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DISERTAČNÍ PRÁCE

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Aproximace, numerická realizace a kvalitativní analýza kontaktních úloh se třením

Katedra numerické matematiky

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Název práce: Aproximace, numerická realizace a kvalitativní analýza kontaktních úloh se třením

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Abstrakt: Tato práce se zabývá teoretickou analýzou a numerickou realizací diskretizovaných kontaktních úloh s Coulombovým třením. Nejprve je pomocí pevněbodového přístupu provedena analýza diskretizovaných 3D statických kontaktních úloh s izotropním a ortotropním Coulombovým třením a koeficienty tření závislými na řešení. Existence alespoň jednoho řešení je dokázána pro koeficienty tření reprezentované omezenými kladnými spojitými funkcemi. Pokud jsou tyto funkce navíc lipschitzovsky spojité a horní meze jejich hodnot spolu s jejich moduly lipschitzovskosti jsou dostatečně malé, je zaručena jednoznačnost tohoto řešení. Dále jsou v případě 2D statických kontaktních úloh s izotropním Coulombovým třením a koeficientem nezávislým na řešení studovány vlastnosti řešení parametrizovaných koeficientem tření nebo vektorem zatížení. S pomocí dvou variant věty o implicitních funkcích jsou ustaveny podmímky, za nichž existuje lokální lipschitzovská větev řešení na okolí daného referenčního bodu. Následně je navržen algoritmus po částech hladké kontinuace, který nám umožňuje sledovat takové větvě řešení numericky. Na závěr je uvedena dobře formulovaná prostorová semidiskretizace dynamických kontaktních úloh s izotropním Coulombovým třením, kde koeficient nezávisí na řešení.

Klíčová slova: konktakní úloha, Coulombovo tření, lokálně lipschitzovská větev řešení, po částech hladká kontinuační metoda, metoda přerozdělení hmotnosti

Title: Approximation, numerical realization and qualitative analysis of contact problems with friction

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Abstract: This thesis deals with theoretical analysis and numerical realization of discretized contact problems with Coulomb friction. First, discretized 3D static contact problems with isotropic and orthotropic Coulomb friction and solution-dependent coefficients of friction are analyzed by means of the fixed-point approach. Existence of at least one solution is established for coefficients of friction represented by positive, bounded and continuous functions. If these functions are in addition Lipschitz continuous and upper bounds of their values together with their Lipschitz moduli are sufficiently small, uniqueness of the solution is guaranteed. Second, properties of solutions parametrized by the coefficient of friction or the load vector are studied in the case of discrete 2D static contact problems with isotropic Coulomb friction and coefficient independent of the solution. Conditions under which there exists a local Lipschitz continuous branch of solutions around a given reference point are established due to two variants of the implicit-function theorem. Consequently, a piecewise smooth continuation algorithm, which enables us to follow such branches of solutions numerically, is proposed. In the end, a well-posed spatial semi-discretization of dynamic contact problems with isotropic Coulomb friction where the coefficient does not depend on the solution is introduced.

Keywords: contact problem, Coulomb friction, local Lipschitz continuous branch of solutions, piecewise smooth continuation method, mass redistribution method

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Notation

General

Vectors, matrices and tensors will be denoted by bold letters. If a constant c will depend on parameters $p_1, \dots, p_s$, this will be indicated by writing $c(p_1, \dots, p_s)$.

Vectors, matrices

$\mathbb{R}_+$, $\mathbb{R}_-$: sets of non-negative and non-positive real numbers, respectively.

$\mathbb{R}^n_+$, $\mathbb{R}^n_-$: sets of all vectors in $\mathbb{R}^n$ with non-negative and non-positive components, respectively.

$\boldsymbol{u} \cdot \boldsymbol{v} = (\boldsymbol{u}, \boldsymbol{v}) = \sum_{1 \le i \le n} u_i v_i$: scalar product of vectors $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in \mathbb{R}^n$.

$\|\boldsymbol{u}\| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$: Euclidean norm in $\mathbb{R}^n$.

$\|\boldsymbol{u}\|_\infty = \max_{i=1,\dots,n} |u_i|$: max-norm in $\mathbb{R}^n$.

$\mathbb{M}^n$: set of all real square matrices of order n.

$\mathbb{M}^n_> = \{\boldsymbol{A} \in \mathbb{M}^n \mid \det \boldsymbol{A} > 0\}$.

$\boldsymbol{I}_n$: identity matrix of order n. (For brevity, we shall sometimes omit the subscript n.)

$\|\boldsymbol{A}\| = \sup_{\boldsymbol{v} \neq \boldsymbol{0}} (\|\boldsymbol{A}\boldsymbol{v}\| / \|\boldsymbol{v}\|)$: matrix norm in $\mathbb{M}^n$.

$\boldsymbol{A} : \boldsymbol{B} = \sum_{1 \le i,j \le n} a_{ij} b_{ij}$: Frobenius product of matrices $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij}) \in \mathbb{M}^n$.

$\mathbf{Cof}\,\boldsymbol{A}$: cofactor matrix of the matrix $\boldsymbol{A}$ ($\mathbf{Cof}\,\boldsymbol{A} = (\det \boldsymbol{A})\boldsymbol{A}^{-T}$ if $\boldsymbol{A}$ is invertible).

$\mathbb{M}^{m,n}$: set of all m-by-n matrices.

$\mathbf{0}_{m,n}$: m-by-n zero matrix (besides $\mathbf{0}_{n,1}$, the n-dimensional zero vector will be written simply $\mathbf{0}$ as well).

Sets

$\overline{G}$: closure of a set G.

$\mathring{G}$: interior of a set G.

∂G: boundary of a set G.

N_G: normal cone of a set G.

$|G|$: number of elements of a set G.

$\operatorname{diam}(G) = \sup\{\|\boldsymbol{x} - \boldsymbol{y}\| \mid \boldsymbol{x}, \boldsymbol{y} \in G\}$: diameter of a set $G \subset \mathbb{R}^n$.

Differential calculus

In what follows, G is an open subset of $\mathbb{R}^n$.

$D^{\boldsymbol{\alpha}} f = \dfrac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$: multi-index notation for partial derivatives of a function $f : G \subset \mathbb{R}^n \to \mathbb{R}$ with $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{N}^n$.

$\nabla f(\bar{\boldsymbol{x}}) = \left(\dfrac{\partial f}{\partial x_i}(\bar{\boldsymbol{x}})\right) \in \mathbb{R}^n$: gradient of a real-valued function $f : G \subset \mathbb{R}^n \to \mathbb{R}$ at $\bar{\boldsymbol{x}} \in G$.

$\boldsymbol{\nabla f}(\bar{\boldsymbol{x}}) = \left(\dfrac{\partial f_i}{\partial x_j}(\bar{\boldsymbol{x}})\right) \in \mathbb{M}^{m,n}$: gradient of a vector-valued function $\boldsymbol{f} : G \subset \mathbb{R}^n \to \mathbb{R}^m$ at $\bar{\boldsymbol{x}} \in G$.

$\mathbf{div}\,\boldsymbol{\sigma}(\bar{\boldsymbol{x}}) = \left(\sum_{1\le j\le n} \dfrac{\partial \sigma_{ij}}{\partial x_j}(\bar{\boldsymbol{x}})\right) \in \mathbb{R}^n$: divergence of a tensor field $\boldsymbol{\sigma} = (\sigma_{ij}) : G \subset \mathbb{R}^n \to \mathbb{M}^n$ at $\bar{\boldsymbol{x}} \in G$ (here and in what follows, we mean by *a tensor* a second-order tensor and we identify the set of all such tensors with the set $\mathbb{M}^n$).

$\boldsymbol{f}'(\boldsymbol{x}; \boldsymbol{y})$: directional derivative of a function $\boldsymbol{f} : G \subset \mathbb{R}^n \to \mathbb{R}^m$ at the point $\boldsymbol{x} \in G$ in the direction $\boldsymbol{y} \in \mathbb{R}^n$.

