# Reasoning with Inconsistent Information

Usuzování s nekonzistentními informacemi

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# Aim of the thesis

The aim of this thesis is to systematically study extensions of the four-valued Belnap–Dunn logic using the methods of abstract algebraic logic.

The idea that extensions of the Belnap–Dunn logic, also known as the logic of first-degree entailment (FDE), form a family which deserves a systematic study in its own right is fairly recent. It was first suggested by Rivieccio [17], who coined the term *super-Belnap logics* for these extensions. The proposal to systematically explore this family of logics was motivated by the discovery of a previously unknown extension of the Belnap–Dunn logic, called the Exactly True Logic by Pietz and Rivieccio [13]. This discovery made it natural to ask what other unknown extensions there are. Rivieccio's paper [17] provided a first glimpse at a territory which was by and large previously uncharted. It is our goal in this thesis to pick up where this paper left off and explore the landscape of super-Belnap logics in more detail in order to provide future researchers with a reasonably comprehensive map. This investigation will be conducted within the framework of abstract algebraic logic [4, 8, 9], which views logics as structural single-conclusion consequence relations and studies them via their matrix semantics.

There are some exceptions to this paucity of information about super-Belnap logics. The *Belnap–Dunn logic* itself has attracted a good deal of attention from logicians, philosophers, and computer scientists since the seminal papers of Dunn [6] and Belnap [2, 3] published over 40 years ago, which argued that it is a natural logic for dealing with inconsistent and incomplete information. The *strong three-valued Kleene logic* [10, 11] and the *Logic of Paradox* [14] are also well-known non-classical logics, which have been used as bases for theories of truth [12] and for proposed solutions to semantic paradoxes such as the Liar paradox. Less attention was paid to what we call, following Rivieccio, *Kleene's logic order*, identified by Dunn [5] as the first-degree fragment of the relevance logic R-Mingle. Of course, *classical logic* belongs to the family of super-Belnap logics too.

Several factors may be responsible for this lack of previous research into super-Belnap logics. Firstly, each of the above logics was introduced with a fairly specific purpose in mind, which it generally serves well. Researchers employing these logics therefore have little need to look for alternative logics in their neighbourhood. (The lack of a systematic understanding of the *expansions* of the Belnap–Dunn logic is much more puzzling in this respect.) Moreover, according some definitions of logics, there indeed are no other extensions of the Belnap–Dunn logic. This is e.g. the case with Dunn's study [7] of these extensions, which essentially builds the proof by cases property (i.e. disjunction introduction in the antecedent) into the definition of a super-Belnap logic. Finally, the investigation of super-Belnap logics involves technical obstacles which do not come up in the study of, say, super-intuitionistic or normal modal logics. This is because super-Belnap logics are, in a precise sense, not algebraizable, therefore their study cannot be reduced to the study of some class of algebras.

There are, on the other hand, also several reasons for investigating this family of logics in more detail, in addition to the intrinsic mathematical interest of the task. Firstly, although most super-Belnap logics may have little use compared to the prominent logics mentioned above (just like most super-intuitionistic logics have little use compared to the prominent ones like the Gödel–Dummett logic), knowing precisely what gives the above logics special status among all super-Belnap logics gives us more insight into these logics. As we shall see, we may even gain more insight into classical logic by studying it in the context of other super-Belnap logics.

Secondly, studying super-Belnap logics contributes to our understanding of so-called non-protoalgebraic logics. In contrast to super-intuitionistic or normal modal logics, which can be studied using Heyting algebras or Boolean algebras with operators, these are logics where the link between logic and algebra is too weak to allow us to study super-Belnap logics directly by studying the corresponding algebras. Moreover, many of the theorems of abstract algebraic logic relating syntactic and semantic properties of logics rely on the assumption of protoalgebraicity. Thus, although it is common nowadays to this study entire families of non-classical logics, as far as we know there has been no systematic investigation of a family of non-protoalgebraic logics comparable to the investigation of super-intuitionistic logics, substructural logics, or normal modal logics.

