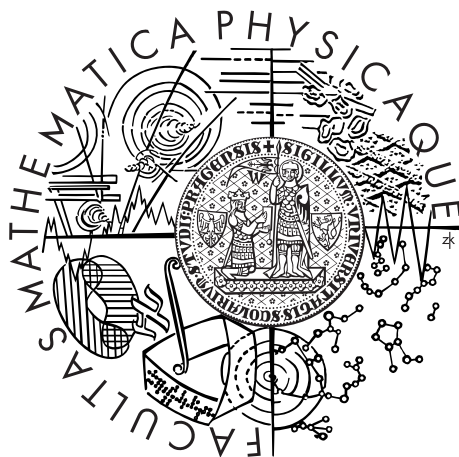


Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE



Jakub Slavík

Nestandardní analýza dynamických systémů

Katedra matematické analýzy

Vedoucí diplomové práce: doc. RNDr. Dalibor Pražák, Ph.D.

Studijní program: Matematika

Studijní obor: Matematická analýza

Praha 2013

Děkuji svému vedoucímu Daliboru Pražákovi za skvělé vedení, množství užitečných postřehů a za jeho ochotu nejen v poslední fázi vzniku této práce.

Prohlašuji, že jsem tuto diplomovou práci vypracoval(a) samostatně a výhradně s použitím citovaných pramenů, literatury a dalších odborných zdrojů.

Beru na vědomí, že se na moji práci vztahují práva a povinnosti vyplývající ze zákona č. 121/2000 Sb., autorského zákona v platném znění, zejména skutečnost, že Univerzita Karlova v Praze má právo na uzavření licenční smlouvy o užití této práce jako školního díla podle §60 odst. 1 autorského zákona.

V dne

Podpis autora

Název práce: Nestandardní analýza dynamických systémů

Autor: Jakub Slavík

Katedra: Katedra matematické analýzy

Vedoucí diplomové práce: doc. RNDr. Dalibor Pražák, Ph.D., Katedra matematické analýzy

Abstrakt: V předložené práci se zabýváme aplikací nestandardní analýzy na dynamické systémy, konkrétně na ω -limitní množinu, stabilitu a globální atraktor. V práci zavádíme pojem elementárního vnoření, podrobně rozebíráme zavedení infinitesimálních reálných čísel a studujeme metrické prostory pomocí nestandardních metod, konkrétně spojitost a kompaktnost, které úzce souvisí s teorií dynamických systémů. Nakonec se věnujeme samotným dynamickým systémům a předkládáme nestandardní charakterizace pojmů jako asymptotická kompaktnost a disipativita a pomocí těchto charakteristik dokážeme jednu ze základních vět této teorie - větu o existenci globálního atraktoru.

Klíčová slova: Dynamický systém, nestandardní analýza, elementární vnoření, globální atraktor, stabilita.

Title: Nonstandard analysis of dynamical systems

Author: Jakub Slavík

Department: Department of Mathematical Analysis

Supervisor: doc. RNDr. Dalibor Pražák, Ph.D., Department of Mathematical Analysis

Abstract: In the presented thesis, we study an application of nonstandard analysis to dynamical systems, in particular to ω -limit set, stability and global attractor. We recall the definition and properties of elementary embedding, in detail explore the introduction of infinitesimals to the real line and study metric spaces using nonstandard methods, in particular continuity and compactness which are closely related to the theory of dynamical systems. Last we attend to dynamical systems and present nonstandard characterizations of some of its properties such as asymptotic compactness and dissipativity and using these characterizations we prove one of the basic results of this theory - existence of a global attractor.

Keywords: dynamical system, nonstandard analysis, elementary embedding, global attractor, stability.

Contents

Preface	2
1 Preliminaries	5
1.1 Formal language	5
1.2 Superstructures	6
1.3 Interpretations	9
2 Nonstandard models	10
2.1 Elementary embeddings	10
2.2 Internal and external sets	14
2.3 Ultrapower models	15
3 Hypernatural and hyperreal numbers	18
3.1 Infinite and infinitesimal numbers	19
3.2 Standard part homomorphism	21
3.3 The permanence principle	22
4 Metric spaces	24
4.1 Basic definitions	24
4.2 Convergence and continuity in metric spaces	27
4.3 Compact and complete metric spaces	30
5 Dynamical systems	33
5.1 Dynamical system	33
5.2 ω -limit set	34
5.3 Stability	37
5.4 Asymptotic compactness and dissipativity	40
5.5 Global attractor	42
Afterword	45
Bibliography	46
List of used symbols	47
Index	48

Preface

Dynamical systems are an abstract model developed with the intention of providing tools that could be used for long term analysis of models in mechanics. However, the generality of the conditions required of dynamical systems allows the use of these tools for any suitable system that can be described by autonomous evolutionary differential equation which has a unique solution and continuous dependence on the initial condition, for example some of the many fluid dynamics equations in physics or reaction-diffusion models in chemistry and biology.

In certain models such as in Lorentz's famous system the long term (or asymptotic) behavior tends to be chaotic. One of the main goals of the theory of dynamical systems is to describe and capture this complex behavior by defining so-called attractors.

Dynamical systems can be divided into several categories by different criteria from which two stand out. The first criterion is the nature of time. We may study both discrete phenomena and continuous problems. The second criterion is determinism. In some situations the stochastic nature of the phenomenon has to be taken into account, for example in settings where the studied equation describes amounts of substances so small that the outcome is heavily influenced by chance. This approach for continuous time is generally related to stochastic differential equations.

In this thesis we present basic results about deterministic dynamical systems with continuous time covering the properties of the so-called ω -limit set, stability and global attractor proved by the methods of nonstandard analysis.

In the 17th century G. W. Leibniz developed the calculus of infinitesimals which we would today consider as a part of nonstandard analysis. In present terminology we could say that the analysis at the end of 17th century was nonstandard. The main disadvantage of infinitesimal approach up to the second half of the 20th century was the fact that the theory did not have solid and rigorous foundations and if not used in a certain safe way it could produce false results. Despite these flaws the infinitesimal calculus was used developed through the 18th century for example by L. Euler but after the introduction of classical ε - δ definition of the limit the mathematicians ceased to use the infinitesimal approach.

The situation changed in 1960 when A. Robinson published his book Nonstandard Analysis. In the book Robinson introduced new numbers to the real line, so-called hyperreal numbers, which contained the infinitesimals as well. He not only gave rigorous justification built on model theory for calculations made by Leibniz and Euler but also used his nonstandard methods to provide new and elegant proofs to many theorems from calculus or topology. After Robinson's pioneering work, there appeared other mathematicians who developed the nonstandard methods further and simplified the process of creating so-called nonstandard embeddings, for example W. Luxemburg who introduced ultrapower models to nonstandard analysis.

The biggest advantage nonstandard analysis has is the fact that nonstandard formulations of different concepts usually requires less or no quantifiers. T. Tao in his book [8], p. 135, writes "One of the features of nonstandard analysis, as opposed to its standard counterpart, is that it efficiently conceals almost all of

the epsilons and deltas that are so prevalent in standard analysis. As a consequence, analysis acquires a much more *algebraic flavour* when viewed through the nonstandard lens."

In this work, we try to utilize the first advantage and instead of studying the asymptotic behavior through limits of certain time dependent sequences, we look at the state of the system in infinite times.

There are many publications discussing the applications of nonstandard methods to dynamical systems and attractors, for example [3] studies limit sets and stability or [1] covers the attractors for stochastic Navier-Stokes equations. In this thesis, we try not only to present nonstandard proofs of the basic theorems but also to give nonstandard characterizations to most of the sets and properties we study.

This book is organized into 5 chapters. The first four chapters provide the apparatus needed in the fifth chapter which contains all the results from dynamical systems theory present in this thesis.

The first chapter covers basics from set theory and logic. In particular, in this chapter the reader will find the classical definitions of formal language, formulas, sentences and generally less known superstructures. All the definitions and theorems (including proofs) in this chapter come from [10].

The second chapter reviews the basic definitions and properties of nonstandard models and presents the three key principles of nonstandard analysis - the transfer principle, the standard definition principle and the internal definition principle. The behavior of functions in nonstandard worlds is studied closely and at the end of the chapter, the existence of nonstandard models is briefly discussed. This chapter closely follows [10], §3 and §4 in particular. Thus similarly as in the previous chapter all the definitions, theorems and proofs can be found in the referenced book.

The third chapter studies hypernatural and hyperreal numbers and introduces infinitesimals and infinite numbers and its properties. A proof of the permanence principle often used in following chapters can also be found there. The presentation again follows the book [10], §5 and §6 in particular.

The fourth chapter provides nonstandard characterizations for most of the basic concepts from the theory of metric spaces like open and closed sets, continuity and compactness and uses these to prove generally known theorems used in the fifth chapter. The proofs of the theorems presented in the first three sections in this chapter come mostly from [4], [10] or [6]. Since in these publications some of the results are treated in more general setting of topological or uniform spaces, we tried to simplify the proofs as much as possible. Since these proofs are classical and can be found in many books covering the theory of metric spaces using nonstandard methods, we refrain from providing each theorem with a reference. We of course do not claim any credit for any of these results.

The fifth chapter presents the basics of the theory of dynamical systems treated using nonstandard methods reviewed in the previous chapters. This chapter covers ω -limit sets and its properties, stability, asymptotic compactness, dissipativity and necessary and sufficient conditions for the existence of a global attractor. The definitions and the theorems come from [2], [7] and [9]. The proofs presented in this chapter are independent work of the author of this text and should be regarded as the main asset of this thesis.

The goal of this work is twofold: to apply nonstandard methods to dynamical systems and to carefully present these methods. We realize that the presentation of the methods takes the larger part of this thesis. Our intention was to cover all the required nonstandard results in real numbers and metric spaces and their proofs in a way that the reader will not need to consult additional sources for anything outside the properties and the construction of the nonstandard embedding.

1. Preliminaries

Before we explore the principles and applications of nonstandard analysis, we need to recall few definitions from logic theory. In this chapter, we first recall the definition of a formal language and syntax which allows us to formulate statements about objects we study. Then we define the superstructure, which is to be the set standing as the domain of so-called nonstandard embedding defined later in Chapter 2. Last we will briefly discuss interpretations to illustrate one of the basic principles of nonstandard analysis.

1.1 Formal language

First of all we will define formal language as a tool for describing properties of the objects of our study.

Definition. *The formal language $\mathcal{L}$ consists of the following symbols:*

- (i) *The variables are symbols from countable set. In this thesis the set of variables will consist of Roman and Greek letters and will be underlined for clearer distinction from the constants.*
- (ii) *The constants are symbols from a set large enough that there is a one-to-one correspondence between constants and the elements of the set under consideration. We denote the set of all constants $\text{cns}(\mathcal{L})$.*
- (iii) *The basic predicates $\in$ and $=$.*
- (iv) *The separation symbols $),)$ and $:$.*
- (v) *The logical connectives $\wedge, \vee, \implies, \iff$ and $\neg$.*
- (vi) *The quantifiers $\forall$ and $\exists$.*

Since there are combinations of the symbols of formal language that do not bear any useful information, we are generally interested only in sequences of these symbols created following a certain set of rules.

Definition. *Let $\mathcal{L}$ be a formal language. We define well-formed formulas (shortly formulas) inductively:*

The atomic formulas $x = y$ and $x \in y$, where x and y are variables or constants, are well-formed formulas.

If α, β are well-formed formulas, then $(\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \implies \beta), (\alpha \iff \beta)$ and $(\neg \alpha)$ are well-formed formulas. If α does not already contain one of $\forall \underline{x}$ or $\exists \underline{x}$, then $(\forall \underline{x} : \alpha)$ and $(\exists \underline{x} : \alpha)$ are also well-formed formulas.

In the last two well-formed formulas we call α the scope of the quantifier $\forall \underline{x}$ or $\exists \underline{x}$.

In cases where there is no risk of ambiguity we will omit the braces and colons. Similarly we will sometimes use abbreviations like $\forall \underline{x}, \underline{y} \in X : \alpha$ instead of $\forall \underline{x} \in X \forall \underline{y} \in X : \alpha$ or $x \notin y$ instead of $\neg(x \in y)$.

Definition. We call the occurrence of a variable x in formula α free, if it is not the occurrence in a quantifier and not in the scope of any quantifier. All the other occurrences are called bound. If there are no free occurrences of any variable in α , then we call α a sentence, otherwise it is called a predicate.

For example the formula $\forall \underline{x} : \underline{x} \in X$ is a sentence provided X is a constant. On the other hand the formula $\forall \underline{x} \in X : \underline{x} \neq \underline{z}$ is a predicate since the occurrence of $\underline{z}$ is free.

If α is a predicate with free occurrence of a variable $\underline{x}$, we will sometimes denote this by $\alpha(\underline{x})$.

Definition. We call a quantifier $\forall \underline{x}$ or $\exists \underline{x}$ transitively bounded if it occurs in the form $\forall \underline{x} : \underline{x} \in X \Rightarrow \alpha$, resp. $\exists \underline{x} : \underline{x} \in X \Rightarrow \alpha$, where X is a constant or a variable. If X is a constant we call the quantifier bounded. A formula is bounded, resp. transitively bounded, if all the quantifiers in the formula are bounded, resp. transitively bounded.

We will write $\forall \underline{x} \in X : \alpha$ instead of $\forall \underline{x} : \underline{x} \in X \Rightarrow \alpha$. The importance of transitively bounded formulas in nonstandard analysis will be discussed in Chapter 2.

1.2 Superstructures

A superstructure is a set we will later use as the set of constants of our formal language $\mathcal{L}$. It is constructed in a way to ensure that all the objects we might want to study are contained in the superstructure.

Definition. Let S be a set. Let $S_0 = S$ and set $S_{n+1} = S_0 \cup \mathcal{P}(S_n)$. We define the superstructure of S by

$$\widehat{S} = \bigcup_{n=1}^{\infty} S_n.$$

We call the elements of S atoms and the elements of $\widehat{S}$ that are not atoms are called entities.

The notion *atom* is chosen to reflect the fact that the sentence $x \in a$ should be false for every atom $a \in S$ and $x \in \widehat{S}$.

Lemma 1.1 ([10]; Theorem 2.1, part 1). *The sets S_n are transitive, i. e. each element $A \in S_n$ which is not an atom is a subset of S_n . In particular, $x \in S_n$ for every $x \in A$.*

Proof. Since $A \in S_n$ is an entity and not an atom, we have $A \in \mathcal{P}(S_{n-1})$ and thus $A \subseteq S_{n-1} \subseteq S_n$. $\square$

As we will see later in Chapter 2 we need to choose S of infinite cardinality, otherwise the nonstandard approach gives no new information about the structure of S .

To show that the superstructure contains all the structures we want to study, let us first recall few definitions from set theory.

Definition. Let $n \in \mathbb{N}, n \geq 2$. Then we define

- a ordered pair (x, y) by $(x, y) = \{\{x\}, \{x, y\}\}$,
- an n -tuple $(x_1, x_2, \dots, x_n)$ as the pair $((x_1, x_2, \dots, x_{n-1}), x_n)$ for $n \geq 3$,
- the Cartesian product $X_1 \times X_2 \times \dots \times X_n$ of sets $X_1, X_2, \dots, X_n$ as the set of such n -tuples $(x_1, x_2, \dots, x_n)$ that $x_i \in X_i$ for all $i \in \{1, \dots, n\}$,
- an n -ary relation Φ over sets $X_1, X_2, \dots, X_n$ as a subset of $X_1 \times X_2 \times \dots \times X_n$.

Let X, Y be sets and Φ be a binary relation over X and Y . Then we define

- the domain of Φ $\text{dom}(\Phi)$ as the set of all $x \in X$ for which there exist some $y \in Y$ satisfying $(x, y) \in \Phi$,
- the range of Φ $\text{rng}(\Phi)$ to be the set of all $y \in Y$ such that $(x, y) \in \Phi$ for some $x \in X$.

We call Φ a function from X to Y if for each $x \in X$ there is exactly one $y \in Y$ satisfying $(x, y) \in \Phi$. We denote the fact that f is a function from X to Y by $f : X \rightarrow Y$.

Since functions play significant role in the theory of metric spaces and dynamical systems, the next proposition shows that relations are contained in the respective superstructure.

Proposition 1.2 ([10]; Theorem 2.1, part 9). *Let $X, Y \in \widehat{S}$ be entities and let $\phi \subseteq X \times Y$ be a relation. Then $\phi \in \widehat{S}$.*

Proof. Let $X \in S_n$ and $Y \in S_m$ for some $n, m \in \mathbb{N}$. Without any loss of generality we may assume $n = m$, otherwise we redefine $n = \max\{n, m\}$ and use the inclusion $S_k \subseteq S_l$ for $k, l \in \mathbb{N}, k < l$. From the definition of the superstructure we have $X \cup Y \in S_{n+1}$ and from the definition of the Cartesian product it follows that $X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$, thus $\phi \in S_{n+3} \subseteq \widehat{S}$. $\square$

Similarly we can show that for a set X all the topologies or σ -algebras over X are contained in the superstructure $\widehat{S}$ and thus the superstructure is big enough to be the set of constants for the respective formal language and thus well suited for subsequent use in nonstandard analysis.

The proper choice of the set S depends on the structures we study. If we are interested in calculus on one dimension, it is enough to take $S = \mathbb{R}$. If we are interested in a metric space X , we choose $S = \mathbb{R} \cup X$ since by Proposition 1.2 any metric on X will be contained in the superstructure $\widehat{S}$.

Next theorem presents a list of abbreviations we will use in this thesis and shows that they can be written in transitively bounded form.

Theorem 1.3 ([10]; Proposition 3.6). *Let $\phi, X, X_1, \dots, X_n$ be constants or variables which do not represent atoms and $x_1, \dots, x_n$ be variables or constants. Then the following predicates are abbreviations of transitively bounded formulas:*

- (i) $X_1 \subseteq X_2, X = X_1 \setminus X_2, X = X_1 \cup X_2, X = X_1 \cap X_2$.
- (ii) $X = \{x_1, \dots, x_n\}$.
- (iii) $X = (x_1, \dots, x_n), (x_1, \dots, x_n) \in X$.

(iv) $X = X_1 \times \cdots \times X_n$.

(v) $\varphi \subseteq X_1 \times \cdots \times X_n$.

(vi) $\varphi : X_1 \rightarrow X_2, x_2 = \varphi(x_1)$.

Proof. The formula $X_1 \subseteq X_2$ is an abbreviation of $\forall \underline{x} \in X_1 : (\underline{x} \in X_2)$. The formula $X = X_1 \setminus X_2$ is a shortcut for

$$(\forall \underline{x} \in X : (\underline{x} \in X_1 \wedge \underline{x} \notin X_2)) \wedge (\forall \underline{x} \in X_1 : (\underline{x} \notin X_2 \Rightarrow \underline{x} \in X)).$$

The formula $X = X_1 \cup X_2$ can be written as

$$(\forall \underline{x} \in X : (\underline{x} \in X_1 \vee \underline{x} \in X_2)) \wedge (\forall \underline{x} \in X_1 : (\underline{x} \in X)) \wedge (\forall \underline{x} \in X_2 : (\underline{x} \in X)),$$

and the formula $X = X_1 \cap X_2$ will be used as an abbreviation for

$$(\forall \underline{x} \in X : (\underline{x} \in X_1 \wedge \underline{x} \in X_2)) \wedge (\forall \underline{x} \in X_1 : (\underline{x} \in X_2 \Rightarrow \underline{x} \in X)) \\ \wedge (\forall \underline{x} \in X_2 : (\underline{x} \in X_1 \Rightarrow \underline{x} \in X)).$$

The formula $X = \{x_1, \dots, x_n\}$ is a shortcut for

$$(\forall \underline{x} \in X : (\underline{x} = x_1 \vee \cdots \vee \underline{x} = x_n)) \wedge (x_1 \in X) \wedge \cdots \wedge (x_n \in X).$$

The formula $X = (x_1, x_2)$ can be rewritten as

$$\exists \underline{x}, \underline{y} \in X : (\underline{x} = \{x_1\} \wedge \underline{y} = \{x_1, x_2\} \wedge X = \{\underline{x}, \underline{y}\}).$$

For $n \geq 3$ we rewrite the respective formula using inductive definition. The formula $(x_1, \dots, x_n) \in X$ is an abbreviation of $\exists \underline{x} \in X : \underline{x} = (x_1, \dots, x_n)$.