∂f: Clarke sub-differential of a real-valued Lipschitz continuous function f.

$\boldsymbol{\partial f}$: generalized Jacobian of a vector-valued Lipschitz continuous function $\boldsymbol{f}$.

$\dot{\boldsymbol{f}}(t, \boldsymbol{x}) = \dfrac{\partial \boldsymbol{f}}{\partial t}(t, \boldsymbol{x})$, $\ddot{\boldsymbol{f}}(t, \boldsymbol{x}) = \dfrac{\partial^2 \boldsymbol{f}}{\partial t^2}(t, \boldsymbol{x})$: the first and the second time derivative of a function $\boldsymbol{f} : (0, T) \times G \to \mathbb{R}^n$ at $(t, \boldsymbol{x}) \in (0, T) \times G$, $T > 0$, $G \subset \mathbb{R}^n$.

Function spaces

$P_k(G)$: space of all polynomial functions from $G \subset \mathbb{R}^n$ into $\mathbb{R}$ of degree up to k.

$C(G)$, $C^k(G)$: space of all continuous and k-times continuously differentiable functions from $G \subset \mathbb{R}^n$ into $\mathbb{R}$.

Let $G \subset \mathbb{R}^n$ be open.

$L^p(G)$: space of all measurable functions $v : G \to \mathbb{R}$ such that $\|v\|_{0,p,G} < +\infty$, where

$$\|v\|_{0,p,G} = \begin{cases} \left(\int_G |v|^p \,\mathrm{d}\boldsymbol{x}\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{\boldsymbol{x}\in G} |v(\boldsymbol{x})| & \text{if } p = \infty. \end{cases}$$

$W^{k,p}(G) = \{v \in L^p(G) \mid D^{\boldsymbol{\alpha}} v \in L^p(G),\ \forall\, |\boldsymbol{\alpha}| \le k\}$: the Sobolev space equipped with the norm:

$$\|v\|_{k,p,G} = \begin{cases} \left(\int_G \sum_{|\boldsymbol{\alpha}|\le k} |D^{\boldsymbol{\alpha}} v|^p \,\mathrm{d}\boldsymbol{x}\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{|\boldsymbol{\alpha}|\le k} \|D^{\boldsymbol{\alpha}} v\|_{0,\infty,G} & \text{if } p = \infty. \end{cases}$$

$H^k(G) = W^{k,2}(G)$, $\|v\|_{k,G} = \|v\|_{k,2,G}$.

$(u, v)_{k,G} = \displaystyle\int_G \sum_{|\boldsymbol{\alpha}|\le k} D^{\boldsymbol{\alpha}} u D^{\boldsymbol{\alpha}} v \,\mathrm{d}\boldsymbol{x}$: scalar product of functions $u, v \in H^k(G)$.

If $X(G)$ stands for a space of real-valued functions defined over G, $X(G; \mathbb{R}^m)$ and $X(G; \mathbb{M}^m)$, or shortly $\boldsymbol{X}(G)$, denote the spaces of vector-valued or tensor-valued mappings whose components belong to $X(G)$, for example,

$$L^p(G;\mathbb{R}^m) = \boldsymbol{L}^p(G) = \{\boldsymbol{v} = (v_i)\,|\,v_i \in L^p(G)\},$$
$$W^{k,p}(G;\mathbb{M}^m) = \boldsymbol{W}^{k,p}(G) = \{\boldsymbol{\sigma} = (\sigma_{ij})\,|\,\sigma_{ij} \in W^{k,p}(G)\}.$$

The associated norms are denoted by the same symbols, for example,

$$\|\boldsymbol{v}\|_{k,p,G} = \begin{cases} \left(\sum_{1\le i\le m}\|v_i\|^p_{k,p,G}\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max_{1\le i\le m}\|v_i\|_{0,\infty,G} & \text{if } p = \infty. \end{cases}$$

Let X be a Banach space and $T > 0$.
$C^1(0,T;X)$: Bochner space of continuously differentiable abstract functions from $[0,T]$ into X.
$H^1(0,T;X)$ Sobolev-Bochner space on $[0,T]$.

Elasticity

Continuous Setting

$\overline{\Omega} \subset \mathbb{R}^n$ $(n = 2,3)$: reference configuration of an elastic body. In what follows, we shall always assume that Ω is a bounded domain with a Lipschitz boundary $\partial\Omega$ which contains three disjoint, (relatively) open subsets Γ_D, Γ_N and Γ_C so that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$.
$\boldsymbol{u}$: displacement field.
$\boldsymbol{\sigma}(\boldsymbol{u})$: stress tensor corresponding to $\boldsymbol{u}$.
$\boldsymbol{\varepsilon}(\boldsymbol{u}) = 1/2(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T)$: linearized strain tensor.
$\boldsymbol{\mathcal{A}}$: the fourth-order elasticity tensor.
$\boldsymbol{f}$: density of volume forces.
$\boldsymbol{h}$: density of surface tractions.
$\mathscr{F}$: coefficient of friction.
$\boldsymbol{\mathscr{F}} = \mathbf{Diag}(\mathscr{F}_1, \mathscr{F}_2)$: matrix of friction coefficients $\mathscr{F}_1$ and $\mathscr{F}_2$ in the case of orthotropic friction.
ρ: mass density.
$\boldsymbol{\nu}$: unit outward normal vector along $\partial\Omega$.
$u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$: normal component of the displacement vector on $\partial\Omega$.
$\sigma_\nu(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu}\cdot\boldsymbol{\nu}$: normal component of the stress vector $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu}$ on $\partial\Omega$.

Let $n = 2$ and $\boldsymbol{\tau}$ be a unit tangent vector along Γ_C (orthonormal to $\boldsymbol{\nu}$).
$u_\tau = \boldsymbol{u}\cdot\boldsymbol{\tau}$: tangential displacement on Γ_C.
$\sigma_\tau(\boldsymbol{u}) = (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu})\cdot\boldsymbol{\tau}$: tangential stress on Γ_C.

Let $n = 3$ and $\boldsymbol{\tau}_1(\boldsymbol{x})$ and $\boldsymbol{\tau}_2(\boldsymbol{x})$ be two unit vectors from the tangent plane to Γ_C at $\boldsymbol{x}$ such that the triplet $\{\boldsymbol{\nu}(\boldsymbol{x}), \boldsymbol{\tau}_1(\boldsymbol{x}), \boldsymbol{\tau}_2(\boldsymbol{x})\}$ forms a local orthonormal basis in $\mathbb{R}^3$ for any $\boldsymbol{x} \in \Gamma_C$.

$\boldsymbol{u}_\tau = (u_{\tau,1}, u_{\tau,2})$, $u_{\tau,i} = \boldsymbol{u} \cdot \boldsymbol{\tau}_i$: tangential displacement on Γ_C.

$\boldsymbol{\sigma}_\tau(\boldsymbol{u}) = (\sigma_{\tau,1}(\boldsymbol{u}), \sigma_{\tau,2}(\boldsymbol{u}))$, $\sigma_{\tau,i} = \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} \cdot \boldsymbol{\tau}_i$: tangential stress on Γ_C.

Discrete Setting

$n_{\boldsymbol{u}}$: number of degrees of freedom for displacements.

n_c: number of nodes on $\overline{\Gamma}_C$ corresponding to the degrees of freedom for displacements.

$\boldsymbol{u}$: vector of degrees of freedom of the discretized displacement.

$\boldsymbol{\lambda}_\nu$, $\boldsymbol{\lambda}_\tau$: discrete normal and tangential Lagrange multipliers, respectively.

$\boldsymbol{A} \in \mathbb{M}^{n_{\boldsymbol{u}}}$: stiffness matrix.

$\boldsymbol{B}_\nu, \boldsymbol{B}_\tau \in \mathbb{M}^{n_c, n_{\boldsymbol{u}}}$: matrices representing the linear mappings which associate with a displacement field its normal and tangential component on the contact zone, respectively.

$\boldsymbol{f}$: load vector.