Thirdly, related to the previous point, the study of super-Belnap logics provides a motivation, as well as a testing ground, for new developments in abstract algebraic logic. One new direction which naturally suggests itself in connection with super-Belnap logics is the study of explosive or anti-axiomatic extensions of logics. Just like axiomatic extensions postulate that certain formulas are always true, explosive extensions postulate that certain sets of formulas are never true. In the case of super-Belnap logics, it is the lattice of explosive extensions rather than the lattice of axiomatic extensions that forms an interesting object of study. Remarkably, it turns out that this lattice is dually isomorphic, give or take an element at the top and bottom, to the lattice of classes of finite graphs closed under homomorphisms.

Finally, in their Gentzen-style formulation super-Belnap logics provide semantics for sequent calculi without the Cut rule and the Identity axiom. Just like substructural logics provide an algebraic semantics for calculi which keep these rules but relax the structural rules of Exchange, Weakening, and

Contraction in the sequent calculi for classical logic, super-Belnap logics keep these rules while relaxing Cut and Identity. Elimination rules, i.e. the inverses of the introduction rules, are part of these calculi. Studying super-Belnap logics therefore amounts to studying cut-free and identity-free Gentzen calculi with elimination rules.

## Outline

We now outline the structure of this thesis. A summary of the main results can be found in the following section. Let us note here that throughout the thesis we restrict our attention to propositional logics.

The preliminary part of the thesis consists of chapters 1–3. Here we review the general algebraic and logical preliminaries (Chapter 1), introduce the variety of De Morgan algebras (Chapter 2), and finally introduce the Belnap–Dunn logic and its best-known extensions (Chapter 3). The material presented in these chapters is, except for some parts of Chapter 3, not new.

The main arc of the thesis consists of chapters 4–8. These chapters build on the preceding ones and should therefore be read in linear order. We first prepare the ground for later chapters by introducing explosive extensions of logics as extensions by antiaxioms and investigating their basic properties (Chapter 4). Explosive parts of logics are also introduced and shown to be helpful when axiomatizing logics determined by products of matrices. This general theory is then applied to obtain a crop of new completeness results for super-Belnap logics (Chapter 5). Several completeness theorems for super-Belnap logics are also proved directly.

The global structure of the lattice of super-Belnap logics is investigated, using so-called splitting pairs of logics (Chapter 6). In particular, we split the lattice of super-Belnap logics into three main parts. We then describe the fine structure of the lattice of super-Belnap logics in terms of finite graphs (Chapter 7). This link between the realms of super-Belnap logics and graph theory is perhaps the most surprising and mathematically pleasing part of this thesis. Finally, metalogical properties of super-Belnap logics are studied, including their classification in the Leibniz and Frege hierarchies and their algebraic counterparts and strong versions (Chapter 8). We show that only very few super-Belnap logics enjoy the desirable properties of the Belnap–Dunn logic.

The final three chapters of the thesis deal with three separate topics, and full acquaintance with the main arc of the thesis is not required in most places. We first develop the rudiments of a Gentzen-style proof theory for super-Belnap logics, including an analogue of the cut elimination theorem, and use this theorem to prove interpolation theorems for super-Belnap logics (Chapter 9). We then consider what changes have to be made to the results of the thesis if we modify our framework by dropping the truth constants from the Belnap–Dunn logic, or moving to multiple-conclusion consequence, or adding an extra predicate to the Belnap–Dunn logic (Chapter 10). In the final chapter, we study the expansion of the Belnap–Dunn logic by the truth operator $\Delta$ and its algebraic counterpart, the variety of De Morgan algebras with $\Delta$ (Chapter 11).

The bulk of this thesis (Chapters 4–7 and Chapter 10) presents material from the unpublished manuscript [16]. Parts of Chapter 3 and Chapter 8, in particular the description of the truth-equational and assertional super-Belnap logics and some of the results on strong versions of super-Belnap logics and strong versions of explosive extensions, are based on joint work with Hugo Albuquerque and Umberto Rivieccio, published in [1]. Moreover, several results proved in this thesis were first obtained by Umberto Rivieccio in his unpublished notes [18]. Proper credit for these will be given at the appropriate places throughout the thesis. Finally, Chapter 9 is entirely based on the paper [15].

# Main results

Let us now briefly summarize the main results or definitions of each chapter of the thesis, skipping the first two preliminary chapters.