Instead of the formula $X = X_1 \times \cdots \times X_n$ we can write

$$(\forall \underline{x} \in X \exists \underline{x}_1 \in X_1 \dots \exists \underline{x}_n \in X_n : \underline{x} = (\underline{x}_1, \dots, \underline{x}_n)) \\ \wedge (\forall \underline{x}_1 \in X_1 \dots \forall \underline{x}_n \in X_n \exists \underline{x} \in X : \underline{x} = (\underline{x}_1, \dots, \underline{x}_n)).$$

The formula $\varphi : X_1 \rightarrow X_2$ is an abbreviation for

$$(\varphi \subseteq X_1 \times X_2) \wedge (\forall \underline{x} \in X_1 \exists \underline{y} \in X_2 : (\underline{x}, \underline{y}) \in \varphi) \\ \wedge (\forall \underline{x} \in X_2 \forall \underline{y}_1, \underline{y}_2 \in X_2 : (((\underline{x}, \underline{y}_1) \in \varphi \wedge (\underline{x}, \underline{y}_2) \in \varphi) \Rightarrow (\underline{y}_1 = \underline{y}_2)))$$

and finally $x_2 = \varphi(x_1)$ is a shortcut for $(x_1, x_2) \in \varphi$. $\square$

The symbol $\mathcal{P}(X)$ denoting the power set of X will never be used in case that X is a variable. We will only use this symbol if X is a constant and not an atom. Then the set $\mathcal{P}(X)$ is again a constant and is an element of the superstructure $\hat{S}$.

1.3 Interpretations

So far we have discussed how the sentences of our language are formulated. In this section we discuss the way the truth value is assigned to a sentence.

Definition. Let $\mathcal{S}$ be a set equipped with binary relations $\in^*$ and $=^*$. An abstract interpretation map I is a one-to-one mapping of a set $\text{dom}(I) \subset \text{cns}(\mathcal{L})$ into $\mathcal{S}$.

Let α be a sentence such that all constants occurring in α are in $\text{dom}(I)$. Given an abstract interpretation map and such sentence α we can define the interpreted sentence $^I\alpha$ by replacing all occurrences of constants $c \in \text{dom}(I)$ by $I(c)$ and all occurrences of $\in$ and $=$ by $\in^*$ and $=^*$ respectively.

The truth value of sentence α under interpretation I is the truth value of the interpreted sentence $^I\alpha$: The formula $x \in^* y$, resp. $x =^* y$ is true if and only if (x, y) belong to $\in^*$ or $=^*$ respectively. The logical connectives have their usual meaning: Let ϕ, ψ be sentences. Then

- $\phi \wedge \psi$ is true if both ϕ and ψ are true,
- $\phi \vee \psi$ is true if at least one of the sentences ϕ and ψ is true,
- $\phi \Rightarrow \psi$ is true if either ψ is true or ϕ is not,
- $\phi \Leftrightarrow \psi$ is true if the sentences ϕ and ψ are either both true or both false.

The sentences $\forall \underline{x} : \beta$, resp. $\exists \underline{x} : \beta$ are true if β is true for all, resp. at least one $x \in \mathcal{S}$ in the sense that β is true if we replace all occurrences of x with $I(x)$.

If $\mathcal{S}$ is a superstructure and $\in^*$ and $=^*$ are defined in the set-theoretical way, we call the map I an interpretation map in set theory.

We often write ^Ix instead of $I(x)$ for an abstract interpretation map and $x \in \text{dom}(I)$.

We emphasize that a sentence can have different truth value under different interpretations. To illustrate this fact, let A be a set, $x, y \in A$, $<$ a binary relation on A and $A, x, y, < \in \text{cns}(\mathcal{L})$. Let α be the sentence $\exists \underline{z} \in A : x < \underline{z} < y$.

Now let I, J be interpretation maps satisfying

$$\begin{array}{ll} ^Ix = 1 & ^Jx = 1 \\ ^Iy = 2 & ^Jy = 2 \\ ^IA = \mathbb{N} & ^JA = \mathbb{Q} \end{array}$$

and such that $^I<$, resp. $^J<$ is the usual order on $\mathbb{N}$ or $\mathbb{Q}$ respectively. Then the sentence α is clearly false under interpretation I and at the same time true under interpretation J .

Since the evaluation of the truth value takes place not in $\text{rng}(I)$ but in $\mathcal{S}$, an interesting situation might occur if the interpretation map I is not surjective. In that case the truth value of the sentence might depend on an element of $\mathcal{S}$ which cannot be described using the original language and under such assumption the quantifiers $\forall \underline{x} \in X$ and $\exists \underline{x} \in X$ run through bigger sets than $I(X)$. This is actually the idea behind nonstandard analysis: introducing new elements into the original set, for example infinitesimal numbers into the real line.

2. Nonstandard models

As we saw in Chapter 1 the truth value of a sentence α depends on the interpretation map we choose. In this chapter we study the properties of interpretation maps that preserve the truth value assigned to α under "standard" set theory interpretation and briefly discuss its existence.

Nonstandard analysis and its underlying principles can be presented at various levels of complexity. Here we chose to introduce the basic definitions to ease any eventual subsequent study of nonstandard models but we omit most of the proofs and present only those that will in our opinion illustrate the use of nonstandard methods required later. For more exhaustive treatment of this topic see [10] or [4].

In the whole chapter let S be a set and $\hat{S}$ the corresponding superstructure.

2.1 Elementary embeddings

Let $\mathcal{L}$ be a formal language and let $I : \text{dom}(I) \rightarrow \hat{S}$ be a surjective interpretation map where $\text{dom}(I) \subseteq \text{cns}(\mathcal{L})$. Let K_0 denote the set of all true sentences whose constants are in $\text{dom}(I)$.

Let T be a set and $I' : \text{dom}(I') \rightarrow \hat{T}$ an interpretation map for the language $\mathcal{L}$ in set theory.

Definition. We call the mapping $* = I' \circ I^{-1} : \hat{S} \rightarrow \hat{T}$ elementary embedding if the two following conditions hold:

- (i) Each transitively bounded sequence from K_0 which is true under the interpretation I is also true under the interpretation I' .
- (ii) $*S = T$

We call the set $\hat{S}$ the standard universe. Every element in the range of $*$ is called standard. We define the standard copy ${}^\sigma A$ of an entity $A \in \hat{S}$ by

$${}^\sigma A = \{^*a; a \in A\}.$$

Naturally we would like the elementary embedding to be a function for which we clearly need $\text{dom}(I) \subseteq \text{dom}(I')$. However, we do not need to include that assumption in the definition explicitly: Let $c \in \text{dom}(I)$. Then the sentence $c = c$ is true and by the definition of $*$ it is true under I' , in particular $c = c$ has an interpretation under I' and hence $c \in \text{dom}(I')$.

Proposition 2.1 ([10], Proposition 3.3). *The definition of elementary embedding $*$ does not depend on $\mathcal{L}$, I or I' . More specifically let $\mathcal{L}_0$ be a formal language, $I_0 : \text{dom}(I_0) \rightarrow \hat{S}$ where $\text{dom}(I_0) \subseteq \text{cns}(\mathcal{L}_0)$ and $I'_0 : \text{dom}(I'_0) \rightarrow \hat{T}$ satisfy $* = I' \circ I^{-1} = I'_0 \circ I_0^{-1}$. Then $*$ is elementary with respect to $\mathcal{L}$, I , I' if and only if it is elementary with respect to $\mathcal{L}_0$, I_0 , I'_0 .*

From now on we assume that I is an identity on $\text{cns}(\mathcal{L})$. This has two important implications. First every element of $\hat{S}$ is a constant in $\mathcal{L}$ and second each formula in $\mathcal{L}$ whose constants are in $\text{cns}(\mathcal{L})$ is also a formula in set theory.

Proposition 2.2 ([10], Lemma 3.5). *Let $*$: $\widehat{S} \rightarrow \widehat{T}$ be an elementary embedding. Then the following holds:*

- (i) *For every $x, y \in \widehat{S}$ we have $x = y, x \in y$, resp. $x \subseteq y$ if and only if $*x = *y, *x \in *y$, resp. $*x \subseteq *y$.*
- (ii) *x is an atom, resp. an entity, in $\widehat{S}$ if and only if $*x$ is an atom, resp. an entity, in $\widehat{*S}$.*
- (iii) *${}^\sigma A \subseteq *A$ for every entity $A \in \widehat{S}$.*

The general idea is that $*$ enlarges the original set S with new elements and thus presents the possibility that the truth value of α depends on an element which cannot be described by the original language $\mathcal{L}$.

Definition. *Let $*$: $\widehat{S} \rightarrow \widehat{*S}$ be an elementary embedding. We call $*$ a nonstandard embedding if for each infinite entity $A \in \widehat{S}$ the condition ${}^\sigma A \subsetneq *A$ holds.*

Definition. *Let $*$: $\widehat{S} \rightarrow \widehat{T}$ be elementary embedding and α a formula in $\mathcal{L}$ with constants in $\widehat{S}$. The $*$ -transform of α is a formula $*\alpha$ which arises from α by changing all the constants c to their respective $*$ values $*c$.*

This allows us to formulate one of the key principles in nonstandard analysis, the transfer principle.

Theorem 2.3 (Transfer principle; [10], Theorem 3.7). *Let $*$: $\widehat{S} \rightarrow \widehat{*S}$ be an elementary embedding. Then a transitively bounded formula α with constants in $\widehat{S}$ is true if and only if its $*$ -transform $*\alpha$ is true.*

There are multiple versions of the transfer principle in the literature that differ in requirements on the quality of the sentences. Some allow more general sentences than transitively bounded sentences considered in [10].

The transfer principle can be used both ways, i. e. to prove $*\alpha$ from α or to prove α from $*\alpha$. In the second case we need to distinguish if a constant $c \in \widehat{*S}$ can be written as $c = *d$ for some $d \in \widehat{S}$. This problem is solved by the standard definition principle.

Theorem 2.4 (Standard definition principle; [10], Theorem 3.8). *Let $*$: $\widehat{S} \rightarrow \widehat{*S}$ be elementary and $A \in \widehat{*S}$ be an entity. Then A is standard if and only if it can be written in the form*

$$A = \{x \in B; \alpha(x)\},$$

where α is a transitively bounded predicate with x being its only free variable and B and all constants occurring in α are standard.

In the rest of this section, we will focus on certain properties of $*$, namely on the way how $*$ operates on the superstructure $\widehat{S}$ and how the properties of functions and relations in $\widehat{S}$ transfer to $\widehat{*S}$.

Definition. *A map $*$: $\widehat{S} \rightarrow \widehat{T}$ is called a superstructure monomorphism if it is one-to-one and if for all entities $A, B \in \widehat{S}$ and for all $x_1, x_2, \dots, x_n \in \widehat{S}$ the following holds:*

- (i) $*$ preserves $\in$: $a \in A$ implies $*a \in *A$.
- (ii) $*$ preserves finite sets: $\{x_1, x_2, \dots, x_n\} = \{x_1, x_2, \dots, x_n\}$.
- (iii) $*$ preserves grouping: $*(x_1, \dots, x_n) = (*x_1, \dots, *x_n)$.
- (iv) $*$ preserves basic set operations:

$$\begin{aligned} *\emptyset &= \emptyset, *(A \cup B) = *A \cup *B, *(A \cap B) = *A \cap *B, *(A \setminus B) = *A \setminus *B, \\ *(A \times B) &= *A \times *B. \end{aligned}$$

- (v) $*$ preserves domains and ranges of relations and commutes with permutations of the variables: Let $\Phi \in \widehat{S}$ be a binary relation. Then $*\Phi$ is a relation such that $\text{dom}(*\Phi) = *\text{dom}(\Phi)$ and $\text{rng}(*\Phi) = *\text{rng}(\Phi)$. Moreover for a relation Ψ satisfying $(x, y) \in \Psi$ if and only if $(y, x) \in \Phi$, then $(w, z) \in *\Psi$ if and only if $(z, w) \in \Phi$. In other words $*(\Phi^{-1}) = (*\Phi)^{-1}$.

Analogous assertion holds for n -ary relations.

- (vi) $*$ preserves the equality relation: $\{(x, x); x \in A\} = \{(x, x); x \in *A\}$.
- (vii) $*$ preserves atomic standard definitions of sets:

$$*\{(x, y); x \in y \in A\} = \{(x, y); x \in y \in *A\}.$$

Theorem 2.5 ([10], Theorem 3.10). Let $* : \widehat{S} \rightarrow *\widehat{S}$ be elementary, then $*$ is a superstructure monomorphism.

Proof. The proof relies on the transfer principle and the standard definition principle. We will also use the abbreviations for transitively bounded formulas from Theorem 1.3.

- (i) Follows from Proposition 2.2.
- (ii) Let the constant C denote the set $\{x_1, \dots, x_n\}$, i. e. the bounded sentence $C = \{x_1, \dots, x_n\}$ is bounded and since $C, x_1, \dots, x_n$ are constants, by transfer principle the $*$ -transform $*C = \{x_1, \dots, x_n\}$ also holds true.
- (iii) Similarly as in the previous part of this proof let C denote the n -tuple $(x_1, \dots, x_n)$. The sentence $C = (x_1, \dots, x_n)$ is transitively bounded and holds true. Again from the transfer principle we obtain $*C = (*x_1, \dots, *x_n)$.
- (iv) Let C denote the entity $A \setminus B$. Following the same argument as in the previous parts we obtain $*C = *A \setminus *B$.
- (v) Let $\Phi \in \widehat{S}$ be a binary relation. Clearly $\Phi \in S_n$ for some $n \in \mathbb{N}$. Since S_n is transitive by Lemma 1.1, from $(x, y) \in \Phi$ it follows that $x, y \in S_n$ and thus $\Phi \subseteq S_n \times S_n$. By Lemma 2.2 we have $*\Phi \subseteq *S_n \times *S_n$ and thus $*\Phi$ is an entity in $*\widehat{S}$.

Define $C = \text{dom}(\Phi)$, i. e. $C = \{x \in S_n; \exists y \in S_n : (x, y) \in \Phi\}$. From the standard definition principle we have $*C = \{x \in *S_n; \exists y \in *S_n : (x, y) \in *\Phi\}$

and thus ${}^*\text{dom}(\Phi) = \text{dom}({}^*\Phi)$. Following the same argument we can show ${}^*\text{rng}(\Phi) = \text{rng}({}^*\Phi)$.

Let Ψ be a binary relation such that $(w, z) \in \Psi$ if and only if $(z, w) \in \Phi$. Then

$$\Psi = \{z \in S_n \times S_n; \exists x, y \in S_n : (z = (y, x) \wedge (x, y) \in \Phi)\}.$$

By the standard definition principle we have

$${}^*\Psi = \{z \in {}^*S_n \times {}^*S_n; \exists x, y \in {}^*S_n : (z = (y, x) \wedge (x, y) \in {}^*\Phi)\}$$

$$\text{and thus } {}^*\Psi = \{(y, x) \in {}^*S_n \times {}^*S_n; (x, y) \in {}^*\Phi\}.$$

As we will not need the remaining assertions in the rest of this thesis, for the rest of the proof we refer to [10]. $\square$

Corollary 2.6 ([10], Corollary 3.11). *Let $*$ be a nonstandard embedding and A be an entity in $\widehat{S}$. Then ${}^*A = {}^\sigma A$ if and only if A is finite.*

The next theorem shows that every function on an entity $A \in \widehat{S}$ can be extended to the standard set ${}^*A$.

Theorem 2.7 ([10], Theorem 3.13). *Let $A, B \in \widehat{S}$ be entities and $f : A \rightarrow B$ a function. Then ${}^*f : {}^*A \rightarrow {}^*B$ and the following holds:*

- (i) *f is injective if and only if ${}^*f$ is injective.*
- (ii) *f is surjective if and only if ${}^*f$ is surjective.*
- (iii) *${}^*(f(a)) = ({}^*f)({}^*a)$ for every $a \in A$.*
- (iv) *${}^*(f|_C) = {}^*f|_{{}^*C}$ for $C \subseteq A$.*
- (v) *${}^*(f(C)) = {}^*f({}^*C)$ for $C \subseteq A$ and ${}^*(f^{-1}(D)) = {}^*f^{-1}({}^*D)$ for $D \subseteq B$.*
- (vi) *Let $g : B \rightarrow C$ for some $C \in \widehat{S}$, then ${}^*(g \circ f) = ({}^*g) \circ ({}^*f)$.*

Proof. By Theorem 1.3 the sentence $f : A \rightarrow B$ is true and transitively bounded, thus from the transfer principle we have ${}^*f : {}^*A \rightarrow {}^*B$.

- (i) The function f is injective if and only if the sentence

$$\forall \underline{x}, \underline{y} \in A : \underline{x} \neq \underline{y} \Rightarrow f(\underline{x}) \neq f(\underline{y})$$

holds true. By the transfer principle the sentence

$$\forall \underline{x}, \underline{y} \in {}^*A : \underline{x} \neq \underline{y} \Rightarrow ({}^*f)(\underline{x}) \neq ({}^*f)(\underline{y})$$

is also holds true if and only if f is injective and thus ${}^*f$ is injective if and only if ${}^*f$ is injective.

- (ii) Following the same argument as in the previous part of the proof, f is surjective if and only if the sentence $\forall \underline{b} \in B \exists \underline{a} \in A : (\underline{a}, \underline{b}) \in f$ is true and by the transfer principle the previous sentence is true if and only if the sentence $\forall \underline{b} \in {}^*B \exists \underline{a} \in {}^*A : (\underline{a}, \underline{b}) \in {}^*f$ holds.

- (iii) Let $c = f(a)$, then the transitively bounded sentence $(a, c) \in f$ is true. Now since $*$ is a superstructure monomorphism, we have $*(a, c) = (*a, *c) \in *f$ and thus $*c = (*f)(*a)$.
- (iv) Clearly $f|_C = \{(x, y) \in f; x \in C\}$ and by the standard definition principle implies $*(f|_C) = \{(x, y) \in *f; x \in *C\}$, hence $*(f|_C) = (*f)|_{*C}$.
- (v) Since $*$ is a structure monomorphism, we have

$$*(f(C)) = *(\text{rng}(f|_C) = \text{rng}(*f|_{*C}) = (*f)(*C).$$

Since $f^{-1}(D) = \{x \in A; f(x) \in D\}$, from the standard definition principle it follows

$$*(f^{-1}(D)) = \{x \in *A; (*f)(x) \in *D\} = (*(f^{-1}))(*D).$$

- (vi) Define $h = g \circ f$. The sentence $\forall \underline{x} \in A : h(\underline{x}) = g(f(\underline{x}))$ is transitively bounded thanks to Theorem 1.3 and holds true. By the transfer principle the $*$ -transform of the previous sentence $\forall \underline{x} \in *A : *h(\underline{x}) = *g(*f(\underline{x}))$ also holds true, thus $*(g \circ f) = *g \circ *f$. $\square$

Let A and B be entities in $\widehat{S}$ and $f : A \rightarrow B$ be a function. Since for every $a \in A$ the equation $*(f(a)) = (*f)(*a)$ holds, in cases with no risk of ambiguity we will from now on not distinguish between the functions on $*A$ and A and will write only f instead of $*f$. We will also follow similar convention for relations.

2.2 Internal and external sets

So far we were successful in translating statements about atoms in $\widehat{S}$ by the transfer principle and some statements about entities and functions in $\widehat{S}$ by the standard definition principle and by showing that $*$ is a superstructure monomorphism.

Problems arise when we start translating statements of subsets of S : Let α be the sentence

$$\forall \underline{A} \in \mathcal{P}(S) : \beta(\underline{A}),$$

where $\beta(\underline{A})$ is suitable transitively bounded predicate, and assume α holds true. The $*$ -transform of α (recalling that $\mathcal{P}(S)$ is a constant)

$$\forall \underline{A} \in *\mathcal{P}(S) : \beta(\underline{A})$$

is also true by the transfer definition principle. If we were able to prove that $*\mathcal{P}(S) = \mathcal{P}(*S)$, then we could say that even the sentences about the subsets of S can be easily translated. However, we will see that it is not the case and following the previous example we already know that the properties of the subsets of S pass to the elements of $*\mathcal{P}(S)$, the so-called internal sets.

Definition. Let $*$: $\widehat{S} \rightarrow *\widehat{S}$ be an elementary embedding. Then all the elements of standard sets are called *internal*. Every $x \in *\widehat{S}$, which is not internal, is called *external*.

We define the nonstandard universe $\mathcal{J}$ as the system of all internal elements in ${}^*\widehat{S}$. In other words

$$\mathcal{J} = \bigcup \{ {}^*A; A \text{ is an entity in } \widehat{S} \}.$$

Proposition 2.8 ([10], Proposition 3.16). *Every standard element is internal. Moreover*

$$\mathcal{J} = \bigcup_{n=1}^{\infty} {}^*S_n.$$

Definition. *Let α be a formula. If all the constants occurring in α are elements of the nonstandard universe $\mathcal{J}$, the formula α is called an internal formula.*

To determine whether an entity in ${}^*\widehat{S}$ is internal we use the internal definition principle.

