to prove it). (Charles, online, translated).

Charles's beliefs about the role of pupils' individual engagement in problem solving, did not allow him to provide a full argument to correct the error, although he would supply warrants when he felt pupils were "stuck". In the observed lessons, Charles was the clear authority on correctness, but he referred to the mathematics to back his verdict. Pupils' incorrect arguments further seemed to broaden his pedagogical knowledge and he adjusted his text and possibly the lesson script to provide suggestions in the form of arguments or counter-arguments, or a sequence of warrants).

The same is actually true about some of the original solutions. For example, in the pedagogical commentary, Charles includes this in the notes about the problem of finding a circle's centre:

> Other methods come up [in the pupils' solutions] – for example, we can choose three points on the circle and find the centre of a circumscribed circle. If this method comes up[63], we compare it to the method used in the [textbook] solution and come to a conclusion that both methods are actually identical in its core (in both cases, we look for axes of line segments, we choose three points, …). (Charles, online, translated)

The text clearly reflects Charles's experience with pupils although it does not include any of the original arguments explored or started in my lesson observations. There must be another mechanism that Charles chooses for selecting the arguments for his pedagogical commentary.

Both phenomena (pupils' original solutions and pupils' errors) are potential triggers for what Schoenfeld (2010) calls unplanned or impromptu excursions. The participants' orientations, goals

[63] This method did not come up during the observed lesson in my investigation.

and resources, especially knowledge, can explain the decision that the participants made in each case. It is likely that Charles and Barbara, teaching their topics for the first time at that school level, were simultaneously extending their pedagogical content knowledge through the unexpected (as we saw in Charles's textbook pedagogical commentary, or when Barbara resumed exploration of the stick problem to include the notion of an infinite number of solutions). Meanwhile, Karen, who had had more opportunities for teaching the topic of percent previously, displayed pedagogical content knowledge that had already solidified.

4.4.3 Textbook influence – justification in text

Karen's case showed us best how a mainstream textbook curriculum can influence arguments in a class. Her beliefs about efficiency determined the modes of representation as well as the choice of methods (warrants) for problem solving and the lack of justification for a shown method.

Karen acted as the decision-maker in choosing what representations were useful in warrants, i.e., efficient, for her class (see Section 4.3.1). The question remains what her choice not to include the textbook's geometrical representation in reasoning was based on. In the authors' view expressed in the teacher's manual, the geometrical representation helps pupils to get a better insight into the problem. This belief about a need to understand the problem through the use of a geometrical (or pictorial) representation seems to collide with Karen's beliefs about what is important for her pupils. Rather, she values efficiency and straightforwardness in problem solving. The utility of such representation is also lost to her, as the problems can be solved without it, using her method, and the textbook (or her supplementary materials) does not further provide problems where such representation is necessary.

It seems that the textbook does not give the representation a utilitarian value, which is very important for Karen. It does not provide opportunities for its direct use in problem solving, in fact, there are no problems that involve this representation directly and thus, it is worthless in Karen's eyes. The authors claim in the teacher's manual that the representation provides pupils with a better insight into the problem, but Karen clearly does not find this convincing enough. Therefore, three factors influenced her decision: a) Karen's pedagogical content belief about the efficiency of a certain type of arguments (methods) and b) the problems (namely, opportunities for arguments rather than argument forms themselves) presented by the textbook authors in the unit, which is key because of c) Karen's utilitarian view of the goals of mathematics education, which mostly consist in being able to solve problems provided by the curriculum.

Another insight comes from the case of the rule-of-three method (Section 4.3.2). The method that Karen presented when she introduced the procedure for finding the percent in a word problem is based on the ratio warrant. This warrant also underlies the rule-of-three method. Karen felt that the justification would lead to an unnecessary excursion (given the present knowledge base of her pupils) and thus, she backed the procedure up with her own authority. We saw in Study 2A that Karen deemed justification of general mathematical statements (including methods) important. On the other hand, she conceded that not all such justifications are accessible to all pupils. Even the textbook authors acknowledge that their warrant used in the justification of the rule-of-three method is outside pupils' immediate knowledge. These are similar narratives in the textbook author’s and Karen’s beliefs. So, in this case, the determining factors were a) Karen's perception of her pupils' abilities and knowledge as well as the belief that b) not justifying is sometimes necessary[64].

Ultimately, it seems that it was the textbook and the tasks in it that influenced Karen’s decision about justification, arguments and warrants.

4.4.4 Textbook influence – the tasks

In Barbara’s case, the textbook material about fractions did not contain any arguments (except for the initial problem about Egyptian fractions, see Figure 4.23). Barbara added problems (warm-up fraction addition, the farmers’ stick problem and the camel problem, as well as the discussion about simplifying fractions) on her own initiative. The arguments were wholly in the teacher’s and pupils’ control: as we saw, though, the representations, the modes of reasoning and the warrants themselves were chosen from the palette embedded in the textbook’s problems. When we look closely at Barbara’s supplementary problems, they can be viewed as preparatory

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[64] It is arguable whether it was Karen’s lack of content knowledge that caused her not justifying the method. I have no evidence of this and having observed her meticulousness in the case of the other two methods and mathematical rigour throughout the lessons, I lean towards discarding the notion.

problems for those that are included later in the book. For example, Barbara added a preparatory problem for the clock-dial problem series in the book. The stick problem is also working with arguments that can be utilised in problems that the book states later, such as "15% of a stick is painted white, $5/12$ yellow and the remaining $2.6\ \mathrm{m}$ red. How long is the stick?" (Barbara, pilot, translated). The solutions and arguments raised in the solution of this problem can be later utilised in solving these. But what motivated Barbara to give this problem to the pupils and allocate time to explore the idea of (infinitely) many solutions was her belief in the richness and the potential for pupils' engagement with concepts and the pictorial representations of (operations on) fractions. This scaffolding (grading, building on warrants from simpler problems to be utilized in more complex ones) of problems is also common in the text.

