**FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University**

BACHELOR THESIS

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Standard and alternative cosmological models

Institute of Theoretical Physics

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Study programme:	Physics
Study branch:	General physics

Prague 2018

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My cordial gratitude belongs to my supervisor Giovanni Acquaviva, Ph.D. for his readiness and willingness to share his knowledge and, of course, for his priceless help. I am thankfull to Dr. Acquaviva for being my guide and companion on my first trip to the breathtaking marvels of the Universe.

I would like to thank my dear grandparents for introducing me to physics and for their infinite love.

I heartily thank all my loved ones, especially Ondřej Novák, for their support and for the help with the editing this thesis.

Title: Standard and alternative cosmological models

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Abstract: The main aim of this thesis is the study of the dependence of the scale factor on the cosmic time for different models of Universe's evolution in the framework of the general theory of relativity. In this thesis we consider the FLRW metric and admit nonzero curvature. The models we consider differ from each other by the equation of state of the source, hence by the composition of the cosmic fluid under study. In this thesis the following models are discussed: ΛCDM (we consider a perfect cosmic fluid consisting of the incoherent dust, radiation and a cosmological constant in a curved space-time), generalized Chaplygin gas, and, also, two kinds of the scalar field (describing separately power-law inflation and the period after recombination). The numerical and analytical results obtained are processed graphically.

Keywords: cosmology, dark matter, dark energy

Contents

Introduction

Conventions

All calculations are performed using the geometrized unit system ($G = c = 1$). Einstein summation convention is used.
We denote by the Greek letters indices which gain all of the $\{t, r, \theta, \phi\}$ values. We use the Roman alphabet only for the spatial coordinates.
Quantities with the subscript 0 denote the present-day values.
Vectors are indicated by a bold font.

Historical overview

We live in a wonderful age when science develops so rapidly that its results change our everyday life. Everyone realizes all the wonders of modern technology, but people, who come in contact with physics can truly appreciate the wealth and beauty that laws of nature offer. This especially applies to gravity, results of observing the Universe and the objects from which it consists - from interstellar and intergalactic gas and asteroids - to stars, galaxies, their clusters, and, of course, black holes. And the general theory of relativity arose - a theory that managed not only to describe and explain a wide range of the already known gravitational phenomena but also to predict new ones. One of the most striking examples are the gravitational waves. Their recent experimental discovery (Abbott et al. [2016]) was a brilliant confirmation of this theory.

The equations of classical mechanics, expressing the law of universal gravitation and Newton's second law, are in aggregate sufficient for predicting the motion of a point particle with respect to given initial conditions - velocities and coordinates. The law of gravity makes it possible to find the force acting on a given object by known distances to other objects. Newton's second law makes it possible to calculate the acceleration, that makes possible (using the integral calculus invented again by Newton!) calculating the velocity vector of the object, new coordinates, etc.

According to the ideas of classical mechanics, the particle, in addition to the gravitational mass entering into Newton's law of gravitation and playing the role of a gravitational charge, also has an "inertial" mass that describes the body's ability to resist acceleration and enters Newton's second law. But there was no such division in Newtonian mechanics. It was assumed that these masses are equal.

The transition to the continual approximation in the the law of the universal gravitation and Newton's second law lead to equations completely analogous to the electrostatic equations (Alekseev [2015]). In this case, the gravitational field at time t at any distance from the system is determined by the mass distribution of this system at the same time t, that is, in this theory, the gravitational field is propagated instantaneously (at infinite speed).

However, the provision on the instantaneous propagation of the field contradicts the experiments and breaks the special theory of relativity (STR). We can summarize that the Newtonian theory is approximately applicable for the motion of particles with not too high velocities in a weak gravitational field.

Exceeding the limits of these restrictions in describing the world became possible only within the framework of the general theory of relativity (GTR). In 1911, Einstein, based on Galileo's statement about the same acceleration of all bodies falling in the gravitational field of the Earth, formulated the principle of equivalence of gravitational and inertial forces. Having in mind two reference frames, one of which rests in a homogeneous gravitational field, and the other moves with an acceleration equal in magnitude to the intensity of the gravitational field, he assumed that the first of them "is possible to consider as being in space without a gravitational field, but then we must consider that this frame undergoes the constant acceleration... From this point of view, the same acceleration of all falling bodies is obvious" (Einstein [1911]). Equality of the inertial and gravitational mass is a mathematical expression of this principle (weak equivalence principle (Alekseev [2015]).

GTR, constructed by Einstein in subsequent years, was developed on the way of generalization of the equivalence principle to the general case of arbitrary gravitational fields. It is based on the strong principle of equivalence, which in the most general form can be formulated as follows: in an arbitrary gravitational field at each point of space-time there exists a locally inertial frame of reference in which, in a sufficiently small neighborhood of the point under consideration, the laws of nature will have the same form, as in non-accelerated Cartesian frames of reference (Alekseev [2015], Einstein [1915]).

In 1915 the theory was basically completed and published Einstein [1915], consisting its main result, the Einstein's equations relates the metric of the curved space-time to the properties of the matter filling it. In this sense, general relativity is often called geometrodynamics. The meaning of the Einstein's equations is well stated in the classical monograph (Misner et al. [1973]): matter indicates space, how to bend. The curved space indicates matter, how to move.

In 1922, A. Friedmann obtained solutions to the Einstein equations of general relativity (without the cosmological constant), from which it followed that the Universe is not stationary and depending on the values of the parameter, associated with the average density of matter, either expands or contracts (Friedmann [1922], Alekseev [2015]). Preliminary estimates of Friedmann gave for the age of the Universe about 10 billion years (Friedmann [1922], Mukhanov [2016]). Thus, ending with the historical overview of theoretical studies, we can summarize that within the framework of general relativity, it was the first time to make scientific predictions about the evolution of the Universe. Next, we will focus on the results of astrophysical observations, which, together with general relativity, lie in the foundation of modern cosmology.

Observations

Hubble

Theoretical predictions of GTR stimulated the development of observational astronomy. In 1929, Hubble at the Mount Wilson Observatory discovered that the spectra of distant galaxies were shifted towards the red side (Hubble [1929]). He interpreted this redshift as a Doppler shift due to the dispersal of galaxies (Hubble

[1929], Mukhanov [2016]). According to Hubble's law, the velocity of the galaxy recession is proportional to the distance to it. The constant of proportionality is called the Hubble constant. According to modern estimates, in the present era, its inverse magnitude corresponds to the age of the Universe about 13.8 billion years (Alekseev [2015]). As noted in (Mukhanov [2016]), the Hubble's discovery, which initiated observational cosmology, can be considered a brilliant confirmation of the theoretical prediction of Friedmann.

CMB

Later, based on the relatively high concentration of helium in the Universe, G. Gamow (Gamow [1946]) and R.A. Alpher with R.C. Herman (Alpher and Herman [1949]) suggested that in the distant past, close to the cosmological singularity, the Universe was super-hot (the Big Bang theory). According to the Big Bang theory, in the early hot Universe, most of the helium was formed. The energy released in the form of "hot" photons was thermalized, and the radiation was cooled inversely proportional to its radius with the expansion of the Universe (Mukhanov [2016]).

At an early hot stage, the radiation was absorbed by electrons that were then still free, and the Universe was opaque to radiation. Under cooling below 3000 K, recombination of hydrogen atoms began, and radiation, practically not absorbed, was free to propagate and has survived to the present day. For this reason, it is also called relic radiation (Alekseev [2015], Mukhanov [2016]).

The main predictions of the Big Bang theory were confirmed in 1965 by A.A. Penzias, R. W. Wilson (Penzias and Wilson [1965], Mukhanov [2016]). In their observations, CMB (Cosmic Microwave Background) appeared as an ineradicable noise in the radio antenna, the intensity of which did not depend on the direction. The main property of the CMB is that it is homogeneous and isotropic with a high degree of accuracy. The more subtle properties of the CMB, from which one can judge the structure and evolution of the Universe, were investigated, in particular, by V.F. Mukhanov and his colleagues in the theory of the quantum origin of inhomogeneities in the early Universe (see Mukhanov [2016]). This theory predicts the Euclidean geometry of the Universe, the adiabaticity of the perturbations and the Gaussianity of the initial inhomogeneities. In Mukhanov [2016], the results of recent observations and experiments are analyzed and it is concluded that they confirm the theoretical results with great accuracy. The *cosmological principle* according to which on a large scale the Universe is homogeneous and isotropic is also confirmed by the latest observations with unprecedented accuracy [16]. Regarding the DMR (Differential Microwave Radiometer) we let ourselves to quote (Mukhanov [2016]) "... for the first time, CMB temperature variations were detected in different directions in the sky, which amounted to about 0.0001 K. Thus, we were finally able to see the seeds of galaxies in the Universe when it was only a few hundred thousand years".

Dark matter

For the first time the need for a hidden mass to explain the motion of galaxies in clusters was substantiated in 1937 by F. Zwicky (Zwicky [1937]). Without this, it was impossible to explain the motion of galaxies in clusters (Zasov et al. [2017]).

In addition, the differential rotation curves of galaxies could not be explained: for stars on the periphery of the galaxy, where the density decreases rapidly, the rotation speed remained almost constant when moving away from the center of the galaxy (Zasov et al. [2017], Alekseev [2015]). The rapid growth of interest in this topic since the 1980s to the present time was caused by the need to explain the motion of gas in galaxies, which has similar features (Zasov et al. [2017]). The masses of visible matter in galaxies, calculated from the red shift and known luminosities of galaxies, were not enough to realize such a movement, and the missing mass (which is not visible in optical telescopes, but finds itself only in gravitational interaction) was called "dark matter" (Zasov et al. [2017], Alekseev [2015]).

There are also alternative approaches in which the observed anomalies are explained without invoking the hypothesis of dark matter. For example, Modified Newtonian Dynamics. But they contain additional free parameters and, most importantly, do not completely agree with the results of observations (Zasov et al. [2017]).

A detailed analysis of the problem of dark matter is contained, for example, in the review (Zasov et al. [2017]). Here we only note that the totality of observations indicates, that according to the standard cosmological model it predominates over ordinary matter (Alekseev [2015]). According to the results of the Planck space mission (Verkhodanov [2016]), the physical density of dark matter in the Universe exceeds the physical density of baryonic matter by more than 5 times.

Acceleration of the expansion of the Universe

However it turned out that dark matter is not enough to adequately describe observational data, since the accuracy of observations increased, and in 1999 convincing experimental evidence emerged that the expansion of the Universe in the modern era is not slow, but accelerated (Riess et al. [1998], Perlmutter et al. [1999]). The review authors (Bolotin et al. [2012]) evaluate this discovery as one of the most important, not only in cosmology, but also in physics in general. The physical reasons for this acceleration of expansion have not yet been found, it is clear only that it can be caused by some agent (effect or substance) opposing gravity. For this agent, the term "dark energy" was introduced (Bolotin et al. [2012]). In the standard cosmological model, the role of the dark energy (effective repulsion) is played by the cosmological constant (Λ-term), and the influence of the dark energy was determined by the sign and the magnitude of this constant.

Due to the fact that the physical nature of dark energy is not clear, and the ΛCDM model does not have well-founded assumptions, alternative cosmological models that do not require the introduction of dark matter and dark energy were proposed (Alekseev [2015], (Zasov et al. [2017]), Bolotin et al. [2012]). There is also some doubt about the description of antigravitation with the help of negative pressure (Novikov et al. [2018]).

ΛCDM cosmological model

ΛCDM (Λ-Cold Dark matter) is a Big Bang cosmological model. It is assumed, that the Universe contains a cosmological constant, associated with the Dark energy (DE), and the Cold Dark Matter (DM). Cold in this context means, that the velocities of the DM particles are nonrelativistic during the period of structure formation (Garrett and Dūda [2011]). It is assumed, that the relative dominance of one dark component over the other drives the character of the Universe's evolution. The ΛCDM model does not contain the inflation, but could be extended by adding, for instance, the other components to the cosmic fluid (such as the scalar field).

1. Derivation of the Einstein's equations

The Einstein's (field) equations provide a connection between a geometry of the space-time and a distribution of matter in the Universe. Having solved these equations one gets a metric tensor (also called a metric), which contains information about physical properties of the space-time regions. There is an analogy between a metric tensor and a gravitational potential of the Newtonian theory of gravity so the metric can be roughly interpreted as a generalization of a gravitational potential. Thus metric is needed for solving the equations of motion and for any predictions of the further evolution of the Universe. Geometrically "metric turns observer-dependent coordinates into invariants" (Cosmology notes Hiranya V. Peiris). Invariants in this context are, for example, distances, which are necessarily needed for describing the world quantitatively.

The main aim of this chapter is to derive Einstein's field equations for the space-time with the given symmetries and describe the evolution of the Universe in case of being filled with the only one of the components such as incoherent dust, radiation, cosmological constant or, eventually, with the "curvature fluid". It could help the reader to get some basic intuition of the Universe's behaviour for the further cosmological problems discussed in this thesis.

