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Entropy as a Measure of Predictability in Financial Time Series

Bachelor Thesis

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Abstract

This work studies stock markets efficiency and predictability using the information-theoretic concepts of approximate entropy (ApEn) and sample entropy (SampEn) and compares them to the estimates of the Hurst exponent. This is assessed together with the property of distinguishing between developing and developed markets. Moreover, an investment strategy based on the value of the sample entropy is tested. ApEn shows very weak relationship with other measures and performs poorly as a measure of efficiency. SampEn and the Hurst exponent clearly confirm lower overall efficiency of developing markets. The sample entropy also forms quite strong downward linear relationship with hit-rates of forecasting models. ARMA shows highest hit-rates in periods with SampEn values around 1.6 – 1.7. This could be considered as an investment strategy with lower risk; however, also as one with potentially lower accumulated returns due to smaller investing windows.

JEL Classification	C22, G11, G14, G17
Keywords	sample entropy, approximate entropy, Hurst exponent, market efficiency, forecasting, investment strategy, econophysics

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Abstrakt

Tato práce studuje efektivnost akciových trhů a jejich prediktabilitu pomocí konceptů z teorie informace, approximate entropy (ApEn) a sample entropy (SampEn), a porovnává jejich vlastnosti s odhadem Hurstova exponentu. U těchto měřítek je také porovnávána jejich schopnost rozlišovat rozvíjející se a rozvinuté trhy. Na závěr je testována investiční strategie postavená na hodnotě sample entropy. ApEn ukazuje velmi slabý vztah jak se SampEn, tak s Hurstovým exponentem a zároveň slabý výkon jako měřítko efektivnosti trhů. Sample entropy a Hurstův exponent jasně rozlišují nížší celkovou efektivnost rozvojových trhů. SampEn také utváří poměrně silný klesající vztah s hit-rates predikčních modelů. ARMA má nejvyšší hit-rate v obdobích, kdy je SampEn v rozmezí 1.6 – 1.7. Toto může být potenciálně využito v investičních strategiích za účelem nižšího risku; nicméně, s tím souvisí i možné nižší celkové zisky z důvodu menších investičních oken.

JEL klasifikace	C22, G11, G14, G17
Klíčová slova	sample entropy, approximate entropy, Hurstův exponent, efektivnost trhů, předpovědi, investiční strategie, ekonofyzika

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Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature, and that this work has not been used to gain any other title.

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Prague, 16 May 2017

Signature

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Bachelor Thesis Proposal

Author	Vladimír Nahodil
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Proposed topic	Entropy and predictability in financial time series

Preliminary content of the work:

During the last decades, simultaneously with a rise of the new scientific field econophysics, the information theory has found its applications in various financial models. Particularly, different concepts of entropy, which originates from the laws of thermodynamics and can be used as a measure of uncertainty, were introduced to models of the optimal portfolio diversification (e.g. maximum entropy as an alternative to Markowitz's mean-variance approach) or predictions of excess returns in a stock market (metric entropy). As entropy is depending on more parameters of a probability distribution than variance and is related to higher-order moments, it may offer a better characterization of uncertainty and better predictions.

The aim of this work is to estimate stock markets efficiency and predictability using the concept of approximate entropy and compare these estimates of the memory property to the predictions of models with Hurst exponent. The calculation of approximate entropy is model-independent; it is not testing for a particular model but quantifies the irregularity of data which can be seen as an advantage of this concept. The investment strategy in these markets will be based on a prediction of the financial time series in the periods of low entropy (therefore lower efficiency and higher predictability) and the results are to be compared with other models of future stock price prediction using Diebold – Mariano test.

Outline:

1. Introduction
2. Theoretical background on entropy
3. Methodology and model
4. Analysis of the results
5. Conclusion

Core bibliography:

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Contents

List of Tables

List of Figures

Acronyms

ADF	Augmented Dickey-Fuller test
AICc	Corrected Akaike Information Criterion
ApEn	Approximate entropy
ARMA	autoregressive-moving-average
DFA	detrended fluctuation analysis
EEG	Electroencephalography
GARCH	generalized autoregressive conditional heteroscedasticity
KLIC	Kullback-Leibler information criteria
KPSS	Kwiatkowski-Phillips-Schmidt-Shin test
M-V	mean-variance
MAIC	Modified Akaike Information Criterion
NN	nearest neighbour
OLS	ordinary least squares
RMSE	root mean squared error
SampEn	Sample entropy
SCM	statistical complexity measure
SD	standard deviation
SE	stock exchange

Introduction

A large part of financial theory is based on the weak-form Efficient Market Hypothesis, which states that stock market prices reflect all information coming from historical prices (Fama, 1970). However, deviations from the hypothesis are present in markets and many researchers already tried to find a suitable model specification which would be able to successfully predict (at least to some extent) future stock prices or returns based on the set of historical values (Campbell et al., 1997).

This goes in hand with studies of market (in)efficiency in different periods and markets in order to find right opportunities for investment. After the influx of papers using the concepts from physics in economics and finance in past years, this thesis is another interdisciplinary attempt adding to the field of econophysics with its testing of two potential measures of market predictability/inefficiency coming originally from the information theory: the approximate entropy and the sample entropy. Both measures are model-independent – they are not testing for a particular model but quantify the irregularity of data, which can be seen as an advantage of this concept.

As was previously explored, emerging markets show higher degrees of autocorrelation and should be less efficient (Di Matteo et al., 2005). Aforementioned entropic measures (together with the Hurst exponent) are used here to compare market efficiency of different geographic regions, on a larger scale than in previous papers on this matter and being the first to bring the sample entropy to the discussion. The aim of this thesis is to select the best estimator of market predictability measured by its relationship with the success ratio of forecasting models. Moreover, investment strategies based on values of the sample entropy are tested to see if the inclusion of entropy-based threshold (where lower value of entropy should imply better predictability) to the strategy brings higher accumulated returns.

The thesis is organized in the following way: Chapter 1 outlines the theoretical background, history and literature review on econophysics and entropy. Methods and datasets used for assessment of market efficiency and for forecasting are described in Chapter 2. Chapter 3 presents the results and the last chapter summarises the findings of work and concludes.

Chapter 1

Theoretical background and Literature Review

This chapter provides a literature review and needed background on concepts used later on in this thesis. It briefly overviews interdisciplinary attempts of mathematicians and physicians to explain mechanics in economics and finance in the first section. Next, it gives a detailed look on the history of entropy. In the last section, it looks on past research on entropy as a measure of predictability.

1.1 Mathematics and Physics in Economics and Finance

1.1.1 Early work

The connotations between laws in mathematics and physics on the one side and in economics and finance on the other started to be of interest of researchers ever since economics became "mathematicised" in the late 19th century (e.g. in the work of Léon Walras on a general equilibrium theory (Walras, 1896)). Many branches of physics like statistical mechanics, non-linear dynamics or phase transitions include powerful concepts like unpredictable time series or the presence of power laws, which could be successful in explaining mechanics in economy and finance (Mantegna & H. E. Stanley, 1999).

One of the first works using this knowledge in social sciences was presented by Vilfredo Pareto who used the power law distribution to describe the distribution of wealth of individuals (Pareto, 1896). In 1900, French physicist and mathematician Louis Bachelier formulated the theory of random walks, in which he used the phenomenon of the movement of grains suspended in water to explain the price fluctuations of options in speculative markets (Bachelier, 1900). This concept of random walks was then rediscovered and popularised by Albert Einstein in his work on Brownian motion (Einstein, 1905). Although their conclusions on financial and economical theories may not seem absolutely correct as of now, these pioneering works had laid ground for others to come in the 20th century.

However, apart from the paper of Majorana on the analogy between statistical laws in physics and in social sciences (Majorana, 1942) the activity in this area had been low for several next decades. The pricing theory has been firstly redefined in 1950s when prices were believed to follow a log-normal distribution (a geometric Brownian motion). But a redefining work uniting the findings of Bachelier and Pareto was written by a French mathematician Benoît Mandelbrot as late as 1963. He found that cotton prices follow a power law distribution, specifically *Lévy stable distribution*. In addition to that he formulated a phenomenon called *a volatility clustering* which states that *"large changes tend to be followed by large changes of either sign and small changes tend to be followed by small changes"* (Mandelbrot, 1963).

Ten years later, the Black-Scholes model (Black & Scholes, 1973) and its option-pricing formula, although in contradiction with a model specified by Mandelbrot due to its normality assumption (and therefore ignoring the heavy tails presented in power law distributions), was presented as another analogy between finance and physics, specifically the diffusion theory (diffusion-advection equation).

1.1.2 Econophysics

Following the Smithsonian Agreement most of the currencies abandoned the fixed exchange rate subsequently in 1971 and 1973. In 1973, currencies began to be traded on financial markets active 24 hours per day. In 1980s, electronic trading was adapted to foreign exchange markets together with the electronic storing of bid and ask quotes, which made available a huge amount of electronically stored financial data for testing and research. This immense amount of data has caught the attention of physicists who could, as in the spirit of experimental physics, analyse real data without having any prior models (Mantegna & H. E. Stanley, 1999).

This increased activity of physicists and a following wave of papers on new approaches derived from those in physical sciences and then used on economic and financial data gave birth to the new field of study, with H. Eugene Stanley being the first one to call it *econophysics* on a conference in Kolkata in 1996 (Chakraborti et al., 2011).

While equilibrium/non-equilibrium statistical mechanics, the theory of stochastic processes or the theory of chaos are still important branches of modern mathematics and physics, they also became prolific tools with strong analogies to financial systems (Chakraborti et al., 2011). Research activities can be sorted into two main groups (Marschinski & Kantz, 2002):

1. *"Microscopic" approach*, which investigates market dynamics from the view of single agents using e.g. analogies with spin systems (Chowdhury & Stauffer, 1999) or self-organised criticality (Lux & Marchesi, 1999); the target is to encompass the macroscopic behaviour of the market by microscopic equations.

2. *"Macroscopic" approach*, which analyses statistical properties of macroscopic behaviour of financial markets using e.g. exit-time statistic (Baviera et al., 2000), random matrix theory (Laloux et al., 1999), power law distributions (Gopikrishnan et al., 1998) or entropic measures for a study of multivariate properties (Darbellay & Wuertz, 2000). Distribution of price changes, temporal memory of series or higher-

order statistical properties are some examples of the areas of study in macroscopic approach.

Moreover, econophysics also comprises studies that focus on other areas than financial markets, e.g. on the income distribution of firms (Okuyama et al., 1999) or statistical properties of their growth rates (M. H. Stanley et al., 1996). For a more complex view and a great introduction to econophysics, please see Mantegna & H. E. Stanley (1999) or Jovanovic & Schinckus (2016).

1.2 Entropy and its definitions

1.2.1 Thermodynamic approach

Although many physical concepts have found their use in economics, *entropy* will be of a main interest of this thesis. The concept of entropy was firstly specified in classical thermodynamics by Rudolf Clausius (Clausius, 1867) as a part of the second law of thermodynamics. A change of entropy ΔS was defined as a change of heat δQ divided by temperature T:

$$\Delta S = \frac{\delta Q}{T}.$$

As heat always transfers from a warmer object to a colder one by the definition of the second law of thermodynamics, an isolated system loses some of its ability to change its state. Change in entropy ΔS quantifies this loss; after the heat transfer $\delta Q_1 + \delta Q_2 = 0$ in an isolated system between two objects of different temperatures T_1 and T_2, entropy of a system is always higher than before the transfer as is shown in the equation (1.1) (at least for irreversible processes which occur naturally; the change may be equal to zero for a theoretical Carnot cycle):

$$\Delta S = \Delta S_1 + \Delta S_2 = \frac{\delta Q_1}{T_1} + \frac{\delta Q_2}{T_2} = (\frac{1}{T_1} - \frac{1}{T_2})\delta Q_1 = \frac{T_2 - T_1}{T_1 T_2}\delta Q_1 \geq 0. \quad (1.1)$$

Therefore, all isolated systems (and also the universe itself) tend to increase their entropy; a maximum entropy is achieved when the system reaches thermodynamic equilibrium and it has no free thermodynamic energy, which leads to a heat death in the case of universe.

Another definition of entropy was established by Ludwig Boltzmann and J. Willard Gibbs (Boltzmann, 1877) as one of central pillars of then newly created field – statistical mechanics. This definition of an absolute amount of entropy is, in contrast to the aforementioned Clausius' entropy, based on the microscopic behaviour in the system. *Gibbs entropy* measures the uncertainty of a system; it increases with a number of possible microstates the system may attain after taking into account its macroscopic properties:

$$S = -k_B \sum_i p_i \ln p_i,$$

where $k_B = 1.38064852 \times 10^{-23} J * K^{-1}$ is Boltzmann constant and p_i is a probability of a microstate i, $0 < p_i \leq 1$ for all i.

1.2.2 Information theoretic approach

Information theory, which studies the communication of information, came with a concept of entropy related to the probabilistic view of statistical mechanics. Claude E. Shannon introduced the *Shannon entropy* H (eq. (1.2)) in his pioneering work of the field called *A Mathematical Theory of Communication* (Shannon, 1948):

$$H = -\sum_i p_i \log_b p_i, \ \ 0 < p_i \leq 1, \qquad (1.2)$$

where p_i is a probability of a state i for all i and b is a base of the logarithm. States with zero probability are ignored in calculation.

Shannon entropy can be seen as a general version of Gibbs entropy (or Gibbs entropy as a particular application of Shannon entropy, see Jaynes (1957)) measuring the uncertainty of a system. From the other perspective, it can also measure the (average) amount of information contained in a message. If we have, for example, a sure outcome of a coin toss (imagine a coin with two heads), then the result is certain and there is no amount of new information from the result of the toss – the result was known already before the toss; therefore, entropy is equal to zero. On the other hand, in a fair coin toss with equiprobable outcomes, the uncertainty and also the information from the result (message) is at its maximum resulting in the highest possible entropy. Generally speaking, entropy is highest for uniformly distributed outcomes and zero for a sure outcome. The unit of entropy depends on a base b of a logarithm used – with the most common base $b = 2$ entropy is measured in *shannons/bits*, with $b = e$ in *nats* and with $b = 10$ in *hartleys*.