The camel problem also involves a discrete model for fractions, which is something that the text does not provide. Barbara utilizes her pedagogical content knowledge to select this problem to expose pupils to this model, even if the book does not.

What we see is that both Barbara's pedagogical content knowledge and her pedagogical knowledge match the piloting text's nature and content. For example, Barbara's textbook also provides options for multiple solutions, contains no general mathematical rules, and provides no examples of arguments.

4.4.5 Discussion of results in other studies

In this section, I will attend to phenomena pointed out by previous studies of argumentation-related practices (see Section 3.1).

Like the teachers observed in Bieda (2010), the three teachers in my Study 2B attended to the mathematics immediately connected with the task, and the pupils' conceptual understanding, less so to broader and meta-skills, or social aspects of argumentation. On the other hand, Barbara and Charles seemed to be aware of the difference between building ways of knowing and building ways of thinking (Harel & Sowder, 2005).

The communicative function of argumentation was important for all three teachers, but there were three different audiences and goals for such argumentation: a) communication for the benefit of the teacher (Karen), b) communication for others to advance the class knowledge as well as individual knowledge (Barbara), and c) expanding on modes of thinking in Charles's case.

When Charles did proof with his class, he was motivated by two goals: a) showing the way of thinking (which he admits some may get and some not, at this stage) and b) showing that nothing comes out of the blue in mathematics. He also used this activity to deepen conceptual understanding (e.g., the relationships between angle sizes in a tringle). In Karen's case, justification was motivated by the second goal, while using conceptual understanding and making connections (but because she does it quickly and herself, that may only be accessible to some pupils). Barbara's tasks were motivated by deepening the pupils' conceptual understanding.

Did the teachers place different expectations on their pupils, regarding the representations in arguments, as reported in Nardi, Biza, and Zachariades (2011)? This phenomenon did not show in any of my data. As long as the solution was correct, both Barbara and Charles would accept it. Karen was also accepting alternative arguments but she qualified them as non-standard and she also requested a written record of the solution steps in a set way.

I interpret Karen's demand for the orderliness of a record as a token of discipline, which each pupil should have (this also relates to the communicating of an argument to an external audience in the paragraph below). In contrast, Barbara and Charles value an original solution over discipline. In Charles's case, I noted that the teacher matched this orderliness with the linear (more conforming) thinking of his female pupils. Both Barbara and Charles admit that this communication to an external audience is a skill separate from problem solving (while Barbara understands the need for the pupils to be learning this as any other skill, Charles seems to be simply stating that they do not have it, as if wondering why). Both speak about it in the context of the stronger (the original solutions) pupil cases, which resonates with Bieda's observation that teachers perceive that pupils' explanations of their reasoning process as impressive and only for pupils who are developmentally ready (see also Section 3.4.2).

The difference between higher order thinking and the communication whereof may also explain why (unlike in Staples et al., 2012) I did not observe that Barbara, Charles, or Karen would engage the more able pupils in justification more often than others. At the same time, I might not have had enough data to observe these differences. In fact, in Study 2A, we saw the interesting result (on surface counter but in reality complementary to Staples et al.'s (2012) findings): we saw that Karen would choose to let weaker pupils justify (easy arguments) to help their confidence. We also witnessed Zack express that letting stronger pupils justify is counterproductive because their argument is incomprehensible to the rest of the class. We witnessed Barbara valuing Pavel's solution and rephrasing it, so that it, in her view, became an argument that was more comprehensible (where Zack would not have asked for the explanation at all). To Victor, being able to justify or explain is inherent to understanding (and the ultimate assessment of understanding).

5 Conclusion

In this dissertation work, I have allowed myself to take an excursion into the Czech mathematics education, exploring the what, how and why of mathematical argumentation, justification and arguments. I started the journey on the publicly accepted context of the national curricula and textbooks and proceeded to individual teachers' professional worlds in order to finally gain insight into actual arguments in three authentic classroom communities.

My original motivation came from the need to make sense of an experience as a teacher in a foreign (literally) cultural context of mathematics education. It probably did not escape the reader that, consequently, it was the teacher who was the central figure in my investigation while it is the national context that matters to me while explaining the observed phenomena. In the following text, I want to summarize my findings, elaborating on a few themes that appeared strong and relevant.

5.1 General mathematical statements and their justification

Recall from the beginning of this text that a particular conversation with my Chicago student, James, made me interested in the explanation of what I perceived as a rule of non-division by zero. This experience further inspired my curiosity about justification of mathematical rules in the context of Czech education. What were my findings?

Top-down, the national curricular document (the FEP) declares that mathematics education "places an emphasis on a thorough understanding of basic ways of thinking, mathematical concepts and their mutual interaction" (MEYS, 2007, p. 28) but leaves much to imagination as to what this embodies. The document does not specifically mention the practice of content discovery or justification in our sense of the term, etc., although it does specify the understanding of the meaning (the real life model) and algorithmic meaning (why algorithms are carried out the way they are) of arithmetical operations as well as of the metrics of geometrical shapes and solids, as a desirable outcome.

As we saw in literature and in Study 1, textbook authors in different countries vary in their ways of justifying (or not), suggesting that the collectively owned mathematics education knowledge[65] is at least partly unique to communities (such as countries). In the Czech Republic, it seems that authors have the need to show why things in mathematics work the way they do. On the other hand, it is hard to discern what this justification accomplishes. Just like teachers' endeavours (as reported in Rendl et al. (2013)), to justify and/or make mathematics rules more accessible by building on less abstract concepts (analogies), thus rendering it more memorable, the textbooks, in general, do not use models systematically to develop pupils' ability to use them for solving problems, but rather, to illustrate or provide meaning of a concept/algorithm on a one-time, isolated, basis.