$\boldsymbol{\mathscr{F}}$: vector characterizing the distribution of the coefficient of friction.

$\boldsymbol{M} \in \mathbb{M}^{n_{\boldsymbol{u}}}$: mass matrix.

Introduction

Contact problems describing behaviour of a system of loaded deformable bodies which may come into mutual contact have been of permanent interest in a few last decades. It is well-known that besides non-penetration conditions, one often has to take into account the influence of friction on contacting zones to get a more realistic model. The most classical model of friction is given by the local Coulomb law of friction. Although its formulation is quite simple, the model of contact problems with Coulomb friction has not been completely understood yet.

In the framework of static linearized elasticity, which the first three chapters of this thesis are mainly devoted to, the first existence result for the continuous problem was obtained in [48] for a coefficient of friction $\mathscr{F}$ independent of the solution. Later, the existence analysis was extended to coefficients which may depend on the solution itself (see [16] and the references therein). Typically, existence of a solution is guaranteed provided that $\mathscr{F}$ is sufficiently small (with additional technical assumptions on the regularity of data). More recently, it has been proved in [52] that if the solution possesses a certain property, $\mathscr{F}$ is small enough and does not depend on the solution, the solution is unique. On the other hand, some examples of non-unique solutions are known for large $\mathscr{F}$ ([30, 31]).

Properties of appropriate finite-element discretizations of the problems discussed above are somewhat more explored. It was shown in [21] that at least one solution exists for $\mathscr{F}$ belonging to a large class of coefficients. Moreover, this solution was shown to be unique if the values of $\mathscr{F}$ are small enough. In [33], existence of a solution was obtained in quite general cases when $\mathscr{F}$ depends on the solution. Nevertheless, the bound on the values of $\mathscr{F}$ ensuring uniqueness of the solution in [21] is mesh-dependent, and it vanishes when the norm of the finite-element partition tends to zero. Even for models with very small number of degrees of freedom, multiple solutions exist and structure of solutions is relatively complicated ([35, 29]). For models with high number of degrees of freedom, bifurcations of solutions have been detected numerically ([25]). In addition, it was observed in these cases that a small change of $\mathscr{F}$ leads to a dramatical change of the solution.

From this point of view, study of local behaviour of solutions seems to be promising. However, as far as we know, the only results of local character have been presented in [32], where existence of local Lipschitz continuous branches of discrete solutions parametrized by $\mathscr{F}$ was established under the assumption that $\mathscr{F}$ is constant.

In the case of elastodynamic contact problems, which we shall focus in the last chapter of this thesis on, some theoretical analysis is also established (see [16]). Nevertheless, we shall confine here to the issue concerning satisfactory approximation of these problems, which seems to be involved as well. Indeed, several strategies for constructing numerical schemes that are as much as possible stable and respect the contact constraint have already been proposed in the literature. In [8, 20], energy dissipative numerical schemes were built by impliciting the contact force. However, the drawback of this method is that the kinetic energy of contacting finite-element nodes is canceled at each impact. On the other hand, energy conserving schemes introduced in [40, 41, 28] either lead to spurious oscillations on the contact boundary or allow a small interpenetration. Even though energy conserving schemes with a penalized contact condition were constructed in ([1, 26, 28]), these still evoke important oscillations of the normal stress. In this context, it was early detected that the key point is satisfaction of the complementarity condition between the velocity and contact pressure in the normal direction, the so-called persistency condition ([34, 40, 1]). But a compromise has to be made between satisfaction of this condition and the non-penetration one.

A common point of all these works is that they are focused on finding a good time discretization scheme. However, in [38] and [53], it has been shown that it is rather obtaining a well-posed and regular spatially semi-discrete problem that allows for stable schemes (see also [19, 27] for further developments). The spatial semi-discretizations proposed there allow use of any reasonable time discretization scheme whereas almost all time discretization schemes are unstable with the standard spatial finite-element semi-discretization. Nevertheless, these works are focused on elastodynamic contact without friction.

In Chapter 1 of this thesis, we generalize the existence and uniqueness results from [21] to discretized three-dimensional (3D) static contact problems with isotropic and orthotropic Coulomb friction in which coefficients of friction depend on the solution. In Chapter 2, we extend the local analysis from [32] to discrete two-dimensional (2D) static contact problems with Coulomb friction where the coefficient $\mathscr{F}$ depends on the spatial variable and we establish also existence of local Lipschitz continuous branches of solutions parametrized by loading in this case. To follow such solution branches, parametrized either by $\mathscr{F}$ or the loading, numerically, we propose an appropriate continuation algorithm and we present its application to 2D quasi-static contact problems in large deformations in Chapter 3. Chapter 4 concerns a spatial well-posed semi-discretization for dynamic contact problems with Coulomb friction, which is based on the mass redistribution method ([38]).

1 3D Static Problems with Solution-Dependent Coefficients of Friction

The aim of this chapter is to study discretized 3D contact problems with orthotropic Coulomb friction in which both coefficients of friction in the directions of the principal axes of orthotropy depend on the magnitudes of the tangential components of contact displacement. The Signorini-type problem is considered, that is, the contact problem between an elastic body and a rigid foundation. As a special case, analysis of problems with isotropic Coulomb friction and a solution-dependent coefficient of friction is attained. The results have been published in [24]. The case of isotropic friction itself was previously studied in [45].

This chapter consists of three sections. In Section 1.1, continuous setting of both problems with orthotropic and isotropic Coulomb friction is presented. A weak solution to the more general problem with orthotropic Coulomb friction is then defined in two ways: as a solution to an implicit variational inequality and as a fixed point of an auxiliary mapping Φ. The discretized form of the problem is then based on an appropriate discretization of Φ (Section 1.2). Section 1.3 presents existence and uniqueness analysis of the discretized problem. We show that at least one solution exists for any positive, bounded and continuous coefficients of friction. Assuming that the coefficients are in addition Lipschitz continuous, we prove that the discretization of Φ is Lipschitz continuous as well. The estimate of its Lipschitz modulus is derived in terms of the bound $\mathscr{F}_{\max}$ on the values of the friction coefficient matrix $\boldsymbol{\mathscr{F}}$, the bound L on the Lipschitz moduli of the components of $\boldsymbol{\mathscr{F}}$, the condition number of $\boldsymbol{\mathscr{F}}$ and the mesh norms of the respective finite-element meshes used to build the discretized model. If $\mathscr{F}_{\max}$ and L are sufficiently small (expressed in terms of the mesh norms), then it is proved that the solution of the discretized problem is unique.

1.1 Problem Formulation

Let us consider a body whose reference configuration is represented by a bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$. Let Γ_D, Γ_N and Γ_C be three disjoint, (relatively) open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$ and the areas of Γ_D and Γ_C are positive. The body is fixed on Γ_D, surface tractions of density $\boldsymbol{h}$ act on Γ_N while a rigid foundation unilaterally supports the body along Γ_C. For the sake of simplicity of our presentation, we shall assume that the rigid foundation is flat and there is no gap between it and Γ_C, that is, Γ_C is a part of a hyper-plane (see

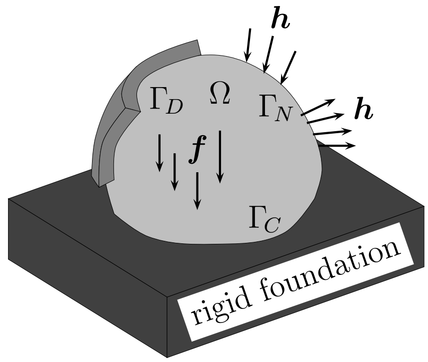


Figure 1.1: Geometry of the problem

Fig. 1.1). In addition, volume forces of density $\boldsymbol{f}$ are applied to Ω. Our aim is to find an equilibrium state of the body.

