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VALUATION TECHNIQUES OF LIFE INSURANCE LIABILITIES

Abstract of Doctoral Thesis

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Branch m5 - Econometrics and Operations Research

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Univerzita Karlova v Praze,
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METODY HODNOCENÍ ZÁVAZKŮ ŽIVOTNÍHO POJIŠTĚNÍ

Autoreferát disertační práce

Mgr. Martin Janeček

Školitel:
Prof. RNDr. Tomáš Cipra, DrSc.
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Obor m5 - Ekonometrie a operační výzkum

Disertační práce byla vypracována na základě výsledků získaných během doktorského studia, které uchazeč absolvoval na Katedře pravděpodobnosti a matematické statistiky MFF UK v Praze v letech 1999 – 2006.

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S disertační prací je možno se seznámit na studijním oddělení doktorského studia MFF UK, Ke Karlovu 3, Praha 2.

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Contents:

Introduction

One of the main challenges for actuaries nowadays is a determination of the 'true' value of liabilities of policy contracts (insurance liabilities). Especially how to define its value and what calculation method should be used. Several methods even with considerably different results are currently used in practice.
Main objectives of this work are:

1) To describe the main principles of currently used liability valuation techniques and their mutual relations not only in a way of general descriptions but in a way of mathematical formulas and numerical examples as well, in the first part (sections 3 – 6).
2) To show a real exercise of a kind of a stochastic Fair Value calculation in the second part (section 7). Fair value will be determined based on interest rates simulations in order to cover the value of interest rate options embedded in policy contracts. All other assumptions (mortality, lapses, expenses, etc.) will not be simulated, remain on their deterministic value.

We will focus on life insurance liabilities only, in this text.

1. Valuation of life liabilities – general introduction

1.1. Several liabilities arise from the policy contract for the insurance company. These liabilities are not on the same level in every policy year but can change during a policy period. Insurance company usually receives more premium than it has to pay in a form of claims at the beginning of the policy period but situation changes during the time and at the end of the policy period claims are much higher than the premium income.

1.2. There are several ways how to evaluate liabilities of policy contracts.
In this text, we'll show following techniques:
   a) Statutory valuation approach
   b) Traditional Embedded Value (EV) methodology with short remarks to current development.
   c) Fair Value (FV) methodology – first a deterministic estimation and then a kind of the stochastic approach

1.3. Policies where a claim benefit is paid in case of death or maturity only will be studied in this text. Such policy products are the most common type of life insurance (at least in the Czech insurance market).

2. Notation

We will define several notations, symbols and terms used in the entire document with an identical meaning if not defined explicitly in another way further in the text.
We will also define a *working example* (an example of a specific policy product what results of numerical examples will be shown for) in this section.
Time notation:
n ....policy period
t or r ....policy year

Index *t* (resp. *r*) used within some of the variables expresses that the variable has occurred at the end of the policy year *t* (resp. *r*).
Index *t*-1 (resp. *r*-1) used within some of the variables expresses that the variable has occurred at the beginning of the policy year *t* (resp. *r*).

Other symbols:

x .... age of an insured person at policy inception

q_x .... probability that a person who is alive at the age of *x* will die before (s)he will reach the age of *x*+1

p_x .... probability that a person who is alive at the age of *x* will reach the age *x*+1 ($p_x = 1 - q_x$)

${}_t p_x$ .... probability that a person who is alive at the age of *x* will survive *t* years from the age *x* (till the age of *x*+*t*)

l_t .... expected number of policies in-force at the beginning of the year *t*+1 (i.e. at the end of the year *t*)

d_t .... expected number of deaths during the year *t* (assumed to occur at the end of the year *t*)

w_t .... expected number of lapses during the year *t* (assumed to occur at the end of the year *t*)

tir .... technical interest rate

$v = (1+tir)^{-1}$ .... one-year discount rate related to technical interest rate

rdr_t .... risk discount rate related to the year *t*

rfr_t .... risk free rate related to the year *t*

z_t .... risk free zero spot interest rate for *t* years

f_t .... one-year forward risk free rate at the year *t*

i_t .... investment return (in %) related to the year *t*

I_t .... investment income (in EUR or CZK or other currencies) based on i_t

$mfee$ .... management fee (i.e. the charge from the investment return exceeding the technical interest rate)

SA_t^{death} .... sum assured paid in case of death in the policy year *t;* the index *t* is omitted if the sum assured is constant through all the policy period

$SA_t^{maturity}$ .... sum assured paid in case of maturity in the policy year *t;* the index *t* is omitted if the sum assured is constant through all the policy period or usually it is paid only if the insured person will survive till the end of the policy period

SAR_t .... sum at risk at the end of the year *t*

Res_t^{Stat} .... value of the Statutory reserve at the end of the policy year *t*

Res_t^{EV} .... value of the Embedded Value reserve at the end of the policy year *t*

FV_t .... Fair Value of liabilities at the end of the policy year *t*

P_{t-1} .... premium paid at the beginning of the year *t*

$risk_{t-1}$ ....risk component (risk premium) of the premium paid at the beginning of the year *t*

ΔRes_{t-1}^{Stat} ....saving component of premium paid at the beginning of the year *t*

λ ....policy charge (as a per cent of premium) – usually used in the *working example*

E_{t-1} ....company expenses paid at the beginning of the year *t*

C_{t-1} ....commissions paid to agents at the beginning of the year *t*

CH_{t-1} ....general policy charges deducted from the policy premium at the beginning of the year *t*

$Surr_t$ ....surrender value paid at the end of the year *t*

$SurrCh_t$ ....surrender charge applied when the surrender is paid (at the end of the year *t*)

GP_t ....gross profit at the end of the year *t*

CF_t ....cash flow at the end of the policy year *t*

DF_t ....discounting factor related to the year *t*

Notation of financial variables with or without decrements:
If a financial variable (as e.g., P_{t-1}, $risk_{t-1}$, ΔRes_{t-1}^{Stat}, E_{t-1}, C_{t-1}, CH_{t-1}, $Surr_t$, $SurrCh_t$, Res_t^{Stat}, etc.) is multiplied by a 'decrement' variable (as e.g. q_x, p_x, ${}_tp_x$, l_t, d_t, w_t) then the financial variable itself means its value under the condition that the financial variable is realized.
For instance:

- in a formula $l_{t-1} \cdot P_{t-1}$, the symbol P_{t-1} expresses the premium of a (one) policy if it is in-force at the beginning of the year *t*
- in a formula $w_t \cdot Surr_t$, the symbol $Surr_t$ expresses the surrender value of a (one) policy if it surrenders at the end of the policy year *t*
- etc.

If a financial variable is not multiplied by a 'decrement' variable then the financial variable means the total (already probability weighted) cash flow.
For instance:

- P_{t-1} expresses the total premium paid for all policies in-force at the beginning of the year *t*
- $Surr_t$ expresses the surrender value of all policies which surrender at the end of the policy year *t*
- etc.

Working example:
We will show the numerical results mainly on the following example of an endowment policy regularly (annually) paid with the following parameters.

ENDOWEMENT - REGULARLY PAID	
TECHNICAL INTEREST RATE	4,5%
CHARGES (λ) – from each premium	15%
POLICY DATA	
Inception	1.1.2000
Entry age	30
Sex	Male
Period	10
Sum assured	300 000
Premium (brutto)	27 820
Valuation date	1.1.2004
Other product features	
Surrender limit	since 3rd policy year
Surrender charge	3% of the surrender payment
Profit share rule	90% of investment surplus

3. Statutory valuation approach

3.1. Statutory valuation approach is the traditional methodology of a liability valuation and still is very often required by local European Insurance Acts. Its main principle is that the value of liabilities (a value of the technical reserve, a value of the reserve) at every time in the policy period has to be based on the future cash flow projection using the same statistical data and the interest rate assumptions as were used for the premium calculation.

3.2. The value of liabilities (the value of statutory technical reserve) at the end of the policy year *t* equals to the present value of future outcomes minus the present value of future incomes at valuation date. Present value is defined as a sum of discounted cash flows at the technical interest rate. We assume, in line with a usual approach used in practice, that premium and charges are cash flows at the beginning of the year while claims are paid at the end of the year.

$$Res_t^{Stat} = PV_{outcomes,t} - PV_{incomes,t} ,$$

where

$$PV_{outcomes,t} = \sum_{r=t+1}^{n} \left({}_{r-t-1}p_{x+t} \cdot \left(q_{x+r-1} \cdot SA_r^{death} \cdot v + CH_{r-1} \right) \cdot v^{r-t-1} \right) + {}_{n-t}p_{x+t} \cdot SA^{maturity} \cdot v^{n-t}$$

and

$$PV_{incomes,t} = \sum_{r=t+1}^{n} {}_{r-t-1}p_{x+t} \cdot P_{r-1} \cdot v^{r-t-1}$$

3.3. Often the Insurance Acts define that if the value of the statutory reserve is negative, then zero value should be accounted. Possible negative part of the statutory reserve value (or part of the negative value) is then sometimes reported as the company assets. We will not use this ‘nullification’ in this text.

3.4. *Profit share*
The Statutory valuation approach assumes, as already mentioned above, that the policyholder's money will be invested with returns on the level of the technical interest rate. Often, if the insurance company investment performance is higher than the technical interest rate, part of the difference between the actual investment return and the technical interest rate level (guaranteed return) is given to policyholder as *profit share*.
We will use a special profit share fund (the fund where annual profit share is cumulated) in this text what is one of the most common practice among insurance products. Value of liabilities then equals to the sum of the value calculated by the formula for Res_t^{Stat} plus the recent value of the profit share fund.

3.5. Real example of the statutory reserve calculation of the working example is presented in the thesis with the result of 104 986.

Pros and cons of the Statutory valuation approach:

Pros:

+ It is easy – especially due to a very simple assumptions value of liabilities can be usually calculated using a simple formula.
+ There is a limited subjective decision included – when the premium is set (technical interest rate, charges, life tables are known) then the value of reserve is given by an explicit formula with locked parameters.

Cons:

- The main negative of this approach is that such a way of liability valuation doesn't reflect to the expectable future evolution of the policy (of the company), e.g. expected expenses, investment returns, lapses, mortalities can be much different than assumed in the calculations.

3.6. The reader can find comprehensive information about this traditional valuation approach in many books, e.g. in [2].

4. Embedded Value approach

Main principles

4.1. Comparing to Statutory valuation approach the Embedded Value approach intends to show the 'real state' of the company – the *best estimation* of the future evolution is usually assumed. The *best estimated* level of the assumptions is understood to be their expected value. The fundamental task of the Embedded Value calculation is to project the expected future gross profits for the policies in-force at the valuation date and to calculate their present value (present value of future profits, PVFP).

4.2. Annual gross profit at the end of the year *t* (GP_t) is defined as

$$GP_t = P_{t-1} - C_{t-1} - E_{t-1} - Claims_t + I_t - \left(Res_t^{Stat} - Res_{t-1}^{Stat} \right) - (PSfund_t - PSfund_{t-1}) .$$

4.3. We will then define the value of liabilities under the EV approach (EV reserves, Res_t^{EV}) as the value of the statutory reserve minus the present value of future gross profits (of policies in-force).

$$Res_t^{EV} = Res_t^{Stat} - PVFP_t \,,$$

where

$$PVFP_t = \sum_{r=t+1}^{n} GP_r \cdot DF_{r-t} \,,$$

where

DF_{r-t} … is the discounting factor relevant to the year *r-t* from the valuation date. More discussion about the discounting factor will continue in this section.

4.4. Unlike the Statutory valuation approach, all expected future events and cash flows should be included in the EV calculation (e.g. lapses are included, expenses and investment returns are assumed, etc.).

Assumptions

4.5. All assumptions used for the cash flow projection under the EV methodology are to be on the level of their *best estimates*.
Some of the possible methods sometimes applied in practice when estimating the *best estimation* of assumptions are shown in the thesis. Mainly mortalities, lapses, commissions, expenses and their inflation, premium and/or sum assured increase, investment return and the discount rate are mentioned. Especially the discount rate in connection with risk and uncertainty coverage in the liability value is further discussed.

4.6. The risk and uncertainty that real future cash flows will differ from the best estimation should be, under the Traditional EV methodology, covered in the discount rate (risk discount rate, *rdr*). Risk discount rate is the interest rate determined as a risk free rate (*rfr*, interest rate of risk free assets) and a margin for risk and uncertainty.
Risk discount rate may for example follow the formula:

$$rdr = rfr + \beta \cdot (R_M - rfr)$$

which has its basis in the capital asset pricing model (CAPM).
CAPM dictates the relationship between risk and expected return where
R_M is the expected market return
$R_M - rfr$ is the market risk premium
β is *measure beta* what can be interpreted as the tendency of returns to respond the swings in the market. For insurance companies beta is usually used on the level of about 1 to 1.5. Single discount rate is usually used in this Traditional EV approach.

4.7. Example of EV reserve calculation of our *working example* is presented in the thesis with the result of 100 037.

4.8. Usually the higher *rdr* (the higher margin), the lower PVFP, the higher EV reserve. However, there is an example shown in the thesis where higher risk discount rate is not conservative, hence does not cover any risk and uncertainty, and even worse, may lead to an underestimation of reserves. The thesis also discusses possible solutions.

Alternative approach

4.9. An alternative definition of gross profit is further discussed in the thesis.
The gross profit at each policy year *t* is alternatively defined as:

$$GP_t = expense\ profit$$
$$+mortality\ profit$$
$$+surrender\ profit$$
$$+investment\ profit$$

This definition allows us to come to PVFP according to different segments ('funds'). This gross profit definition is called as *'fund way'* for the purposes of this text.

4.10. In the following, we'll show that results of both gross profit definitions ('accounting' and 'fund' way) should give the same results. We again will use the typical endowment policy (see more description in the thesis).

4.11. Accounting gross profit is:

$$GP_t^{acounting} =$$
$$l_{t-1} \cdot (\lambda \cdot P_{t-1} + q_{x+t-1} \cdot SAR_t \cdot v) \cdot (1+s_t)$$
$$+l_{t-1} \cdot \Delta Res_{t-1}^{Stat} \cdot (1+i_t)$$
$$-l_{t-1} \cdot E_{t-1} \cdot (1+s_t) - l_{t-1} \cdot C_{t-1} \cdot (1+s_t)$$
$$+l_{t-1} \cdot Res_{t-1}^{Stat} \cdot i_t + l_{t-1} \cdot PSfund_{t-1} \cdot i_t$$
$$-d_t \cdot (Res_t^{Stat} + SAR_t + PSfund_t) - w_t \cdot (Res_t^{Stat} + PSfund_t - SurrCh_t)$$
$$-l_t \cdot Res_t^{Stat} + l_{t-1} \cdot Res_{t-1}^{Stat} - l_t \cdot PSfund_t + l_{t-1} \cdot PSfund_{t-1} =$$

where s_t is the investment return of investing the premium, expenses and commissions – usually assume to be some short term rate or even zero.

It holds:

$$l_t + w_t + d_t = l_{t-1}$$

$$= l_{t-1} \cdot (\lambda \cdot P_{t-1} + q_{x+t-1} \cdot SAR_t \cdot v) \cdot (1+s_t) + l_{t-1} \cdot \Delta Res_{t-1}^{Stat} \cdot (1+i_t)$$
$$-l_{t-1} \cdot E_{t-1} \cdot (1+s_t) - l_{t-1} \cdot C_{t-1} \cdot (1+s_t) + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) \cdot i_t$$
$$-l_{t-1} \cdot (Res_t^{Stat} + PSfund_t)$$
$$-d_t \cdot SAR_t + w_t \cdot SurrCh_t + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) =$$

It holds:

$$Res_t^{Stat} + PSfund_t = (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + i_t - (i_t - tir) \cdot mfee) \text{ if } i_t \geq tir$$

and

$$Res_t^{Stat} + PSfund_t = (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + tir)$$
$$= (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + i_t + (tir - i_t)) \text{, if } i_t < tir .$$

Thus, summarized into one formula:

$$Res_t^{Stat} + PSfund_t = (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + i_t - \min(i_t - tir, (i_t - tir) \cdot mfee))$$

$$= l_{t-1} \cdot (\lambda \cdot P_{t-1} + q_{x+t-1} \cdot SAR_t \cdot v) \cdot (1+s_t) + l_{t-1} \cdot \Delta Res_{t-1}^{Stat} \cdot (1+i_t)$$
$$-l_{t-1} \cdot E_{t-1} \cdot (1+s_t) - l_{t-1} \cdot C_{t-1} \cdot (1+s_t) + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) \cdot i_t$$
$$-l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + i_t - \min(i_t - tir, (i_t - tir) \cdot mfee))$$
$$-d_t \cdot SAR_t + w_t \cdot SurrCh_t + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) =$$

$$
\begin{aligned}
&= l_{t-1}\cdot(\lambda\cdot P_{t-1}+q_{x+t-1}\cdot SAR_t\cdot v)\cdot(1+s_t)\\
&-l_{t-1}\cdot E_{t-1}\cdot(1+s_t)-l_{t-1}\cdot C_{t-1}\cdot(1+s_t)\\
&+l_{t-1}\cdot(Res_{t-1}^{Stat}+PSfund_{t-1}+\Delta Res_{t-1}^{Stat})\cdot(i_t-(i_t-\min(i_t-tir,(i_t-tir)\cdot mfee)))\\
&-d_t\cdot SAR_t+w_t\cdot SurrCh_t=
\end{aligned}
$$

$$
\begin{aligned}
&= l_{t-1}\cdot(\lambda\cdot P_{t-1}-E_{t-1}-C_{t-1})\cdot(1+s_t)\text{.......... row 1}\\
&+l_{t-1}\cdot q_{x+t-1}\cdot SAR_t\cdot v\cdot(1+s_t)-d_t\cdot SAR_t\text{.......... row 2}\\
&+w_t\cdot SurrCh_t\text{.......... row 3}\\
&+l_{t-1}\cdot(Res_{t-1}^{Stat}+PSfund_{t-1}+\Delta Res_{t-1}^{Stat})\cdot i_t\text{.......... row 4}\\
&-l_{t-1}\cdot(Res_{t-1}^{Stat}+PSfund_{t-1}+\Delta Res_{t-1}^{Stat})\cdot(i_t-\min(i_t-tir,(i_t-tir)\cdot mfee)))=\text{...... row 5}
\end{aligned}
$$

$$
\begin{aligned}
&= expense\ profit/loss\ (row\ 1)\\
&+mortality\ profit/loss\ (row\ 2)\\
&+surrender\ profit/loss\ (row\ 3)\\
&+investment\ profit/loss\ (row\ 4\ and\ 5)\\
&= GP_t^{fund}
\end{aligned}
$$

Other use of the Embedded Value

4.12. Embedded Value is used not only for the purposes of valuation of liabilities but very often as an estimation of the value of the company as well.
More details and examples of EV calculations are described in the thesis.

Pros and cons of the Traditional Embedded Value methodology

4.13. Positives:

\+ In comparison to Statutory valuation approach the EV methodology is closer to the reality. Expected (best estimation of) assumptions are used.

4.14. Negatives:

- It is a difficult task to get the best estimation of the assumptions. Many subjective decisions usually have to be made.
- Construction of the risk discount rate is subjective and may not cover risk and uncertainty in a satisfactory way.
- No value of embedded options is calculated in the Traditional EV approach. Only deterministic scenarios are usually used in the traditional approach => especially wrong estimation of future investment return (if higher or lower than technical rate) could cause significant changes in the EV reserve.
- Investment risk is considered very purely. For example if more equities are assumed to be in the asset portfolio, the higher investment return is usually assumed, this causes the higher PVFP and the lower EV reserve. However, in the Traditional EV approach, pure or no investment risk margin is usually considered.

Current development

4.15. There are two main improvements of this traditional approach nowadays - ‘European Embedded Value Approach’ presented by a forum of Chief Financial Officers of a number of the largest European life offices (CFO forum) – see [6] and ‘Market Consistent Embedded Value’. Short description about these two approaches is mentioned in the thesis.

5. Fair Value – deterministic estimation

Main principles

5.1. The Fair Value reporting standard is based on a definition of the fair value of assets and liabilities. This definition is as follows:
Fair value is the amount for which an asset could be exchanged and a liability settled between knowledgeable, willing parties in an arm's length transaction.

5.2. Fair value of assets is understood to be its market value (if exists). But, trading with liabilities is nearly none comparing to assets market. Thus market value of liabilities is very rarely available. Alternative approach, what will match the asset and liability valuation methodology has to be used.

5.3. We will show a *deterministic estimation* of the liability FV in this section. We use the term *deterministic estimation* of the fair value since more precious approach what would meet the FV definition better would be the stochastic approach rather than the deterministic. The stochastic valuation may better cover the risk and uncertainty included in the future cash flow projections and the value of options embedded in policy contracts would be priced more properly than using one deterministic scenario.

5.4. The basis for a determination of the fair value of liabilities is cash flows projection of policies in-force at valuation date to the future. We define the annual cash flows at the policy year *r* as follows (index *t* expresses the policy year at the valuation date):

$$CF_r = \left(P_{r-1} - C_{r-1} - E_{r-1}\right) \cdot (1 + rfr_{r-t}) - Claims_r ,$$

where rfr_{r-t} is the (risk free) interest rate related to policy year *r* which is applied to the cash flows at the beginning of the year in order to express their value at the end of the policy year.
We again assume that premiums, commissions and expenses are cash flows at the beginning of the year and claims payments are the cash flows at the end of the year – the same approach as all above (Statutory and EV approaches).

5.5. Like EV approach, all expected future cash flows (expected events) should be included in the projection.

5.6. The present value of such cash flows (denoted as $FV_t^{\det}$) as at the valuation date is studied. Hence,

$$FV_t^{\det} = \sum_{r=t+1}^{n} CF_r \cdot DF_{r-t} ,$$

where DF_{r-t} is the discounting factor relevant to the year *r-t* from the valuation date (see further the paragraphs 5.8 and 5.11).

Assumptions, risk and uncertainty

5.7. The same structure of assumptions as of the EV approach is used. The difference is in how the risk and uncertainty is evaluated.

5.8. Generally, *risk free rate (rfr)* should be used for all interest rate assumptions. Risk free rate uses to be defined as market yield at the valuation date on *risk free assets*.

5.9. Risk free rate is, under the deterministic FV approach, used for:

a) discounting the future cash flows and

b) an annual investment income of the investment of the statutory reserves.

5.10. Risk free rate (rfr_{r-t}) in the paragraph 5.4 expresses a risk free rate at the policy year *r* known at valuation date. Thus that rfr_{r-t} is usually assumed to be the forward rate determined from the risk free rate yield curve as at the valuation date (expressed by the policy year *t*) with maturity *r-t.*

5.11. DF_{r-t} in the paragraph 5.6 is then defined as:

a) $DF_{r-t} = \prod_{j=1}^{r-t}(1+f_j)^{-1}$ (or $DF_{r-t} = \prod_{j=1}^{r-t}(1+rfr_j)^{-1}$ respectively) when using forward rates f_j (or rfr_j respectively)

b) $DF_{r-t} = (1+z_{r-t})^{-(r-t)}$ where z_{r-t} expresses corresponding zero (risk free) spot rates for the maturity *r – t* years from the valuation date.

5.12. It is not an easy task to determine what interest rate should be used for the investment return assumption in the deterministic estimation of FV. We know that it should be risk free and should not depend on the structure of current portfolio of assets (financial instrument).

One should realize that profit sharing rules (based on the investment returns) could be defined differently in different companies and their policy conditions.

Some of the examples of the profit sharing definitions may be:

- Investment return is based on the accounting investment performance. Then other questions have to be answered, e.g.:
  - Does the investment performance depend on the accounting class of assets (i.e. are the assets classified as available for sale, held to maturity, etc. under the accounting standard)?
  - Will only an investment profit which is accounted in P&L accounts be included? Should the investment profit what is accounted through equity (in the balance sheet) be included as well?
  - Will unrealized gains/losses (e.g. changes in market value of assets thanks to market conditions) be taken into account or only realized profits/losses are considered?
  - etc.
- Investment return is based on the fixed prescribed formula in policy conditions – e.g. investment return is the average of 10Y zero rates of government bonds during last 5 years or similar.
- Investment return is on the management discretion – then probably some ‘reasonable policyholder expectation’ will be assumed.

5.13. We will use one-year forward rate as an annual investment income of the investment of the Statutory reserves in this section. This is now a generally accepted risk free estimation of the future investment returns in the Czech insurance market when calculating the FV under the deterministic approach.

We will use the ‘reasonable policyholder expectation’ in the section 7 when we will discuss the stochastic FV approach.

5.14. Unlike EV approach, assumptions are not used on the best estimation level, but are adjusted by *market value margin* (*MVM*). MVM is the adjustment of the best estimation level of the assumption and should cover the risk and uncertainty of its future evolution.

5.15. How to come to a proper MVM is a difficult and individual task. Usually data for deep statistical analysis (mortality experience, lapses, expenses, etc.) are not available or not sufficient, thus the final decision very much depends on the actuarial judgment ('feeling of the risk') of the company actuary (and/or the management) when determining the FV via the deterministic estimation.

5.16. Two examples of MVM is shown in the thesis.

a) Czech Society of Actuaries guideline:

Czech Society of Actuaries in its guideline (see [4]) suggests MVMs to be the adjustments of the best estimates shown in the table 5.1 here (the table 5.3 in the thesis) which cause an increase of the present value of future cash flows.

Risk	***MVM as per cents of best estimate***
Mortality	10%
Lapses	10%
Expenses	10%
Expense inflation	10%

Table 5.1 – MVM suggested by the Czech Society of Actuaries

These parameters are suggested but not required. Final value of the MVM should be the decision of the company, individually according to their risks.

Relevant discussion could for instance be whether MVMs should not be increasing in time (uncertainty in long-term horizon is probably higher than for short-term) or how to group policies when evaluating the present value changes (increase or decrease), etc.

b) Other MVM example – mortality MVM

An example of mortality MVM is shown in the thesis.

5.17. Example of the deterministic FV calculation of our *working example* and the MVM determination as suggested by Czech Society of Actuaries is shown in the thesis with the result of 100 318.

Pros and cons of the deterministic FV approach

5.18. Positives:

+ In comparison to Statutory valuation approach, the deterministic estimation of the fair value of liabilities is closer to the reality. Assumptions used are based on their best estimation and adjusted by MVM. Lapses and all other expected events are included – similar to EV methodology.
+ Comparing to EV approach, slightly less subjectivism is involved when determining interest rate assumptions (risk free rate should be used).

5.19. Negatives:

- Subjectivism in a determination of best estimation of assumptions – the same negatives as for EV methodology.
- Subjectivism in the settling of market value margins. (Similar to subjectivism of risk discount rate determination under the EV approach.)
- No value of options is added when using the deterministic approach.

Sometimes the value of embedded options in the deterministic approach is estimated using adjustments of the interest rate assumptions (e.g. decrease of the discount rate of 25 bps as suggested in [4]).

6. Comparison of the Statutory valuation approach, the Embedded Value and the Fair Value approaches

6.1. A comparison of the valuation methodologies in a schematic way is shown in the table 6.1.

	Statutory approach	***Embedded Value***	***Fair Value***
Methodology	Deterministic.	Deterministic. (EEV and MCEV - interest rate options should be evaluated via stochastic simulations)	Should be fully stochastic. Deterministic simplification is sometimes used.
Financial flows to be discounted	Projected cash flows.	Projected gross profits.	Projected cash flows.
Future events included	Death and maturity only. No lapses, no other events.	All expected.	All expected.
Investment return	Technical interest rate.	Best estimation based on assets held.	Risk free rate.
Discount rate	Technical interest rate.	Risk discount rate. (MCEV market consistent rates)	Risk free rate.
Other assumptions *(mortality, lapses, expenses, etc.)*	'First order' level – same as in the premium calculation.	Best estimation level.	Stochastic approach - assumptions are based on the probability distributions taking into account their correlations. Deterministic approach - best estimation adjusted by market value margin.
Risk and uncertainty	Not explicitly.	In the discount rate (risk discount rate)	In assumptions (stochastic or MVMs are applied)
Value of options	No	No (EEV and MCEV yes, through the stochastic valuation)	Through stochastic valuations. (sometimes adjustments of investment returns and/or discount rates is used in the deterministic approach)
Value of liabilities	Direct result from a formula.	Indirectly as the value of statutory reserves minus the present value of gross profits.	Direct result as the expected value of present value of projected cash flows under stochastic (deterministic) scenarios.

Table 6.1 – Schematic comparison of liability valuation methods

6.2. It is further shown in the thesis, in a way of comparison of mathematical formulas for deterministic FV and EV calculation, that using EV formulas with FV assumptions gives the same results of reserve value as if FV formulas are used. And this theoretical result is then shown in the numerical example. Both approaches give the same result of the value of liabilities (for our *working example*) on the level of 100 318.

7. Stochastic Fair Value approach

In this section, we will make an effort to calculate the fair value of life liabilities based on stochastic simulations of future interest rates in order to include the value of interest rates options embedded in policy contracts.