My investigation among teachers confirmed Rendl et al.'s (2013) observation that Czech teachers also believe that they should show that in mathematics, nothing comes "out of nowhere" and the participants felt strongly it is one of the most defining aspects of mathematics. Still, as we saw in Study 2, this belief itself can coexist within diverse mathematics education orientations and practices (recall the teachers in Study 2B). The mainstream teachers in Study 2A (Jenny, Karen, Victor and Zack) admitted that not everything needed to be or could be justified. So even if they do share this notion of non-arbitrary mathematics with their pupils while justifying general rules or methods, they must be doing so with the knowledge that there will be situations when they will not be able to justify (as we saw in Karen's case) and will need to admit that mathematics can also be a toolbox of tools that are "black boxes".

So, is justification of general truths even relevant? Of course, it is only relevant within a curriculum where general truths and rules are present. Because general statements and truths, or methods, are part of the curricula or at least its language, the teachers feel that the justification is relevant. They believe that deducing a rule is helpful either from the point of conceptual understanding (or, as it seems, the connection to real-world models and situations), or of retention, as well as a representation of a way of thinking (or a mode of reasoning, likely a

[65] If we view textbook and published materials as a type of collective knowledge (content or pedagogical content).

generalization). As we saw in Barbara's case (or at least as the data allowed us to see), if sense making is the ultimate goal and problem solving only a vehicle to making meaning, generalization does not have to be formalized into a statement or method.

5.2 Argumentation in problem solving: sense making and efficiency

What became clear during Study 2 was that the teachers' orientations towards problem solving and their beliefs about the goals of their teaching (or learning mathematics) clearly determined the arguments that took place in the classroom. We had noted that the national curriculum allows for practices that are varied but it does not provide guidance to alternative approaches. Nor does it provide guidance for teaching practices (such as example outcomes) regarding argumentation and arguments in mathematics. The influence of textbooks as teachers' resources and collectively accepted knowledge is then paramount. In fact, it has been reported in research (Son & Senk, 2010) that not only do textbooks affect what happens in the classroom but also have the following function:

> Textbooks serve as intermediaries in turning intentions into reality. An examination of textbooks informs policy makers of how societal visions and educational objectives seen in national policies and official documents as the intended curriculum are potentially embodied in classrooms. (p. 118)

It seems that the Czech textbooks' perpetuation of the Old Humanist or Industrial Trainer (Ernest, 1991) traditions goes hand in hand with some of the language of the FEP (precise language, use of tested methods, efficiency).

On the other hand, we observed that even the authors attempt at promoting declaredly sense-making representations and methods. The authors of Karen's textbook promote the idea of pupils' getting an insight from representations (such as a rectangular model) or from using the rule-of-three method. But we saw that this was thwarted in Karen's classroom. Her own experience and experience-formed and experience-tested convictions, her resistance to or ignorance of the textbook author's recommendation, which the book did not validate (in her sense of efficiency) by clear applications (e.g., problems to be solved) or valuable outcomes (such as more efficient problem solving), made her leave these representations and methods out. Thus, the textbook's prescribed way of forming "preliminary analysis[66] of the problem and forming a general idea of what the results may be" (Odvárko & Kadleček, 1999, p. 8) in itself, and without broader implementation, is not convincing enough for teachers with utilitarian orientations.

As Charles so poignantly expressed in our interviews: textbooks justify or even provide problems to justify but the fundamental premise is wrong (i.e., discovery and challenging problem solving activity need to be at the heart of the selected activities). Thus, what I chose to observe in Study 2B were also argumentation practices belonging to two teachers (Barbara and Charles) who broke away from the collectively known (textbook) curriculum.

What stood out about these teachers was the fact that they clearly prioritised individual sense making over efficient problem solving (use of methods), and put emphasis on the effort each individual makes in doing mathematics, that is in the thinking and reasoning rather than in solving a number of problems. In his seminal work, Schoenfeld (1992) says about sense making in mathematics education:

> If one hopes for students to achieve the goals specified here -- in particular, to develop the appropriate mathematical habits and dispositions of *interpretation and sense-making* as well as the *appropriately mathematical modes of thought* -- then the communities of practice in which they learn mathematics must reflect and support those ways of thinking. That is, classrooms must be communities in which mathematical sense-making, of the kind we hope to have students develop, is practiced. (p. 34, emphasis added)

In Study 2A, I interviewed and observed both teachers who are fine with the system and make it work best they can (Karen, Victor, Zack, Jenny) and others (Barbara and Charles) who believe in the necessity and viability of sense making and problem solving in Schoenfeld's sense and had resorted to writing (or co-writing) their own textbooks to support their teaching. In that sense, their teaching approaches and the selection of tasks (problems) satisfied Schoenfeld's necessary condition (and the teachers' choices were deliberate).

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[66] In Czech, "Zamyšlení se".

Interestingly, Charles conforms to the curriculum content, knowing that his pupils will need to succeed in the final mathematics exams and be ready for university, but still feels a conflict with the system, especially when discussing the problem of retention (to satisfy the curriculum, the expected outcomes based on knowledge and routinizing certain methods) and questions his method in front of the quantity of mathematical content, at least on the upper secondary level. Barbara, on the other hand, is not worried about the formal assessment at all, but is willing to concede that her "pupils [are not going to be] able to solve it as quickly as others but will solve it" (from an interview with Barbara). Note that quickness is an issue, in the competitive nature of external examinations and tests.

5.3 Argumentation: ways of thinking

The results of Study 1 and Study 2A confirmed that in the interviewed Czech teachers' and general public beliefs as well as in the FEP, the continuous development of logical thinking is one of the aims of primary and secondary mathematics education. At the same time, I noted vagueness and ambiguity concerning thinking, logical thinking and reasoning, and mathematical thinking and reasoning in the declared aims. Recall from Chapter 1 that one of argumentation's central roles in a mathematics classroom can be the development of advanced thinking skills (or Harel and Sowder's (2005) ways of thinking), which is likely what teachers have in mind when they speak about logical thinking, and what perhaps Schoenfeld (1992) calls "appropriately mathematical modes of thought" above.