1.3 Entropy as a measure of stock predictability

Since the Shannon's work in 1940s, many related entropies found their use in other fields of science. This section lists notable works in finance, which used some of the concepts based on information theory and Shannon entropy for measuring predictability/uncertainty of financial time series. For other uses of entropy in finance, please see Appendix D.

There are two ways in which time series deviate from constancy which could potentially scare investors – they either show high deviation from their mean or they are highly unpredictable. Generally speaking, deviation from centrality (for which the appropriate measure is a standard deviation) is not as feared – if even high swings from centrality are easily predictable, these foreseen changes could be taken in account while planning future strategy. On the other hand, the extent of irregularity or uncertainty in future price movement could be frightening; therefore, it is of interest to be able to measure a degree of stock uncertainty.

1.3.1 Approximate entropy

The family of entropy statistics can be helpful in capturing stock uncertainty without a need to impose any limitations on the theoretical probability distribution. The *Approximate entropy* (ApEn) has been proposed as such measure for statistics of system regularity (Pincus, 1991). It is a model-independent measure of sequential irregularity, which evaluates the logarithmic likelihood that patterns that are close remain close on next incremental comparisons. In its early use, several researchers in various fields used ApEn to study serial irregularity – from serial EEG data to voltage data of a power generator, where high system instability/irregularity can lead to faults and blackouts (Pincus & Huang, 1992).

The approximate entropy has several useful properties in evaluating system irregularity. ApEn applies to both infinite and finite sequences, even very short in length. Moreover, ApEn can detect even slight differences of regularity in highly irregular time series, whereas statistics like linear correlation or power spectrum often fail to discriminate one series from another or from a random walk as ApEn is a more robust measure of feature persistence. ApEn can be also applied to evaluate and reject/validate econometric models such as a random walk, Black-Scholes diffusion, ARMA, GARCH and others or can be applied as a marker of system stability, where significantly increased ApEn values may foreshadow state changes (Pincus & Kalman, 2004). For mathematical derivation and interpretation of the approximate entropy, please refer to the Section 2.2.

Although most financial analyses focus on price return or price increment data series thanks to their better properties like stationarity, most empirical applications of ApEn have been directly to price time series – apart from measuring the irregularity of series, it can reveal shifts in serial characteristics.

Pincus & Kalman (2004) tested DJIA prices taken at 10-minute intervals. They found highly significant variation from the maximum approximate entropy over the time; however, autocorrelation and spectra analysis were not able to distinguish time series from random. Moreover, substantial deviations from the maximum ApEn suggested that the data violated the Black-Scholes model of option pricing which implies that security prices obey a geometric Brownian motion with drift – log-ratio series should then be i.i.d. and normally distributed and ApEn nearly maximal. Authors also tested other potential property of ApEn – to forecast rapid changes, e.g. market crashes. ApEn was calculated for running 120-point long daily incremental price series of Hong Kong's Hang Seng index between 1992 and 1998. The ApEn value rapidly increased to its highest value in the chosen period immediately before the November 1997 market crash. It was anticipated that with less crude data better inference could be provided – it is of interest that crash was predicted even on daily data.

Eom et al. (2008) were empirically testing the connection between market efficiency/uncertainty and predictability. They used two measures of market efficiency – the approximate entropy and *the Hurst exponent* (see more about the Hurst exponent in the Section 2.2). The analysis was taken on the return time series of daily indices of 27 stock markets. Both Hurst exponent and ApEn showed correlation with market predictability measured by a hit-rate of forecasts (percentage of forecasts that successfully predicted the price change direction), with the Hurst exponent showing slightly better results in this regard.

1.3.2 Shannon entropy

Other authors used entropy in an approach to distinguish the developed and developing markets by estimating their efficiency. The Hurst parameter was not found as a universal measure and the *complexity-entropy causality plane* was proposed as an alternative (Zunino et al., 2010).

This method plots the normalised Shannon entropy $\mathcal{H}_S = H(P)/H_{max}$ against a *statistical complexity measure* (SCM) $\mathcal{C}_{JS}$ (Equation (1.3), P_e being a uniform distribution), which combines the normalised Shannon entropy $\mathcal{H}_S(P)$ with a disequilibrium $\mathcal{Q}_J$ (Equations (1.4)) as proposed by Lamberti et al. (2004):

$$\mathcal{C}_{JS}(P) = \mathcal{Q}_J(P, P_e)\ \mathcal{H}_S(P), \tag{1.3}$$

where a disequilibrium $\mathcal{Q}_J$ is:

$$\begin{aligned} \mathcal{Q}_J(P, P_e) &= \mathcal{Q}_0\ \mathcal{J}(P, P_e), \\ \mathcal{J}(P, P_e) &= H\left(\frac{P + P_e}{2}\right) - \frac{H(P)}{2} - \frac{H(P_e)}{2}, \\ \mathcal{Q}_0 &= \frac{1}{max\mathcal{J}(P, P_e)}. \end{aligned} \tag{1.4}$$

SCM is able to detect essential details of the dynamics and to differentiate different degrees of periodicity and chaos; perfect order and maximal randomness have both $\mathcal{C}_{JS} = 0$ because they possess no structure.

Paper evaluates the normalised Shannon entropy and the intensive SCM using the permutation probability distribution found by *Bandt & Pompe symbolization method*, which takes into account the time causality of the system dynamics. The complexity-entropy causality plane was successful in differentiating developing and developed markets (as classified by Morgan Stanley Capital Index) on daily price data of 32 equity indices. It should be noted that this method was also used to measure the efficiency of commodity markets one year later (Zunino et al., 2011).

From other entropic measures, a normalisation of a *metric entropy* was used to test nonlinear, nonparametric dependence of the monthly excess returns of S&P 500 in order to discover potential market inefficiency (Maasoumi & Racine, 2002). Other notable papers use again Shannon entropy (Risso, 2008) and the approximate entropy (Oh et al., 2007) as a measure of market predictability.

Chapter 2

Methodology

This chapter focuses on the methodology used to obtain the results. The first section outlines the underlying tested data. In the second part, three measures of uncertainty/predictability and their calculation are described in detail. Next, tests of stationarity and two models used to get the forecasts of series are explained in the section 3.

The statistical software R version 3.3.2 was used for method implementation. For a detailed list of used software and R packages see Appendix A.

2.1 Data

Two datasets of daily close prices were analysed in this thesis, both obtained from the Thomson Reuters Wealth Manager. Both sets contained a mix of series from both developing and developed markets/countries in order to compare their market predictability.

First dataset comprised *28 market indices*, with 3 in Western Europe, 5 in Central & Eastern Europe, 4 in North America, 5 in Central & South America, 6 in Asia, 4 in Africa and 1 in Australia. For a detailed list of selected indices, please see Table 2 in Appendix B.

Second dataset contained *stock prices of 49 banks* with the largest market capitalisation in their respective stock exchanges as indicated by the software. 9 of these banks are located in Western Europe, 4 in Central & Eastern Europe, 3 in North America, 7 in Central & South America, 7 in Middle

East, 10 in Asia and 9 in Africa. For a detailed list of selected bank stocks, please see Table 3 in Appendix B.

2.1.1 Period

Close prices were obtained for up to twenty-years-long period from March 31st 1997 to March 30th 2017. This period length gave a sufficiently large set of values with 2015 being the lowest number of observations in the first dataset (for Zimbabwe SE Industrial Index) and 1778 the lowest number in the second dataset (for National Microfinance Bank PLC in Tanzania), while skipping weekends and holidays (non-trading days).

2.1.2 Return time series

The return time series of market index/bank stock was calculated for each time series of daily close prices. The expected stationarity of the return series is a useful property for forecasting index/stock returns. The return series R was calculated by a logarithmic change in price series P:

$$R(t) = \ln P(t) - \ln P(t-1),$$

where $P(t)$ is a close price at time/observation t.

2.2 Measures of uncertainty/predictability

2.2.1 Approximate entropy

A first measure of predictability of time series used in this thesis was the approximate entropy as proposed by Pincus (1991). Given a time series of length N $u(1), u(2), ...u(N)$, ApEn(m,r,N) takes two input parameters, m being a block or length of compared runs of data and r being a tolerance window. As mentioned in the Section 1.3, ApEn measures the logarithmic frequency that runs of patterns that are close within r for m contiguous observations remain close within the same tolerance window r on the next incremental comparison (Pincus & Kalman, 2004).

Approximate entropy is calculated as follows:

1. For a fixed positive integer m (run length) and a series $u(1), u(2), ..., u(N)$, a sequence of vectors $\{\mathbf{x}_m(i) \in \mathbb{R}^m, i = 1, 2, ..., N - m + 1\}$ is created, where:
$$\mathbf{x}_m(i) = [u(i), u(i+1), ..., u(i+m-1)].$$
2. A distance d between two vectors of sequence is defined as a maximum value of distances of scalars of these vectors from the first values up to m-th:
$$d[\mathbf{x}_m(i), \mathbf{x}_m(j)] = \max_{k=1,2,...,m} |u(i+k-1) - u(j+k-1)|.$$
3. Then, for a positive real tolerance window r and for each i, $i = 1, 2, ..., N - m + 1$, $B_i^m(r)$ is defined as:
$$B_i^m(r) \equiv \text{number of } \mathbf{x}_m(j) \text{ such that } d[\mathbf{x}_m(i), \mathbf{x}_m(j)] \leq r,$$
$$j = 1, 2, ..., N - m + 1.$$
4. A share of vectors meeting the condition is calculated for each i:
$$\mathrm{C_i^m(r)} = \frac{B_i^m(r)}{N - m + 1}.$$
5. Then, a magnitude of repeated pattern occurrences $\Phi^m(r)$ is defined as:
$$\Phi^m(r) = (N - m + 1)^{-1} \sum_{i=1}^{N-m+1} \ln(\mathrm{C_i^m(r)}).$$
6. And finally:
$$\mathrm{ApEn}(m, r, N) = \Phi^m(r) - \Phi^{m+1}(r) \geq 0.$$

ApEn assigns a non-negative number to the time series of interest with larger values for a greater irregularity and smaller values for a higher number of recognisable patterns in series. If patterns occur with the same frequency for both dimensions m and $m + 1$, then $\mathrm{ApEn}(m, r, N) = 0$.

Widely used (e.g. in Eom et al. (2008), Pincus & Kalman (2004) or Oh et al. (2007)) parameters $m = 2$ and $r = 20\%$ of standard deviation

(SD) were selected for calculations – relating a tolerance window r to SD of time series makes ApEn scale invariant. ApEn was measured for a moving window of $N = 200$ observations and average ApEn was then calculated as $\overline{\mathrm{ApEn_j}} = \frac{1}{T-200}\sum_{t=1}^{T-200}\mathrm{ApEn_{j,t}}$ for each stock time series j, where T is a number of observations in time series.

2.2.2 Sample entropy

The *Sample entropy* (SampEn) was proposed as an alternative entropic measure of a regularity of system. A goal in creating sample entropy was to eliminate the bias observed in ApEn. As ApEn counts each pattern as matching itself (in order to avoid having logarithms of zero in the procedure), it makes ApEn consistency heavily dependent on sample length and having uniformly lower, biased value (Richman & Moorman, 2000).

SampEn(m,r,N) is the negative natural logarithm of the conditional probability that two patterns similar for m points remain similar for $m+1$ points, without counting self-matching in the probability calculation. A lower value again indicates higher similarity of patterns (Richman & Moorman, 2000).

Sample entropy requires less computational time than ApEn due to its simpler algorithm. Furthermore, based on empirical findings, SampEn is less sensitive to changes in sample length and demonstrates fewer problems with relative consistency (Yentes et al., 2013).

Sample entropy is calculated as follows:

1. For a fixed positive integer m (run length) and a series $u(1), u(2), ..., u(N)$, sequences of vectors $\{\mathbf{x}_m(i) \in \mathbb{R}^m, i = 1, 2, ..., N-m\}$ and $\{\mathbf{x}_{m+1}(i) \in \mathbb{R}^{m+1}, i = 1, 2, ..., N-m\}$ are created, where:

$$\mathbf{x}_m(i) = [u(i), u(i+1), ..., u(i+m-1)],$$
$$\mathbf{x}_{m+1}(i) = [u(i), u(i+1), ..., u(i+m-1), u(i+m)].$$

2. A distance d between two vectors of a sequence is defined as a maximum value of distances of scalars of these vectors from the first values up to

the last:

$$d[\mathbf{x}_m(i), \mathbf{x}_m(j)] = \max_{k=1,2,\ldots,m} |u(i+k-1) - u(j+k-1)|,$$
$$d[\mathbf{x}_{m+1}(i), \mathbf{x}_{m+1}(j)] = \max_{k=1,2,\ldots,m+1} |u(i+k-1) - u(j+k-1)|.$$

3. Then, for a positive real tolerance window r and for each i, a share of vectors meeting the distance condition is calculated:

$$B_i^m(r) \equiv \frac{\text{number of } \mathbf{x}_m(j) : d[\mathbf{x}_m(i), \mathbf{x}_m(j)] \leq r}{N-m-1},$$
$$A_i^m(r) \equiv \frac{\text{number of } \mathbf{x}_{m+1}(j) : d[\mathbf{x}_{m+1}(i), \mathbf{x}_{m+1}(j)] \leq r}{N-m-1},$$
$$j = 1, 2, ..., N-m, j \neq i.$$

4. Averages across the whole series are calculated:

$$B^m(r) = \frac{\sum_{i=1}^{N-m} B_i^m(r)}{N-m},$$
$$A^m(r) = \frac{\sum_{i=1}^{N-m} A_i^m(r)}{N-m}.$$

5. And finally:

$$\text{SampEn}(m, r, N) = -\ln \frac{A^m(r)}{B^m(r)}.$$

The same parameters as in the case of ApEn were used: $m = 2$ and $r = 20\%\text{SD}$. Sample entropy was also measured for a moving window of $N = 200$ observations and average SampEn was calculated as $\overline{\text{SampEn}_\text{j}} = \frac{1}{T-200}\sum_{t=1}^{T-200} \text{SampEn}_{\text{j,t}}$ for a time series j, where T is a number of observations in the series.