1.1 FLRW flat metric

The metric tensor which satisfies the cosmological principle can have only a time-dependence. Translation and rotation symmetries in space lead us to the following form of metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix} \tag{1.1}$$

This metric tensor is called the Friedmann-Lamaître-Robertson-Walker metric (abbreviated FLRW). The orthogonality of the covariant metric tensor implies that its inverse metric is simply given by

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2a^2(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2a^2(t)\sin^2\theta} \end{pmatrix} \tag{1.2}$$

The derivatives of metric are needed for the following calculations:

$$\begin{aligned}
g_{rr,t} &= 2a(t)\dot{a}(t) \\
g_{\theta\theta,t} &= 2a(t)\dot{a}(t)r^2 \\
g_{\theta\theta,r} &= 2a^2(t)r \\
g_{\phi\phi,t} &= 2a(t)\dot{a}(t)r^2\sin^2\theta \\
g_{\phi\phi,r} &= 2a^2(t)r\sin^2\theta \\
g_{\phi\phi,\theta} &= 2a^2(t)r^2\sin\theta\cos\theta
\end{aligned} \tag{1.3}$$

1.1.1 Christoffel symbols

In flat space-time we use normal derivatives; they commute, mathematically that means, that $V^\alpha_{,\mu\nu} - V^\alpha_{,\nu\mu} = 0$. According to the definition, a derivative is proportional to the difference between the studied quantity in two infinitesimally close points. In the curved space, unlike the Euclidean one, the quantity nearby is not necessarily the element of the same space as the original quantity (for instance, the tangent vector could belong to another cotangent bundle). It breeds a problem with derivatives on the curved manifolds and the solutions are so-called covariant derivatives: the normal derivatives are modified by the addition of the Christoffel symbols: $V^\mu_{;\nu} = V^\mu_{,\nu} + \Gamma^\mu{}_{\rho\nu}V^\rho$. The new derivatives do not commute and are commonly denoted by a semicolon. Moreover, the commutator of covariant derivatives defines the Riemann tensor: $V^\alpha_{;\mu\nu} - V^\alpha_{;\nu\mu} = R^\alpha{}_{\sigma\mu\nu}V^\sigma$. Hence the Christoffel symbols encode the effect of curvature of the space-time.

The Christoffel symbols of the second kind are defined in the following way:

$$\Gamma_{\alpha\ \beta}^{\ \mu} = \frac{1}{2}g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}). \tag{1.4}$$

According to the diagonal form of matrix representing FRW metric in the eq. (1.4) the only non-zero Christoffel symbols we get are when $\mu = \nu$

$$\Gamma_{\alpha\ \beta}^{\ \mu} = \frac{1}{2}g^{\mu\mu}(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu}) \tag{1.5}$$

Due to the symmetry of the metric tensor we gain 3 types of the Christoffel symbols

$$\Gamma_{\mu\beta}^{\ \mu} = \frac{1}{2}g^{\mu\mu}g_{\mu\mu,\beta} \tag{1.6}$$

$$\Gamma_{\alpha\alpha}^{\ \mu} = \frac{1}{2}g^{\mu\mu}(-g_{\alpha\alpha,\mu}) \tag{1.7}$$

$$\Gamma_{\beta\mu}^{\ \mu} = \frac{1}{2}g^{\mu\mu}g_{\beta\beta,\mu} \tag{1.8}$$

After comparing these formulas we can mention one more the symmetry

$$\Gamma_{\alpha\ \beta}^{\ \alpha} = \Gamma_{\alpha\ \beta}^{\ \beta} \tag{1.9}$$

Having put the specific components of the metric tensor (1.1) and (1.2) to the eq. (1.6) we get

$$
\begin{aligned}
\Gamma^{\ t}_{r\ r} = \frac{1}{2}g^{tt}(-g_{rr,t}) &= a(t)\dot{a}(t) \\
\Gamma^{\ t}_{\theta\ \theta} &= \frac{1}{2}g^{tt}(-g_{\theta\theta,t}) = r^2 a(t)\dot{a}(t) \\
\Gamma^{\ t}_{\phi\ \phi} &= \frac{1}{2}g^{tt}(-g_{\phi\phi,t}) = r^2 a(t)\dot{a}(t)\sin^2\theta \\
\Gamma^{\ r}_{\theta\ \theta} &= \frac{1}{2}g^{rr}(-g_{\theta\theta,r}) = -r \\
\Gamma^{\ r}_{\phi\ \phi} &= \frac{1}{2}g^{rr}(-g_{\phi\phi,r}) = -r\sin^2\theta \\
\Gamma^{\ \theta}_{\phi\ \phi} &= \frac{1}{2}g^{\theta\theta}(-g_{\phi\phi,\theta}) = -sin\theta\cos\theta
\end{aligned}
\tag{1.10}
$$

If we put the same components into the eq. (1.7) and (1.8) using (1.9) we get

$$
\begin{aligned}
\Gamma^{\ r}_{r\ t} = \frac{1}{2}g^{rr}(g_{rr,t}) &= \frac{\dot{a}(t)}{a(t)} \\
\Gamma^{\ \theta}_{\theta\ t} = \frac{1}{2}g^{\theta\theta}(g_{\theta\theta,t}) &= \frac{\dot{a}(t)}{a(t)} \\
\Gamma^{\ \theta}_{\theta\ r} = \frac{1}{2}g^{\theta\theta}(g_{\theta\theta,r}) &= \frac{1}{r} \\
\Gamma^{\ \phi}_{\phi\ t} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,t}) &= \frac{\dot{a}(t)}{a(t)} \\
\Gamma^{\ \phi}_{\phi\ r} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,r}) &= \frac{1}{r} \\
\Gamma^{\ \phi}_{\phi\ \theta} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,\theta}) &= \cot\theta
\end{aligned}
\tag{1.11}
$$

1.1.2 Ricci curvature tensor

One may calculate the Ricci tensor as a trace of the Riemann curvature tensor

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\kappa\nu} \tag{1.12}$$

so we get

$$R_{\mu\nu} = \Gamma^{\ \alpha}_{\mu\ \nu,\alpha} - \Gamma^{\ \alpha}_{\mu\ \alpha,\nu} + \Gamma^{\ \alpha}_{\beta\alpha}\Gamma^{\ \beta}_{\mu\nu} - \Gamma^{\ \alpha}_{\beta\nu}\Gamma^{\ \beta}_{\mu\alpha} \tag{1.13}$$

After the direct calculations we obtain, that the Ricci tensor is diagonal (μ and ν are the same). Specifically, the off-diagonal components of the third and the fourth terms of the Ricci tensor cancel after the summation according to the eq. (1.13); the first and the second terms have no off-diagonal components by themselves. For this reason we do not list the cancelled terms below.

For the different values of μ the first term of eq. (1.13) is represented by the following terms:

$$\Gamma^{\ \alpha}_{r\ r,\alpha} = \Gamma^{\ t}_{r\ r,t} = \dot{a}^2(t) + a(t)\ddot{a}(t) \tag{1.14}$$

$$\Gamma^{\ \alpha}_{\theta\ \theta,\alpha} = \Gamma^{\ t}_{\theta\ \theta,t} + \Gamma^{\ r}_{\theta\ \theta,r} = r^2(\dot{a}^2(t) + a(t)\ddot{a}(t)) - 1 \tag{1.15}$$

$$\Gamma_{\phi\ \phi,\alpha}^{\ \alpha} = \Gamma_{\phi\ \phi,t}^{\ t} + \Gamma_{\phi\ \phi,r}^{\ r} + \Gamma_{\phi\ \phi,\theta}^{\ \theta} = r^2\sin^2\theta(\dot{a}^2(t) + a(t)\ddot{a}(t)) - \cos^2\theta \quad (1.16)$$

For the second term we gain the following formulas

$$\Gamma_{t\ \alpha,t}^{\ \alpha} = \Gamma_{t\ r,t}^{\ r} + \Gamma_{t\ \theta,t}^{\ \theta} + \Gamma_{t\ \phi,t}^{\ \phi} = 3\frac{a(t)\ddot{a} - \dot{a}^2(t)}{a^2(t)} \quad (1.17)$$

$$\Gamma_{r\ \alpha,r}^{\ \alpha} = \Gamma_{r\ \theta,r}^{\ \theta} + \Gamma_{r\ \phi,r}^{\ \phi} = -\frac{2}{r^2} \quad (1.18)$$

$$\Gamma_{\theta\ \alpha,\theta}^{\ \alpha} = \Gamma_{\theta\ \phi,\theta}^{\ \phi} = -\frac{1}{sin^2\theta} \quad (1.19)$$

The components of the third term are

$$\Gamma_{\beta\ \alpha}^{\ \alpha}\Gamma_{r\ r}^{\ \beta} = \Gamma_{t\ \theta}^{\ \theta}\Gamma_{r\ r}^{\ t} + \Gamma_{t\ \phi}^{\ \phi}\Gamma_{r\ r}^{\ t} = 2\dot{a}^2(t) \quad (1.20)$$

$$\Gamma_{\beta\ \alpha}^{\ \alpha}\Gamma_{\theta\ \theta}^{\ \beta} = \Gamma_{t\ \phi}^{\ \phi}\Gamma_{\theta\ \theta}^{\ t} + \Gamma_{r\ \phi}^{\ \phi}\Gamma_{\theta\ \theta}^{\ r} + \Gamma_{t\ r}^{\ r}\Gamma_{\theta\ \theta}^{\ t} = 2\dot{a}^2(t)r^2 - 1 \quad (1.21)$$

$$\Gamma_{\beta\ \alpha}^{\ \alpha}\Gamma_{\phi\ \phi}^{\ \beta} = \Gamma_{t\ r}^{\ r}\Gamma_{\phi\ \phi}^{\ t} + \Gamma_{t\ \theta}^{\ \theta}\Gamma_{\phi\ \phi}^{\ t} + \Gamma_{r\ \theta}^{\ \theta}\Gamma_{\phi\ \phi}^{\ r} = 2\dot{a}^2(t)r^2\sin^2\theta - \sin^2\theta \quad (1.22)$$

The components of the fourth term can be written like

$$\Gamma_{\beta\ t}^{\ \alpha}\Gamma_{t\ \alpha}^{\ \beta} = \Gamma_{r\ t}^{\ r}\Gamma_{t\ r}^{\ r} + \Gamma_{\theta\ t}^{\ \theta}\Gamma_{t\ \theta}^{\ \theta} + \Gamma_{\phi\ t}^{\ \phi}\Gamma_{t\ \phi}^{\ \phi} = 3\frac{\dot{a}^2(t)}{a^2(t)} \quad (1.23)$$

$$\Gamma_{\beta\ r}^{\ \alpha}\Gamma_{r\ \alpha}^{\ \beta} = \Gamma_{t\ r}^{\ r}\Gamma_{r\ r}^{\ t} + \Gamma_{\theta\ r}^{\ \theta}\Gamma_{r\ \theta}^{\ \theta} + \Gamma_{\phi\ r}^{\ \phi}\Gamma_{r\ \phi}^{\ \phi} = 2\dot{a}^2(t) + \frac{2}{r^2} \quad (1.24)$$

$$\Gamma_{\beta\ \theta}^{\ \alpha}\Gamma_{\theta\ \alpha}^{\ \beta} = \Gamma_{t\ \theta}^{\ \theta}\Gamma_{\theta\ \theta}^{\ t} + \Gamma_{r\ \theta}^{\ \theta}\Gamma_{\theta\ \theta}^{\ r} + \Gamma_{\phi\ \theta}^{\ \phi}\Gamma_{\theta\ \phi}^{\ \phi} = r^2\dot{a}^2(t) - 1 + \cot^2\theta \quad (1.25)$$

$$\Gamma_{\beta\ \phi}^{\ \alpha}\Gamma_{\phi\ \alpha}^{\ \beta} = \Gamma_{\phi\ \phi}^{\ t}\Gamma_{\phi\ t}^{\ \phi} + \Gamma_{\phi\ \phi}^{\ r}\Gamma_{\phi\ r}^{\ \phi} + \Gamma_{\phi\ \phi}^{\ \theta}\Gamma_{\phi\ \theta}^{\ \phi} = r^2\dot{a}^2(t)\sin^2\theta - \sin^2\theta - \cos^2\theta \quad (1.26)$$

Putting together the components corresponding to the same value of μ according to the formula (1.13) we finally get the components of the Ricci tensor:

$$R_{tt} = -3\frac{\ddot{a}(t)}{a}(t) \quad (1.27)$$

$$R_{rr} = 2\dot{a}^2(t) + a(t)\ddot{a}(t) \quad (1.28)$$

$$R_{\theta\theta} = r^2(2\dot{a}^2(t) + a(t)\ddot{a}(t)) \quad (1.29)$$

$$R_{\phi\phi} = \sin^2\theta\ r^2(2\dot{a}^2(t) + a(t)a\ddot{(t)}) \quad (1.30)$$

1.1.3 Ricci scalar

The Ricci scalar is obtained by metric contraction of the Ricci tensor

$$R = g^{\mu\nu} R_{\mu\nu} \tag{1.31}$$

The off-diagonal components of the metric tensor are zero so we easily get a specific expression

$$R = 6\frac{\dot{a}^2(t) + a(t)\ddot{a}(t)}{a^2(t)} \tag{1.32}$$

1.1.4 Einstein's equations

At this place, we would like to define the Einstein's tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{1.33}$$