Let us consider the following quote from Pólya and Szegö (1925) as quoted in Schoenfeld (1992):

> General rules which could prescribe in detail the most useful discipline of thought are not known to us. Even if such rules could be formulated, they could not be very useful. […] One must have them assimilated in flesh and blood. […] The independent solving of challenging problems will aid the reader far more than the aphorisms which follow, although as a start these can do him no harm. (p. 16)

In this sense, again, we could observe the emphasis on independent problem solving, and could say that this is what both Charles's and Barbara's texts reflected. The notion of not being able to prescribe in detail the discipline of thought is also interesting to us, because one could say that this is why the Czech national curriculum is vague about this. Yet, we know that in the NCTM documents, there has been an effort to describe at least some of these processes. The question remains whether a standardization of these would lead to formal teaching and learning of them (as feared in the quote above) although they certainly cannot do harm. It seems that they can help "as a start" but do not suffice.

In my investigation, we have seen a few examples of attempts to teach to think in a particular way. Take, for example, Karen and Jenny's belief that showing pupils a certain way of organizing their thoughts is useful or perhaps the aforementioned case of a textbook authors' recommendation for a particular representation. The effect on individual pupils is not clear: would it lead to imitative or creative problem solving? That is a completely different dissertation topic. What I did observe was that even when teachers believe in pupils' learning through the practice of problem solving, what is a "challenging problem" is understood in various ways. For example, in a survey among Czech lower secondary teachers, 92% of the 244 respondents agreed that showing examples of "type" problems (typical problems) is important for (their pupils') solving word problems (Vondrová et al., 2015). This may suggest that Karen's approach to problem solving is also adopted in many classrooms across the country.

I also observed one example when a particular way of thinking (proving a hypothesis) was the subject of a lesson. The teacher's expectation of pupils' struggle with enacting this type of proof was based on his knowledge of the pupils' previous exposure to similar experiences and his judgment on whether this was sufficient for them to have assimilated this way of thinking. Teachers consider their pupils' knowledge all the time when they plan, and he prepared the proof stages (the teacher-led start of the proof) accordingly. He acknowledged that pupils were not cognitively ready to carry the proof out on their own but understood it his responsibility to provide pupils with opportunities for assimilating this way of thinking. It is also true that he was sceptical in his expectations of all of his pupils to succeed in this (at least within their careers of secondary education).

Finally, I do not want to give the impression that the other interviewed teachers did not cultivate communities where an enculturation in Schoenfeld's sense could happen; however, in this study, I simply have no evidence of this. Karen's practice of efficiency and her goals of preparing her pupils for being able to solve curriculum-set problems efficiently, however, seem to contrast with Schoenfeld's position on sense making.

5.4 Argumentation: problem solving, understanding and justifying

One thing that became clear during study 2A was the distinction teachers made between being able to understand or solve a problem, and explaining the solution. All six teachers professed or demonstrated that there was a group of pupils whom they perceive as strong in mathematics (i.e., the pupils are able to find valid and original solutions, they "see it right away") but who lack skills in explaining or showing their solution to a particular audience. Although their reasoning is mathematically valid, they are not able to produce an argument that can be accepted by the interlocutor or reader, which, in turn, means that they are not able to justify their solutions. In the case of these mathematically able pupils, this seemed to be a minor issue for all teachers. Even those who felt that these pupils should learn how to justify (Charles, Victor, and Barbara) did not express any forethought strategies for this learning. Karen, in this case, was an exception, as her strategy for assessment included teaching pupils how to communicate or display their solutions to her, as an external (or pseudo-external) audience. It would make an interesting study to see how and whether strong pupils from a classroom where the teacher uses Karen's strategy for communicating their solutions develops skills to present their problem's justification to an external audience. What the teachers noted, though, was that these pupils usually felt dislike of such a formal extension of their engagement with mathematics. For them, the internal satisfaction and conviction was what mattered. It seems that developing justification communication skills is, in the teacher's beliefs, important only for some pupils while others can get away with "seeing" it. I believe that this phenomenon would require more attention and investigation in the future.

On the subject of communication, I also observed three different approaches to the use of language in argumentation. In Study 1, we saw that the FEP stresses the efficiency and precision of language and arguments. What we saw in Karen's class was an example of how this efficiency may play out in the classroom. In terms of publicly displayed arguments, Barbara's pupils' contributions were exhibiting formal mathematical terminology the least (in fact, recall the communication breakdown between Barbara and the pupils in Section 4.3.2 when they attempted to use standard terminology used in factoring numbers), Karen's and Charles's more so. In Karen's case, the terminology was simply part of the teacher's discourse. In Charles's classes, there were tasks and problems that used specific terminology to show the usefulness of distinguishing between various terms (such as disk and circle).

5.5 Final remarks

It is important to acknowledge that the study of argumentation in the Czech mathematical context did not include a salient picture of the pupil. I focused on the national, textbook and teachers' intended and implemented curriculum. We cannot judge what was actually accepted by individual pupils, i.e., what the attained curriculum looked like. The nature of the data and the nature of the classroom communication patters and social norms restricted our view into pupils' minds. For example, I noticed that in Barbara's classroom, pupils were not accepting certain general arguments that Barbara wanted them to discover because they did not state them in the particular whole-class activity. What happened in the cases when a teacher took over and supplied warrants or a sequence of warrants in a classroom where norms were not set up for pupils to express their understanding other than a teacher's "right?", is of course a subject to a different investigation. What we did observe was that their contributions of arguments did matter in the classroom communities, in that they informed the teachers' knowledge, affecting especially future arguments.