Theorem 2.9 (The internal definition principle; [10], Theorem 3.17). *Let $A \in {}^*\widehat{S}$ be an entity. Then A is internal if and only if A can be written in the form*

$$A = \{x \in B; \alpha(x)\},$$

where B is an internal entity and α is a transitively bounded internal predicate with x being the only free variable in α .

The internal sets play an important role in nonstandard analysis, which will become clear later, especially in Chapter 3. Many important sets in ${}^*\widehat{S}$ will turn out to be external, although that is not a disadvantage. On the contrary, the fact that certain sets are external has important consequences.

Following theorems describe various properties of internal and external sets which we will need later in the following chapters.

Theorem 2.10 ([10]; Theorem 3.19). *Let $A, B \in \mathcal{J}$ be entities, then $A \cup B$, $A \cap B$, $A \setminus B$, $A \times B \in \mathcal{J}$.*

Theorem 2.11 ([10], Theorem 3.21). *Let $A \in \widehat{S}$ be an entity. Then*

$${}^*\mathcal{P}(A) = \{B \subseteq {}^*A; B \text{ is internal}\}.$$

Theorem 2.12 ([10]; Theorem 3.22). *Let $A \in \widehat{S}$ be an infinite entity. Then ${}^\sigma A$ and $\mathcal{P}({}^*A)$ are external and ${}^\sigma\mathcal{P}(A) \subsetneq {}^*\mathcal{P}(A) \subsetneq \mathcal{P}({}^*A)$.*

Theorem 2.13 ([10], Corollary A.6). *Let X_i , I and $\mathcal{X} = \{X_i; i \in I\}$ be entities in $\widehat{S}$. Then*

$${}^*\left(\bigcup_{i \in I} X_i\right) = \bigcup_{i \in {}^*I} {}^*X_i.$$

2.3 Ultrapower models

There are two common approaches to nonstandard analysis. The first, which we use in this thesis, relies on model theory and has been introduced by Robinson in 1960. The existence of such models can be proved using various methods. In this section we will outline some of the principles behind so-called ultrapower models.

The second approach follows completely different route. Rather than introducing new numbers to the real line, it modifies the axioms of set theory to create *IST*, internal set theory. This concept was introduced by Nelson in [5].

The idea behind ultrapower models is not complicated: For our set S , we find an index set J big enough and define an equivalence on the set of all mappings $f : J \rightarrow \widehat{S}$ such that two functions from $\widehat{S}^J$ are in the same equivalence class if they are equal "almost everywhere". To implement this equivalence, we need to clarify the concept of the "almost everywhere" property. For this purpose, we define ultrafilters.

Definition. Let X be a set. We call a set $\mathcal{F} \subseteq \mathcal{P}(X)$ a filter, if the following holds:

- (i) $\emptyset \notin \mathcal{F}$.
- (ii) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
- (iii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

We call $\mathcal{F}$ an ultrafilter, if in addition the following condition is satisfied:

- (iv) For $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ holds.

Definition. We call a filter $\mathcal{F}$ free, if $\bigcap \mathcal{F} = \emptyset$. If there exists a countable set $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\bigcap \mathcal{F}_0 \notin \mathcal{F}$, the filter $\mathcal{F}$ is called δ -incomplete.

If the set X is infinite, there always exist a free ultrafilter constructed as the set of all subsets $A \subseteq X$ such that $X \setminus A$ is finite, although the proof relies on the *axiom of choice* (or one of its less restrictive forms called the *maximal ideal theorem*) (see [10], p. 47, and the reference there).

It can be proved that every δ -incomplete ultrafilter is free (see for example [10], Proposition 4.12). The implication that every free ultrafilter is δ -incomplete, also known as *Ulam's measure problem*, is consistent with the axioms of ZF set theory, but it is not provable whether the negation of Ulam's measure problem is consistent with ZF set theory. For more information on this topic, we refer to [10], p. 48, and the reference cited there.

Let $\mathcal{L}$ be a formal language and I be a surjective interpretation map onto $\widehat{S}$. Let J be an infinite set, $\mathcal{F}$ ultrafilter on J and $\widehat{S}^J$ the set of all functions $f : J \rightarrow \widehat{S}$. Let $f, g \in \widehat{S}^J$. Then we define the equivalence $\sim$ on $\widehat{S}^J$ by

$$f \sim g \stackrel{\text{def}}{\iff} J_{f,g} = \{j \in J; f(j) = g(j)\} \in \mathcal{F}.$$

Let $\mathcal{S}$ be the set of equivalence classes with respect to the equivalence $\sim$ and let us denote the class containing $f \in \widehat{S}^J$ by $[f]$. Then we can define the interpretation map $I_0 : \widehat{S} \rightarrow \mathcal{S}$ such that every $c \in \widehat{S}$ is associated with the class containing the constant function f_c defined by $f_c(j) = c$ for every $j \in J$. The interpretation map needs to be equipped with relations $\in_{\mathcal{F}}$ and $=_{\mathcal{F}}$. For $[f], [g] \in \mathcal{S}$, we define

$$\begin{aligned} [f] \in_{\mathcal{F}} [g] &\stackrel{\text{def}}{\iff} f(j) \in g(j) \text{ for almost all } j \in J, \\ [f] =_{\mathcal{F}} [g] &\stackrel{\text{def}}{\iff} f(j) = g(j) \text{ for almost all } j \in J. \end{aligned}$$

Now it remains to construct a map ϕ such that the interpretation map $I' = \phi \circ I_0$ maps $\widehat{S}$ into a superstructure ${}^*\widehat{S}$. This process is too broad and technical to be fully covered in this thesis and therefore we refer the reader to [10], p. 48 - 57. The results proved there rely on theorem proved by Łoś and Luxemburg stating that a sentence in $\mathcal{L}$ is true under the interpretation map I if and only if it is true under the interpretation map I_0 .

The following theorem sums up the properties of ultrapower models.

Theorem 2.14 ([10], Theorem 4.20). *The map $*$ constructed using ultrapower models is an elementary embedding which maps S_n into ${}^*S_n$.*

Moreover, if the ultrafilter $\mathcal{F}$ is δ -incomplete, then $*$ is a nonstandard embedding.

3. Hypernatural and hyperreal numbers

In this chapter let $*$ be a nonstandard embedding and let $\mathbb{R} \in \widehat{S}$ be an entity.

Definition. We call the elements of ${}^*\mathbb{N}$ hypernatural numbers and the elements of ${}^*\mathbb{R}$ hyperreal numbers.

This chapter covers results concerning the hypernatural and the hyperreal numbers, in particular the properties of infinitesimal and infinite numbers.

Let us first focus on order and arithmetic on ${}^*\mathbb{R}$.

Proposition 3.1 ([10]; Proposition 5.4). $*$ $\leq$ is a total order on ${}^*\mathbb{R}$.

Proof. Since $\leq$ is a total order on $\mathbb{R}$, the following sentences hold:

$$\begin{aligned} \forall \underline{x} \in \mathbb{R} : \underline{x} \leq \underline{x}, & \quad \forall \underline{x}, \underline{y} \in \mathbb{R} : (\underline{x} \leq \underline{y} \wedge \underline{y} \leq \underline{x}) \Rightarrow (\underline{x} = \underline{y}), \\ \forall \underline{x}, \underline{y} \in \mathbb{R} : \underline{x} \leq \underline{y} \vee \underline{y} \leq \underline{x}, & \quad \forall \underline{x}, \underline{y}, \underline{z} \in \mathbb{R} : (\underline{x} \leq \underline{y} \wedge \underline{y} \leq \underline{z}) \Rightarrow \underline{x} \leq \underline{z}. \end{aligned}$$

From the transfer principle applied on the previous sentences we immediately obtain that $*$ $\leq$ is a total order on ${}^*\mathbb{R}$. $\square$

Similarly as in the previous proposition we can prove that $*$ $<$ is a strict order on ${}^*\mathbb{R}$.

Recall $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$. Since $\mathbb{R} \in \widehat{S}$, by Theorem 2.7 there exists $|\cdot| : {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$ in $\widehat{S}$ such that $|r| = |s|$ for $r \in {}^\sigma\mathbb{R}$, $s \in \mathbb{R}$ satisfying $r = *s$.

Theorem 3.2 ([10]; Theorem 5.5). ${}^*\mathbb{R}$ and ${}^\sigma\mathbb{R}$ are ordered fields and ${}^\sigma\mathbb{R}$ is isomorphic to $\mathbb{R}$ with respect to $|\cdot|$.

Proof. Apply the transfer principle to the following sentences

$$\begin{aligned} \forall \underline{a}, \underline{b}, \underline{c} \in \mathbb{R} : (\underline{a} + \underline{b}) + \underline{c} &= \underline{a} + (\underline{b} + \underline{c}), & \forall \underline{a}, \underline{b}, \underline{c} \in \mathbb{R} : (\underline{a} \cdot \underline{b}) \cdot \underline{c} &= \underline{a} \cdot (\underline{b} \cdot \underline{c}), \\ \forall \underline{a}, \underline{b} \in \mathbb{R} : \underline{a} + \underline{b} &= \underline{b} + \underline{a}, & \forall \underline{a}, \underline{b} \in \mathbb{R} : \underline{a} \cdot \underline{b} &= \underline{b} \cdot \underline{a}, \\ \forall \underline{a}, \underline{b}, \underline{c} \in \mathbb{R} : \underline{a} \cdot (\underline{b} + \underline{c}) &= \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}, & \forall \underline{a}, \underline{b}, \underline{c} \in \mathbb{R} : (\underline{a} + \underline{b}) \cdot \underline{c} &= \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{c}, \\ \exists \underline{0} \in \mathbb{R} \forall \underline{a} \in \mathbb{R} : \underline{a} + \underline{0} &= \underline{a} = \underline{0} + \underline{a}, & \exists \underline{1} \in \mathbb{R} \forall \underline{a} \in \mathbb{R} : \underline{1} \cdot \underline{a} &= \underline{a} = \underline{a} \cdot \underline{1} \end{aligned}$$

to show that ${}^*\mathbb{R}$ is a field. The set ${}^*\mathbb{R}$ is an ordered field since the $*$ -transforms of the sentences

$$\begin{aligned} \forall \underline{a}, \underline{b}, \underline{c} \in \mathbb{R} : (\underline{a} \leq \underline{b}) &\Rightarrow (\underline{a} + \underline{c} \leq \underline{b} + \underline{c}), \\ \forall \underline{a}, \underline{b}, \underline{c} \in \mathbb{R} : (\underline{a} \leq \underline{b} \wedge \underline{c} \geq 0) &\Rightarrow (\underline{a} \cdot \underline{c} \leq \underline{b} \cdot \underline{c}). \end{aligned}$$

hold true by the transfer principle.

${}^\sigma\mathbb{R}$ is isomorphic to $\mathbb{R}$ by Theorem 2.7. $\square$

Following a similar convention for functions and relations we will often not distinguish between real numbers written actually as numbers and their hyperreal counterparts from ${}^\sigma\mathbb{R}$, for example we will write 1 instead of $*1$.

3.1 Infinite and infinitesimal numbers

In this section we define the infinite and the infinitesimal numbers and using the apparatus from the previous chapter we show that these numbers actually exist in ${}^*\mathbb{R}$.

Definition. We call a number $r \in {}^*\mathbb{R}$

- finite if $|r| < n$ for some $n \in {}^\sigma\mathbb{N}$,
- infinite if $|r| > n$ for every $n \in {}^\sigma\mathbb{N}$,
- infinitesimal if $|r| < n^{-1}$ for every $n \in {}^\sigma\mathbb{N}$.

We denote the set of all finite numbers by $\text{fin}({}^*\mathbb{R})$, the set of all infinitesimal numbers by $\text{inf}({}^*\mathbb{R})$ and the set of all infinite hyperreal numbers by $\mathbb{R}_\infty$.

Since in the next chapters we will work mostly with positive numbers, it is convenient that we similarly denote the set of all positive finite numbers by $\text{fin}({}^*\mathbb{R}^+)$, the set of all positive infinitesimal numbers by $\text{inf}({}^*\mathbb{R}^+)$ and the set of all positive infinite hyperreal numbers by $\mathbb{R}_\infty^+$.

Proposition 3.3 ([10], Proposition 5.9). *There exists at least one hypernatural number $N \in {}^*\mathbb{N}$ such that $N \notin {}^\sigma\mathbb{N}$.*

Moreover every $N \in {}^*\mathbb{N} \setminus {}^\sigma\mathbb{N}$ is infinite.

Proof. Since $\mathbb{N}$ is infinite, from Corollary 2.6 it follows that ${}^*\mathbb{N} \setminus {}^\sigma\mathbb{N} \neq \emptyset$.

Let $n \in \mathbb{N}$. Then the sentence

$$\forall \underline{m} \in \mathbb{N} : (\underline{m} \neq 1 \wedge \underline{m} \neq 2 \wedge \cdots \wedge \underline{m} \neq n) \Rightarrow \underline{m} > n$$

holds. By transfer principle the $*$ -transform

$$\forall \underline{m} \in {}^*\mathbb{N} : (\underline{m} \neq {}^*1 \wedge \underline{m} \neq {}^*2 \wedge \cdots \wedge \underline{m} \neq {}^*n) \Rightarrow \underline{m} > {}^*n$$

also holds. Now choose $N \in {}^*\mathbb{N} \setminus {}^\sigma\mathbb{N}$. Then N cannot be written in the form $N = {}^*k$ for any $k \in \mathbb{N}$ and thus $N > k$ for all $k \in {}^\sigma\mathbb{N}$. $\square$

Definition. We denote the set of all infinite hypernatural numbers by $\mathbb{N}_\infty$.

Corollary 3.4. *The set $\text{inf}({}^*\mathbb{R})$ is nonempty.*

Proof. Take $N \in \mathbb{N}_\infty$. Then $r = N^{-1}$ is infinitesimal by definition. $\square$

Corollary 3.5 ([10]; Proposition 5.15). *Let $r \in {}^*\mathbb{R} \setminus \{0\}$. Then $r \in \text{inf}({}^*\mathbb{R})$ if and only if $r^{-1} \in \mathbb{R}_\infty$.*

Proof. Using the respective definitions we have

$$r \in \text{inf}({}^*\mathbb{R}) \Leftrightarrow \forall \underline{n} \in {}^\sigma\mathbb{N} : |r| < \underline{n}^{-1} \Leftrightarrow \forall \underline{n} \in {}^\sigma\mathbb{N} : |r|^{-1} > \underline{n} \Leftrightarrow r^{-1} \in \mathbb{R}_\infty. \quad \square$$

Proposition 3.6. *Let $r \in \text{fin}({}^*\mathbb{R})$.*

(i) *Let $x \in \text{inf}({}^*\mathbb{R})$. Then $r \cdot x \in \text{inf}({}^*\mathbb{R})$.*

(ii) Let $x \in \mathbb{R}_\infty$. Then $x - r, x \cdot r^{-1} \in \mathbb{R}_\infty$.

Proof. Let $|r| \leq k$ for some $k \in {}^\sigma\mathbb{N}$.

(i) Let $x \in \inf({}^*\mathbb{R})$ and choose $m \in {}^\sigma\mathbb{N}$. Then $|r \cdot x| < k \cdot n^{-1}$ for every $n \in {}^\sigma\mathbb{N}$ and for $n \geq k \cdot m$ in particular. Then $|r \cdot x| \leq m^{-1}$ and thus $r \cdot x \in \inf({}^*\mathbb{R})$.

(ii) Let $x \in \mathbb{R}_\infty$ and choose $m \in {}^\sigma\mathbb{N}$. Then $|x - r| \geq |x| - |r| \geq n - k - 1$ holds for every $n \in {}^\sigma\mathbb{N}$, especially for $n \geq m + k + 1$. Then $|x - r| \geq m$ and $x - r \in \mathbb{R}_\infty$.

The last assertion is proved in similar manner: choose $m \in {}^\sigma\mathbb{N}$. Then we have $|x \cdot r^{-1}| \geq n \cdot k^{-1}$ for every $n \in \mathbb{N}$ and for $n \geq k \cdot m$ in particular. Then $|x \cdot r^{-1}| \geq m$, hence $x \cdot r^{-1} \in \mathbb{R}_\infty$. $\square$

So far all the properties of $\mathbb{R}$ have translated into ${}^*\mathbb{R}$ using only the transfer principle. Let us now look on situations where internal sets are needed.

Theorem 3.7 ([10], Theorem 5.12). *The set $\mathbb{N}_\infty$ has no smallest element. However, every internal subset of ${}^*\mathbb{N}$ has the smallest element.*

Proof. For contradiction assume that $N \in \mathbb{N}_\infty$ is the smallest element of $\mathbb{N}_\infty$. Then $N > n$ for all $n \in {}^\sigma\mathbb{N}$ and $N - 1 > n$ for all $n \in {}^\sigma\mathbb{N}$. Since the sentence $\exists \underline{m} \in {}^\sigma\mathbb{N} \forall \underline{n} \in {}^\sigma\mathbb{N} : \underline{m} > \underline{n}$ is false, we have $N - 1 \in \mathbb{N}_\infty$, a contradiction.

Since $\mathbb{N}$ is well-ordered, the sentence

$$\forall \underline{A} \in \mathcal{P}(\mathbb{N}) : (\underline{A} \neq \emptyset) \Rightarrow (\exists \underline{n} \in \underline{A} \forall \underline{m} \in \underline{A} : \underline{n} \leq \underline{m})$$

holds true. By the transfer principle the $*$ -transform

$$\forall \underline{A} \in {}^*\mathcal{P}(\mathbb{N}) : (\underline{A} \neq \emptyset) \Rightarrow (\exists \underline{n} \in {}^*\underline{A} \forall \underline{m} \in {}^*\underline{A} : \underline{n} \leq \underline{m})$$

also holds true. Thus every nonempty element of ${}^*\mathcal{P}(\mathbb{N})$ has a smallest element and from Theorem 2.11 it follows that every nonempty internal subset of ${}^*A$ has a smallest element. $\square$

Following analogous argument, a similar result can be achieved for internal subsets of ${}^*\mathbb{R}$.

Theorem 3.8 ([10]; Theorem 5.14). *Let $A \subseteq {}^*\mathbb{R}$ be internal and bounded from above in ${}^*\mathbb{R}$. Then A has the least upper bound.*

Proof. Since every set $A \in \mathcal{P}(\mathbb{R})$ bounded from above has the least upper bound, the sentence

$$\forall \underline{A} \in \mathcal{P}(\mathbb{R}) : (\underline{A} \neq \emptyset \wedge \alpha(\underline{A}, \mathbb{R}) \Rightarrow \beta(\underline{A}, \mathbb{R}),$$

where

$$\alpha(\underline{A}, \mathbb{R}) = \exists \underline{x} \in \mathbb{R} \forall \underline{a} \in \underline{A} : \underline{a} \leq \underline{x}$$

is a transitively bounded predicate for " A is bounded from above" and

$$\beta(\underline{A}, \mathbb{R}) = \exists \underline{s} \in \mathbb{R} : (\forall \underline{a} \in \underline{A} : \underline{a} \leq \underline{s}) \wedge (\forall \underline{x} \in \mathbb{R} \exists \underline{a} \in \underline{A} : (\underline{x} < \underline{s}) \Rightarrow (\underline{x} < \underline{a}))$$

is a transitively bounded predicate meaning " A has the least upper bound", holds and is transitively bounded. From the transfer principle we receive that every $A \in {}^*\mathcal{P}(\mathbb{R})$ that is bounded from above in ${}^*\mathbb{R}$ has the least upper bound in ${}^*\mathbb{R}$ and by Theorem 2.11 we know that the elements of ${}^*\mathcal{P}(\mathbb{R})$ are precisely the internal subsets of ${}^*\mathbb{R}$. $\square$

3.2 Standard part homomorphism

Definition. Let $x, y \in {}^*\mathbb{R}$. We call x and y infinitely close, if $|x - y| \in \inf({}^*\mathbb{R}^+)$. We write $x \simeq y$.

Theorem 3.9 ([10]; Theorem 5.19). Let $x \in \text{fin}({}^*\mathbb{R})$. Then there exists unique $\tilde{x} \in {}^\sigma\mathbb{R}$ such that $x \simeq \tilde{x}$.

Proof. First we show the existence. Set $A = \{y \in {}^\sigma\mathbb{R}; y < x\}$. Since $x \in \text{fin}({}^*\mathbb{R})$, the set A is bounded from above and since $\mathbb{R}$ and ${}^\sigma\mathbb{R}$ are isomorphic by Theorem 3.2 it follows that there exists a least upper bound $\tilde{x} \in {}^\sigma\mathbb{R}$ of the set A . It remains to show $\tilde{x} \simeq x$, i. e. $|x - \tilde{x}| < \frac{1}{n}$ for every $n \in {}^\sigma\mathbb{N}$.