Confining ourselves to the framework of linearized elasticity, we seek the displacement vector $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ satisfying the following partial differential equation and boundary conditions:

(*equilibrium equation*)

$$-\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

where

(*Hook's law*)

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{1.2}$$

(*boundary condition of place*)

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{1.3}$$

(*boundary condition of traction*)

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{h} \quad \text{on } \Gamma_N, \tag{1.4}$$

(*unilateral condition*)

$$u_\nu \le 0, \quad \sigma_\nu(\boldsymbol{u}) \le 0, \quad u_\nu \sigma_\nu(\boldsymbol{u}) = 0 \quad \text{on } \Gamma_C. \tag{1.5}$$

Here, $\boldsymbol{\sigma}(\boldsymbol{u})$ is a stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u}) \equiv 1/2(\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T)$ is the linearized strain tensor, and $\boldsymbol{\mathcal{A}}$ is the fourth-order elasticity tensor. Further, $\boldsymbol{\nu}$ is the unit outward normal vector along $\partial\Omega$, and $u_\nu \equiv \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\sigma_\nu(\boldsymbol{u}) \equiv \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} \cdot \boldsymbol{\nu}$ stand for the normal component of the displacement vector $\boldsymbol{u}$ and the stress vector $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu}$, respectively.

The introduced problem has to be supplied by a frictional condition on the contact zone. Here, we shall use the local orthotropic Coulomb friction law. To this end, let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be principal axes of orthotropic friction on the tangent plane to Γ_C such that the triplet $\{\boldsymbol{\nu}(\boldsymbol{x}), \boldsymbol{\tau}_1(\boldsymbol{x}), \boldsymbol{\tau}_2(\boldsymbol{x})\}$ forms a local orthonormal basis in $\mathbb{R}^3$

for any $\boldsymbol{x} \in \Gamma_C$. By $\boldsymbol{u}_\tau$, $\boldsymbol{\sigma}_\tau(\boldsymbol{u})$ we denote the vectors whose components are the coordinates of $\boldsymbol{u}$, $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu}$ with respect to $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, that is, $\boldsymbol{u}_\tau = (u_{\tau,1}, u_{\tau,2})$, $\boldsymbol{\sigma}_\tau(\boldsymbol{u}) = (\sigma_{\tau,1}(\boldsymbol{u}), \sigma_{\tau,2}(\boldsymbol{u}))$ with $u_{\tau,i} \equiv \boldsymbol{u} \cdot \boldsymbol{\tau}_i$, $\sigma_{\tau,i} \equiv \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} \cdot \boldsymbol{\tau}_i$, $i = 1, 2$. Finally, let $\mathscr{F}_1$ and $\mathscr{F}_2$ be coefficients of friction in the directions $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, respectively, which may depend on the magnitudes of $u_{\tau,1}$ and $u_{\tau,2}$ on Γ_C, that is, $\mathscr{F}_i = \mathscr{F}_i(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|)$, $\boldsymbol{x} \in \Gamma_C$, $i = 1, 2$. We set

$$\boldsymbol{\mathscr{F}}(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|) := \begin{pmatrix} \mathscr{F}_1(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|) & 0 \\ 0 & \mathscr{F}_2(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|) \end{pmatrix}, \quad \boldsymbol{x} \in \Gamma_C$$

(we shall write also $\boldsymbol{\mathscr{F}}(|u_{\tau,1}|, |u_{\tau,2}|)$ for short). *The orthotropic Coulomb friction law* with a solution-dependent matrix of friction coefficients reads as follows:

$$\left.\begin{aligned} &\boldsymbol{u}_\tau(\boldsymbol{x}) = \boldsymbol{0} \implies \|\boldsymbol{\mathscr{F}}^{-1}(\boldsymbol{x}, 0, 0)\boldsymbol{\sigma}_\tau(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\| \le -\sigma_\nu(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})), \quad \boldsymbol{x} \in \Gamma_C, \\ &\boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0} \implies \boldsymbol{\mathscr{F}}^{-1}(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|)\boldsymbol{\sigma}_\tau(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \\ &\qquad = \sigma_\nu(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \frac{\boldsymbol{\mathscr{F}}(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|)\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{\mathscr{F}}(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|)\boldsymbol{u}_\tau(\boldsymbol{x})\|}, \quad \boldsymbol{x} \in \Gamma_C. \end{aligned}\right\} \tag{1.6}$$

Remark 1.1. If $\mathscr{F}_1$ coincides with $\mathscr{F}_2$, orthotropic friction reduces to isotropic one, which can be described by one coefficient $\mathscr{F} := \mathscr{F}_1 = \mathscr{F}_2$. *The isotropic Coulomb law of friction* with a solution-dependent coefficient of friction then reads as follows:

$$\left.\begin{aligned} &\boldsymbol{u}_\tau(\boldsymbol{x}) = \boldsymbol{0} \implies \|\boldsymbol{\sigma}_\tau(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\| \le -\mathscr{F}(\boldsymbol{x}, 0, 0)\sigma_\nu(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})), \quad \boldsymbol{x} \in \Gamma_C, \\ &\boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0} \implies \boldsymbol{\sigma}_\tau(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \\ &\qquad = \mathscr{F}(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|)\sigma_\nu(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \frac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|}, \quad \boldsymbol{x} \in \Gamma_C. \end{aligned}\right\} \tag{1.7}$$

Let us note that in this case, it is reasonable to assume that the coefficient of friction $\mathscr{F}$ depends on the Euclidean norm of $\boldsymbol{u}_\tau$ on Γ_C, that is,

$$\mathscr{F}(\boldsymbol{x}, |u_{\tau,1}(\boldsymbol{x})|, |u_{\tau,2}(\boldsymbol{x})|) = \hat{\mathscr{F}}(\boldsymbol{x}, (|u_{\tau,1}(\boldsymbol{x})|^2 + |u_{\tau,2}(\boldsymbol{x})|^2)^{1/2}), \quad \boldsymbol{x} \in \Gamma_C, \tag{1.8}$$

for some $\hat{\mathscr{F}}$ defined on $\Gamma_C \times \mathbb{R}_+$.

The classical formulation of our problem is represented by (1.1)–(1.6). To give the weak one, we introduce the following spaces and sets:

$$\begin{aligned} V &:= \{v \in H^1(\Omega) \mid v = 0 \text{ a.e. on } \Gamma_D\}, \\ \boldsymbol{V} &:= V \times V \times V, \\ \boldsymbol{K} &:= \{\boldsymbol{v} \in \boldsymbol{V} \mid v_\nu \le 0 \text{ a.e. on } \Gamma_C\}, \end{aligned}$$

$$\begin{aligned} X &:= \{\varphi \in L^2(\Gamma_C) \,|\, \exists\, v \in V :\, \varphi = v \text{ a.e. on } \Gamma_C\}, \\ X_\nu &:= \{\varphi \in L^2(\Gamma_C) \,|\, \exists\, \boldsymbol{v} \in \boldsymbol{V} :\, \varphi = v_\nu \text{ a.e. on } \Gamma_C\}, \\ X_{\nu+} &:= \{\varphi \in X_\nu \,|\, \varphi \geq 0 \text{ a.e. on } \Gamma_C\}, \\ \boldsymbol{X}_{\tau+} &:= \{\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Gamma_C) \,|\, \exists\, \boldsymbol{v} \in \boldsymbol{V} :\, \boldsymbol{\varphi} = (|v_{\tau,1}|, |v_{\tau,2}|) \text{ a.e. on } \Gamma_C\}, \end{aligned}$$

and endow X_ν with the norm

$$\|\varphi\|_{X_\nu} := \inf_{\substack{\boldsymbol{v} \in \boldsymbol{V} \\ v_\nu = \varphi \text{ on } \Gamma_C}} \|\boldsymbol{v}\|_{1,\Omega}.$$

By X'_ν we shall denote the (topological) dual of X_ν, and $\langle .,. \rangle_\nu$ will be used for the corresponding duality pairing.