2.2.3 Hurst exponent

The *Hurst exponent* H, which quantitatively measures the long-range dependence in time series (Zunino et al., 2010), was selected as a control measure of predictability. In their work, Eom et al. (2008) found the Hurst exponent as a measure with a higher correlation to a measure of predictive ability than approximate entropy.

$0 \leq \mathrm{H} < 0.5$ corresponds to a short-term anti-persistent memory in time series, $\mathrm{H} = 0.5$ means no autocorrelation in series and with $\mathrm{H} > 0.5$ there is a long-term memory in time series. However, Bassler et al. (2006) argued that in special cases even $\mathrm{H} \neq \frac{1}{2}$ is perfectly consistent with Efficient Market Hypothesis and no memory in time series. For an appropriate evidence of autocorrelation, time series increments need to be stationary in addition to $\mathrm{H} \neq \frac{1}{2}$.

Several methods (e.g. Higuchi, Detrended Fluctuation Analysis, Detrended Moving Average) for estimation of the Hurst exponent are already implemented in R (Zunino et al., 2011) and *the Detrended Fluctuation Analysis* (DFA) was found to provide good results (Weron, 2002); therefore, this method was used also in this thesis.

The Hurst exponent estimated by DFA is calculated in this way:

1. First of all, the mean of the original time series x of length N is subtracted from the series:

$$y(i) = x(i) - \bar{x}, \ \ i = 1, ..., N.$$

2. Time series y, which measures deviations from the mean of series x, is then accumulated to series z:

$$z(k) = \sum_{i=1}^{k} y(i), \ \ k = 1, ..., N.$$

3. Accumulated series z is divided into ν boxes of length n, where $\nu n = N$. OLS is estimated for each box and accumulated time series $z(k)$ is then detrended by trend line estimates $z_n(k)$. Fluctuation around the trend for a given box length n is calculated as:

$$F(n) = \sqrt{\frac{1}{N} \sum_{k=1}^{N} (z(k) - z_n(k))^2}.$$

4. This process is repeated over set box lengths n (equally spaced on a logarithmic scale) to provide a scaling relationship between $F(n)$ and n:

$$F(n) \sim n^{\mathrm{H}}.$$

5. The scaling exponent (the Hurst exponent) is then obtained as a slope parameter in the OLS regression of logarithms:

$$\log F(n) \sim \mathrm{H} \log n.$$

Unlike approximate and sample entropies, the Hurst exponent was not calculated for moving windows of size $N = 200$ for each time series - it primarily measures long-term memory and it is subject to statistical errors with short windows (Peng et al., 1995). Therefore, whole series were used to estimate the Hurst exponent, with a lower bound for box sizes set at 4 and an upper bound at $N/4$, where N is the length of series.

2.3 Forecasting models

2.3.1 Testing for stationarity

Return time series of indices and banks were tested for stationarity before forecasting, which is an assumption of ARMA forecasts. *Augmented Dickey-Fuller test* (ADF) together with the *Kwiatkowski-Phillips-Schmidt-Shin test* (KPSS) were selected as methods for confirming stationarity.

ADF(p) is a modified version of Dickey-Fuller test, which includes p lags of differences of time series y as regressors (Said & Dickey, 1984). In its widest form, which includes also a constant and a time trend as independent variables, its underlying model is:

$$\Delta y_t = \alpha + \beta t + \gamma y_{t-1} + \delta_1 \Delta y_{t-1} + ... + \delta_{p-1} \Delta y_{t-p+1} + \epsilon_t, \tag{2.1}$$

where it tests null hypothesis of a unit root in time series (which implies non-stationary process) $\mathrm{H_0} : \gamma = 0$ against alternative hypothesis of no unit root (meaning stationary or trend-stationary process) $\mathrm{H_A} : \gamma < 0$.

A maximum number of lags in consideration was determined for each series by the formula proposed in Schwert (1989):

$$p_{max} = 12 * \sqrt[4]{\frac{N}{100}},$$

where N is the number of observations.

Testing followed *Perron's sequential testing procedure* (Ng & Perron, 1995). However, a number of lags included in final model for each step of procedure was not selected by the general-to-specific strategy (starting with p_{max} and subsequent decreases in number of lags by 1 if the last lag is not significant at 5% significance level); this would add too much additional calculation complexity given the number of series tested. Number of lags was rather selected by the procedure mentioned in Ng & Perron (2001), where from all possible models with $p = 0, 1, ..., p_{max}$ is selected one with the lowest value of *Modified Akaike Information Criterion* (MAIC).

KPSS test, as proposed by Kwiatkowski et al. (1992), tests the null hypothesis of level or trend stationarity against the alternative of non-stationary time series. This test was used to confirm the results provided by ADF test. The option selecting a level stationarity as the null hypothesis was selected in *kpss.test()* function provided by *tseries* package in R.

2.3.2 ARMA

The *autoregressive moving average* model (ARMA) was chosen to retrospectively predict the set of returns for each time series. ARMA(p,q), is composed of two parts, where p is the order of autoregressive part and q is the order of moving average part. Autoregressive part implies that dependent variable is regressed on p lags of itself. Moving average part involves modelling of the error term as a linear combination of previous q contemporaneously calculated white noise error terms.

Given a time series y, ARMA(p,q) model is:

$$y_t = c + \sum_{i=1}^{p} \alpha_i y_{t-i} + \sum_{i=1}^{q} \theta_i \epsilon_{t-i} + \epsilon_t,$$

where c is a constant term and ϵ_t is the error term in time t.

The *Corrected Akaike Information Criterion* (AICc) was used as a criterion to select the right combination of p and q for each regression. This method was advised by Brockwell & Davis (1991) and confirmed by McQuarrie & Tsai (1998) on extensive simulation work. R function *auto.arima()* from package *forecast* compared all models with $p, q = 0, 1, 2, 3$ and selected the one with lowest AICc for each regression. Regression coefficients were then estimated by a Maximum Likelihood Estimation method.

ARMA model was estimated on a moving window of 200 observations in each case and this fitted model was then used to obtain a one-step-ahead forecast. Most of the (possible) serial correlation in close price series should have been already eliminated by taking their difference and creating return series (Wooldridge, 2015). Furthermore, the main focus was on point forecasts, which should still be consistent even in the case of serial correlation of residuals. Therefore, no special treatment was implemented to remove potential serial correlation from return series.

2.3.3 Nearest neighbour prediction

The nearest neighbour prediction method (NN) was selected as a second model for computation of one-step-ahead forecasts. This non-parametric method does not impose any assumptions on underlying series; it analyses price patterns in the past which are similar to the current pattern to predict current price change.

The nearest neighbour method proved to be useful for predictions within a relatively short time frame. Eom et al. (2008) used the nearest neighbour prediction method on the return series data in order to assess the relationship between efficiency and predictability. Similar method was also utilised by Bajo-Rubio et al. (1992) and Soofi & Cao (1999) for forecasting of Spanish Peseta–U.S. Dollar exchange rate.

As packages already implemented in R for k-nearest neighbour prediction did not exactly match the need of this thesis (they are more suitable for discrete predictions), own version had to be implemented. It uses patterns of length 2 and finds 3 nearest neighbours.

For predicting the value of return y_t:

1. First of all, current pattern of previous actual returns $current = [y_{t-2}, y_{t-1}]$ is stored.

2. A pattern series V, composed of patterns of returns up to the one before current pattern, is created:

$$V = \{[y_1, y_2], [y_2, y_3], ..., [y_{t-3}, y_{t-2}]\}.$$

3. An Euclidean distance d to the current pattern is calculated for all patterns in V, where a distance of pattern $x = [y_{x1}, y_{x2}] \in V$ is:

$$d(x,\ current) = \sqrt{(y_{x1} - y_{t-2})^2 + (y_{x2} - y_{t-1})^2}.$$

4. Three patterns in V closest to the current pattern are found; if two patterns have the same distance to the current pattern and battle for the third closest position, then a pattern located further in time series (and therefore, closer to current pattern) has a priority.

5. Values next to three selected patterns $q = [y_{q1}, y_{q2}]$, $r = [y_{r1}, y_{r2}]$, $s = [y_{s1}, y_{s2}]$ (their "forecasts") are stored: $y_{q2+1}, y_{r2+1}, y_{s2+1}$.

6. If these forecasts are all different in their values, then their arithmetic mean is selected as a prediction of y_t. However, if any value occurs at least twice, it is chosen as a forecast by a majority vote.

Predictions of the time series of returns started by forecasting y_{201}, for which the pattern series V was filled by patterns $\{[y_1, y_2], [y_2, y_3], ..., [y_{198}, y_{199}]\}$. For forecasts of next values $y_{202}, ..., y_N$, where N is the length of the return series, one pattern of actual returns was always added to pattern series in a for-loop.

2.4 Measures of predictive ability

After obtaining retrospective forecasts of returns for all indices/bank stocks, it was essential to find measures, which would help to compare and capture:

- the accuracy of forecasts of ARMA/nearest neighbour,
- the relationship between entropic measures/the Hurst exponent and a successful prediction,
- differences in geopolitical areas.

Hit-rate

Hit-rate was chosen as a first measure of predictive ability. It measures the rate of consistency between the direction of the actual price change and the predicted price change. Therefore, its value lies between 0 and 1.

Own version of a hit-rate measure was written in R as, unlike RMSE below, it was not yet part of any package. For actual values of returns y_t, forecasts $\hat{y}_t$ $(t = 1, ..., N)$, hit-rate is calculated as:

$$\text{hit-rate} = N^{-1} \sum_{t=1}^{N} z_t,$$

with z_t values given in Table 2.1.

Table 2.1: Value of z_t given combination of y_t and $\hat{y}_t$

Variable	Sign of the price change								
y_t	+	+	+	0	0	0	–	–	–
$\hat{y}_t$	+	0	–	+	0	–	+	0	–
z_t	1	1	0	1	1	0	0	0	1

Source: Author.

RMSE

The *root mean squared error* (RMSE) was selected as a representative of traditional out-of-sample criteria. This criterion is widely used (e.g. in Maasoumi & Racine (2002)) and is essentially the sample standard deviation of forecast errors without an adjustment for degrees of freedom (Wooldridge, 2015).

For a time series y_t and its predictions $\hat{y}_t$, where $t = 1, ..., N$, RMSE is:

$$\text{RMSE} = \sqrt{N^{-1} \sum_{t=1}^{N} (y_t - \hat{y}_t)^2}.$$

Chapter 3

Results

This chapter sets out the key results of the thesis. Results of tests of stationarity are presented in the first section. Then, the predictive ability of forecasting models is reviewed in the Section 3.2. In the next two sections, the relationship of uncertainty measures and their ability to discriminate efficient and inefficient markets is shown. Finally, the potential of the sample entropy for investment strategies is briefly tested in the Section 3.5.

3.1 Stationarity of returns

Return series were firstly tested for a presence of unit root with the ADF test. As stated in the Perron's testing procedure, models that have a constant and a time trend included were tested for a unit root in the first step (see Eq. (2.1) for an underlying model).

After comparing ADF test statistic with MacKinnon critical values (MacKinnon, 2010), the null hypothesis of a unit root was rejected at 1% significance level for all return series of indices and banks. With a rejection already in the first step of testing procedure in a model with time trend, models without trend/trend and constant did not have to be tested and it was concluded that logarithmic returns are integrated of order 0.

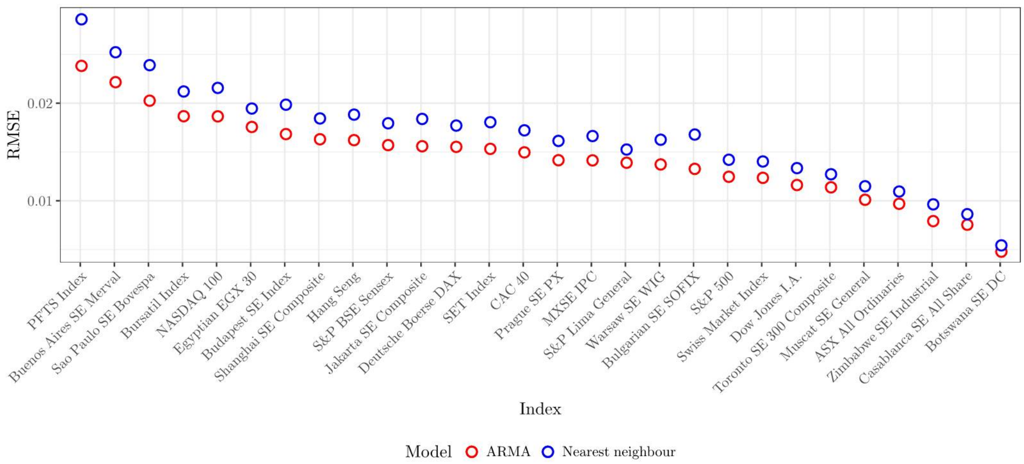
KPSS test did not reject the null hypothesis of stationarity at 5% significance level for 25 out of 28 returns of indices and for 47 out of 49 returns of bank stocks. Therefore, as the results of tests for five series were ambiguous, their ARMA fits could have been inconsistent and were taken with caution. For a list of approximate p-values of all ADF and KPSS tests, please see Tables 4 and 5 in Appendix C.

3.2 Predictive ability

The accuracy of forecasting of both ARMA and nearest neighbour models is analysed in this section.

3.2.1 RMSE

The root-mean-squared error values were calculated for all time series in both datasets – one value for errors of ARMA and one for NN each time.

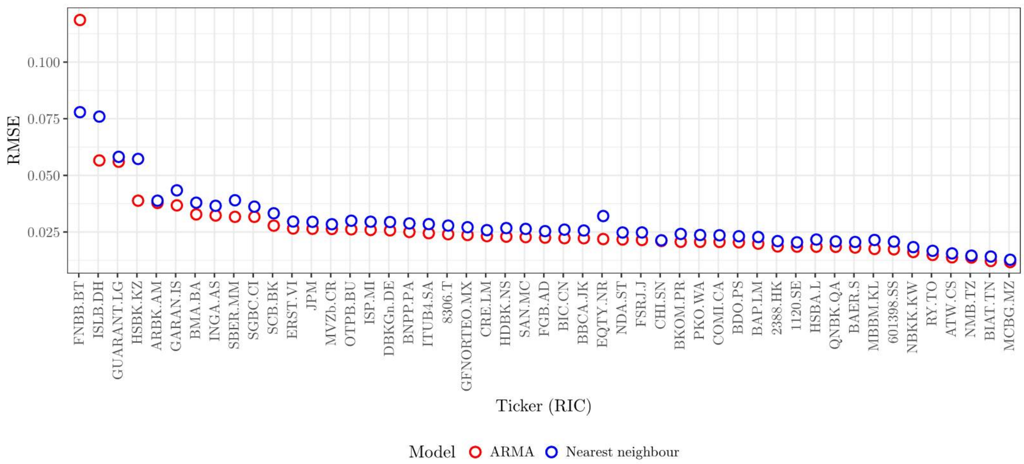


Source: Author.