Methodology

General formulas

7.1. We understand the fair value of liabilities (FV) to be the expected value of the present values of stochastic simulations of future cash flows (cash flows are defined in the same way as in the paragraph 5.4).
Hence the FV as at the end of the year *t* is expressed as:

$$FV_t = E\{PV_t \mid \mathcal{F}_t\}$$

where

E ..... is the expected value in the risk neutral world

PV_t ... is the present value of future generated cash flows.

$\mathcal{F}_t$ ..... is the filtration expressing the situation in and before the year *t*

And, similarly to notation in the section 5 it is:

$$PV_t = \sum_{r=t+1}^{n} CF_r \cdot DF_{r-t}$$

where

$$CF_r = \left(P_{r-1} - C_{r-1} - E_{r-1}\right) \cdot (1 + rfr_{r-t}) - Claims_r$$

and

$$DF_{r-t} = \prod_{j=1}^{r-t} \frac{1}{1 + rfr_j}.$$

Risk and uncertainty

7.2. The FV should cover risks and uncertainties related to future cash flow evolution to a certain level. Several ways how to cover risks and uncertainties in the calculations are possible to use. E.g.:

a) Adjusting the assumptions used (e.g. using market value margin (MVM) to adjust the best estimated level of assumptions). The discount rate is then assumed at the risk free rate level.

b) Using best estimates of assumptions and adjusting the discount rates (similar to EV approach).

c) Mix of a) and b).

We will use the approach a) in the rest of this text.

It especially means that assumptions like mortality, lapses, expenses, etc. will not be simulated stochastically and remain on the same level as was used in the deterministic calculations before (i.e. best estimate + MVM). MVM will be assumed to be on the level as suggested in [4].

We therefore assume that MVM sufficiently cover the risk and uncertainty of these parameters for the purposes of our work.

Interest rate options

7.3. In order to price the interest rate options (technical interest rate guarantee and a profit sharing) usually embedded in policy contracts we will:

- simulate 10000 sets of interest rates.

Two types of future interest rates were generated:

1. the first interest rates will be assumed to be future annual investment returns (we have chosen 1Y and 5Y zero rates) and
2. the next one will be used for discounting (future 1Y rates).

- Then we will calculate the present value of future cash flows (assuming simulated investment return and using simulated discount rates) for each of the set of interest rates.
- FV will then be determined as the expected value (mean) of the results of such present values (as defined in 7.1).

7.4. In order to be able to run our stochastic fair value calculations we will need:

- interest rate scenarios (used as investment returns and discount rates)
- description of the product type what will be modeled
- specify the policy contracts to be modeled (modelpoints)
- set assumptions
- build liability modeling tool and the cash flow model

The thesis describes all these steps in details. We will make a hi-level abstract of this process with a focus to interest rate scenario generator in this text.

Interest rates scenarios

Short term interest rate

7.5. Basic definitions and related formulas:

The simplest interest rate contract is to pay some money today in exchange to receive a different amount (usually larger) later. This moment in the future is called maturity and we will denote it by T. We can regard a promise to receive one dollar at time T as an asset – a risk-free discount bond, which can be evaluated during the life of such contract. We will denote its price at time t by $P(t,T)$. It is clear from the definition that $P(T,T) = 1$. Suppose that the continuously compounded interest rates are constant at rate $R(t,T)$.

Then we have

$$P(t,T) = \exp\{-R(t,T)\cdot(T-t)\}, \qquad t \in [0,T]$$

and therefore

$$R(t,T) = -\frac{\log P(t,T)}{T-t}, \qquad t \neq T \,,$$

where log is a natural logarithm function.

$R(t,T)$ can be viewed as an average interest rate offered by the bond until its maturity and is a function of time and maturity. It is useful to derive a single number r_t representing the current rate of interest for $T \to t_+$.

We obtain

$$r_t = \lim_{T\to t+} R(t,T) = \lim_{T\to t+} -\frac{\log P(t,T)}{T-t} = -\frac{\partial}{\partial T}\log P(t,T)$$

We will call r_t the instantaneous rate or the short rate.

The forward yield for period from T_1 to T_2 is then derived in the thesis as:

$$F(t,T_1,T_2) = -\frac{\log P(t,T_2) - \log P(t,T_1)}{T_2 - T_1}, \qquad t < T_1 < T_2 \,.$$

And the forward instantaneous rate at the time T is

$$f(t,T)=\lim_{\Delta t\to 0+}-\frac{\log P(t,T+\Delta t)-\log P(t,T)}{\Delta t}=-\frac{\partial}{\partial T}\log P(T,t).$$

And equivalently

$$P(t,T)=\exp\{-\int_t^T f(t,z)dz\},\qquad t\in[0,T]$$

The reader can find more details e.g. in [1].

7.6. Basic framework of the short term interest rate models:

Let $\{W_t, t\geq 0\}$ be a Wiener process. Then we will call $\{\mathcal{F}_t,\ t\geq 0\}$ a filtration generated by this process (i.e. W_t is measurable with respect to $\mathcal{F}_t$ for all *t*).

One possible way of how to model the term structure of interest rates is to model short rate process $\{r_t\}$ defined above in 7.5. It is assumed that this process follows a stochastic differential equation (SDE), which in general is of the form:

$$dr_t=\theta(t,r_t)dt+\sigma(t,r_t)dW_t,\qquad t\in[0,T].$$

This equation is the equivalent transcription of the form:

$$r_t=r_0+\int_0^t\theta(u,r_u)du+\int_0^t\sigma(u,r_u)dW_u\ .$$

The stochastic process $\{r_t\}$ fulfilling this equation is the solution of the SDE. The bond prices are given by

$$P(t,T)=E[\ \exp\{-\int_t^T r_u du\}\,|\,\mathcal{F}_t\ ].$$

If the solution $\{r_t\}$ is unique, this process is also a Markov process. Then we can write

$$P(t,T)=E[\ \exp\{-\int_t^T r_u du\}\,|\,r_t\ ],$$

where E denotes expected value in risk-neutral world. And for the pricing at time *t* of the arbitrary derivative security that produces a single payment of *X* at maturity date *T*:

$$V(t)=E[\ X\exp\{-\int_t^T r_u du\}\,|\,r_t\ ],$$

where again E denotes expectation with respect to risk neutral measure.

The reader can find such information e.g. in [21] or [24].

7.7. Scenario generating:

Our task now is to generate interest rate scenarios which should describe possible but uncertain development of future economic environment.

There are many known parametric models for short term interest rate modeling.

The reader can find a good text e.g. in [3], [5], [7], [8], [9], [10], [11], [12], [22], [24] and [26].

After a short discussion (in the thesis) about practical properties of the interest rate model finally single-factor Hull-White (HW) model will be used.

Single-factor Hull White model

7.8. Introduction:

HW model is described by the following stochastic differential equation (SDE):

$$dr_t = (\theta_t - ar_t)dt + \sigma\, dW_t,$$

using the same notation as in the paragraphs above.

If θ_t satisfies the equation:

$$\theta_t = \frac{\partial}{\partial T} f(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1-e^{-2at}),$$

then the model fits the current structure of interest rates with given a and σ.

Authors (Hull & White) also suggest using trinomial trees for calibration purposes (parameters a and σ are to be find) – see [10] and [11].

7.9. Practical issues:

We show only a short description of this model referring to the text in [9], [10] and [11] and make use of very good personal comments of Jan Šrámek on this topic.

Basic formulas and ideas are:

- The basic SDE is:

$$dr_t = (\theta_t - ar_t)dt + \sigma\, dW_t.$$

- Setting:

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a}$$

and

$$A(t,T) = \int_t^T \left(\frac{\sigma^2}{2} B^2(x,T) - \theta(x)B(x,T) \right) dx$$

it is possible to express the bond price at time *t* (with maturity at time *T*) as:

$$P(t,T) = e^{A(t,T)-B(t,T)r_t}$$

and the corresponding investment return (continuously compounded) as:

$$Y(t,T) = \frac{B(t,T)}{T-t} r_t - \frac{A(t,T)}{T-t}.$$

- Using arbitrage-free formula for θ_t:

$$\theta_t = \frac{\partial}{\partial T} f(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1-e^{-2at})$$

it is possible to write the simulation equation for r_t as:

$$r_t = e^{-a(t-s)} r_s + f(0,t) - e^{-a(t-s)} f(0,s) + \frac{\sigma^2}{2a^2}\left(1 - e^{-a(t-s)} + e^{-2at} - e^{-a(t+s)}\right) + \sqrt{\frac{\sigma^2}{2a}(1-e^{-2a(t-s)})}\,\varepsilon$$

and $A(t,T)$ is possible to rewrite into more practical form as:

$$A(t,T) = \log\frac{P(0,T)}{P(0,t)} + B(t,T) f(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at}) B^2(t,T)$$

Now $Y(t,T)$ is possible to express in a practical form (for simulating purposes) as well.

- Finally, especially for *s*=*t*-1, it is

$$r_t = e^{-a} r_{t-1} + f(0,t) - e^{-a} f(0,t-1) + \frac{\sigma^2}{2a^2}\left(1 - e^{-a} + e^{-2at} - e^{-a(2t-1)}\right) + \sqrt{\frac{\sigma^2}{2a}(1 - e^{-2a})}\varepsilon$$

- For calibration, trinomial tree is used, as suggested by Hull and White (see [10] and/or [11]).
  The best fit of the value of floor options evaluated by the HW tree to the market value of these options were used. Market value of such options is calculated using Black-Scholes formula and based on the corresponding market volatilities.

Interest rate scenarios as at December 31, 2005 EUR data using HW model

7.10. Methodology:

- We examined the model for the December 31, 2005 EUR data.
- We have used the EUR interest rate swap par rates (mid) as the market risk free yield curve.
- Calibration was done using trinomial tree (see [10] and/or [11]).
- Best fit (minimizing of the square deviations) of the value of floor options with maturities 2, 3, 4, 5, 7, and 10 years evaluated by the HW tree to the market value of these options were used.

7.11. Inputs:

Inputs to our interest rate model are:

1) EUR interest rate swap par rates (mid):
2) Floor options volatilities (mid)

The determination of the EUR IRS rates and calibration is presented in the thesis.

7.12. Results of the interest rates further used for stochastic valuations:

We will study 2 versions of the investment return assumptions as mentioned in 7.3. Both will express the level of the ‘reasonable policyholder expectation’.

A) investment return is assumed to be at the level of <u>1Y zero rate</u>
B) investment return is be assumed to be at the level of <u>5Y zero rate</u>

Future 1Y zero rates will be used for discounting in both cases (A and B).

We have generated 10 000 sets of the interest rates:

- 1Y rate in the case A used for investment returns assumption as well as for discounting
- 5Y zero rates assumed to be investment returns and 1Y rate used for discounting in the case B.

One set of interest rates is the set of investment returns and discount rates as at the end of the years 2005, 2006, …, 2032 based on the same r_t trajectory.

Of course, in the case A, the investment returns are equal to the discount rates (both are 1Y rates). In the case B, the two rates are generally different.

Results of the generated rates are shown in the thesis.

Product modeled

7.13. A typical endowment policy has been modeled. Details of the product are included in the thesis.

Modelpoints

7.14. We will run our calculations on the following modelpoints (policies):

Valuation date: 31.12.2005
Policy type: ....Typical endowment
Age at policy inception: ..30 years
Sex: ....Male
Sum assured: 100 000
Policy period: ....30 years
Payment frequency: ....Regular (annually) and single
Policy inceptions:

1981=> 5 year till the end of the policy period
1986 => 10 year till the end of the policy period
1991 => 15 year till the end of the policy period
1996 => 20 year till the end of the policy period
2001 => 25 year till the end of the policy period

Technical interest rates: 2%, 3%, 4%, 5% and 6%.
Hence, we obtained 50 endowment polices to be tested.

Assumptions

7.15. Assumptions used in our calculations are shown in the table 7.9 in the thesis.

Liability model

7.16. The tool:
We have built the annual cash flow model in the actuarial system *Sophas* prepared by the JL Soft company (see [20]).

7.17. Formulas – details of the formula used in the liability model are described in the thesis.

Results of the stochastic liability fair value calculation

7.18. Results of the stochastic fair value for above described product are shown in the thesis in the table 7.10. One can notice that:

- FV for regularly paid policies is lower than for single, what is natural, since future premium income is expected when a policy premium is paid regularly and no future premium will be paid in case of a single premium payment.
- FV is higher when 5Y rates are assumed comparing to 1Y rate assumption. This is due to the fact that 5Y rates are higher then 1Y rates in our case (EUR rates as at December 31, 2005) and hence more profit share is paid in the 5Y rate case, thus higher outflow occurs what makes higher liabilities for the insurance company. The level of discount rates is the same in both cases – future 1Y rates are used for discounting for both investment returns versions (1Y and 5Y rate).

7.19. Histograms of the distribution of the present value of future cash flows (i.e. result of one scenario) for several modelpoints are shown in the thesis (graph 7.8 in the thesis). We have chosen regularly paid policy with the inception year equal to 1996 and the technical interest rate on the levels 2%, 4% and 6% for both versions of the level of investment returns (1Y and 5Y). One can see that:

- figures (present values on the x-axis) in ‘5Y rate cases’ are slightly higher than in ‘1Y rates’; this is caused by the same fact as discussed in 7.18 (first item).
- the higher technical interest rate the higher spread between the minimum and the maximum of the present values – histograms are ‘wider’ (i.e. more different present values are generates for higher technical rates).

7.20. We add a comparison of the stochastic FV results with 2 deterministic ones – results are shown in the table 7.3 in the thesis. First deterministic approach assumes investment returns as well as discount rates to be on the level of 1Y forward rate determined from the current risk free yield curve. The other deterministic approach assumes investment returns to be on the level of 1Y forward rate while discount rates to be on the level of 1Y forward rate minus 0.25% (25 bps) as suggested in [4] in order to cover the value of interest rate options.

Remarks:

- We show this comparison because such deterministic approaches are widely used in the Czech insurance market.
- However, we do not believe that the decrease of the discount rate is relevant to a valuation of interest rate options embedded in policy contracts since manipulation with the discount rate affects not only cash flows depending on interest rates but the cash flows which do not depend on the interest rates as well – e.g. expense outflow, commissions paid, premium income, etc. – what may destroy the results.

  Maybe more relevant solution could be adjusting not the discount rate but the investment return assumption or using adjusted discount rates only to the cash flows which depend on the interest rates (profit sharing paid). What should be the correct adjustment and what level of such adjustments should be used remains an open issued at this moment.

One can see that the change of the approaches could make even more than 10% underestimation of the liability value.

Alternative approach of calculation formulas, computation tool

7.21. An alternative approach (formulas and the real procedure) which allows, under certain circumstances, using common tools (like MS Excel) for stochastic calculation is shown in the thesis.

Open issues and space for further improvements

7.22. We mention here several open issues of such calculations which requires further analysis and improvements.

- Interest rate model - we have chosen the single-factor HW model. What would be the results if a different model was used?
- Level of the future investment returns - we have used 1Y and 5Y zero rates as the reasonable policyholder expectation. What would be the results if different levels (e.g. 3Y, 7Y, 10Y zero rates.) were used?
- Other assumptions - we have calculated the fair value of life liabilities via interest rate simulations; we did not simulate other assumptions (mortality, lapses, expenses, etc.) and their correlations. What is the effect and what would be the results if the other assumptions (mortality, lapses, expenses,

etc.) were simulated as well? What are correlations among these assumptions?

- Volatility of the interest rates volatilities - it is quite difficult to determine the 'real' market volatilities, since interest rate options (evaluated on the basis of the interest rate volatilities) are not frequent deals and depends significantly on the final agreement between the business parties. How are the FV results sensitive to the market volatilities?
- Estimation of the value of interest rate option via deterministic approach - we have mentioned our doubts about the current suggested deterministic approach of estimation of the interest rate options embedded in policy contracts (in [4]). We have mentioned that probably adjusting investment returns rather than the discount rates would probably be more relevant. But, what is the right adjustment? Some other analyses and many calculations are still required.

8. Conclusions

The intention of this work was to contribute to the current actuarial discussion about the valuation of life liabilities with some summary of current most frequent valuation methodologies.

We have started with the most traditional one (Statutory valuation approach), gone through the more developed (deterministic Embedded Value approach and the deterministic estimation of the Fair Value) to the most recent one – stochastic Fair Value approach via simulating the future interest rates.

We have intended to give a more detailed overview to the liability valuation methodologies not only in a way of a general description but in a way of the specific mathematical formulas and numerical examples as well in order to see their mutual relations and similarities, their positives and negatives.

At the second part of this text, we have showed the real process of the stochastic liability fair value calculation as at December 31, 2005 under the interest rate simulations. We went through all the procedure explaining all important issues and steps and finally presented the results. We also have mentioned that many unexplored issues still remain and large space for further studies is open.

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Content:

Introduction

One of the main challenges for actuaries nowadays is a determination of the 'true' value of liabilities of policy contracts (insurance liabilities). Especially how to define its value and what calculation method should be used.
The value of insurance liabilities is a crucial economic figure for insurance companies since it very often represents the major part of a balance sheet amount. Therefore even relatively small change in the value of liabilities can cause a significant change e.g. in profit and loss account, adequacy of assets backed insurance liabilities, solvency of the company etc.
Several methods even with considerably different results are currently used in practice.
Special issue of the liability valuation is how options embedded in policy contracts (especially interest rate options) should be priced.

We will focus on life insurance liabilities only, in this text.

We will discuss some of the valuation methodologies applied starting with a traditional Statutory valuation approach[1] following to more developed ones, deterministic Embedded Value and deterministic Fair Value concept, concluding with a modern one, stochastic Fair Value approach.

We will give more attention to Embedded Value (EV) and Fair Value (FV) principles than to traditional Statutory valuation approach since the EV and FV methodologies represent the current trend in life liability valuations.

Main objectives

Main objectives of this work are:

1) To describe the main principles of currently used liability valuation techniques and their mutual relations not only in a way of general descriptions but in a way of mathematical formulas and numerical examples as well, in the first part (sections 3 – 6).
2) To show a real exercise of a kind of a stochastic Fair Value calculation in the second part (section 7). Fair value will be determined based on interest rates simulations in order to cover interest rate options embedded in policy contracts. All other assumptions (mortality, lapses, expenses, etc.) will not be simulated, remain on their deterministic value.

[1] This name is used only for the purposes of this text. It has its basis in the fact that still is often required by the Insurance Acts (especially in European countries) for the statutory reporting.

Document structure

This document is structured as follows:

Section 1
We will start with a general introduction to a valuation of life liabilities in the section 1. Simple motivating example will be included.

Section 2
Notations, symbols and some of the terms used throughout the text are set in this section. We will also define a *working example* (an example of a policy product what results of numerical examples will be shown for) in this section.

Section 3
We will then start discussing liability valuation methods using the traditional Statutory valuation approach in the section 3. Its main principles, general formulas, a numerical example and pros and cons will be discussed there.

Section 4
The Embedded Value methodology will be discussed in the section 4. We will concentrate mainly to Traditional Embedded Value principles although remarks regarding the current development will be added at the end of the section.

Section 5
Principles of the deterministic estimation of the fair value of liabilities will be discussed in the section 5.
Assumptions used and risk and uncertainty coverage will be discussed especially and compared to EV methodology.

Section 6
Comparison of the Statutory, the deterministic Embedded Value and the deterministic Fair Value approach will be presented in the section 6. We will focus especially to a comparison between the deterministic Embedded Value and the deterministic Fair Value methodology in a way of mathematical formulas as well as numerical example.

Section 7
In the section 7, we will make an effort to calculate the fair value of life liabilities under the stochastic simulations of future interest rates.
We will refer to definitions and formulas of short term interest rates and their models first. Single-factor Hull-White model will then be used for the interest rates simulations. Results of the simulations based on the EUR data as at December 31, 2005 will be presented.
Description of the specific insurance product modeled, modelpoints and assumptions used, liability computation tool and related formulas of the calculation will be introduced.
Results of the stochastic FV calculation will be then presented and discussed.
An alternative approach which allows using common calculation tools for stochastic Fair Value calculation will shortly be shown here.

Open issues which should be further discussed and solved will be mentioned at the end of this section.

Section 8
Conclusions will form the last section 8.

Literature
List of related literature is presented at the end of this document.

1. Valuation of life liabilities – general introduction

General introduction to the valuation of life liabilities is presented in this section. Illustrative (motivating) example will be shown as well.

1.1. Life insurance companies make contracts with their clients (policyholders). In such contracts, the company promises that in case of a claim of insured person (that can be different than policyholder) such as death during agreed period of time (policy period), survival till the end of the policy period, critical or other illness, disability etc., it will pay agreed benefit to beneficiary (usually to insured person or to his (her) relatives).
The policyholder is obliged to pay a premium to the company (either single, i.e. once at policy inception or regular, i.e. usually annually, semi- annually, quarterly, monthly).

1.2. Several liabilities arise from the policy contract for the insurance company.
The company is obliged:

- to pay a benefit in case of a claim,
- to cover all expenses related to the policy contract such as agent commissions, administrative costs (e.g. salaries to the company staff, to externals, etc.),
- to invest and valorize the policyholder's money (often the insurance company promises valorization of the policyholder's money at some agreed minimum interest rate – technical interest rate),
- to accept the premium paid

1.3. These liabilities are not on the same level in every policy year but can change during a policy period.

Illustrative example:
Let's assume a group of 10 000 of identical policies (same age at entry = 30, sex = male, annual premium = 1 370 CZK, sum assured = 100 000 CZK, policy period = 40 years) which cover death during the policy period (i.e. the sum assured is paid when the insured person dies).
For such a group of policies different number of claims (deaths) is to expect in each year, this number will probably increase as clients (insured persons) are getting older (the older person the higher probability of death).

Table 1.1 shows:

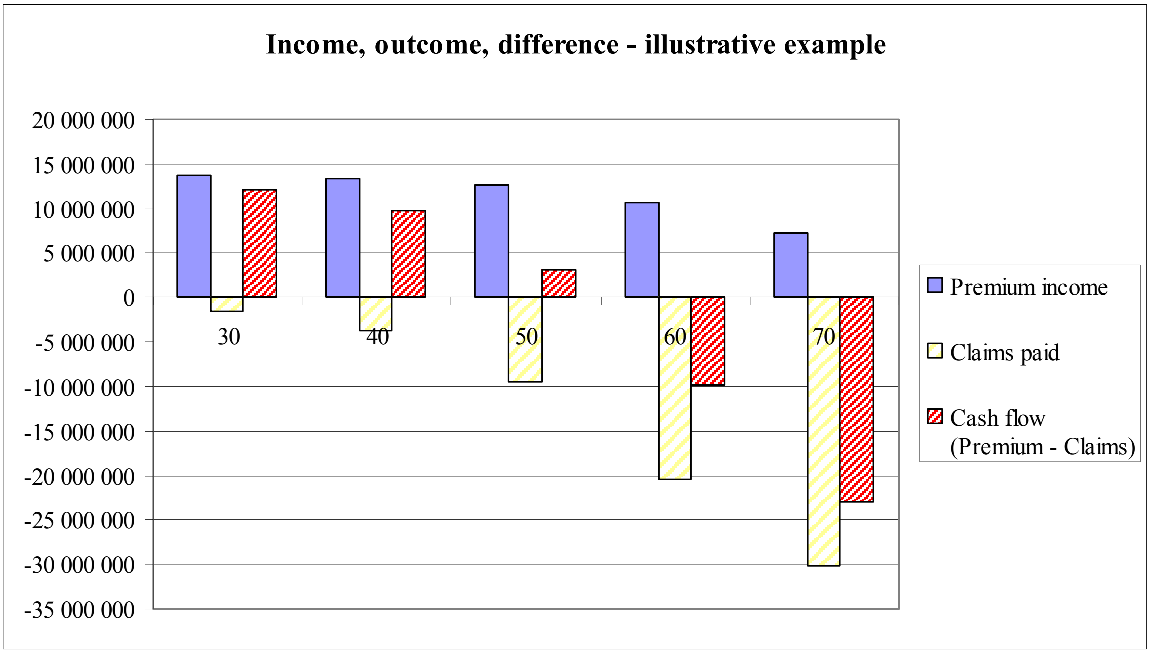
- number of in-force policies (i.e. insured person is alive at that time)
- total premium income
- total number of deaths expected
- total claims paid and
- total cash flow being the difference between the premium income and claims outcome

for our illustrative example.

Age	*Number of lives*	*Premium income*	*Number of deaths*	*Claims paid*	*Cash flow (Premium - Claims)*
30	10 000	13 700 000	16	-1 613 880	12 086 120
40	9 776	13 393 120	37	-3 700 000	9 693 120
50	9 185	12 583 450	94	-9 400 000	3 183 450
60	7 816	10 707 920	205	-20 500 000	-9 792 080
70	5 260	7 206 200	302	-30 200 000	-22 993 800

Table 1.1 – Illustrative example

Graph 1.1 shows premium incomes, claims paid and their differences (i.e. the cash flow) of the illustrative example (figures from the table 1.1) in a graphical form.



Graph 1.1 – Income, outcome and the difference for illustrative example

One can see that the insurance company receives more premium than it has to pay in a form of claims at the beginning of the policy period but situation changes during the time and at the end of the policy period claims are much higher than the premium income.

This is why the insurance company must not be only focused on the current year situation but must take into account all the future evolution of its cash flows (outcomes and incomes). If necessary it has to accumulate a capital (create technical reserves) already from the inception of policy to be able to pay claims in the remaining time when the policy is in-force.

1.4. There are several ways how to evaluate liabilities of policy contracts.
In this text, we'll show following techniques:

a) Statutory valuation approach
b) Traditional Embedded Value (EV) methodology with short remarks to current development to:
    - European Embedded Value and
    - Market Consistent Embedded Value
c) Fair Value (FV) methodology – first a deterministic estimation and then a kind of the stochastic approach

1.5. Policies where a claim benefit is paid in case of death or maturity only will be studied in this text. Such policy products are the most common type of life insurance (at least in the Czech insurance market). Extension to other claim benefits (critical illness, disability, etc.) is possible using similar (often the same) principles as discussed in this text.

2. Notation

We will define several notations, symbols and terms used in the entire document with an identical meaning if not defined explicitly in another way further in the text.
We will also define a *working example* (an example of a specific policy product what results of numerical examples will be shown for) in this section.

Time notation:

n .... policy period
t or r .... policy year

Index *t* (resp. *r*) used within some of the variables expresses that the variable has occurred at the end of the policy year *t* (resp. *r*).
Index *t*-1 (resp. *r*-1) used within some of the variables expresses that the variable has occurred at the beginning of the policy year *t* (resp. *r*).

Other symbols:

x .... age of an insured person at policy inception
q_x .... probability that a person who is alive at the age of *x* will die before (s)he will reach the age of *x*+1
p_x .... probability that a person who is alive at the age of *x* will reach the age *x*+1 ($p_x = 1 - q_x$)
${}_tp_x$ .... probability that a person who is alive at the age of *x* will survive *t* years from the age *x* (till the age of *x*+*t*)
l_t .... expected number of policies in-force at the beginning of the year *t*+1 (i.e. at the end of the year *t*)
d_t .... expected number of deaths during the year *t* (assumed to occur at the end of the year *t*)
w_t .... expected number of lapses during the year *t* (assumed to occur at the end of the year *t*)
tir .... technical interest rate
$v = (1 + tir)^{-1}$ .... one-year discount rate related to technical interest rate
rdr_t .... risk discount rate related to the year *t*
rfr_t .... risk free rate related to the year *t*
z_t .... risk free zero spot interest rate for *t* years
f_t .... one-year forward risk free rate at the year *t*

i_t .... investment return (in %) related to the year *t*

I_t .... investment income (in EUR or CZK or other currencies) based on i_t

$mfee$ .... management fee (i.e. the charge from the investment return exceeding the technical interest rate)

SA_t^{death} .... sum assured paid in case of death in the policy year *t;* the index *t* is omitted if the sum assured is constant through all the policy period

$SA_t^{maturity}$ .... sum assured paid in case of maturity in the policy year *t;* the index *t* is omitted if the sum assured is constant through all the policy period or usually it is paid only if the insured person will survive till the end of the policy period

SAR_t .... sum at risk at the end of the year *t*

Res_t^{Stat} .... value of the Statutory reserve at the end of the policy year *t*

Res_t^{EV} .... value of the Embedded Value reserve at the end of the policy year *t*

FV_t .... Fair Value of liabilities at the end of the policy year *t*

P_{t-1} .... premium paid at the beginning of the year *t*

$risk_{t-1}$ .... risk component (risk premium) of the premium paid at the beginning of the year *t*

ΔRes_{t-1}^{Stat} .... saving component of premium paid at the beginning of the year *t*

λ .... policy charge (as a per cent of premium) – usually used in the *working example*

E_{t-1} .... company expenses paid at the beginning of the year *t*

C_{t-1} .... commissions paid to agents at the beginning of the year *t*

CH_{t-1} .... general policy charges deducted from the policy premium at the beginning of the year *t*

$Surr_t$ .... surrender value paid at the end of the year *t*

$SurrCh_t$ .... surrender charge applied when the surrender is paid (at the end of the year *t*)

GP_t .... gross profit at the end of the year *t*

CF_t .... cash flow at the end of the policy year *t*

DF_t .... discounting factor related to the year *t*

Notation of financial variables with or without decrements:

If a financial variable (as e.g., P_{t-1}, $risk_{t-1}$, ΔRes_{t-1}^{Stat}, E_{t-1}, C_{t-1}, CH_{t-1}, $Surr_t$, $SurrCh_t$, Res_t^{Stat}, etc.) <u>is multiplied by a 'decrement' variable</u> (as e.g. q_x, p_x, ${}_tp_x$, l_t, d_t, w_t) then the financial variable itself means its value under the condition that the financial variable is realized.