Chapter 3

- The basic properties of the Belnap–Dunn logic $\mathcal{BD}$, the strong Kleene logic $\mathcal{K}$, the Logic of Paradox $\mathcal{LP}$, Kleene's logic of order $\mathcal{KO}$, the Exactly True Logic $\mathcal{ETL}$, and classical logic $\mathcal{CL}$ are reviewed.

- Completeness theorems are proved for these logics.

Chapter 4

- An explosive extension is defined as an extension by antiaxioms, which postulate that a certain set of formulas cannot be jointly designated.

- The explosive part of an extension $\mathcal{L}$ of a base logic $\mathcal{B}$ is defined as the largest explosive extension of $\mathcal{B}$ lying below $\mathcal{L}$.

- Computing the explosive parts of the logics determined by $\mathbb{M}$ and $\mathbb{N}$ is helpful when axiomatizing the logic determined by $\mathbb{M} \times \mathbb{N}$.

Chapter 5

- A completeness theorem for the logic $\mathcal{ECQ}$, which extends $\mathcal{BD}$ by the principle of *ex contradictione quodlibet* $p, -p \vdash q$.
- A completeness theorem for the logic $\mathcal{K}_-$, which is the strongest extension of $\mathcal{ETL}$ strictly below $\mathcal{K}$.

Chapter 6

- The lattice of non-trivial super-Belnap logics splits into the three disjoint intervals $[\mathcal{BD}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $[\mathcal{ETL}, \mathcal{CL}]$.
- The lattice of non-trivial super-Belnap logics also splits into the three disjoint intervals $[\mathcal{BD}, \mathcal{ETL}]$, $[\mathcal{LP} \cap \mathcal{ECQ}_2, \mathcal{K}_-]$, $[\mathcal{KO}, \mathcal{CL}]$.

Chapter 7

- Finite reduced models of $\mathcal{BD}$ correspond precisely to triples $\langle G, H, k \rangle$, where $G$ and $H$ are finite graphs and $k \in \omega$.
- Finitary super-Belnap logics in $[\mathcal{ETL}, \mathcal{ETL}_\omega]$ correspond precisely to classes of finite graphs closed under surjective homomorphisms, disjoint unions, and contracting isolated edges (loops are allowed).
- Finitary explosive extensions of $\mathcal{BD}$ correspond precisely to classes of finite graphs closed under homomorphisms.
- There is a non-finitary explosive extension of $\mathcal{BD}$.

Chapter 8

- The logics $\mathcal{BD}$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$ are the only well-behaved super-Belnap logics from several points of view.
- With one exception, the algebraic counterpart of a super-Belnap logic $\mathcal{L}$ is a (quasi)variety if and only if $\mathcal{L} \in [\mathcal{BD}, \mathcal{ETL}]$ or $\mathcal{L} \in [\mathcal{KO}, \mathcal{CL}]$.

Chapter 9

- Each super-Belnap logic has an equivalent Gentzen counterpart, which is axiomatized by adding elimination rules to a standard calculus for classical logic and relaxing Cut and Identity.
- A normal form for proofs in these calculi is defined and a normalization theorem is proved. For classical proofs from an empty set of premises this theorem essentially reduces to the cut elimination theorem.

- Extensions of $\mathcal{BD}$ by a set of so-called generalized cut rules, such as the logic $\mathcal{ETL}$, are shown enjoy the interpolation property.
- A new syntactic proof is provided of an interpolation theorem which splits consequence in $\mathcal{CL}$ between $\mathcal{K}$ and $\mathcal{LP}$.

Chapter 10

- The lattice of super-Belnap logics remains essentially the same whether the truth constants are included in the signature or not.
- The multiple-conclusion versions of the logics $\mathcal{BD}$, $\mathcal{KO}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$ are the only extensions the multiple-conclusion version of $\mathcal{BD}$.

Chapter 11

- A structure theory for De Morgan algebras with $\Delta$, the (quasi)variety generated by the four-element De Morgan algebra expanded by the truth operator $\Delta$, is developed.
- The expansion of the Belnap–Dunn logic by the truth operator $\Delta$ is studied and axiomatized.

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