Figure 3.1: RMSE of ARMA and NN forecasts for indices.

RMSE values for stock indices ranged between 0.005 and 0.024 for ARMA model. The lowest root-mean-squared error of forecasts occurred for Botswanian BSE Domestic Company Index, followed by another two African indices from Zimbabwe and Morocco. Highest RMSE values were found for a benchmark index PFTS of Ukraine Stock Exchange, followed by an Argentinian MERVAL index. Values for NN model were between 0.006 and 0.029, with highest and lowest values for the same indices as in the case of ARMA RMSE values. ARMA fared better in all 29 indices based on this measure (Figure 3.1). Although close in most indices, there was a larger gap between ARMA and NN accuracy for Bulgarian SOFIX index and for aforementioned indices on the upper end of RMSE range.

For bank stocks, RMSE ranged between 0.011 (MCB Group Ltd – Mauritius) and 0.119 (First National Bank of Botswana Ltd) for ARMA and between 0.013 and 0.078 for NN with the same banks having the best and worst results. ARMA again fared better in all cases but one (see Figure 3.2), where NN strongly outperformed ARMA in the lonely case of Botswanian National Bank.



Source: Author.

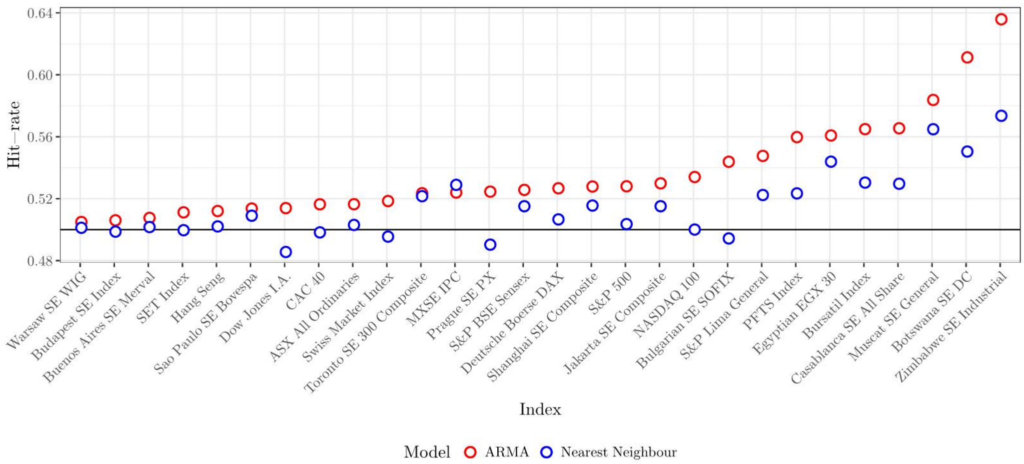
Figure 3.2: RMSE of ARMA and NN forecasts for bank stocks.

3.2.2 Hit-rate

Hit-rate values were calculated in the same way as RMSE – for all stock indices and bank stocks for both ARMA and NN.

In indices, hit percentage ranged between 50.53% (Polish WIG index) and 63.62% (Zimbabwe SE Industrial Index) for ARMA. Therefore, all ARMA models performed better than a coin-toss on average denoted by a horizontal line at 50% in Figure 3.3). NN hit-rate was from 48.59% (Dow Jones Industrial Average) to 57.39% (Zimbabwe SE Industrial Index). Nearest neighbour showed lower than 50% accuracy for 6 indices, with 18 indices under 52% (10 in the case of ARMA).

Measured by hit-rate, ARMA again fared better in forecasting indices. However, in one case of Mexican MXSE IPC Index, NN outperformed ARMA (52.93% vs 52.43%).

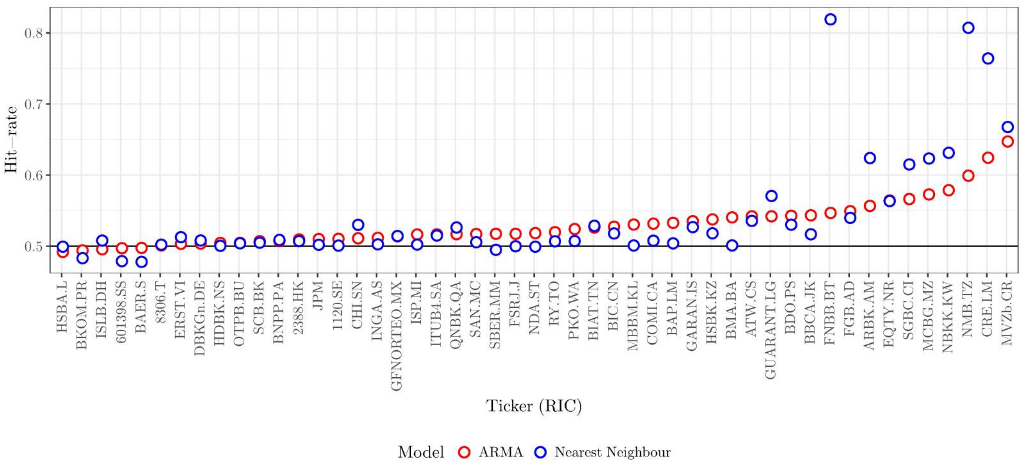


Source: Author.

Figure 3.3: Hit-rate of ARMA and NN forecasts for indices; black horizontal line at $y = 0.5$ indicates the hit-rate of a coin toss.

The situation was different in the case of bank stocks. Although a proportionally higher amount of bank stocks showed quite low predictability (25 and 32 out of 49 stocks had hit-rates under 52%, for ARMA and NN respectively), several stocks were successfully predicted in over 60% of forecasts.

ARMA hit-rates ranged from 49.26% (HSBC Holdings PLC – United Kingdom) to 64.79% (Mercantil Servicios Financieros CA – Venezuela). NN hit the right direction of stock value change in 47.86% in the worst case (Julius Baer Gruppe AG – Switzerland) but in 81.97% for the First National Bank of Botswana Ltd (Figure 3.4). Despite inconclusive overall results in the case of bank stocks, NN outperformed ARMA in 8 out of 10 cases with the highest predictability.



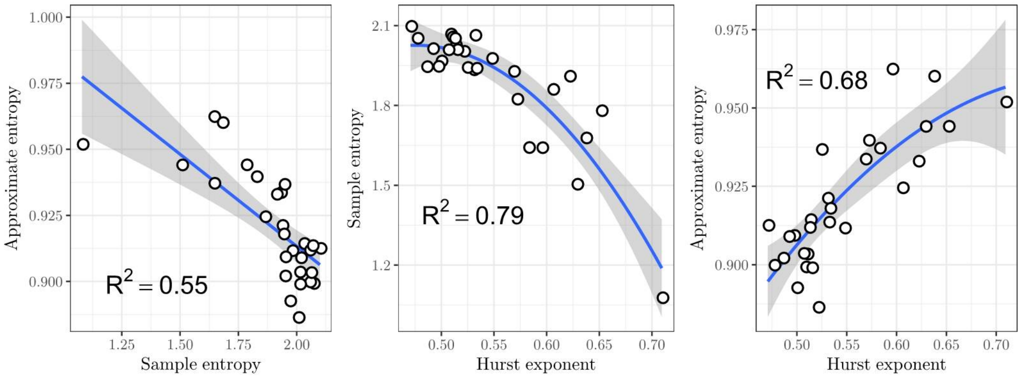
Source: Author.

Figure 3.4: Hit-rate of ARMA and NN forecasts for bank stocks; black horizontal line at $y = 0.5$ indicates the hit-rate of a coin toss.

Although ARMA provided more successful forecasts overall (lower RMSE in 76 out of 77 cases) than the nearest neighbour method, NN showed its ability for simpler time series with repeating patterns (or no price changes), where it provided more accurate predictions of a price movement direction.

3.3 Relationship of uncertainty measures

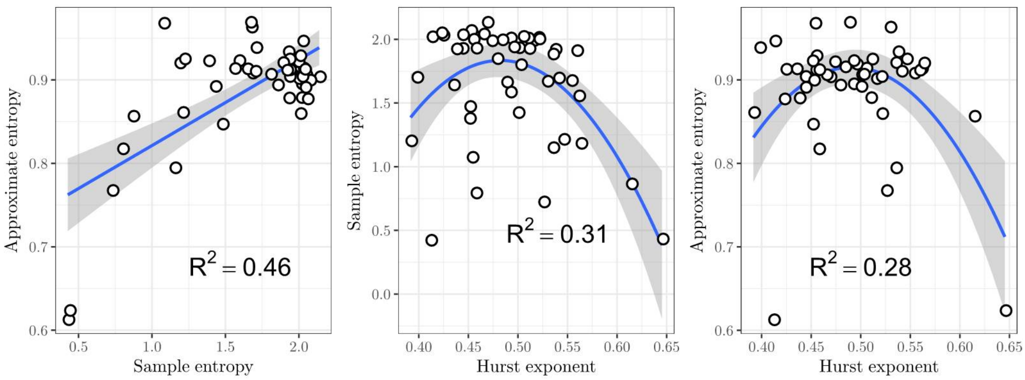
The overall relationship of three uncertainty measures was tested for both datasets. From a theoretical point of view, the approximate entropy and the sample entropy should show a positive correlation – both are lower for series with more repeated patterns and have a maximum value for a uniformly distributed series. On the other hand, the Hurst exponent tends to go up from 0.5 for series with a long-term memory – it should be negatively correlated with both entropic measures (Eom et al. (2008) confirmed this negative correlation with the approximate entropy).



Source: Author.

Figure 3.5: The relationship between the approximate entropy, the sample entropy and the Hurst exponent for indices; blue line is a fitted line of a regression (linear in the case of ApEn-SampEn relationship, quadratic in relationships with the Hurst exponent), grey area represents 95% confidence interval.

For average values of measures for indices, only the relationship between the sample entropy and the Hurst exponent followed theoretical assumptions (Figure 3.5). Values of the approximate entropy were very similar for all indices and probably did not accurately measured the long-term memory. For bank stocks, the results were consistent with theory (positive relationship of ApEn and SampEn; highest values of entropy around H = 0.5). However, relationships did not seem very strong (see Figure 3.6).



Source: Author.

Figure 3.6: The relationship between the approximate entropy, the sample entropy and the Hurst exponent for bank stocks; blue line is a fitted line of a regression (linear in the case of ApEn-SampEn relationship, quadratic in relationships with the Hurst exponent), grey area represents 95% confidence interval.

3.4 Difference in markets

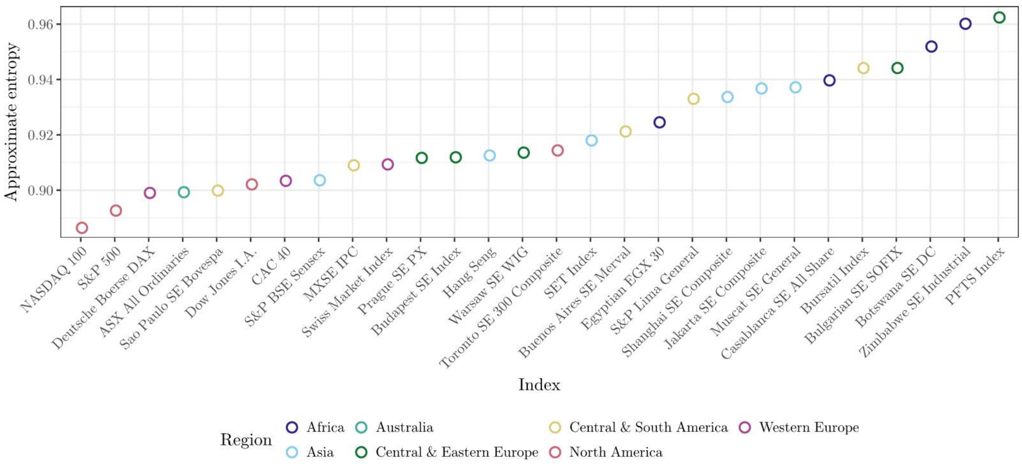
This section provides results for ability of measures of uncertainty to measure predictability (which is proxied by hit-rates). Moreover, their ability to differentiate between developed and developing markets is tested.

3.4.1 Indices

Approximate entropy

As was mentioned in the previous section, the approximate entropy was suspected to measure the long-term memory inaccurately in the case of stock indices. Approximate entropy values were opposite to what was expected. Return series of indices, such as S&P 500 or NASDAQ 100, had the lowest approximate entropy, which should suggest their higher predictability because of repeated patterns. On the other hand, African indices were among those with the highest entropy (Figure 3.7).

However, in the case of developed markets, to which aforementioned U.S. indices clearly belong, markets should be efficient (or very close to efficient) – therefore, with a very low number of repeated patterns.



Source: Author.

Figure 3.7: Average approximate entropy of indices (coloured by regions).

Figure 3.8 confirms the low predictability of American and Western European indices. Among the most predictable indices are those with the highest ApEn, mostly African and South American. This quite strong positive correlation between ApEn and hit-rates is rather dubious and is not consistent with the theory.