Now the Einstein's equations have the following form:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.34}$$

Tensor on the right-hand side (RHS) denotes the tensor of energy and momentum. In this thesis we consider the Universe filled with a perfect fluid

$$T_{\mu\nu} = (\rho(t) + p(t)) u_\mu u_\nu + p(t) g_{\mu\nu} \tag{1.35}$$

Here ρ denotes energy density, p denotes pressure and u^μ denotes four-velocity of the fluid's flow. Considering comoving grid $u^\mu = (1, 0, 0, 0)$ the eq. (1.35) will be simplified to the form

$$\begin{aligned} T_{tt} &= \rho(t) \\ T_{jj} &= p(t) g_{jj} \end{aligned} \tag{1.36}$$

We finally get the specific components of the Einstein's equations (1.34)

$$\begin{aligned} 3\frac{\dot{a}^2(t)}{a^2(t)} &= 8\pi\rho(t) && \text{for } \mu = \nu = \text{t} \\ -2a(t)\ddot{a}(t) - \dot{a}^2(t) &= 8\pi p(t) a^2(t) && \text{for } \mu, \nu \in \{r, \theta, \phi\} \end{aligned} \tag{1.37}$$

In the terms of Hubble function $H(t) = \frac{\dot{a}(t)}{a(t)}$ we can formulate the Friedmann equation

$$G_{tt} = 3H(t)^2 = 8\pi\rho(t) \tag{1.38}$$

Einstein's equations for the spatial diagonal components of the Einstein's tensor

$$G_{jj} = -3H(t)^2 - 2\dot{H}(t) = 8\pi p(t) \tag{1.39}$$

1.1.5 The conservation law

It is possible to express the conservation of energy-momentum as follows:

$$T^\mu{}_{\nu;\mu} = T^\mu{}_{\nu,\mu} + \Gamma^\mu{}_{\rho\mu} T^\rho{}_\nu - \Gamma^\rho{}_{\nu\mu} T^\mu{}_\rho = 0 \tag{1.40}$$

where the semicolon denotes the covariant derivative. The object $T^{\mu}_{\ \nu}$ we get by rising the first index of the energy-momentum tensor using the metric

$$T^{\mu}_{\ \nu} \ = \ g^{\mu a} T_{a\nu} \tag{1.41}$$

Christoffel symbols for the eq. (1.40) we obtain from the equations (1.6), (1.7) and (1.8) considering

$$\Gamma^{\mu}_{\ \rho\nu} \ = \ g^{\mu\alpha} g_{\beta\rho} \Gamma_{\alpha\ \nu}^{\ \beta} \tag{1.42}$$

Using formula (1.41) we obtain

$$\begin{aligned} T^{t}_{\ t} \ &= \ -\rho(t) \\ T^{j}_{\ j} \ &= \ p(t) \end{aligned} \tag{1.43}$$

Finally from the eq. (1.40) we get the continuity equation for the general relativity.

$$T^{\mu}_{\ \nu;\mu} \ = \ -\dot{\rho}(t) - 3H(t)\left(\rho(t) + p(t)\right) = 0 \tag{1.44}$$

Comparing to the classical continuity equation one can mention, that within the framework of the GTR there is an additional term, which contains pressure. That is, pressure also acts as a source for the gravitational field.

The conservation of energy and momentum is an important concept in physics, so it is reasonable to require the RHS to be zero. For the Einstein's equations was needed the tensor, which would contain the information about the geometry of the space-time and ”which will guarantee the automatic conservation of the source” (Misner et al. [2017]). Einstein's tensor turned out to be the appropriate one - the Bianchi indentities provides the required conservation.

1.1.6 Energy conditions

The energy-momentum tensor, apart from being conserved, is not a priori constrained. Physically speaking, it is natural to require some conditions on its behaviour.

At this place, we will pay attention to two of the standard energy conditions, that are the most useful for the purposes of this thesis. The conditions below are stated in accordance with Hawking and Ellis [1973].
Let M be the C^r *n-dimensional manifold*, $q \in M$ and T_qM is a *cotangent bundle* of M at the point q.

- Weak energy condition (WEC)

  To an observer whose world-line at q has unit tangent timelike vector $\mathbf{V} \in T_qM$, the local energy density appears to be $T_{\mu\nu}V^{\mu}V^{\nu}$ and the WEC one can write like this

  $$\rho(t) \ = \ T_{\mu\nu}V^{\mu}V^{\nu} \geq 0$$

  Hence this condition one can reformulate as follows: the energy density measured by any observer is always non-negative.

As we approximate a cosmic fluid by a perfect fluid we apply this condition on the eqs. (1.36) and obtain the specific representation of the WEC for our model

$$\rho(t) + p(t) \geq 0 \tag{1.45}$$

Note, that this condition allows the negative values of the pressure. While considering singularities, one of the most important consequence of the WEC is that matter always has a non-diverging effect on the general congruences of geodesics.

- Strong energy condition (SEC)

  We shall say, that $T_{\mu\nu}$ fulfillls the SEC if it obeys the inequality

$$\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) W^{\mu}W^{\nu} \geq 0$$

  for any timelike vector $\mathbf{W}$. For the perfect fluid condition looks as follows:

$$\rho(t) + 3p(t) \geq 0 \tag{1.46}$$

This condition could be violated by a negative value of the energy density or by a large negative pressure (see fig. 3.2). Let us draw your attention to the fact, that despite the names the fulfilllment of the strong condition doesn't imply the fulfilllment of the weak condition in general.

It is easy to demonstrate, that any "normal matter" (the matter we have daily experience with) satisfies any of the standard energy condition (Visser [1997]).

Accelerated expansion

According to the observations, our Universe expands with acceleration. The aim of this subsection is to find the condition for the accelerated expansion of the perfect fluid. The equations (1.38), (1.39) and (1.44) give us

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{8\pi}{6}\left(\rho(t) + 3p(t)\right) a(t)$$

$$\text{Clearly} \quad \ddot{a}(t) > 0 \Longleftrightarrow \rho(t) + 3p(t) < 0$$

Therefore for the accelerated expansion we derived the condition should be respected

$$p(t) < -\frac{1}{3}\rho(t) \tag{1.47}$$

Taking into account the consequence of the strong energy condition (1.46) it is obvious, that in order to have accelerated expansion the SEC has to be violated. As we will see hereafter, there are such models of fluids.

Energy conditions and the singularity theorem

In the following we will formulate a sketch of a proof of the specific case of the Penrose–Hawking singularity theorems (for the exact formulation see Hawking and Ellis [1973]). Let us assume

- The FLRW metric

  Combining the equations (1.38), (1.39) and (1.44) we get

$$\dot{H}(t) + H^2(t) + \frac{8\pi}{6}\left(\rho(t) + 3p(t)\right) = 0 \tag{1.48}$$

- The fulfilllment of the SEC: $\rho(t) + 3p(t) \geq 0 \implies$

$$\dot{H}(t) + H^2(t) \leq 0 \implies \frac{d}{dt}\left(\frac{1}{H}\right) \geq 1$$

- The non-contracting Universe: $H(t) \geq 0 \implies$

$$\int_H^{H_0} d\left(\frac{1}{\tilde{H}}\right) \geq \int_t^{t_0} d\tilde{t} \implies \frac{1}{H} \leq t - \left(t_0 - \frac{1}{H_0}\right)$$

  Hence, a homogeneous and isotropic Universe containing matter satisfying the SEC had a Big Bang singularity at finite time $t = t_0 - \frac{1}{H_0}$. This theorem provides one more theoretical criterion for evaluation of the results of the cosmological models. Thanks to it cosmologists know what can be expected from the solutions.

1.1.7 Calculating the system of equations

For simplifying the look of the equations we now avoid writing an argument of some functions explicitly but keep in mind, that density, pressure, scale factor and, accordingly, a Hubble function are the functions of time only.

The next step will be solving the Einstein's equations considering the conservation law. Formulas (1.38), (1.39) and (1.44) (after specifying the state equation) contains all the necessary information to describe the time evolution of a metric:

$$\begin{aligned} G_{tt} &= 3H^2 = 8\pi\rho \\ G_{jj} &= -3H^2 - 2\dot{H} = 8\pi p \\ \dot{\rho} &= -3H(\rho + p) \end{aligned} \tag{1.49}$$

The first equation in this system is called Friedmann equation and the second one is Raychaudhuri equation. From G_{tt} one might obtain the differential equation for the scale factor a:

$$\dot{a} = a\sqrt{\frac{8}{3}\pi\rho} \tag{1.50}$$

In the following paragraphs we will consider different state equations.

1.1.8 Linear barotropic equation of state

One of the simplest models describing a relation of a pressure to a density of the cosmic fluid is so-called linear barotropic equation of state (EoS)

$$p = w\rho \tag{1.51}$$

with a parameter w (the meaning is to be discussed beneath). The last equation of the system (1.49) can be written in the following form:

$$\dot{\rho} = -3H\rho(1+w) \tag{1.52}$$

In this chapter we solve Einstein's equations for the three cases:

1. Considering $w = 0$ for the Universe filled with the incoherent dust and equation (1.52) we get

   $$\dot{\rho}_d + 3H\rho_d = 0$$
   $$\rho_d = \exp\int_t^{t_0} 3H(\tilde{t})d\tilde{t} = \exp(3\ln a_0) - \exp(3\ln a)$$

   Hence the energy density of dust scales as is

   $$\rho_d = \rho_{d0}\left(\frac{a_0}{a}\right)^3 \tag{1.53}$$

   Using the first equation of the system (1.49) we get

   $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi\rho_{d0}\left(\frac{a_0}{a}\right)^3 \tag{1.54}$$
   $$\tag{1.55}$$

   This differential equation can be solved by separation of variables method:

   $$\frac{da}{dt} = \left(\frac{8}{3}\pi\rho_{0d}\frac{a_0^3}{a}\right)^{\frac{1}{2}}$$
   $$\int a^{\frac{1}{2}}da = \sqrt{\frac{8}{3}\pi\rho_{d0}a_0^3}\int dt$$
   $$\frac{2}{3}a^{\frac{3}{2}} = \sqrt{\frac{8}{3}\pi\rho_{d0}a_0^3}(t-t_0)$$

   Hence the time dependence of the scale factor for the dust-filled Universe is

   $$a = \left(\sqrt{6\pi\rho_{d0}}(t-t_0)+1\right)^{\frac{2}{3}} \tag{1.56}$$

   In fact, after extracting the square root of the eq. (1.54) we get two possible signs: the solution of the equation with the plus ($\dot{a} > 0$) is eq. (1.56) and corresponds to expanding scale factor, while the equation with the minus would give a contracting solution ($\dot{a} < 0$). Here and in the following cases we will perform the calculations for only the expanding case since the other solution is contradicted by the observations. Moreover, it might happen that the differential equations will give several other solutions which are unphysical: in the rest of the thesis we simply disregard those.

2. $w = \frac{1}{3}$ for the Universe filled with radiation

$$\dot{\rho}_r + 4H\rho_r = 0$$

Similarly to the "dust case" we obtain

$$\rho_r = \rho_{r0}\left(\frac{a_0}{a}\right)^4 \tag{1.57}$$

The scale factor then

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi\rho_{r0}\left(\frac{a_0}{a}\right)^4$$
$$\int_a^{a_0} \tilde{a}\, d\tilde{a} = a_0^2\sqrt{\frac{8}{3}\pi\rho_{r0}}\int_t^{t_0} d\tilde{t}$$

The scale factor for the Universe filled with the radiation could be expressed as follows:

$$a = 1 + 2\left(\frac{2}{3}\pi\rho_{r0}\right)^{\frac{1}{4}}(t - t_0)^{\frac{1}{2}} \tag{1.58}$$

3. $w = -1$ for the Universe dominated by the cosmological constant

$$\dot{\rho}_\Lambda = 0$$

then

$$\rho_\Lambda = \rho_{\Lambda 0} = constant \tag{1.59}$$

Starting from the equation below we calculate the scale factor using a separation of variables as we did before

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi\rho_{\Lambda 0}$$

implies

$$\frac{da}{dt} = a\sqrt{\frac{8}{3}\pi\rho_{\Lambda 0}}$$

Finally for the Universe dominated by the cosmological constant the scale factor evolution is described by the equation

$$a = \exp\left((t - t_0)\sqrt{\frac{8}{3}\pi\rho_{\Lambda 0}}\right) \tag{1.60}$$

In the following we illustrate the role of the cosmological constant using a tensor of energy and momentum for the perfect fluid in Einstein's equations

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 0 \implies$$
$$G_{\mu\nu} = -g_{\mu\nu}\Lambda \equiv 8\pi(\rho + p)u_\mu u_\nu + 8\pi p g_{\mu\nu}$$

Hence this identification leads to

$$p = -\frac{\Lambda,}{8\pi}, \quad \rho = -p \tag{1.61}$$

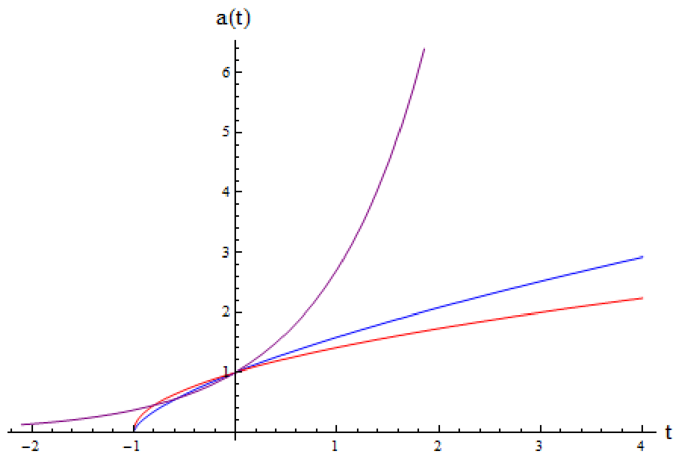

Figure 1.1: Scale factor as a function of time considering linear barotropic equation of state for the flat FLRW metric. Calculations were performed for the Universe filled with the incoherent dust (blue curve), radiation (red curve) and the cosmological constant (purple curve). This and the following graphs describe the results only qualitatively.