There are many other questions about argumentation that this study has not answered. I was not able to get an insight into the local issues, related to mathematical content. Methodology that would involve teachers teaching the same topic is able to uncover further nuances. Also, observing how teacher's perception of pupils' abilities affect their decision making about argumentation also deserves further attention (for example, does the same teacher change their expectations and routines depending on the group of pupils?). Gauging an impact of a standard textbook on the arguments on two or more teachers' argumentation practices would also help us separate the teacher and textbook authors' participation on arguments in the classroom.

Finally, I trust this work will serve as an introductory exploration into Czech teachers' beliefs and practices and that it will inspire investigations that will further deepen the understanding of the identity of the Czech mathematics education.

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Appendix list

APPENDIX A

Philosophies of Mathematics Education – typology descriptors

Table A1:

Typology descriptors adopted from Ernest (1991).

	INDUSTRIAL TRAINER	TECHNOLOGICAL PRAGMATIST	OLD HUMANIST	PROGRESSIVE EDUCATOR	PUBLIC EDUCATOR
Political ideology	radical right, black and white, true or false established by authority	meritocratic, conservative, unquestioning the existing structures, valuing practical outcomes of intellect and ethics	truth, fairness and 'blind' justice for all	liberal, progressive and liberal but without questioning status quo	democratic socialist
View of mathematics	value-free, mathematics is a body of true facts, skills and theories. knowledge stems from authority (the Bible, experts)	absolutist, no best method application, choices between approaches are made on the utilitarian value	platonic, queen of the sciences, perfect, absolute, elegant, abstract truth	absolutist in humanistic and personalized terms, mathematics as a language, create and human side but absolutist innate knowledge, recreated by individuals	mathematics not existing outside human mind, is made or remade inside each person's mind, value-laden, culture-bound, knowledge is access to power, mathematics based on human agreement, not separated from reality, political value

	INDUSTRIAL TRAINER	TECHNOLOGICAL PRAGMATIST	OLD HUMANIST	PROGRESSIVE EDUCATOR	PUBLIC EDUCATOR
Moral values	freedom, individualism, inequality, competition in the "market place". Strict regulation is needed because humans are fallible	utility, expediency, pragmatism, self- or group interest	principles of fairness and justice, reason, rationality, logic, purist aesthetic, objectivity in reasoning	relate, nurture, comfort, expression, style, experience, subcultures, child-centred, creativity, feelings, subjectivity and dynamic growth	Western liberalism: justice, rights, feelings, sense-making, equality, liberty, fraternity (social justice), individuality, all people and cultures, equal opportunities, democracy
Theory of society	classes that differentiate by amount of virtue and ability, strong national unique heritage that should be preserved, monoculture views	experts, technocrats, bureaucracies are high up and run the existing structures	conservation of West and high culture, hierarchical structures inherited from the past, gentle vs common folk, cultured elite vs masses	a nurturing and supportive but also with ills that require responses to individuals, progressive and liberal but without questioning status quo	society is an extended family, importance of political, social and economic values, masses without knowledge therefore not in power, the sleeping giant,
Theory of child	undisciplined, naughty lazy and playful	empty vessel, need to be filled with facts and skills, experience is the source of skill, blunt tool for future deployment	fallen angels and empty vessels but with enough disposition can be cultured	Piaget: innately curious explorers, growing flower and noble savage; given the proper environment, everyone will grow to their full potential, essentially what the child has been given at birth	equal gifts and potential, clay to be moulded,

	INDUSTRIAL TRAINER	TECHNOLOGICAL PRAGMATIST	OLD HUMANIST	PROGRESSIVE EDUCATOR	PUBLIC EDUCATOR
Theory of ability	different ability is innate, differentiation (selectiveness) of schools according to the ability no holding back of the better-able, elite in education, mathematics theories are complex and should be reserved for more able	inherited (fixed) but needs nurturing to be realized	talent and ability inherited, mathematics ability equals high intelligence	innate inherited differences leading to difference in rate of readiness for further mathematics development, but definitely the need for a set of experiences	ability is a cultural product (experience and how one perceives oneself) and is not fixed at birth
Aims of mathematics education	functional numeracy and obedience, essential learning of basics, set minimum standards in basic knowledge and technique	equip pupils (skills and knowledge) for employment, Further technological progress (application), certification to aid employment selection	transmit pure math per se, central part of human heritage, culture and intellectual achievement, emphasis on structure, abstract, conceptual level and rigor; elitist, no application, pure mathematics with beauty elegance, harmony, balance, depth, education future pure mathematicians	grow a human being, develop creativity and self-learning through learning experience, fostering autonomous inquirer, individual's self-efficacy	enabling an individual to think, take control of their life, participate fully and critically in democratic society, move towards more just society but also accepting alternative perspectives, growing problem-solvers and posers of problems springing from social context, understand the social institution of mathematics

	INDUSTRIAL TRAINER	TECHNOLOGICAL PRAGMATIST	OLD HUMANIST	PROGRESSIVE EDUCATOR	PUBLIC EDUCATOR
Theory of learning	through hard work, practice and application (of oneself)	through practical experience, not just watch and/or repeat but do for yourself	receive and understand a large, logically structured body of knowledge, and modes of thought, solve puzzles and mathematics problems, different methods according to ingenuity	investigation, discovery, projects, discussion, exploration, experience in order to (re)create own knowledge; individual concrete experiences following by abstractions; they pass through similar stages at their own time/rate, shield from failure and negative experiences, self-expression encouragement, ideas and mathematics projects valued,	discussing mathematics embedded in their lives and environments as well as broader social context, assumptions must be articulated and confronted with others, allow development of critical thinking, language and social interaction important, internal constructions resulting from social interaction and negotiation of meaning, learning by creation