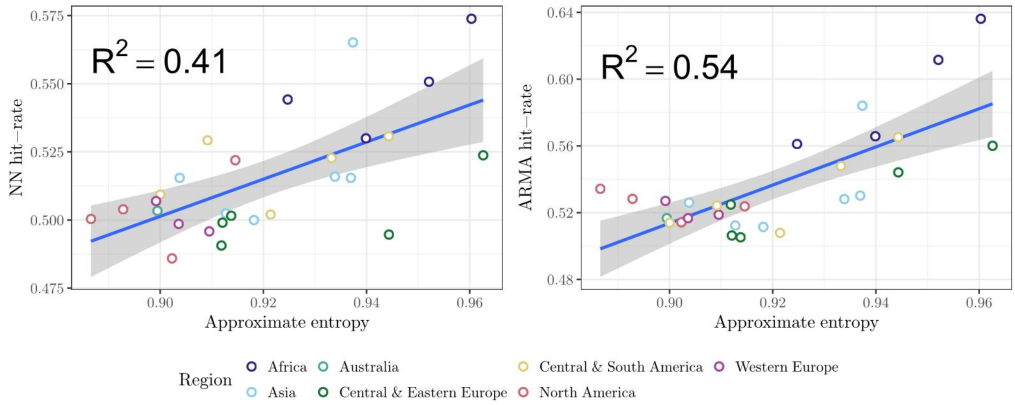


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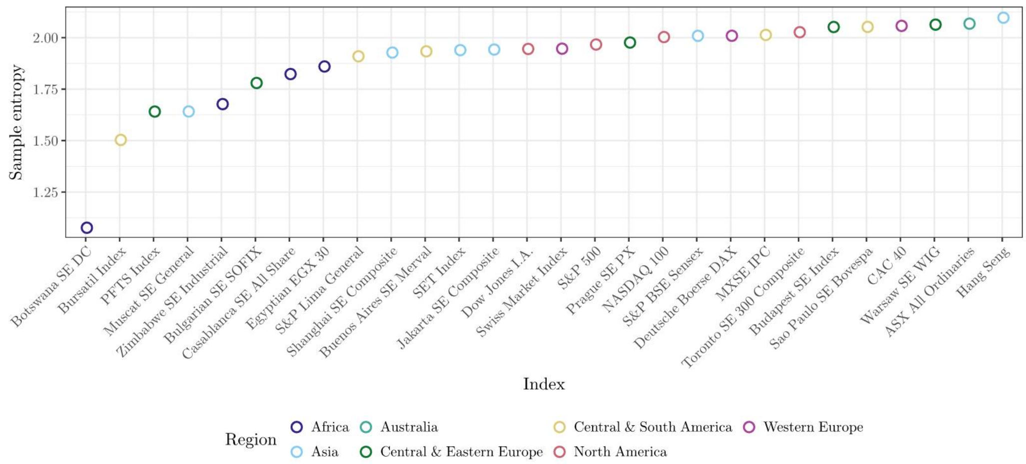
Figure 3.8: The relationship between average approximate entropy of indices and hit-rates of NN and ARMA (coloured by regions); blue line is a fitted line of a linear regression, grey area represents 95% confidence interval.

Sample entropy

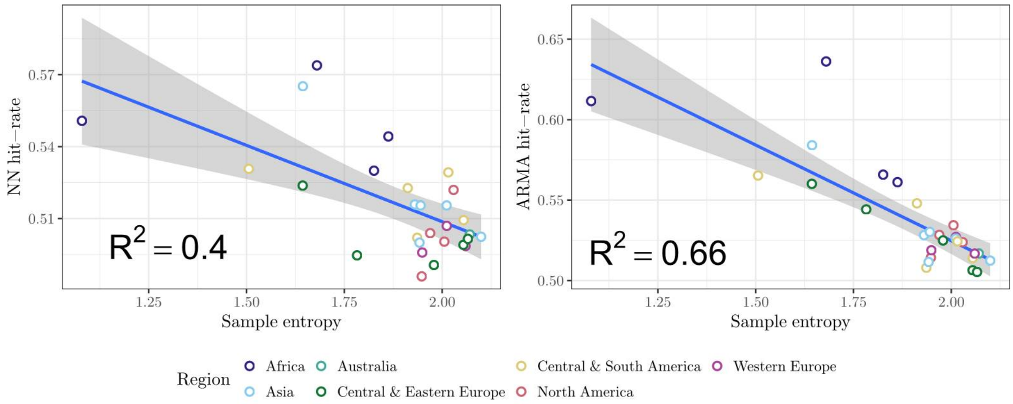


Source: Author.

Figure 3.9: Average sample entropy of indices (coloured by regions).

The sample entropy showed probably a better view on the reality, with 13 indices with the lowest sample entropy being either from Africa, South America, Asia or Eastern Europe (Figure 3.9). Values are also spread more widely, with the lowest value of 1.08 for Botswana Stock Exchange DC Index and the highest value 2.10 for Hang Seng Index of Hong Kong. SampEn therefore acts as a good measure for differentiation of markets.

Sample entropy also evidenced quite strong negative correlation with hit-rates, especially in the case of ARMA ($\rho = -0.81$, the right plot in Figure 3.10). For all values of SampEn up to 1.91, hit-rate of ARMA was 54.42% at minimum, which shows good properties of SampEn as a measure of predictability.

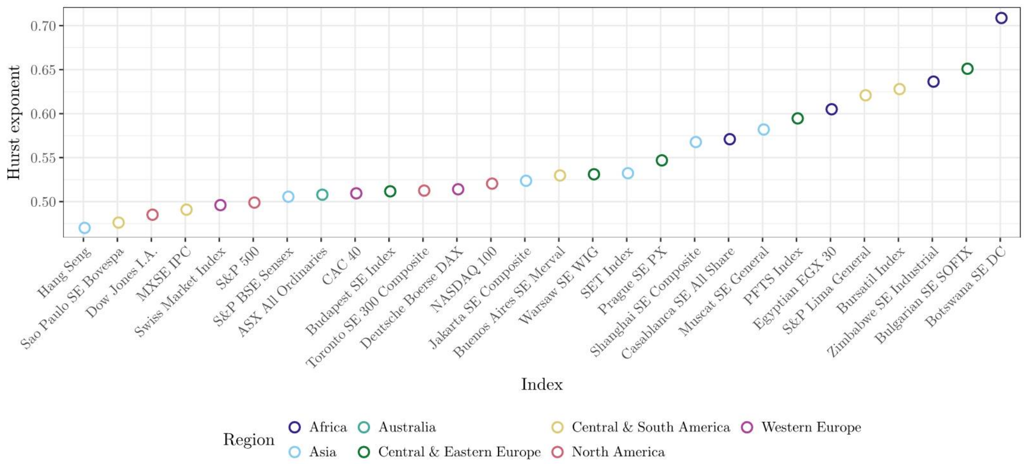


Source: Author.

Figure 3.10: The relationship between average sample entropy of indices and hit-rates of NN and ARMA (coloured by regions); blue line is a fitted line of a linear regression, grey area represents 95% confidence interval.

Hurst exponent

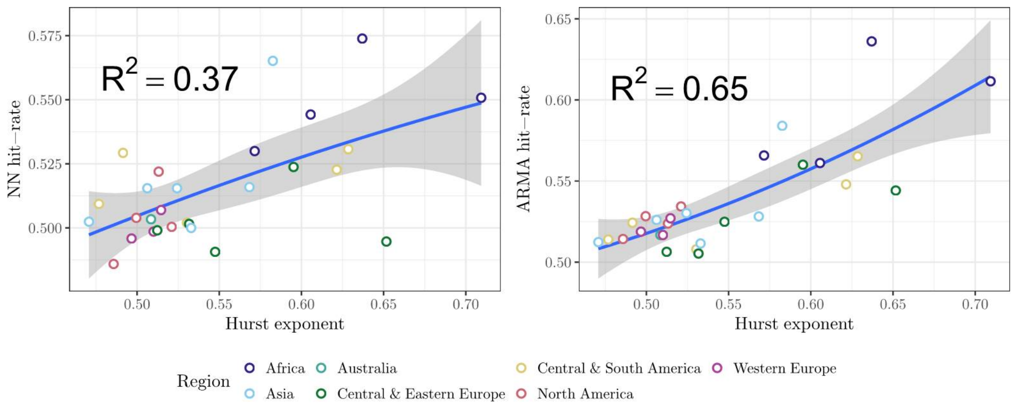
The Hurst exponent, selected as a control measure in addition to the entropic measures, showed a very strong differentiation of markets. All North American and Western European indices are in the range close to H = 0.5 (Figure 3.11). On the upper side of the range, top three positions are occupied by two African indices and Eastern European SOFIX Index.



Source: Author.

Figure 3.11: The Hurst exponent of indices (coloured by regions).

The Hurst exponent appeared to have a good and linear relationship with hit-rates, especially with ARMA (Figure 3.12). Indices with a low hit-rate values had also H close to 0.5 with that being mostly series from economically developed areas. With Hurst exponent values over 0.57, hit-rates of ARMA were consistently above 54%.



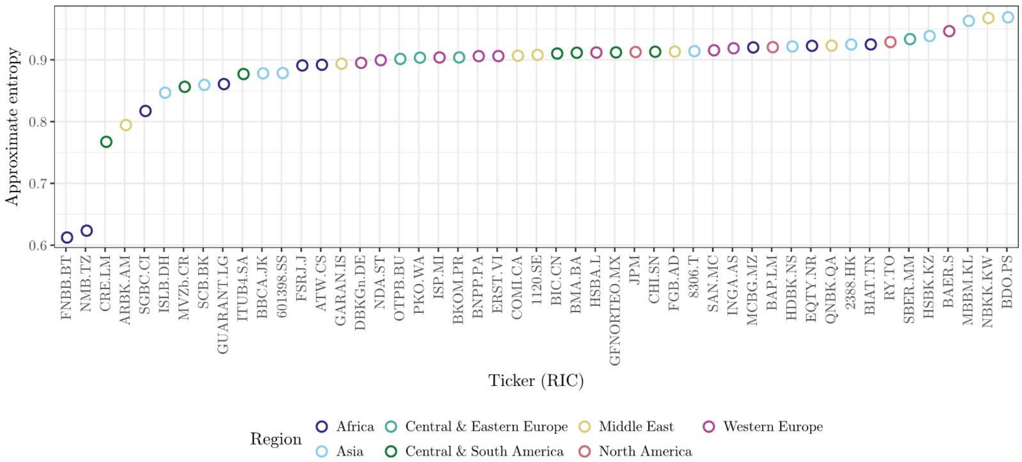
Source: Author.

Figure 3.12: The relationship between Hurst exponent of indices and hit-rates of NN and ARMA (coloured by regions); blue line is a fitted line of a quadratic regression, grey area represents 95% confidence interval.

3.4.2 Banks

Approximate entropy

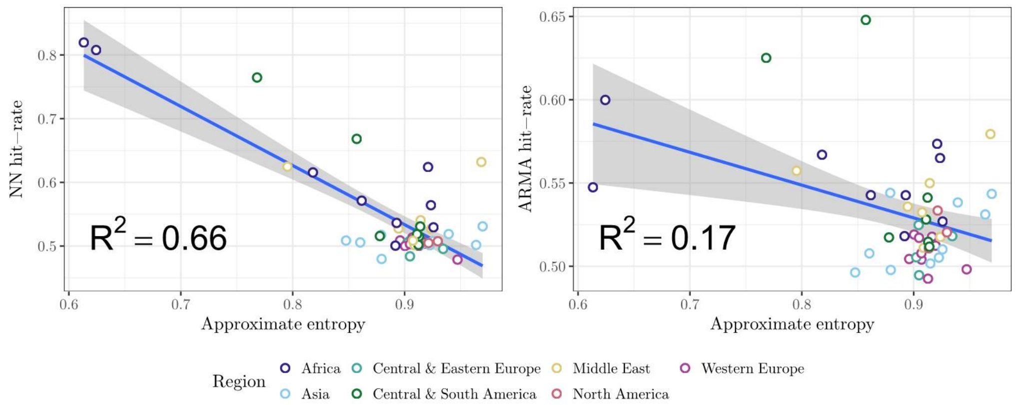
The same hypotheses and relationships were tested also for values of bank stocks. Very low ApEn values, compared to other bank stocks, were observed for two African stocks, First National Bank of Botswana Ltd and Tanzanian National Microfinance Bank PLC (Figure 3.13). These two stocks were clearly differentiated from the rest, with another three stocks having also a somewhat recognisable lower ApEn value than remaining stocks. No clear pattern of market recognition could be seen in the rest of values.



Source: Author.

Figure 3.13: Average approximate entropy of bank stocks (coloured by regions).

The relationship between ApEn and hit-rate is displayed in Figure 3.14. The approximate entropy correctly predicted improved predictability of aforementioned five stocks with the lowest ApEn; however, no clear relationship can be seen with ARMA hit-rate. Although the fitted line of the linear regression has a negative slope as predicted by theory, it can be argued that this relationship is just driven by few points in the left part of the graph.

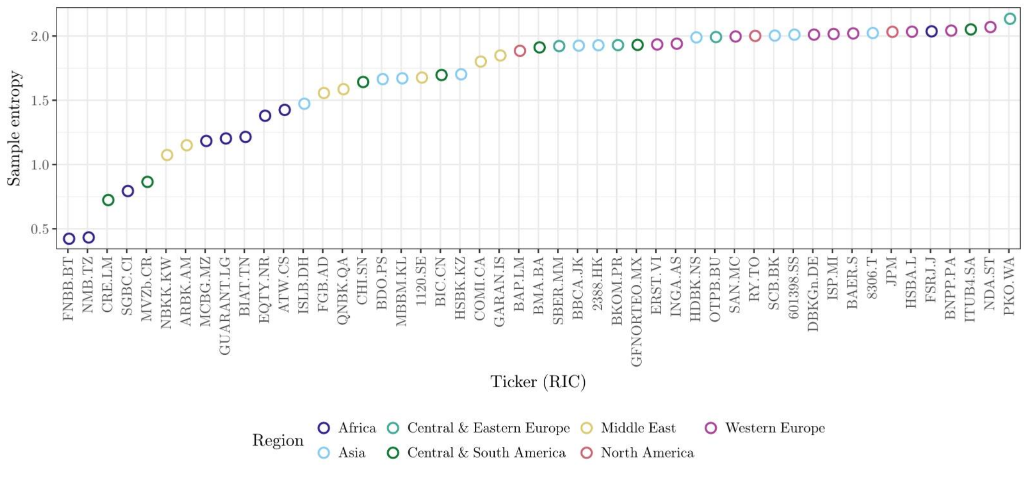


Source: Author.

