DOCTORAL THESIS REVIEW

MAXWELL-TYPE VISCOELASTICITY IN SMALL AND LARGE DEFORMATIONS OF PLANETARY MANTLES

BY VOJTĚCH PATOČKA

1. Overview

The objective of the thesis is to "investigate footprints of elasticity in the context of mantle convection". In particular, the author investigates several important geophysical problems, and in the modelling of the corresponding flows he uses *Maxwell-type viscoelastic fluid models* in place of the standard *Navier–Stokes viscous fluid model*.

The first chapter is devoted to the analysis of Maxwell viscoelastic fluid model, while the emphasis is placed on the concept of objective tensor rates and material frame-indifference. The chapter is concluded by a brief overview of thermodynamical considerations underpinning the Maxwell model.

The second chapter is focused on the role of elasticity in the glacial isostatic adjustment (GIA), and it is *based on paper Patočka et al. (2018) written by the author with his collaborators*. Here the fluid model is an integral type model, and the model can be seen as an approximation of the standard rate-type Maxwell model. The flow problem however constitutes only a part of the full glacial isostatic adjustment problem. In fact, the core mechanism of interest is the coupling between GIA and Earth's rotation. It is shown that the coupling between GIA and Earth's rotation is strong and that it must be handled carefully using fully nonlinear Liouville equation. Besides the numerical solution of the corresponding governing equations the authors also discuss the net energy balance in a rotating self-gravitating body. The balance of energy is claimed to be a useful tool for the verification of numerical solutions to the coupled GIA/rotation problem. The second chapter of the thesis clearly shows that the author is able to built on his earlier results, see Patočka (2016), and that he is able to design an numerically solve mathematical models for complex phenomena.

The third chapter deals with the role of elasticity in the stagnant lid convection problem, and it is *based on paper Patočka et al. (2017) written by the author with his collaborators*. Here the fluid model is a viscoelastic rate-type model with the Jaumann rate. Unlike in the previous chapter, the main focus is on the implementation of the proposed numerical scheme using the code-base developed by Tackley (2008). The main outcome of numerical experiments done with the developed code is that the viscoelastic rheology leads, in comparison with the standard viscous rheology, to a complex stress structure in the lid.

The fourth chapter is devoted to the role of elasticity in the problem of the onset of plate tectonics on Earth. It is again based on the analysis of numerical experiments based on a visco-elasto-plastic model. Unfortunately, the author does not specify what model is used in the numerical simulations. (No constitutive relation for the stress is given in Section 4.2.) Is it the same viscoelastic model as in the previous chapter but now with added yield stress? The results are compared to Crameri and Tackley (2016), who have used a visco-plastic model only.

2. Major remarks/questions

- Since the first chapter is devoted to a through discussion of the concept of objective time derivative, and to a brief discussion of thermodynamical underpinnings of Maxwell model, I would expect that this introductory chapter will be substantially used in the rest of the thesis. In particular, in the second chapter the author works with the elastic energy and the dissipation in a viscoelastic rate-type model, but he uses only *approximate* formulae for these quantities. I have no doubt that this is fully acceptable in the setting he is interested in, but it might not be the case in other geophysical applications.

  Consequently, it would be worthwhile to see the explicit exact formulae for the energy and the dissipation in the Maxwell type model as well as the corresponding temperature evolution equation. (Especially if the author has spent some time on studying thermodynamically based theories that give direct access to this piece of information, see for example the quoted reference Málek and Průša (2017) or a newer contribution Hron et al. (2017) written by one of author's colleagues in the department.) This might be interesting since the author enthusiastically writes about the paper by Jaquet et al. (2016), where the temperature evolution plays a significant role, but yet it is treated only in an approximate way. Could the author comment on these issues during the thesis defense?

  I would also like to note that the choice of the objective time derivative has also impact on the (exact) formula for the energy, hence it provides yet another guideline for the choice of the objective time derivative.
- In the third chapter the author uses a viscoelastic rate-type model with the Jaumann derivative. Why is the Jaumann derivative preferred to other objective rates? In rheology, the choice of the objective time derivative is often motivated by the behaviour predicted by the corresponding model in the simple shear flow or in the extensional flow, see for example Szabó and Balla (1989) and many other similar works. Concerning the geophysical literature, are there some field-specific arguments that are in favour of one specific choice of the objective time derivative?
- During the defense, the author should explicitly show the governing equations for the visco-elasto-plastic model that has been used in the numerical experiments.

3. Minor remarks/questions

- On page 35, the author claims that the "density ρ and $\det \hat{\mathbb{B}}_\phi$ are not mutually independent state variables". I think that the opposite is true. Here, one can not use the standard balance of mass in the form

  $$\rho \det \mathbb{F} = \rho_{\mathrm{R}},$$

  where ρ denotes the referential density and $\mathbb{F}$ is the total deformation gradient. Tensor $\hat{\mathbb{B}}_\phi$ is defined as $\hat{\mathbb{B}}_\phi =_{\mathrm{def}} \hat{\mathbb{F}}_\phi \hat{\mathbb{F}}_\phi^{\top}$, where the deformation gradient $\hat{\mathbb{F}}_\phi$ describes only a part of the total deformation. The relation between the total deformation gradient $\mathbb{F}$ and the deformation gradient associated to the elastic part of the deformation $\hat{\mathbb{F}}_\phi$ reads $\mathbb{F} = \hat{\mathbb{F}}_\phi \tilde{\mathbb{F}}_\phi$. Consequently, if one knows $\det \hat{\mathbb{B}}_\phi$, then one also knows $\det \hat{\mathbb{F}}_\phi$, but $\det \tilde{\mathbb{F}}_\phi$ is unknown, and the balance of mass

  $$\rho \det \hat{\mathbb{F}}_\phi \det \tilde{\mathbb{F}}_\phi = \rho_{\mathrm{R}}$$

  can not be used for the identification of the density in terms of $\hat{\mathbb{B}}_\phi$ only.
- Suppose that one wants to develop an objective counterpart of the integral on the right-hand side of (2.19)? Would it be sufficient to keep the integral as it is?

- The term "dissipative energy" used in equation (2.34) is strange. Either it should be denoted as the thermal energy or one should work with its time derivative and call it dissipation. Is this the standard nomenclature used in the field?
- Are the "frozen" stress structures discussed in the third chapter robust with respect to the discretisation of the corresponding partial differential equations? (Do they stay the same if the grid size is changed?) Do they appear in other scenarios than in the 1:1 box geometry?

4. Decision

The obtained results are new and relevant for the scientific community on the international level. The thesis is written in a clear readable form and the mathematical methods used are well described. The issues pointed out above only document the complexity and richness of the topics being studied in the thesis, and they by no means diminish the scientific value of the thesis. Vojtěch Patočka has clearly proved the ability to independently and creatively solve complex scientific problems and to work on long term projects. The thesis documents that he is able to handle equally well theoretical as well as implementation challenges. I strongly recommend Vojtěch Patočka to be awarded by PhD degree.

Praha, 8th June 2018

Vít Průša

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