Furthermore, we shall assume that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{h} \in \boldsymbol{L}^2(\Gamma_N)$, and $\boldsymbol{\mathcal{A}} = (a_{ijkl})$ with $a_{ijkl} \in L^\infty(\Omega)$, $i,j,k,l = 1,2,3$, satisfies the usual symmetry and ellipticity conditions

$$\left.\begin{aligned} a_{ijkl} = a_{jikl} = a_{klij} \quad \text{a.e. in } \Omega,\ \forall\, i,j,k,l = 1,2,3, \\ \exists\, a_0 > 0 : \quad \boldsymbol{\mathcal{A}}\boldsymbol{\xi} : \boldsymbol{\xi} \geq a_0 \boldsymbol{\xi} : \boldsymbol{\xi} \quad \text{a.e. in } \Omega,\ \forall\, \boldsymbol{\xi} \in \mathbb{M}^3,\ \boldsymbol{\xi} = \boldsymbol{\xi}^T. \end{aligned}\right\} \tag{1.9}$$

We shall also suppose that

$$\text{the mapping } \boldsymbol{x} \mapsto (\boldsymbol{\tau}_1(\boldsymbol{x}), \boldsymbol{\tau}_2(\boldsymbol{x})) \text{ belongs to } W^{1,\infty}(\Gamma_C; \mathbb{R}^6), \tag{1.10}$$

and the coefficients of friction $\mathscr{F}_1$ and $\mathscr{F}_2$ are positive, continuous and bounded:

$$\left.\begin{aligned} \mathscr{F}_i \in C(\Gamma_C \times \mathbb{R}^2_+), \quad i = 1,2, \\ 0 < \mathscr{F}_i(\boldsymbol{x}, \boldsymbol{\xi}) \leq \mathscr{F}_{\max}, \quad \forall\, \boldsymbol{x} \in \Gamma_C,\ \forall\, \boldsymbol{\xi} \in \mathbb{R}^2_+,\ i = 1,2, \end{aligned}\right\} \tag{1.11}$$

where $0 < \mathscr{F}_{\max}$ is given.

The weak formulation of (1.1)–(1.6) is given by the following *implicit* variational inequality:

$$\left.\begin{aligned} &\text{Find } \boldsymbol{u} \in \boldsymbol{K} \text{ such that} \\ &a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \langle \sigma_\nu(\boldsymbol{u}), \|\boldsymbol{\mathscr{F}}(|u_{\tau,1}|, |u_{\tau,2}|)\boldsymbol{v}_\tau\| \rangle_\nu \\ &\qquad + \langle \sigma_\nu(\boldsymbol{u}), \|\boldsymbol{\mathscr{F}}(|u_{\tau,1}|, |u_{\tau,2}|)\boldsymbol{u}_\tau\| \rangle_\nu \geq \ell(\boldsymbol{v} - \boldsymbol{u}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{K}, \end{aligned}\right\} \tag{$\mathscr{P}$}$$

where

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &:= \int_\Omega \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}, \\ \ell(\boldsymbol{v}) &:= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{h} \cdot \boldsymbol{v} \,\mathrm{d}S, \quad \boldsymbol{v} \in \boldsymbol{V}. \end{aligned}$$

Owing to (1.9) and Korn's inequality, a is a symmetric bilinear form that is $\boldsymbol{V}$-elliptic and continuous on $\boldsymbol{V} \times \boldsymbol{V}$:

$$\exists\, \alpha > 0: \quad a(\boldsymbol{v}, \boldsymbol{v}) \geq \alpha \|\boldsymbol{v}\|_{1,\Omega}^2, \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{1.12}$$

$$\exists\, M > 0: \quad |a(\boldsymbol{u}, \boldsymbol{v})| \leq M \|\boldsymbol{u}\|_{1,\Omega} \|\boldsymbol{v}\|_{1,\Omega}, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{1.13}$$

Remark 1.2. In the case of the model with isotropic Coulomb's law of friction (1.7), the coefficient of friction $\mathscr{F} (= \mathscr{F}_1 = \mathscr{F}_2)$ may even vanish, and the assumption (1.11) can be replaced by

$$\mathscr{F} \in C(\Gamma_C \times \mathbb{R}_+^2), \quad 0 \leq \mathscr{F}(\boldsymbol{x}, \boldsymbol{\xi}) \leq \mathscr{F}_{\max}, \quad \forall\, \boldsymbol{x} \in \Gamma_C,\ \forall\, \boldsymbol{\xi} \in \mathbb{R}_+^2 \tag{1.14}$$

in all the results established in this chapter. If we consider in addition (1.8) (the coefficient of friction depending on $\|\boldsymbol{u}_\tau\|$), this is obviously guaranteed by the following assumption:

$$\hat{\mathscr{F}} \in C(\Gamma_C \times \mathbb{R}_+), \quad 0 \leq \hat{\mathscr{F}}(\boldsymbol{x}, \xi) \leq \mathscr{F}_{\max}, \quad \forall\, \boldsymbol{x} \in \Gamma_C,\ \forall\, \xi \in \mathbb{R}_+.$$

Remark 1.3. If $\boldsymbol{u}$ is smooth enough, then applying Green's formula to $(\mathscr{P})$, we recover (1.1)–(1.6). Let us notice, however, that to make sense of the duality terms in $(\mathscr{P})$, one needs an additional smoothness of $\boldsymbol{u}$, $\boldsymbol{\mathscr{F}}$ and the mapping $\boldsymbol{x} \mapsto (\boldsymbol{\tau}_1(\boldsymbol{x}), \boldsymbol{\tau}_2(\boldsymbol{x}))$, $\boldsymbol{x} \in \Gamma_C$, ensuring that $\|\boldsymbol{\mathscr{F}}(|u_{\tau,1}|, |u_{\tau,2}|)\boldsymbol{v}_\tau\| \in X_\nu$ for any $\boldsymbol{v} \in \boldsymbol{V}$ (see [16]). To overcome this difficulty, we shall suppose that $\sigma_\nu(\boldsymbol{u}) \in L^2(\Gamma_C)$ in what follows. Then the duality pairing $\langle ., . \rangle_\nu$ can be replaced by the $L^2(\Gamma_C)$-scalar product $(., .)_{0,\Gamma_C}$, so (1.10) and (1.11) are sufficient.

Below, we introduce a fixed-point formulation of $(\mathscr{P})$, on which the finite-element discretization will be based. To start with, we associate with any $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \boldsymbol{X}_{\tau+}$, $g \in L_+^2(\Gamma_C)$ the following *auxiliary* problem:

$$\left.\begin{array}{l} \text{Find } \boldsymbol{u} = \boldsymbol{u}(\varphi_1, \varphi_2, g) \in \boldsymbol{K} \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\varphi_1, \varphi_2, g, \boldsymbol{v}_\tau) - j(\varphi_1, \varphi_2, g, \boldsymbol{u}_\tau) \geq \ell(\boldsymbol{v} - \boldsymbol{u}), \\ \hfill \forall\, \boldsymbol{v} \in \boldsymbol{K}. \end{array}\right\} \quad (\mathscr{P}(\varphi_1, \varphi_2, g))$$

Here,

$$j(\varphi_1, \varphi_2, g, \boldsymbol{v}_\tau) := (g, \|\boldsymbol{\mathscr{F}}(\varphi_1, \varphi_2)\boldsymbol{v}_\tau\|)_{0,\Gamma_C}, \quad (\varphi_1, \varphi_2) \in \boldsymbol{X}_{\tau+},\ g \in L_+^2(\Gamma_C),\ \boldsymbol{v} \in \boldsymbol{V},$$

and $L_+^2(\Gamma_C)$ stands for the set of all non-negative functions from $L^2(\Gamma_C)$.