For instance:

- in a formula $l_{t-1} \cdot P_{t-1}$, the symbol P_{t-1} expresses the premium of a (one) policy if it is in-force at the beginning of the year *t*
- in a formula $w_t \cdot Surr_t$, the symbol $Surr_t$ expresses the surrender value of a (one) policy if it surrenders at the end of the policy year *t*
- etc.

If a financial variable is not multiplied by a 'decrement' variable then the financial variable means the total (already probability weighted) cash flow.

For instance:

- P_{t-1} expresses the total premium paid for all policies in-force at the beginning of the year *t*
- $Surr_t$ expresses the surrender value of all policies which surrender at the end of the policy year *t*
- etc.

Several terms definition:

Several terms used throughout the text is defined here.

Endowment
Endowment is a standard insurance product which covers deaths and maturity. In case of death or maturity the insurance company pays the agreed benefit. Policyholder pays single or regular premium.

Pure endowment
Pure endowment is a pure saving product, where the insurance company pays a benefit only in case of maturity.
Policyholder pays single or regular premium.

Term insurance
Term insurance is an insurance product, where the insurance company pays a benefit only in case of insured person's death during a policy period.
Policyholder pays single or regular premium.

Sum at risk is a difference between the sum assured payable in case of death and the value of netto (statutory) reserve of the policy.

Risk premium is a part of a premium which is calculated to cover the sum at risk paid in case of deaths.

Technical interest rate is the interest rate assumed to be the investment return and the discount rate under the Statutory liability valuation approach.

Working example:

We will show the numerical results mainly on the following example of an endowment policy regularly (annually) paid with the following parameters.

ENDOWEMENT - REGULARLY PAID

TECHNICAL INTEREST RATE	4,5%
CHARGES (λ) – from each premium	15%
POLICY DATA	
Inception	1.1.2000
Entry age	30
Sex	Male
Period	10
Sum assured	300 000
Premium (brutto)	27 820
Valuation date	1.1.2004
Other product features	
Surrender limit	since 3rd policy year
Surrender charge	3% of the surrender payment
Profit share rule	90% of investment surplus

We will call it the *working example* for the purposes of this document.

Such a policy product is a frequent one on the Czech insurance market.
Although in practice the charges are usually related not only to the premium payment but to the sum assured as well. We will use such a simplified approach anyway, since it makes the understanding of the text easier and doesn't affect the generality of the corresponding results and conclusions.

3. Statutory valuation approach

Statutory valuation approach will be described in this section.
Main principles, general formulas, the numerical example and pros and cons will be discussed here.

3.1. Statutory valuation approach is the traditional methodology of a liability valuation and still is very often required in local European Insurance Acts.

3.2. Its main principle is that the value of liabilities (a value of the technical reserve, a value of the reserve) at every time in the policy period has to be based on the future cash flow projection using the same statistical data and the interest rate assumptions as were used for the premium calculation. Sometimes this requirement is referred to as locking-in of assumptions or 'first order' assumptions.
This especially means that:

- insured person is assumed to die exactly according to probabilities set in life tables which are determined by the insurer at the inception of the policy (or even sooner) and used for the premium calculation; these tables are usually more conservative than the expected reality,
- no lapses are usually taken into account,
- investment returns and discount rates are assumed to be on the level of technical interest rate in each year (flat rate) and
- future expenses of the company are assumed to be exactly equal to policy charges applied.

3.3. Using such assumptions policy cash flows for a general endowment policy (generally with different sum assured in case of death at every future year and a different sum assured in case of maturity at the end of the policy period) are as shown in the table 3.1.
We assume, in line with a usual approach used in practice, that premium and charges are cash flows at the beginning of the year while claims are paid at the end of the year.

Policy year	*Outcome*		*Income*
	Claims	***Expenses (equal to policy charges)***	***Premium***
1	$q_x \cdot SA_1^{death}$	CH_1	P_1
2	$p_x \cdot q_{x+1} \cdot SA_2^{death}$	$p_x \cdot CH_2$	$p_x \cdot P_2$
3	$p_x \cdot p_{x+1} \cdot q_{x+2} \cdot SA_3^{death}$	$p_x \cdot p_{x+1} \cdot CH_3$	$p_x \cdot p_{x+1} \cdot P_3$
…	…	…	…
n	$\prod_{j=0}^{n-2} p_{x+j} \cdot q_{x+n-1} \cdot SA_n^{death} + \prod_{j=0}^{n-1} p_{x+j} \cdot SA_n^{maturity}$	$\prod_{j=0}^{n-2} p_{x+j} \cdot CH_n$	$\prod_{j=0}^{n-2} p_{x+j} \cdot P_n$

Table 3.1 – Cash flow formulas – Endowment policy– Statutory valuation

3.4. Premium under this approach is usually calculated so that the present value of outcomes at the beginning of the policy period equals to the present value of incomes at the beginning of the policy period both weighted by corresponding probabilities (equivalency principle).
Present value is defined as a sum of discounted cash flows at the technical interest rate.

3.5. Therefore, for life policies where a benefit is paid in case of death during the policy period or in case of survival till the end of the policy period the formulas are as follows:

Present value of outcomes (at the policy inception, at time 0) is:

$$PV_{outcomes,0} = \sum_{t=1}^{n} \left({}_{t-1}p_x \cdot \left(q_{x+t-1} \cdot SA_t^{death} \cdot v + CH_{t-1} \right) \cdot v^{t-1} \right) + {}_n p_x \cdot SA^{maturity} \cdot v^n ,$$

Present value of incomes is:

$$PV_{incomes,0} = \sum_{t=1}^{n} {}_{t-1}p_x \cdot P_{t-1} \cdot v^{t-1} ,$$

Premium is then derived to fulfill the condition:

$$PV_{incomes,0} = PV_{outcomes,0}$$

3.6. The value of liabilities (the value of statutory technical reserve) at the end of the policy year *t* is then calculated using the same idea.
Its value equals to the present value of future outcomes minus the present value of future incomes at valuation date.
Therefore the value of liabilities under this Statutory valuation approach at the end of the year *t* (Res_t^{Stat}) equals to:

$$Res_t^{Stat} = PV_{outcomes,t} - PV_{incomes,t} ,$$

where

$$PV_{outcomes,t} = \sum_{r=t+1}^{n} \left({}_{r-t-1}p_{x+t} \cdot \left(q_{x+r-1} \cdot SA_r^{death} \cdot v + CH_{r-1} \right) \cdot v^{r-t-1} \right) + {}_{n-t}p_{x+t} \cdot SA^{maturity} \cdot v^{n-t}$$

and

$$PV_{incomes,t} = \sum_{r=t+1}^{n} {}_{r-t-1}p_{x+t} \cdot P_{r-1} \cdot v^{r-t-1}$$

3.7. Often the Insurance Acts define that if the value of the statutory reserve (calculated by the formulas shown in the paragraph 3.6) is negative, then zero value should be accounted. Possible negative part of the statutory reserve value (or part of the negative value) is then sometimes reported as the company assets. We will not use this 'nullification' in this text.

3.8. *Profit share*

The Statutory valuation approach assumes, as already mentioned above, that the policyholder's money will be invested with returns on the level of the technical interest rate. Often, if the insurance company investment performance is higher than the technical interest rate, part of the difference between the actual investment return and the technical interest rate level (guaranteed return) is given to policyholder as *profit share*.

Profit share could have many forms, e.g.:

- real cash payment every year,
- increase in the sum assured,
- an increase of the policyholder's separate account,
- etc.

We will use the form of a special profit share fund in this text what is one of the most common practice among insurance products. Value of liabilities then equals to the sum of the value calculated by the formula for Res_t^{Stat} from the paragraph 3.6 plus the recent value of the profit share fund.

3.9. *Example*

Numerical results of the value of liabilities under the Statutory valuation approach (Statutory reserve) related to our *working example* are shown in tables 3.2 and 3.3.
(BoY = beginning of the year, EoY = end of the year)

Cash flow						
Year	*Premium (BoY)*	*Expenses (BoY)*	*Claims (EoY)*	*TOTAL INCOMES (EoM)*	*TOTAL OUTCOMES (EoM)*	*TOTAL CASH FLOW (EoY)*
2004	27 820	4 173	606	**29 072**	**4 967**	**25 468**
2005	27 764	4 165	669	**29 014**	**5 021**	**25 352**
2006	27 703	4 155	736	**28 949**	**5 078**	**25 228**
2007	27 635	4 145	812	**28 878**	**5 143**	**25 088**
2008	27 560	4 134	897	**28 800**	**5 217**	**24 933**
2009	27 477	4 122	297 581	**28 713**	**301 888**	**-271 829**

Table 3.2 – Statutory reserve of the working example – cash flow

Reserve	
PV outcomes	254 119
PV incomes	149 133
Statutory reserve	**104 986**

Table 3.3 – Statutory reserve of the working example – result

Pros and cons of the Statutory valuation approach:

3.10. We now summarize several pros and cons of the Statutory valuation approach.

Pros:

+ It is easy – especially due to a very simple assumptions reserve can be usually calculated using a simple formula; no robust cash flow models are required.
+ There is a limited subjective decision included – when the premium is set (technical interest rate, charges, life tables are known) then the value of reserve is given by an explicit formula with locked parameters.

Cons:

- The main negative of this approach is that such a way of liability valuation doesn't reflect to the expectable future evolution of the policy (of the company), e.g. expected expenses, investment returns, lapses, mortalities can be much different than assumed in the calculations.

  If for instance current market interest rates are lower than the technical interest rate, actual company expenses are higher than the policy charges, other loss experience (mortality, etc.) are worse than used for the premium calculation, then the value of liabilities determined by this Statutory valuation approach could be underestimated significantly.

3.11. The reader can find comprehensive information about this traditional valuation approach in many books, e.g. in [2].

4. Embedded Value approach

Traditional Embedded Value (EV) approach is mainly described in this section. Short remarks about the current development of the EV methodology are added at the end of this section.

- We'll explain the **main principles** of this methodology first.
- **Assumptions** used for EV calculations will be discussed next. Several practical issues will be included.
- Covering the **risk and uncertainty** will then be discussed especially.
- We will further show an **alternative approach** of the EV calculation (named 'fund' way for the purposes of this text) and compare it in a form of mathematical formulas as well as numerical examples with the typically used 'accounting' definition.
- Since EV calculation has its very important applications in the **analysis of the value of the company**, we will add remarks of this usage (headed 'Other use of the EV') as well.
- **Pros and cons** of the Traditional EV approach will continue.
- Short remarks about the **current development** of the EV methodology (especially European Embedded Value and Market Consistent Embedded Value principles) are added at the end of this section.

Main principles

4.1. Comparing to Statutory valuation approach the Embedded Value approach intends to show the 'real state' of the company – the *best estimation* (for the definition see further in the paragraph 4.8) of the future evolution is usually assumed.

4.2. The fundamental task of the Embedded Value calculation is to project the expected future gross profits for the policies in-force at the valuation date and to calculate their present value (present value of future profits, PVFP).

4.3. Annual gross profit at the end of the year *t* (GP_t) consists of:

- premium paid by the policyholder to the insurance company (P_{t-1})
- commissions paid by the company to agents (C_{t-1})
- expenses what need to be covered by the insurance company (E_{t-1})
- claims (deaths, maturities, surrenders, etc.) paid ($Claims_t$)
- investment incomes (I_t)
- the change of Statutory reserves $\left(Res_t^{Stat} - Res_{t-1}^{Stat}\right)$ and
- the change of the profit share fund $(PSfund_t - PSfund_{t-1})$.

We again assume premiums, commissions and expenses to be the cash flows occurred at the beginning of the year and claims, investment incomes and change of the Statutory reserve and profit share fund to be the financial flows occurred at the end of the year.
We also assume (same as in the section 3) that annual profit share is cumulated in a separate policyholder’s fund.

In line with the notation defined in the section 2, all financial flows (mentioned in this paragraph 4.3) are already probability weighted, i.e. all of the symbols express the total (probability weighted) expected financial flows.
E.g.:

- P_{t-1} means the premium income of all policies in-force at the beginning of the year *t*,
- C_{t-1} (resp. E_{t-1}) are commissions (resp. expenses) paid for all policies in-force at the beginning of the year *t,*
- $Claims_t$ expresses the claim cash flows which are paid to all policies which have a claim during the year *t,*
- I_t is the investment income obtained by the company at the end of the year *t,*
- *etc.*

Hence it is,

$$GP_t = P_{t-1} - C_{t-1} - E_{t-1} - Claims_t + I_t - \left(Res_t^{Stat} - Res_{t-1}^{Stat}\right) - \left(PSfund_t - PSfund_{t-1}\right).$$

4.4. We will then define the value of liabilities under the EV approach (EV reserves, Res_t^{EV}) as the value of the statutory reserve minus the present value of future gross profits (of policies in-force).

$$Res_t^{EV} = Res_t^{Stat} + PSfund_t - PVFP_t,$$

where

$$PVFP_t = \sum_{r=t+1}^{n} GP_r \cdot DF_{r-t},$$

where
DF_{r-t} … is the discounting factor relevant to the year *r-t* from the valuation date*.*
More discussion about the discounting factor will continue in the paragraph 4.9.8.

4.5. Notice that if the ‘first order’ assumptions, as under the Statutory valuation approach, were used then $GP_t = 0$ for all *t.*
Hence

$$Res_t^{EV} = Res_t^{Stat} \text{ for all } t.$$

Therefore, the present value of future gross profits (PVFP) indicates whether the current statutory reserve is over-or underestimated.

4.6. Unlike the Statutory valuation approach, all expected future events and cash flows should be included in the EV calculation (e.g. lapses are included, expenses and investment returns are assumed, etc.). This complexity is the main reason why the present value of future gross profits (PVFP) and thus the EV reserve is not easy to calculate by a simple formula, but robust cash flow models on per policy basis are usually required.

4.7. We have introduced the main principles and formulas of the EV calculation. We will now discuss several practical issues regarding the assumptions used for the calculations.

Assumptions

4.8. All assumptions used for the cash flow projection under the EV methodology are to be on the level of their *best estimates*. The *best estimated* level of the assumptions is understood to be their expected value.

4.9. Since reliable probabilistic distributions of the underlying assumptions are not always available simplified estimations are usually used instead. We will show some of the possible methods sometimes applied in practice when estimating the *best estimation* of assumptions. We do not pretend the presented processes are the only possible or the only correct ones, rather the opposite, many other ways may be used, but the reader may notice some of the points related to each of the assumption type.

4.9.1. **Mortality**

Assumption about future mortalities should be based on (to take into account) the last experience of the company (mortality experience).
Mortality assumption might be expressed for example as:

a) ratio between the expected mortality and the mortality charged in a premium or
b) ratio between the mortality expected and the mortality of a general population or
c) if the company has long history enough, it could be able to make its own experience life tables.

Especially, for policies where the risk of death is insured, mortalities used for the premium calculation are usually higher than is the expected reality. On the contrary, for policies where the main risk is survival of a certain period (e.g. pure endowments, life pensions) the charged mortalities are usually lower than the expected reality.

Mortality experience may be split according to:

- sex of the insured person
- age of the insured person
- policy type
- smoker status
- policy year
- etc.

However, usually insurance companies suffer from a lack of relevant data to be able to construct a detailed distribution, hence at least the distinction between males and females and the risk (death × survival) is usual assumed.

Often the *initial selection period* is considered. The initial selection period indicates that due to initial medical underwriting (for the policies where the risk is the death) it is expectable that the expected mortality at the first few policy years will be lower than in following years.
The mortality experience (as per cents of mortality charged) may have a form as shown in the table 4.1:

Policy year	*Mortality experience Males*	*Mortality experience Females*
1	25%	20%
2	35%	30%
3	45%	40%
4	55%	50%
5 and others	65%	60%

Table 4.1 – Mortality experience taking into account the initial selection period

4.9.2. **Lapses**

Lapse rates are understood as a probability of lapse according to policy month or policy year.
Similar to mortality experience, lapse rates should be based on the most recent and reliable analysis of the company experience.

It is useful to build such an analysis at least according to:

- calendar year (month) of the policy inception – this may capture the quality of sales in each calendar year separately,

- lapse year (month) – at least in the first policy year where usually special lapse behavior due to initial underwriting, unpaid premium, etc. is possible to observe, monthly evolution is useful;
- product type – usually there is different lapse experience for single and regularly paid policies, for long-term policies as universal life and short-term such as children or term insurance.

Again, data usually are the core problem.
They may not be statistically significant (especially for a detailed distribution). This is why certain grouping is usually applied. Very often the lapse experience for a longer period of time is missing.

Results of a lapse analysis of a certain group of products may be in a form as shown in the table 4.2 (annual rates only):

Policy inception year	***Policy year***							
	1	***2***	***3***	***4***	***5***	***6***	***7***	***8***
1995	20%	10%	7%	10%	5%	***7%***	5%	5%
1996	**25%**	**14%**	**11%**	**13%**	***11%***	**8%**	**7%**	
1997	17%	12%	8%	***14%***	6%	4%		
1998	18%	9%	***12%***	11%	5%			
1999	19%	***15%***	8%	10%				
2000	***25%***	9%	8%					
2001	22%	10%						
2002	20%							

Table 4.2 – Example of a lapse analysis

Figures in the table 4.2 for example show a probable extraordinary situation of the company in the year 2000 (the diagonal in italics – company troubles?) and not very good sale quality in the year 1996 (all lapse rates are relatively higher than for the other inception years).

Based on this analysis usually one series of lapse rates is chosen and applied. This series may be e.g. derived as:

- (weighted) average for each of the policy years – weights may be for instance the volume of policies in-force at the beginning of each policy year, higher weight may be set to a more recent figures (to a most recent diagonal), etc.
- using last diagonal as the most recent figures
- etc.

The final set of figures is usually subjective and qualified judgment is required.

Further not an easy task is how to estimate lapse rates for policy years where no company statistics exist – in our example it is the policy year 9 and followings.

Some solutions could be:

- assuming a stable evolution after the last year and using the last year lapse rate (8th year in our example) for all the other policy years,
- make use of time series analysis and applying e.g. modified exponential trend,
- using a qualified judgment taking into account all possible future events,
- etc.

4.9.3. **Commissions**

There are usually two main types of commissions in life products

- initial (paid in the first one to five policy years) and
- renewal (paid after the end of a payment of initial commissions, usually until the end of policy period).

Initial commissions

The most proper way of initial commission modeling is including the initial commission schedule relevant to each policy modeled separately.
If it is not possible, then for example the average initial commission structure may be calculated and applied but at least for each policy types and for each calendar year of inception separately (commission policy of the company might change during the time).

Renewal commissions

Again, the most appropriate way is including the renewal commission rate according to each policy modeled.
Possible increase in renewal commission rates in future should be covered as well since the company could change the renewal commission policy in future.

Commission claw back

Commission claw back expresses the rules and the value of commissions which has to be retuned back from the agent to the company if a policyholder withdraws the policy agreement. It usually regards mainly to initial commissions in the first and second policy year. Significant part of initial commissions is usually required to be returned back to the company.
However, in reality, not 100 per cent of required commissions are really returned – e.g. insurance broker that falls in bankruptcy before claw back payment, etc. – hence some lower amount of the required sum should be taken into account.

4.9.4. **Expenses**

Expected future expenses of the company should be split at least between the initial and renewal (maintenance) ones.

Initial expenses regard to one-off expenses related to the policy inception – such as medical underwriting, policy forms, product advertisement, remuneration of sale forces (except commissions), etc.
Renewal (maintenance) expenses should cover the expenses needed to be able to administrate and keep the policies in-force. They are usually further split at least between claims handling expenses and others.

Of course, there are several not easily distributed company expenses (e.g. management salary, buildings and many others). Some proper distribution criteria have to be chosen to split them individually.
The company expense analysis should be made at least according to each group of policy types – expenses of single or regularly paid policies are expected to be much different, differences are expected also for flexible products where a lot of changes are allowed (e.g. universal life) and products where nearly no changes are possible (such as term insurance, etc.).

Split of expenses according to what variable they depend on should be assumed as well. Some expenses relate to sum assured (underwriting, claims handling, etc.), some to premium (general renewals, etc.), some to value of reserves (asset management fees, etc.) and some may be fixed per contract.

4.9.5. **Expense inflation**

The increase of expenses in time should be considered as well.
The expenses naturally increase at least thanks to the inflation.
But, since usually large part of expenses relates to salaries, not only consumer price index (CPI) but wage index should be taken into account when projecting the expense inflation. Example of how estimating the expense inflation is shown in the table 4.3.

Year	*CPI*	*Salary inflation*	*Salary part of total expenses*	*Final expense inflation as a weighted avg.*
2003	3.80%	7.02%	60%	**5.73%**
2004	3.30%	6.40%	60%	**5.16%**
2005	3.10%	6.61%	60%	**5.20%**
2006	2.90%	6.40%	60%	**5.00%**
2007	2.50%	5.58%	60%	**4.35%**
2008	2.00%	4.04%	60%	**3.22%**
2009+	2.00%	4.04%	60%	**3.22%**

Table4.3 – Expense inflation

For example:
Final expense inflation in the year 2003 is:

$$5.73\% = 60\% \cdot 7.02\% + (1\text{-}60\%) \cdot 3.80\%$$

since it is assumed that 60% of the total company expenses relates to salaries and 40% will grow in line with CPI.

4.9.6. **Premium (or sum assured) indexation**

Sometimes the policyholder has right to increase the sum assured and/or premium every year based on some agreed index. It should be considered whether the premium (sum assured) indexation is automatic (no policyholder action is required) or if policyholder has to ask for it individually when estimating the level of indexation. At least in the latter case (policyholder has to ask for) some probability of indexation accepted should be used.

4.9.7. **Investment return**

Investment return assumption should, under the EV approach, reflect the expected future returns on the assets held at valuation date. Assumptions of reinvestments of future positive cash flows should be based on the expected future investment strategy agreed within the company. In markets where longer-term fixed interest markets are underdeveloped, investment return assumptions should be based on an assessment of longer-term economic conditions, or other similar markets.

4.9.8. **Discount rate**

All projected future gross profits are discounted to the valuation date using *risk discount rate* (*rdr*). Further discussion is made in the paragraph 4.10. In this paragraph we only show a formula for the discounting factor.
It is:

$$DF_t = \prod_{r=1}^{t} \frac{1}{1 + rdr_r} \, .$$

4.9.9. **Other assumptions**

Other assumptions (events) should be considered as well.
E.g.:

- ad-hoc premium payments
- partial withdrawals (i.e. policyholder has right to withdraw part of his money before the end of the policy period)

- premium holidays (i.e. policyholder has right to stop paying premium for certain period of time)
- etc.

Risk and uncertainty

4.10. The risk and uncertainty that real future cash flows will differ from the best estimation should be, under the Traditional EV methodology, covered in the discount rate (risk discount rate, *rdr*). Risk discount rate is the interest rate determined as a risk free rate (*rfr*, interest rate of risk free assets) and a margin for risk and uncertainty.
Risk discount rate may for example follow the formula:

$$rdr = rfr + \beta \cdot (R_M - rfr)$$

which has its basis in the capital asset pricing model (CAPM).
CAPM dictates the relationship between risk and expected return where
R_M is the expected market return
$R_M - rfr$ is the market risk premium
β is *measure beta* what can be interpreted as the tendency of returns to respond the swings in the market. Beta equal to 1 indicates that the security price will move with the market. Beta less than 1 indicates that the security price will be less volatile than the market. Beta more than 1 indicates that the security price will be more volatile than the market. For example, if a stock's beta is *1.2* it is theoretically 20% more volatile than the market.
For insurance companies beta is usually used on the level of about 1 to 1.5.

Single discount rate is usually used in this Traditional EV approach.
Hence, $rdr_r = rdr$ (used in the formula in 4.9.8) for all relevant policy years r.

4.11. *Example:*

We'll show an example of calculation of EV reserve for our *working example* based on the following assumptions (best estimation).

MODEL ASSUMPTIONS		
Mortality experience	80%	of mortality charged
Annual lapse rate	5,0%	all the policy period
Investment return	5,00% p.a.	
Risk discount rate	12% p.a.	
Expenses		
Initial (1st year)	1 500	Annual fixed
Renewal	1 000	Annual fixed
	+ 5%	from each premium

Expense inflation	4% p.a.	
Commissions		
Initial 1st year	50%	from annual premium
Initial 2nd year	20%	from annual premium
Renewal (starting 3rd year)	3%	from annual premium

The cash flow projection is then as shown in the table 4.4.
(BoY = begin of the year, EoY = end of the year).

Cash flow						
Year	***Premium (BoY)***	***Expenses and commissions (BoY)***	***Investment return (EoY)***	***Change of reserves (EoY)***	***Claims (EoY)***	***TOTAL GROSS PROFIT (EoY)***
2004	27 820	3 226	6 479	22 738	7 009	***1 327***
2005	26 387	3 097	7 551	21 441	8 130	***1 270***
2006	25 023	2 975	8 561	20 209	9 187	***1 213***
2007	23 725	2 857	9 512	19 035	10 188	***1 157***
2008	22 490	2 745	10 408	17 917	11 135	***1 101***
2009	21 314	2 637	11 250	-206 325	235 208	***1 044***

Table 4.4 – EV reserve calculation of the working example – cash flow

The EV reserve result is as shown in the table 4.5.

Reserve	
Statutory reserve	104 986
PVFP	4 949
TOTAL RESERVE	**100 037**

Table 4.5 – EV reserve of the working example – result

Notice that the value of the statutory reserve is the same as in the example shown in the paragraph 3.9.

4.12. Usually the higher *rdr* (the higher margin), the lower PVFP, the higher EV reserve. We'll show now the results of EV reserves (PVFP results) depending on the value of *rdr* in the table 4.6. The results correspond to the same example as in the previous paragraph 4.11.

Other results		
RDR	***PVFP***	***Total reserve***
6%	5 875	99 111
8%	5 538	99 448
10%	5 230	99 756
12%	**4 949**	**100 037**
14%	4 692	100 294
16%	4 456	100 530
18%	4 239	100 747

Table 4.6 – EV reserve according to rdr

4.13. However, let's look at a similar example as in the last paragraphs 4.11 and 4.12. It is the same policy product, the same policy data, but with different assumptions.
Mortality experience assumption is higher, investment return is lower, expenses and expense inflation are higher and renewal commissions are higher as well.

MODEL ASSUMPTIONS		
Mortality experience	90%	of mortality charged
Annual lapse rate	5,0%	all the policy period
Investment return	4,00% p.a.	
Risk discount rate	12%	
Expenses		
Initial (1st year)	2000	annual fixed
Renewal	1200	annual fixed
	+ 5%	from each premium
Expense inflation	4,5%	
Commissions		
Initial 1st year	50%	from annual premium
Initial 2nd year	20%	from annual premium
Renewal (starting 3rd year)	4%	from annual premium

The cash flow projection, the EV reserve result and its sensitivity to *rdr* are shown in the tables 4.7, 4.8 and 4.9.

Cash flow						
Year	***Premium (BoY)***	***Expenses and commissions (BoY)***	***Investment return (EoY)***	***Change of reserves (EoY)***	***Claims (EoY)***	***TOTAL GROSS PROFIT (EoY)***
2004	27 820	3 704	5 164	22 165	7 038	**78**
2005	26 382	3 563	5 999	20 769	8 126	**-78**
2006	25 012	3 429	6 780	19 444	9 144	**-225**
2007	23 709	3 301	7 511	18 182	10 101	**-364**
2008	22 469	3 178	8 193	16 981	10 999	**-496**
2009	21 288	3 060	8 830	-202 527	230 206	**-622**

Table 4.7 – EV reserve calculation of the working example – cash flow

Reserve	
Statutory reserve	104 986
PVFP	-980
TOTAL RESERVE	**105 966**

Table 4.8 – EV reserve of the working example – result

Other results		
RDR	***PVFP***	***Total reserve***
6%	-1 281	106 267
8%	-1 170	106 155
10%	-1 070	106 055
12%	**-980**	**105 966**
14%	-899	105 885
16%	-827	105 813
18%	-761	105 747

Table 4.9 – EV reserve according to rdr

In this example, higher risk discount rate is not conservative, hence does not cover any risk and uncertainty, and even worse, may lead to an underestimation of reserves. It is of course caused by the negative gross profit patterns shown in the table 4.7. In order to cover risk and uncertainty lower risk discount rate (negative margin) is more conservative in this case.

4.14. Other approaches sometimes may be applied to avoid this exceptionality – e.g. risk discount rate is applied to positive profits GP_t and risk free rate (no margin) for negative GP_t, etc.

But still the main question remains: what level should the profits be grouped on (per each individual policy, per group of similar policies, per each month, year, etc.).