Cosmological constant in vacuum then causes the same effect as a negative pressure and can explain the accelerated expansion of the Universe. Now let us consider the Universe, filled with dust, radiation and cosmological constant, which don't interact with each other. The analysis of the dependencies which were obtained gives us the "eras" of dominating of the certain components of the cosmic fluid at different time periods.

Assuming the expansion during all time we see that at early times dominates a "radiation term"- a density of the radiation has the greatest value. On the other hand it is the steepest function so another one will necessarily prevail. There are two possible scenarios of the following evolution. If the dust density function intersects the radiation density curve at the value greater than the cosmological constant density in a certain time the matter era will start. Since the cosmological constant has a tiny, but non-zero constant value, as we know from the observations, this variant is quite plausible. The other option is the beginning of the era of a cosmological constant right after the radiation era. In any of these cases the cosmological constant era is the final one. The described results are plotted in fig.1.2.

1.2 FLRW maximally symmetric metric

In the following chapter, we consider that the curvature of the space-time is not necessarily zero. The requirement of isotropy in spatial directions restricts the metric tensor by an assumption of the constant curvature in the whole Universe. There are the only three constant curvature manifolds, analogous to the two-dimensional sphere, Euclidean space and hyperboloid. One could represent those manifolds using the following metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-\kappa r^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix} \tag{1.62}$$

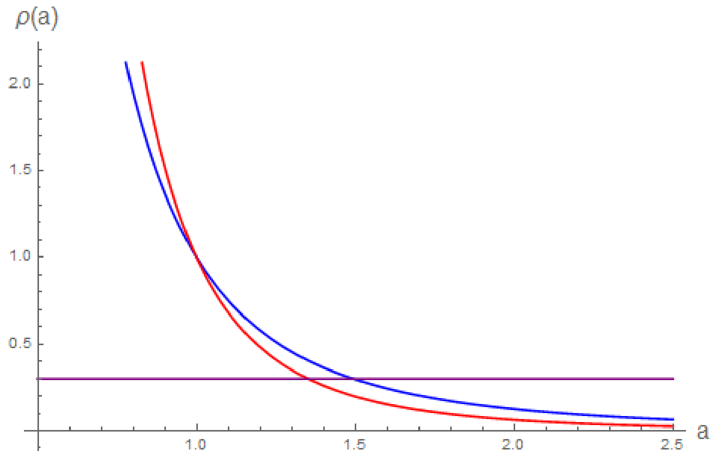

Figure 1.2: The energy density as a function of the scale factor considering linear barotropic equation of state for the flat FLRW metric. The functions were plotted for the three components of the cosmic fluid: an incoherent dust (blue curve), radiation (red curve) and cosmological constant (purple curve).

where κ denotes curvature. The inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-\kappa r^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2 a^2(t)} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 a^2(t)\sin^2\theta} \end{pmatrix} \tag{1.63}$$

and as before we provide the relevant derivatives

$$\begin{aligned} g_{rr,t} &= \frac{2a(t)\dot{a}(t)}{1-\kappa r^2} \\ g_{rr,r} &= \frac{2\kappa r a^2(t)}{(1-\kappa r^2)^2} \\ g_{\theta\theta,t} &= 2a(t)\dot{a}(t)r^2 \\ g_{\theta\theta,r} &= 2a^2(t)r \\ g_{\phi\phi,t} &= 2a(t)\dot{a}(t)r^2\sin^2\theta \\ g_{\phi\phi,r} &= 2a^2(t)r\sin^2\theta \\ g_{\phi\phi,\theta} &= 2a^2(t)r^2\sin(\theta)\cos(\theta) \end{aligned} \tag{1.64}$$

1.2.1 Christoffel symbols

The new set of the Christoffel symbols after adding a curvature looks as follows:

$$\begin{aligned} \Gamma_{r\ r}^{\ r} &= \frac{1}{2}g^{rr}(-g_{rr,r}) = \frac{\kappa r}{1-\kappa r^2} \\ \Gamma_{r\ r}^{\ t} &= \frac{1}{2}g^{tt}(-g_{rr,t}) = \frac{a(t)\dot{a}(t)}{1-\kappa r^2} \\ \Gamma_{\theta\ \theta}^{\ r} &= \frac{1}{2}g^{rr}(-g_{\theta\theta,r}) = -r(1-\kappa r^2) \\ \Gamma_{\phi\ \phi}^{\ r} &= \frac{1}{2}g^{rr}(-g_{\phi\phi,r}) = -r\sin^2\theta(1-\kappa r^2) \end{aligned} \tag{1.65}$$

Other symbols underwent no changes

$$
\begin{aligned}
\Gamma_{\theta\ \theta}^{\ t} &= \frac{1}{2}g^{tt}(-g_{\theta\theta,t}) = a(t)\dot{a}(t)r^2 \\
\Gamma_{\phi\ \phi}^{\ t} &= \frac{1}{2}g^{tt}(-g_{\phi\phi,t}) = a(t)\dot{a}(t)\sin^2\theta r^2 \\
\Gamma_{\phi\ \phi}^{\ \theta} &= \frac{1}{2}g^{\theta\theta}(-g_{\phi\phi,\theta}) = -sin(\theta)\cos(\theta) \\
\Gamma_{r\ t}^{\ r} = \frac{1}{2}g^{rr}(g_{rr,t}) &= \frac{\dot{a}(t)}{a(t)} \\
\Gamma_{\theta\ t}^{\ \theta} = \frac{1}{2}g^{\theta\theta}(g_{\theta\theta,t}) &= \frac{\dot{a}(t)}{a(t)} \\
\Gamma_{\theta\ r}^{\ \theta} = \frac{1}{2}g^{\theta\theta}(g_{\theta\theta,r}) &= \frac{1}{r} \\
\Gamma_{\phi\ t}^{\ \phi} &= \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,t}) = \frac{\dot{a}(t)}{a(t)} \\
\Gamma_{\phi\ r}^{\ \phi} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,r}) &= \frac{1}{r} \\
\Gamma_{\phi\ \theta}^{\ \phi} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,\theta}) &= \cot\theta
\end{aligned}
\tag{1.66}
$$

1.2.2 Ricci tensor and scalar

The equations (1.13) are to be used again considering the Christoffel symbols calculated above. In the first term the only non-zero elements are

$$\Gamma_{r\ r,\alpha}^{\ \alpha} = \Gamma_{r\ r,t}^{\ t} + \Gamma_{r\ r,r}^{\ r} = \frac{\dot{a}^2(t) + a(t)\ddot{a}}{1-\kappa r^2} + \frac{\kappa(1-\kappa r^2) + 2\kappa^2 r^2}{(1-\kappa r^2)^2} \tag{1.67}$$

$$\Gamma_{\theta\ \theta,\alpha}^{\ \alpha} = \Gamma_{\theta\ \theta,t}^{\ t} + \Gamma_{\theta\ \theta,r}^{\ r} = r^2(\dot{a}^2(t) + a(t)\ddot{a}(t)) - 1 + 3\kappa r^2 \tag{1.68}$$

$$\Gamma_{\phi\ \phi,\alpha}^{\ \alpha} = \Gamma_{\phi\ \phi,t}^{\ t} + \Gamma_{\phi\ \phi,r}^{\ r} + \Gamma_{\phi\ \phi,\theta}^{\ \theta} = r^2\sin^2\theta(\dot{a}^2(t) + a(t)\ddot{a}(t) + 3\kappa) - \cos^2\theta \tag{1.69}$$

The components for the second term turn out to be the same as in the flat case (eq. (1.19)) except

$$\Gamma_{r\ \alpha,r}^{\ \alpha} = \Gamma_{r\ r,r}^{\ r} + \Gamma_{r\ \theta,r}^{\ \theta} + \Gamma_{r\ \phi,r}^{\ \phi} = \frac{\kappa(1-\kappa r^2) + 2\kappa^2 r^2}{(1-\kappa r^2)^2} - \frac{2}{r^2} \tag{1.70}$$

As we did in the previous section the off-diagonal terms are not listed here because they are either zero or cancelled after the summation of the all terms of the Ricci tensor. The third term components are

$$
\begin{aligned}
\Gamma_{\beta\ \alpha}^{\ \alpha}\Gamma_{r\ r}^{\ \beta} &= \Gamma_{t\ r}^{\ r}\Gamma_{r\ r}^{\ t} + \Gamma_{r\ r}^{\ r}\Gamma_{r\ r}^{\ r} + \Gamma_{t\ \theta}^{\ \theta}\Gamma_{r\ r}^{\ t} + \Gamma_{r\ \theta}^{\ \theta}\Gamma_{r\ r}^{\ r} + \\
\Gamma_{t\ \phi}^{\ \phi}\Gamma_{r\ r}^{\ t} + \Gamma_{r\ \phi}^{\ \phi}\Gamma_{r\ r}^{\ r} &= 3\frac{\dot{a}^2(t)}{1-\kappa r^2} + \frac{2\kappa^2 r^2}{(1-\kappa r^2)^2} + \frac{2\kappa}{1-\kappa r^2}
\end{aligned}
\tag{1.71}
$$

$$
\begin{aligned}
\Gamma_{\beta\ \alpha}^{\ \alpha}\Gamma_{\theta\ \theta}^{\ \beta} &= \Gamma_{r\ r}^{\ r}\Gamma_{\theta\ \theta}^{\ r} + \Gamma_{t\ r}^{\ r}\Gamma_{\theta\ \theta}^{\ t} + \Gamma_{t\ \theta}^{\ \theta}\Gamma_{\theta\ \theta}^{\ t} + \Gamma_{r\ \theta}^{\ \theta}\Gamma_{\theta\ \theta}^{\ r} + \\
\Gamma_{t\ \phi}^{\ \phi}\Gamma_{\theta\ \theta}^{\ t} + \Gamma_{r\ \phi}^{\ \phi}\Gamma_{\theta\ \theta}^{\ r} &= \kappa r^2 + 3\dot{a}^2(t)r^2 - 2
\end{aligned}
\tag{1.72}
$$

$$\Gamma^{\alpha}_{\beta\ \alpha}\Gamma^{\beta}_{\phi\ \phi} = \Gamma^{r}_{t\ r}\Gamma^{t}_{\phi\ \phi} + \Gamma^{r}_{r\ r}\Gamma^{r}_{\phi\ \phi} + \Gamma^{\theta}_{t\ \theta}\Gamma^{t}_{\phi\ \phi} + \Gamma^{\theta}_{r\ \theta}\Gamma^{r}_{\phi\ \phi} + \Gamma^{\phi}_{t\ \phi}\Gamma^{t}_{\phi\ \phi} + \Gamma^{\phi}_{r\ \phi}\Gamma^{r}_{\phi\ \phi} + \Gamma^{\phi}_{\theta\ \phi}\Gamma^{\theta}_{\phi\ \phi} = 3\dot{a}^2(t)r^2\sin^2\theta - 2\sin^2\theta(1-\kappa r^2) - \kappa r^2\sin^2\theta - \cos^2\theta \tag{1.73}$$