Choose $n \in {}^\sigma\mathbb{N}$ arbitrary. The inequality $\tilde{x} \geq x - \frac{1}{n}$ holds since otherwise the inequality $\tilde{x} < x - \frac{1}{n}$ would imply $\tilde{x} + \frac{1}{n} \in A$ and thus $\tilde{x}$ could not be an upper bound, a contradiction. Analogously $\tilde{x} \leq x - \frac{1}{n}$ holds, since the converse would imply $\tilde{x} - \frac{1}{n} > x$, therefore $\tilde{x} - \frac{1}{n}$ is an upper bound of A strictly smaller than $\tilde{x}$ which contradicts the choice of $\tilde{x}$.

To show the uniqueness, let $\tilde{x}_1, \tilde{x}_2 \in {}^\sigma\mathbb{R}$ such that $\tilde{x}_1 \simeq x \simeq \tilde{x}_2$. Then we have $|\tilde{x}_1 - \tilde{x}_2| < \frac{1}{n}$ for every $n \in {}^\sigma\mathbb{N}$ and thus $\tilde{x}_1 = \tilde{x}_2$. $\square$

Definition. Let $\text{st} : \text{fin}({}^*\mathbb{R}) \rightarrow {}^\sigma\mathbb{R}$ be the map $x \rightarrow \tilde{x}$ defined in Theorem 3.9. We call $\text{st}(x)$ the standard part of x and the mapping st a standard part homomorphism.

Definition. Let $x \in {}^\sigma\mathbb{R}$. We define the monad of x by

$$\text{mon}(x) = \{y \in {}^*\mathbb{R}; y \simeq x\}.$$

Proposition 3.10 ([10], Proposition 5.23). The set $\text{fin}({}^*\mathbb{R})$ is a disjoint union of all monads, i. e.

$$\text{fin}({}^*\mathbb{R}) = \bigcup_{r \in {}^\sigma\mathbb{R}} \text{mon}(r).$$

Proof. Let $r \in \text{fin}({}^*\mathbb{R})$. Clearly $r \in \text{mon}(\text{st}(r))$ and thus $\text{fin}({}^*\mathbb{R}) \subseteq \bigcup_{r \in {}^\sigma\mathbb{R}} \text{mon}(r)$.

Conversely let $\tilde{r} \in {}^\sigma\mathbb{R}$ and $r \in \text{mon}(\tilde{r})$. We have $|\tilde{r}| \leq n$ for some $n \in {}^\sigma\mathbb{N}$ and thus $|r| < |\tilde{r}| + 1 \leq n + 1$, hence $r \in \text{fin}({}^*\mathbb{R})$.

To prove that the monads are disjoint, let $r, s \in {}^\sigma\mathbb{R}, r \neq s$. For contradiction let us assume there exists $x \in \text{mon}(r) \cap \text{mon}(s)$. Then $\text{st}(x) = r$ and $\text{st}(x) = s$, thus $r = s$, a contradiction. $\square$

Considering the permanence principle proved in the following section it is important to determine whether the sets defined in this chapter are internal. In the next theorem we show that the sets $\text{fin}({}^*\mathbb{R})$, $\text{inf}({}^*\mathbb{R})$ and $\text{mon}(x)$ are in fact external.

Theorem 3.11 ([10]; Theorem 5.24). The sets $\text{fin}({}^*\mathbb{R})$, $\text{inf}({}^*\mathbb{R})$ and $\text{mon}(x)$ for $x \in {}^*\mathbb{R}$ are external.

Proof. To prove that $\text{inf}({}^*\mathbb{R})$ is external, by Theorem 3.8 it suffices to prove that it has no least upper bound. For contradiction assume that $x \in {}^*\mathbb{R}$ is a least upper bound of $\text{inf}({}^*\mathbb{R})$. If $x \in \text{inf}({}^*\mathbb{R})$, then $2x \in \text{inf}({}^*\mathbb{R})$, which would imply that x is not an upper bound. Thus $x \notin \text{inf}({}^*\mathbb{R})$ and $\frac{x}{2} \notin \text{inf}({}^*\mathbb{R})$. Since every

infinite number is smaller than every not-infinite number, $\frac{x}{2}$ is an upper bound of $\inf(*\mathbb{R})$, a contradiction.

If the set $\text{fin}(*\mathbb{R})$ were internal, then the set

$$\inf(*\mathbb{R}) = \{x \in *\mathbb{R}; x \neq 0 \wedge \frac{1}{x} \notin \text{fin}(*\mathbb{R})\} \cup \{0\}$$

would be internal by the internal definition principle, which contradicts the previous part of the proof.

Similarly if the set $\text{mon}(x)$ were internal for some $x \in {}^\sigma\mathbb{R}$, the set

$$\inf(*\mathbb{R}) = \{y \in *\mathbb{R}; \exists z \in \text{mon}(x) : y = z - x\}$$

would have to be internal by the internal definition principle, a contradiction. $\square$

3.3 The permanence principle

In this section we prove the permanence principle, sometimes called the Cauchy principle. It states that, roughly speaking, if something holds for sufficiently large natural numbers, it also holds for some infinite hypernatural number and, under additional assumptions, vice versa. The permanence principle and its immediate corollary will be often used in the following chapters to provide nonstandard characterizations for various qualities of metric spaces and dynamical systems linking the standard definitions with infinitesimal and infinite hypernatural and hyperreal numbers.

Theorem 3.12 (The permanence principle for $*\mathbb{N}$; [10], Theorem 6.1). *Let $\alpha(\underline{n})$ be an internal predicate with n being its only free variable.*

(i) *Let $\alpha(\underline{n})$ hold for all sufficiently large natural numbers, i. e. let there exist $n_0 \in {}^\sigma\mathbb{N}$ such that $\alpha(\underline{n})$ holds for all $n \in {}^\sigma\mathbb{N}, n \geq n_0$. Then there exists $N \in \mathbb{N}_\infty$ such that $\alpha(\underline{n})$ holds for every $n \in *\mathbb{N}, n_0 \leq n \leq N$. In particular, $\alpha(\underline{n})$ holds for some $N \in \mathbb{N}_\infty$.*

(ii) *Let $\alpha(\underline{n})$ hold for every $n \in \mathbb{N}_\infty$. Then there exists $n_0 \in {}^\sigma\mathbb{N}$ such that $\alpha(\underline{n})$ holds for every $n \in *\mathbb{N}, n_0 \leq n$.*

Proof. (i) The set $M = \{n \in *\mathbb{N}; \forall m \in \mathbb{N} : (n_0 \leq m \leq n) \Rightarrow \alpha(m)\}$ is internal by the internal definition principle and ${}^\sigma\mathbb{N} \subseteq M$. Since ${}^\sigma\mathbb{N}$ is external by Theorem 2.12 and M is internal, $M \setminus {}^\sigma\mathbb{N}$ is nonempty and thus we can find $N \in \mathbb{N}_\infty$ such that $\alpha(n)$ holds for $n_0 \leq n \leq N$.

(ii) The set $M = \{n \in *\mathbb{N}; \forall m \in *\mathbb{N} : n \leq m \Rightarrow \alpha(m)\}$ is internal. From the assumptions we have $\mathbb{N}_\infty \subseteq M$. Since ${}^\sigma\mathbb{N}$ is external by Theorem 2.12, from Theorem 2.10 it follows that $\mathbb{N}_\infty$ is also external and thus $M \setminus \mathbb{N}_\infty$ is nonempty. Therefore we can find $n_0 \in {}^\sigma\mathbb{N}$ such that $\alpha(n)$ holds for every $n \in *\mathbb{N}, n_0 \leq n$. $\square$

Corollary 3.13 (The permanence principle for $*\mathbb{R}^+$; [10], Corollary 6.2). *Let $\alpha(\underline{\varepsilon})$ be an internal predicate with ε being the only free variable in α .*

- (i) Let $\alpha(\underline{\varepsilon})$ hold for all sufficiently small positive real numbers, i. e. let there exist $\varepsilon_0 \in {}^\sigma\mathbb{R}^+$ such that $\alpha(\underline{\varepsilon})$ holds for every $\varepsilon \in {}^\sigma\mathbb{R}^+, \varepsilon < \varepsilon_0$. Then there exists $c \in \inf({}^*\mathbb{R}^+)$ such that $\alpha(c)$ holds.
- (ii) Let $\alpha(\underline{c})$ hold for every $c \in \inf({}^*\mathbb{R}^+)$. Then there exists $\varepsilon_0 \in {}^\sigma\mathbb{R}^+$ such that $\alpha(\underline{\varepsilon})$ holds for every $\varepsilon \in {}^*\mathbb{R}^+, \varepsilon < \varepsilon_0$.
- (iii) Let $\alpha(\underline{r})$ hold for all sufficiently big real numbers, in other words let there exists $r_0 \in {}^\sigma\mathbb{R}^+$ such that $\alpha(\underline{r})$ holds for every $r \in {}^\sigma\mathbb{R}, r \geq r_0$. Then $\alpha(\underline{R})$ holds for some $R \in \mathbb{R}_\infty^+$.
- (iv) Let $\alpha(\underline{R})$ hold for all $R \in \mathbb{R}_\infty^+$. Then there exists $r_0 \in {}^\sigma\mathbb{R}^+$ such that $\alpha(\underline{r})$ holds for every $r \in {}^*\mathbb{R}^+, r \geq r_0$.

Proof. (i) From the assumptions we know that $\alpha(n^{-1})$ holds for all sufficiently large $n \in \mathbb{N}$. By Theorem 3.12 $\alpha(N^{-1})$ for some $N \in \mathbb{N}_\infty$ and the desired claim follows by setting $c = N^{-1}$.

(ii) Let $\beta(\underline{\varepsilon})$ be the internal predicate

$$\forall \underline{r} \in {}^*\mathbb{R} : 0 < \underline{r} \leq \frac{1}{\underline{\varepsilon}} \Rightarrow \alpha(\underline{r}).$$

Now $\beta(\underline{N})$ holds for every $N \in \mathbb{N}_\infty$ and by Theorem 3.12 $\beta(\underline{n})$ holds for some $n \in {}^\sigma\mathbb{N}$. Now it remains to set $\varepsilon_0 = n^{-1}$.

(iii) Follows from (i) by taking $\alpha(r^{-1})$.

(iv) Since $R \in \mathbb{R}_\infty$ if and only if $R^{-1} \in \inf({}^*\mathbb{R})$ by Corollary 3.5, the assertion follows from (ii) for $\alpha(r^{-1})$. $\square$

4. Metric spaces

In this chapter we review the classical results in metric spaces theory proved by nonstandard methods. We introduce infinitesimally close point to the metric space similarly as we did with real numbers and study the properties of the space and functions acting on it using the relation of infinitesimal closeness. In particular, we discuss different types of continuity, convergence and compactness.

In the whole chapter let X be a set and an entity in $\widehat{S}$, $\mathbb{R}_0^+ \in \widehat{S}$ be an entity and $* : \widehat{S} \rightarrow \widehat{*S}$ be a nonstandard embedding.

Definition. Let $d : X \times X \rightarrow \mathbb{R}_0^+$. We call the couple (X, d) a metric space, if the following conditions hold:

- (i) $d(x, y) = d(y, x)$ for every $x, y \in X$.
- (ii) $d(x, y) = 0$ if and only if $x = y$.
- (iii) (the triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \in X$.

We call the mapping d a metric.

In the following text we will often use the symbol X for the whole metric space.

From Theorem 2.7 and the transfer principle it follows that the mapping $*d : *X \times *X \rightarrow *\mathbb{R}_0^+$ satisfies all the requirements we have on metric for every $x, y, z \in *X$, the only difference being that $*d$ attains values from $*\mathbb{R}$.

4.1 Basic definitions

In this section the nonstandard characterizations of open, closed and bounded sets are proved.

Definition. Let $x \in X$ and $R \in \mathbb{R}_0^+$. Then we define open ball with the center x and radius R by $B(x, R) = \{y \in X; d(x, y) < R\}$.

Definition. We call a set $U \subset X$ a neighborhood of x if there exists $\delta \in \mathbb{R}^+$ such that $B(x, \delta) \subseteq U$.

Definition. Let $A \subseteq X$. We call a point $x \in A$ an interior point, if there exists a neighborhood $U \subseteq X$ of x such that $U \subseteq A$.

The set A is called open if every $x \in A$ is an interior point.

Definition. Points $y, z \in *X$ are called infinitely close if $d(y, z) \simeq 0$. We write $y \simeq z$.

Let $x \in *X$. We define a monad of x by

$$\text{mon}(x) = \{y \in *X; x \simeq y\} \quad (4.1)$$

For given $x \in X$ the set $\text{mon}(*x)$ can be characterized by

$$\text{mon}(*x) = \bigcap_{n \in \mathbb{N}} *B(x, \frac{1}{n}).$$

since for $y \in *X$ we have

$$y \in \text{mon}(x) \Leftrightarrow x \simeq y \Leftrightarrow \forall \underline{n} \in {}^\sigma\mathbb{N} : d(x, y) < \frac{1}{\underline{n}} \Leftrightarrow \forall \underline{n} \in {}^\sigma\mathbb{N} : y \in *B(x, \frac{1}{\underline{n}}).$$

Proposition 4.1. *Let $A \subseteq X$ and $x \in A$. The point x is an interior point of A if and only if $\text{mon}(x) \subseteq *A$.*

Proof. Let $x \in A$ be an interior point of A , i.e. $B(x, r) \subseteq A$ for some $r \in \mathbb{R}^+$. By (4.1) we get $\text{mon}(*x) \subseteq *B(x, r) \subseteq *A$.

Conversely let $\text{mon}(*x) \subseteq A$. Define $B(\delta) = \{y \in *X; d(*x, y) < \delta\}$. The predicate $B(\delta) \subseteq *A$ is internal and holds for all $\delta \in \inf(*\mathbb{R}^+)$. From the permanence principle there exists $\delta_0 \in \mathbb{R}$ such that $B(*\delta_0) \subseteq *A$ holds. By the standard definition principle we have $*B(x, \delta_0) = B(*\delta_0)$ and thus from the transfer principle we receive $B(x, \delta_0) \subseteq A$. $\square$

The nonstandard characterization of open sets follows immediately from the previous proposition.

Corollary 4.2. *A set $A \subseteq X$ is open if and only if $\bigcup_{x \in A} \text{mon}(x) \subseteq *A$.*

Definition. *Let $A \subseteq X$. We define closure of A by*

$$\overline{A} = \{x \in X; \forall r \in \mathbb{R}^+ : B(x, r) \cap A \neq \emptyset\}.$$

We call the set A closed if $A = \overline{A}$.

Definition. *Let $x \in *X$. We call x a standard point if $x = *y$ for some $y \in X$ and a nearstandard point if $x \simeq *y$ for some $y \in X$. We denote the set of all nearstandard points in $*X$ by $\text{ns}(*X)$.*

The standard part relation $\text{st} \subseteq *X \times X$ is defined by

$$\text{st} = \{(y, x) \in *X \times X; y \simeq *x\}.$$

We define the standard part of $y \in *X$ by

$$\text{st}(y) = \{x \in X; (y, x) \in \text{st}\}$$

and for $A \subseteq *X$ we define

$$\text{st}(A) = \bigcup_{a \in A} \text{st}(a).$$

Let $x, y \in X$ satisfy $*x \simeq z \simeq *y$ for some $z \in *X$. Then from $d(*x, *y) \simeq 0$ and Theorem 3.9 it immediately follows $d(x, y) = 0$ and therefore $x = y$. Thus st is a function on $\text{ns}(*X)$.

We emphasize that by the previous definition $\text{st}(y) = \emptyset$ for $y \notin \text{ns}(*X)$.

Proposition 4.3. *Let $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if $*x \simeq a$ for some $a \in *A$.*

Proof. Let $x \in \overline{A}$. From the definition the sentence

$$\forall \underline{n} \in \mathbb{N} \exists \underline{a} \in A : d(x, \underline{a}) < \frac{1}{\underline{n}}$$

holds true. By the transfer principle the $*$ -transform of the previous sentence holds also true. Choose $n \in \mathbb{N}_\infty$ arbitrary to find $a \in {}^*A$ such that $d({}^*x, a) \simeq 0$, in other words ${}^*x \simeq a$.

Conversely let $x \in X$ and $a \in {}^*A$ satisfy ${}^*x \simeq a$. Choose $n \in \mathbb{N}$. Since $a \in \text{mon}({}^*x) \subseteq {}^*B(x, \frac{1}{n})$, the sentence ${}^*A \cap {}^*B(x, \frac{1}{n}) \neq \emptyset$ holds true. From the transfer principle we obtain $A \cap B(x, \frac{1}{n}) \neq \emptyset$ and since n was arbitrary, we have $x \in \overline{A}$. $\square$

Corollary 4.4. *A set $A \subseteq X$ is closed if and only if for every $a \in {}^*A \cap \text{ns}({}^*X)$ we can find $\tilde{a} \in {}^\sigma A$ such that $a \simeq \tilde{a}$.*

Proof. Let A be closed and $a \in {}^*A \cap \text{ns}({}^*X)$. Find $x \in {}^\sigma X$ such that $x \simeq a$. By the previous proposition $x \in {}^\sigma \overline{A} = {}^\sigma A$.

Conversely let $x \in \overline{A}$. From the previous proposition there exists $a \in {}^*A$ such that ${}^*x \simeq a$. From the assumption it follows $a \simeq \tilde{a}$ for some $\tilde{a} \in {}^\sigma A$ and since ${}^*x, \tilde{a} \in {}^\sigma X$ and ${}^*x \simeq \tilde{a}$, we have ${}^*x = \tilde{a}$, hence $x \in A$. $\square$

Proposition 4.5. *Let (X, d) be a metric space.*

- (i) *Let $\mathcal{I}$ be a set and let $G_i \subseteq X$ be open for every $i \in \mathcal{I}$. Then $\mathcal{G} = \bigcup_{i \in \mathcal{I}} G_i$ is open.*
- (ii) *Let $\mathcal{J}$ be a set and let $F_j \subseteq X$ be closed for every $j \in \mathcal{J}$. Then $\mathcal{F} = \bigcap_{j \in \mathcal{J}} F_j$ is closed.*

Proof. Let $x \in \mathcal{G}$. Then $x \in G_i$ for some $i \in \mathcal{I}$. Since $\text{mon}({}^*x) \subseteq {}^*G_i \subseteq {}^*\mathcal{G}$, the set $\mathcal{G}$ is open.

Let $x \in {}^*\mathcal{F} \cap \text{ns}({}^*X)$. The sentence $\forall \underline{x} \in \mathcal{F} \forall \underline{j} \in \mathcal{J} : \underline{x} \in F_{\underline{j}}$ holds and the transfer principle gives us $x \in {}^*F_j \cap \text{ns}({}^*X)$ for every $j \in {}^*\mathcal{J}$. Since F_j are closed, for every $j \in \mathcal{J}$ we can find $x_j \in {}^\sigma F_j$ such that $x_j \simeq x$ for every $j \in \mathcal{J}$. From $x_j \simeq x_k$ and $x_j, x_k \in {}^\sigma X$ for every $j, k \in \mathcal{J}$, we obtain $x_k = x_j$ for every $j, k \in \mathcal{J}$. Thus for $j \in \mathcal{J}$ we have $x_j \in {}^\sigma \mathcal{F}$ and the set $\mathcal{F}$ is closed by Proposition 4.4. $\square$

Theorem 4.6. *Let $A \subseteq {}^*X$ be an internal set. Then $\text{st}(A)$ is closed.*

Proof. Let $x \in \overline{\text{st}(A)}$. Choose $n \in \mathbb{N}$ arbitrary. From the definition we find $y \in \text{st}(A)$ satisfying $d(x, y) < \frac{1}{2n}$. Since $y \in \text{st}(A)$, there exists $a \in A$ such that $d({}^*y, a) < \frac{1}{2^n}$. Thus, using the triangle inequality, the predicate

$$\exists \underline{a} \in A : d({}^*x, \underline{a}) < \frac{1}{\underline{n}}$$

holds for every $n \in {}^\sigma \mathbb{N}$ and by the permanence principle it also holds for some $N \in \mathbb{N}_\infty$. Then we have $d({}^*x, a) \simeq 0$ for some $a \in A$, in other words $x \in \text{st}(A)$. $\square$

Definition. *We call a set $A \subseteq X$ bounded if $A \subseteq B(x, r)$ for some $x \in X$ and $r \in \mathbb{R}^+$.*

Definition. We call a point $x \in {}^*X$ finite, if there exists $y \in {}^\sigma X$ such that $d(x, y) \in \text{fin}({}^*\mathbb{R}_0^+)$. We denote the set of all finite points of ${}^*X$ by $\text{fin}({}^*X)$.