Alternative approach

4.15. As it has already been mentioned above, annual gross profit for PVFP calculation is usually defined as (see paragraph 4.3):

$$GP_t = P_{t-1} - C_{t-1} - E_{t-1} - Claims_t + I_t - \left(Res_t^{Stat} - Res_{t-1}^{Stat}\right) - (PSfund_t - PSfund_{t-1})$$

This definition is very close to the way how the gross profit is accounted in the accounting books. Let's call this definition as *'accounting way'* of gross profit definition for the purposes of this text.

However, very often, especially managers want to know not only the total value of PVFP but also the split of the PVFP according to its sources.

Usually it is required to compare:
charges vs. commissions and expenses .... expense profit
mortality charges (risk premium) vs.
sum at risk paid in case of death.. mortality profit
surrender charges when policy lapses................................ surrender profit
investment incomes vs. valorization of reserves .... investment profit.

The gross profit at each policy year *t* is then defined as:

$$\begin{aligned}
GP_t &= expense\ profit \\
&+ mortality\ profit \\
&+ surrender\ profit \\
&+ investment\ profit
\end{aligned}$$

This definition allows us to come to PVFP according to different segments ('funds'). Let's call this gross profit definition as *'fund way'* for the purposes of this text.

4.16. In the following, we'll show that results of both gross profit definitions ('accounting' and 'fund' way) should give the same results.
We again will use the endowment policy, where:

- sum assured plus the value of the profit share fund is paid in case of a death or maturity
- charges are based on each premium within all the policy period
- profit share is distributed to policyholders' separate funds
- technical interest rate is guaranteed even for the profit share fund.

Our *working example* is one of such type of product.

4.17. Accounting gross profit then is:

$$\begin{aligned}
&GP_t^{accounting} = \\
&l_{t-1} \cdot P_{t-1} \cdot (1+s_t) - l_{t-1} \cdot E_{t-1} \cdot (1+s_t) - l_{t-1} \cdot C_{t-1} \cdot (1+s_t) + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) \cdot i_t \quad , \\
&-d_t \cdot (SA + PSfund_t) - w_t \cdot Surr_t - l_t \cdot Res_t^{Stat} + l_{t-1} \cdot Res_{t-1}^{Stat} - l_t \cdot PSfund_t + l_{t-1} \cdot PSfund_{t-1}
\end{aligned}$$

---

where s_t is the investment return of investing the premium, expenses and commissions – usually assume to be some short term rate or even zero.

Notice, that now the cash flows $P_{t-1}, E_{t-1}, C_{t-1}, Res_{t-1}^{Stat}, Res_t^{Stat}, PSfund_{t-1}, PSfund_t, Surr_t$ are the cash flows of a (one) policy under the condition that the financial flow is realized in the time *t* or *t-1* respectively.

It holds:

$SA = SAR_t + Res_t^{Stat}$

$Surr_t = Res_t^{Stat} + PSfund_t - SurrCh_t$

$w_n = l_{n-1} - d_n$, (*n* … policy period) since $l_n = l_{n-1} - d_n - w_n = 0$.

Remark:
This is a usual definition of the accounting gross profit.
However in order to be accurate we should split the premium payment between saving part (ΔRes_{t-1}^{Stat}) and non-saving part (risk premium ($q_{x+t-1} \cdot SAR_t \cdot v$) plus charges ($\lambda \cdot P_{t-1}$)). Saving part of the premium is then assumed to be invested in the same way (with the same investment return) as the technical (statutory) reserves while non-saving part of the premium is assumed to be invested with the investment return s_t.

Thus:

$$\begin{aligned}
&GP_t^{acounting} = \\
&l_{t-1} \cdot (\lambda \cdot P_{t-1} + q_{x+t-1} \cdot SAR_t \cdot v) \cdot (1+s_t) \\
&+l_{t-1} \cdot \Delta Res_{t-1}^{Stat} \cdot (1+i_t) \\
&-l_{t-1} \cdot E_{t-1} \cdot (1+s_t) - l_{t-1} \cdot C_{t-1} \cdot (1+s_t) \\
&+l_{t-1} \cdot Res_{t-1}^{Stat} \cdot i_t + l_{t-1} \cdot PSfund_{t-1} \cdot i_t \\
&-d_t \cdot (Res_t^{Stat} + SAR_t + PSfund_t) - w_t \cdot (Res_t^{Stat} + PSfund_t - SurrCh_t) \\
&-l_t \cdot Res_t^{Stat} + l_{t-1} \cdot Res_{t-1}^{Stat} - l_t \cdot PSfund_t + l_{t-1} \cdot PSfund_{t-1} =
\end{aligned}$$

It holds:

$$l_t + w_t + d_t = l_{t-1}$$

$$\begin{aligned}
&= l_{t-1} \cdot (\lambda \cdot P_{t-1} + q_{x+t-1} \cdot SAR_t \cdot v) \cdot (1+s_t) + l_{t-1} \cdot \Delta Res_{t-1}^{Stat} \cdot (1+i_t) \\
&-l_{t-1} \cdot E_{t-1} \cdot (1+s_t) - l_{t-1} \cdot C_{t-1} \cdot (1+s_t) + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) \cdot i_t \\
&-l_{t-1} \cdot (Res_t^{Stat} + PSfund_t) \\
&-d_t \cdot SAR_t + w_t \cdot SurrCh_t + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) =
\end{aligned}$$

It holds:

$$Res_t^{Stat} + PSfund_t = (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + i_t - (i_t - tir) \cdot mfee) \text{ if } i_t \geq tir$$

and

$$\begin{aligned}
Res_t^{Stat} + PSfund_t &= (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + tir) \\
&= (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + i_t + (tir - i_t)) \text{, if } i_t < tir.
\end{aligned}$$

Thus, summarized into one formula:

$$Res_t^{Stat} + PSfund_t = (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + i_t - \min(i_t - tir, (i_t - tir) \cdot mfee))$$

$$\begin{aligned}
&= l_{t-1} \cdot (\lambda \cdot P_{t-1} + q_{x+t-1} \cdot SAR_t \cdot v) \cdot (1+s_t) + l_{t-1} \cdot \Delta Res_{t-1}^{Stat} \cdot (1+i_t) \\
&- l_{t-1} \cdot E_{t-1} \cdot (1+s_t) - l_{t-1} \cdot C_{t-1} \cdot (1+s_t) + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) \cdot i_t \\
&- l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (1 + i_t - \min(i_t - tir, (i_t - tir) \cdot mfee)) \\
&- d_t \cdot SAR_t + w_t \cdot SurrCh_t + l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1}) =
\end{aligned}$$

$$\begin{aligned}
&= l_{t-1} \cdot (\lambda \cdot P_{t-1} + q_{x+t-1} \cdot SAR_t \cdot v) \cdot (1+s_t) \\
&- l_{t-1} \cdot E_{t-1} \cdot (1+s_t) - l_{t-1} \cdot C_{t-1} \cdot (1+s_t) \\
&+ l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (i_t - (i_t - \min(i_t - tir, (i_t - tir) \cdot mfee))) \\
&- d_t \cdot SAR_t + w_t \cdot SurrCh_t =
\end{aligned}$$

$$\begin{aligned}
&= l_{t-1} \cdot (\lambda \cdot P_{t-1} - E_{t-1} - C_{t-1}) \cdot (1+s_t) \ldots\ldots \textit{row 1} \\
&+ l_{t-1} \cdot q_{x+t-1} \cdot SAR_t \cdot v \cdot (1+s_t) - d_t \cdot SAR_t \ldots\ldots \textit{row 2} \\
&+ w_t \cdot SurrCh_t \ldots\ldots \textit{row 3} \\
&+ l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot i_t \ldots\ldots \textit{row 4} \\
&- l_{t-1} \cdot (Res_{t-1}^{Stat} + PSfund_{t-1} + \Delta Res_{t-1}^{Stat}) \cdot (i_t - \min(i_t - tir, (i_t - tir) \cdot mfee)) = \ldots\ldots \textit{row 5} \\
&\qquad = \textit{expense profit/loss } (\textit{row 1}) \\
&\qquad + \textit{mortality profit/loss } (\textit{row 2}) \\
&\qquad + \textit{surrender profit/loss } (\textit{row 3}) \\
&\qquad + \textit{investment profit/loss } (\textit{row 4 and 5}) \\
&\qquad = GP_t^{fund}
\end{aligned}$$

4.18. *Example:*

Let's look to the PVFP calculation of our *working example* using the 'fund way' of the gross profit definition.

MODEL ASSUMPTIONS		
Mortality experience	80%	of mortality charged
Annual lapse rate	5,0%	all the policy period
Investment return	5,00% p.a.	
Risk discount rate	12% p.a.	
Expenses		
Initial (1st year)	1500	annual fixed
Renewal	1000	annual fixed
	+ 5%	from each premium
Expense inflation	4% p.a.	

Commissions		
Initial 1st year	50%	from annual premium
Initial 2nd year	20%	from annual premium
Renewal (starting 3rd year)	3%	from annual premium

The cash flow projection is then as shown in the table 4.10.

Cash flow					
Year	***Expense profit (EoY)***	***Mortality profit (EoY)***	***Investment profit (EoY)***	***Surrender profit (EoY)***	***TOTAL GROSS PROFIT - fund way (EoY)***
2004	995	69	64	199	***1 327***
2005	904	59	75	232	***1 270***
2006	818	47	85	263	***1 213***
2007	737	34	95	292	***1 157***
2008	660	18	104	319	***1 101***
2009	588	0	112	344	***1 044***

Table 4.10 – EV reserve calculation of the working example – fund way – cash flow

The EV reserve result is shown in the table 4.11.

Reserve	
Statutory reserve	104 986
PVFP	**4 949**
TOTAL RESERVE	**100 037**

Table 4.11 – EV reserve calculation of the working example – fund way – result

4.19. We can see (comparing to the results in the paragraph 4.11 – tables 4.4 and 4.5) that the results of gross profits projection => PVFP => EV reserves really are the same.

Moreover a split of the total PVFP could be done for each source of the profit as shown in the table 4.12.

Profit source	***PVFP***
Expense profit/loss	3 331
Mortality profit/loss	174
Investment profit/loss	353
Surrender profit/loss	1 090
PVFP Total	**4 949**

Table 4.12 – Split of the total PVFP into sources of profit

Other use of the Embedded Value

4.20. Embedded Value is used not only for the purposes of valuation of liabilities but very often as an estimation of the value of the company as well. By the term *Embedded Value* is then usually meant a sum of the present value of future *distributable earnings* of policies in-force at valuation date and a value of the company *free assets*.

Distributable earnings (DE) are usually defined as a part of annual profits which could be released from the company without jeopardizing the company economy (e.g. what could be paid for example in a form of dividends to shareholders). DE is usually determined as a gross profit minus a company tax payment minus a change of a required solvency capital.

Free assets is usually understood as a capital and surplus allocated to, but not required to support, the in-force business at the valuation date – the main part of *free assets* are the assets not backing liabilities and required solvency margin at valuation date.

All the basic terms mentioned in this paragraph are shown in an illustrative form in the tables 4.13, 4.14 and 4.15.

Embedded Value:

Free assets	**Embedded Value**
PV of distributable earnings	

Table 4.13 – Embedded Value

Free assets:

Reserves	Total assets
Required solvency capital	
Free assets	

Table 4.14 – Free assets

Distributable earnings:

Tax	Gross profit
Change of required solvency capital	
Distributable earnings	

Table 4.15 – Distributable earnings

4.21. The value of the EV itself and especially its changes during the time is a strong steering tool for the management of insurance companies. Standardized EV reports usually include:

- value of the EV
- sensitivity analysis of the EV – i.e. changes of the EV due to changes of assumptions
- analysis of movements = analysis of a change of the EV during the last period (what assumptions has/has not been matched and why, what is the impact of a new business, etc.).

4.22. *Example:*
Let's assume a theoretical insurance company having 10 000 identical life policies, of the same type as in paragraph 4.11 and let's assume the same calculation assumptions as in the paragraph 4.11.
Let's study the present value of future gross profits and its sensitivity.
We will test each of parameters to a relative change of ±20% while the rest of assumptions stay on the base case level – see the table 4.16.
We assume all parameters to be independent which doesn't have to be always true. There may be some correlation for example between investment return and lapses (high interest rates can cause higher lapses), investment return and expense inflation etc.
All figures in the table 4.16 are presented in mln (e.g. CZK).

PVFP sensitivity	*-20%*		*Base case*	*+20%*	
	PVFP	*% of change*		*% of change*	*PVFP*
Mortality experience	50,9	3%	49,5	-3%	48,1
Lapse rates	48,3	-2%		2%	50,6
Investment return	10,5	-79%		15%	57,1
Risk discount rate	52,9	7%		-6%	46,4
Renewal expenses fixed	57,9	17%		-17%	41,1
Renewal exp. as % from premium	60,2	22%		-22%	38,7
Expense inflation	50,2	1%		-1%	48,8
Renewal commission	55,9	13%		-13%	43,0

Table 4.16 – Sensitivity analysis

It seems that the most sensitive assumption in this case is the investment return, because when it falls down a big change of PVFP occurs. We can realize that this significant change in PVFP is caused especially by the decrease of investment return under the technical interest rate. Other important assumptions seem to be expenses and renewal commissions.

Of course, the importance of parameters depends significantly on many other circumstances, e.g.:

- level of the base case of the assumption (especially for investment return),
- policy data (e.g. sum assured for mortality profit, surrender penalty for surrender profit, etc.),
- the level of the base case PVFP (if it is small then each relative change can be huge)
- etc.

One should also understand that changes in different parameters (of e.g. 20%) are not equally likely.

4.23. In addition to the Embedded Value sometimes Appraisal Value is calculated as another measure of the value of the life company. Appraisal Value consists of the EV and the present value of distributable earnings of estimated future new business (sometimes referred to as Goodwill). This value is usually a basis for the value of the company estimation in mergers and acquisitions. Of course, what will be the structure of the new business and for how long it should be modeled are subjective decisions and depend on the agreement between the business parties.

Pros and cons of the Traditional Embedded Value methodology

4.24. Positives:

\+ In comparison to Statutory valuation approach the EV methodology is closer to the reality. Expected assumptions are used; lapses and all other expected events are included in the projection.

4.25. Negatives:

- It is a difficult task to get the best estimation of the assumptions. Many subjective decisions usually have to be made.
- Construction of risk discount rate is subjective and may not cover risk and uncertainty in a satisfactory way – see the example in the paragraph 4.13.
- No value of embedded options is calculated in the Traditional EV approach. Only deterministic scenarios are usually used in the traditional approach => especially wrong estimation of future investment return (if higher or lower than technical rate) could cause significant changes in the EV reserve (see e.g. the sensitivity to investment return in the example in paragraph 4.22).
- Investment risk is considered very purely. For example if more equities are assumed to be in the asset portfolio, the higher investment return is usually assumed, this causes the higher PVFP and the lower EV reserve.

However, in the Traditional EV approach, pure or no investment risk margin is usually considered.

Current development

4.26. So far we have described the Traditional EV approach. There are two main improvements of this traditional approach nowadays. First is the 'European Embedded Value Approach' presented by a forum of Chief Financial Officers of a number of the largest European life offices (CFO forum) – see [6]. The latter one is the development usually called as 'Enhanced' or 'Market Consistent Embedded Value', which uses concepts of financial economics.

European Embedded Value (EEV) concept

4.27. The main principles of this approach are defined is the document with the same name (European Embedded Value, EEV) prepared by the CFOs (chief financial officers) forum and issued on May 5, 2004 (see [6]). It consists of 12 'Principles', providing a framework for the derivation of valuation assumptions, calculation and reporting of the Embedded Value results. CFO forum terms of reference were to produce guidance which allows for consistent application between comparable companies, allows for appropriate valuation of guarantees and options and prescribes a minimum level of disclosure including sensitivity analysis. EEV approach is currently accepted and applied in the majority of the European insurance companies.

4.28. There is only a limited number of issues changed comparing to the Traditional EV approach.
Some of the most interesting are:

- Risk discount rate doesn't have to be only one for all calculations but may vary among product groups and territories. Risk discount rate should reflect different risks included.
- Option and guarantees:
  - Traditional EV: Deterministic integration of these risks in the risk discount rate is applied. This often leads to a significant underestimation of the option risk.
  - EEV: Stochastic simulations should be used for options measurement, though no guidelines are included in the document. The EEV allow considerable scope in the choice of methodology and assumptions, and it appears unlikely that comparison among companies will be straightforward, particularly where the approach chosen is not market consistent and the result is not benchmarked against a market-consistent valuation.
- Risk margin in the risk discount rate:
  - Traditional EV: All risks should be covered by the risk margin.
  - EEV: Risk premium is partially included, since e.g. options and guarantees are to be measured stochastically.

Market Consistent Embedded Value (MCEV) concept

4.29. Another development of the Embedded Value calculation is the concept of Market Consistent Embedded Value. Within MCEV framework, assets and liabilities are valued in line with market prices and consistently with each other. In principle, each cash flow is valued using the discount rate consistent with that applied to such a cash flow in capital markets. Thus, the value of assets is their market value. The value of liabilities is the value of comparable assets cash flow (or the value of a replicating portfolio).

4.30. The main improvements comparing to Traditional EV approach are:

- Risk discount rate is based on observable market rates of return at the valuation date.
- The costs of options and guarantees are valued objectively and explicitly, using stochastic option pricing techniques consistent with the market price of options.

4.31. MCEV is the one among the EV approaches that is closest to Fair Value principles (see the next sections). Some of the European companies have already accepted and used this approach.

5. Fair Value – deterministic estimation

Deterministic estimation of the fair value (FV) of liabilities will be discussed here in this section.

- We'll start with **short history overview**.
- **Main principles** of a deterministic Fair Value calculation will continue then.
- **Assumptions** used and **risk and uncertainty** coverage will further be discussed especially and compared to EV methodology.
- **Pros and cons** of this deterministic estimation of the FV will end this section.

Short history overview

5.1. In 1997, International Accounting Standard Board (IASB) (originally named as International Accounting Standard Committee – IASC) was established in order to develop an official international standard for the reporting of insurance contracts.

5.2. Brief history of IASB project:

- 1997 – start of the project – International Accounting Standard Committee (IASC) established,
- 12/1999 – Steering Committee of IASC published an Issues Paper for discussion purposes,
- 6/2001 – Steering Committee of IASC published the Draft Statements of Principles (DSOP) setting out the principles of the fair value which should be applied to insurance business.

The plan of International Standard Board at that time (in 2001) was to finalize the standard during the year 2003 and to use it officially since the year 2005.

But then other complications arose and many unresolved issues still remained so that the project was split in two phases in the year 2003.

- First phase, which doesn't change a lot from local accounting principles, was finalized in March 2004 (standard IFRS4).
- The deadline for final 2nd phase is currently (at January 2006) even not exactly known.

Main principles

5.3. The Fair Value reporting standard is based on a definition of the fair value of assets and liabilities. The definition is as follows:
Fair value is the amount for which an asset could be exchanged and a liability settled between knowledgeable, willing parties in an arm's length transaction.

5.4. Fair value of assets is understood to be its market value (if exists). But, trading with liabilities is nearly none comparing to assets market. Thus market value of liabilities is very rarely available. Alternative approach, what will match the asset and liability valuation methodology has to be used.

5.5. We will focus mainly on the fair value of liabilities in this section.

5.6. We will show a *deterministic estimation* of the FV in this section. We use the term *deterministic estimation* of the fair value since probably more precious approach than what would meet the definition of the FV better would be the stochastic approach rather than the deterministic. The stochastic valuation (i.e. all assumptions are simulated according to their probabilistic distribution and their correlations are considered as well) may better cover the risk and uncertainty included in the future cash flow projections and the value of options embedded in policy contracts would be priced more properly than using one deterministic scenario. We will show an example of the stochastic approach (only a valuation of interest rate option via interest rate simulations) further in the section 7. The term *deterministic estimation of the FV* and *the deterministic FV* will further be used with the identical meaning in this section.

5.7. The basis for a determination of the fair value of liabilities is cash flows projection of policies in-force at valuation date to the future. We define the annual cash flows at the policy year *r* as follows (index *t* expresses the policy year at the valuation date):

$$CF_r = \left(P_{r-1} - C_{r-1} - E_{r-1}\right) \cdot (1 + rfr_{r-t}) - Claims_r \,,$$

where
rfr_{r-t} … is the (risk free) interest rate related to policy year *r* which is applied to the cash flows at the beginning of the year in order to express their value at the end of the policy year (see further the paragraph 5.13 and 5.24).
We again assume that premiums, commissions and expenses are cash flows at the beginning of the year and claims payments are the cash flows at the end of the year – the same approach as all above (Statutory and EV approaches).

5.8. *Remark*:
Notice that comparing to gross profit projection used when calculating the Embedded Value no change of the Statutory reserve and no investment income from the statutory reserve are explicitly included – only the expected inflow (premium) and outflow (commissions, expenses and claims) are assumed.

5.9. Like EV approach, all expected future cash flows (expected events) should be included in the projection.

5.10. The present value of such cash flows as at the valuation date is studied. Let's denote this present value at the end of the year *t* as $FV_t^{\det}$ (meaning the deterministic Fair Value, not Future Value).

Hence,

$$FV_t^{\det} = \sum_{r=t+1}^{n} CF_r \cdot DF_{r-t} \,,$$

where

DF_{r-t} … is the discounting factor relevant to the year *r-t* from the valuation date (see further the paragraphs 5.13 and 5.25).

5.11. *Remark:*

Notice that such a definition of $FV_t^{\det}$ is similar to the statutory reserve definition. Both are defined as the present value of future cash flows.

But we can see the differences in mainly two aspects:

a) all expected future cash flows are assumed when calculating $FV_t^{\det}$ (lapses, etc.)

b) assumptions used for $FV_t^{\det}$ calculation are based on their expected level – see below.

Assumptions, risk and uncertainty

5.12. The same structure of assumptions as of the EV approach is used. The difference is in how the risk and uncertainty is evaluated.

5.13. Generally, *risk free rate (rfr)* should be used for all interest rate assumptions. Risk free rate uses to be defined as market yield at the valuation date on *risk free assets*. We understand the risk free assets as those with readily observable market prices whose cash flows are least variable for a given maturity and currency.

5.14. Risk free rate is, under the deterministic FV approach, used for:

a) discounting the future cash flows and

b) an annual investment income of the investment of the statutory reserves. This investment income is not explicitly expressed as a cash flow in the CF_r definition, but is implicitly included in the $Claim_r$ cash flow. Realize that surrenders, maturities or death benefits are based on the value of the statutory reserves plus the value of the profit share which is explicitly based on the investment incomes.

5.15. *Remarks*:

a) Notice that the risk discount rate is used for discounting and expected investment return is used for investment income under the EV approach. Risk free rate is used for both cases under the FV approach.

b) Under the FV methodology the discount rate should not be depending on assets held – risk free rate of return is assumed.

5.16. We now make a short excursion to interest rate structure theory since different interest rates will further be used in the text. The reader can find a good text e.g. in [1]. After this excursion we will follow in the deterministic FV description.

Brief interest rates structure theory

5.17. There are several types of interest rates (yield curves) – e.g. coupon × zero coupon, spot × forward, etc. We'll discuss the interest rate generally now and conclude with what rates are usually used for the deterministic estimation of the FV.

5.18. The *n*-year *zero rate* (zero-coupon rate) is the rate of interest earned on an investment that starts today and lasts for *n* years. All the interests and principal is realized at the end of *n* years. The *n*-year *zero rate* is also referred to as the *n*-year spot rate. Suppose the five year zero rate is quoted as 5% per annum. This means that 100, if invested at the risk free rate for five years, would grow to

$$100 \cdot (1+0.05)^5 = 127,63$$

5.19. Many of interest rates observable in the market are not pure zero rates. Consider a five year government bond that provides a 6% coupon. The price of this bond does not exactly determine the five year zero rate because some of the return on the bond is realized in the form of coupons prior to the end of five years.

5.20. The *par yield* for a certain maturity is the coupon rate that causes the bond (coupon bond) price to equal its face value. *Par yields* are usually quoted in the market.

Usually the bond is assumed to provide annual or semiannual coupons.

Suppose that the coupon on a two-year bond is semiannual at *c%* per annum (i.e. $\frac{1}{2} \cdot c$ per six months).

Further assume the zero rates as in the table 5.1.

Maturity (years)	***Zero rate (%)***
0.5	2.5
1.0	2.8
1.5	3.1
2.0	3.5

Table 5.1 – Zero rates – example

The value of the bond is then equal to its face value when

$$\tfrac{c}{2}\cdot(1+0.025)^{-0.5}+\tfrac{c}{2}\cdot(1+0.028)^{-1.0}+\tfrac{c}{2}\cdot(1+0.031)^{-1.5}+(1+\tfrac{c}{2})\cdot(1+0.035)^{-2.0}=1$$

This equation is solved giving the c=3.45%. The two-year par yield is therefore 3.45% per annum (with semiannual compounding).

5.21. More generally, if d is the present value of 1 received at the maturity of the bond, A is the value of an annuity that pays 1 in each coupon payment date , and m is the number of coupon payments per year, the par yield c must satisfy

$$1 = A\cdot\frac{c}{m}+d$$

so that

$$c=\frac{(1-d)\cdot m}{A}$$

In our example, $m=2, d=(1+0.035)^{-2}=0.933511$, and

$$A=(1+0.025)^{-0.5}+(1+0.028)^{-1.0}+(1+0.031)^{-1.5}+(1+0.035)^{-2.0}=3.849242$$

5.22. *Forward interest rates* are the rates implied by current zero rates for periods of time in the future. To illustrate how they are calculated, we suppose the zero rates as in the table 5.2.

Maturity (n years)	*Zero rate (% p.a.) for an n-year investment*	*Forward rate(% p.a.) for n-th year*
1	2.8	
2	3.5	4.2
3	4.1	5.3

Table 5.2 – Forward rates – example

The forward rate for the year 2 is 4.2% per annum. This is the rate of interest that is implied by the zero rates for the period of time between the end of the first year and the end of the second year. It can be calculated from the one-year zero interest rate of 2.8% per annum and the two-year zero interest rate of 3.5% per annum. It is the rate of interest for year 2, when combined with 2.8% per annum for year 1, gives 3.5% overall for the two years. To show that the correct answer is 4.2% per annum, suppose that 100 is invested. A rate of 2.8% for the first year and 4.2% for the second year yields

$$100\cdot(1+0.028)\cdot(1+0.042)=107.123$$

at the end of second year. A rate of 3.5% per annum for two years yield

$$100 \cdot (1+0.035)^2 = 107.123.$$

5.23. More *generally*, if f_t is the forward rate for the *t*-th year and the z_t (resp. z_{t-1}) is the zero rate for *t* (resp. *t*-1) years, the forward rate f_t must satisfy

$$(1+z_{t-1})^{t-1} \cdot (1+f_t) = (1+z_t)^t$$

so that

$$f_t = \frac{(1+z_t)^t}{(1+z_{t-1})^{t-1}} - 1$$

5.24. Risk free rate (rfr_{r-t}) in the paragraph 5.7 expresses a risk free rate at the policy year *r* known at valuation date. Thus that rfr_{r-t} is usually assumed to be the forward rate determined from the risk free rate yield curve as at the valuation date (expressed by the policy year *t*) with maturity *r-t.*

5.25. DF_{r-t} in the paragraph 5.10 is then defined as:

a) $DF_{r-t} = \prod_{j=1}^{r-t}(1+f_j)^{-1}$ (or $DF_{r-t} = \prod_{j=1}^{r-t}(1+rfr_j)^{-1}$ respectively) when using forward rates f_j (or rfr_j respectively)

b) $DF_{r-t} = (1+z_{r-t})^{-(r-t)}$ where z_{r-t} expresses corresponding zero spot rates for the maturity *r – t* years from the valuation date.

5.26. It is not an easy task to determine what interest rate should be used for the investment return assumption in the deterministic estimation of FV. We know that it should be risk free and should not depend on the structure of current portfolio of assets (financial instrument).
One should realize that profit sharing rules (based on the investment returns) could be defined differently in different companies and their policy conditions.
Some of the examples of the profit sharing definitions may be:

- Investment return is based on the accounting investment performance. Then other questions have to be answered, e.g.:
    - Does the investment performance depend on the accounting class of assets (i.e. are the assets classified as available for sale, held to maturity, etc. according to IFRS)?
    - Will only an investment profit which is accounted in P&L accounts be included? Should the investment profit what is accounted against equity (in the balance sheet) be included as well?
    - Will unrealized gains/losses (e.g. changes in market value of assets thanks to market conditions) be taken into account or only realized profits/losses are considered?
    - etc.

- Investment return is based on the fixed prescribed formula in policy conditions – e.g. investment return is the average of 10Y zero rates of government bonds during last 5 years or similar.
- Investment return is on the management discretion – then probably some 'reasonable policyholder expectation' will be included.

5.27. We will use one-year forward rate as an annual investment income of the investment of the Statutory reserves in this section. This is now a generally accepted risk free estimation of the future investment returns in the Czech insurance market when calculating the FV under the deterministic approach.
We will use the 'reasonable policyholder expectation' in the section 7 when we will discuss the stochastic FV approach.