The components of the fourth term are

$$\Gamma^{\alpha}_{\beta\ t}\Gamma^{\beta}_{t\ \alpha} = \Gamma^{r}_{r\ t}\Gamma^{r}_{t\ r} + \Gamma^{\theta}_{\theta\ t}\Gamma^{\theta}_{t\ \theta} + \Gamma^{\phi}_{\phi\ t}\Gamma^{\phi}_{t\ \phi} = 3\frac{\dot{a}^2(t)}{a^2(t)} \tag{1.74}$$

$$\Gamma^{\alpha}_{\beta\ r}\Gamma^{\beta}_{r\ \alpha} = \Gamma^{t}_{r\ r}\Gamma^{r}_{r\ t} + \Gamma^{r}_{t\ r}\Gamma^{t}_{r\ r} + \Gamma^{r}_{r\ r}\Gamma^{r}_{r\ r} + \Gamma^{\theta}_{\theta\ r}\Gamma^{\theta}_{r\ \theta} + \Gamma^{\phi}_{\phi\ r}\Gamma^{\phi}_{r\ \phi} = 2\frac{\dot{a}^2(t)}{1-\kappa r^2} + \frac{2\kappa^2 r^2}{(1-\kappa r^2)^2} + \frac{2}{r^2} \tag{1.75}$$

$$\Gamma^{\alpha}_{\beta\ \theta}\Gamma^{\beta}_{\theta\ \alpha} = \Gamma^{t}_{\theta\ \theta}\Gamma^{\theta}_{\theta\ t} + \Gamma^{r}_{\theta\ \theta}\Gamma^{\theta}_{\theta\ r} + \Gamma^{\theta}_{t\ \theta}\Gamma^{t}_{\theta\ \theta} + \Gamma^{\theta}_{r\ \theta}\Gamma^{r}_{\theta\ \theta} + \Gamma^{\phi}_{\phi\ \theta}\Gamma^{\phi}_{\theta\ \phi} = 2r^2\dot{a}^2(t) + 2(\kappa r^2 - 1) + \cot^2\theta \tag{1.76}$$

$$\Gamma^{\alpha}_{\beta\ \phi}\Gamma^{\beta}_{\phi\ \alpha} = \Gamma^{t}_{\phi\ \phi}\Gamma^{\phi}_{\phi\ t} + \Gamma^{r}_{\phi\ \phi}\Gamma^{\phi}_{\phi\ r} + \Gamma^{\theta}_{\phi\ \phi}\Gamma^{\phi}_{\phi\ \theta} + \Gamma^{\phi}_{t\ \phi}\Gamma^{t}_{\phi\ \phi} + \Gamma^{\phi}_{r\ \phi}\Gamma^{r}_{\phi\ \phi} + \Gamma^{\phi}_{\theta\ \phi}\Gamma^{\theta}_{\phi\ \phi} = 2r^2\dot{a}^2(t)\sin^2\theta + 2\sin^2\theta(\kappa r^2 - 1) - 2\cos^2\theta \tag{1.77}$$

Summarizing terms above we obtain the following components of the Ricci tensor:

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{1.78}$$

$$R_{rr} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1-\kappa r^2} \tag{1.79}$$

$$R_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa) \tag{1.80}$$

$$R_{\phi\phi} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\sin^2\theta \tag{1.81}$$

According to the eq. (1.31). The Ricci scalar then

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right] \tag{1.82}$$

1.2.3 Einstein's equations and the conservation law

As it was in previous section we consider Einstein's equations in form eq. (1.34), matter is approximated by the perfect fluid and the comoving grid is used. Hence the general formulas for the $T_{\mu\nu}$ stays the same as in eqs. (1.36).

Then the particular field equations for the maximally symmetric curved FLRW metric looks as follows:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} &= \frac{8}{3}\pi\rho && \text{for } \mu = \nu = \mathrm{t} \\ \frac{2a\ddot{a} + \dot{a}^2 + \kappa}{a^2} &= -8\pi p && \text{for } \mu,\nu \in \{r,\theta,\phi\} \end{aligned} \tag{1.83}$$

In terms of Hubble's function

$$\begin{aligned} H^2 + \frac{\kappa}{a^2} &= \frac{8}{3}\pi\rho && \text{for } \mu = \nu = \mathrm{t} \\ 2\dot{H} + 3H^2 + \frac{\kappa}{a^2} &= -8\pi p && \text{for } \mu,\nu \in \{r,\theta,\phi\} \end{aligned} \tag{1.84}$$

For the conservation law one may obtain the same equation as it was in the flat FLRW metric

$$T^{\mu}_{t;\mu} \; = \; -\dot{\rho} - 3H\left(\rho + p\right) = 0 \tag{1.85}$$

1.2.4 Calculating the system of equations

On the basis of the above the evolution equations for the curved FLRW metric looks as follows:

$$\begin{aligned} G_{tt} &= H^2 + \frac{\kappa}{a^2} \; = \; \frac{8}{3}\pi\rho \\ G_{jj} &= 2\dot{H} + 3H^2 + \frac{\kappa}{a^2} \; = \; -8\pi p \\ \dot{\rho} &= -3H(\rho + p) \end{aligned} \tag{1.86}$$

From G_{tt} one might obtain the differential equation for the scale factor a. In the following paragraphs we will consider different state equations.

1.2.5 Linear barotropic equation of state

In this part of the thesis we will introduce one useful cosmological function which in this case will help us to solve the differential equations. The **conformal time** is defined as a following function of the cosmic time:

$$\eta \; = \; \int_{t^*}^{t} \frac{dt'}{a(t')} \tag{1.87}$$

All quantities denoted by * corresponds to the values at the Big Bang.

The physical interpretation of this quantity is easier to understand in the geometrized units ($c = 1$), where η also defines so-called **causal horizon** - the total comoving distance that light could have reached since the Big Bang. Two regions of space separated by the distance $l > \eta$ on the comoving grid could have never been in a causal contact.

The system of the equations (1.86) as a function of the conformal time considering (1.51) looks as follows:

$$h^2 + \kappa \; = \; \frac{8}{3}\pi\rho a^2 \tag{1.88}$$

$$2h' + h^2 + \kappa \; = \; -8\pi p a^2 \tag{1.89}$$

$$\rho'(\eta) = -3h(1+w)\rho \tag{1.90}$$

where we defined $h = \frac{a'}{a}$. Primes denotes derivatives with respect to the conformal time η.

In the following part of the thesis, we will find the solutions of the system of the equations (1.88) - (1.90) for the different cosmic fluids taking into account the sign of the curvature. The reader can compare our results to Griffiths and Podolský [2009] and Ellis et al. [2012] .

As we mentioned above, the equation following from the conservation law applies for both curved and flat FLRW metrics. Hence the equations (1.53), (1.57) and (1.59) do not change for non-zero curvature. It remains only to find dependence of the scale-factor on time.

Density as a function of the scale factor can be described by a single formula

$$\rho(a) = \rho_0 a^{-3(1+w)}$$

So then for our purposes we have to solve the Friedmann equation

$$h^2 + \kappa \;=\; \frac{8}{3}\pi\rho_0 a^{-3(1+w)} \tag{1.91}$$

considering different cosmic fluids.

1. Incoherent dust: $w = 0$

   - $\kappa = 1$:
     Computation of the eq. (1.91) in the Wolfram Mathematica provides

     $$a(\eta) = \frac{8}{3}\pi\rho_0 \sin^2\left(\frac{\eta + C_{0p}}{2}\right) \tag{1.92}$$

     where $C_{0p} = 2\arcsin\left(\sqrt{\frac{3}{8\pi}}\right)$. As it is customary in cosmology, we consider $a_0 = a(\eta_0) = 1$. From the definition (1.87) follows

     $$d\eta \;=\; \frac{dt}{a}$$

     then

     $$\int_{\eta}^{\eta_0} a(\eta) d\eta \;=\; \int_t^{t_0} dt$$

     which allows us to find the relation between cosmic and conformal times

     $$\boxed{t(\eta) \;=\; t_{0p} + \tfrac{4}{3}\pi\rho_0\left(\eta - \sin\left(C_{0p} + \eta\right)\right)}$$

     Using a parametric plot we obtain the scale factor as a function of a cosmic time (see fig. 1.3). For all graphs in this chapter we use values $\rho_0 = 1$ and for the curved cases $|\kappa| = 1$.

   - $\kappa = 0$: For the scale factor we get

     $$a(\eta) \;=\; \frac{1}{3}\left(3 - 2\sqrt{6\pi}\eta + 2\pi\eta^2\right) \tag{1.93}$$

     From the definition (1.87) we obtain

     $$\boxed{t(\eta) \;=\; t_{00} + \tfrac{1}{3}\left(3\eta - \sqrt{6\pi}\eta^2 + \tfrac{2}{3}\pi\eta^3\right)}$$

   - $\kappa = -1$:

     $$a(\eta) = \frac{8}{3}\pi\rho_0 \sinh^2\left(\frac{1}{2}(\eta + C_{0n})\right) \tag{1.94}$$

     where $C_{0n} = 2\,\text{arcsinh}\sqrt{\frac{3}{8\pi}}$.
     Finally we derive the connection between the cosmic time and the conformal time

     $$\boxed{t(\eta) \;=\; t_{0n} + \tfrac{8}{3}\pi\rho_0\left(-\tfrac{\eta}{2} + \tfrac{1}{2}\sinh\left(\eta + C_{0n}\right)\right)}$$

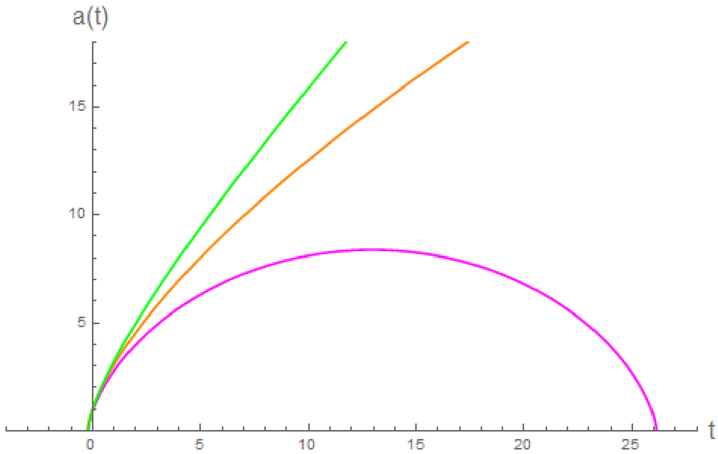

Figure 1.3: Scale factor as a function of time considering linear barotropic equation of state for the curved FLRW metric. Calculations were performed for the Universe filled with the incoherent dust for the cases of positive (pink curve), zero (orange curve) and negative (green curve) curvatures. Used parameters: $t_{0p} = 2.75, t_{00} = -0.47, t_{0n} = -3.05$

We get only decelerating solutions, one corresponds to the recollapsing Universe and the others imply the ever lasting expansion. For all solutions the Universe begins with a singularity in a finite time, that is in line with the predictions of the SEC.

2. Radiation: $w = \frac{1}{3}$
In this paragraph the calculations are done in the same way.

- $\kappa = 1:$
Solution of the eq. (1.91) is

$$a(\eta) = \sqrt{\frac{8\pi}{3}}\rho_0 \sin\left(\eta + C_{\frac{1}{3}p}\right) \tag{1.95}$$

where $C_{\frac{1}{3}p} = \arcsin\left(\sqrt{\frac{3}{8\pi}}\right)$. From the definition (1.87) follows

$$\boxed{t(\eta) \;=\; t_{\frac{1}{3}p} - \rho_0\left(\sqrt{-1+\tfrac{8\pi}{3}}\cos\eta + \sin\eta\right)}$$

- $\kappa = 0:$ For the scale factor we get

$$a(\eta) \;=\; \frac{1}{3}\rho_0\left(3 + 2\sqrt{6\pi}\eta\right) \tag{1.96}$$

From the definition (1.87) we obtain

$$\boxed{t(\eta) \;=\; t_{\frac{1}{3}0} + \rho_0\left(\eta + \sqrt{\tfrac{2\pi}{3}}\eta^2\right)}$$

- $\kappa = -1:$

$$a(\eta) = \sqrt{\frac{8\pi}{3}}\rho_0 \sinh\left(\eta + C_{\frac{1}{3}n}\right) \tag{1.97}$$

where $C_{\frac{1}{3}n} = \text{arcsinh}\sqrt{\frac{3}{8\pi}}$.
The connection between the cosmic time and the conformal time then

$$\boxed{t(\eta) \;=\; t_{\frac{1}{3}n} + \rho_0\left(\sqrt{1+\tfrac{8\pi}{3}}\cos\eta + \sin\eta\right)}$$

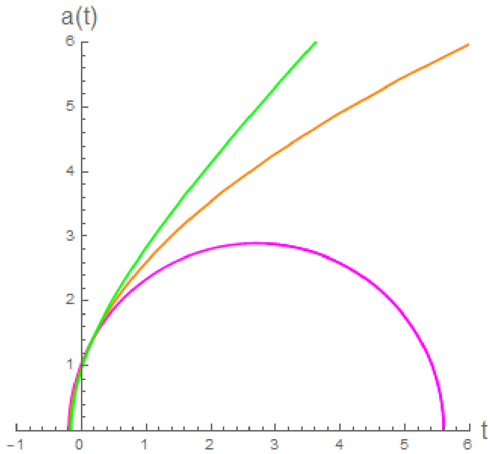

Figure 1.4: Scale factor as a function of time considering linear barotropic equation of state for the curved FLRW metric. Calculations were performed for the Universe filled with a radiation for the cases of positive (pink curve), zero (orange curve) and negative (green curve) curvatures. Used parameters: $t_{0p} = 2.7, t_{00} = 0, t_{0n} = -3.05$

Qualitatively, our results are similar to the results obtained for the incoherent dust, since the EoS for the radiation satisfies conditions of the singularity theorem as well.