We emphasize that every finite point $x \in \text{fin}({}^*X)$ is an element of ${}^*B(x_0, r)$ for some $x_0 \in X$ and $r \in \mathbb{R}^+$.

Proposition 4.7. A set $A \subseteq X$ is bounded if and only ${}^*A \subseteq \text{fin}({}^*X)$.

Proof. Let A be bounded. Then we can find $x \in X$ such $d(x, a) < r$ for every $a \in A$ and some $r \in \mathbb{R}^+$. From the transfer principle we obtain $d({}^*x, a) < {}^*r$ for every $a \in {}^*A$.

Conversely let A be unbounded. Choose $x \in X$. The sentence

$$\forall \underline{n} \in \mathbb{N} \exists \underline{a} \in A : d(x, \underline{a}) \geq \underline{n}$$

holds true. Using the transfer principle we find $a \in {}^*A$ such that $d({}^*x, a) \geq n$ for some $n \in \mathbb{N}_\infty$. Now for every $y \in X$ the triangle inequality implies

$$d({}^*y, a) \geq d({}^*x, a) - d({}^*x, {}^*y) \geq n - d({}^*x, {}^*y)$$

and since $n - d({}^*x, {}^*y) \in \mathbb{R}_\infty^+$ by Proposition 3.6, we have $a \notin \text{fin}({}^*X)$. $\square$

4.2 Convergence and continuity in metric spaces

This section covers convergence, limit points, Cauchy sequences, continuity, equicontinuity and uniform continuity.

Definition. We say that a sequence $\{x_n\}_{n=1}^\infty$ of elements of X converges to $x \in X$ in (X, d) if $d(x_n, x) \rightarrow 0$. We write $x_n \rightarrow x$.

Recall that a sequence $\{x_n\}_{n=1}^\infty$ in X is actually a function $x : \mathbb{N} \rightarrow X$. By Theorem 2.7 the nonstandard extension of x is a function $x : {}^*\mathbb{N} \rightarrow {}^*X$ and therefore indexes from ${}^*\mathbb{N}$ are admissible.

Proposition 4.8. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X and $x \in X$. Then $x_n \rightarrow x$ if and only if $x_N \simeq {}^*x$ for every $N \in \mathbb{N}_\infty$.

Proof. Let $x_n \rightarrow x$. Then for $\varepsilon \in \mathbb{R}^+$ given we find $n_0 \in \mathbb{N}$ such that the sentence

$$\forall \underline{N} \in \mathbb{N} : \underline{N} \geq n_0 \Rightarrow d(x_{\underline{N}}, x) < \varepsilon$$

holds true. From the transfer principle we obtain $d(x_N, {}^*x) < {}^*\varepsilon$ for every $N \in \mathbb{N}_\infty$ and $\varepsilon \in \mathbb{R}^+$, hence $x_N \simeq {}^*x$ for every $N \in \mathbb{N}_\infty$.

Conversely let $x_N \simeq {}^*x$ for every $N \in \mathbb{N}_\infty$. The predicate $d(x_N, {}^*x) < {}^*\varepsilon$ is true for every $N \in \mathbb{N}_\infty$. By the permanence principle we find $n_0 \in {}^\sigma \mathbb{N}$ such that $d(x_n, {}^*x) < {}^*\varepsilon$ for every $n \in {}^*\mathbb{N}, n \geq n_0$. To prove $x_n \rightarrow x$ it remains to apply the transfer principle. $\square$

Definition. We call a sequence $\{x_n\}_{n=1}^\infty$ in X a Cauchy sequence if for every $\varepsilon \in \mathbb{R}^+$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ holds for every $n, m \in \mathbb{N}, n \geq n_0, m \geq n_0$.

Proposition 4.9. *A sequence $\{x_n\}_{n=1}^\infty$ in X is a Cauchy sequence if and only if $x_N \simeq x_M$ for every $N, M \in \mathbb{N}_\infty$.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence. Choose $\varepsilon \in \mathbb{R}^+$ arbitrary and find $n_0 \in \mathbb{N}$ such that the sentence

$$\forall \underline{n}, \underline{m} \in \mathbb{N} : (n \geq n_0 \wedge m \geq n_0) \Rightarrow d(x_{\underline{n}}, x_{\underline{m}}) < \varepsilon$$

holds. By the transfer principle the $*$ -transform of the previous sentence

$$\forall \underline{n}, \underline{m} \in {}^*\mathbb{N} : (\underline{n} \geq {}^*n_0 \wedge \underline{m} \geq {}^*n_0) \Rightarrow d(x_{\underline{n}}, x_{\underline{m}}) < {}^*\varepsilon$$

also holds. Choose $N, M \in \mathbb{N}_\infty$, then $N \geq n_0$ for every $n_0 \in {}^\sigma\mathbb{N}$ (similarly for M) and since ε is arbitrary, we obtain $x_N \simeq x_M$.

Conversely let $x_N \simeq x_M$ for every $N, M \in \mathbb{N}_\infty$. Choose $\varepsilon \in \mathbb{R}^+$. Then the internal predicate

$$\forall \underline{N}, \underline{M} \in {}^*\mathbb{N} : (\underline{N} \geq \underline{n} \wedge \underline{M} \geq \underline{n}) \Rightarrow d(x_{\underline{N}}, x_{\underline{M}}) < {}^*\varepsilon$$

holds for every $n \in \mathbb{N}_\infty$. By the permanence principle we find $n_0 \in {}^\sigma\mathbb{N}$ such that the previous predicate holds for $n \in {}^*\mathbb{N}$, $n \geq n_0$. To show that $\{x_n\}_{n=1}^\infty$ is Cauchy, it remains to use the transfer principle. $\square$

Definition. *A sequence $\{x_n\}_{n=1}^\infty$ in X has a limit point $x \in X$ if there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ in X such that $x_{n_k} \rightarrow x$ for $k \rightarrow \infty$.*

Proposition 4.10. *A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ has a limit point $x \in X$ if and only if $x_N \simeq x$ for some $N \in \mathbb{N}_\infty$.*

Proof. Let $x \in X$ be a limit point. Then the sentence

$$\forall \underline{\varepsilon} \in \mathbb{R}^+ \forall \underline{n} \in \mathbb{N} \exists \underline{m} \in \mathbb{N} : \underline{m} \geq \underline{n} \wedge d(x_{\underline{m}}, x) < \underline{\varepsilon}$$

holds true. By the transfer principle the $*$ -transform of the previous sentence holds. Now it remains to choose $n \in \mathbb{N}_\infty$ and $\varepsilon \in \inf({}^*\mathbb{R}^+)$.

Conversely let $x_N \simeq x$ for some $N \in \mathbb{N}_\infty$. Chose $\varepsilon \in \mathbb{R}^+$ and $m \in \mathbb{N}$. The sentence

$$\exists \underline{n} \in {}^*\mathbb{N} : \underline{n} \geq {}^*m \wedge d(x_{\underline{n}}, x) < {}^*\varepsilon$$

holds for every $\varepsilon \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$. By the transfer principle the sentence

$$\exists \underline{n} \in \mathbb{N} : \underline{n} \geq m \wedge d(x_{\underline{n}}, x) < \varepsilon$$

also holds and since $\varepsilon \in \mathbb{R}^+$ and $m \in \mathbb{N}$ are arbitrary, x is a limit point of $\{x_n\}_{n=1}^\infty$. $\square$

Definition. *Let (X, d_X) and (Y, d_Y) be metric spaces and $x_0 \in X$ and $f : X \rightarrow Y$. The mapping f is continuous at x_0 if for every $\varepsilon \in \mathbb{R}^+$ we can find $\delta \in \mathbb{R}^+$ such that $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$. The mapping f is continuous on X if it is continuous at every $x \in X$.*

Proposition 4.11. *Let (X, d_X) and (Y, d_Y) be metric spaces and $x_0 \in X$. A mapping $f : X \rightarrow Y$ is continuous at x_0 if and only if $f(\text{mon}({}^*x_0)) \subseteq \text{mon}(f({}^*x_0))$, i.e. for every $x \in {}^*X$ the relation ${}^*x_0 \simeq x$ implies $f({}^*x_0) \simeq f(x)$.*

Proof. Let f be continuous at x_0 and $x \in \text{mon}(*x_0)$. For $\varepsilon \in \mathbb{R}^+$ arbitrary we find $\delta_\varepsilon \in \mathbb{R}^+$ such that $d(f(y), f(x_0)) < \varepsilon$ for every $y \in B(x_0, \delta_\varepsilon)$. Since $x \in \text{mon}(*x_0) \subseteq {}^*B(x_0, \delta)$ for every $\delta \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+$ is arbitrary, we obtain $d(f(x), f(*x_0)) \in \inf(*\mathbb{R}^+)$ and thus $f(x) \simeq f(*x_0)$.

Conversely let $\varepsilon \in \mathbb{R}^+$ be given. Then the internal predicate

$$\forall \underline{x} \in {}^*X : d_X(\underline{x}, *x_0) < \underline{\delta} \Rightarrow d_Y(f(\underline{x}), f(*x_0)) < *\varepsilon \quad (4.2)$$

holds for every $\delta \in \inf(\mathbb{R}^+)$. By the permanence principle there exists $\delta_0 \in {}^\sigma\mathbb{R}^+$ such that (4.2) holds for δ_0 . Now we apply the transfer principle to (4.2) with $\delta = \delta_0$ and obtain that f is continuous at x . $\square$

Definition. Let (X, d_X) and (Y, d_Y) be metric spaces, $x_0 \in X$ and $\mathcal{F}$ be a family of functions from X to Y . We call the family $\mathcal{F}$ equicontinuous at x_0 if for every $\varepsilon \in \mathbb{R}^+$ there exists $\delta \in \mathbb{R}^+$ such that $f(x) \in B_Y(f(x_0), \varepsilon)$ holds for every $x \in B_X(x_0, \delta)$ and $f \in \mathcal{F}$.

Let $A \subseteq X$. Then $\mathcal{F}$ is equicontinuous on A if it is equicontinuous in every $x \in A$.

Proposition 4.12. Let (X, d_X) and (Y, d_Y) be metric spaces, $x_0 \in X$ and $\mathcal{F}$ be a family of functions from X to Y . Then the family $\mathcal{F}$ is equicontinuous at x_0 if and only if $f(\text{mon}(*x_0)) \subseteq \text{mon}(f(*x_0))$ for every $f \in {}^*\mathcal{F}$.

Proof. Let $\mathcal{F}$ be equicontinuous in x_0 . Choose $\varepsilon_0 \in \mathbb{R}^+$ arbitrary and find $\delta_0 \in \mathbb{R}^+$ such that the sentence

$$\forall \underline{x} \in X \forall \underline{f} \in \mathcal{F} : \underline{x} \in B_X(x_0, \delta_0) \Rightarrow \underline{f}(\underline{x}) \in B_Y(\underline{f}(x_0), \varepsilon_0)$$

holds true and so does its $*$ -transform

$$\forall \underline{x} \in {}^*X \forall \underline{f} \in {}^*\mathcal{F} : \underline{x} \in {}^*B_X(x_0, \delta_0) \Rightarrow \underline{f}(\underline{x}) \in {}^*B_Y(\underline{f}(x_0), \varepsilon_0)$$

by the transfer principle. Now let $x \in \text{mon}(*x_0)$, thus $x \in {}^*B_X(x_0, \delta)$ for every $\delta \in \mathbb{R}^+$. By the previous sentence $f(x) \in {}^*B_Y(f(x_0), \varepsilon)$ and similarly as in the proof of Proposition 4.11 we acquire $f(\text{mon}(*x_0)) \subseteq \text{mon}(f(*x_0))$ for every $f \in \mathcal{F}$.

Now let $f(\text{mon}(x_0)) \subseteq \text{mon}(f(x_0))$ for every $f \in {}^*\mathcal{F}$. Choose $\varepsilon \in \mathbb{R}^+$ arbitrary. The predicate

$$\forall \underline{x} \in {}^*B_X(x_0, \underline{\delta}) \forall \underline{f} \in {}^*\mathcal{F} : \underline{f}(\underline{x}) \in {}^*B_Y(\underline{f}(x_0), \varepsilon)$$

is internal and holds for every $\delta \in \inf(*\mathbb{R}^+)$. By the permanence principle there exists $\delta_0 \in {}^\sigma\mathbb{R}^+$ such that the previous predicate holds for every $\delta \in *\mathbb{R}^+$, $\delta < \delta_0$. The proof is finished by using the transfer principle. $\square$

Thus the difference between a family of continuous functions and a family of equicontinuous functions is very distinct in nonstandard terms. Let $\mathcal{F}$ be a family of continuous functions at $x_0 \in X$. Then the sentence

$$\forall \underline{f} \in {}^\sigma\mathcal{F} : \underline{f}(\text{mon}(*x_0)) \subseteq \text{mon}(\underline{f}(*x_0))$$

is true. Moreover if the functions in $\mathcal{F}$ are equicontinuous at x_0 then a stronger sentence

$$\forall \underline{f} \in {}^*\mathcal{F} : \underline{f}(\text{mon}(*x_0)) \subseteq \text{mon}(\underline{f}(*x_0))$$

holds.

Definition. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \rightarrow Y$ is uniformly continuous on X if for every $\varepsilon \in \mathbb{R}^+$ there exists $\delta \in \mathbb{R}^+$ such that $d_Y(f(x), f(y)) < \varepsilon$ holds for every $x, y \in X$ satisfying $d_X(x, y) < \delta$.

Proposition 4.13. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \rightarrow Y$ a mapping. Then f is uniformly continuous on X if and only if for every $x, y \in {}^*X$ the relation $x \simeq y$ implies $f(x) \simeq f(y)$.

Proof. Let f be uniformly continuous. For $\varepsilon \in \mathbb{R}^+$ arbitrary we find $\delta \in \mathbb{R}^+$ such that the sentence

$$\forall \underline{x}, \underline{y} \in X : d_X(\underline{x}, \underline{y}) < \delta \Rightarrow d_Y(f(\underline{x}), f(\underline{y})) < \varepsilon$$

holds. By the transfer principle the $*$ -transform of the previous sentence

$$\forall \underline{x}, \underline{y} \in {}^*X : d_X(\underline{x}, \underline{y}) < \delta \Rightarrow d_Y(f(\underline{x}), f(\underline{y})) < \varepsilon$$

also holds true. We proceed similarly as in the proof of Proposition 4.11. Let $x, y \in {}^*X$ such that $x \simeq y$. We observe that $d_X(x, y) < \delta$ for every $\delta \in \mathbb{R}^+$ and since ε is arbitrary, we obtain $d_Y(f(x), f(y)) \simeq 0$.

Conversely let $\varepsilon \in \mathbb{R}^+$ be arbitrary. The predicate

$$\forall \underline{x}, \underline{y} \in {}^*X : d_X(\underline{x}, \underline{y}) < \underline{\delta} \Rightarrow d(f(\underline{x}), f(\underline{y})) < {}^*\varepsilon$$

is internal and holds for every $\delta \in \inf({}^*\mathbb{R}^+)$. By the permanence principle we find $\delta_0 \in {}^\sigma\mathbb{R}^+$ such that the previous predicate holds true for $\delta \in {}^*\mathbb{R}^+$, $\delta < \delta_0$. Now it remains to fix δ_0 and apply the transfer principle. $\square$

4.3 Compact and complete metric spaces

In this section we present Robinson's compactness criterion which will often be used in the last chapter. As we will see in the rest of this section many proofs from standard theory of metric spaces might appear easier than its standard counterparts and thus the criterion is of interest on its own.

Definition. We call a set $A \subseteq X$ compact if every sequence $\{x_n\}_{n=1}^\infty$ in A has a limit point in A .

Theorem 4.14 (Robinson's criterion). Let $A \subseteq X$. Then A is compact if and only if

$$\forall \underline{x} \in {}^*A \exists \underline{a} \in {}^\sigma A : x \simeq a. \quad (4.3)$$

In other words A is compact if and only if ${}^*A \subseteq \text{ns}({}^*X)$.

Proof. Let us first assume that (4.3) does not hold. Our goal is to construct a sequence in A which does not have any limit point in A . Let us find $x \in {}^*A$ such that $x \notin \text{mon}(a)$ for every $a \in {}^\sigma A$. Only two possible situations can occur:

$$\exists \underline{\varepsilon} \in \mathbb{R}^+ \forall \underline{a} \in {}^\sigma A : d(x, \underline{a}) \geq {}^*\underline{\varepsilon}, \quad (4.4a)$$

$$\forall \underline{\varepsilon} \in \mathbb{R}^+ \exists \underline{a} \in {}^\sigma A : d(x, \underline{a}) < {}^*\underline{\varepsilon}. \quad (4.4b)$$

Let (4.4a) hold and let $\varepsilon \in \mathbb{R}^+$ such that $d(x, a) \geq {}^*\varepsilon$ for every $a \in {}^\sigma A$. Choose $a_1 \in A$ arbitrary. The sentence

$$\exists \underline{y} \in {}^*A : d({}^*a_1, \underline{y}) \geq {}^*\varepsilon$$

holds true for $y = x$. By the transfer principle we can find $a_2 \in A$ such that $d(a_1, a_2) \geq \varepsilon$.

We proceed by induction. Let $n \in \mathbb{N}$ and let $a_1, \dots, a_n \in A$ satisfy $d(a_i, a_j) \geq \varepsilon$ for every $i, j \in \{1, \dots, n\}, i \neq j$. The sentence

$$\exists \underline{y} \in {}^*A : d({}^*a_1, \underline{y}) \geq \varepsilon \wedge \dots \wedge d({}^*a_n, \underline{y}) \geq {}^*\varepsilon$$

holds true by taking $y = x$. Using the transfer principle we can find $a_{n+1} \in A$ such that $d(a_{n+1}, a_k) \geq \varepsilon$ for every $k \in \{1, \dots, n\}$. Thus the sequence $\{a_n\}_{n=1}^\infty$ has no limit point in A .

Now let (4.4b) hold. Then we can find a sequence $\{a_n\}_{n=1}^\infty$ in A such that $d({}^*a_n, x) < {}^*\frac{1}{n+1}$ for every $n \in \mathbb{N}$.

For contradiction let us assume that the sequence $\{a_n\}_{n=1}^\infty$ has a limit point $y \in A$. Since $x \notin \text{mon}(y)$, we can find $m \in {}^\sigma \mathbb{N}$ such that $d({}^*y, x) \geq \frac{1}{m}$. Since y is a limit point of $\{a_n\}_{n=1}^\infty$, there exists $n \in {}^\sigma \mathbb{N}$, $n > 2m$, such that $d({}^*a_n, {}^*y) < \frac{1}{2m}$. Now using the triangle inequality we obtain

$$\frac{1}{m} \leq d(x, {}^*y) \leq d(x, {}^*a_n) + d({}^*a_n, {}^*y) < \frac{1}{2m+1} + \frac{1}{2m} < \frac{1}{m},$$

a contradiction, and the first implication is therefore proved.

Conversely let (4.3) hold and let $\{a_n\}_{n=1}^\infty$ be a sequence in A . By the transfer principle $a_N \in {}^*A$ for every $N \in {}^*\mathbb{N}$. Choose $N \in \mathbb{N}_\infty$ arbitrary. From (4.3) it follows that $a_N \simeq {}^*a$ for some $a \in A$ and by Proposition 4.10 a is a limit point of $\{a_n\}_{n=1}^\infty$, hence A is compact. $\square$

Corollary 4.15. *Let $A \subseteq X$ be compact and $B \subseteq A$ closed. Then $B \subseteq A$ is compact.*

Proof. Let $b \in {}^*B$. Since A is compact, we can find $\tilde{a} \in {}^\sigma A$ satisfying $b \simeq \tilde{a}$ and thus $b \in \text{ns}({}^*X)$. Since B is closed, there exists $\tilde{b} \in {}^\sigma B$ such that $b \simeq \tilde{b}$ and therefore B is compact by Theorem 4.14. $\square$

Corollary 4.16. *Let (X, d_X) be a compact metric space, (Y, d_Y) a metric spaces and $f : X \rightarrow Y$ be a continuous function on X . Then f is uniformly continuous on X .*

Proof. Let $x, y \in {}^*X$ satisfy $x \simeq y$. Since X is compact, there exists $\tilde{x} \in {}^\sigma X$ such that $x \simeq \tilde{x} \simeq y$. From the continuity of f at $\tilde{x}$ it follows that $f(x) \simeq f(\tilde{x}) \simeq f(y)$, hence f is uniformly continuous on X . $\square$

Corollary 4.17. *Let (X, d_X) be a compact metric space, (Y, d_Y) a metric spaces and $f : X \rightarrow Y$ be a continuous function on X . Then $f(X)$ is compact.*

Proof. Let $x \in {}^*X$ and $y \in {}^*Y$ such that $y = f(x)$. Since X is compact, we can find $\tilde{x} \in {}^\sigma X$ satisfying $x \simeq \tilde{x}$. Since f is continuous at $\tilde{x}$, we immediately obtain $y = f(x) \simeq f(\tilde{x}) \in {}^\sigma f(X)$ and the set $f(X)$ is compact by Theorem 4.14. $\square$

Definition. We call a set $A \subseteq X$ relatively compact, if $\overline{A}$ is compact, in other words if every sequence $\{a_n\}_{n=1}^\infty$ in A has a limit point.