5.28. Unlike EV approach, assumptions are not used on the best estimation level, but are adjusted by so called *market value margin* (*MVM*). MVM is the adjustment of the best estimation level of the assumption and should express the risk and uncertainty of its future evolution.

5.29. How to come to a proper MVM is a difficult and individual task. Usually data for deep statistical analysis (mortality experience, lapses, expenses, etc.) are not available or not sufficient, thus the final decision very much depends on the actuarial judgment ('feeling of the risk') of the company actuary (and/or the management) when determining the FV via the deterministic estimation.

5.30. We'll show two examples of MVM in this paragraph.

a) Czech Society of Actuaries recommendation:

Czech Society of Actuaries in its guideline (see [4]) suggests MVMs to be the adjustments of the best estimates shown in the table 5.3 which causes an increase of the present value of future cash flows.

Risk	***MVM as per cents of best estimate***
Mortality	10%
Lapses	10%
Expenses	10%
Expense inflation	10%

Table 5.3 – MVM suggested by the Czech Society of Actuaries

These parameters are suggested but not required. Final value of the MVM should be the decision of the company, individually according to their risks.
Relevant discussion could for instance be whether MVMs should not be increased in time (uncertainty in long-term horizon is probably higher than for short-term) or how to group policies when evaluating the present value changes (increase or decrease), etc.

b) Other MVM example – mortality MVM

Sometimes expected mortality improvement in a future is assumed as the best estimation, e.g.

$$q_{x,t} = coef \cdot q_{x,t-1},$$

where

$q_{x,t}$ … expresses the mortality of *x*-aged person in the calendar year *t*

$coef$ … shows a decrease of mortality in one year.

Let's assume the best estimation of $coef$ to be 99% (based on historical data). MVM may then be expressed e.g. in a way that:

- $coef = 99.5\%$ … for policies where the risk is death and
- $coef = 98.5\%$ … for policies where the risk is survival.

5.31. *Example*:

We'll show several examples of calculation of present value of cash flow under the deterministic FV methodology for our *working example*.

5.31.1. Present value of cash flows if the best estimation level of assumptions is applied.

MODEL ASSUMPTIONS – best estimates		
Mortality experience	80%	of mortality charged
Annual lapse rate	5,0%	all the policy period
Risk free rate	5% p.a.	
Expenses		
Initial (1st year)	1500	annual fixed
Renewal	1000	annual fixed
	+ 5%	from each premium
Expense inflation	4% p.a.	
Commissions		
Initial 1st year	50%	from annual premium
Initial 2nd year	20%	from annual premium
Renewal (starting 3rd year)	3%	from annual premium

Under such assumptions the cash flow projection is as shown in the table 5.4.

Cash flow					
Year	*Premium (BoY)*	*Expenses (BoY)*	*Interest on premium and expenses (EoY)*	*Claims (EoY)*	*CASH FLOW (EoY)*
2004	27 820	3 226	1 230	7 009	***18 816***
2005	26 387	3 097	1 164	8 130	***16 324***
2006	25 023	2 975	1 102	9 187	***13 964***
2007	23 725	2 857	1 043	10 188	***11 723***
2008	22 490	2 745	987	11 135	***9 597***
2009	21 314	2 637	934	235 208	***-215 598***

Table 5.4 – Cash flow projection – best estimation of assumptions

This results to the value of the present value of future cash flows equal to -98 929, therefore the value of liabilities is +**98 929.**

Notice:
Negative value of the present value means that (remember the CF_t definition from the paragraph 5.7) the present value of outcomes (claims, expenses) are higher than the present value of incomes (premium), hence the value of reserve is then positive.

5.31.2. We'll further show (in the table 5.5) the results of the present value of the cash flows if MVMs according to guidelines of Czech Society of Actuaries are applied. Increase or decrease the best estimation of assumptions of ±10% is applied.

Risk	*-10%*	*best estimation*	*10%*
Mortality	98 848	**98 929**	**99 011**
Expenses	97 757		**100 102**
Expense inflation	98 884		**98 976**
Lapses	**98 997**		98 864

Table 5.5 – MVM effect

One can see that for all assumptions except lapses, MVM is the increase of the best estimation level which was to expect.
MVM for lapses is the decrease of the best estimation level. This means that if lapses are lower than their best estimation the deterministic FV is higher, therefore more conservative approach is decreasing the expected lapse rates.
MVM shift of lapse rates is usually affected especially by the value of surrender penalties (higher penalty can cause that higher lapses show higher profit => lower reserve) and general profitability of the product (in a less profitable product higher lapses can cause lower deficits => lower reserve).

Hence, if MVM is applied then the results of the deterministic FV are as in the table 5.6.

Cash flow					
Year	***Premium (BoY)***	***Expenses (BoY)***	***Interest on premium and expenses (EoY)***	***Claims (EoY)***	***CASH FLOW (EoY)***
2004	27 820	3 465	1 218	6 404	***19 169***
2005	26 521	3 349	1 159	7 456	***16 875***
2006	25 278	3 238	1 102	8 459	***14 683***
2007	24 089	3 131	1 048	9 422	***12 583***
2008	22 950	3 029	996	10 346	***10 572***
2009	21 859	2 930	946	241 259	***-221 383***

Table 5.6 – FV calculation if MVM used

This results to a value of present value of cash flows equal to **–**100 318 => the value of liabilities under this deterministic FV estimation is +**100 318**.

Pros and cons of the deterministic FV approach

5.32. Positives:

+ In comparison to Statutory valuation approach, the deterministic estimation of the fair value of liabilities is closer to the reality. Assumptions used are based on their best estimation (and adjusted by MVM). Lapses and all other expected events are included – similar to EV methodology.
+ Comparing to EV approach, slightly less subjectivism is involved when determining interest rate assumptions (risk free rate should be used).

5.33. Negatives:

- Subjectivism in a determination of best estimation of assumptions – the same negatives as for EV methodology.
- Subjectivism in the settling of market value margins. (Similar to subjectivism of risk discount rate determination under the EV approach.)
- No value of options is added when using the deterministic approach.
  Stochastic valuations should be applied (some example is shown in the section 7).
  Sometimes the value of embedded options in the deterministic approach is estimated using adjustments of the interest rate assumptions (e.g. decrease of the discount rate of 25 bps as suggested in [4]). We will discuss this approach further in the section 7.

6. Comparison of the Statutory valuation approach, the Embedded Value and the Fair Value approaches

Comparison of the Statutory, the Embedded Value and the Fair Value and methodologies is included in this section.

- We'll give a schematic overview of several aspects of all of these methodologies first.
- We will then concentrate especially to the comparison of EV and FV techniques in a way of mathematical formulas as well as in a way of examples.
- Finally, we'll present the numerical example which will show the different risk and uncertainty coverage applied under the EV and the FV methodology

6.1. Let's compare several aspects of the valuation methodologies in a schematic way first – see the table 6.1.

	Statutory approach	*Embedded Value*	*Fair Value*
Methodology	Deterministic.	Deterministic. (EEV and MCEV - interest rate options should be evaluated via stochastic simulations)	Should be fully stochastic. Deterministic simplification is sometimes used.
Financial flows to be discounted	Projected cash flows.	Projected gross profits.	Projected cash flows.
Future events included	Death and maturity only. No lapses, no other events.	All expected.	All expected.
Investment return	Technical interest rate.	Best estimation based on assets held.	Risk free rate.
Discount rate	Technical interest rate.	Risk discount rate. (MCEV market consistent rates)	Risk free rate.
Other assumptions *(mortality, lapses, expenses, etc.)*	'First order' level – same as in the premium calculation.	Best estimation level.	Stochastic approach - assumptions are based on the probability distributions taking into account their correlations. Deterministic approach - best estimation adjusted by market value margin.
Risk and uncertainty	Not explicitly.	In the discount rate (risk discount rate)	In assumptions (stochastic or MVMs are applied)
Value of options	No	No (EEV and MCEV yes, through the stochastic valuation)	Through stochastic valuations. (sometimes adjustments of investment returns and/or discount rates is used in the deterministic approach)
Value of liabilities	Direct result from a formula.	Indirectly as the value of statutory reserves minus the present value of gross profits.	Direct result as the expected value of present value of projected cash flows under stochastic (deterministic) scenarios.

Table 6.1 – Schematic comparison of liability valuation methods

6.2. We have described the FV and EV techniques (mainly what financial flows are used and what present values are calculated) in the previous sections. We'll now try to compare the mathematical formulas of both methods.

6.3. When calculating fair value, cash flows defined in the paragraph 5.7 is studied. It is:

$$CF_r = \left(P_{r-1} - C_{r-1} - E_{r-1}\right) * (1 + rfr_{r-t}) - Claims_r ,$$

Present value of such cash flows (being the deterministic estimation of the fair value) has been defined in the paragraph 5.10:

$$FV_t^{\det} = \sum_{r=t+1}^{n} CF_r \cdot DF_{r-t}$$

where

$$DF_{r-t} = \prod_{j=1}^{r-t} \frac{1}{1 + rfr_j}$$

Embedded Value reserve has been defined in the paragraph 4.4 as a difference between the value of the statutory reserve and the present value of future gross profits (PVFP). It is:

$$Res_t^{EV} = Res_t^{Stat} + PSfund_t - PVFP_t = Res_t^{Stat} + PSfund_t - \sum_{r=t+1}^{n} GP_r * DF_{r-t}$$

where

$$GP_r = P_{r-1} - C_{r-1} - E_{r-1} - Claims_r + I_r - \left(Res_r^{Stat} + PSfund_r - Res_{r-1}^{Stat} - PSfund_{r-1}\right)$$

and

$$DF_{r-t} = \prod_{j=1}^{r-t} \frac{1}{1 + rdr_j}$$

Now we will show that if the assumptions for the EV and the FV approach are the same, then the results of the value of liabilities are the same under both techniques.

Using of the same assumptions especially means that:

- mortalities, lapses and expenses for EV reserve calculation are on the same level as for the FV calculations (the best estimation adjusted by MVM) and
- investment returns and discount rates are on the level of risk free rate.

It then is:

$$PVFP_t = \sum_{r=t+1}^{n} GP_r * DF_{r-t} =$$

$$= \frac{GP_{t+1}}{1+rfr_1} + \frac{GP_{t+2}}{(1+rfr_1)\cdot(1+rfr_2)} + ... + \frac{GP_n}{(1+rfr_1)\cdot(1+rfr_2)\cdot ... \cdot(1+rfr_{n-t})} =$$

According to our assumptions, rfr_j are forward risk free rates for the *j*-th year from the valuation date and are used both for a discounting as well as for the investment return assumption.

$$= \frac{CF_{t+1}}{1+rfr_1} + \frac{CF_{t+2}}{(1+rfr_1)\cdot(1+rfr_2)} + ... + \frac{CF_n}{(1+rfr_1)\cdot(1+rfr_2)\cdot ... \cdot(1+rfr_{n-t})}$$

$$- \frac{Res_{t+1}^{Stat} + PSfund_{t+1} - (Res_t^{Stat} + PSfund_t)\cdot(1+rfr_1)}{1+rfr_1}$$

$$- \frac{Res_{t+2}^{Stat} + PSfund_{t+1} - (Res_{t+1}^{Stat} + PSfund_{t+1})\cdot(1+rfr_2)}{(1+rfr_1)\cdot(1+rfr_2)}$$

$$- ... - \frac{Res_n^{Stat} + PSfund_n - (Res_{n-1}^{Stat} + PSfund_{n-1})\cdot(1+rfr_{n-t})}{(1+rfr_1)\cdot(1+rfr_2)\cdot ... \cdot(1+rfr_{n-t})} =$$

It further holds:

$$Res_n^{Stat} = 0$$

and

$$PSfund_n = 0$$

One obtains:

$$= \frac{CF_{t+1}}{1+rfr_1} + \frac{CF_{t+2}}{(1+rfr_1)\cdot(1+rfr_2)} + ... + \frac{CF_n}{(1+rfr_1)\cdot(1+rfr_2)\cdot ... \cdot(1+rfr_{n-t})} + Res_t^{Stat} + PSfund_t =$$

$$= FV_t^{\det} + Res_t^{Stat} + PSfund_t$$

so that,

$$FV_t^{\det} = -Res_t^{Stat} + PSfund_t + \sum_{r=t+1}^{n} GP_r \cdot DF_{r-t}$$

and finally

$$-FV_t^{\det} = Res_t^{Stat} + PSfund_t - \sum_{r=t+1}^{n} GP_r \cdot DF_{r-t} = Res_t^{EV}$$

Notice:
This is a similar remark as in the paragraph 5.31.1.
Negative value of $FV_t^{\det}$ means (due to a CF_t definition) a positive value of the fair value of life liabilities which is the same convention as for Res_t^{EV} and Res_t^{Stat}.

6.4. We have shown that using EV formulas with FV assumptions gives the same results as if FV formulas are used.

6.5. *Example – FV and EV results under the same assumptions:*
We will now show a comparison of the FV reserve and the EV reserve for our *working example:*

MODEL ASSUMPTIONS
(best estimation adjusted by MVMs)

Mortality experience	88%	of mortality charged
Annual lapse rate	4.5%	all the policy period
Discount rate (RFR)	5% p.a.	
Investment return (RFR)	5% p.a	
Expenses		
Initial (1st year)	1650	annual fixed
Renewal	1100	annual fixed
	+ 5.5%	from each premium
Expense inflation	4.4% p.a.	
Commissions		
Initial 1st year	50%	from annual premium
Initial 2nd year	20%	from annual premium
Renewal (starting 3rd year)	3%	from annual premium

a) Cash flows under the deterministic FV approach and their present value are shown in the table 6.2 (same example as in the paragraph 5.31.2).

Cash flow					
Year	***Premium (BoY)***	***Expenses (BoY)***	***Interest on premium and expenses (EoY)***	***Claims (EoY)***	***CASH FLOW (EoY)***
2004	27 820	3 465	1 218	6 404	**19 169**
2005	26 521	3 349	1 159	7 456	**16 875**
2006	25 278	3 238	1 102	8 459	**14 683**
2007	24 089	3 131	1 048	9 422	**12 583**
2008	22 950	3 029	996	10 346	**10 572**
2009	21 859	2 930	946	241 259	**-221 383**

Table 6.2 – Fair Value – cash flow

This results to a value of FV equal to **–**100 318 => the value of liabilities is then +**100 318**.

b) Gross profits under EV approach and the total EV reserve are shown in the table 6.3 and 6.4. (Value of the Statutory reserve is the same as in the example in the paragraph 3.9).

Cash flow						
Year	***Premium (BoY)***	***Expenses and commissions (BoY)***	***Investment return (EoY)***	***Change of reserves (EoY)***	***Claims (EoY)***	***TOTAL GROSS PROFIT (EoY)***
2004	27 820	3 465	6 467	23 390	6 404	**1 029**
2005	26 521	3 349	7 577	22 312	7 456	**982**
2006	25 278	3 238	8 636	21 282	8 459	**935**
2007	24 089	3 131	9 646	20 292	9 422	**890**
2008	22 950	3 029	10 609	19 340	10 346	**844**
2009	21 859	2 930	11 527	-211 602	241 259	**799**

Table 6.3 – Embedded Value methodology – cash flow

Reserve	
Statutory reserve	104 986
PV of gross profit	4 668
TOTAL RESERVE	**100 318**

Table 6.4 – Embedded Value methodology – result

One can see that the results of the value of liabilities are really the same under both approaches.

6.6. *Example – EV and FV comparison (what RDR to use to obtain the same results under both approaches):*

As mentioned above in this section, the main difference between FV and EV deterministic approaches is the level of assumptions used and the way how risk and uncertainty is covered.

We'll show now, on our *working example*, what level of the risk discount rate (the way of risk and uncertainty cover under the EV methodology) has to be assumed in order to obtain the value of the EV reserve on the same level as under the deterministic FV approach.

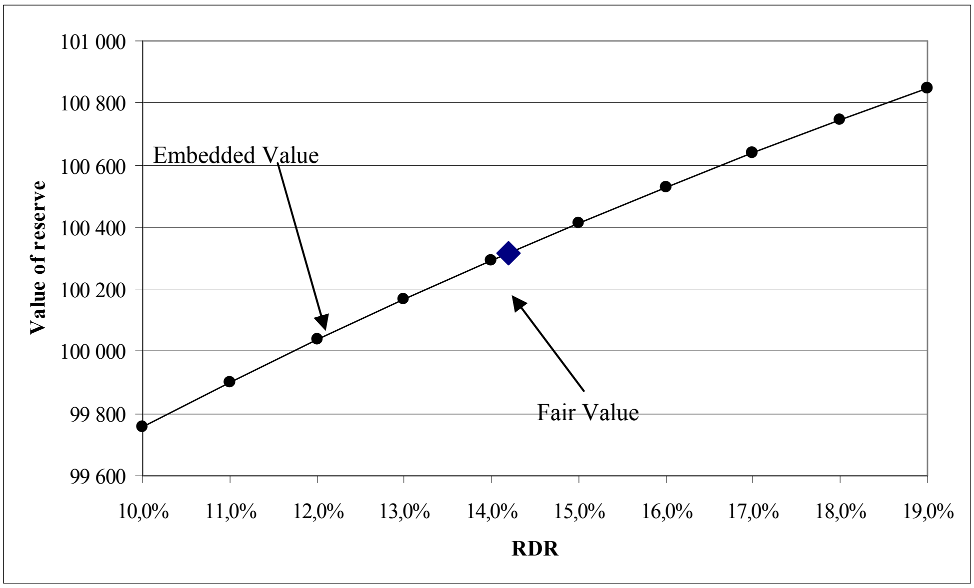
For FV calculation, we will use the best estimate assumptions adjusted by MVM (on the level as suggested by Czech Society of Actuaries – [4]), and the risk free interest rates for investment returns and discount rates. The FV results are then as in the paragraph 5.31.2, where the value of liabilities was +100 318.

We will use the best estimated assumptions without any MVM (as in the paragraph 4.11) for the EV reserve calculation and will try to find the level of risk discount rate (RDR) in order to obtain EV reserve result equal to the FV one.

The results are given in the table 6.5 and the graph 6.1.

RDR	***EV reserve***
10,0%	99 756
11,0%	99 900
12,0%	100 037
13,0%	100 168
14,0%	100 294
14,2%	**100 318**
15,0%	100 414
16,0%	100 530
17,0%	100 640
18,0%	100 747
19,0%	100 849

Table 6.5 – Embedded Value according to RDR



Graph 6.1 – Embedded Value according to RDR

We can conclude that in order to obtain EV reserve of the same value as the FV reserve for our *working example* and under the assumptions set in this paragraph, risk discount rate used for EV valuation should be on the level of 14.2%.

7. Stochastic Fair Value approach

In this section, we will make an effort to calculate the fair value of life liabilities based on stochastic simulations of future interest rates in order to include the value of interest rates options embedded in policy contracts. Other assumptions will remain on their best estimation level adjusted by MVM (same approach as was used in the section 5). Therefore, for the purposes of this text, we assume that risks and uncertainties related to assumptions are covered properly by using the MVMs.

- We'll start with a short methodology description.
- We will then be working on a preparation of 10 000 sets of future interest rates started from the end of the year 2005.
  - We will begin with the theory of short term interest rates and their modeling. Definitions and formulas related to short term interest rates will be reminded.
  - We will then continue with the basic framework of interest rates modeling and discuss practical properties of the models.
    It is not the intention of this work to make a deep description of known models. Only basic ideas and references will be mentioned here in order to introduce the final model for further use as the economic scenarios generator.
  - We'll finally choose a single-factor Hull-White model. We will calibrate it for the December 31, 2005 EUR data and use it for our calculations (10 000 sets of interest rate scenarios will be prepared).
- The description of the liability computation tool and formulas used for the calculation will continue.
- Then we'll run the stochastic FV calculations using the actuarial system *Sophas* and present the results.
- An alternative modeling approach which under certain conditions can be calculated even in not specialized software is added although using professional systems is recommended anyway.
- At the end of this section we will discuss open issues which should further be taken into account, discussed and solved.

Methodology

General formulas

7.1. We understand the fair value of liabilities (FV) to be the expected value of the present values of stochastic simulations of future cash flows (cash flows are defined in the same way as in the paragraph 5.7).

Hence the FV as at the end of the year *t* is expressed as:

$$FV_t = E\{PV_t \mid \mathcal{F}_t\}$$

where
E ..... is the expected value in the risk neutral world
PV_t ... is the present value of future generated cash flows.
$\mathcal{F}_t$ ..... is the filtration expressing the situation in and before the year *t* – see further (e.g. in 7.8).

And, similarly to notation in the section 5 it is:

$$PV_t = \sum_{r=t+1}^{n} CF_r \cdot DF_{r-t}$$

where

$$CF_r = \left(P_{r-1} - C_{r-1} - E_{r-1}\right) \cdot (1 + rfr_{r-t}) - Claims_r$$

and

$$DF_{r-t} = \prod_{j=1}^{r-t} \frac{1}{1 + rfr_j} .$$

Risk and uncertainty

7.2. The FV should cover risks and uncertainties related to future cash flow evolution to a certain level. The level of such a cover still is not finally defined and is under a wide international discussion.

7.3. Several ways how to cover risks and uncertainties in the calculations are possible to use. E.g.:

a) Adjusting the assumptions used (e.g. using market value margin (MVM) to adjust the best estimated level of assumptions). The discount rate is then assumed at the risk free rate level.
b) Using best estimates of assumptions and adjusting the discount rates (similar to EV approach).
c) Mix of a) and b).

We will use the approach a) in the rest of this text.
It especially means that assumptions like mortality, lapses, expenses, inflation, etc. will not be simulated stochastically and remain on the same level as was used in the deterministic calculations before (i.e. best estimate + MVM). MVM will be assumed to be on the level as suggested in [4] (see also the paragraphs 5.30 a) or) 5.31.2).

We therefore assume that MVM sufficiently cover the risk and uncertainty of these parameters for the purposes of our work.

Interest rate options

7.4. In order to price the interest rate options (technical interest rate guarantee and a profit sharing) usually embedded in policy contracts we will:

- simulate 10000 sets of interest rates.
  Each set is based on one generated trajectory of r_t and 2 types of future interest rates were calculated:
  1. the first interest rates will be assumed to be future annual investment returns (we have chosen 1Y and 5Y zero rates – see further in 7.16) and
  2. the second one will be used for discounting (future 1Y rates).
- Then we will calculate the present value of future cash flows (assuming simulated investment return and using simulated discount rates) for each of the set of interest rates.
- FV will be then determined as the expected value (mean) of the results of such present values (as defined in 7.1).

7.5. Notice that:

- rfr_{r-t} in the formulas in n 7.1 are the discount rates related to the policy year *r* (*r* – *t* years from the valuation date)
- investment returns relevant to the policy year *r* are included in the cash flow $Claims_r$, since they affect the value of the final profit share paid together with the guaranteed benefit in case of death, maturity or surrender.

7.6. In order to be able to run our stochastic fair value calculations we will need:

- interest rate scenarios (used as investment returns and discount rates)
- description of the product type what will be modeled
- specify the policy contracts to be modeled (modelpoints)
- set assumptions
- build liability modeling tool and the cash flow model

Interest rates scenarios

Short term interest rate

7.7. Basic definitions and related formulas:

The simplest interest rate contract is to pay some money today in exchange to receive a different amount (usually larger) later. This moment in the future is called maturity and we will denote it by T. We can regard a promise to receive one dollar at time T as an asset – a risk-free discount bond, which can be evaluated during the life of such contract. We will denote its price at time t by $P(t,T)$. It is clear from the definition that $P(T,T) = 1$, i.e. the price of the bond is one dollar on its maturity. Suppose that the continuously compounded interest rates are constant at rate $R(t,T)$.

Then we have

$$P(t,T) = \exp\{-R(t,T)\cdot(T-t)\}, \qquad t \in [0,T]$$

and therefore

$$R(t,T) = -\frac{\log P(t,T)}{T-t}, \qquad t \neq T\ ,$$

where log is a natural logarithm function.

R(t,T) can be viewed as an average interest rate offered by the bond until its maturity and is a function of time and maturity. If we fix t and look at $R(t,T)$ as a function of maturity it represents a yield curve at time t.

It is useful to derive a single number r_t representing the current rate of interest for $T \to t_+$.

We obtain

$$r_t = \lim_{T\to t+} R(t,T) = \lim_{T\to t+} -\frac{\log P(t,T)}{T-t} = -\frac{\partial}{\partial T}\log P(t,T)$$

We will call r_t the instantaneous rate or the short rate. The processes $R(t,T)$ and $P(t,T)$ can be derived one from the other (the one-to-one mapping is shown in the equation above). The process r_t is not one-to-one mapping of neither $P(t,T)$ nor $R(t,T)$ and therefore none of these processes can be recovered from r_t without any additional information.

Consider now a forward bond contract. It is an agreement at time t to pay some money at the future date T_1 and to receive a different amount at even later moment T_2. The forward price of a bond maturing at T_2 must be $P(t, T_2)/P(t, T_1)$, otherwise there exists an arbitrage opportunity.

Suppose the forward price $x< P(t, T_2)/P(t, T_1)$. Consider the following strategy: At the present moment (time t) an investor sells one T_2-bond and simultaneously buys $P(t,T_2)/P(t,T_1)$ units of a T_1-bond. He also buys a forward contract to buy T_2-bond at time T_1 for x dollars, which has nil value at time t. The initial value of this operation is zero:

$$P(t,T_2) - \frac{P(t,T_2)}{P(t,T_1)} P(t,T_1) = 0 .$$

Then at the moment T_1 one of the bonds matures and the investor receives $P(t,T_2)/P(t, T_1)$ dollars. For this amount he buys T_2-bond at the agreed price of x. As a result he owns more than one T_2-bond:

$$y = \frac{P(t,T_2)}{x \cdot P(t,T_1)} > 1 .$$

When the maturity of this bond comes, the investor has to pay one dollar for the bond he has sold at the beginning. On the other side he receives y dollars from the bonds he has bought. As a result he realizes a profit of y-1 dollars with zero initial investment. It can be shown that there is an arbitrage opportunity also when $x > P(t, T_2)/P(t, T_1)$. Therefore the forward price to buy a T_2-bond at time T_1 must be $P(t, T_2)/P(t, T_1)$.

The forward yield for period from T_1 to T_2 is

$$F(t,T_1,T_2) = -\frac{\log P(t,T_2) - \log P(t,T_1)}{T_2 - T_1}, \qquad t < T_1 < T_2 .$$

And the forward instantaneous rate at the time T is

$$f(t,T) = \lim_{\Delta t \to 0+} -\frac{\log P(t,T+\Delta t) - \log P(t,T)}{\Delta t} = -\frac{\partial}{\partial T} \log P(T,t) .$$

And equivalently

$$P(t,T) = \exp\{-\int_t^T f(t,z)dz\}, \qquad t \in [0,T]$$

It is clear that

$$f(t,t) = r_t .$$

The reader can find more details e.g. in [1].

7.8. Basic framework of the short term interest rate models:

Let $\{W_t, t \geq 0\}$ be a Wiener process. Then we will call $\{\mathcal{F}_t, t \geq 0\}$ a filtration generated by this process (i.e. W_t is measurable with respect to $\mathcal{F}_t$ for all *t*).
One possible way of how to model the term structure of interest rates is to model short rate process $\{r_t\}$ defined above in 7.7. It is assumed that this process follows a stochastic differential equation (SDE), which in general is of the form:

$$dr_t = \theta(t, r_t)dt + \sigma(t, r_t)dW_t, \qquad t \in [0, T].$$

This equation is the equivalent transcription of the form:

$$r_t = r_0 + \int_0^t \theta(u, r_u)du + \int_0^t \sigma(u, r_u)dW_u \ .$$

The stochastic process $\{r_t\}$ fulfilling this equation is the solution of the SDE. The bond prices are given by

$$P(t,T) = E[\ \exp\{-\int_t^T r_u du\} \mid \mathcal{F}_t\ ].$$

If the solution $\{r_t\}$ is unique, this process is also a Markov process. Then we can write

$$P(t,T) = E[\ \exp\{-\int_t^T r_u du\} \mid r_t\ ],$$

where E denotes expected value in risk-neutral world. And for the pricing at time *t* of the arbitrary derivative security that produces a single payment of *X* at maturity date *T*:

$$V(t) = E[\ X \exp\{-\int_t^T r_u du\} \mid r_t\ ],$$

where again E denotes expectation with respect to risk neutral measure.

The reader can find such information e.g. in [21] or [24].

7.9. Scenario generating:

Our task now is to generate interest rate scenarios which should describe possible but uncertain development of future economic environment.
Generally, when modeling such uncertainties, scenarios are assumed to be the atoms of the discrete probability distribution used to approximate the underlying probability distribution which describes the uncertainty.
The main recognized random factor which drives the prices and returns of bonds (=> interest rates) is the evolution of the short term interest rates.
There are many known parametric models for short term interest rate modeling.
The reader can find a good text e.g. in [3], [5], [7], [8], [9], [10], [11], [12], [22], [24] and [26].
Here, we will only discuss the properties of such models that would be practical and finally, we'll choose one of the models what will be used for further calculations.