	INDUSTRIAL TRAINER	TECHNOLOGICAL PRAGMATIST	OLD HUMANIST	PROGRESSIVE EDUCATOR	PUBLIC EDUCATOR
Theory of teaching mathematics	transmission, strict discipline, facts and skills learned and applied, rote learning, memorization, practice of skills, rejection of progressive education, no salesmanship, no enthusiastic teaching, no attractive teaching materials	skill instruction and motivating through work-relevant (applied) problem solving, teaching is the art of the art of applying mathematics	lecture and explain, communicate the structure meaningfully, enriching curriculum with additional activities, enchanting and enthusiasm, benign master-pupil relationship; teaching math, not children	teach children not curriculum, problem solving, generalizing, multi-level, circus of activities, encouragement, integration of other subjects (Montessori), non-intrusive guidance and management of mathematics activities, shielding from conflict and negative	genuine discussion, cooperative group work and projects, autonomous projects and problem posing for self-direction, creativity, learner questioning the course content, pedagogy etc., socially relevant material include race, gender and math, social engagement and empowerment
Theory of resources	teacher above equipment, paper and pencil (warning against too much other resources), restriction of calculator use	experimental and practical learning resources, computers, video etc.	models, examples but not 'hands-on' explorations because not pure/real M; heuristics only for lower achievers	access to resources self-determined by child, a rich environment, structural apparatus and other equipment to support conceptual understanding, and representation	practical resources, authentic resources to socially engage, self-access to materials

	INDUSTRIAL TRAINER	TECHNOLOGICAL PRAGMATIST	OLD HUMANIST	PROGRESSIVE EDUCATOR	PUBLIC EDUCATOR
Theory of assessment in mathematics	competition orally necessary, standardized exams are a must, errors are castigated as the failure of self-application, clear and simple targets are necessary, survival of the fittest, no protection from failure	records of achievement and certification of skills for external (employment) purposes	summative assessment must be rigorous, separating talent from the rest, formative assessment has a wider range	informal, criterion-based without labelling as incorrect, correction of errors is avoided or softened. Conflict with absolutist view of mathematics	social importance of certification but assessment should be informal on variety of forms
Theory of social diversity	social issues are math independent, racism, classicism etc. irrelevant to mathematics, and the system of national values is built on non-recognition of these.	mathematics is neutral, as long as race, gender etc. are not employment barriers, no concern with good of all individuals, the good of society as a whole	mathematics is pure and unrelated to social diversity	only positive aspects of multiculturalism are emphasized, diversity as source for learning, individualistic, build self-esteem, avoid conflict, deny reality of social conflict and racism	curriculum must be barrier-void, diverse historical, cultural and geographical locations

APPENDIX B

Argumentation in the FEP and sample SEPs

The pupil:

- makes independent observations and experiments; compares the pieces of knowledge so gained, assesses them critically and draws conclusions from them for future use;
- recognizes and understands problems; considers discrepancies and their causes; considers and plans ways to address/solve problems based on his or her own reasoning and experience;
- seeks for information suitable for solving problems; identifies identical, similar and different features of pieces of information; makes use of acquired knowledge to discover/identify various ways to solve problems;
- uses logical, mathematical and empirical methods to address/solve problems;
- tests practically the adequacy of approaches to problem solving and applies proven methods when addressing similar or new problems;
- formulates and expresses his or her ideas and opinions in a logical sequence; his or her oral or written expression is apt, coherent and cultivated;
- defends his or her opinion and uses appropriate arguments; contributes to discussions within a small group as well as to debate in the classroom.

Figure B1: Key competencies involving aspects of argumentation. Adopted from Ministry of Education, Youth and Sports (2008[67]).

[67] This is the only existing version in English but the relevant text was not changed in the updated Czech document in 2013.

➢ Education [mathematics] places an emphasis on a thorough understanding of basic ways of thinking, mathematical concepts and their mutual interaction. ➢ Three components of arithmetic operations: the ability to perform operations, algorithmic understanding (why an operation is performed in the manner presented), and understanding meaning (the ability to relate an operation to real-life situations). ➢ Understanding that reality is more complex than any mathematical model, that one model may be applied various situations and that one situation may be demonstrated using various models. ➢ Non-Standard Application Exercises and Problems, which may to a large degree be solved independently from the pupil's mathematical knowledge and skills, but which require logical thinking.
➢ Analysing problems and planning solutions, choosing the proper approach to resolving a problem, evaluating results for correctness with a view towards the nature of the task or problem. ➢ Expressing themselves precisely and succinctly by using the language of mathematics, including mathematical symbols and by performing analyses and keeping records during problem-solving and for perfecting their graphic abilities. ➢ Learning to co-operate while solving problems and applied tasks which reflect situations form everyday life, and subsequently applying the solution in practice; learning about the possibilities of mathematics in real life, and the fact that results may be arrived at in several different ways. ➢ Learning to trust in their own problem-solving skills and abilities, systematic self-evaluation at each step of the solution process, developing a systematic approach, determination and precision and the ability to express hypotheses on the basis of experience or experiment and to verify them or reject them using counterexamples. ➢ Developing combinatory and logical thinking, critical judgment and comprehensible and factual argumentation by solving mathematical problems.

Figure B2: Argumentation in mathematics subject description and outcomes. Adopted from Ministry of Education, Youth and Sports (2007).