3. Cosmological constant $w = -1$.
In this subsection we calculate the scale factor from the system (1.86) directly without using a conformal time. We consider $\Lambda > 0$.

- $\kappa > 0$: The eqs. (1.88) and (1.89) yield

$$\dot{a}^2(t) + |\kappa| \;=\; \frac{8}{3}\pi\rho_{0\Lambda}a^2(t) =: ba^2(t)$$

Having used the separation of variables method

$$\int_a^{a_0} \frac{da}{\sqrt{ba^2(t) - |\kappa|}} \;=\; \int_t^{t_0} dt$$

we can express time as a function of the scale factor

$$t \;=\; t_0 + \frac{\log\left(\sqrt{\beta^2a^2(t) - 1} + \beta a(t)\right)}{\sqrt{b}} + C_{-1p} \qquad (1.98)$$

Let us introduce the notation that we are using for making our calculations more clear

$$\begin{aligned}
\beta &= \sqrt{\frac{b}{|\kappa|}} \\
C_{-1p} &= \frac{1}{\sqrt{b}} \log\left(\sqrt{\beta^2 - 1} + \beta\right) \\
t_0 &= C_{-1p} - t0 - \frac{\log\sqrt{b|\kappa|}}{\sqrt{b}}
\end{aligned}$$

Appying identities (1.61) and

$$\log\left(\sqrt{\beta^2 a^2(t) - 1} + \beta a(t)\right) = \operatorname{arccosh}\left(\beta a(t)\right)$$

one gets

$$\boxed{a(t) = \sqrt{\tfrac{3|\kappa|}{\Lambda}} \cosh\left[(t - t_0)\sqrt{\tfrac{\Lambda}{3|\kappa|}}\ \right]} \tag{1.99}$$

- $\kappa = 0$: In accordance with previous notation

$$\dot{a}(t) = \frac{8}{3}\pi\rho_{0\Lambda} a(t) =: \sqrt{b} a(t)$$

Hence

$$\boxed{a(t) = \exp\left(\sqrt{\tfrac{\Lambda}{3}}(t - t_0)\right)} \tag{1.100}$$

- $\kappa < 0$: In this case we have the differential equation of the following form

$$\dot{a}^2(t) - |\kappa| = \frac{8}{3}\pi\rho_{0\Lambda} a^2(t)$$

Using the same notation we can express a cosmic time as a function of the scale factor as it was done above

$$t = t_0 + \frac{\log\left(\sqrt{\beta^2 a^2(t) + 1} + \beta a(t)\right)}{\sqrt{b}} + C_{-1n} \tag{1.101}$$

The useful identities for this case are (1.61) and

$$\log\left(\sqrt{\beta^2 a^2(t) + 1} + \beta a(t)\right) = \operatorname{arcsinh}\left(\beta a(t)\right)$$

The result for the scale factor is

$$\boxed{a(t) = \sqrt{\tfrac{3|\kappa|}{\Lambda}} \sinh\left[(t - t_0)\sqrt{\tfrac{\Lambda}{3|\kappa|}}\ \right]}$$

We should mention, that not every point a (t) = 0 necessarily corresponds to a physical singularity. Depending on the choice of coordinates, so-called coordinate singularities can arise. A detailed analysis of the singularities goes beyond the scope of this thesis.

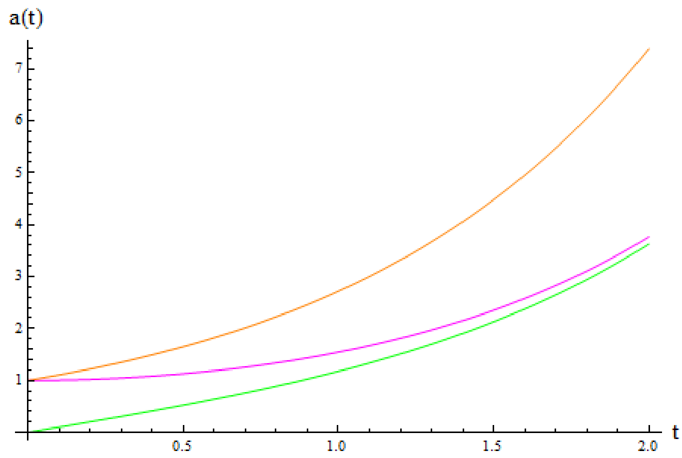

Figure 1.5: Scale factor as a function of time considering linear barotropic equation of state for the curved FLRW metric. Calculations were performed for the Universe dominated by the cosmological constant for the cases of positive (pink curve), zero (orange curve) and the negative (green curve) curvatures. Used parameters: $b = 1, t_0 = 0$.

4. "Curvature fluid" $w = -\frac{1}{3}$
   Here we would like to introduce one interesting correspondence of the constant negative curvature in the empty FLRW space to the cosmic fluid in the flat FLRW space. But firstly we will perform the auxiliary calculation. Now we are interested in the solution of the system of the equations for the flat case (1.86) considering the following EoS:

$$p = -\frac{1}{3}\rho \tag{1.102}$$

The continuity equation then gives

$$\dot{\rho} = -2H\rho$$

The solution for density as a function of a scale factor is the result we needed

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^2 \tag{1.103}$$

On the other hand, the Friedmann equation (1.83) in vacuum one can write using the assumption $\kappa < 0$ like this

$$H^2 = \frac{|\kappa|}{a^2}$$

Comparing to the eq.(1.103) we see, that it has the same form as the Friedmann equation for the flat metric. Then for $|\kappa| = \rho_0 a_0^2$ the RHS is proportional to the energy density so we can treat the negative curvature term as a component of the density of some cosmic fluid. We call it the "curvature fluid" to underline, that it is the curvature term that can play the same role as the cosmic fluid under some conditions.

The time evolution of the scale factor for this case is

$$a(t) = a_0\sqrt{\frac{8}{3}\pi\rho_0}\; t + C$$

2. ΛCDM model with radiation in curved FLRW metric

One of the most common ways of describing the Universe is to consider several components, the role of which increases or decreases with the expansion of the Universe. This is what we do in this chapter. In addition to matter, we assume the presence of radiation, curvature of the space-time and also a non-vanishing cosmological constant. Hence, according to the accepted model the Universe is uniformly filled with these components of the cosmic fluid, the energy density is now the sum of the densities of the individual components. The Friedmann equation then looks as follows:

$$H^2 + \frac{\kappa}{a^2} = \frac{8}{3}\pi \sum_X \rho_X$$

where $X \in \{m, r, \Lambda, \kappa\}$. We denoted the energy density of matter, radiation, cosmological constant and curvature by ρ_m, ρ_r, ρ_Λ and ρ_κ respectively. One can write the equation in the dimensionless form

$$1 = \frac{8\pi}{3H^2}\rho_m + \frac{8\pi}{3H^2}\rho_r + \frac{8\pi}{3H^2}\rho_\Lambda - \frac{\kappa}{a^2H^2}$$

So we can denote it as

$$1 = \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_\kappa \tag{2.1}$$

According to the results obtained by the Planck mission (Ade et al. [2016]) the curvature of the observable Universe is very close to the zero value ($|\kappa| < 0.005$) and the errorbar allows both positive and negative values of the curvature. As it was stipulated in the previous chapter the negative curvature can be interpreted as a component of the cosmic fluid (the "curvature fluid"). In case of the positive curvature, such an interpretation is not possible, but we can all still use the resulting equation (2.1) "leaving Ω_κ on the left-hand side" and perceive it not as a source, but as an element of geometry.

We consider non-interacting components, which makes it impossible to describe the early Universe within the framework of this model. For this purpose, the scalar field model described in the next chapter can be used.

Despite the unimaginable accuracy of the measuring instruments that modern scientists have at their disposal, determination of values of certain physical quantities remains challenging. In order to reduce the number of parameters with a non-zero measurement error, we use the derived equation (2.1) and express Ω_κ in terms of the other omega factors.

Let us remind readers of the results obtained in the previous chapter

$$\rho_m = \rho_{m0}\left(\frac{a}{a_0}\right)^{-3} \qquad \rho_r = \rho_{r0}\left(\frac{a}{a_0}\right)^{-4} \qquad \rho_\kappa = \rho_{\kappa 0}\left(\frac{a}{a_0}\right)^{-2} \qquad \rho_\Lambda = \rho_{\Lambda 0}$$

For our purposes let us consider $x = \frac{a}{a_0}$ and rewrite the Friedmann equation in another useful dimensionless form

$$H^2(x) = \left(\frac{1}{x}\frac{dx}{dt}\right)^2 = H_0^2\left(\Omega_{m0}\,x^{-3} + \Omega_{r0}\,x^{-4} + \Omega_{\Lambda 0} + [1 - \Omega_{m0} - \Omega_{r0} - \Omega_{\Lambda 0}]\,x^{-2}\right) \tag{2.2}$$

Separation of variables gives

$$t(a) = \frac{1}{H_0} \int_0^{\frac{a}{a_0}} \frac{d\tilde{x}}{\sqrt{(\Omega_{m0}\,\tilde{x}^{-3} + \Omega_{r0}\,\tilde{x}^{-4} + \Omega_{\Lambda 0} + [1 - \Omega_{m0} - \Omega_{r0} - \Omega_{\Lambda 0}]\,\tilde{x}^{-2})}} \tag{2.3}$$

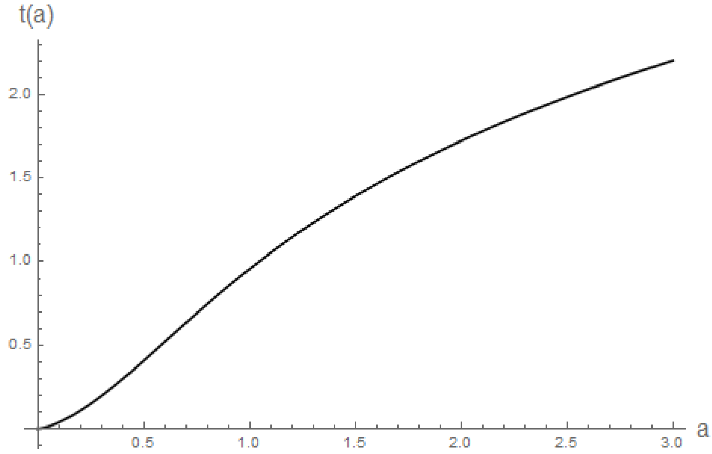

Figure 2.1: The evolution of the scale factor for the curved FLRW metric considering linear barotropic EoS for the dustlike matter, radiation and cosmological constant. Used parameters: $\Omega_{m0} = 0.3089$, $\Omega_{\Lambda 0} = 0.6914$ (Ade et al. [2016]), $\Omega_{r0} = 9 \cdot 10^{-5}$ (Jeong [2017]).

This model is a combination of the solutions discussed in the first chapter. At the beginning of evolution, we observe a slow growth, which corresponds to the domination of matter and radiation. As is shown on the fig. 1.2, the cosmological constant will eventually dominate and the expansion will become accelerated. This model describes the hot past of the dense Universe that we know through the analysis of CMB, provides a theoretical description of Hubble's observations and of a wide range of other observational evidences.

But the ΛCDM model is not unique in its kind. Nowadays scientists are actively trying to find alternative models that would not contain as many quantities which are not described by modern physics yet.

Furthermore, ΛCDM model needs to be complemented by a theory for describing the early Universe - the time period between the Big Bang and the so-called photon decoupling (the time when the opaque plasma that filled the early Universe cooled down enough to the formation of matter and allowed the radiation resulting from the release of binding energy to spread in space freely). Such a description lies on the shoulders of particle physics, quantum field theory and, perhaps, on the yet unformed theory of quantum gravity. Cosmologists, in turn, are trying to offer a model that would complement the ΛCDM for describing the expected behaviour of the primordial Universe. This aspect, however, is outside the scope of the present thesis.

The models which try to describe the dark sector of the ΛCDM model in alternative ways are discussed in the next chapter.

3. Extensions and alternatives to ΛCDM model

According to observations, ΛCDM model describes the observable Universe with remarkable accuracy. In the framework of this model, there is a need for the existence of the so-called ”dark sector”: about 26% of the total matter in the Universe is occupied by the dark matter and 69% by the dark energy. Only 5% remains to the ”normal” (baryonic) matter that is well known to us, and since modern science is not yet able to provide a detailed description of the nature and properties of the ”dark quantities” physicists are actively developing alternative models in an attempt to ”reduce the number of unknowns”. We bring to your attention several such alternative models.

3.1 Generalized Chaplygin gas

Chaplygin gas is such a substance which is described by the equation of state first introduced by S. Chaplygin in 1904 in the context of hydromechanical engineering. In the context of cosmology, it belongs to the class of the so-called Unifying Dark Fluids. The concept is based on the ambition to describe dark energy and dark matter not as separate components, but as a single substance. The equation of state reads:

$$p = -\frac{A}{\rho}$$

where A is a positive constant.
Nowadays one of the main objects of study in the non-standard cosmology is so-called generalized Chaplygin gas (gCg), which equation of state can be formulated like

$$p = -\frac{A}{\rho^{\alpha}} \tag{3.1}$$

where α is a positive constant. The positiveness is a consequence of the restriction given by the requirement for the sound velocity to be lower then the light has:

$$v_s = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S} = \sqrt{\alpha \frac{A}{\rho^{\alpha+1}}} < 1$$

where S denotes entropy, which is constant automatically since we consider an isolated system. Derived condition gives

$$\alpha A < \rho^{\alpha+1}$$

The weak energy condition then implies the positiveness of both parameters. In the scientific literature is commonly used the restriction on this parameter $0 < \alpha < 1$, but in this thesis we consider any positive value.