7.10. Practical properties of interest rate models:

We assume the practical properties which should be displayed by the interest rate model to be the following:

1. The model should be flexible and simple as much as possible in order to be able to be prepared and run in a reasonable time period.
2. It should be well specified so that it could be calibrated using observable inputs; moreover good fit to data should be possible.
3. Consistence with the economic theory is expected.
4. The results should be realistic; negative or large values of interest rates are not relevant, hence, models which display the mean reversion property are considered more appropriate.

In order to fill in the most of the above-mentioned conditions we finally choose the single-factor Hull-White (HW) model.
Since HW model:

1. is easy.
2. is possible to calibrate either using historical data (if they are consistent) or current market data (implied volatilities of options).
3. is arbitrage free – the analytic formula for the risk free bond price even exists.
4. is mean reverting.

The negative of this model is that less than zero values of interest rates could occur with small but non-zero probability.

Since none of the known model fulfills perfectly all the above-stated conditions and we believe that positives of HW models predominate over its negatives, we finally will make use it for further calculations. Any negative interest rates were replaced by zero.

Single-factor Hull White model

7.11. Introduction:

Single-factor HW model (HW model) was proposed in 1990 by Hull & White and is sometimes referred to as extended Vasicek model (see [9]).
HW model is described by the following stochastic differential equation (SDE):

$$dr_t = (\theta_t - ar_t)dt + \sigma\, dW_t\,,$$

using the same notation as in the paragraphs above.

This model is an extension of both Ho-Lee and Vasicek model.

Notice
SDE of Ho-Lee model is: $dr_t = \theta_t dt + \sigma\, dW_t$
Vašiček SDE formula is: $dr_t = (\theta - ar_t)dt + \sigma\, dW_t$
HW model adds mean reversion feature to the former and the time dependent reversion level to the latter.

If θ_t satisfies the equation:

$$\theta_t = \frac{\partial}{\partial T} f(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})\,,$$

then the model fits the current structure of interest rates with given a and σ.

Authors (Hull & White) also suggest using trinomial trees for calibration purposes (parameters a and σ are to be find) – see [10] and [11].

7.12. Practical issues:

We'll show only a short description of this model (focused on practical issues of generating scenarios) without an appropriate formula derivation in this work.
We refer to the text in [9], [10] and [11] and make use of very good personal comments of Jan Šrámek on this topic.

Basic formulas and ideas are:

- The basic SDE is:

$$dr_t = (\theta_t - ar_t)dt + \sigma\, dW_t\,.$$

- r_t could be derived as:

$$r_t = e^{-a(t-s)} r_s + e^{-at} \int_s^t e^{ax} \theta(x) dx + \sqrt{\frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})} \varepsilon \ ,$$

where *0 < s < t* and *ε* ~ N(0,1).

- Setting:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

and

$$A(t,T) = \int_t^T \left(\frac{\sigma^2}{2} B^2(x,T) - \theta(x) B(x,T) \right) dx$$

it is possible to express the bond price at time *t* (with maturity at time *T*) as:

$$P(t,T) = e^{A(t,T) - B(t,T) r_t}$$

and the corresponding investment return (continuously compounded) as:

$$Y(t,T) = \frac{B(t,T)}{T-t} r_t - \frac{A(t,T)}{T-t} \ .$$

- Using arbitrage-free formula for θ_t:

$$\theta_t = \frac{\partial}{\partial T} f(0,t) + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

it is possible to write the simulation equation for r_t as:

$$r_t = e^{-a(t-s)} r_s + f(0,t) - e^{-a(t-s)} f(0,s) + \frac{\sigma^2}{2a^2}\left(1 - e^{-a(t-s)} + e^{-2at} - e^{-a(t+s)}\right) + \sqrt{\frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})} \varepsilon$$

and $A(t,T)$ is possible to rewrite into more practical form as:

$$A(t,T) = \log \frac{P(0,T)}{P(0,t)} + B(t,T) f(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2at}) B^2(t,T)$$

Now $Y(t,T)$ is possible to express in a practical form (for simulating purposes) as well.

- Finally, especially for *s*=*t*-1, it is

$$r_t = e^{-a} r_{t-1} + f(0,t) - e^{-a} f(0,t-1) + \frac{\sigma^2}{2a^2}\left(1 - e^{-a} + e^{-2at} - e^{-a(2t-1)}\right) + \sqrt{\frac{\sigma^2}{2a}(1 - e^{-2a})}\varepsilon$$

- For calibration, trinomial tree is used, as suggested by Hull and White (see [10] and/or [11]).
  The best fit of the value of floor options evaluated by the HW tree to the market value of these options were used. Market value of such options is calculated using Black-Scholes formula and based on the corresponding market volatilities.

Interest rate scenarios as at December 31, 2005 EUR data using HW model

7.13. Methodology:

- We examined the model for the December 31, 2005 EUR data.
- We have used the EUR interest rate swap par rates (mid) as the market risk free yield curve.
- Calibration was done using trinomial tree (see [10] and/or [11]).
- Best fit (minimizing of the square deviations) of the value of floor options with maturities 2, 3, 4, 5, 7, and 10 years evaluated by the HW tree to the market value of these options were used. Market value of such options was calculated using Black-Scholes formula and based on the corresponding market volatilities.
- We make use of MS Excel spreadsheet model originally prepared by Jan Šrámek for generating and calibrating the HW model with his kind approval. We have checked it and adjusted to our purposes.

7.14. Inputs:

Inputs to our interest rate model are:
1) EUR interest rate swap par rates (mid):
2) Floor options volatilities (mid)

Ad1) EUR interest rate swap par rates (mid) are shown in the table 7.1.

Maturity (years) (t)	***EUR IRS par rate (mid) (IRS_t)***	***cont. (t)***	***(IRS_t)***
1	2,86%	***16***	3,70%
2	3,03%	***17***	3,72%
3	3,11%	***18***	3,75%
4	3,16%	***19***	3,77%
5	3,21%	***20***	3,79%
6	3,26%	***21***	3,80%
7	3,31%	***22***	3,81%
8	3,36%	***23***	3,82%
9	3,41%	***24***	3,83%
10	3,46%	***25***	3,83%
11	3,51%	***26***	3,83%
12	3,56%	***27***	3,83%
13	3,60%	***28***	3,83%
14	3,64%	***29***	3,83%
15	3,68%	***30***	3,83%

Table 7.1 – EUR interest rate swap pare rates (mid)

We have fitted this IRS par rates using Nelson-Siegel approach (see [23]) and calculated corresponding spot rates.

Nelson-Siegel formula for spot continuously compounded yield curve is in a form:

$$Y(0,t) = \beta_0 + (\beta_1 + \beta_2)\frac{(1-e^{-\frac{t}{\gamma}})}{-\frac{t}{\gamma}} - \beta_2 e^{-\frac{t}{\gamma}} \text{ for } t > 0 \text{ and } t \leq 30.$$

In order to fit the coupon rate determined from the spot rate modeled (using Nelson-Siegel) to market IRS data as at December 31, 2005 we obtain the following estimates of parameters:

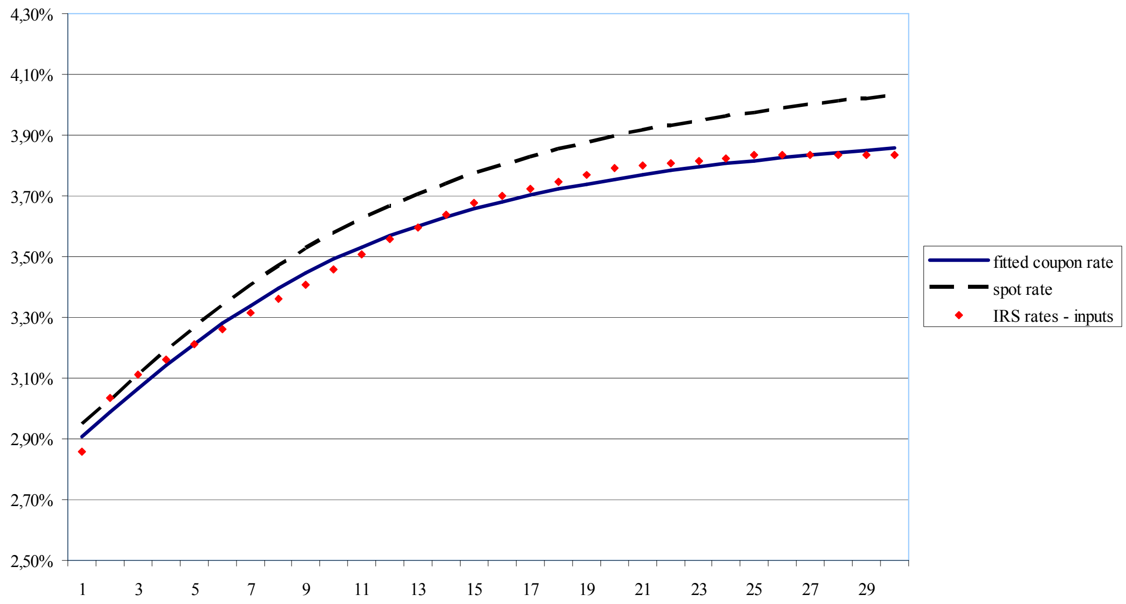
- $\hat{\beta}_0$=0.041 825
- $\hat{\beta}_1$= – 0.013 870
- $\hat{\beta}_2 = -0.008\ 893$
- $\hat{\gamma} = 3.530\ 323$

Results of observed IRS rates , fitted coupon rate and corresponding zero rates are shown in the table 7.2.

t	*IRS t* *31.12.2005 - mid*	*Coupon rate (fitted)*	*Zero rate*
1	2,86%	2,91%	2,95%
2	3,03%	2,99%	3,03%
3	3,11%	3,06%	3,11%
4	3,16%	3,14%	3,19%
5	3,21%	3,21%	3,27%
6	3,26%	3,28%	3,34%
7	3,31%	3,34%	3,41%
8	3,36%	3,39%	3,47%
9	3,41%	3,45%	3,53%
10	3,46%	3,49%	3,58%
11	3,51%	3,53%	3,62%
12	3,56%	3,57%	3,67%
13	3,60%	3,60%	3,71%
14	3,64%	3,63%	3,74%
15	3,68%	3,66%	3,77%
16	3,70%	3,68%	3,80%
17	3,72%	3,70%	3,83%
18	3,75%	3,72%	3,85%
19	3,77%	3,74%	3,87%
20	3,79%	3,75%	3,89%
21	3,80%	3,77%	3,91%
22	3,81%	3,78%	3,93%
23	3,82%	3,79%	3,95%
24	3,83%	3,81%	3,96%
25	3,83%	3,82%	3,97%
26	3,83%	3,83%	3,99%
27	3,83%	3,83%	4,00%
28	3,83%	3,84%	4,01%
29	3,83%	3,85%	4,02%
30	3,83%	3,86%	4,03%

Table 7.2 – IRS, coupon rate and zero rate as at Dec 31, 2005

And the same results in a graphical form are shown in the graph 7.1.



Graph 7.1 – IRS, coupon rate and zero rate as at Dec 31, 2005

Now we are able to calculate and generate the short interest rate r_t following the formula:

$$r_t = e^{-a} r_{t-1} + f(0,t) - e^{-a} f(0,t-1) + \frac{\sigma^2}{2a^2}\left(1 - e^{-a} + e^{-2at} - e^{-a(2t-1)}\right) + \sqrt{\frac{\sigma^2}{2a}(1-e^{-2a})}\varepsilon ,$$

where $f(0,t)$ follows the Nelson-Siegel formula

$$f(0,t) = \beta_0 + \beta_1 e^{-t/\gamma} + \beta_2 \frac{t}{\gamma} e^{-t/\gamma} .$$

Then we are able to calculate:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

and

$$A(t,T) = \log \frac{P(0,T)}{P(0,t)} + B(t,T) f(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2at}) B^2(t,T) ,$$

hence, finally the investment return may be obtained as

$$Y(t,T) = \frac{B(t,T)}{T-t} r_t - \frac{A(t,T)}{T-t} .$$

The next thing we still need to do is to calibrate the HW model, i.e. to set a and σ parameters in order to fit market price of floor options.

Ad2) Floor options volatilities:

Floor options volatilities have been used to calibrate HW model (to find the parameters a and σ).
Floor options' (2, 3, 4, 5, 7, 10 Y) market prices have been calculated based on quoted market volatilities and using Black-Scholes formula.

The relevant volatilities (mid) related to EUR floor options are given in the table 7.3.

Duration	***2Y***	***3Y***	***4Y***	***5Y***	***7Y***	***10Y***
Volatility	18,7	20,1	20,5	20,6	20,4	19,7

Table 7.3 – EUR floor options volatilities (mid) as at December 31, 2005

7.15. Calibration results:

Applying above mentioned inputs and the methodology (fitting the value of options calculated by trinomial tree to their value determined by Black-Scholes formula) the estimations of a and σ are

- $\hat{a} = 0.007\ 675\ 918$
- $\hat{\sigma} = 0.006\ 784\ 426$.

7.16. Results of the interest rates further used for stochastic valuations:

We will study 2 versions of the investment return assumptions as mentioned in 7.4. Both will express the level of the 'reasonable policyholder expectation'.[2]

A) investment return is assumed to be at the level of 1Y zero rate
B) investment return is be assumed to be at the level of 5Y zero rate

Future 1Y zero rates will be used for discounting in both cases (A and B).

[2] We assumed that the profit share is based on the investment return what is expected by policyholders. We suppose the company is (e.g. from the competition reasons) 'obliged' to pay such interest rates. This 'obligation' is sometimes referred as a *constructive obligation*.

We have generated 10 000 sets of the interest rates:

- 1Y rate in the case A used for investment returns assumption as well as for discounting
- 5Y zero rates assumed to be investment returns and 1Y rate used for discounting in the case B.

One set of interest rates is the set of investment returns and discount rates as at the end of the years 2005, 2006, …, 2032 based on the same r_t trajectory.

Thus one set of interest rates has a structure as shown in the table 7.4.

	investment return	*discount rate*
2005	2,95	3,27
2006	3,34	3,61
2007	4,25	4,42
…	…	
2032	5,12	5,31

Table 7.4 – example of a one set of interest rates

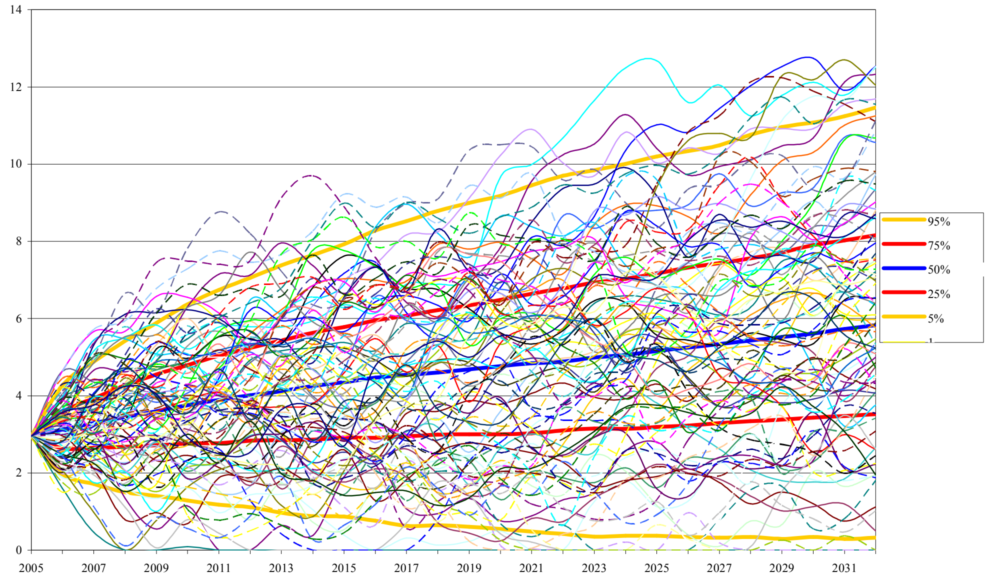
Of course, in the case A, the investment returns are equal to the discount rates (both are 1Y rates). In the case B, the two columns are generally different.

AdA) Results when policyholder expectations is on the level of 1Y rate

We generated 10 000 sets of 1Y zero rates ($Y(t,1)$ in our notation) at the end of the years 2005, 2006, …, 2032.

Following tables and graphs show the results (some of the percentiles) of 1Y rates and their comparison with 1Y forward rates derived from the current yield curve (i.e. EUR yield curve as at December 31, 2005).

Graph 7.2 shows the result of the first 100 simulations (for illustrative purposes) where 5%, 25%, 50%, 75% and 95% percentiles are presented especially.



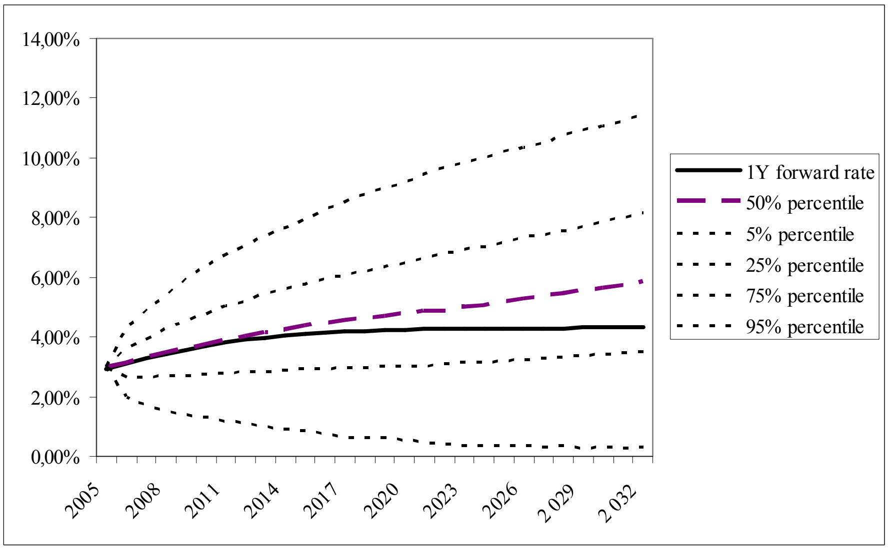
Graph 7.2 – 1Y rate simulations

Table 7.5 shows a comparison of some of the percentiles with 1Y forward rate.

Year	*1Y fw rate*	*50% percentile*	*5% percentile*	*25% percentile*	*75% percentile*	*95% percentile*
2005	**2,95%**	**2,95%**	2,95%	2,95%	2,95%	2,95%
2006	**3,11%**	**3,13%**	2,02%	2,67%	3,59%	4,28%
2007	**3,28%**	**3,31%**	1,74%	2,67%	3,96%	4,95%
2008	**3,43%**	**3,49%**	1,53%	2,69%	4,29%	5,49%
2009	**3,58%**	**3,66%**	1,42%	2,74%	4,60%	5,97%
2010	**3,70%**	**3,81%**	1,33%	2,77%	4,84%	6,37%
2011	**3,81%**	**3,93%**	1,17%	2,81%	5,10%	6,75%
2012	**3,90%**	**4,08%**	1,13%	2,84%	5,30%	7,08%
2013	**3,98%**	**4,15%**	1,03%	2,86%	5,51%	7,40%
2014	**4,04%**	**4,28%**	0,94%	2,94%	5,68%	7,71%
2015	**4,09%**	**4,37%**	0,87%	2,97%	5,88%	7,98%
2016	**4,14%**	**4,47%**	0,85%	2,99%	6,00%	8,28%
2017	**4,17%**	**4,52%**	0,77%	3,00%	6,15%	8,55%
2018	**4,20%**	**4,64%**	0,74%	2,99%	6,29%	8,77%
2019	**4,22%**	**4,72%**	0,60%	3,03%	6,39%	8,99%
2020	**4,24%**	**4,78%**	0,62%	3,06%	6,57%	9,20%
2021	**4,25%**	**4,89%**	0,64%	3,05%	6,70%	9,34%
2022	**4,27%**	**4,95%**	0,60%	3,09%	6,82%	9,52%
2023	**4,27%**	**5,03%**	0,52%	3,16%	6,95%	9,78%
2024	**4,28%**	**5,12%**	0,46%	3,20%	7,09%	9,98%
2025	**4,29%**	**5,20%**	0,41%	3,22%	7,24%	10,21%
2026	**4,29%**	**5,28%**	0,47%	3,25%	7,34%	10,31%
2027	**4,30%**	**5,37%**	0,45%	3,30%	7,48%	10,48%
2028	**4,30%**	**5,46%**	0,41%	3,37%	7,61%	10,70%
2029	**4,30%**	**5,58%**	0,45%	3,40%	7,71%	10,88%

Table 7.5 – 1Y forward rate and some of 1Y rate percentiles

Comparison of the percentiles and the 1Y forward rate also shows the graph 7.3.



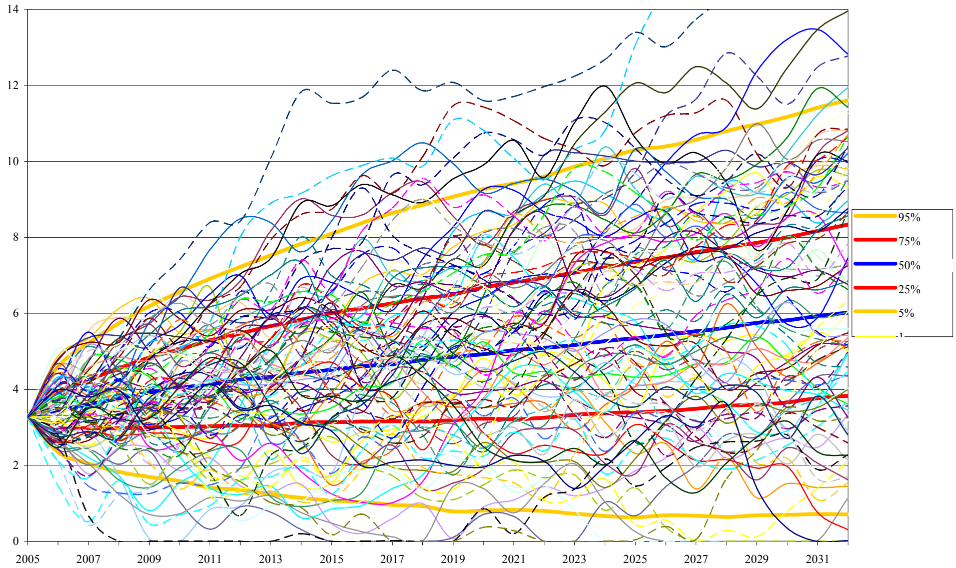
Graph 7.3 – 1Y forward rate and some of 1Y rate percentiles

AdB) Results when policyholder expectations is on the level of 5Y zero rate

In this case, we have generated 10 000 sets of 1Y and 5Y zero rates at the end of the years 2005, 2006, …, 2032. 5Y rate will be used as the investment return assumption, 1Y rate will be used for a discounting.

Following tables and graphs show the results (some of the percentiles) of 5Y rates and their comparison with 5Y forward rates derived from the current yield curve (i.e. EUR yield curve as at December 31, 2005).

Graph 7.4 shows the result of first 100 simulations of 5Y zero rates (for illustrative purposes) where 5%, 25%, 50%, 75% and 95% percentiles are presented especially.



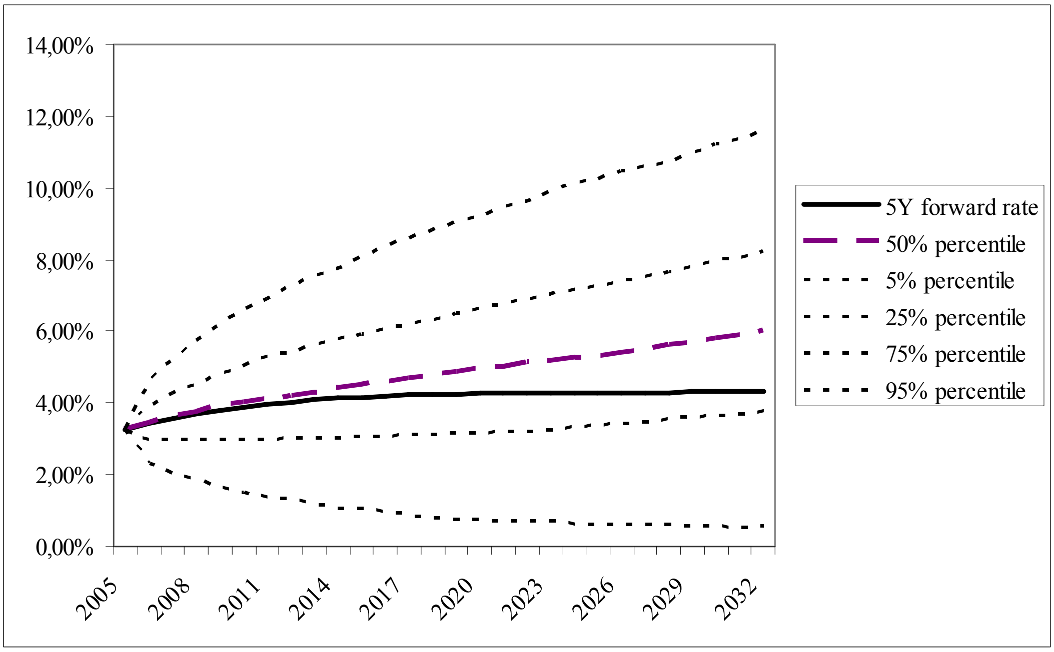
Graph 7.4 – 5Y rate simulations

Table 7.6 shows a comparison of some of the percentiles with 5Y forward rate.

Year	***5Y fw rate***	***50% percentile***	***5% percentile***	***25% percentile***	***75% percentile***	***95% percentile***
2005	**3,27%**	**3,27%**	3,27%	3,27%	3,27%	3,27%
2006	**3,42%**	**3,45%**	2,35%	2,99%	3,90%	4,59%
2007	**3,56%**	**3,61%**	2,05%	2,98%	4,25%	5,22%
2008	**3,68%**	**3,76%**	1,83%	2,98%	4,55%	5,74%
2009	**3,79%**	**3,91%**	1,70%	3,00%	4,84%	6,19%
2010	**3,89%**	**4,04%**	1,59%	3,01%	5,05%	6,57%
2011	**3,96%**	**4,13%**	1,41%	3,03%	5,29%	6,91%
2012	**4,03%**	**4,26%**	1,36%	3,05%	5,47%	7,23%
2013	**4,08%**	**4,32%**	1,24%	3,05%	5,67%	7,53%
2014	**4,13%**	**4,44%**	1,15%	3,12%	5,83%	7,83%
2015	**4,16%**	**4,52%**	1,07%	3,14%	6,01%	8,09%
2016	**4,19%**	**4,61%**	1,04%	3,15%	6,13%	8,38%
2017	**4,22%**	**4,66%**	0,96%	3,16%	6,27%	8,64%
2018	**4,24%**	**4,78%**	0,93%	3,15%	6,40%	8,85%
2019	**4,25%**	**4,86%**	0,79%	3,19%	6,50%	9,07%
2020	**4,26%**	**4,92%**	0,81%	3,23%	6,68%	9,27%
2021	**4,27%**	**5,03%**	0,83%	3,21%	6,81%	9,42%
2022	**4,28%**	**5,09%**	0,80%	3,26%	6,94%	9,60%
2023	**4,29%**	**5,17%**	0,73%	3,33%	7,07%	9,86%
2024	**4,29%**	**5,27%**	0,67%	3,37%	7,21%	10,07%
2025	**4,30%**	**5,35%**	0,63%	3,40%	7,36%	10,30%
2026	**4,30%**	**5,43%**	0,69%	3,43%	7,47%	10,39%
2027	**4,30%**	**5,53%**	0,68%	3,49%	7,61%	10,57%
2028	**4,30%**	**5,62%**	0,64%	3,57%	7,74%	10,80%
2029	**4,30%**	**5,75%**	0,69%	3,60%	7,85%	10,98%

Table 7.6 – 5Y forward rate and some of 5Y rate percentiles

Comparison of some of the percentiles and the 5Y forward rate shows the graph 7.5.