Using Proposition 4.10 we observe that $A \subseteq X$ is relatively compact if and only if for every sequence $\{a_n\}_{n=1}^\infty$ in A we have $x_N \in \text{ns}(*X)$ for some $N \in \mathbb{N}_\infty$.

Definition. We call a metric space (X, d) complete, if every Cauchy sequence is convergent in X .

Definition. Let $x \in *X$. We call x a pre-nearstandard point if there exists $y \in {}^\sigma X$ such that $d(x, y) < {}^*\varepsilon$ for every $\varepsilon \in \mathbb{R}^+$. We denote the set of all pre-nearstandard points in $*X$ by $\text{pns}(*X)$.

Theorem 4.18. A metric space (X, d) is complete if and only if the inclusion $\text{pns}(*X) \subseteq \text{ns}(*X)$ holds.

Note that the inclusion $\text{ns}(*X) \subseteq \text{pns}(*X)$ always holds and therefore the completeness of X is equivalent to $\text{pns}(*X) = \text{ns}(*X)$.

Proof. Let X be complete and $x \in \text{pns}(*X)$. There exists a sequence $\{x_n\}_{n=1}^\infty$ in X such that the predicate $d(*x_{\underline{n}}, x) < \frac{1}{\underline{n}}$ holds for every $n \in {}^\sigma \mathbb{N}$. By the permanence principle it also holds for some $N \in \mathbb{N}_\infty$. The sequence $\{x_n\}_{n=1}^\infty$ is clearly a Cauchy sequence and from the completeness of X there exists $y \in X$ such that $x_n \rightarrow y$. From Proposition 4.8 we obtain $x \simeq x_N \simeq {}^*y$, hence $x \in \text{ns}(*X)$.

Conversely let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in X . Fix $\varepsilon \in \mathbb{R}^+$. Since $\{x_n\}_{n=1}^\infty$ is Cauchy, we can find $m \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for every $n \in \mathbb{N}, n \geq m$. The transfer principle gives that the sentence

$$\forall \underline{n} \in {}^*\mathbb{N} : \underline{n} \geq {}^*m \Rightarrow d(*x_m, x_{\underline{n}}) < {}^*\varepsilon$$

implies that $x_N \in \text{pns}(*X) \subseteq \text{ns}(*X)$ for every $N \in \mathbb{N}_\infty$ and from Proposition 4.9 we know that $x_N \simeq x_M$ for every $N, M \in \mathbb{N}_\infty$. Thus all the points $\{x_N; N \in \mathbb{N}_\infty\}$ must be infinitely close to one point $*x$ for some $x \in X$. From Proposition 4.8 it follows that $x_n \rightarrow x$, hence X is complete. $\square$

5. Dynamical systems

In this chapter let (X, d) be a complete metric space, $X, \mathbb{R}_0^+ \in \widehat{S}$ be entities and $* : \widehat{S} \rightarrow \widehat{*S}$ be a nonstandard embedding.

The theory of dynamical systems was developed to provide a tool for analyzing the behavior of some systems at large times. Classical mathematics tends to use considerably high number of quantifiers to describe this behavior. Nonstandard analysis on the other hand offers a simpler way: investigating only the behavior at infinite times. In this chapter we formalize this concept using the apparatus reviewed in the previous chapters.

5.1 Dynamical system

Definition. Let (X, d) be a complete metric space and $S(t) : X \rightarrow X$ for every $t \in \mathbb{R}_0^+$. We call the couple $(X, S(t))$ a dynamical system, if

- (i) $S(0) = \text{Id}$, where Id is an identity on X ,
- (ii) $S(t) \circ S(s) = S(t + s)$ for every $t, s \in \mathbb{R}_0^+$, (semigroup property)
- (iii) the mapping $(t, x) \rightarrow S(t)x$ is continuous.

The set X is called phase space and the mapping $S(\cdot)x : \mathbb{R}_0^+ \rightarrow X$ a trajectory.

Dynamical systems are closely related to evolutionary differential equations. Let Y be Banach space and $F : Y \rightarrow Y$ be such an operator that for every $u_0 \in Y$ the equation

$$\begin{aligned} \frac{d}{dt}u(t) &= F(u(t)) \text{ for all } t \in \mathbb{R}_0^+, \\ u(0) &= u_0 \end{aligned} \tag{5.1}$$

has a unique solution $u : \mathbb{R}_0^+ \rightarrow Y$ that continuously depends on the initial condition and is continuous in time. We define $S(t)u_0$ to be the value of solution u of (5.1) in time t . Then the couple $(Y, S(t))$ is a dynamical system.

The asymptotic behavior of dynamical systems is studied through time invariant sets. The concept is mathematically described as follows.

Definition. Let $A \subseteq X$. We call the set A positively invariant if $S(t)A \subseteq A$ holds for every $t \in \mathbb{R}^+$. Set A is called negatively invariant if $S(t)A \supseteq A$ holds for every $t \in \mathbb{R}^+$.

We call the A invariant if $S(t)A = A$ holds for every $t \in \mathbb{R}_0^+$.

The results presented in this chapter can be divided into two groups: First group contains nonstandard characterizations of important sets and properties in the theory of dynamical systems, the second group offers nonstandard proofs for standard theorems using these characterizations and the apparatus introduced in the previous chapters.

5.2 ω -limit set

Definition. Let $A \subseteq X$. We define ω -limit set of A by

$$\omega(A) = \{x \in X; \exists \{a_n\}_{n=1}^\infty \subseteq A \exists \{t_n\}_{n=1}^\infty : t_n \rightarrow \infty \wedge S(t_n)a_n \rightarrow x\}.$$

Proposition 5.1. Let $A \subseteq X$. Then

$$\omega(A) = \{x \in X; \exists a \in {}^*A \exists T \in \mathbb{R}_\infty^+ : S(T)a \simeq {}^*x\}. \quad (5.2)$$

Proof. Let us denote the right hand side of (5.2) with $\tilde{\omega}(A)$.

Let $x \in \omega(A)$, $\{t_n\}_{n=1}^\infty \subseteq \mathbb{R}$ and $\{a_n\}_{n=1}^\infty \subseteq A$ such that $t_n \rightarrow \infty$ and $S(t_n)a_n \rightarrow x$. It holds that

$$\forall \underline{n} \in \mathbb{N} \exists \underline{m} \in \mathbb{N} : \underline{t}_m > \underline{n} \wedge \underline{m} \geq \underline{n} \wedge d(S(\underline{t}_m)a_{\underline{m}}, x) < \frac{1}{\underline{n}}.$$

By the transfer principle the $*$ -transform of the previous sentence

$$\forall \underline{n} \in {}^*\mathbb{N} \exists \underline{m} \in {}^*\mathbb{N} : \underline{t}_m > \underline{n} \wedge \underline{m} \geq \underline{n} \wedge d(S(\underline{t}_m)a_{\underline{m}}, {}^*x) < \frac{1}{{}^*\underline{n}}$$

also holds true. Choose $n \in \mathbb{N}_\infty$ arbitrary. From the $*$ -transform it follows that $t_m \in \mathbb{R}_\infty^+$ and $d(S(t_m)a_m, {}^*x) \simeq 0$ for some $m \in \mathbb{N}_\infty$, therefore $S(t_m)a_m \simeq {}^*x$ and $\omega(A) \subseteq \tilde{\omega}(A)$.

Conversely let $x \in \tilde{\omega}(A)$, then there exists $T \in \mathbb{R}_\infty^+$ and $a \in {}^*A$ satisfying $S(T)a \simeq {}^*x$. Choose $t_1 \in \mathbb{R}_0^+$ and $a_1 \in A$ arbitrary. The sentence

$$\exists \underline{\tau} \in {}^*\mathbb{R}^+ \exists \underline{b} \in {}^*A : \underline{\tau} > {}^*t_1 + 1 \wedge d(S(\underline{\tau})\underline{b}, {}^*x) < 1$$

holds for $\tau = T$ and $b = a$. Using the transfer principle we see that the sentence

$$\exists \underline{\tau} \in \mathbb{R}^+ \exists \underline{b} \in A : \underline{\tau} > t_1 + 1 \wedge d(S(\underline{\tau})\underline{b}, x) < 1$$

holds for some $\tau \in \mathbb{R}^+$ and $b \in A$. Set $t_2 = \tau, a_2 = b$.

We proceed by induction. Suppose that for fixed $n \in \mathbb{N}$ we have sequences $t_1, \dots, t_n \in \mathbb{R}^+$ and $a_1, \dots, a_n \in A$ satisfying $t_k > t_{k-1} + 1$ and $d(S(t_k)a_k, x) < {}^*(k^{-1})$ for every $k \in \{1, \dots, n\}$. The sentence

$$\exists \underline{\tau} \in {}^*\mathbb{R}^+ \exists \underline{b} \in {}^*A : \underline{\tau} > {}^*t_n + 1 \wedge d(S(\underline{\tau})\underline{b}, {}^*x) < \frac{1}{{}^*n},$$

holds for $\tau = T$ and $b = a$. The transfer principle implies that the sentence

$$\exists \underline{\tau} \in \mathbb{R}^+ \exists \underline{b} \in A : \underline{\tau} > t_n + 1 \wedge d(S(\underline{\tau})\underline{b}, x) < \frac{1}{n},$$

also holds for some $\tau \in \mathbb{R}^+$ and $b \in A$. Set $t_{n+1} = \tau$ and $a_{n+1} = b$.

The sequences $\{t_n\}_{n=1}^\infty$ and $\{a_n\}_{n=1}^\infty$ satisfy $t_n \rightarrow \infty$ and $S(t_n)a_n \rightarrow x$ as $n \rightarrow \infty$, hence $x \in \omega(A)$ and $\tilde{\omega}(A) \subseteq \omega(A)$. $\square$

Using the previous proposition we can easily prove that the ω -limit set is fully invariant.

Corollary 5.2. The ω -limit set invariant.

Proof. Let $A \subseteq X$, $x \in \omega(A)$ and $S(T)a \simeq {}^*x$ for some $T \in \mathbb{R}_\infty^+$ and $a \in {}^*A$. Fix $t \in \mathbb{R}$. Then ${}^*t + T \in \mathbb{R}_\infty^+$ and using continuity of $S(t)$ at x and Proposition 4.11 we obtain

$${}^*(S(t)x) \simeq S({}^*t)S(T)a = S({}^*t + T)a$$

and thus $S(t)x \in \omega(A)$. □

In the rest of this section we will consider only ω -limit sets of a point of the phase space X . With slight abuse of notation we will write $\omega(x)$ instead of $\omega(\{x\})$ for $x \in X$.

Definition. Let $x_0 \in X$. We define positive orbit of x_0 by

$$\gamma^+(x_0) = \{S(t)x_0; t \in \mathbb{R}_0^+\}.$$

Let $x_0 \in X$. In the following proofs we will need to determine whether a point $y \in {}^*X$ lies in ${}^*\gamma^+(x_0)$. From the standard definition principle it follows that

$${}^*\gamma^+(x_0) = \{S(t){}^*x_0; t \in {}^*\mathbb{R}_0^+\}.$$

Lemma 5.3. Let $x_0 \in X$, then

$$\omega(x_0) = \bigcap_{t \in {}^\sigma\mathbb{R}_0^+} \text{st}(\{S(t+\tau){}^*x_0; \tau \in {}^*\mathbb{R}_0^+\}) = \bigcap_{t \in \mathbb{R}_0^+} \text{st}({}^*\gamma^+(S(t)x_0)). \quad (5.3)$$

Moreover, the set $\text{st}(\{S(t+\tau){}^*x_0; \tau \in {}^*\mathbb{R}_0^+\})$ is closed for every $t \in {}^\sigma\mathbb{R}_0^+$.

Proof. Define $\tilde{\omega}(x_0) = \bigcap_{t \in {}^\sigma\mathbb{R}_0^+} \text{st}(\{S(t+\tau){}^*x_0; \tau \in {}^*\mathbb{R}_0^+\})$.

Let $x \in \omega(x_0)$ and $T \in \mathbb{R}_\infty^+$ such that $S(T){}^*x_0 \simeq {}^*x$. Choose $t \in {}^\sigma\mathbb{R}_0^+$ and set $\tau = T - t$, then ${}^*x \simeq S(T){}^*x_0 = S(t+\tau){}^*x_0$, hence $x \in \tilde{\omega}(x_0)$.

Conversely let $x \in \tilde{\omega}(x_0)$. Only two possible situations can occur: either we can find $t \in {}^\sigma\mathbb{R}^+$ and $\tau \in \mathbb{R}_\infty$ such that $S(t+\tau){}^*x_0 \simeq {}^*x$, which makes this case trivial as then $t+\tau \in \mathbb{R}_\infty$ and $x \in \omega(x_0)$, or for every $t \in {}^\sigma\mathbb{R}^+$ there exists $\tau_t \in \text{fin}({}^*\mathbb{R}^+)$ satisfying $S(t+\tau_t){}^*x_0 \simeq {}^*x$. In this case it suffices to restrict ourselves to natural numbers. We know that for every $n \in {}^\sigma\mathbb{N}$ there we can find $\tau_n \in \text{fin}({}^*\mathbb{R}^+)$ such that $S(n+\tau_n){}^*x_0 \simeq {}^*x$. Define $t_n = \text{st}(\tau_n)$. Since $S(n+t_n)$ are continuous operators, we see that $S(n+t_n)x_0 = x$, i.e.

$$\forall \underline{n} \in \mathbb{N} \exists \underline{t}_n \in \mathbb{R}^+ : S(\underline{n} + \underline{t}_n)x_0 = x.$$

By the transfer principle the $*$ -transform of the previous sentence also holds. Now we choose $n \in \mathbb{N}_\infty$ and observe ${}^*x = S(n+t_n){}^*x_0 \in \omega(x_0)$ since $n+t_n \in \mathbb{R}_\infty^+$.

Let $t \in {}^\sigma\mathbb{R}_0^+$. Then the set $\{S(t+\tau){}^*x_0; \tau \in {}^*\mathbb{R}_0^+\}$ is internal by the internal definition principle (Theorem 2.9) and closed by Theorem 4.6. □

Theorem 5.4. Let $x_0 \in X$. Then $\omega(x_0)$ is closed.

Moreover if $\gamma^+(x_0)$ is relatively compact, then $\omega(x_0)$ is nonempty and compact.

Proof. Using Lemma 5.3 we see that $\omega(x_0)$ is an intersection of closed sets, thus by Proposition 4.5 $\omega(x_0)$ is closed.

Let $\gamma^+(x_0)$ be relatively compact and let $\{t_n\}_{n=1}^\infty \subseteq \mathbb{R}^+$ satisfy $t_n \rightarrow \infty$. We can assume $t_n > n$ for every $n \in \mathbb{N}$, otherwise we pass to a subsequence. Using the definition of positive orbit the sentence

$$\forall \underline{n} \in \mathbb{N} : t_{\underline{n}} > \underline{n} \wedge S(t_{\underline{n}})x_0 \in \gamma^+(x_0)$$

holds true. By the transfer principle the $*$ -transform

$$\forall \underline{n} \in {}^*\mathbb{N} : t_{\underline{n}} > \underline{n} \wedge S(t_{\underline{n}})^*x_0 \in {}^*\gamma^+(x_0)$$

also holds true. Now choose $n \in \mathbb{N}_\infty$ arbitrary. Since $\gamma^+(x_0)$ is relatively compact, there exists $x \in \overline{\gamma^+(x_0)}$ such that $S(t_n)^*x_0 \simeq x$. Clearly $t_n \in \mathbb{R}_\infty^+$ and thus $x \in \omega(x_0)$.

To show that $\omega(x_0)$ is compact, choose $x \in {}^\sigma\omega(x_0)$. Then from Proposition 5.1 we have $x \simeq S(T)^*x_0 \in {}^*\gamma(x_0)$ for some $T \in \mathbb{R}_\infty^+$. From Proposition 4.3 it follows that $x \in {}^\sigma\overline{\gamma^+(x_0)}$ and $\omega(x_0) \subseteq \overline{\gamma^+(x_0)}$, i. e. $\omega(x_0)$ is a closed subset of a compact set. By Corollary 4.15 the set $\omega(x_0)$ is compact. $\square$

Definition. Let $x \in X$. We say that x lies on a periodic cycle if there exists $t \in \mathbb{R}^+$ such that $S(t)x = x$. If $S(t)x = x$ for every $t \in \mathbb{R}_0^+$, we call x a stationary point.

Let $x \in X$ be a stationary point. By the transfer principle the predicate $S(t)^*x = {}^*x$ holds true for every $t \in {}^*\mathbb{R}^+$ and thus $x \in \omega(x) = \{x\}$. The following theorem gives the characterization of limit cycles.

Theorem 5.5. Let $x_0 \in X$. Then x_0 lies on a periodic orbit if and only if $\gamma^+(x_0) = \omega(x_0)$ and $\gamma^+(x_0)$ is compact.

Proof. Suppose x_0 lies on a periodic orbit, i. e. there exists $\tau \in \mathbb{R}^+$ satisfying $\forall \underline{N} \in \mathbb{N} : S(\tau \underline{N})x_0 = x_0$. From the transfer principle it follows that the sentence

$$\forall \underline{N} \in {}^*\mathbb{N} : S({}^*\tau \underline{N})^*x_0 = {}^*x_0 \quad (5.4)$$

holds true. Let $x \in \gamma^+(x_0)$ and $t \in \mathbb{R}_0^+$ such that $S(t)x_0 = x$. Choose $N \in \mathbb{N}_\infty$, then using (5.4) we get ${}^*x = (S(t)x_0) = S({}^*t)S({}^*\tau N)^*x_0 = S({}^*t + {}^*\tau N)^*x_0$. Clearly ${}^*t + {}^*\tau N \in \mathbb{R}_\infty^+$ and therefore $x \in \omega(x_0)$ and $\gamma^+(x_0) \subseteq \omega(x_0)$.

Assume $x \in \omega(x_0)$, i. e. $S(T)^*x_0 \simeq x$ for some $T \in \mathbb{R}_\infty^+$. If we can find $N \in \mathbb{N}_\infty$ such that $T - {}^*\tau N \in \text{fin}({}^*\mathbb{R}^+)$ then using (5.4) and continuity of the mapping $t \rightarrow S(t)x_0$ we have

$${}^*x \simeq S(T)^*x_0 = S(T - {}^*\tau N)S({}^*\tau N)^*x_0 = S(T - {}^*\tau N)^*x_0 \simeq (S(t)x_0)$$

for $t = st(T - \tau N)$. Now ${}^*x, (S(t)x_0) \in {}^\sigma X$ implies $x = S(t)x_0$ and thus $x \in \gamma^+(x_0)$ and $\gamma^+(x_0) = \omega(x_0)$.

First we find $M \in \mathbb{N}_\infty$ such that $0 \leq T - M < 1$. Such M clearly exists since the sentence $\forall \underline{T} \in \mathbb{R}^+ \exists \underline{M} \in \mathbb{N} : 0 \leq \underline{T} - \underline{M} < 1$ holds true. The sentence

$$\forall \underline{M} \in \mathbb{N} \forall \underline{\tau} \in \mathbb{R}^+ \exists \underline{N} \in \mathbb{N} : 0 \leq \underline{M} - \underline{\tau} \underline{N} < \underline{\tau}$$

also holds true (consider $N = \lfloor \frac{M}{\tau} \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function) and using the transfer principle we can find $N \in \mathbb{N}_\infty$ satisfying $0 \leq M - {}^*\tau N < {}^*\tau$. Then

$$T - {}^*\tau N = T - M + M - {}^*\tau N < 1 + {}^*\tau \in \text{fin}({}^*\mathbb{R}^+)$$

and $\gamma^+(x_0) = \omega(x_0)$.

Since x_0 lies on a periodic orbit we have $\gamma^+(x_0) = \{S(t)x_0; t \in [0, \tau]\}$. Also $\gamma^+(x_0)$ is a continuous image of a compact set and thus the compactness follows from Corollary 4.17.