3.1.1 FLRW flat metric

At this place, we pursue the same goal - to get the relation between the scale factor and the cosmic time. As in previous chapter, we find the density as a function of the scale factor first. The last equation of system 1.49 now has a following form

$$\dot{\rho} \; = \; -3H\left(\rho - A\rho^{-\alpha}\right)$$

Using separation of variables method and the boundary condition $a(0) = 1$ gives

$$\int_{\rho}^{\rho_0} \frac{d\tilde{\rho}}{\rho - A\rho^{-\alpha}} \; = \; -\int_{t}^{t_0} 3H d\tilde{t}$$

After the integration using the dependence of the Hubble's function on a scale factor, we get the equality beneath (where we also relabel some constants):

$$\ln a^{3(1+\alpha)} \; = \; \ln\left(\frac{\rho_0^{\alpha+1} - A}{\rho^{\alpha+1} - A}\right) =: \ln\left(\frac{C_0}{\rho^{\alpha+1} - A}\right)$$

We further have

$$\rho(a) \; = \; \left(\frac{C_0}{a^{3(1+\alpha)}} + A\right)^{\frac{1}{1+\alpha}} \tag{3.2}$$

Using again a separation of variables method for the eq. (1.50) we put the integrals to Wolfram Mathematica so we get a relation between a cosmic time and a scale factor:

$$t(a) \; = \; \frac{2}{3}\sqrt{\frac{3}{8\pi \mathrm{G}}}\left(A + C_0 a^{-3(1+\alpha)}\right)^{-\frac{1}{2(1+\alpha)}}\left(1 + \frac{A}{C_0}a^{3(1+\alpha)}\right)^{\frac{1}{2(1+\alpha)}}$$
$$_2F_1\left(\frac{1}{2\alpha+2}, \frac{1}{2\alpha+2}; 1 + \frac{1}{2\alpha+2}; -\frac{A}{C_0}a^{3(\alpha+1)}\right)$$

One can mention, that in the limit of the early times the equation (3.2) implies that the energy density diverges. Using this fact in the eq. (3.1) we got the EoS for the incoherent dust (see eq. (1.53)). Later for the great values of the scale factor the density has an almost constant value, as in the results we gained for the cosmological constant (eq. (1.59)). As can be seen from the graph (fig. 3.1) there is an inflexion point and after the deceleration epoch follows accelerating one, which we observe nowadays.

There is one more way to analyze the results were obtained. On diagram 3.2 we plot the linear barotropic EoS and generalized Chaplygin gas EoS. The time evolution of the systems, filled with a corresponding cosmic fluid, is going along these curves in the direction shown with arrows. On this figure is illustrated the result, derived in the section about the energy conditions. The "normal matter" such as the dust and the radiation fulfills the SEC and, according to the (1.47) can not ensure the accelerated expansion (see fig. 1.3 and 1.4). On the other hand, there is a fluid from the dark sector, which violates the SEC and provide the exponential expansion (see fig. 1.5). The generalized Chaplygin gas has more dynamic evolution: starting in the same range as the dust on the diagram 3.2 the expansion decelerates so long as it crosses the border of the fulfillment of the

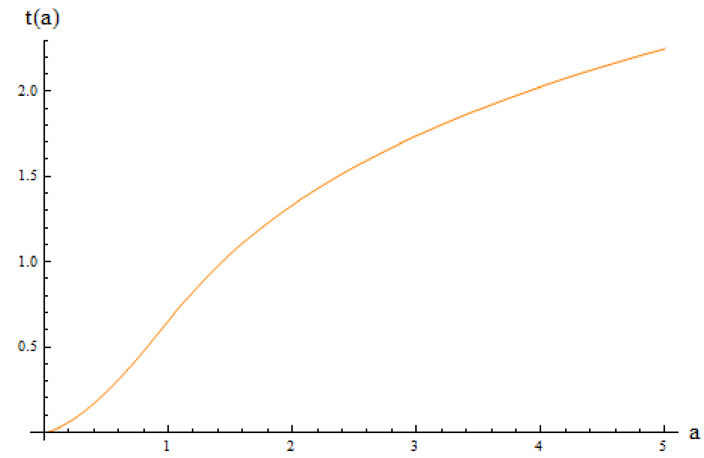

Figure 3.1: Scale factor as a function of time considering the equation of state for the generalized Chaplygin gas for the flat FLRW metric. Used parameters $\alpha = 2, C_0 = 1, A = 1$.

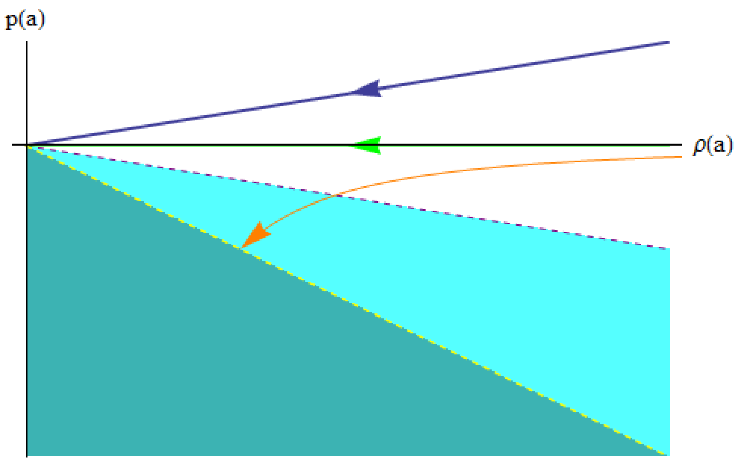

Figure 3.2: Here the area filled with lighter (darker) blue colour corresponds to the violation of the SEC (WEC). The EoS of the cosmological constant is denoted by a yellow dashed line, incoherent dust by the green, radiation by the blue and gCg by the orange curve. The arrows indicate the directions of the increasing scale factor. Parameters for the gCg: $\alpha = 2, C_0 = 1, A = 1$.

SEC (on figs. 3.1 and 3.3 - the inflexion points). Hereafter follows the expansion with the non-constant acceleration. At the limit of the infinite time, the EoS converges to the cosmological constant one.

3.1.2 FLRW maximally symmetric metric

We consider the same EoS as for the previous section so the result (3.2) is still valid. Having put it to the Friedmann equation we aim to solve the following problem

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi}{3}\left(\frac{C_0}{a^{3(1+\alpha)}} + A\right)^{\frac{1}{1+\alpha}} \tag{3.3}$$

Considering cases of the positive and negative curvatures separately we find out that the type of the resulting function does not depend on the sign of the curvature. Using Wolfram Mathematica we introduce the result for the both cases (fig. 3.3).

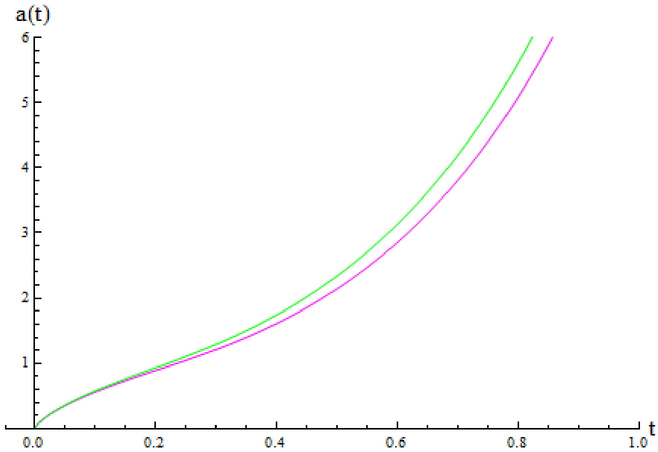

Figure 3.3: Scale factor as a function of time considering the EoS for the generalized Chaplygin gas for the FLRW metric with a positive (pink curve) and negative (green curve) curvatures. Used parameters $\alpha = 2, C_0 = 1, A = 1$

The curves do not differ qualitatively from each other. Moreover, the generalized Chaplygin gas for all signs of curvature gives a result similar to the ΛCDM model predictions. The advantage of the gCg is the only one fluid, which provides such a behaviour. Nevertheless, this equation of state does not answer the question of nature of such substance.

3.2 Scalar field

In order to describe certain models of gravitation one has to introduce "fields" as sources of Einstein's equations: a well-known example, not in cosmology, is the inclusion of electromagnetic fields in black hole models, giving Reissner-Nordstrom solution; in cosmology, one needs fields to describe for instance the primordial inflation. Fields are obviously necessary if one wants to include quantum effects in cosmology, such as the formation and evolution of CMB fluctuations (however this quantum approach is outside the scope of the thesis). Scalar fields, under certain conditions, can be equivalent to normal fluids, but they usually have more general behavior: this makes them suitable for extending the standard model. The main problem is to use physically reasonable fields whose particles could be experimentally found. In this thesis, we would like to introduce simple scalar fields which behave like the perfect fluids and can give rise to accelerated expansion and the "dark matter" behavior.

3.2.1 Inflation

Motivation

Inflation theory claims, that there was the epoch of the rapid expansion of the Universe in early times. One of the main problems, which lead to the inflation theory is a *horizon problem*. As was mentioned in the introduction, we observe the very homogeneous and isotropic Universe on its huge scales. It raises the question, how it is possible, if, according to standard cosmology, such distant space regions have never been in causal contact.

For further purposes, we now introduce the essential cosmological concept - a **comoving Hubble radius**. The small modification of the definition of the conformal time (1.87) and the relation $\frac{da}{adt} = H$ leads us to the so-called **particle horizon**:

$$\chi_{ph}(\eta) = \int_{t^*}^{t} \frac{d\tilde{t}}{a(\tilde{t})} = \int_{a^*}^{a} \frac{d\tilde{a}}{\tilde{a}\dot{\tilde{a}}} = \int_{\ln a^*}^{\ln a} \frac{d\ln\tilde{a}}{\tilde{a}H(\tilde{a})} \tag{3.4}$$

Physically, the particle horizon at a certain conformal time η is the border of the space region from where the light signal could reach us from the beginning of time. Obviously, regardless of the dynamics of the Universe, this value can not decrease with time. On the light cone in the comoving frame, the particle horizon is represented by the intersection of the past of the light cone with the moment $\eta = \eta^*$.

The integrand on RHS determines the above-mentioned Hubble radius, which indicates whether two points in space are in causal contact *now*. For the Universe filled with a cosmic fluid with a constant linear barotropic EoS one can get

$$(a\,H)^{-1} = \frac{1}{H_0} a^{\frac{1}{2}(1+3w)}$$

Hence, according to the SEC for the perfect fluid (1.46) for the "normal matter", the comoving Hubble radius grows with time as the Universe expands.

Let us introduce the horizon problem using fig. 3.4, where letters p and q denotes two distant objects, which we observe nowadays (they are in our past light cone after the photon decoupling). From their past light cones one can deduce, that they have never been in causal contact according to evaluation based on predictions of the standard cosmological models.

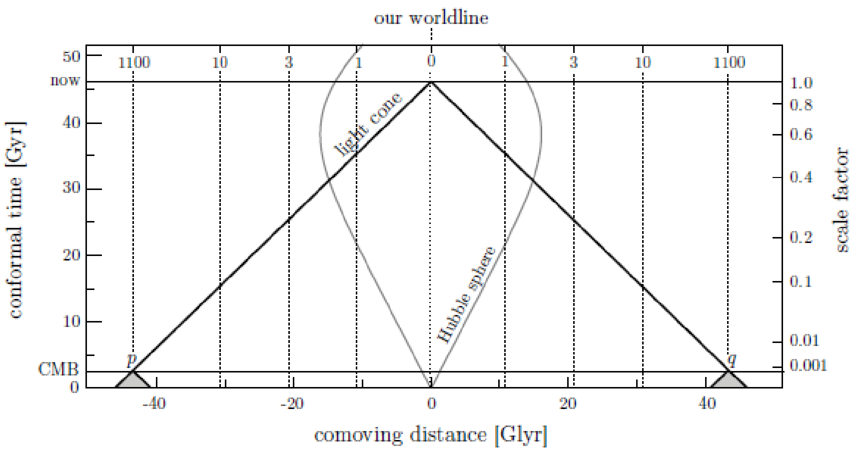

Figure 3.4: Here horizontal line $\eta = 0$ is the beginning of the time, line denoted as *CMB* is a period of recombination and photon decoupling. It is assumed, that the primordial Universe had the non-zero size. Then past light cones of the events p and q do not intersect - they have never been in the causal contact. There are difficulties in explaining the isotropy and homogeneity of the Universe. The figure is taken from Baumann [2015].