Figure 3.14: The relationship between average approximate entropy of bank stocks and hit-rates of NN and ARMA (coloured by regions); blue line is a fitted line of a linear regression, grey area represents 95% confidence interval.

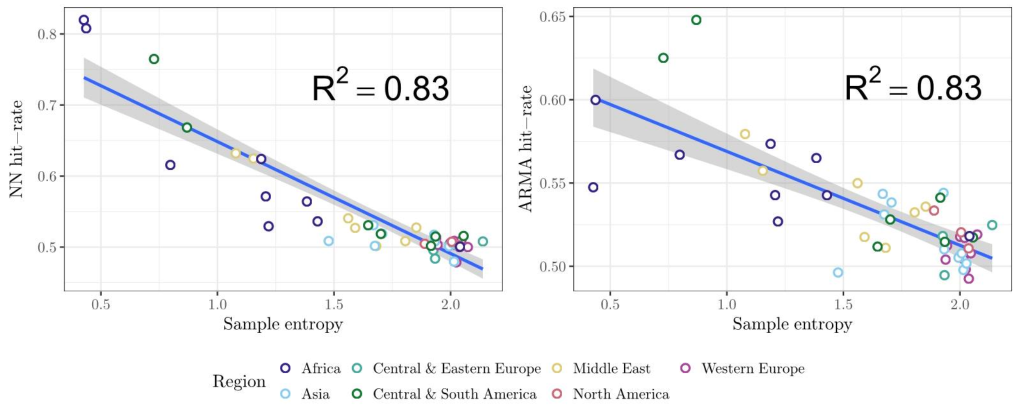
Sample entropy

The sample entropy measure separated stocks more distinctively than ApEn. Values ranged from 0.4268 to 2.1384. All African stocks but one were in the quarter of stocks with lowest SampEn; on the other side, all Western European and North American stocks stood in the upper half (see Figure 3.15). Therefore, SampEn seemed to follow the hypothesis of developed markets having higher entropy.



Source: Author.

Figure 3.15: Average sample entropy of bank stocks (coloured by regions).



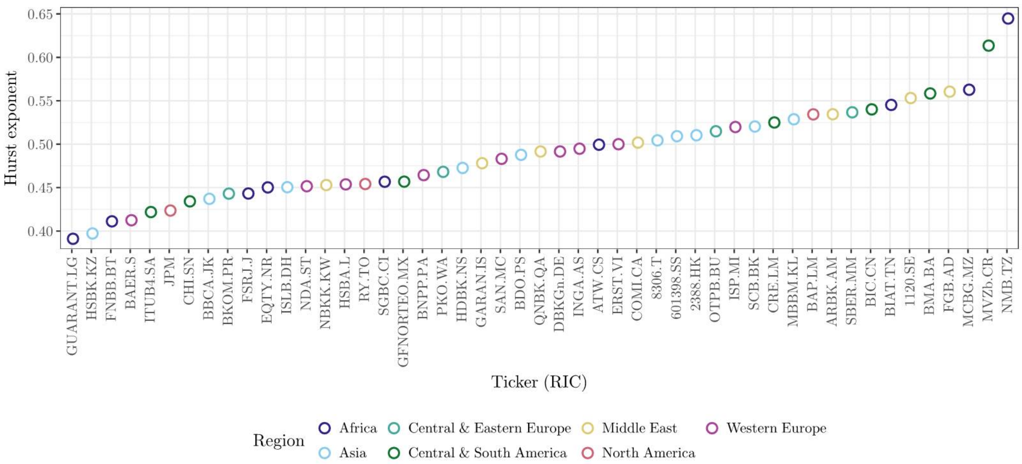
Source: Author.

Figure 3.16: The relationship between average sample entropy of bank stocks and hit-rates of NN and ARMA (coloured by regions); blue line is a fitted line of a linear regression, grey area represents 95% confidence interval.

The sample entropy showed a strong negative relationship with hit-rate measures for bank stocks (Figure 3.16). In particular, the relationship with the nearest neighbour method hit-rate was short of any major outliers. However, SampEn no longer discriminated different levels of hit-rates if its value was close to 2.0.

Hurst exponent

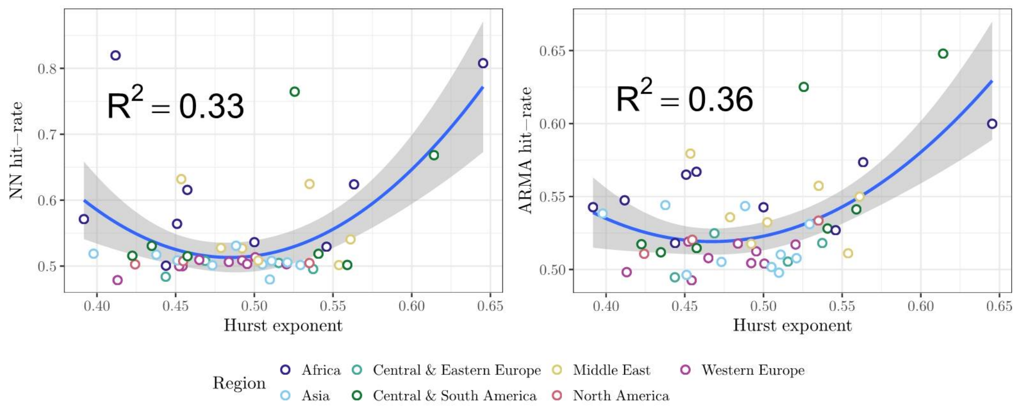
Hurst exponents of bank stocks spanned over a different, wider region of values than in the case of stock indices. More than a half of stocks had their estimated Hurst exponent under 0.5, indicating a presence of a short-term memory. Geographic regions were not grouped around a common value (Figure 3.17), e.g. African stocks occupied both end of range meaning either a short-term or long-term memory in their time series. Also, not all of the Western European and North American stocks had their values close to 0.5, with Swiss Julius Baer Gruppe AG and U.S. JPMorgan Chase & Co being the outliers (with H = 0.413 and H = 0.424, respectively).



Source: Author.

Figure 3.17: The Hurst exponent of bank stocks (coloured by regions).

As displayed in Figure 3.18, the relationship between the Hurst exponent and hit-rate did not appear to be linear, which is consistent with a theoretical breaking point at H = 0.5. A convex parabola with a minimum at a level of Hurst exponent between 0.47 and 0.5 provided a better fit than a linear regression fitted line. However, apart from the extreme cases on both side, Hurst exponent did not differentiate stocks with a good predictability from the rest.

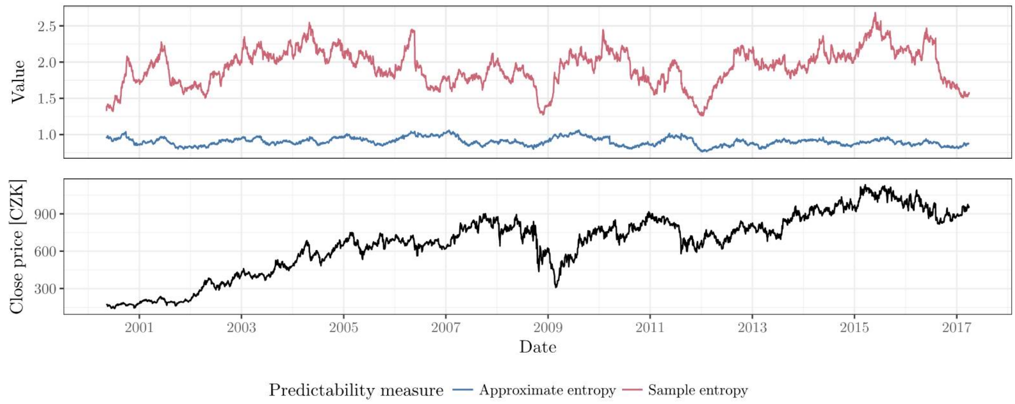


Source: Author.

Figure 3.18: The relationship between Hurst exponent of bank stocks and hit-rates of NN and ARMA (coloured by regions); blue line is a fitted line of a quadratic regression, grey area represents 95% confidence interval.

3.5 Investment strategies based on entropy

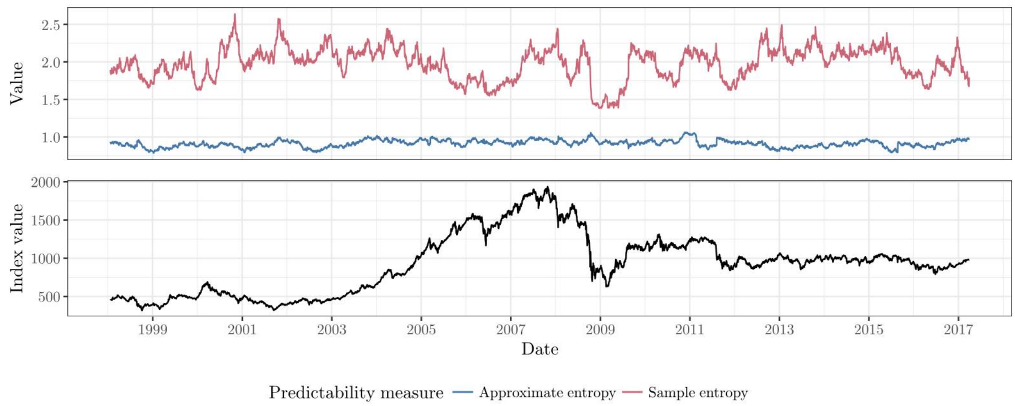
This section analyses the possibility of investment strategies based on values of the approximate entropy and the sample entropy. The main motivation to implement ApEn and/or SampEn to the process was their theoretical ability to find periods of time during which we can forecast more precisely. Two datasets were used to test this ability - historical close prices of Komercni banka a.s. (KB) and historical values of PX Index of Prague Stock Exchange.



Source: Author.

Figure 3.19: Daily close price in CZK of KB stock between 11/05/2000 and 30/03/2017 (bottom graph), ApEn and SampEn values for KB stock during the same period (top graph).

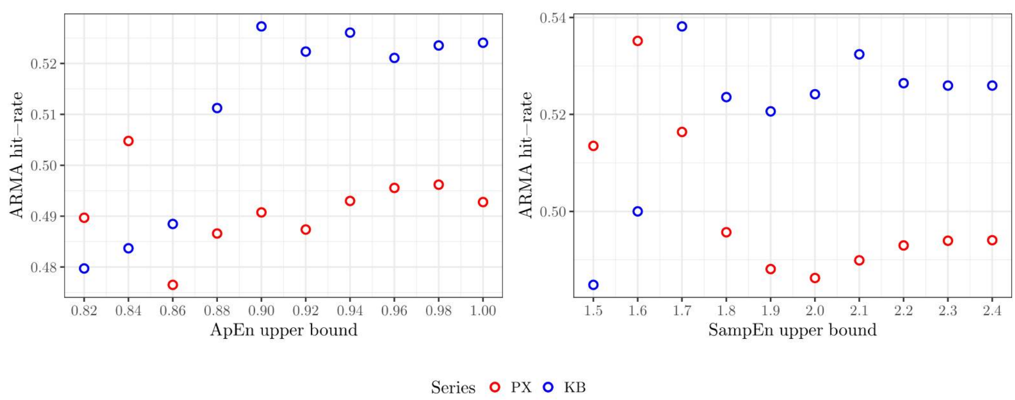
First 201 values of both level datasets were only included as parts for rolling windows (200 observations of returns) of ARMA, NN, ApEn and SampEn procedures explained in Methodology and no estimation of forecasts/entropic measures was done for their period. Therefore, these values are omitted in figures for the rest of investment analysis. Entropy values were calculated for a period between May 2000 and March 2017 for the stock of KB (Figure 3.19) and for a period between January 1998 and March 2017 for the PX index (Figure 3.20). ApEn values ranged between 0.76 and 1.06 for KB and between 0.79 and 1.06 for PX. SampEn for KB was from 1.26 to 2.68 and from 1.38 to 2.64 in the case of PX index.



Source: Author.

Figure 3.20: Daily index value of PX index between 22/01/1998 and 30/03/2017 (bottom graph), ApEn and SampEn values for PX index during the same period (top graph).

Figure 3.21 shows the relationship between the hit-rate of ARMA and the upper bound of entropy measure at which are the forecasts considered. For example, for the upper bound of SampEn = 1.8 are compared forecasts and real values only in periods with SampEn ≤ 1.8. Theoretically, hit-rate should be higher in periods with lower entropy. However, this theoretical negative relationship seemed to show up only in the case of SampEn and PX index and declined in other cases. As ApEn did not show any promising results, strategies below are based only on the sample entropy.



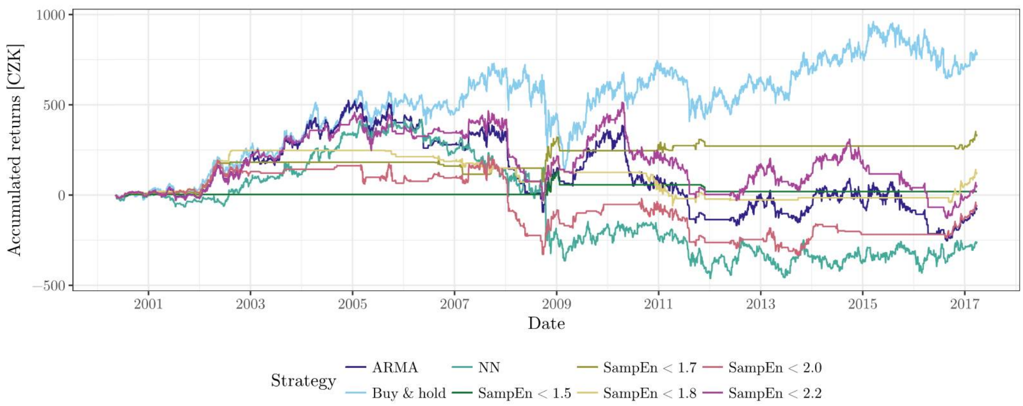
Source: Author.

Figure 3.21: Average ARMA hit-rates in periods selected by ApEn (left graph) and SampEn (right graph) values lower than the value on x axis.