Problem $(\mathscr{P}(\varphi_1, \varphi_2, g))$ is a weak formulation of a contact problem with orthotropic friction of *Tresca* type and the *fixed* matrix $\boldsymbol{\mathscr{F}}(\varphi_1, \varphi_2)$ of friction coefficients. The existence of a unique solution is guaranteed for any $(\varphi_1, \varphi_2) \in \boldsymbol{X}_{\tau+}$, $g \in L_+^2(\Gamma_c)$ due to the equivalence of this problem to a convex minimization problem (see [17, Chapter II]). This enables us to define the mapping $\Phi : \boldsymbol{X}_{\tau+} \times L^2_+(\Gamma_C) \to \boldsymbol{X}_{\tau+} \times X'_\nu$ by

$$\Phi(\varphi_1, \varphi_2, g) := (|u_{\tau,1}|, |u_{\tau,2}|, -\sigma_\nu(\boldsymbol{u})), \quad (\varphi_1, \varphi_2) \in \boldsymbol{X}_{\tau+},\ g \in L^2_+(\Gamma_c),$$

where $\boldsymbol{u}$ solves $(\mathscr{P}(\varphi_1, \varphi_2, g))$ and $\sigma_\nu(\boldsymbol{u})$ is the corresponding normal contact stress. By comparing problems $(\mathscr{P})$ and $(\mathscr{P}(\varphi_1, \varphi_2, g))$, it is readily seen that if the triplet $(|u_{\tau,1}|, |u_{\tau,2}|, -\sigma_\nu(\boldsymbol{u}))$ is a fixed point of Φ in $\boldsymbol{X}_{\tau+} \times L^2_+(\Gamma_C)$, then $\boldsymbol{u}$ is a solution to $(\mathscr{P})$.

Let $(\varphi_1, \varphi_2) \in \boldsymbol{X}_{\tau+}$ and $g \in L^2_+(\Gamma_C)$ be fixed and Λ_ν be the cone of non-negative elements in X'_ν:

$$\Lambda_\nu := \{\mu \in X'_\nu \,|\, \langle \mu, \varphi \rangle_\nu \geq 0,\ \forall\, \varphi \in X_{\nu+}\}.$$

To release the unilateral constraint $\boldsymbol{u} \in \boldsymbol{K}$, we introduce the following *mixed* formulation of $(\mathscr{P}(\varphi_1, \varphi_2, g))$:

$$\left.\begin{aligned} &\text{Find } (\boldsymbol{u}, \lambda_\nu) = (\boldsymbol{u}(\varphi_1, \varphi_2, g), \lambda_\nu(\varphi_1, \varphi_2, g)) \in \boldsymbol{V} \times \Lambda_\nu \text{ such that} \\ &a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\varphi_1, \varphi_2, g, \boldsymbol{v}_\tau) - j(\varphi_1, \varphi_2, g, \boldsymbol{u}_\tau) \\ &\qquad\qquad\qquad \geq \ell(\boldsymbol{v} - \boldsymbol{u}) - \langle \lambda_\nu, v_\nu - u_\nu \rangle_\nu, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ &\langle \mu_\nu - \lambda_\nu, u_\nu \rangle_\nu \leq 0, \quad \forall\, \mu_\nu \in \Lambda_\nu. \end{aligned}\right\} (\mathscr{M}(\varphi_1, \varphi_2, g))$$

It is known that $(\mathscr{M}(\varphi_1, \varphi_2, g))$ has a unique solution for any $(\varphi_1, \varphi_2) \in \boldsymbol{X}_{\tau+}$, $g \in L^2_+(\Gamma_C)$. Moreover, $\boldsymbol{u}$ solves $(\mathscr{P}(\varphi_1, \varphi_2, g))$ and $\lambda_\nu = -\sigma_\nu(\boldsymbol{u})$ as follows from the Green formula ([2]). This gives an equivalent expression for the mapping Φ:

$$\Phi(\varphi_1, \varphi_2, g) = (|u_{\tau,1}|, |u_{\tau,2}|, \lambda_\nu), \quad \forall\, (\varphi_1, \varphi_2) \in \boldsymbol{X}_{\tau+},\ \forall\, g \in L^2_+(\Gamma_C) \tag{1.15}$$

with $(\boldsymbol{u}, \lambda_\nu)$ being the solution of $(\mathscr{M}(\varphi_1, \varphi_2, g))$.

1.2 Finite-Element Discretization

This section deals with an approximation of problem $(\mathscr{P})$, which is based on a fixed-point formulation for an appropriate discretization of the mapping Φ. This is done with the aid of (1.15) and a mixed finite-element discretization of $(\mathscr{M}(\varphi_1, \varphi_2, g))$.

Let V^h, L^H be the following polynomial Lagrange finite-element spaces corresponding to some partitions $\mathscr{T}^h_\Omega$ and $\mathscr{T}^H_{\Gamma_C}$ of $\overline{\Omega}$ and $\overline{\Gamma}_C$, respectively:

$$\begin{aligned} V^h &:= \{v^h \in C(\overline{\Omega}) \,|\, v^h|_{\mathcal{T}} \in P_{K_1}(\mathcal{T}),\ \forall\, \mathcal{T} \in \mathscr{T}^h_\Omega \ \&\ v^h = 0 \text{ on } \Gamma_D\}, \\ L^H &:= \{\mu^H \in L^2(\Gamma_C) \,|\, \mu^H|_{\mathcal{R}} \in P_{K_2}(\mathcal{R}),\ \forall\, \mathcal{R} \in \mathscr{T}^H_{\Gamma_C}\}. \end{aligned}$$

Here, $K_1 \geq 1$, $K_2 \geq 0$ are integers and h, H stand for the norms of the partitions $\mathscr{T}^h_\Omega$ and $\mathscr{T}^H_{\Gamma_C}$, respectively. Only what we suppose at this moment is that $\mathscr{T}^h_\Omega$ is compatible with the decomposition of $\partial\Omega$ into Γ_D, Γ_N and Γ_C. In general, $\mathscr{T}^H_{\Gamma_C}$ is different from $\mathscr{T}^h_\Omega|_{\overline{\Gamma}_C}$, but the case when they equal each other is not excluded. Further, we set

$$\begin{aligned}
\boldsymbol{V}^h &:= V^h \times V^h \times V^h,\\
X^h &:= \{\varphi^h \in C(\overline{\Gamma}_C) \,|\, \exists\, v^h \in V^h : \varphi^h = v^h \text{ on } \overline{\Gamma}_C\},\\
X^h_+ &:= \{\varphi^h \in X^h \,|\, \varphi^h \geq 0 \text{ on } \overline{\Gamma}_C\},\\
\Lambda^H_\nu &:= \{\mu^H \in L^H \,|\, \mu^H \geq 0 \text{ on } \Gamma_C\}.
\end{aligned}$$

Clearly, $\boldsymbol{V}^h$ and Λ^H_ν will serve as natural approximations of $\boldsymbol{V}$ and Λ_ν, respectively. In what follows, we shall suppose that the following condition is satisfied:

$$(\mu^H \in L^H \quad \& \quad (\mu^H, v^h_\nu)_{0,\Gamma_C} = 0, \;\; \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h) \quad \Longrightarrow \quad \mu^H = 0. \tag{1.16}$$

This makes it possible to endow the spaces L^H and $X^h \times X^h \times L^H$ with the following (mesh-dependent) norms:

$$\begin{aligned}
\|\mu^H\|_{*,h} &:= \sup_{\boldsymbol{0}\neq \boldsymbol{v}^h \in \boldsymbol{V}^h} \frac{(\mu^H, v^h_\nu)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,\Omega}},\\
\|(\varphi^h_1, \varphi^h_2, \mu^H)\|_{X^h\times X^h\times L^H} &:= \|(\varphi^h_1, \varphi^h_2)\|_{0,\Gamma_C} + \|\mu^H\|_{*,h}.
\end{aligned}$$

Remark 1.4. Let us briefly mention two examples of the discretizations posited above.

(FE1) $\mathscr{T}^H_{\Gamma_C} = \mathscr{T}^h_\Omega|_{\overline{\Gamma}_C}$, $K_2 = K_1$, $L^H = X^h$.
Condition (1.16) is always satisfied.

(FE2) $K_1 = 1$, $K_2 = 0$.
In this case, (1.16) is fulfilled provided that the ratio H/h is sufficiently large, that is, the partition $\mathscr{T}^H_{\Gamma_C}$ is coarser than $\mathscr{T}^h_\Omega|_{\overline{\Gamma}_C}$ (see [22]).