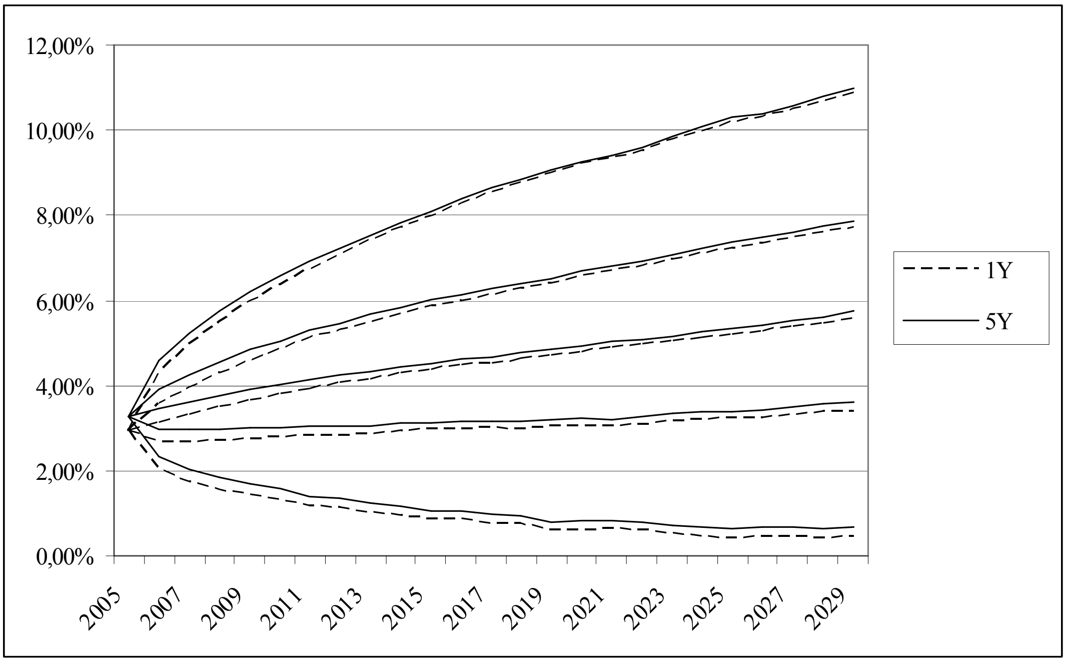
Graph 7.5 – 5Y forward rate and some of 5Y rate percentiles

We add the comparison of 1Y and 5Y rates simulated together in the table 7.7.

Year	*forward rate*		*50% percentile*		*5% percentile*		*25% percentile*		*75% percentile*		*95% percentile*	
	1Y	*5Y*	*1Y*	*5Y*	*1Y*	*5Y*	*1Y*	*5Y*	*1Y*	*5Y*	*1Y*	*5Y*
2005	**2,95%**	**3,27%**	2,95%	3,27%	2,95%	3,27%	2,95%	3,27%	2,95%	3,27%	2,95%	3,27%
2006	**3,11%**	**3,42%**	3,13%	3,45%	2,02%	2,35%	2,67%	2,99%	3,59%	3,90%	4,28%	4,59%
2007	**3,28%**	**3,56%**	3,31%	3,61%	1,74%	2,05%	2,67%	2,98%	3,96%	4,25%	4,95%	5,22%
2008	**3,43%**	**3,68%**	3,49%	3,76%	1,53%	1,83%	2,69%	2,98%	4,29%	4,55%	5,49%	5,74%
2009	**3,58%**	**3,79%**	3,66%	3,91%	1,42%	1,70%	2,74%	3,00%	4,60%	4,84%	5,97%	6,19%
2010	**3,70%**	**3,89%**	3,81%	4,04%	1,33%	1,59%	2,77%	3,01%	4,84%	5,05%	6,37%	6,57%
2011	**3,81%**	**3,96%**	3,93%	4,13%	1,17%	1,41%	2,81%	3,03%	5,10%	5,29%	6,75%	6,91%
2012	**3,90%**	**4,03%**	4,08%	4,26%	1,13%	1,36%	2,84%	3,05%	5,30%	5,47%	7,08%	7,23%
2013	**3,98%**	**4,08%**	4,15%	4,32%	1,03%	1,24%	2,86%	3,05%	5,51%	5,67%	7,40%	7,53%
2014	**4,04%**	**4,13%**	4,28%	4,44%	0,94%	1,15%	2,94%	3,12%	5,68%	5,83%	7,71%	7,83%
2015	**4,09%**	**4,16%**	4,37%	4,52%	0,87%	1,07%	2,97%	3,14%	5,88%	6,01%	7,98%	8,09%
2016	**4,14%**	**4,19%**	4,47%	4,61%	0,85%	1,04%	2,99%	3,15%	6,00%	6,13%	8,28%	8,38%
2017	**4,17%**	**4,22%**	4,52%	4,66%	0,77%	0,96%	3,00%	3,16%	6,15%	6,27%	8,55%	8,64%
2018	**4,20%**	**4,24%**	4,64%	4,78%	0,74%	0,93%	2,99%	3,15%	6,29%	6,40%	8,77%	8,85%
2019	**4,22%**	**4,25%**	4,72%	4,86%	0,60%	0,79%	3,03%	3,19%	6,39%	6,50%	8,99%	9,07%
2020	**4,24%**	**4,26%**	4,78%	4,92%	0,62%	0,81%	3,06%	3,23%	6,57%	6,68%	9,20%	9,27%
2021	**4,25%**	**4,27%**	4,89%	5,03%	0,64%	0,83%	3,05%	3,21%	6,70%	6,81%	9,34%	9,42%
2022	**4,27%**	**4,28%**	4,95%	5,09%	0,60%	0,80%	3,09%	3,26%	6,82%	6,94%	9,52%	9,60%
2023	**4,27%**	**4,29%**	5,03%	5,17%	0,52%	0,73%	3,16%	3,33%	6,95%	7,07%	9,78%	9,86%
2024	**4,28%**	**4,29%**	5,12%	5,27%	0,46%	0,67%	3,20%	3,37%	7,09%	7,21%	9,98%	10,07%
2025	**4,29%**	**4,30%**	5,20%	5,35%	0,41%	0,63%	3,22%	3,40%	7,24%	7,36%	10,21%	10,30%
2026	**4,29%**	**4,30%**	5,28%	5,43%	0,47%	0,69%	3,25%	3,43%	7,34%	7,47%	10,31%	10,39%
2027	**4,30%**	**4,30%**	5,37%	5,53%	0,45%	0,68%	3,30%	3,49%	7,48%	7,61%	10,48%	10,57%
2028	**4,30%**	**4,30%**	5,46%	5,62%	0,41%	0,64%	3,37%	3,57%	7,61%	7,74%	10,70%	10,80%
2029	**4,30%**	**4,30%**	5,58%	5,75%	0,45%	0,69%	3,40%	3,60%	7,71%	7,85%	10,88%	10,98%

Table 7.7 – Comparison of 1Y and 5Y rates

And the comparison of percentiles relate to 1Y resp. to 5Y are shown in the graph 7.6.



Graph 7.6 – Comparison of percentile of 1Y and 5Y rates

7.17. Now we have prepared 2 versions (A and B – different investment returns – different policyholders' expectations – 1Y rate or 5Y rate) of 10000 sets of interest rates which will be used in our calculations.

Where are we? See 7.6.

7.18. In order to be able to run our fair value calculations we will need:

- ~~interest rate scenarios (used as investment returns and for discounting)~~
- description of the product type what will be modeled
- specify the policy contracts to be modeled (modelpoints)
- set assumptions
- build liability modeling tool and the cash flow model

7.19. We will now continue with a description of the insurance product type what will be modeled.

Product modeled

7.20. We will make our calculations on a typical endowment policy paid regularly or single with the following features:

- agreed fixed sum assured (*SA*) plus the value of the profit share fund is paid in case of death during policy period
- fix sum assured plus the value of the profit share fund in case of survival all the policy period is paid; sum assured is the same as the sum assured in case of death
- surrender value (equals to the part of the accounting statutory reserve (brutto reserve using zillmerization in our case)) plus the value of the profit share fund at the lapse time; surrender charge ($SurrCh$) is applied;
- the policyholder has right to have the surrender value from the 3rd policy year
- technical interest rate guarantee is applied
- all profit share is accumulated in the special profit share fund
- technical interest rate is guaranteed even for profit share fund
- $mfee$ is applied – $mfee$ is the management fee taken as the company margin out of the investment surplus
- product charges structure is:
  $\alpha = 5\%$
  $\beta = 0.4\%$
  $\gamma = 6\%$
  $mfee = 15\%$ and
  $SurrCh = 3\%$
- netto premium for a unit of sum assured is calculated using a standard formula:

  - regular:

$$Netto\_premium = \frac{A_{xn}}{\ddot{a}_{xn}}$$

  - single:

$$Netto\_premium = A_{xn}$$

where

$$A_{xn} = \frac{d_x \cdot v + d_{x+1} \cdot v^2 + d_{x+2} \cdot v^3 + ... + d_{x+n-1} \cdot v^n + l_{x+n} \cdot v^n}{l_x}$$

and

$$\ddot{a}_{xn} = \frac{l_x + l_{x+1} \cdot v + l_{x+2} \cdot v^2 + ... + l_{x+n-1} \cdot v^{n-1}}{l_x},$$

l_x (resp. d_x) are the number of lives at the age of *x* (resp. number of deaths in the age of x) according to life tables and $v = (1 + tir)^{-1}$.

- brutto premium is then derived as:
  - regular:

$$brutto\_premium = \frac{(netto\_premium + \frac{\alpha}{\ddot{a}_{xn}} + \beta)}{1-\gamma} \cdot SA$$

  - single:

$$brutto\_premium = (netto\_premium + \alpha + \beta \cdot \ddot{a}_{xn}) \cdot SA$$

- netto statutory reserve as at the end of the year *t* is calculated as:
  - regular:

$$R_t = (A_{x+t,n-t} - Netto\_premium \cdot \ddot{a}_{x+t,n-t}) \cdot SA$$

  - single:

$$R_t = A_{x+t,n-t} \cdot SA$$

- brutto statutory (accounting) reserve as at the end of the year *t* is calculated as:
  - regular:

$$R_t^{acc} = \max(0, R_t - \frac{\alpha \cdot SA}{\ddot{a}_{x,n}} \cdot \ddot{a}_{x+t,n-t})$$

  - single:

$$R_t^{acc} = R_t + \beta \cdot SA \cdot \ddot{a}_{x+t,n-t}$$

Notice:
We have chosen such an insurance product since it is still very frequent in the Czech insurance market and includes significant saving part what is the topic of our interest.

Where are we? See 7.6 and 7.18

7.21. In order to be able to run our fair value calculations we will need:

- ~~interest rate scenarios (used as investment returns and for discounting)~~
- ~~description of the product type what will be modeled~~
- specify the policy contracts to be modeled (modelpoints)
- set assumptions
- build liability modeling tool and the cash flow model

7.22. Let's now choose the policies (policy contracts, modelpoints) of such a product what will be used in our calculations.

Modelpoints

7.23. We will run our calculations on the following modelpoints (policies):
Valuation date: 31.12.2005
Policy type:.....................Endowment described in the paragraph 7.20
Age at policy inception: ..30 years
Sex:................................Male
Sum assured:.................100 000
Policy period:..................30 years
Payment frequency: ....Regular (annually) and single
Policy inceptions:

a) 1981
=> 25th policy year at valuation date
=> 5 year till the end of the policy period
b) 1986
=> 20th policy year at valuation date
=> 10 year till the end of the policy period
c) 1991
=> 15th policy year at valuation date
=> 15 year till the end of the policy period
d) 1996
=> 10th policy year at valuation date
=> 20 year till the end of the policy period
e) 2001
=> 5th policy year at valuation date
=> 25 year till the end of the policy period

Technical interest rates: 2%, 3%, 4%, 5% and 6%.
Hence, we obtained 50 endowment polices to be tested – see the table 7.8.

Modelpoint number	Technical interest rate	Payment frequency	Policy inception	Age at entry	Sex	Policy period	Sum assured
1	2%	REGULAR	1981	30	Male	30	100 000
2	2%	REGULAR	1986	30	Male	30	100 000
3	2%	REGULAR	1991	30	Male	30	100 000
4	2%	REGULAR	1996	30	Male	30	100 000
5	2%	REGULAR	2001	30	Male	30	100 000
6	3%	REGULAR	1981	30	Male	30	100 000
7	3%	REGULAR	1986	30	Male	30	100 000
8	3%	REGULAR	1991	30	Male	30	100 000
9	3%	REGULAR	1996	30	Male	30	100 000
10	3%	REGULAR	2001	30	Male	30	100 000
11	4%	REGULAR	1981	30	Male	30	100 000
12	4%	REGULAR	1986	30	Male	30	100 000
13	4%	REGULAR	1991	30	Male	30	100 000
14	4%	REGULAR	1996	30	Male	30	100 000
15	4%	REGULAR	2001	30	Male	30	100 000
16	5%	REGULAR	1981	30	Male	30	100 000
17	5%	REGULAR	1986	30	Male	30	100 000
18	5%	REGULAR	1991	30	Male	30	100 000
19	5%	REGULAR	1996	30	Male	30	100 000
20	5%	REGULAR	2001	30	Male	30	100 000
21	6%	REGULAR	1981	30	Male	30	100 000
22	6%	REGULAR	1986	30	Male	30	100 000
23	6%	REGULAR	1991	30	Male	30	100 000
24	6%	REGULAR	1996	30	Male	30	100 000
25	6%	REGULAR	2001	30	Male	30	100 000
26	2%	SINGLE	1981	30	Male	30	100 000
27	2%	SINGLE	1986	30	Male	30	100 000
28	2%	SINGLE	1991	30	Male	30	100 000
29	2%	SINGLE	1996	30	Male	30	100 000
30	2%	SINGLE	2001	30	Male	30	100 000
31	3%	SINGLE	1981	30	Male	30	100 000
32	3%	SINGLE	1986	30	Male	30	100 000
33	3%	SINGLE	1991	30	Male	30	100 000
34	3%	SINGLE	1996	30	Male	30	100 000
35	3%	SINGLE	2001	30	Male	30	100 000
36	4%	SINGLE	1981	30	Male	30	100 000
37	4%	SINGLE	1986	30	Male	30	100 000
38	4%	SINGLE	1991	30	Male	30	100 000
39	4%	SINGLE	1996	30	Male	30	100 000
40	4%	SINGLE	2001	30	Male	30	100 000
41	5%	SINGLE	1981	30	Male	30	100 000
42	5%	SINGLE	1986	30	Male	30	100 000
43	5%	SINGLE	1991	30	Male	30	100 000
44	5%	SINGLE	1996	30	Male	30	100 000
45	5%	SINGLE	2001	30	Male	30	100 000
46	6%	SINGLE	1981	30	Male	30	100 000
47	6%	SINGLE	1986	30	Male	30	100 000
48	6%	SINGLE	1991	30	Male	30	100 000
49	6%	SINGLE	1996	30	Male	30	100 000
50	6%	SINGLE	2001	30	Male	30	100 000

Table 7.8 – Modelpoints tested

Where are we? See 7.6 and 7.18 and 7.20.

7.24. In order to be able to run our fair value calculations we will need:

- ~~interest rate scenarios (used as investment returns and for discounting)~~
- ~~description of the product type what will be modeled~~
- ~~specify the policy contracts to be modeled (modelpoints)~~
- set assumptions
- build liability modeling tool and the cash flow model

7.25. Let's look at the assumptions now.

Assumptions

7.26. Assumptions used in our calculations are shown in the table 7.9.

Mortality experience	80% of mortalities used for premium calculation.
Lapse rate	5% p.a.
Investment return	**2 versions⁺:** **a) simulated 1Y zero spot rate** **b) simulated 5Y zero spot rate**
Discount rates	**2 versions:** **a) simulated 1Y zero spot rate – same as for the case a) for the investment return assumption** **b) simulated 1Y zero spot rate – based on the same r_t trajectory as used for the investment return assumption in the case b)**
Initial expenses in the first policy year Renewal expenses starting from the 2nd policy year Expense inflation	1 500 CZK as at the valuation date 1 000 CZK as at the valuation date 4% p.a.
Initial commission in the first policy year Initial commission in the second policy year Renewal commission starting from the 3rd policy year	50% from premium 20% from premium 3% from premium
Market value margins (MVM) Mortality Expenses Lapses	 +10% +10% -10%

Table 7.9 –Assumptions used

Interest rates for both versions are generated in a way as described in the paragraph7.16.

Where are we? See 7.6 and 7.18 and 7.20 and 7.24.

7.27. In order to be able to run our fair value calculations we will need:

- ~~interest rate scenarios (used as investment returns and for discounting)~~
- ~~description of the product type what will be modeled~~
- ~~specify the policy contracts to be modeled (modelpoints)~~
- ~~set assumptions~~
- build liability modeling tool and the cash flow model

Liability model

7.28. The tool:

We have built the annual cash flow model in the actuarial system *Sophas* prepared by the JL Soft company (see [20]).
This system is especially prepared for quick calculation and easy preparation of cash flow models and is used in the Czech and Slovak insurance market since the year 2003. The main advantage of this system for the purposes of our work is that it is able to compute many scenarios in a short period of time.

7.29. Formulas:

We will show the mathematical formulas of the main cash flows projected in our *Sophas* model in this part.

- We model cash flow as at the end of the year to be:

$$CF_r = \left(P_{r-1} - C_{r-1} - E_{r-1}\right)(1 + rfr_{r-t}) - Deaths_r - Surr_r - Mat_r \,,$$

where

n .... is policy period in years
t .... is th epoicy year at the valuation date
r .... is the policy year, $0 \le r \le n$
P_{r-1} .... is the premium income assumed to happen at the beginning of the policy year *r*
C_{r-1} .... are commissions paid to agents (or other sales network) assumed to be paid at the beginning of the policy year *r*
E_{r-1} .... is the expense outflow again assumed to happen at the beginning of the policy year *r*
rfr_{r-t} ..... represents the discounting rate (future 1Y rate) in the policy year *r* (*r*-*t*) from the valuation date
i_r .... is the investment return at the year *r*

$Deaths_r$. is the outflow representing the deaths benefit assumed to be paid at the end of the policy year *r*

$Surr_r$ ..... are the surrenders paid again assumed to be paid at the end of the policy year *r*

Mat_r ..... is the outflow of maturity benefits, again assumed to be payable at the end of the policy year *r*

- Having the following inputs:

eq_x .... expected mortality of the person at age *x* (=person at the age of *x* alive at the beginning of the year will die in the 1 year)

w_r .... probability of lapse at the policy year *r* (= policy in-force at the beginning of the year *r* will lapse in 1 year)

- we define:

l_r .... number of policies in-force at the end of the year *r*

$l_r = l_{r-1} - d_r - wthd_r$, where

d_r .... number of deaths at the end of the year *r*

$d_r = l_{r-1} \cdot eq_x$

$wthd_r$ .... number of lapses at the end of the year *r*

$wthd_r = l_{r-1} \cdot (1 - eq_x) \cdot w_r$

m_r .... number of maturities at the end of the year *r*

$m_r = 0$ for *r* < *n*

$m_r = l_{r-1} - d_r - wthd_r$ for *r* = *n*

p_{r-1} .... the premium of 1 policy when the policy is in-force at the beginning of the policy year *r*

c_{r-1} .... commissions paid for 1 policy if the policy is in-force at the beginning of the policy year *r*

e_{r-1} .... the expense outflow related to 1 in-force at the beginning of the policy year *r*

SA .... the death benefit of 1 policy if the policyholder dies in the policy period

$surr\_ch_r$ the surrender charge (as percentage from statutory reserve) of 1 policy lapsed in the policy year *r*

R_r .... netto statutory reserve at the end of the year *r,* of policy in-force at the end of the year *r*

R_r^{acc} .... accounted statutory reserve at the end of the year *r,* of policy in-force at the end of the year *r*

$PSfund_r$ value of profit share fund at the end of the year *r* for the policy in-force at the end of the year *r*

- Therefore, we'll set:

$$P_{r-1} = l_{r-1} \cdot p_{r-1}$$
$$C_{r-1} = l_{r-1} \cdot c_{r-1}$$
$$E_{r-1} = l_{r-1} \cdot c_{r-1}$$
$$Deaths_r = d_r \cdot (SA + PSfund_r)$$
$$Surr_r = wthd_r \cdot (1 - surr\_ch_r) \cdot (R_r^{acc} + PSfund_r)$$
$$Mat_r = m_r \cdot (R_r + PSfund_r)$$

- Finally we have

$$CF_r = \left(P_{r-1} - C_{r-1} - E_{r-1}\right)(1 + rfr_{r-t}) - Deaths_r - Surr_r - Mat_r =$$
$$= l_{r-1} \cdot (p_{r-1} - c_{r-1} - e_{r-1})(1 + rfr_{r-t}) - d_r \cdot (k + PSfund_r)$$
$$- wthd_r \cdot (1 - surr\_ch_r)(R_r^{acc} + PSfund_r) - m_r \cdot (R_r + PSfund_r)$$

- We now calculate present value of the cash flow projection discounted by rfr_{r-t}.

Therefore it is:

$$PV_t = \frac{CF_{t+1}}{(1 + rfr_1)} + \frac{CF_{t+2}}{(1 + rfr_2)(1 + rfr_1)} + \ldots + \frac{CF_{t+n}}{(1 + rfr_{n-t}) \cdot \ldots \cdot (1 + rfr_2) \cdot (1 + rfr_1)} = \sum_{i=t+1}^{n} \frac{CF_i}{\prod_{j=1}^{i-t} (1 + rfr_j)}$$

Where are we? See 7.6 and 7.18 and 7.21 and 7.24 and 7.27.

7.30. In order to be able to run our fair value calculations we will need:

- ~~interest rate scenarios (used as investment returns and for discounting)~~
- ~~description of the product type what will be modeled~~
- ~~specify the policy contracts to be modeled (modelpoints)~~
- ~~set assumptions~~
- ~~build liability modeling tool and the cash flow model~~

7.31. We are now ready for our stochastic calculations of the present values of future cash flows under the simulations of the investment returns and discount rates and based on these results to determine the stochastic FV as defined in 7.1 – 7.4.

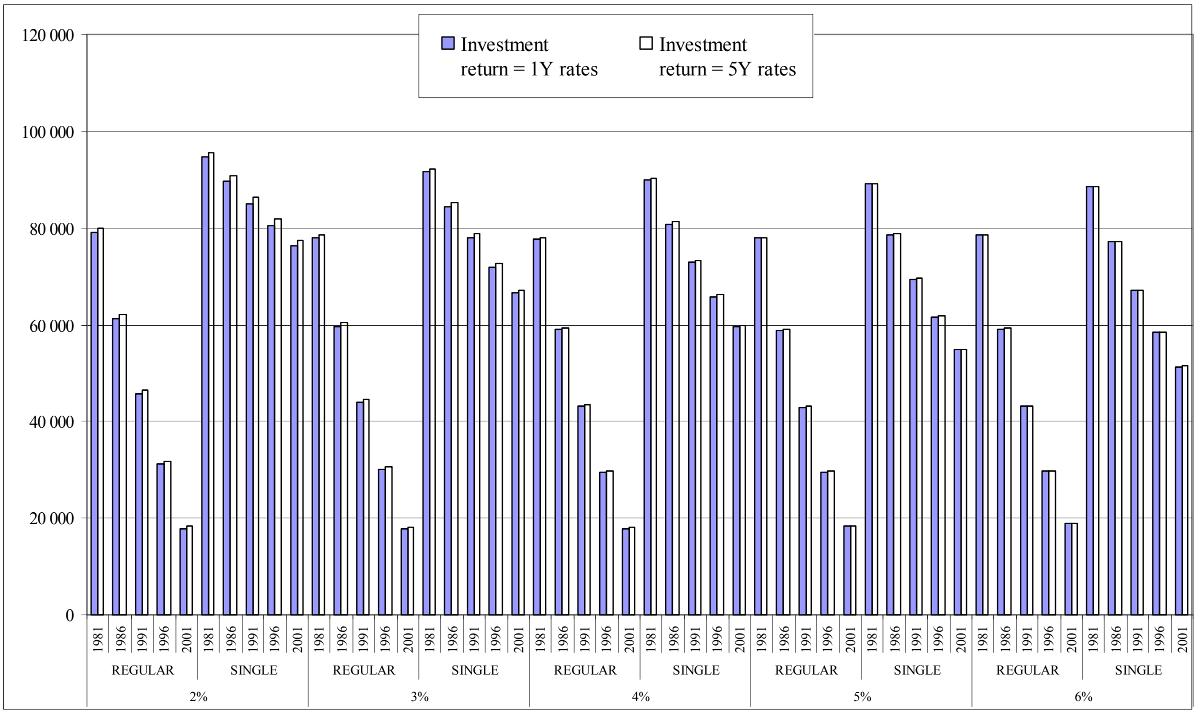
Results of the stochastic liability fair value calculation

7.32. We will show the results of the stochastic fair value for above described product and modelpoints using the assumptions set above for both version of investment return assumptions (1Y rate and 5Y rate) – see the table 7.10.

Stochastic fair value results								
Technical interest rate	*Payment frequency*	*Policy inception*	*Age at entry*	*Sex*	*Policy period*	*Sum assured*	*Investment return = 1Y rates*	*Investment return = 5Y rates*
2%	REGULAR	1981	30	Male	30	100 000	**79 069**	**79 850**
		1986	30	Male	30	100 000	**61 317**	**62 223**
		1991	30	Male	30	100 000	**45 543**	**46 374**
		1996	30	Male	30	100 000	**31 164**	**31 853**
		2001	30	Male	30	100 000	**17 896**	**18 410**
	SINGLE	1981	30	Male	30	100 000	**94 594**	**95 433**
		1986	30	Male	30	100 000	**89 672**	**90 771**
		1991	30	Male	30	100 000	**85 018**	**86 201**
		1996	30	Male	30	100 000	**80 547**	**81 758**
		2001	30	Male	30	100 000	**76 230**	**77 442**
3%	REGULAR	1981	30	Male	30	100 000	**77 910**	**78 461**
		1986	30	Male	30	100 000	**59 664**	**60 308**
		1991	30	Male	30	100 000	**43 951**	**44 538**
		1996	30	Male	30	100 000	**30 102**	**30 584**
		2001	30	Male	30	100 000	**17 763**	**18 122**
	SINGLE	1981	30	Male	30	100 000	**91 678**	**92 263**
		1986	30	Male	30	100 000	**84 423**	**85 185**
		1991	30	Male	30	100 000	**77 898**	**78 699**
		1996	30	Male	30	100 000	**71 935**	**72 732**
		2001	30	Male	30	100 000	**66 451**	**67 225**
4%	REGULAR	1981	30	Male	30	100 000	**77 588**	**77 830**
		1986	30	Male	30	100 000	**58 897**	**59 226**
		1991	30	Male	30	100 000	**43 133**	**43 448**
		1996	30	Male	30	100 000	**29 622**	**29 893**
		2001	30	Male	30	100 000	**17 954**	**18 168**
	SINGLE	1981	30	Male	30	100 000	**89 928**	**90 180**
		1986	30	Male	30	100 000	**80 842**	**81 215**
		1991	30	Male	30	100 000	**72 860**	**73 263**
		1996	30	Male	30	100 000	**65 813**	**66 217**
		2001	30	Male	30	100 000	**59 570**	**59 962**
5%	REGULAR	1981	30	Male	30	100 000	**77 899**	**77 974**
		1986	30	Male	30	100 000	**58 865**	**59 000**
		1991	30	Male	30	100 000	**42 955**	**43 100**
		1996	30	Male	30	100 000	**29 610**	**29 745**
		2001	30	Male	30	100 000	**18 388**	**18 503**
	SINGLE	1981	30	Male	30	100 000	**89 075**	**89 151**
		1986	30	Male	30	100 000	**78 591**	**78 739**
		1991	30	Male	30	100 000	**69 440**	**69 613**
		1996	30	Male	30	100 000	**61 550**	**61 731**
		2001	30	Male	30	100 000	**54 769**	**54 950**
6%	REGULAR	1981	30	Male	30	100 000	**78 399**	**78 417**
		1986	30	Male	30	100 000	**59 151**	**59 201**
		1991	30	Male	30	100 000	**43 086**	**43 147**
		1996	30	Male	30	100 000	**29 821**	**29 884**
		2001	30	Male	30	100 000	**18 914**	**18 973**
	SINGLE	1981	30	Male	30	100 000	**88 611**	**88 629**
		1986	30	Male	30	100 000	**77 085**	**77 139**
		1991	30	Male	30	100 000	**67 004**	**67 074**
		1996	30	Male	30	100 000	**58 455**	**58 534**
		2001	30	Male	30	100 000	**51 287**	**51 369**

Table 7.10 – Stochastic FV results

7.33. We add the FV results – both versions of investment returns (1Y and 5Y) in the graph 7.7.

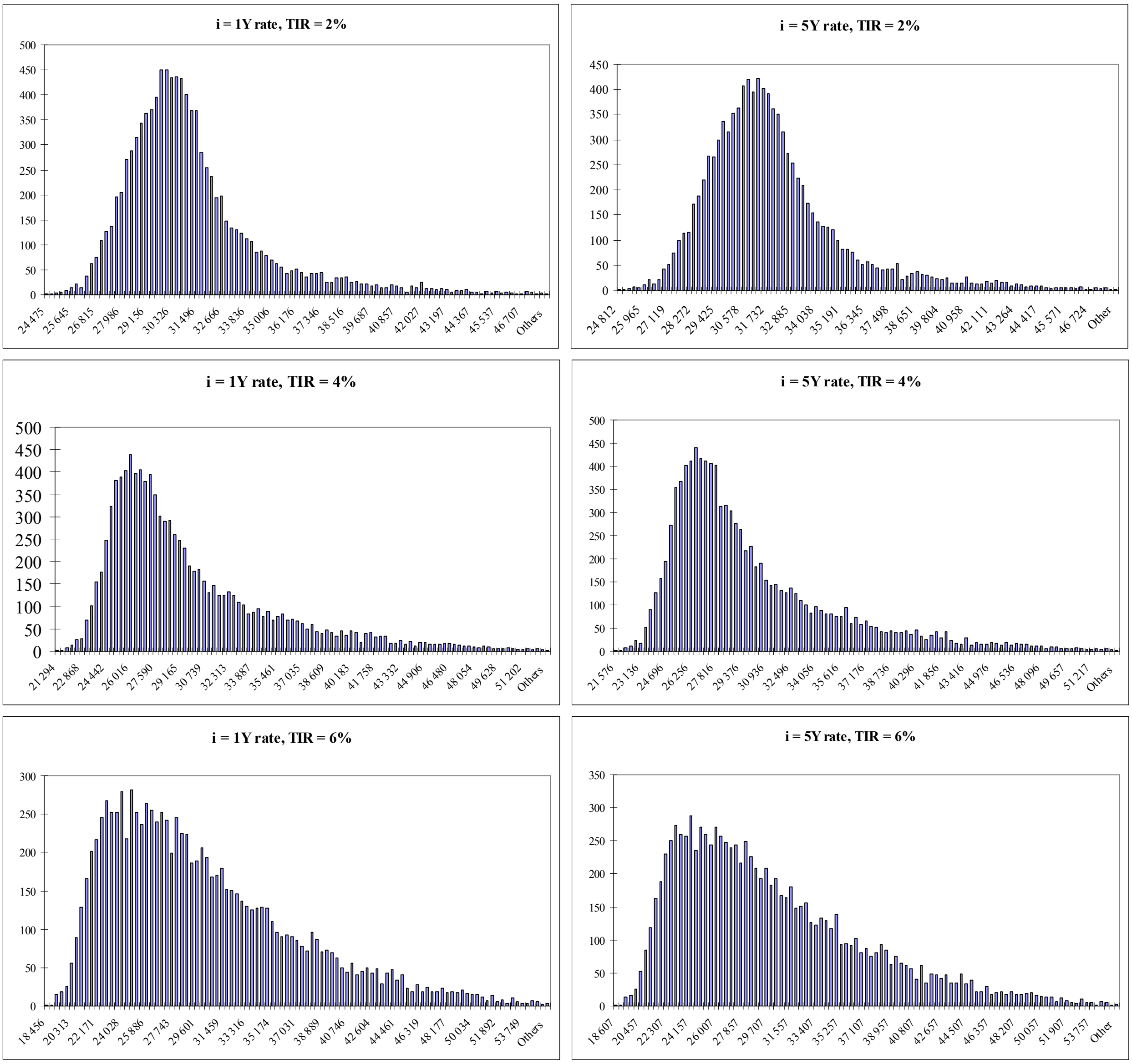


Graph 7.7 – Stochastic FV results

7.34. One can notice that:

- FV for regularly paid policies is lower than for single, what is natural, since future premium income is expected when a policy premium is paid regularly and no future premium will be paid in case of a single premium payment.
- FV is higher when 5Y rates are assumed comparing to 1Y rate assumption. This is due to the fact that 5Y rates are higher then 1Y rates in our case (EUR rates as at December 31, 2005) and hence more profit share is paid in the 5Y rate case, thus higher outflow occurs what makes higher liabilities for the insurance company. The level of discount rates is the same in both cases – future 1Y rates are used for discounting for both investment returns versions (1Y and 5Y rate).