Several investment strategies were tested for both datasets, with transactions being buying/selling of one "unit" of stock/index per day:

- *Buy & hold*: Stock was bought in the first period and held until the last (or bought each day and sold the day after).
- *ARMA*: If ARMA predicted positive return , stock was bought and sold the next day (repeated in every period).
- *NN*: If NN predicted positive return, stock was bought and sold the next day (repeated in every period).
- $SampEn < x$: If the value of SampEn was lower than x, then the prediction of ARMA was checked. When it was positive, stock was bought and then sold the next day (repeated in every period).

No transaction costs were assumed for buying or selling stocks. Short selling was not allowed as it was forbidden in several countries during the selected period and therefore, it was not a universally applicable strategy. Moreover, it would bring additional assumptions with the setting of borrowing fees.



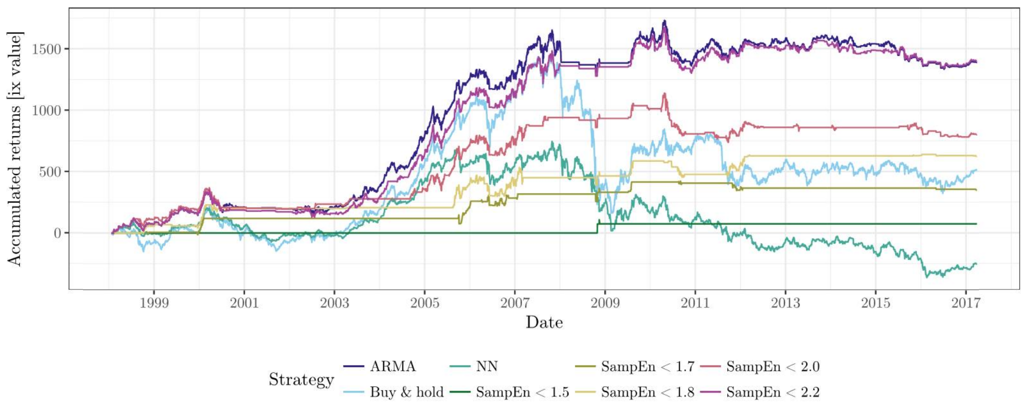
Source: Author.

Figure 3.22: Accumulated returns (in CZK) of different investment strategies on Komercni banka a.s. stock.

Simple buy & hold strategy was the most powerful in the case of KB stock with final accumulated returns of 773.80 CZK (Figure 3.22). Both

ARMA and NN ended with a loss (–82.50 CZK, –269.34 CZK, respectively) and ARMA was improved by SampEn restrictions, with SampEn < 1.7 being the second best strategy overall (the return of 324.98 CZK) strongly outperforming ARMA in the period from 05/2010 to 12/2001.

In the case of the PX index, ARMA performed very strongly (return of 1388.82 index points), outperforming buy & hold strategy by more than 870 points (see Figure 3.23). However, the least restrictive SampEn strategy topped ARMA by a small margin (return of 1400.19 ix points). Other strategies showed worse performance (higher restriction meant lower profitability).



Source: Author.

Figure 3.23: Accumulated returns (in index value) of different investment strategies on PX index.

No entropy-based restriction strategy vastly improved ARMA forecasting in both cases and only the least restrictive version SampEn < 2.2 showed slight improvement in both cases. However, a strategy based on SampEn values in the range of $1.6 - 1.7$ could be seen as a "safe option" with improved hit-rates over unrestricted ARMA (but resulting in investing in much less periods, especially in developed markets).

Conclusion

The purpose of this thesis was to empirically assess the relationship between entropic measures of market efficiency and market predictability as estimated by hit-rates of two models, ARMA and the nearest neighbour. This was tested together with the property of distinguishing between developing and developed markets. Moreover, an out-of-sample analysis of investments strategies, which incorporated a decision point based on the value of the sample entropy, was done on two datasets. The scope of the study is larger in comparison to the previous work done in this area, with a total of 77 time series tested, two forecasting models and three measures of market efficiency; while being the first study to test the sample entropy as intensively for this purpose.

Firstly, both forecasting models, ARMA and NN, were tested for their accuracy. ARMA showed better results overall by both accuracy measures MRSE and hit-rate, with the exception of the simplest bank stock series, where the pattern repetition strategy of NN showed some merit.

The relationship of market efficiency measures ApEn, SampEn and the Hurst exponent was checked for being consistent with theory. The approximate entropy had the relationship with other measures either very weak or with the opposite direction than predicted, whereas the sample entropy and the Hurst exponent were in line with theory. Hence, ApEn was not accepted as a good measure of efficiency.

This was further confirmed by its weak results in differentiation of developed and developing markets. SampEn and the Hurst exponent clearly showed lower overall efficiency of African indices and bank stocks relatively to the western world. The sample entropy also formed quite strong downward linear relationship with hit-rates of forecasting models, especially in the case of bank stocks. This shows some potential of the sample entropy as an estimator of predictability of markets. The Hurst exponent relationship with hit-rates was estimated to be parabolic with a vertex close to the value $H = 0.5$, which is consistent with theory. However, the fit was not as strong as in the case of SampEn.

Investment strategies with the SampEn threshold were dependent on the performance of the underlying forecasting ARMA model. ARMA showed highest hit-rates with SampEn values around 1.6 – 1.7. This could be seen as a safer investment strategy than to rely only on the simple ARMA model; however, also as one with potentially lower accumulated returns due to smaller investing windows. This result could provide useful in determining the investment strategy with the right proportion of risk to establish a well-rounded portfolio. Nevertheless, the result is still preliminary and it could be of interest to undertake further research with more extensive forecasting methods and on larger sample of data. Furthermore, different parameters m and r for SampEn and ApEn could be tested.

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Appendix

A Software

Calculations were done in Microsoft R Open 3.3.2, which offers additional functionalities to the base R 3.3.2, e.g. multi-threaded math libraries. A list of used R packages is in Table 1. RStudio v1.0.136 was used as a primary IDE, with some work dome also in Microsoft Visual Studio 2015 Community with built-in add-on R Tools for Visual Studio 1.0RC2.

Table 1: Additionally installed R packages

Package name	Used function	Description
pracma	approx_entropy	Calculates the approximate entropy of a time series.
	sample_entropy	Calculates the sample entropy of a time series.
fractal	DFA	Estimates the Hurst exponent using the Detrended Fluctuation Analysis.
forecast	accuracy	Returns measures of the forecast accuracy (e.g. RMSE).
	auto.arima	Fits the best ARIMA model (selected by AICc) to a time series.
	forecast	Forecasts time series using (in this case) ARIMA models.
CADFtest	CADFtest	Augmented Dickey-Fuller test with an automatic model selection.
tseries	kpss.test	Computes the (KPSS) test for the null hypothesis of stationarity.
ggplot2	ggplot	Graphs data using ggplot visualisation.
cowplot	plot_grid	Combines several graphs into one.
beepr	beep	Plays a selected sound (e.g. after a long calculation).
extrafont	font_import	Imports system fonts to database.
	loadfonts	Registers fonts with R (for their use in figures).

Source: Author.

B Details of datasets

Table 2: Details of the dataset of market indices

Name	Country of Issuer	Observations	First	Last
Western Europe				
CAC 40 Index	France	5103	01/04/1997	30/03/2017
Deutsche Boerse DAX Index	Germany	5079	01/04/1997	30/03/2017
Swiss Market Index	Switzerland	5035	01/04/1997	30/03/2017
Central & Eastern Europe				
Bulgarian Stock Exchange SOFIX Index	Bulgaria	4060	20/10/2000	30/03/2017
PX Prague SE Index	Czechia	5021	01/04/1997	30/03/2017
Budapest SE Index	Hungary	4994	01/04/1997	30/03/2017
Warsaw SE WIG Poland Index	Poland	5010	01/04/1997	30/03/2017
PFTS Index	Ukraine	4661	03/10/1997	30/03/2017
North America				
S&P 500 Index - CBOE	United States	5035	31/03/1997	30/03/2017
NASDAQ 100 Index	United States	5035	31/03/1997	30/03/2017
Dow Jones Industrial Average Index	United States	5035	31/03/1997	30/03/2017
TSX-Toronto Stock Exchange 300 Composite Index	Canada	5027	31/03/1997	30/03/2017
Central & South America				
Buenos Aires SE Merval Index	Argentina	4926	31/03/1997	30/03/2017
Sao Paulo SE Bovespa Index	Brazil	4956	31/03/1997	30/03/2017
Bursatil Index	Venezuela	4838	31/03/1997	30/03/2017
S&P Lima General Index	Peru	4997	31/03/1997	30/03/2017
MXSE IPC Index	Mexico	5034	31/03/1997	30/03/2017
Asia				
Jakarta SE Composite Index	Indonesia	4882	01/04/1997	30/03/2017
Shanghai SE Composite Index	China (Mainland)	4849	31/03/1997	30/03/2017
Hang Seng Index	Hong Kong	4934	01/04/1997	30/03/2017
S&P BSE Sensex Index	India	4948	31/03/1997	30/03/2017
Muscat SE General Index	Oman	4750	31/03/1997	30/03/2017
SET Index	Thailand	4895	31/03/1997	30/03/2017
Africa				
EGX 30 Index	Egypt	4699	01/01/1998	30/03/2017
Casablanca SE All Share Index	Morocco	3803	02/01/2002	30/03/2017
Zimbabwe SE Industrial Index	Zimbabwe	2015	19/02/2009	30/03/2017
Botswana Stock Exchange DC Index	Botswana	3805	30/04/2001	30/03/2017
Australia				
ASX All Ordinaries Index	Australia	5064	01/04/1997	30/03/2017

Source: Author.

Table 3: Details of the dataset of bank stocks

Name	Ticker	Country	Observations	First	Last
Western Europe					
Deutsche Bank AG	DBK	Germany	5072	01/04/1997	30/03/2017
Erste Group Bank AG	EBS	Austria	4909	01/04/1997	30/03/2017
Julius Baer Gruppe AG	BAER	Switzerland	1887	01/10/2009	30/03/2017
HSBC Holdings PLC	HSBA	United Kingdom	5055	01/04/1997	30/03/2017
Banco Santander SA	SAN	Spain	5068	31/03/1997	30/03/2017
BNP Paribas SA	BNP	France	5102	01/04/1997	30/03/2017
ING Groep NV	INGA	Netherlands	5111	01/04/1997	30/03/2017
Nordea Bank AB	NDA SEK	Sweden	5021	01/04/1997	30/03/2017
Intesa Sanpaolo SpA	ISP	Italy	5076	01/04/1997	30/03/2017
Central & Eastern Europe					
Powszechna Kasa Oszczednosci Bank Polski SA	PKO	Poland	3105	10/11/2004	30/03/2017
OTP Bank Nyrt	OTP	Hungary	4993	01/04/1997	30/03/2017
Komercni Banka a.s.	KOMB	Czechia	4440	01/07/1999	30/03/2017
Sberbank Rossii PAO	SBER	Russia	4723	22/01/1998	30/03/2017
North America					
JPMorgan Chase & Co	JPM	United States	5035	31/03/1997	30/03/2017
Royal Bank of Canada	RY	Canada	5030	31/03/1997	30/03/2017
Credicorp Ltd	BAP	Bermuda	4889	31/03/1997	30/03/2017
Central & South America					
Grupo Financiero Banorte SAB de CV	GFNORTEO	Mexico	4415	07/06/1999	30/03/2017
Mercantil Servicios Financieros CA	MVZ.B	Venezuela	3382	04/01/2000	30/03/2017
Itau Unibanco Holding SA	ITUB4	Brazil	4954	31/03/1997	30/03/2017
Banco de Credito del Peru	CREDITC1	Peru	4018	31/03/1997	30/03/2017
Banco de Chile	CHILE	Chile	4689	26/12/1997	30/03/2017
Bancolombia SA	BCOLOMBIA	Colombia	4522	31/03/1997	30/03/2017
Banco Macro SA	BMA4	Argentina	4761	31/03/1997	30/03/2017
Middle East					
Qatar National Bank SAQ	QNBK	Qatar	4125	08/12/1999	30/03/2017
Al Rajhi Banking & Investment Corporation SJSC	1120	Saudi Arabia	4797	10/01/1999	30/03/2017
First Gulf Bank PJSC	FGB	United Arab Emirates	3827	29/06/2002	30/03/2017
National Bank of Kuwait SAKP	NBK	Kuwait	4897	31/03/1997	30/03/2017
Turkiye Garanti Bankasi AS	GARAN	Turkey	5001	31/03/1997	30/03/2017
Arab Bank PLC	ARBK	Jordan	4882	31/03/1997	29/03/2017
Commercial International Bank Egypt SAE	COMI	Egypt	4880	31/03/1997	30/03/2017
Asia					
Industrial and Commercial Bank of China Ltd	601398	China (Mainland)	2513	27/10/2006	30/03/2017
Mitsubishi UFJ Financial Group Inc	8306	Japan	3922	02/04/2001	30/03/2017
HDFC Bank Ltd	HDFCBANK	India	4981	31/03/1997	30/03/2017
BOC Hong Kong Holdings Ltd	2388	Hong Kong	3627	25/07/2002	30/03/2017
Bank Central Asia Tbk PT	BBCA	Indonesia	4101	31/05/2000	30/03/2017
Malayan Banking Bhd	MAYBANK	Malaysia	4921	31/03/1997	30/03/2017
Siam Commercial Bank PCL	SCB	Thailand	4896	31/03/1997	30/03/2017
BDO Unibank Inc	BDO	Philippines	3616	21/05/2002	30/03/2017
Halyk Bank AO	HSBK	Kazakhstan	2300	20/11/1998	30/03/2017
Islami Bank Bangladesh Ltd	ISLAMIBANK	Bangladesh	2234	25/11/2007	30/03/2017
Africa					
FirstRand Ltd	FSR	South Africa	4997	01/04/1997	30/03/2017
Attijariwafa Bank SA	ATW	Morocco	4723	31/03/1997	30/03/2017
Guaranty Trust Bank PLC	GUARANTY	Nigeria	4789	18/07/1997	30/03/2017
MCB Group Ltd	MCBG	Mauritius	4792	31/03/1997	30/03/2017
Equity Group Holdings Ltd	EQTY	Kenya	2663	27/06/2001	30/03/2017
Societe Generale de Banques en Cote d'Ivoire SA	SGBC	Ivory Coast	2545	01/04/1997	30/03/2017
Banque International Arabe de Tunisie SA	BIAT	Tunisia	4733	31/03/1997	30/03/2017
First National Bank of Botswana Ltd	FNBB	Botswana	2941	01/04/1997	30/03/2017
National Microfinance Bank PLC	NMB	Tanzania	1778	06/11/2008	28/03/2017

Source: Author.