For $(\varphi^h_1, \varphi^h_2, g^H) \in X^h_+ \times X^h_+ \times \Lambda^H_\nu$ given, we introduce the following discretization of problem $(\mathscr{M}(\varphi_1, \varphi_2, g))$:

$$\left.\begin{aligned}
&\text{Find } (\boldsymbol{u}^h, \lambda^H_\nu) = (\boldsymbol{u}^h(\varphi^h_1, \varphi^h_2, g^H), \lambda^H_\nu(\varphi^h_1, \varphi^h_2, g^H))\\
&\qquad \in \boldsymbol{V}^h \times \Lambda^H_\nu \text{ such that}\\
&a(\boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) + j(\varphi^h_1, \varphi^h_2, g^H, \boldsymbol{v}^h_\tau) - j(\varphi^h_1, \varphi^h_2, g^H, \boldsymbol{u}^h_\tau)\\
&\qquad\qquad \geq \ell(\boldsymbol{v}^h - \boldsymbol{u}^h) - (\lambda^H_\nu, v^h_\nu - u^h_\nu)_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h,\\
&(\mu^H_\nu - \lambda^H_\nu, u^h_\nu)_{0,\Gamma_C} \leq 0, \quad \forall\, \mu^H_\nu \in \Lambda^H_\nu.
\end{aligned}\right\} \quad (\mathscr{M}_{hH}(\varphi^h_1, \varphi^h_2, g^H))$$

By reformulating $(\mathscr{M}_{hH}(\varphi^h_1, \varphi^h_2, g^H))$ as a saddle-point problem, condition (1.16) ensures that $(\mathscr{M}_{hH}(\varphi^h_1, \varphi^h_2, g^H))$ has a unique solution $(\boldsymbol{u}^h, \lambda^H_\nu)$ for any $(\varphi^h_1, \varphi^h_2, g^H) \in X^h_+ \times X^h_+ \times \Lambda^H_\nu$ (see [17, Chapter VI]).

Furthermore, by inserting $\mu_\nu^H := 0, 2\lambda_\nu^H$ into the second inequality of problem $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$, it is readily seen that

$$(\lambda_\nu^H, u_\nu^h)_{0,\Gamma_C} = 0 \quad \& \quad u_\nu^h \in \boldsymbol{K}^{hH} := \{\boldsymbol{v}^h \in \boldsymbol{V}^h \,|\, (\mu_\nu^H, v_\nu^h)_{0,\Gamma_C} \leq 0,\ \forall\, \mu_\nu^H \in \Lambda_\nu^H\}.$$

Therefore, $\boldsymbol{u}^h$ solves the following variational inequality (confer $(\mathscr{P}(\varphi_1, \varphi_2, g))$):

$$\left.\begin{aligned} &\text{Find } \boldsymbol{u}^h = \boldsymbol{u}^h(\varphi_1^h, \varphi_2^h, g^H) \in \boldsymbol{K}^{hH} \text{ such that} \\ &a(\boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) + j(\varphi_1^h, \varphi_2^h, g^H, \boldsymbol{v}_\tau^h) - j(\varphi_1^h, \varphi_2^h, g^H, \boldsymbol{u}_\tau^h) \\ &\qquad\qquad\qquad\qquad \geq \ell(\boldsymbol{v}^h - \boldsymbol{u}^h), \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{K}^{hH}. \end{aligned}\right\} \quad (\mathscr{P}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$$

Notice that $\boldsymbol{K}^{hH}$ is an external approximation of $\boldsymbol{K}$, that is, $\boldsymbol{K}^{hH} \not\subset \boldsymbol{K}$. On the other hand, Λ_ν^H is an internal approximation of Λ_ν.

Next, we define a discretization of Φ. Let $r_h : H^1(\Gamma_C) \to X^h$ be a linear interpolation operator preserving positivity:

$$(\varphi \in H^1(\Gamma_C)\ \&\ \varphi \geq 0 \text{ a.e. on } \Gamma_C) \quad \Longrightarrow \quad r_h\varphi \in X_+^h \tag{1.17}$$

and possessing the following approximation property:

$$\exists\, c_{r_h} > 0: \quad \|\varphi - r_h\varphi\|_{0,\Gamma_C} \leq c_{r_h} h_{\Gamma_C} \|\varphi\|_{1,\Gamma_C}, \quad \forall\, \varphi \in H^1(\Gamma_C) \cap X, \tag{1.18}$$

where $h_{\Gamma_C} := \max_{F \in \mathscr{T}_\Omega^h|_{\overline{\Gamma}_C}} \operatorname{diam}(F)$. With such r_h at hand, we introduce the mapping $\Phi_{hH} : X_+^h \times X_+^h \times \Lambda_\nu^H \to X_+^h \times X_+^h \times \Lambda_\nu^H$ by

$$\Phi_{hH}(\varphi_1^h, \varphi_2^h, g^H) := (r_h|u_{\tau,1}^h|, r_h|u_{\tau,2}^h|, \lambda_\nu^H),$$

where $(\boldsymbol{u}^h, \lambda_\nu^H)$ solves $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$.

Definition 1.1. Any couple $(\boldsymbol{u}^h, \lambda_\nu^H) \in \boldsymbol{V}^h \times \Lambda_\nu^H$ is called a solution of the discretized contact problem with orthotropic Coulomb friction and solution-dependent coefficients of friction if $(r_h|u_{\tau,1}^h|, r_h|u_{\tau,2}^h|, \lambda_\nu^H)$ is a fixed point of Φ_{hH}, that is, $(\boldsymbol{u}^h, \lambda_\nu^H)$ solves $(\mathscr{M}_{hH}(r_h|u_{\tau,1}^h|, r_h|u_{\tau,2}^h|, \lambda_\nu^H))$.

1.3 Theoretical Analysis of the Discretized Problem

We shall establish existence as well as uniqueness of a solution to the discretized problem introduced in the previous section. In addition, we shall investigate, how the uniqueness result depends on the size of the problem.

1.3.1 Existence Result

The existence of a discrete solution will be done by using fixed-point arguments. First, we state two auxiliary results, the first one being a minor modification of Lemma 3.3 in [43]. Recall that $\boldsymbol{\xi}_\tau = (\xi_{\tau,1}, \xi_{\tau,2})$ with $\xi_{\tau,i} = \boldsymbol{\xi} \cdot \boldsymbol{\tau}_i$, $i = 1, 2$.

Lemma 1.1. *If $\xi \in H^1(\Gamma_C) \cap X$ then $|\xi| \in H^1(\Gamma_C) \cap X$ and*

$$\| \, |\xi| \, \|_{1,\Gamma_C} \le \|\xi\|_{1,\Gamma_C}.$$

Lemma 1.2. *If (1.10) is satisfied, then $\boldsymbol{\xi}_\tau \in H^1(\Gamma_C; \mathbb{R}^2)$ for any $\boldsymbol{\xi} \in H^1(\Gamma_C; \mathbb{R}^3)$ and there exists a constant $c_\tau > 0$ such that*

$$\|\boldsymbol{\xi}_\tau\|_{1,\Gamma_C} \le c_\tau \|\boldsymbol{\xi}\|_{1,\Gamma_C}, \quad \forall \boldsymbol{\xi} \in H^1(\Gamma_C; \mathbb{R}^3).$$

Proof. Since Γ_C is supposed to be a flat part of $\partial\Omega$, we may assume without loss of generality that $\Gamma_C \subset \mathbb{R}^2 \times \{0\}$ (otherwise, one can introduce an appropriate orthonormal transformation of coordinates). The proof is then straightforward. □

With these results at our disposal, we shall show by using the Brouwer fixed-point theorem that Φ_{hH} has at least one fixed point in the set

$$\mathscr{C}(R_1, R_2) := \left\{ (\varphi_1^h, \varphi_2^h, \mu^H) \in X_+^h \times X_+^h \times \Lambda_\nu^H \;\middle|\; \|(\varphi_1^h, \varphi_2^h)\|_{0,\Gamma_C} \le R_1 \,\&\, \|\mu^H\|_{*,h} \le R_2 \right\}$$

for appropriate $R_1, R_2 > 0$.