The main hypotheses stipulate, that either the primordial Universe had an extremely high homogeneity, or the distant regions actually *were* in the causal contact (see fig. 3.5).

Requiring the fulfillment of any of the conditions below for the early Universe allows us to gain the interesting result: the Hubble sphere will shrink, but the particle horizon will not change its size. In order to observe homogeneity and isotropy on such a large scale, a sufficiently long inflation is necessary, so one can obtain $\chi_{ph} \gg (a\,H)^{-1}$.

Decreasing Hubble radius	Accelerated expansion	SEC violation
$\frac{d}{dt}\left(\frac{1}{a\,H}\right) < 0$	$\frac{d^2 a}{dt^2} > 0$	$\rho + 3p < 0$

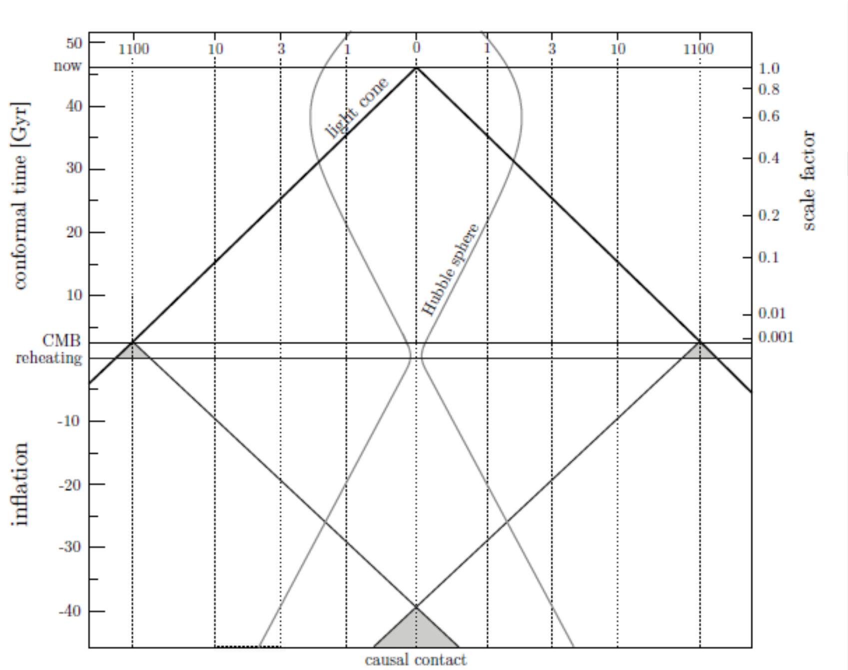

Figure 3.5: Comparing to the fig. 3.4 the Universe's history is extended towards the past. The Big Bang singularity takes place earlier on the timeline and on its original place there is the period of the transition from inflation to the "standard" evolution. Thus, as can be seen from the past light cones, any two events are connected by a common causal past. The figure is taken from Baumann [2015].

Inflation dynamics

One of the simplest models of the inflation can be constructed using the scalar field. The theory is written in terms of the scalar field $\phi(t, \mathbf{x})$ - the so-called inflaton - and the potential V(ϕ). Quantum Field Theory provides the expression of the tensor of energy and momentum in terms of these quantities (derived in Tong [2006]):

$$T_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi\, \partial_\beta \phi \right)$$

In accordance with the cosmological principle described in the introduction we require homogeneity and isotropy, that is $\phi = \phi(t)$. with the help of the symmetry of the FLRW metric, one can obtain

$$
\begin{aligned}
\rho(\phi) &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\
p(\phi) &= \frac{1}{2}\dot{\phi}^2 - V(\phi)
\end{aligned}
\tag{3.5}
$$

Thus we see, that the energy density is given by the sum of the kinetic and the potential energy of the scalar field.

Proceeding from this, the continuity equation takes the form

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{3.6}$$

For the derivation see for instance Baumann [2015].

Power-law inflation

In this section, we would like to introduce to your attention a procedure that will show what kind of the potential we should consider in order to get the required Universe's behaviour (Ellis and Madsen [1991]). We provide our calculations for the flat FLRW metric. Let us assume, that the relation between the scale factor and a cosmic time can be represented by the following formula:

$$a(t) = a_0\, t^p \tag{3.7}$$

where $p > 1$ (in order to obtain the accelerated expansion) and $a_0 > 0$ are parameters. Indeed, this kind of the evolution could arrange the shrinking of the Hubble sphere.

Combining the Friedmann and the Raychaudhuri equations from the system (1.49) we obtain

$$
\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a} &= 8\pi\dot{\phi}^2(t) \\
\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 &= 8\pi V(t)
\end{aligned}
\tag{3.8}
$$

Having solved the first equation with the assumption eq. (3.7) we can invert the result:

$$\phi(t) = C_1 \pm \frac{\sqrt{p}\log t}{2\sqrt{\pi}} \implies t(\phi) = \exp\left(\pm 2\sqrt{\frac{\pi}{p}}(\phi - C_1)\right)$$

Then from the second eq. (3.8) we obtain the following form of the potential:

$$V(\phi) = V_0\, e^{\pm\lambda\,\phi} \tag{3.9}$$

Where we denoted

$$\lambda = 4\sqrt{\frac{\pi}{p}}$$

$$V_0 = \frac{3p(p-1)}{8\pi} e^{\mp\lambda C1}$$

Summarizing, it should be noted that this potential, one of the few, is simple enough to get an analytical solution. Even considering the non-zero curvature will greatly complicate the posed problem. But the importance of the scalar field concept is hard to overestimate since it allows us to describe an impressive part of the history of the Universe before the recombination. The power-law model is the simple model of inflation, but it does not fit the data available in the best way. One of the main problems of inflation models is the difficulty in choosing the potential that would provide to inflation the finite duration and "give way" to other components of the cosmic fluid.

Such result is discussed in detail in Liddle and Lyth [2000].

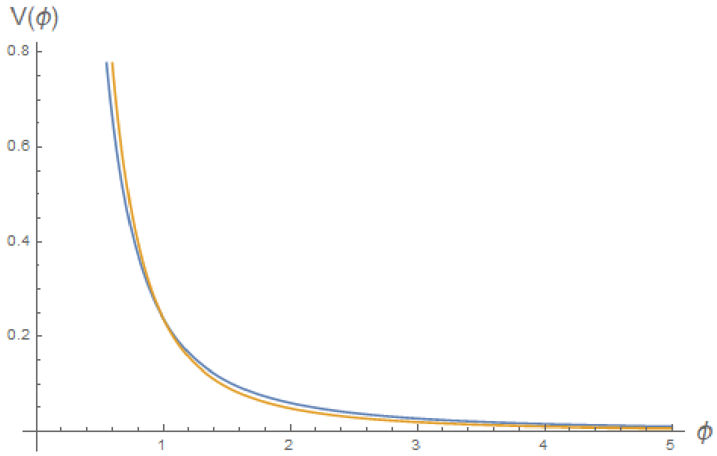

Figure 3.6: The inflationary potential as a function of the scalar field for the FLRW metric for the different values of the parameters. Used parameters are $c_1 = 1$ for both curves, $p = 3$ (blue curve), $p = 4$ (orange curve).

3.2.2 Scalar field as alternative to the ΛCDM model

In the last section of this thesis, we are taking a direct calculation of the scale factor evolution in the curved FLRW metric considering the potential mentioned in Acquaviva and Lukes-Gerakopoulos [2017]

$$V(\phi) = c_1 \cosh^2\left(\sqrt{\frac{3}{8}}\phi\right) + c_2 \tag{3.10}$$

In order to get the non-negative mass of the field, we should choose the appropriate parameters, satisfying $c1 > 0, c_1 + c_2 \geq 0$.

For the description of the field dynamic we use the eq. (3.6), which together with the Raychaudhuri equation forms a system of equations that contains all the necessary information to solve our problem. Substituting the potential in the above equations, we use Wolfram Mathematica for the numerical solution. We used the boundary condition $a(0) = \dot{a}(0) = \phi(0) = \dot{\phi}(0) = 1$.

It can be seen that the fig. 3.8 we obtain agrees with the results of calculations for the Chaplygin gas or for the ΛCDM model with the multi-component Universe: expansion throughout all evolution; decelerated at first and from a certain moment to infinity expands with acceleration. Therefore, this model could be able to describe the Universe as well as the models specified earlier.

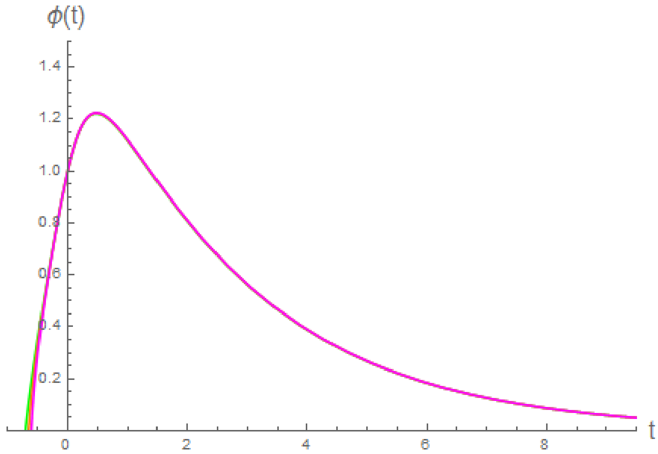

Figure 3.7: Scale factor as a function of time considering the equation of state for the scalar field for the FLRW metric for the positive (pink), zero (orange) and negative (green) curvatures. Used parameters are $c_1 = 1, c_2 = -\frac{1}{3}$.

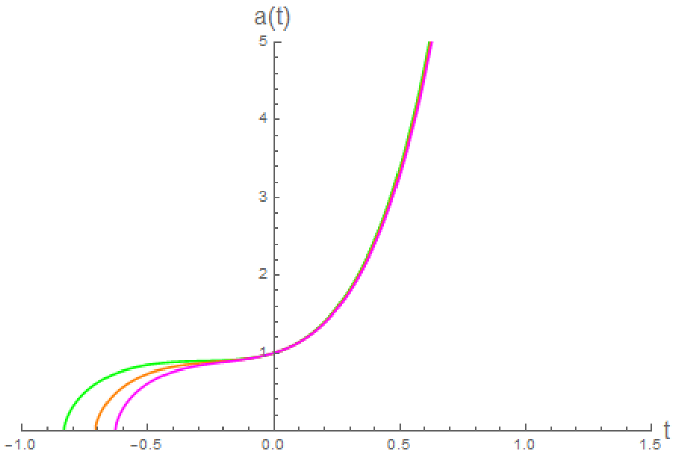

Figure 3.8: Scale factor as a function of time considering the equation of state for the scalar field for the FLRW metric for the positive (pink), zero (orange) and negative (green) curvatures. Used parameters are $c_1 = 1, c_2 = -\frac{1}{3}$.

Conclusion

From the calculations carried out in the first chapter it is clear that normal matter is not able to describe the whole behaviour of our Universe, so here prevails the substance unknown to us yet.

There is also an open issue with the correctness of the SEC: our expectations, which are based on rich experience with the ordinary matter, fail. However, this result depends on the metric and one of the possible solutions is the hypothetical deviation of the real Universe's metric from the FLRW so that the SEC can be satisfied even during the accelerated expansion of the Universe.

In the context of the standard ΛCDM model, the combination of cosmic fluids in the FLRW metric using the observational data provides plausible results for the Universe's evolution - we get an accelerated expansion at the present time and the finite age of the Universe. But this theory, as we have seen, is not the only one able to provide similar results. Generalized Chaplygin gas also copes with this task, describing DM and DE as the interconnected components of one whole. As can be seen from the graph (3.3), the presence of the non-zero curvature has no remarkable influence on the evolution of the Universe. Scalar field with the potential (3.10) gives similar results as gCg and ΛCDM.

Theoretical physics has gone further and is already able to look beyond the veil of an optically not observed period before the recombination considering the EoS for the scalar field. An actual problem is choosing the appropriate potential that would ensure inflation (reduction of the Hubble sphere) in the early stages of the development of the Universe, but later gives way to the remaining component.

The models discussed here do not have a compelling physical theory behind it yet. Nowadays our data from observations is not enough for us to be able to choose the "right one", but greater forces are thrown at finding the unique theoretical predictions about each of the theories that could be confirmed experimentally.

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List of Figures

List of Abbreviations

- GTR - General Theory of Relativity
- STR - Special Theory of Relativity
- CMB - Cosmic Microwave Background
- ΛCDM - Λ-Cold Dark Matter. (The Big Bang cosmological model)
- DM - Dark matter
- DE - Dark energy
- FLRW metric - Friedmann - Lemaître - Robertson - Walker metric. (Describes the homogeneous and isotropic Universe)
- RHS - Right-Hand Side
- LHS - Left-Hand Side
- C^r - class of the r times differentiable functions
- M - manifold
- T_pM - cotangent bundle of M at the point $p \in M$.
- EoS - equation of state
- SEC - Strong Energy Condition
- WEC - Weak Energy Condition
- gCg - generalized Chpalygin gas