School A
Teach pupils to: - not be afraid of problems and to look for own solutions; - formulate correctly the problem's solution and be able to justify and defend them; - be aware of the importance of checking the solution as well as the number of possible solutions; - perceive errors as important work tools; - openly express their opinion based on logical arguments; - value the ability to not only solve a problem but also to explain the solution clearly to others; - express themselves in mathematical language. Emphasise that all methods (ways) that lead to a solution are correct Enable pupils where possible to "discover" on their own Encourage original (non-traditional) solutions to problems
School B
Through solving of relevant problem tasks and logical problems, mathematics riddles and puzzles, the development of abstract and logical thinking is supported. The pupil - solves problems that have more solutions and lead to causal thinking, awareness of relationships, patterns and connections, - is provided the opportunity to try out different ways of problem solving; - looks for and creates new problems that can be solved using a particular method they learned; - carries out solution checks and verification; - learns to argue factually, to communicate during problem solving and by formulating their ideas, to deepen their understanding of concepts, awareness of connections and relationships, the nature of phenomena etc.; - learns to express themselves exactly and concisely, using mathematical language including mathematical symbols.

Figure B3: Examples of argumentation related descriptors in the SEPs. Translated from school internal documents (SEP A and SEP B), accessed online or shared with permission to reprint.

Table B1

Mathematical content specific outcomes involving argumentation in a school mathematics curricular document (the school education program - SEP).

FEP output	SEP Specific outcomes	Topic
Justify and apply the positional and metric properties of basic two-dimensional figures when solving tasks and simple practical problems	- states the difference between circle and disk [68] - states the difference between the radius and diameter (the relationship between them) - calculates the perimeter and the area of a circle and the length of a circumference using formulas. - determines the positional relationship of a circumference and a straight line and that of two circumferences - determines and constructs a tangent, a secant and a chord - draws a circumference with a given centre and radius. - draws the inscribed and escribed circle in a triangle - solves real-life word problems using knowledge related to circumference, includes a picture, mathematisation of the problem, its solution and verification of the result.	Disk and circle
	works actively with error is able to explain the problem's solution	several topics in SEP A
	compares and evaluates various options for the problem's solution process	non-standard problems
	calculates the area and surface area using formulas, writes the problem's solution with emphasis on precision, clarity and the use of mathematical symbol	area of polygons and surface area of solids
	estimates what the result could be and verifies the estimate	Pythagoras theorem

[68] The Czech language distinguishes between two concepts: *Circle* is a closed plane curve consisting of all points at a given distance from a point within it called center. *Disk* is defined as the portion of a plane bounded by such a curve.

APPENDIX C

Textbook use: survey results in GACR 2010

In the GACR (GA ČR P407/11/1740) project, teachers of lower secondary schools were addressed to participate in an online questionnaire about their practices, resources beliefs, etc. One of the questions was about the textbook use in their teaching practice. Table C1 shows the frequency of textbooks mentioned in this questionnaire.

Table C1

Use of textbooks in lower secondary schools. Results from an online questionnaire conducted in 2014, with codes for specific series used in Study 1.

CODE	TEXTBOOK SERIES	FREQUENCY*
A	FORTUNA	24
B	FRAUS	29
C	NOVÁ ŠKOLA	23
D	PRODOS	11
E	PROMETHEUS	122
F	PROMETHEUS GYMNÁZIUM	--
G	STÁTNÍ PEDAGOGICKÉ NAKLADATELSTVÍ	28
	OTHER	22

*Number of respondents: 240

APPENDIX D

Textbooks analysed in Study 1

Binterová, H., Fuchs, E., & Tlustý, P. (2007). *Matematika 6*: *učebnice pro základní školy a víceletá gymnázia*. 1. vyd. Plzeň: Fraus.

Binterová, H., Fuchs, E., & Tlustý, P. (2008). *Matematika 7*: *učebnice pro základní školy a víceletá gymnázia*. 1. vyd. Plzeň: Fraus.

Binterová, H., Fuchs, E., & Tlustý, P. (2009). *Matematika 8*: *učebnice pro základní školy a víceletá gymnázia*. 1. vyd. Plzeň: Fraus.

Binterová, H., Fuchs, E., & Tlustý, P. (2010). *Matematika 9*: *učebnice pro základní školy a víceletá gymnázia*. 1. vyd. Plzeň: Fraus.

Coufalová, J. (2007). *Matematika pro 6. ročník základní školy*. 2. upr. vyd. Praha: Fortuna.

Coufalová, J. (2007). *Matematika pro 7. ročník základní školy*. 2., upr. vyd. Praha: Fortuna.

Coufalová, J. (2007). *Matematika pro 8. ročník základní školy*. 2., upr. vyd. Praha: Fortuna.

Coufalová, J. (2007). *Matematika pro 9. ročník základní školy*. 2., upr. vyd. Praha: Fortuna.

Herman, J., Chrápavá, V., Jančovičová, E., & Šimša, J. (2004). *Matematika*: *racionální čísla, procenta* 2. vyd. Praha: Prometheus.

Herman, J., Chrápavá, V., Jančovičová, E., & Šimša, J. (1998). *Matematika*: *kladná a záporná čísla: prima*. 1. vyd. Praha: Prometheus.

Herman, J., Chrápavá, V., Jančovičová, E., & Šimša, J. (1997). *Matematika*: *výrazy [2] : Tercie*. 1. vyd. Praha: Prometheus.

Herman, J., Chrápavá, V., Jančovičová, E., & Šimša, J. (1995). *Matematika: výrazy.* 1. vyd. Praha: Prometheus.

Herman, J., Chrápavá, V., Jančovičová, E., & Šimša, J. (1997). *Matematika*: *úměrnosti : tercie*. 1. vyd. Praha: Prometheus.

Molnár, J., Emanovský, P., Lepík, L., Lišková, H., & Slouka, J. (1998). *Matematika 6*. Olomouc: Prodos.

Molnár, J., Emanovský, P., Lepík, L., Lišková, H., & Slouka, J. (1999). *Matematika 7*. Olomouc: Prodos.

Molnár, J., Emanovský, P., Lepík, L., Lišková, H., & Slouka, J. (2000). *Matematika 8*. Olomouc: Prodos.

Molnár, J., Emanovský, P., Lepík, L., Lišková, H., & Slouka, J. (2001). *Matematika 9*. Olomouc: Prodos.