Conversely let $\gamma^+(x_0) = \omega(x_0)$ and $\gamma^+(x_0)$ be compact. Since $x_0 \in \omega(x_0)$ there exists $T \in \mathbb{R}_\infty^+$ such that $S(T)^*x_0 \simeq x_0$. Choose $t \in \mathbb{R}^+$ arbitrary. Then $S(T-t)^*x_0 \in \gamma^+(x_0)$ and from the compactness of $\gamma^+(x_0)$ it follows that $S(T-t)^*x_0 \simeq x$ for some $x \in \gamma^+(x_0)$. Now since $S(t)$ is continuous at x we have

$$S(t)^*x \simeq S(t)S(T-t)^*x_0 = S(T)x_0 \simeq x_0,$$

thus $S(t)x = x_0$ and x_0 lies on a periodic orbit. $\square$

5.3 Stability

Definition. Let $A, B \subseteq X$. We define

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Definition. Let $A \subseteq X$ and $\delta \in \mathbb{R}^+$. We define open δ -neighborhood of A in X by

$$B(A, \delta) = \{x \in X; \exists a \in A : d(a, x) < \delta\}.$$

Similarly for $B \subseteq {}^*X$ and $\eta \in {}^*\mathbb{R}^+$ we define open η -neighborhood of B in ${}^*X$ by

$$\mathcal{B}(B, \eta) = \{x \in {}^*X; \exists b \in B : d(b, x) < \eta\}.$$

We also define

$$\text{mon}(B) = \{x \in {}^*X; \exists b \in {}^*B : x \simeq b\}.$$

Let $A \subseteq X$ and $\delta \in \mathbb{R}^+$. Then by the standard definition principle it holds

$${}^*(B(A, \delta)) = {}^*(\{x \in X; \exists a \in A : d(a, x) < \delta\}) = \mathcal{B}({}^*A, {}^*\delta).$$

Similarly as in Chapter 4 we obtain

$$\text{mon}(A) = \bigcap_{n \in \mathbb{N}} {}^*B(A, \frac{1}{n}).$$

Let $x \in {}^*X$. From the definition of a monad for points in Chapter 4 we observe

$$\text{mon}(\{x\}) = \text{mon}(x).$$

Definition. Let $A \subseteq X$ and let A be positively invariant. Then A is called

- Lyapunov stable, if for every $t_0 \in \mathbb{R}^+$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that the condition $x \in B(A, \delta) \Rightarrow S(t)x \in B(A, \varepsilon)$ holds for every $x \in X$ and every $t \geq t_0$,
- unstable, if it is not stable,

- a local attractor, if there exists $\delta \in \mathbb{R}^+$ such that

$$\text{dist}(S(t)B(A, \delta), A) \rightarrow 0,$$

- uniformly asymptotically stable, if it is stable and a local attractor.

Lemma 5.6. *Let $A \subseteq X$ be compact and $B \subseteq X$. Then $\text{dist}(S(t)B, A) \rightarrow 0$ as $t \rightarrow \infty$ if and only if*

$$\forall \underline{b} \in {}^*B \forall \underline{T} \in \mathbb{R}_\infty^+ \exists \underline{a} \in {}^*A : S(\underline{T})\underline{b} \simeq \underline{a}.$$

Proof. Assume $\text{dist}(S(t)B, A) \rightarrow 0$. Then the sentence

$$\forall \underline{\varepsilon} \in \mathbb{R}^+ \exists \underline{t}_0 \in \mathbb{R}^+ \forall \underline{t} \in \mathbb{R}^+ : \underline{t} \geq \underline{t}_0 \Rightarrow \text{dist}(S(\underline{t})B, A) < \underline{\varepsilon}$$

holds. Since $\inf_{a \in A} d(S(t)b, a) \leq \text{dist}(S(t)B, A)$ for every $b \in B$ and $t \in \mathbb{R}^+$, the sentence

$$\forall \underline{\varepsilon} \in \mathbb{R}^+ \exists \underline{t}_0 \in \mathbb{R}^+ \forall \underline{t} \in \mathbb{R}^+ \forall \underline{b} \in B \exists \underline{a}_t \in A : \underline{t} \geq \underline{t}_0 \Rightarrow d(S(\underline{t})\underline{b}, \underline{a}_t) < \underline{\varepsilon}$$

also holds.

From the previous sentence it follows that the sentence

$$\forall \underline{n} \in \mathbb{N} \exists \underline{t}_n \in \mathbb{R}^+ \forall \underline{t} \in \mathbb{R}^+ \forall \underline{b} \in B \exists \underline{a}_{n,t} \in A : \underline{t} \geq \underline{t}_n \Rightarrow d(S(\underline{t})\underline{b}, \underline{a}_{n,t}) < \frac{1}{\underline{n}}$$

holds true. Fix $n \in \mathbb{N}$. Using the transfer principle we obtain that the sentence

$$\forall \underline{t} \in {}^*\mathbb{R}^+ \forall \underline{b} \in {}^*B \exists \underline{a}_{n,t} \in {}^*A : \underline{t} \geq {}^*\underline{t}_n \Rightarrow d(S(\underline{t})\underline{b}, \underline{a}_{n,t}) < {}^*\frac{1}{n}$$

also holds true and consequently the sentence

$$\forall \underline{T} \in \mathbb{R}_\infty^+ \forall \underline{b} \in {}^*B \exists \underline{a}_{n,T} \in {}^*A : d(S(\underline{T})\underline{b}, \underline{a}_{n,T}) < {}^*\frac{1}{n}$$

also holds. Choose $T \in \mathbb{R}_\infty^+$ and $b \in {}^*B$ and define $a_n = a_{n,T}$. Using the compactness of A we can assume $a_n \in {}^\sigma A$ since the displacement is at most infinitesimal. The compactness of A also implies that there exists a subsequence which we also denote a_n and $a \in {}^\sigma A$ such that $\text{st}(a_n) \rightarrow \text{st}(a)$. Now we have

$$d(S(T)b, a) \leq d(S(T)b, a_n) + d(a_n, a) < 2 \cdot {}^*\frac{1}{n}$$

and since we chose n arbitrarily, it follows that $S(T)b \simeq a$.

To show the converse implication fix $\varepsilon \in \mathbb{R}^+$ and $B \subseteq X$. Then for every $b \in B$ and $T \in \mathbb{R}_\infty^+$ we can find $a \in {}^*A$ such that $d(S(T)b, a) < {}^*\varepsilon$. By the permanence principle there exists $t_0 \in \mathbb{R}^+$ such that

$$\forall \underline{t} \in {}^*\mathbb{R}^+ \exists \underline{a} \in {}^*A : \underline{t} \geq {}^*t_0 \Rightarrow d(S(\underline{t})b, \underline{a}) < {}^*\varepsilon$$

By the transfer principle the sentence

$$\forall \underline{t} \in \mathbb{R}^+ \exists \underline{a} \in A : \underline{t} \geq t_0 \Rightarrow d(S(\underline{t})b, \underline{a}) < \varepsilon$$

holds true as well and therefore $\text{dist}(S(t)B, A) \rightarrow 0$. □

Theorem 5.7. *Let $A \subseteq X$ be positively invariant. Then A is Lyapunov stable if and only if $S(t) \text{mon}(*A) \subseteq \text{mon}(*A)$ for every $t \in {}^*\mathbb{R}^+$.*

Moreover, if A is compact then A is a local attractor if and only if there exists $\delta \in \mathbb{R}^+$ such that $S(T)^*B(A, \delta) \subseteq \text{mon}(*A)$ for every $T \in \mathbb{R}_\infty^+$. In other words

$$\forall \underline{x} \in {}^*B(A, \delta) \forall \underline{T} \in \mathbb{R}_\infty^+ \exists \underline{a} \in {}^*A : S(\underline{T})\underline{x} \simeq \underline{a}.$$

Proof. The proof of the first assertion is similar as the proof of Proposition 4.12. Let A be Lyapunov stable. Fix $t_0 \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+$ and find $\delta \in \mathbb{R}^+$ such that the sentence

$$\forall \underline{x} \in B(A, \delta) \forall \underline{t} \in \mathbb{R}^+ : \underline{t} \geq t_0 \Rightarrow S(\underline{t})\underline{x} \in B(S(t)A, \varepsilon)$$

holds true and so does its $*$ -transform

$$\forall \underline{x} \in {}^*B(A, \delta) \forall \underline{t} \in {}^*\mathbb{R}^+ : \underline{t} \geq {}^*t_0 \Rightarrow S(\underline{t})\underline{x} \in \mathcal{B}(S(\underline{t})^*A, {}^*\varepsilon).$$

Let $x \in \text{mon}(*A)$, then $x \in {}^*B(A, \delta)$ for every $\delta \in \mathbb{R}^+$. From the previous sentence we obtain $S(t)x \in {}^*B(S(t)A, \varepsilon)$ and since we chose ε arbitrarily, we have $S(t)x \in \text{mon}(S(t)^*A)$ for every $t \in {}^*\mathbb{R}^+$ satisfying $t \geq t_0$. Since A is positively invariant, we have $S(t)x \in \text{mon}(*A)$. Since t_0 was also arbitrary, we obtain $S(t)x \in \text{mon}(*A)$ for every $t \in {}^*\mathbb{R}^+$.

Conversely let $S(t) \text{mon}(*A) \subseteq \text{mon}(*A)$ for every $t \in {}^*\mathbb{R}^+$. Choose $\varepsilon \in \mathbb{R}^+$ and $t_0 \in \mathbb{R}^+$ arbitrary. The predicate

$$\forall \underline{x} \in {}^*B(A, \delta) \forall \underline{t} \in {}^*\mathbb{R}^+ : \underline{t} \geq {}^*t_0 \Rightarrow S(\underline{t})\underline{x} \in \mathcal{B}(S(\underline{t})^*A, {}^*\varepsilon)$$

is internal and holds for every $\delta \in \inf({}^*\mathbb{R}^+)$. Using the permanence principle we find $\delta_0 \in \mathbb{R}^+$ such that the previous predicate holds for every $\delta \in {}^*\mathbb{R}^+$, $\delta < {}^*\delta_0$. Fix δ_0 . The transfer principle now implies that A is Lyapunov stable.

The second statement follows immediately from Lemma 5.6. $\square$

Corollary 5.8. *Let $x_0 \in X$ be stationary point. Then x_0 is stable if and only if $\{S(t); t \in \mathbb{R}^+\}$ are equicontinuous in x_0 .*

Proof. The proof immediately follows from Theorem 5.7, Proposition 4.12 and the fact that $\text{mon}(*x_0) = \text{mon}(\{^*x_0\})$. $\square$

Analogous characterization does not hold for stability of an arbitrary positively invariant set $A \subseteq X$. However, we can easily prove one implication for A compact.

Proposition 5.9. *Let $A \subseteq X$ be compact and positively invariant set and let $\{S(t); t \in \mathbb{R}^+\}$ be equicontinuous on A . Then A is Lyapunov stable.*

Proof. The sentence

$$\forall \underline{a} \in {}^\sigma A \forall \underline{t} \in {}^*\mathbb{R}^+ : S(\underline{t}) \text{mon}(\underline{a}) \subseteq \text{mon}(S(\underline{t})\underline{a}) \subseteq \text{mon}(*A)$$

holds true since A is positively invariant and $\{S(t); t \in \mathbb{R}^+\}$ are equicontinuous on A . From the compactness of A it follows that $\text{mon}(*A) = \bigcup_{a \in {}^\sigma A} \text{mon}(a)$ and thus the sentence

$$\forall \underline{t} \in {}^*\mathbb{R}^+ : S(\underline{t}) \text{mon}(*A) \subseteq \text{mon}(*A) \tag{5.5}$$

holds and A is Lyapunov stable. $\square$

5.4 Asymptotic compactness and dissipativity

In this section we give nonstandard characterizations of two properties of dynamical systems, namely asymptotic compactness and dissipativity, closely related to the existence of global attractor (as we will see in the next section).

Definition. We call the dynamical system $(X, S(t))$ asymptotically compact if the set $\{S(t_n)x_n; n \in \mathbb{N}\}$ is relatively compact for every bounded sequence $\{x_n\}_{n=1}^\infty$ of elements in X and every $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}_0^+$ satisfying $t_n \rightarrow \infty$.

In the following theorem we allow (X, d) not to be complete. In case the metric space is complete, the first part of the proof is sufficient to prove the theorem since by Theorem 4.18 we have $\text{pns}(*X) = \text{ns}(*X)$.

Theorem 5.10. Let (X, d) be a metric space. Then dynamical system $(X, S(t))$ is asymptotically compact if and only if

$$\forall \underline{x} \in \text{fin}(*X) \forall \underline{T} \in \mathbb{R}_0^+ : S(\underline{T})\underline{x} \in \text{ns}(*X). \quad (5.6)$$

Proof. Let us assume $S(T)x \notin \text{ns}(*X)$ for some $x \in \text{fin}(*X)$ and $T \in \mathbb{R}_0^+$. To show that $(X, S(t))$ is not asymptotically compact we will construct a bounded sequence $\{x_n\}_{n=1}^\infty$ in X and $\{t_n\}_{n=1}^\infty$ such that the set $\{S(t_n)x_n; n \in \mathbb{N}\}$ has no limit points. Since we allowed the metric space (X, d) not to be complete, only two possibilities might occur: Either $S(T)x \notin \text{pns}(*X)$, or $S(T)x \in \text{pns}(*X) \setminus \text{ns}(*X)$.

Let $S(T)x \notin \text{pns}(*X)$, then we can find $\varepsilon \in \mathbb{R}^+$ such that $d(S(T)x, y) \geq * \varepsilon$ for every $y \in {}^s X$. Choose $x_0 \in X$ arbitrary and $t_0 \in \mathbb{R}^+$ arbitrary. Since $x \in \text{fin}(*X)$, $\text{st}(d(*x_0, x))$ is well defined and $R = \text{st}(d(*x_0, x)) + 1 \in \mathbb{R}^+$. The sentence

$$\exists \underline{y} \in *X \exists \underline{\tau} \in *\mathbb{R}^+ : d(*S(t_0)x_0, S(\underline{\tau})\underline{y}) \geq * \varepsilon \wedge \underline{\tau} > *t_0 + 1 \wedge \underline{y} \in *B(x_0, R)$$

holds true (consider $y = x$ and $\tau = T$). By the transfer principle the sentence

$$\exists \underline{y} \in X \exists \underline{\tau} \in \mathbb{R}^+ : d(S(t_0)x_0, S(\underline{\tau})\underline{y}) \geq \varepsilon \wedge \underline{\tau} > t_0 + 1 \wedge \underline{y} \in B(x_0, R)$$

also holds true. Set $x_1 = y, t_1 = \tau$.

We proceed by induction. Choose $n \in \mathbb{N}$ arbitrary and suppose that there exist $t_0, t_1, \dots, t_n \in \mathbb{R}^+$ and $x_0, x_1, \dots, x_n \in X$ such that $x_1, \dots, x_n \in B(x_0, R)$, $d(*S(t_k)x_k, S(T)x) \geq * \varepsilon$ and $t_k > t_0 + k$ for every $k = 0, 1, \dots, n$. The sentence

$$\begin{aligned} \exists \underline{y} \in *X \exists \underline{\tau} \in *\mathbb{R}^+ : d(*S(t_0)x_0, S(\underline{\tau})\underline{y}) \geq * \varepsilon \wedge \dots \wedge d(*S(t_n)x_n, S(\underline{\tau})\underline{y}) \geq * \varepsilon \\ \wedge \underline{\tau} > *t_n + 1 \wedge \underline{y} \in *B(x_0, R) \end{aligned}$$

holds true for $y = x$ and $\tau = T$. Using the transfer principle we may find $t_{n+1} \in \mathbb{R}^+$ and $x_{n+1} \in B(x_0, R)$ satisfying $d(S(t_k)x_k, S(t_{n+1})x_{n+1}) \geq \varepsilon$ for every $k \in \{1, \dots, n\}$, $x_{n+1} \in B(x_0, R)$ and $t_{n+1} > t_0 + n + 1$. Now we observe the sequence $\{S(t_n)x_n\}_{n=1}^\infty$ has no limit points since from construction of the sequence it follows that $d(S(t_m)x_m, S(t_n)x_n) \geq \varepsilon$ for every $m, n \in \mathbb{N}$.

Suppose $S(T)x \in \text{pns}(*X) \setminus \text{ns}(*X)$. First we construct sequences $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}^+$ and $\{x_n\}_{n=1}^\infty$ in X such that $t_n \rightarrow \infty$ for $n \rightarrow \infty$, $\{x_n\}_{n=1}^\infty$ is bounded and $d(S(T)x, *S(t_n)x_n) < *\frac{1}{n}$ for every $n \in \mathbb{N}$. Since $S(T)x \in \text{pns}(*X)$ there exists

bounded sequence $\{z_n\}_{n=1}^\infty$ in X such that $d(S(T)x, {}^*z_n) < \frac{1}{2(n+1)}$. Now from $x \in \text{fin}({}^*X)$ it follows $d(x, {}^*\tilde{x}) \in \text{fin}({}^*\mathbb{R}^+)$ for some $\tilde{x} \in X$. Then $\text{st}(d(x, {}^*\tilde{x}))$ is well defined and $R = \text{st}(d(x, {}^*\tilde{x})) + 1 \in \text{fin}(\mathbb{R}^+)$. Fix $n \in \mathbb{N}$. The sentence

$$\exists \underline{y} \in {}^*X \exists \underline{\tau} \in {}^*\mathbb{R}^+ : \underline{\tau} > {}^*n \wedge d(S(\underline{\tau})\underline{y}, {}^*z_n) < \frac{1}{2(n+1)} \wedge \underline{y} \in {}^*B(\tilde{x}, R)$$

holds true for $y = x, \tau = T$. Following the argument used in previous part of the proof we construct $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}^+$ and $\{x_n\}_{n=1}^\infty$ in X such that $t_n > n$, $x_n \in B(\tilde{x}, R)$ and $d(S(t_n)x_n, z_n) < \frac{1}{2(n+1)}$ for every $n \in \mathbb{N}$. We observe that $t_n \rightarrow \infty$ and $\{x_n\}_{n=1}^\infty$ is bounded. It remains to verify whether the sequences meet our requirement. Using triangle inequality we obtain

$$d(S(T)x, {}^*(S(t_n)x_n)) \leq d(S(T)x, {}^*z_n) + d({}^*z_n, {}^*(S(t_n)x_n)) < \frac{1}{n+1}.$$

Now for contradiction let us assume that $\{S(t_n)x_n\}_{n=1}^\infty$ has a limit point $y \in X$, i. e. there exists a subsequence also denoted $S(t_n)x_n$ such that $S(t_n)x_n \rightarrow y$. Since $S(T)x \notin \text{ns}({}^*X)$ it holds $d({}^*y, S(T)x) \geq \frac{1}{m}$ for some $m \in \mathbb{N}$. From $S(t_n)x_n \rightarrow y$ there exists $n \in \mathbb{N}$ such that $n > 2m$ and $d(S(t_n)x_n, y) < \frac{1}{2m}$. Then we have

$$\frac{1}{m} \leq d(S(T)x, {}^*y) \leq d(S(T)x, {}^*(S(t_n)x_n)) + d({}^*(S(t_n)x_n), {}^*y) < \frac{1}{m},$$

a contradiction, hence $\{S(t_n)x_n\}_{n=1}^\infty$ has no limit point and $(X, S(t))$ is not asymptotically compact.

Conversely let (5.6) hold and let $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}_0^+$ satisfy $t_n \rightarrow \infty$ and let $\{x_n\}_{n=1}^\infty$ in X be bounded. By passing to a subsequence we can freely assume $t_n \geq n$. Choose $N \in \mathbb{N}_\infty$ arbitrary. Proposition 4.7 implies $x_N \in \text{fin}({}^*X)$ and from (5.6) we have $S(t_N)x_N \in \text{ns}({}^*X)$. Using Proposition 4.10 we observe that $\{S(t_n)x_n\}_{n=1}^\infty$ has a limit point and therefore $(X, S(t))$ is asymptotically compact. $\square$

Definition. We call the dynamical system $(X, S(t))$ dissipative, if there exists bounded set $W \subseteq X$ such that for every bounded set $B \subseteq X$ there exists $t_0 \in \mathbb{R}_0^+$ satisfying $S(t)B \subseteq W$ for every $t \geq t_0$. The set W is called an absorbing set.

Proposition 5.11. Let $W \subseteq X$ be an absorbing set of $(X, S(t))$. Then there exists a closed positively invariant absorbing set $\widetilde{W}$.

Proof. Since W is bounded using the definition of absorbing set we find $t_0 \in \mathbb{R}^+$ such that $S(t)W \subseteq W$ for $t \geq t_0$. Define

$$\widetilde{W} = \overline{\bigcup_{t \geq t_0} S(t)W}.$$

The set $\widetilde{W}$ is clearly bounded and closed. To show that $\widetilde{W}$ absorbs bounded sets, let $B \subseteq X$ be bounded and let $t_B \in \mathbb{R}^+$ be such that $S(t)B \subseteq W$ for $t \geq t_B$. Then $S(t)B \subseteq \widetilde{W}$ for $t \geq t_0 + t_B$.