C Summary tables

Table 4: Summary of results for market indices

	p-value					Hit-rate		RMSE	
Name	ADF	KPSS	ApEn	SampEn	Hurst	ARMA	NN	ARMA	NN
Western Europe									
CAC 40 Index	< 0.001	> 0.1	0.904	2.060	0.510	0.517	0.499	0.015	0.017
Deutsche Boerse DAX Index	< 0.001	> 0.1	0.899	2.012	0.515	0.527	0.507	0.016	0.018
Swiss Market Index	< 0.001	> 0.1	0.910	1.949	0.497	0.519	0.496	0.012	0.014
Eastern Europe									
Bulgarian Stock Exchange SOFIX Index	< 0.001	0.021	0.944	1.782	0.652	0.544	0.495	0.013	0.017
PX Prague SE Index	< 0.001	> 0.1	0.912	1.979	0.547	0.525	0.491	0.014	0.016
Budapest SE Index	< 0.001	> 0.1	0.912	2.054	0.512	0.506	0.499	0.017	0.020
Warsaw SE WIG Poland Index	< 0.001	> 0.1	0.914	2.066	0.532	0.505	0.502	0.014	0.016
PFTS Index	< 0.001	> 0.1	0.963	1.643	0.595	0.560	0.524	0.024	0.029
North America									
S&P 500 Index - CBOE	< 0.001	> 0.1	0.893	1.969	0.500	0.528	0.504	0.013	0.014
NASDAQ 100 Index	< 0.001	> 0.1	0.887	2.005	0.521	0.534	0.500	0.019	0.022
Dow Jones Industrial Average Index	< 0.001	> 0.1	0.902	1.948	0.486	0.514	0.486	0.012	0.013
TSX-Toronto Stock Exchange 300 Composite Index	< 0.001	> 0.1	0.915	2.029	0.513	0.524	0.522	0.011	0.013
Central & South America									
Buenos Aires SE Merval Index	< 0.001	> 0.1	0.921	1.936	0.530	0.508	0.502	0.022	0.025
Sao Paulo SE Bovespa Index	< 0.001	> 0.1	0.900	2.055	0.477	0.514	0.509	0.020	0.024
Bursatil Index	< 0.001	> 0.1	0.944	1.506	0.628	0.565	0.531	0.019	0.021
S&P Lima General Index	< 0.001	> 0.1	0.933	1.912	0.621	0.548	0.523	0.014	0.015
MXSE IPC Index	< 0.001	> 0.1	0.909	2.016	0.491	0.524	0.529	0.014	0.017
Asia									
Jakarta SE Composite Index	< 0.001	> 0.1	0.937	1.945	0.524	0.530	0.515	0.016	0.018
Shanghai SE Composite Index	< 0.001	> 0.1	0.934	1.930	0.568	0.528	0.516	0.016	0.018
Hang Seng Index	< 0.001	> 0.1	0.913	2.100	0.471	0.512	0.502	0.016	0.019
S&P BSE Sensex Index	< 0.001	> 0.1	0.904	2.011	0.506	0.526	0.515	0.016	0.018
Muscat SE General Index	< 0.001	> 0.1	0.937	1.644	0.583	0.584	0.565	0.010	0.012
SET Index	< 0.001	> 0.1	0.918	1.942	0.533	0.512	0.500	0.015	0.018
Africa									
EGX 30 Index	< 0.001	> 0.1	0.925	1.862	0.606	0.561	0.544	0.018	0.020
Casablanca SE All Share Index	< 0.001	0.075	0.940	1.826	0.572	0.566	0.530	0.008	0.009
Zimbabwe SE Industrial Index	< 0.001	> 0.1	0.960	1.680	0.637	0.636	0.574	0.008	0.010
Botswana Stock Exchange DC Index	< 0.001	> 0.1	0.952	1.079	0.709	0.612	0.551	0.005	0.005
Australia									
ASX All Ordinaries Index	< 0.001	> 0.1	0.899	2.070	0.508	0.517	0.503	0.010	0.011

Source: Author.

Table 5: Summary of results for bank stocks

	p-value					Hit-rate		RMSE	
Name	ADF	KPSS	ApEn	SampEn	Hurst	ARMA	NN	ARMA	NN
Western Europe									
Deutsche Bank AG	< 0.001	> 0.1	0.896	2.015	0.492	0.504	0.509	0.026	0.030
Erste Group Bank AG	< 0.001	> 0.1	0.907	1.938	0.501	0.504	0.513	0.027	0.030
Julius Baer Gruppe AG	< 0.001	> 0.1	0.947	2.025	0.413	0.498	0.479	0.018	0.021
HSBC Holdings PLC	< 0.001	> 0.1	0.913	2.038	0.454	0.493	0.500	0.019	0.022
Banco Santander SA	< 0.001	> 0.1	0.916	2.001	0.484	0.518	0.506	0.023	0.027
BNP Paribas SA	< 0.001	> 0.1	0.907	2.046	0.465	0.508	0.509	0.025	0.029
ING Groep NV	< 0.001	> 0.1	0.920	1.945	0.495	0.512	0.503	0.033	0.037
Nordea Bank AB	< 0.001	> 0.1	0.900	2.074	0.452	0.519	0.500	0.022	0.025
Intesa Sanpaolo SpA	< 0.001	> 0.1	0.905	2.020	0.520	0.517	0.503	0.026	0.030
Central & Eastern Europe									
Powszechna Kasa Oszczednosci Bank Polski SA	< 0.001	> 0.1	0.904	2.138	0.469	0.525	0.508	0.021	0.024
OTP Bank Nyrt	< 0.001	> 0.1	0.902	1.996	0.516	0.505	0.505	0.026	0.030
Komercni Banka a.s.	< 0.001	> 0.1	0.905	1.934	0.444	0.495	0.484	0.021	0.024
Sberbank Rossii PAO	< 0.001	> 0.1	0.934	1.926	0.537	0.518	0.496	0.032	0.039
North America									
JPMorgan Chase & Co	< 0.001	> 0.1	0.913	2.037	0.424	0.511	0.503	0.027	0.030
Royal Bank of Canada	< 0.001	> 0.1	0.930	2.005	0.455	0.520	0.508	0.015	0.017
Credicorp Ltd	< 0.001	> 0.1	0.921	1.889	0.535	0.533	0.505	0.020	0.023
Central & South America									
Grupo Financiero Banorte SAB de CV	< 0.001	> 0.1	0.913	1.935	0.458	0.515	0.515	0.024	0.027
Mercantil Servicios Financieros CA	< 0.001	0.012	0.857	0.869	0.614	0.648	0.668	0.027	0.029
Itau Unibanco Holding SA	< 0.001	> 0.1	0.878	2.056	0.422	0.517	0.516	0.025	0.029
Banco de Credito del Peru	< 0.001	> 0.1	0.768	0.728	0.526	0.625	0.765	0.023	0.026
Banco de Chile	< 0.001	> 0.1	0.914	1.646	0.435	0.512	0.531	0.021	0.022
Bancolombia SA	< 0.001	> 0.1	0.911	1.701	0.541	0.528	0.519	0.022	0.026
Banco Macro SA	< 0.001	0.021	0.912	1.916	0.559	0.541	0.502	0.033	0.038
Middle East									
Qatar National Bank SAQ	< 0.001	> 0.1	0.924	1.591	0.492	0.518	0.527	0.019	0.021
Al Rajhi Banking & Investment Corporation SJSC	< 0.001	> 0.1	0.909	1.681	0.554	0.511	0.502	0.019	0.021
First Gulf Bank PJSC	< 0.001	> 0.1	0.914	1.561	0.561	0.550	0.541	0.023	0.026
National Bank of Kuwait SAKP	< 0.001	> 0.1	0.969	1.078	0.454	0.579	0.632	0.016	0.019
Turkiye Garanti Bankasi AS	< 0.001	> 0.1	0.895	1.853	0.479	0.536	0.528	0.037	0.044
Arab Bank PLC	< 0.001	> 0.1	0.795	1.154	0.535	0.557	0.625	0.038	0.039
Commercial International Bank Egypt SAE	< 0.001	> 0.1	0.907	1.806	0.502	0.532	0.508	0.021	0.024
Asia									
Industrial and Commercial Bank of China Ltd	< 0.001	> 0.1	0.879	2.015	0.510	0.498	0.480	0.018	0.021
Mitsubishi UFJ Financial Group Inc	< 0.001	> 0.1	0.915	2.028	0.505	0.502	0.503	0.024	0.028
HDFC Bank Ltd	< 0.001	> 0.1	0.922	1.994	0.473	0.505	0.501	0.023	0.027
BOC Hong Kong Holdings Ltd	< 0.001	> 0.1	0.926	1.932	0.511	0.510	0.508	0.019	0.021
Bank Central Asia Tbk PT	< 0.001	> 0.1	0.879	1.929	0.438	0.544	0.517	0.022	0.026
Malayan Banking Bhd	< 0.001	> 0.1	0.964	1.675	0.529	0.531	0.502	0.018	0.022
Siam Commercial Bank PCL	< 0.001	> 0.1	0.860	2.008	0.521	0.508	0.506	0.028	0.033
BDO Unibank Inc	< 0.001	> 0.1	0.970	1.669	0.488	0.543	0.531	0.021	0.023
Halyk Bank AO	< 0.001	> 0.1	0.939	1.706	0.398	0.538	0.519	0.039	0.057
Islami Bank Bangladesh Ltd	< 0.001	> 0.1	0.848	1.477	0.451	0.496	0.509	0.057	0.076
Africa									
FirstRand Ltd	< 0.001	> 0.1	0.892	2.041	0.444	0.518	0.501	0.022	0.025
Attijariwafa Bank SA	< 0.001	> 0.1	0.893	1.429	0.500	0.543	0.536	0.014	0.016
Guaranty Trust Bank PLC	< 0.001	> 0.1	0.862	1.207	0.392	0.543	0.571	0.056	0.058
MCB Group Ltd	< 0.001	> 0.1	0.921	1.188	0.563	0.574	0.624	0.012	0.013
Equity Group Holdings Ltd	< 0.001	> 0.1	0.923	1.384	0.451	0.565	0.564	0.022	0.032
Societe Generale de Banques en Cote d'Ivoire SA	< 0.001	> 0.1	0.818	0.798	0.457	0.567	0.616	0.032	0.036
Banque International Arabe de Tunisie SA	< 0.001	> 0.1	0.926	1.219	0.546	0.527	0.529	0.012	0.014
First National Bank of Botswana Ltd	< 0.001	> 0.1	0.613	0.427	0.412	0.547	0.820	0.119	0.078
National Microfinance Bank PLC	< 0.001	0.097	0.624	0.437	0.645	0.600	0.808	0.014	0.015

Source: Author.

D Other uses of entropy in Finance

Optimal portfolio diversification

In *optimal portfolio diversification* problem widely used methods such as Markowitz's mean-variance optimization tend to, among other problems, optimize portfolios by heavily concentrating on few assets which is in contradiction of a diversification purpose itself. Bera & Park (2008) proposed maximising the *Kullback-Leibler information criteria* as the objective function:

$$KLIC(p, q) = \sum_{i=1}^{N} p_i \ln \frac{p_i}{q_i}.$$

The Kullback-Leibler information criteria is defined as a "distance" between two probabilistic distributions – portfolio weights in case of portfolio diversification. If the distribution q is set as a uniform distribution, KLIC is equal to Shannon entropy and maximising the objective function drives portfolio weights towards the uniform distribution, solving the issue with over-inclination to few assets.

Markowitz's M-V portfolio selection also often leads to negative values for some asset weights, which implies often forbidden short-selling. This issue is also solved as asset weight is essentially a probability in entropy optimization, therefore it is always positive. Also a portfolio selection model, which adds an entropy measure to the mean-variance-skewness model has been investigated by several authors (e.g. see Usta & Kantar (2011)). Dionisio et al. (2006) used the normal entropy (estimated entropy of a normal distribution) to show that the uncertainty of portfolio tends to fall with a higher number of assets in that portfolio. They advocated the usage of entropy against variance as a measure of uncertainty because entropy takes into account the higher-order moments of the empirical probability distributions (variance uses only the second moment); therefore, it is a more general measure of uncertainty than variance.

Flow of information

Transfer entropy was proposed as a non-parametric way to quantify the flow of information between two time series revealing all types of temporal correlations (Schreiber, 2000). Marschinski & Kantz (2002) introduced a modified estimator called effective transfer entropy, which behaves better in smaller samples. Authors were able to confirm a significant information flow between the Dow Jones and DAX indices on the log-returns of one-minute index values, which can be explained by a number of agents trading on both markets simultaneously.

Cross-sample entropy was proposed to analyse the degree of asynchrony between two time series, with larger values corresponding to higher asynchrony. This type of entropy measure was originally used to analyse the similarity of two cardiovascular data sets (Richman & Moorman, 2000). Shi & Shang (2013) used it alongside the aforementioned transfer entropy to analyse correlations and information flows between the return time series of American and Chinese indices before and after the Asian currency crisis.

Option pricing

Entropy has found its use also in option pricing. If an expectation pricing model was set to constrain the maximum entropy distribution, authors were able to extract a unique asset probability distribution, which was able to fit the data accurately (Buchen & Kelly, 1996). In other research, Gulko (1997) introduced the Entropy Pricing Theory, which provided similar results as the Black-Scholes formula and the Sharpe-Lintner capital asset pricing model.

Incremental entropy was used to measure the speed of capital increment (Ou, 2005) and three types of entropy – Shannon, *Rényi* and *Tsallis* were chosen to investigate the long memory and volatility clustering of stock indices (Bentes et al., 2008). For a great review of other applications of entropy in finance, please see Zhou et al. (2013).