Odvárko, O., Kadleček, J. (2004). *Matematika pro 6. ročník základní školy*. 2. vyd. Praha: Prometheus.

Odvárko, O., Kadleček, J. (2004). *Matematika pro 7. ročník základní školy*. 2. vyd. Praha: Prometheus.

Odvárko, O., Kadleček, J. (2004). *Matematika pro 8. ročník základní školy*. 2. vyd. Praha: Prometheus.

Odvárko, O., Kadleček, J. (2004). *Matematika pro 9. ročník základní školy*. 2. vyd. Praha: Prometheus.

Půlpán, Z. (2008). *Matematika 7 pro základní školy: aritmetika.* Praha: SPN - pedagogické nakladatelství.

Rosecká, Z., & Čuhajová, V. (1997). *Aritmetika*: *učebnice pro 6. Ročník*. Brno: Nová škola.

Rosecká, Z., & Čuhajová, V. (1998). *Aritmetika*: *učebnice pro 7. Ročník*. Brno: Nová škola.

Rosecká, Z. (1999). *Algebra*: *učebnice pro 8. ročník*. Brno: Nová škola.

Rosecká, Z. (1999). *Algebra*: *učebnice pro 9. ročník*. Brno: Nová škola.

Trejbal, J., Jirotková, D., & Sýkora, V. (1997). *Matematika pro 6. ročník základní školy*. 1. vyd. Praha: SPN - pedagogické nakladatelství.

Trejbal, J., Jirotková, D., & Sýkora, V. (2004). *Matematika pro 7. ročník základní školy*. 2. vyd. Praha: SPN - pedagogické nakladatelství.

Trejbal, J. (1998). *Matematika pro 8. ročník základní školy*. 1. vyd. Praha: SPN - pedagogické nakladatelství.

Trejbal, J. (2003). *Matematika pro 9. ročník základní školy*. 2. vyd. Praha: SPN - pedagogické nakladatelství.

APPENDIX E

Teachers' orientations in Ernest's (1991) framework

I analysed interviews with each teacher using the descriptors in Appendix A and marked where they fell into each category and type: Industrial Trainer (I.T.), Technological Pragmatist (T.P.), Old Humanist (O.H.), Progressive Educator (Pr.E.) and Public Educator (Pub.E.)

Yes

No

CHARLES	I. T.	T. P.	O. H.	Pr. E.	Pub. E.
Political Ideology	No	No	No	Yes	Yes
View of Mathematics	No	No	No	Yes	No
Moral Values	No	No	Yes	Yes	Yes
Theory of Society	No	No	No	Yes	Yes
Theory of Child	No	No	Yes	No	Yes
Theory of Ability	No	No	No	No	Yes
Aims of Math Education	No	No	No	Yes	Yes
Theory of Learning	Yes	Yes	No	Yes	Yes
Theory of Teaching Mathematics	No	No	No	No	Yes
Theory of Resources	No	No	Yes	Yes	Yes
Theory of Assessment in Mathematics	No	No	No	Yes	Yes
Theory of Social Diversity	No	No	No	No	No

Figure E1: Charles's mathematical education beliefs in Ernest's typology.

BARBARA	I. T.	T. P.	O. H.	Pr. E.	Pub. E.
Political Ideology	No	No	No	No	No
View of Mathematics	No	No	No	Yes	No
Moral Values	No	No	No	Yes	Yes
Theory of Society	No	No	No	No	No
Theory of Child	No	No	No	Yes	Yes
Theory of Ability	No	No	No	Yes	Yes
Aims of Math Education	No	No	No	Yes	Yes
Theory of Learning	No	No	No	Yes	Yes
Theory of Teaching Mathematics	No	No	No	Yes	Yes
Theory of Resources	No	No	No	Yes	No
Theory of Assessment in Mathematics	No	No	No	Yes	Yes
Theory of Social Diversity	No	No	No	No	No

Figure E2: Barbara's mathematical education beliefs in Ernest's typology.

VICTOR	I. T.	T. P.	O. H.	Pr. E.	Pub. E.
Political Ideology					
View of Mathematics					
Moral Values					
Theory of Society					
Theory of Child					
Theory of Ability					
Aims of Math Education					
Theory of Learning					
Theory of Teaching Mathematics					
Theory of Resources					
Theory of Assessment in Mathematics					
Theory of Social Diversity					

Figure E3: Victor's mathematical education beliefs in Ernest's typology.

JENNY	I. T.	T. P.	O. H.	Pr. E.	Pub. E.
Political Ideology					
View of Mathematics					
Moral Values					
Theory of Society					
Theory of Child					
Theory of Ability					
Aims of Math Education					
Theory of Learning					
Theory of Teaching Mathematics					
Theory of Resources					
Theory of Assessment in Mathematics					
Theory of Social Diversity					

Figure E4: Jenny's mathematical education beliefs in Ernest's typology.

ZACK	I. T.	T. P.	O. H.	Pr. E.	Pub. E.
Political Ideology					
View of Mathematics					
Moral Values					
Theory of Society					
Theory of Child					
Theory of Ability					
Aims of Math Education					
Theory of Learning					
Theory of Teaching Mathematics					
Theory of Resources					
Theory of Assessment in Mathematics					
Theory of Social Diversity					

Figure E5: Zack’s mathematical education beliefs in Ernest’s typology.

KAREN	I. T.	T. P.	O. H.	Pr. E.	Pub. E.
Political Ideology					
View of Mathematics					
Moral Values					
Theory of Society					
Theory of Child					
Theory of Ability					
Aims of Math Education					
Theory of Learning					
Theory of Teaching Mathematics					
Theory of Resources					
Theory of Assessment in Mathematics					
Theory of Social Diversity					

Figure E6: Karen’s mathematical education beliefs in Ernest’s typology.