Let $x \in \widetilde{W}$. From Theorem 2.13 it follows that ${}^*(\bigcup_{t \geq t_0} S(t)W) = \bigcup_{t \geq {}^*t_0} S(t){}^*W$ and by Proposition 4.3 ${}^*x \simeq S(\tau)w$ for some $\tau \in \mathbb{R}^+, t \geq {}^*t_0$ and $w \in {}^*W$. Choose $t \in \mathbb{R}^+$. Then since $S(t)$ is continuous at x , from Proposition 4.11 we obtain $S({}^*t){}^*x \simeq S({}^*t)S(\tau)w = S({}^*t + \tau)w$ and therefore $S(t)x \in \widetilde{W}$. $\square$

Proposition 5.12. *Dynamical system $(X, S(t))$ is dissipative if and only if there exists $W \subseteq X$ bounded such that*

$$\forall \underline{b} \in \text{fin}(*X) \forall \underline{T} \in \mathbb{R}_\infty^+ : S(\underline{T})\underline{b} \in *W. \quad (5.7)$$

Proof. Let $b \in \text{fin}(*X)$, then $b \in *B(x, R)$ for some $x \in X$ and $R \in \mathbb{R}^+$. Since $B(x, R)$ is bounded we can find $t_0 \in \mathbb{R}^+$ such that $S(t)B(x, R) \subseteq W$ for every $t \in \mathbb{R}^+, t \geq t_0$. The transfer principle gives $\forall \underline{T} \in \mathbb{R}_\infty^+ : S(\underline{T})^*B(x, R) \subseteq *W$ and thus (5.7) holds.

Conversely let $B \subseteq X$ be bounded. Then from Proposition 4.7 it follows that $*B \subseteq \text{fin}(*X)$. By (5.7) we know that $\forall \underline{T} \in \mathbb{R}_\infty^+ : S(\underline{T})^*B \subseteq *W$. From the permanence principle there exists $t_0 \in \mathbb{R}$ such that $S(\underline{t})^*B \subseteq *W$ holds for every $t \in {}^*\mathbb{R}, t \geq {}^*t_0$. It remains to use the transfer principle to obtain $\forall t \in \mathbb{R} : t \geq t_0 \Rightarrow S(t)B \subseteq W$. $\square$

5.5 Global attractor

Definition. *A set $\mathcal{A} \subseteq X$ is called a global attractor of the dynamical system $(X, S(t))$ if*

- (i) $\mathcal{A}$ is compact,
- (ii) $\mathcal{A}$ is invariant,
- (iii) $\mathcal{A}$ attracts bounded sets, i. e. $\text{dist}(S(t)B, \mathcal{A}) \rightarrow 0$ holds for every bounded $B \subseteq X$ as $t \rightarrow \infty$.

Proposition 5.13. *Let $A \subseteq X$ be compact. The $\text{dist}(S(t)B, A) \rightarrow 0$ for every $B \subseteq X$ bounded if and only if*

$$\forall \underline{b} \in \text{fin}(*X) \forall \underline{T} \in \mathbb{R}_\infty^+ \exists \underline{a} \in *A : S(\underline{T})\underline{b} \simeq \underline{a}. \quad (5.8)$$

Proof. Let $b \in \text{fin}(*X)$. Then $b \in *B(x, R)$ for some $x \in X$ and $R \in \mathbb{R}^+$. The sentence (5.8) holds true by Lemma 5.6.

Let conversely $B \subseteq X$ be bounded. By Proposition 4.7 we have $*B \subseteq \text{fin}(*X)$. The rest again follows immediately from Lemma 5.6 $\square$

Theorem 5.14. *Dynamical system $(X, S(t))$ has a global attractor if and only if it is asymptotically compact and dissipative.*

Let $W \subseteq X$ be an absorbing set. Then we have a nonstandard characterization of the global attractor

$$\mathcal{A} = \{x \in W; \exists w \in {}^*W \exists T \in \mathbb{R}_\infty^+ : S(T)w \simeq {}^*x\}. \quad (5.9)$$

In other words

$$\mathcal{A} = \text{st}(\{S(T)^*W; T \in \mathbb{R}_\infty^+\}).$$

Proof. Suppose that the dynamical system $(X, S(t))$ has a global attractor $\mathcal{A}$. To prove the asymptotic compactness using (5.6) fix $b \in \text{fin}(*X)$ and $T \in \mathbb{R}_\infty^+$. Since $\mathcal{A}$ attracts bounded sets, Proposition 5.13 gives that there exists $a \in {}^*\mathcal{A}$ such that $S(T)b \simeq a$. Using the compactness of the global attractor and Theorem 4.14 we obtain $S(T)b \in \text{ns}(*X)$.

Set $W = \{x \in X; \exists a \in A : d(x, a) < 1\}$, then W is bounded since $\mathcal{A}$ is bounded. It remains to show that W absorbs bounded sets. By Proposition 5.12 and the standard definition principle it suffices to prove

$$\forall \underline{b} \in \text{fin}(*X) \forall \underline{T} \in \mathbb{R}_\infty^+ \exists \underline{a} \in *\mathcal{A} : d(S(\underline{T})\underline{b}, \underline{a}) < 1$$

which follows immediately from Proposition 5.13.

To prove the nonstandard characterization let us denote the right-hand side of (5.9) with $\tilde{\mathcal{A}}$. Let $x \in {}^\sigma\mathcal{A}$. Then the invariance of $\mathcal{A}$ and the transfer principle give $S(T)a = x$ for some $T \in \mathbb{R}_\infty^+$ and $a \in *\mathcal{A}$. From $*\mathcal{A} \subseteq *W$ it follows $x \in {}^\sigma\tilde{\mathcal{A}}$.

Conversely let $x \in {}^\sigma\tilde{\mathcal{A}}$. Then $S(T)w \simeq x$ for some $w \in *W$ and $T \in \mathbb{R}_\infty^+$ and using the attracting property (5.8) we find $a \in *\mathcal{A}$ such that $S(T)w \simeq a$. Using the compactness of $\mathcal{A}$ we can assume $a \in {}^\sigma\mathcal{A}$. Combining these we get $x \simeq S(T)w \simeq a \in {}^\sigma\mathcal{A}$ and since $x, a \in {}^\sigma X$ we get $x \in {}^\sigma\tilde{\mathcal{A}}$, hence $\mathcal{A} = \tilde{\mathcal{A}}$.

Now suppose that $(X, S(t))$ is asymptotically compact and there exists a closed positively invariant absorbing set W . We define $\mathcal{A}$ by (5.9) and show that $\mathcal{A}$ is a global attractor of the dynamical system $(X, S(t))$. Since $\mathcal{A} \subseteq W$, we immediately see that $\mathcal{A}$ is bounded.

Let $a \in {}^\sigma\mathcal{A}$ and $t \in {}^\sigma\mathbb{R}^+$. Using (5.9) we find $x \in *W$ and $T \in \mathbb{R}_\infty^+$ satisfying $S(T)x \simeq a$. Since a is standard, $S(t)$ is continuous at a and thus we obtain $S(t)a \simeq S(t)S(T)x = S(t+T)x$. From Proposition 3.6 we know that $t+T \in \mathbb{R}_\infty^+$, hence $S(t)a \in {}^\sigma\mathcal{A}$.

To show the compactness of $\mathcal{A}$ let $a \in *\mathcal{A}$ and choose $T \in \mathbb{R}_\infty^+$. Since $\mathcal{A}$ is invariant we have $S(T)x = a$ for some $x \in *\mathcal{A}$. From the dissipativity of $(X, S(t))$ it follows that $S(T)x \in *W \subseteq \text{fin}(*X)$ and using asymptotic compactness we have $S(T)x \in \text{ns}(*X)$. Since W is closed, by Corollary 4.4 there exists $w \in {}^\sigma W$ such that $S(T)x \simeq w$, therefore $w \in {}^\sigma\mathcal{A}$ and $\mathcal{A}$ is compact.

It remains to prove that $\mathcal{A}$ attracts bounded sets. Let $b \in \text{fin}(*X)$ and $T \in \mathbb{R}_\infty^+$. Using the dissipativity and asymptotic compactness of $(X, S(t))$ we get $S(T)b \in *W \cap \text{ns}(*X)$ and since W is closed we have $S(T)b \simeq w$ for some $w \in {}^\sigma W$. From Proposition 3.6 we have $\frac{T}{2} \in \mathbb{R}_\infty^+$ and the dissipativity gives $S(\frac{T}{2})b \in *W$. Thus $S(T)b = S(\frac{T}{2})S(\frac{T}{2})b \simeq w$ and $w \in {}^\sigma\mathcal{A}$. The assertion follows from Proposition 5.13. $\square$

Let us look more closely on the relation between absorbing set and global attractor.

Theorem 5.15. *Let $\mathcal{A} \subseteq X$ be a global attractor of the dynamical system $(X, S(t))$ and let $W \subseteq X$ be an absorbing set. Then $\omega(W) = \mathcal{A}$ and we have the characterization*

$$\mathcal{A} = \{x \in X; \exists \{x_n\}_{n=1}^\infty \subseteq X \exists \{t_n\}_{n=1}^\infty \subseteq \mathbb{R}^+ : \{x_n\}_{n=1}^\infty \text{ bounded} \wedge t_n \rightarrow \infty \wedge S(t_n)x_n \rightarrow x\}. \quad (5.10)$$

Proof. From Proposition 5.1 it follows that

$$\omega(W) = \{x \in X; \exists w \in *W \exists T \in \mathbb{R}_\infty^+ : S(T)w \simeq x\}.$$

Let $x \in \omega(W)$, i. e. $S(T)w \simeq x$ for some $w \in *W$ and $T \in \mathbb{R}_\infty^+$. From the attracting property and the compactness of $\mathcal{A}$ it follows that $x \simeq S(T)w \simeq a$ for some $a \in \mathcal{A}$ and thus $x = a$ and $x \in W$. Now we observe that

$$\omega(W) = \{x \in W; \exists w \in *W \exists T \in \mathbb{R}_\infty^+ : S(T)w \simeq x\} = \mathcal{A}. \quad (5.11)$$

Let us denote the right hand side of (5.10) by $\tilde{\mathcal{A}}$ and assume that the absorbing set W is closed and positively invariant. The inclusion $\mathcal{A} = \omega(W) \subseteq \tilde{\mathcal{A}}$ follows immediately from (5.11) and the classical definition of $\omega(W)$. Conversely let $x \in \tilde{\mathcal{A}}$, i. e. $S(t_n)x_n \rightarrow x$ for some $\{x_n\}_{n=1}^\infty$ bounded in X and $\{t_n\}_{n=1}^\infty$ in $\mathbb{R}^+$ satisfying $t_n \rightarrow \infty$. Let us assume $t_n > n$ for all $n \in \mathbb{N}$, otherwise we pass to a subsequence.

We proceed similarly as in the previous proof. Choose $N \in {}^*\mathbb{N}_\infty$ arbitrary, then $t_N \in \mathbb{R}_\infty^+$. From Proposition 4.8 we know $S(t_N)x_N \simeq {}^*x$. By Proposition 4.7 we have $x_N \in \text{fin}({}^*X)$. Since $(X, S(t))$ is dissipative and $\frac{t_N}{2} \in \mathbb{R}_\infty^+$ by Proposition 3.6, it follows that $w = S(\frac{t_N}{2})x_N \in {}^*W$. From the asymptotic compactness and the positive invariance of $\mathcal{A}$ and Theorem 5.10 we obtain $S(\frac{t_N}{2})w \in \text{ns}({}^*X) \cap {}^*W$. Finally since W is closed, Corollary 4.4 gives $x \in W$ and thus $\mathcal{A} = \tilde{\mathcal{A}}$. $\square$

Theorem 5.16. *Let $\mathcal{A}$ be a global attractor. Then $\mathcal{A}$ is uniformly asymptotically stable.*

Proof. By Proposition 5.9 it is sufficient to prove that the family $\{S(t); t \in \mathbb{R}^+\}$ is equicontinuous on $\mathcal{A}$. Choose $\tilde{a} \in {}^\sigma\mathcal{A}$ and $a \in \text{mon}(\tilde{a})$.

Let us first consider the case $t \in \text{fin}(\mathbb{R}_0^+)$. From Proposition 3.10 we know that $t \simeq \tilde{t}$ for some $\tilde{t} \in {}^\sigma\mathbb{R}_0^+$. From continuity of $(s, x) \rightarrow S(s)x$ we obtain $S(t)a \simeq S(\tilde{t})\tilde{a}$, therefore $S(t)a \in \text{mon}(S)$.

Now let $t \in \mathbb{R}_\infty^+$. From the attracting property (5.8) it follows that $S(t)a \simeq \tilde{a}$ for some $\tilde{a} \in {}^*\mathcal{A}$, hence $S(t)a \in \mu({}^*\mathcal{A})$. The set $\mathcal{A}$ is Lyapunov stable by Theorem 5.7.

To show that $\mathcal{A}$ is a local attractor let $\delta \in \mathbb{R}^+, T \in \mathbb{R}_\infty^+$ and $a \in {}^*B(\mathcal{A}, \delta)$. Since $B(\mathcal{A}, \delta)$ is bounded, from Proposition 4.7 and the attracting property (5.8) it follows $S(t)a \simeq \tilde{a}$ for some $\tilde{a} \in {}^*\mathcal{A}$ and by Theorem 5.7 the global attractor is asymptotically stable. $\square$

Afterword

In this work we discussed a nonstandard approach to dynamical systems. First we reviewed the definition and properties of nonstandard embedding, then introduced infinitesimal and infinite numbers to the real line and studied metric spaces using nonstandard methods. Last we used this apparatus to prove nonstandard characterizations of ω -limit set (Proposition 5.1), stability (Theorem 5.7), asymptotic compactness (Theorem 5.10), dissipativity (Proposition 5.12) and global attractor (Theorem 5.14) and showed that global attractor exists if and only if the dynamical system is asymptotically compact and dissipative (see the previous reference) and that a global attractor is uniformly asymptotically stable (Theorem 5.16).

We realize that the presentation of nonstandard methods and principles presented in Chapter 2 was far from complete, for example we completely omitted saturation which is one of the most important concepts in nonstandard models. However, since we worked with only metric spaces, the saturation was not necessary.

We also realize that the theorems proved in Chapter 5 cover only a small part of the theory of dynamical systems. An eventual extension of this work could apply the results from this thesis to concrete differential equations and offer more theorems about the existence of global attractors, for example for α -contracting dissipative dynamical systems.

Other results worth pursuing are the following two assertions (similarly as in Chapter 5 let X be a complete metric space and $(X, S(t))$ a dynamical system):

- (i) Let $x_0 \in X$ and $\gamma^+(x_0)$ is relatively compact. Then $\omega(x_0)$ limit set is connected.
- (ii) Let $W \subseteq X$ be a connected absorbing set of asymptotically compact and dissipative dynamical system $(X, S(t))$. Then the respective global attractor $\mathcal{A}$ is connected.

The proofs of both of these assertions require an elegant nonstandard characterization of connected sets and more detailed study of ${}^*\omega(x_0)$ which we will try to carry out in the future.

Bibliography

- [1] CUTLAND, Nigel J., KIESLER, H. Jerome. *Attractors and neo-attractors for 3D stochastic Navier-Stokes equations*. Stochastics and Dynamics, Vol. 5, No. 4: p. 487 - 533. World Scientific Publishing Company, 2005. DOI: 10.1142/S0219493705001559.
- [2] FIEREISL, Eduard, PRAŽÁK, Dalibor. *Asymptotic Behavior of Dynamical Systems in Fluid Mechanics*. American Institute of Mathematical Sciences, 2010. ISBN 9781601330031.
- [3] HURD, Albert E. *Nonstandard Analysis of Dynamical Systems. I: Limit motions, stability*. Transactions of the American Mathematical Society, Volume 160. American Mathematical Society, 1971. DOI: <http://dx.doi.org/10.1090/S0002-9947-1971-0280827-6>.
- [4] HURD, Albert E., LOEB, Peter A. *An Introduction to Nonstandard Real Analysis*. Elsevier Science, 1985. ISBN 9780080874371.
- [5] NELSON, Edward. *Internal set theory: A new approach to non-standard analysis*. Bulletin of the American Mathematical Society 83(6): p. 1165 - 1198. American Mathematical Society, 1977. DOI: <http://dx.doi.org/10.1090/S0002-9904-1977-14398-X>.
- [6] ROBINSON, Abraham. *Non-Standard Analysis*. Princeton University Press, 1996. ISBN 9780691044903.
- [7] SELL, George R., YOU, Yuncheng. *Dynamics of Evolutionary Equations*. Springer, 2002. ISBN 9780387983479.
- [8] TAO, Terence. *Compactness and contradiction*. American Mathematical Society, 2013. ISBN-10: 0-8218-9492-7.
- [9] TEMAM, Roger. *Infinite Dimensional Dynamical Systems in Mechanics and Physics*. Springer, 1997. ISBN 9780387948669.
- [10] VÄTH, Martin. *Nonstandard Analysis*. Birkhäuser, 2007. ISBN 9780817677732.

List of used symbols

$\mathbb{N}$	natural numbers	
$\mathbb{R}$	real numbers	
$\mathbb{R}^+$	positive real numbers	
$\mathbb{R}_0^+$	non-negative real numbers	
$\mathcal{P}(X)$	set of all subsets of a set X	
$\mathcal{L}$	formal language	5
$\text{cns}(\mathcal{L})$	set of all constants of formal language $\mathcal{L}$	5
$\widehat{S}$	superstructure	6
$*$	elementary or nonstandard embedding	10, 11
$\mathcal{J}$	nonstandard universe	14
${}^*\mathbb{N}$	hypernatural numbers	18
$\mathbb{N}_\infty$	infinite hypernatural numbers	19
${}^*\mathbb{R}$	hyperreal numbers	18
${}^*\mathbb{R}^+$	positive hyperreal numbers	19
$\mathbb{R}_\infty$	infinite hyperreal numbers	19
$\mathbb{R}_\infty^+$	positive infinitive hyperreal numbers	19
$\inf({}^*\mathbb{R})$	infinitesimal hyperreal numbers	19
$\inf({}^*\mathbb{R}^+)$	positive hyperreal numbers	19
$\text{fin}({}^*\mathbb{R})$	finite hyperreal numbers	19
$\text{fin}({}^*\mathbb{R}^+)$	positive finite hyperreal numbers	19
st	standard part homomorphism or relation	21, 25
mon	monad	21, 24, 37
$B(x, r)$	open ball with a center in x and radius R	24
$\text{ns}({}^*X)$	nearstandard points in ${}^*X$	25
$\text{fin}({}^*X)$	finite points in ${}^*X$	27
$\text{pns}({}^*X)$	pre-nearstandard points in ${}^*X$	32
$\omega(B)$	ω -limit set of $B \subseteq X$	34
$\gamma^+(x)$	positive orbit of $x \in X$	35
$\text{dist}(A, B)$	distance of a set $A \subseteq X$ from $B \subseteq X$	37
$\mathcal{A}$	global attractor	42

Index

- *-transform, 11
- $\simeq$ (infinitely close)
 - numbers, 21
 - points, 24
- atom, 6
- attractor
 - global, 42
 - local, 38
- dynamical system, 33
 - asymptotically compact, 40
 - dissipative, 41
- element
 - external, 14
 - internal, 14
 - standard, 10
- embedding
 - elementary, 10
 - nonstandard, 11
- entity, 6
- formal language, 5
- formula, 5
 - internal, 15
- interpretation, 9
- monad
 - of a number, 21
 - of a point, 24
- number
 - finite, 19
 - hypernatural, 18
 - hyperreal, 18
 - infinite, 19
 - infinitesimal, 19
- open δ -neighborhood, 37
- periodic cycle, 36
- point
 - finite, 27
 - limit, 28
 - nearstandard, 25
 - pre-nearstandard, 32
 - standard, 25
 - stationary, 36
- positive orbit, 35
- predicate, 6
- principle
 - internal definition, 15
 - permanence, 22
 - standard definition, 11
 - transfer, 11
- scope of the quantifier, 5
- sentence, 6
- set
 - absorbing, 41
 - invariant, 33
 - Lyapunov stable, 37
 - negatively invariant, 33
 - positively invariant, 33
 - unstable, 37
- standard copy, 10
- standard part
 - of a number, 21
 - of a point, 25
- superstructure, 6
- superstructure monomorphism, 11
- transitively bounded formula, 6
- universe
 - nonstandard, 15
 - standard, 10