7.35. We will show now histograms of the distribution of the present value of future cash flows (i.e. result of one scenario) for several modelpoints. We have chosen regularly paid policy with the inception year equal to 1996 and the technical interest rate on the levels 2%, 4% and 6% for both versions of the level of investment returns (1Y and 5Y).



Graph 7.8 – Histograms of the present values

7.36. One can see that:

- figures (present values on the x-axis) in ‘5Y rate cases’ are slightly higher than in ‘1Y rates’; this is caused by the same fact as discussed in 7.34.
- the higher technical interest rate the higher spread between the minimum and the maximum of the present values – histograms are ‘wider’ (i.e. more different present values are generates for higher technical rates).

7.37. We add a comparison of the stochastic FV results with 2 deterministic ones – see the table 7.11.

- First deterministic approach assumes investment returns as well as discount rates to be on the level of 1Y forward rate determined from the current risk free yield curve.
- The other deterministic approach assumes investment returns to be on the level of 1Y forward rate while discount rates to be on the level of 1Y forward rate minus 0.25% (25 bps) as suggested in [4] in order to cover the value of interest rate options.

Remarks:

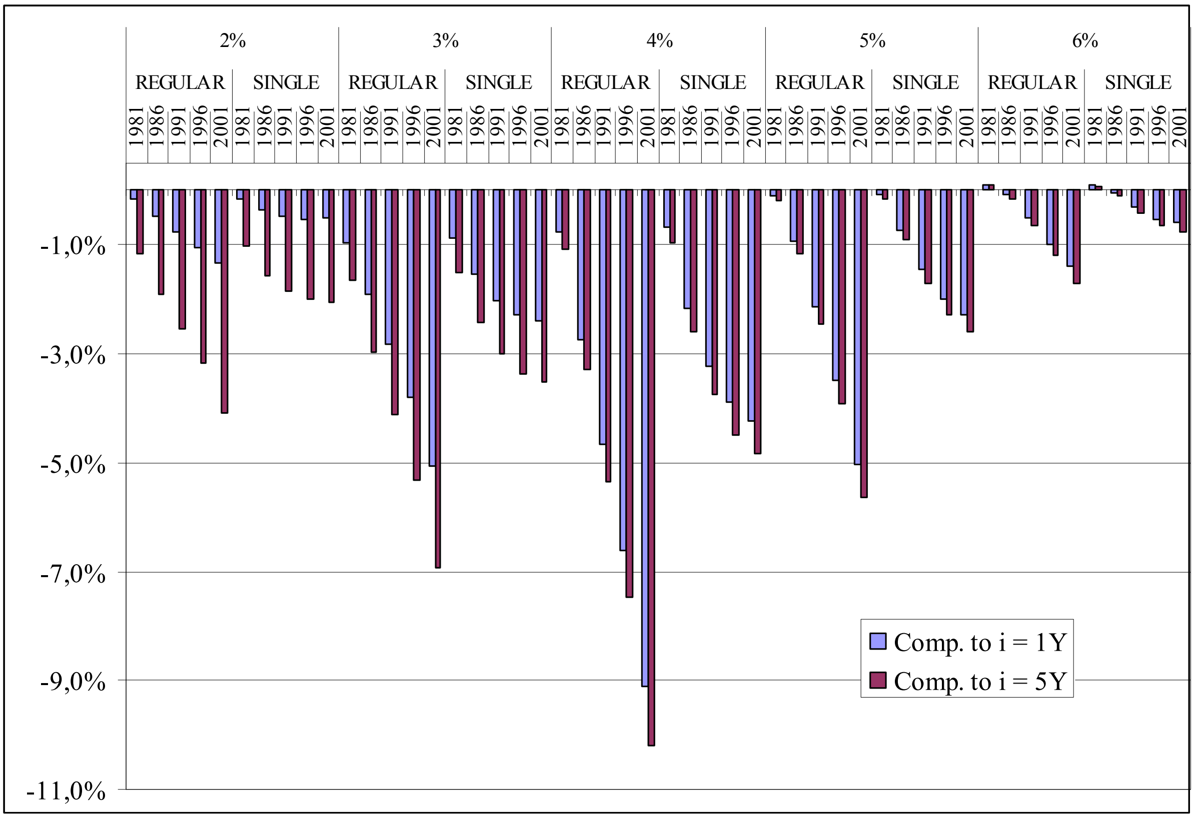
- We show this comparison because such deterministic approaches are widely used in the Czech insurance market.
- However, we do not believe that the decrease of the discount rate is relevant to a valuation of interest rate options embedded in policy contracts since manipulation with the discount rate affects not only cash flows depending on interest rates but the cash flows which do not depend on the interest rates as well – e.g. expense outflow, commissions paid, premium income, etc. – what may destroy the results.
  Maybe more relevant solution could be adjusting not the discount rate but the investment return assumption or using adjusted discount rates only to the cash flows which depend on the interest rates (profit sharing paid). What should be the correct adjustment and what level of such adjustments should be used remains an open issued at this moment.

Modelpoints			Stochastic fair value		Deterministic results					
Technical interest rate (1)	*Payment frequency (2)*	*Policy inception (3)*	*Investment return = 1Y rates (4)*	*Investment return = 5Y rates (5)*	*Inv. return = 1Y fw Dicount rate = 1Y fw (6)*	*(6)/(4)-1*	*(6)/(5)-1*	*Inv. return = 1Y fw Dicount rate = 1Y fw -25bps (7)*	*(7)/(4)-1*	*(7)/(5)-1*
2%	REGULAR	1981	79 069	79 850	78 932	**-0,2%**	**-1,1%**	79 851	**1,0%**	**0,0%**
		1986	61 317	62 223	61 029	**-0,5%**	**-1,9%**	62 380	**1,7%**	**0,3%**
		1991	45 543	46 374	45 196	**-0,8%**	**-2,5%**	46 689	**2,5%**	**0,7%**
		1996	31 164	31 853	30 841	**-1,0%**	**-3,2%**	32 291	**3,6%**	**1,4%**
		2001	17 896	18 410	17 658	**-1,3%**	**-4,1%**	18 944	**5,9%**	**2,9%**
	SINGLE	1981	77 910	78 461	77 158	**-1,0%**	**-1,7%**	78 047	**0,2%**	**-0,5%**
		1986	59 664	60 308	58 519	**-1,9%**	**-3,0%**	59 792	**0,2%**	**-0,9%**
		1991	43 951	44 538	42 706	**-2,8%**	**-4,1%**	44 083	**0,3%**	**-1,0%**
		1996	30 102	30 584	28 960	**-3,8%**	**-5,3%**	30 277	**0,6%**	**-1,0%**
		2001	17 763	18 122	16 864	**-5,1%**	**-6,9%**	18 027	**1,5%**	**-0,5%**
3%	REGULAR	1981	77 588	77 830	76 988	**-0,8%**	**-1,1%**	77 868	**0,4%**	**0,0%**
		1986	58 897	59 226	57 286	**-2,7%**	**-3,3%**	58 511	**-0,7%**	**-1,2%**
		1991	43 133	43 448	41 127	**-4,6%**	**-5,3%**	42 420	**-1,7%**	**-2,4%**
		1996	29 622	29 893	27 663	**-6,6%**	**-7,5%**	28 878	**-2,5%**	**-3,4%**
		2001	17 954	18 168	16 317	**-9,1%**	**-10,2%**	17 379	**-3,2%**	**-4,3%**
	SINGLE	1981	77 899	77 974	77 824	**-0,1%**	**-0,2%**	78 707	**1,0%**	**0,9%**
		1986	58 865	59 000	58 314	**-0,9%**	**-1,2%**	59 548	**1,2%**	**0,9%**
		1991	42 955	43 100	42 038	**-2,1%**	**-2,5%**	43 337	**0,9%**	**0,6%**
		1996	29 610	29 745	28 580	**-3,5%**	**-3,9%**	29 797	**0,6%**	**0,2%**
		2001	18 388	18 503	17 461	**-5,0%**	**-5,6%**	18 525	**0,7%**	**0,1%**
4%	REGULAR	1981	78 399	78 417	78 483	**0,1%**	**0,1%**	79 369	**1,2%**	**1,2%**
		1986	59 151	59 201	59 107	**-0,1%**	**-0,2%**	60 349	**2,0%**	**1,9%**
		1991	43 086	43 147	42 870	**-0,5%**	**-0,6%**	44 180	**2,5%**	**2,4%**
		1996	29 821	29 884	29 527	**-1,0%**	**-1,2%**	30 758	**3,1%**	**2,9%**
		2001	18 914	18 973	18 650	**-1,4%**	**-1,7%**	19 732	**4,3%**	**4,0%**
	SINGLE	1981	94 594	95 433	94 453	**-0,1%**	**-1,0%**	95 443	**0,9%**	**0,0%**
		1986	89 672	90 771	89 356	**-0,4%**	**-1,6%**	90 973	**1,5%**	**0,2%**
		1991	85 018	86 201	84 605	**-0,5%**	**-1,9%**	86 632	**1,9%**	**0,5%**
		1996	80 547	81 758	80 120	**-0,5%**	**-2,0%**	82 413	**2,3%**	**0,8%**
		2001	76 230	77 442	75 856	**-0,5%**	**-2,0%**	78 316	**2,7%**	**1,1%**
5%	REGULAR	1981	91 678	92 263	90 885	**-0,9%**	**-1,5%**	91 835	**0,2%**	**-0,5%**
		1986	84 423	85 185	83 120	**-1,5%**	**-2,4%**	84 616	**0,2%**	**-0,7%**
		1991	77 898	78 699	76 331	**-2,0%**	**-3,0%**	78 144	**0,3%**	**-0,7%**
		1996	71 935	72 732	70 289	**-2,3%**	**-3,4%**	72 277	**0,5%**	**-0,6%**
		2001	66 451	67 225	64 864	**-2,4%**	**-3,5%**	66 936	**0,7%**	**-0,4%**
	SINGLE	1981	89 928	90 180	89 314	**-0,7%**	**-1,0%**	90 248	**0,4%**	**0,1%**
		1986	80 842	81 215	79 100	**-2,2%**	**-2,6%**	80 521	**-0,4%**	**-0,9%**
		1991	72 860	73 263	70 510	**-3,2%**	**-3,8%**	72 174	**-0,9%**	**-1,5%**
		1996	65 813	66 217	63 249	**-3,9%**	**-4,5%**	65 020	**-1,2%**	**-1,8%**
		2001	59 570	59 962	57 056	**-4,2%**	**-4,8%**	58 853	**-1,2%**	**-1,8%**
6%	REGULAR	1981	89 075	89 151	89 001	**-0,1%**	**-0,2%**	89 933	**1,0%**	**0,9%**
		1986	78 591	78 739	78 019	**-0,7%**	**-0,9%**	79 430	**1,1%**	**0,9%**
		1991	69 440	69 613	68 431	**-1,5%**	**-1,7%**	70 064	**0,9%**	**0,6%**
		1996	61 550	61 731	60 323	**-2,0%**	**-2,3%**	62 036	**0,8%**	**0,5%**
		2001	54 769	54 950	53 525	**-2,3%**	**-2,6%**	55 240	**0,9%**	**0,5%**
	SINGLE	1981	88 611	88 629	88 700	**0,1%**	**0,1%**	89 631	**1,2%**	**1,1%**
		1986	77 085	77 139	77 053	**0,0%**	**-0,1%**	78 456	**1,8%**	**1,7%**
		1991	67 004	67 074	66 794	**-0,3%**	**-0,4%**	68 407	**2,1%**	**2,0%**
		1996	58 455	58 534	58 147	**-0,5%**	**-0,7%**	59 827	**2,3%**	**2,2%**
		2001	51 287	51 369	50 983	**-0,6%**	**-0,8%**	52 652	**2,7%**	**2,5%**

Table 7.11 – Stochastic and deterministic results

7.38. For a better illustration we present the results from the table 7.11 also in the graphical form.

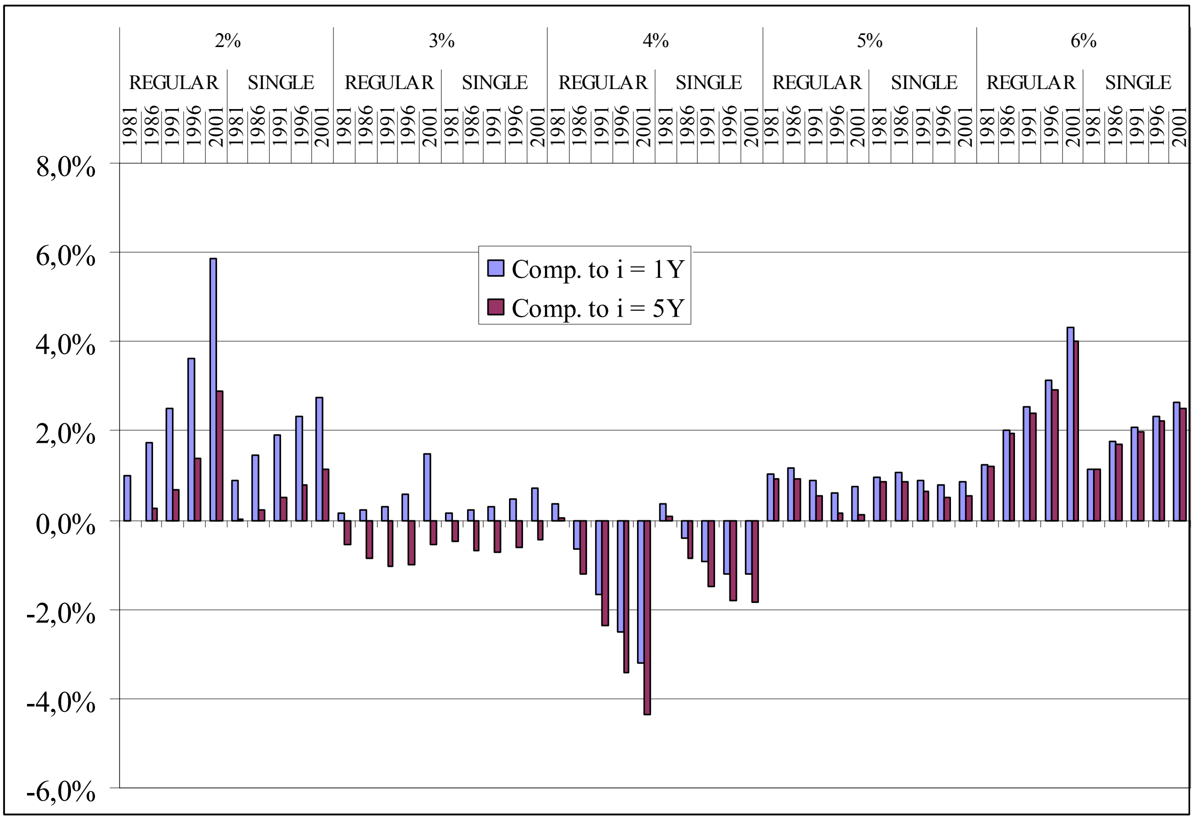
7.39. Comparisons between the deterministic results without any interest rates shifts (column (6)) and the stochastic results (investment returns = 1Y rates – column (4) and investment returns = 5Y rates (column (5)) are shown in the in the graph 7.9.



Graph 7.9 – Stochastic and deterministic (no shifts) results

7.40. One can see that the change of the approaches could make even more than 10% underestimation of the liability value.

7.41. Comparisons between the deterministic results with 25 bps shift on the discount rates (column (7)) and the stochastic results (investment return = 1Y rate – column (4) and investment return = 5Y rate (column (5)) are shown in the in the graph 7.10.



Graph 7.10 – Stochastic and deterministic (25 bps shift) results

7.42. One can see that the change of the approaches could make liability value changes from approx. -4% to +6%.

Alternative approach of calculation formulas, computation tool

7.43. As we have already discussed above, we use the *Sophas* tool for our calculations especially because it is possible to compute a lot of scenarios in a reasonable run time.
If an insurance company has not such an investment in its disposal it has to use ordinary available tools such as MS Excel or others. Their huge disadvantage is that it is practically impossible to run stochastic scenarios without any adjustments since the run times are enormous.
We'll now show here one an example of how the calculation formulas could be adjusted in order to obtain similar results in MS Excel in a reasonable time.

Formulas derivation

We have shown in the paragraph 7.29 that the cash flow at the end of the year *r* is defined as:

$$
\begin{aligned}
&CF_r = \\
&l_{r-1} \cdot (p_{r-1} - c_{r-1} - e_{r-1})(1 + rfr_{r-t}) \\
&-d_r \cdot (SA + PSfund_r) \\
&-wthd_r \cdot (1 - surr\_ch_r)(R_r^{acc} + PSfund_r) \\
&-m_r \cdot (R_r + PSfund_r)
\end{aligned}
$$

---

using:

- $1 + rfr_{r-t} = 1 + tir + rfr_{r-t} - tir$
- sum at risk is $SAR_r = k - R_r$ => $k = SAR_r + R_r$
- $R_r^{acc} = R_r - (R_r - R_r^{acc})$

$$
\begin{aligned}
&= l_{r-1} \cdot (p_{r-1} - c_{r-1} - e_{r-1}) \cdot (1 + tir) \\
&+l_{r-1} \cdot (p_{r-1} - c_{r-1} - e_{r-1}) \cdot (rfr_{r-t} - tir) \\
&-d_r \cdot (SAR_r + R_r + PSfund_r) \\
&-wthd_r \cdot (1 - surr\_ch_r) \cdot \left(R_r - (R_r - R_r^{acc})\right) \\
&-wthd_r \cdot (1 - surr\_ch_r) \cdot PSfund_r \\
&-m_r \cdot (R_r + PSfund_r)
\end{aligned}
$$

---

remember that:

$$P_{r-1} = l_{r-1} \cdot p_{r-1}$$
$$C_{r-1} = l_{r-1} \cdot c_{r-1}$$
$$E_{r-1} = l_{r-1} \cdot c_{r-1}$$

$= (P_{r-1} - C_{r-1} - E_{r-1}) \cdot (1 + tir)$ .... *r. 1*

$-d_r \cdot SAR_r$ .... *r. 2*

$-\left(d_r + wthd_r \cdot (1 - surr\_ch_r) + m_r\right) \cdot R_r$ .... *r. 3*

$+wthd_r \cdot (1 - surr\_ch_r) \cdot (R_r - R_r^{acc})$ .... *r. 4*

$+(P_{r-1} - C_{r-1} - E_{r-1}) \cdot (rfr_{r-t} - tir)$ .... *r. 5*

$-\left(d_r + wthd_r \cdot (1 - surr\_ch_r) + m_r\right) \cdot PSfund_r$ .... *r. 6*

Notice that:

1. The first four rows of the formula **do not depend on the real investment return**.
   Let's name it as the *guaranteed cash flow* and mark it as $CF_r^{guaranteed}$.
   Thus, it is:

$$CF_r^{guaranteed} = (P_{r-1} - C_{r-1} - E_{r-1}) \cdot (1 + tir)$$
$$-d_r \cdot SAR_r - \left(d_r + wthd_r \cdot (1 - surr\_ch_r) + m_r\right) \cdot R_r$$
$$+wthd_r \cdot (1 - surr\_ch_r) \cdot (R_r - R_r^{acc})$$

2. The row 5 contains known cash flows $P_{r-1}, C_{r-1}, E_{r-1}$ from the *guaranteed cash flow*, tir is known product parameter, hence **the only variable is the** rfr_{r-t} **.**

3. The row 6 shows the value of profit share fund paid at the end of the year *r*.
   Let's mark it as $PSfund_r^{paid}$
   Then it is:

$$PSfund_r^{paid} = \left(d_r + wthd_r \cdot (1 - surr\_ch_r) + m_r\right) \cdot PSfund_r .$$

   Remember that $PSfund_r$ is the value of profit share fund of one policy in-force at the end of the year *r*.

Notice further that the row 3 is similar to the row 6.
The formula in the row 3 shows value of guaranteed netto statutory reserve paid at the end of the year *r*.
Let's name it R_r^{paid}.

Hence, it is:

$$R_r^{paid} = \left(d_r + wthd_r \cdot (1 - surr\_ch_r) + m_r\right) \cdot R_r .$$

Thus, the row 6 ($PSfund_r^{paid}$) is possible to rewrite using the row 3 (R_r^{paid}) as:

$$PSfund_r^{paid} = \frac{PSfund_r}{R_r} \cdot R_r^{paid} .$$

Remember again, that R_r is a value of the netto statutory reserve as at the end of the year *t* of one policy in-force at the end of the year *r*.
We know R_r from the model. It does not depend on a real investment return.
We also know R_t^{paid} from the model. It does not depend on a real investment return as well.
Therefore we need to determine the value of $PSfund_r$ (the value of profit share fund of one policy in-force at the end of the year *r*).

It is:

$$PSfund_r = PSfund_{r-1} + newPS_r + InvSurplusPSfund_r ,$$

where $newPS_r$ is the new profit share arisen from statutory (accounted) reserves
$InvSurplusPSfund_r$ is the profit share arisen from the investment surplus of the profit share fund.
Now the task is to determine $newPS_r$ and $InvSurplusPSfund_r$.

It is

$$InvSurplusPSfund_r = PSfund_{r-1} \cdot tir + PSfund_{r-1} \cdot \max\left(0, (1 - mfee) \cdot (i_r - tir)\right)$$

and

$$newPS_r = (P_{r-1}^{netto} + R_r^{acc}) \cdot \max\left(0, (1 - mfee) \cdot (i_r - tir)\right),$$

where P_{r-1}^{netto} means netto premium – it is the premium after risk and charges deductions.

Remind that **the only variable** in both formulas for $newPS_r$ and $InvSurplusPSfund_r$ **is** i_r. All the other parameters are known from the model.

Now we can write:

$$CF_r = CF_r^{guaranteed} + (P_{r-1} - E_{r-1} - C_{r-1}) \cdot (rfr_{r-t} - tir) + PSfund_r^{paid}$$
$$= CF_r^{guaranteed} + (P_{r-1} - E_{r-1} - C_{r-1}) \cdot (rfr_{r-t} - tir) + \frac{PSfund_r}{R_r} \cdot R_r^{paid}$$
$$= CF_r^{guaranteed} + (P_{r-1} - E_{r-1} - C_{r-1}) \cdot (rfr_{r-t} - tir)$$
$$+ \frac{PSfund_{r-1} + newPS_r + InvSurplusPSfund_r}{R_r} \cdot R_r^{paid}$$
$$= CF_r^{guaranteed} + (P_{r-1} - E_{r-1} - C_{r-1}) \cdot (rfr_{r-t} - tir)$$
$$+ \frac{PSfund_{r-1} + PSfund_{r-1} \cdot tir + (P_{r-1}^{netto} + R_r^{acc} + PSfund_{r-1}) \cdot \max\left(0, (1 - mfee) \cdot (i_r - tir)\right)}{R_r} \cdot R_r^{paid}$$

The last formula shows that CF_r could be expressed using cash flows obtainable from the only 1 run of 1 scenario of the model and then the only parameters for generating other scenarios what need to be changed are i_r and rfr_{r-t}.

Therefore, the procedure could be as follows:

1. Run 1 scenario (with $i_r = tir$) and save the required fixed cash flows.
2. Input scenario for i_r and rfr_{r-t} and calculate the present value of the cash flow projection obtained.
3. Input another scenario for i_r and rfr_{r-t} and calculate the present value again
4. …and again
5. Finally we have as many PV results as were our inputs, which is the same result as was described above via using professional actuarial software.

7.44. Conclusion:

We have shown that under some circumstances, it is possible to use common tools, but heavy adjustments have to be done.
This, of course, brings the potential space for modeling (=> results) mistakes and therefore using such an approach is not practical. This is why we would still recommend using professional actuarial tools for such purposes anyway.

Open issues and space for further improvements

7.45. We mention here several open issues of such calculations which ask for further analysis and improvements.

- Interest rate model:
  - We have chosen the single-factor HW model. What would be the results if a different model was used?

- Level of the future investment returns:
  - We have used 1Y and 5Y zero rates as the reasonable policyholder expectation. What would be the results if different levels (e.g. 3Y, 7Y, 10Y zero rates.) were used?

- Other assumptions:
  - We have calculated the fair value of life liabilities via interest rate simulations; we did not simulate other assumptions (mortality, lapses, expenses, etc.) and their correlations. What is the effect and what would be the results if the other assumptions (mortality, lapses, expenses, etc.) were simulated as well? What are correlations among these assumptions?

- Volatility of the interest rates volatilities:
  - It is quite difficult to determine the ‘real’ market volatilities, since interest rate options (evaluated on the basis of the interest rate volatilities) are not frequent deals and depends significantly on the final agreement between the business parties.
    How are the FV results sensitive to the market volatilities?

- Estimation of the value of interest rate option via deterministic approach.
  - We have mentioned our doubts about the current suggested deterministic approach of estimation of the interest rate options embedded in policy contracts (in [4]). But, what is the better one? We have mentioned that probably adjusting investment returns rather than the discount rates would probably be more relevant. But, what is the right adjustment? Some other analyses and many calculations are still required.

8. Conclusions

The intention of this work was to contribute to the current actuarial discussion about the valuation of life liabilities with some summary of current most frequent valuation methodologies.
We have started with the most traditional one (Statutory valuation approach), gone through the more developed (deterministic Embedded Value approach and the deterministic estimation of the Fair Value) to the most recent one – stochastic Fair Value approach via simulating the future interest rates.
We have intended to give a more detailed overview to the liability valuation methodologies not only in a way of a general description but in a way of the specific mathematical formulas and numerical examples as well in order to see their mutual relations and similarities, their positives and negatives.
At the second part of this text, we have showed the real process of the stochastic liability fair value calculation as at December 31, 2005 under the interest rate simulations in order to include the price of interest rate options embedded in policy contracts. We went through all the procedure explaining all important issues and steps and finally presented the results. We also have mentioned that many unexplored issues still remain and large space for further studies is open.

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