Lemma 1.3. *Let (1.9)–(1.11) be satisfied. Then there exist $R_1, R_2 > 0$ such that Φ_{hH} maps $X_+^h \times X_+^h \times \Lambda_\nu^H$ into $\mathscr{C}(R_1, R_2)$.*

Proof. Let $(\varphi_1^h, \varphi_2^h, g^H) \in X_+^h \times X_+^h \times \Lambda_\nu^H$ be given and $(\boldsymbol{u}^h, \lambda_\nu^H)$ be the solution to $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$. Inserting $\boldsymbol{v}^h := 0, 2\boldsymbol{u}^h \in \boldsymbol{K}^{hH}$ into $(\mathscr{P}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$, we get

$$a(\boldsymbol{u}^h, \boldsymbol{u}^h) + j(\varphi_1^h, \varphi_2^h, g^H, \boldsymbol{u}_\tau^h) = \ell(\boldsymbol{u}^h), \tag{1.19}$$

which together with the non-negativeness of j implies that

$$\|\boldsymbol{u}^h\|_{1,\Omega} \le \frac{\|\ell\|_{*,\Omega}}{\alpha}. \tag{1.20}$$

Here $\|.\|_{*,\Omega}$ stands for the dual norm in $(\boldsymbol{H}^1(\Omega))'$ and α is the constant from (1.12). Invoking (1.18), Lemmas 1.1 and 1.2,

$$\begin{aligned}
&\|(r_h|u_{\tau,1}^h|, r_h|u_{\tau,2}^h|)\|_{0,\Gamma_C} \\
&\qquad \le \|(r_h|u_{\tau,1}^h| - |u_{\tau,1}^h|, r_h|u_{\tau,2}^h| - |u_{\tau,2}^h|)\|_{0,\Gamma_C} + \|(|u_{\tau,1}^h|, |u_{\tau,2}^h|)\|_{0,\Gamma_C} \\
&\qquad \le c_{r_h} h_{\Gamma_C} \|(|u_{\tau,1}^h|, |u_{\tau,2}^h|)\|_{1,\Gamma_C} + \|\boldsymbol{u}_\tau^h\|_{0,\Gamma_C} \\
&\qquad \le c_{r_h} h_{\Gamma_C} \|\boldsymbol{u}_\tau^h\|_{1,\Gamma_C} + \|\boldsymbol{u}_\tau^h\|_{0,\Gamma_C} \le c_{r_h} c_\tau h_{\Gamma_C} \|\boldsymbol{u}^h\|_{1,\Gamma_C} + \|\boldsymbol{u}^h\|_{0,\Gamma_C} \\
&\qquad \le (c_{\text{inv}}^{(1,0)} c_{r_h} c_\tau + 1) \|\boldsymbol{u}^h\|_{0,\Gamma_C} \\
&\qquad \le c_{\text{tr}}^{(2)} (c_{\text{inv}}^{(1,0)} c_{r_h} c_\tau + 1) \|\boldsymbol{u}^h\|_{1,\Omega},
\end{aligned} \tag{1.21}$$

where $c_{\rm tr}^{(2)}$ is the norm of the trace mapping from $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{L}^2(\partial\Omega)$ and $c_{\rm inv}^{(1,0)}$ is the constant from the inverse inequality between the $\boldsymbol{H}^1(\Gamma_C)$ and $\boldsymbol{L}^2(\Gamma_C)$-norms for functions belonging to the finite-dimensional space $X^h \times X^h \times X^h$:

$$\|\boldsymbol{\psi}^h\|_{1,\Gamma_C} \le \frac{c_{\rm inv}^{(1,0)}}{h_{\Gamma_C}} \|\boldsymbol{\psi}^h\|_{0,\Gamma_C}, \quad \forall\, \boldsymbol{\psi}^h \in X^h \times X^h \times X^h. \tag{1.22}$$

In view of (1.20) and (1.21), the radius R_1 is of the form

$$R_1 = R_1(c_{\rm inv}^{(1,0)}, c_{r_h}, c_{\rm tr}^{(2)}, c_\tau, \alpha, \ell) := \frac{c_{\rm tr}^{(2)}(c_{\rm inv}^{(1,0)} c_{r_h} c_\tau + 1)}{\alpha} \|\ell\|_{*,\Omega}.$$

Furthermore, one can see from $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ and (1.19) that

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) + j(\varphi_1^h, \varphi_2^h, g^H, \boldsymbol{v}_\tau^h) \ge \ell(\boldsymbol{v}^h) - (\lambda_\nu^H, v_\nu^h)_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h.$$

Introducing the subspace

$$\boldsymbol{V}_0^h := \{\boldsymbol{v}^h \in \boldsymbol{V}^h \mid \boldsymbol{v}_\tau^h = \boldsymbol{0} \text{ on } \Gamma_C\},$$

one obtains

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) = \ell(\boldsymbol{v}^h) - (\lambda_\nu^H, v_\nu^h)_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}_0^h,$$

from which, (1.13) and (1.20),

$$\frac{(\lambda_\nu^H, v_\nu^h)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,\Omega}} = \frac{\ell(\boldsymbol{v}^h) - a(\boldsymbol{u}^h, \boldsymbol{v}^h)}{\|\boldsymbol{v}^h\|_{1,\Omega}} \le \Big(1 + \frac{M}{\alpha}\Big) \|\ell\|_{*,\Omega}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}_0^h. \tag{1.23}$$

To complete the proof, one may assume without loss of generality that $\Gamma_C \subset \mathbb{R}^2 \times \{0\}$ (otherwise, one can introduce an orthonormal transformation $\mathscr{O} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathscr{O}(\Gamma_C) \subset \mathbb{R}^2 \times \{0\}$ and proceed with $\mathscr{O}\boldsymbol{v}^h$) and set

$$\boldsymbol{V}_{00}^h := \{\boldsymbol{v}^h = (v_1^h, v_2^h, v_3^h) \in \boldsymbol{V}^h \mid v_1^h = v_2^h = 0 \text{ in } \Omega\} \subset \boldsymbol{V}_0^h.$$

Then

$$\begin{aligned}\|\lambda_\nu^H\|_{*,h} &= \sup_{\boldsymbol{0}\ne\boldsymbol{v}^h\in\boldsymbol{V}^h} \frac{(\lambda_\nu^H, v_\nu^h)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,\Omega}} \le \sup_{\boldsymbol{0}\ne\boldsymbol{v}^h\in\boldsymbol{V}^h} \frac{(\lambda_\nu^H, v_3^h)_{0,\Gamma_C}}{\|v_3^h\|_{1,\Omega}} \\ &= \sup_{\boldsymbol{0}\ne\boldsymbol{v}^h\in\boldsymbol{V}_{00}^h} \frac{(\lambda_\nu^H, v_\nu^h)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,\Omega}} \le \sup_{\boldsymbol{0}\ne\boldsymbol{v}^h\in\boldsymbol{V}_0^h} \frac{(\lambda_\nu^H, v_\nu^h)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,\Omega}}.\end{aligned}$$

From this and (1.23), one can take

$$R_2 = R_2(M, \alpha, \ell) := \Big(1 + \frac{M}{\alpha}\Big) \|\ell\|_{*,\Omega}.$$

□

Remark 1.5. Let us notice that at this moment the partitions $\mathscr{T}_\Omega^h$ and $\mathscr{T}_{\Gamma_C}^H$ are fixed and the constants c_{r_h} and $c_{\rm inv}^{(1,0)}$ in (1.18) and (1.22) may depend on h. Later on, we shall consider $\mathscr{T}_\Omega^h$ and $\mathscr{T}_{\Gamma_C}^H$ as elements of systems $\{\mathscr{T}_\Omega^h\}$, $\{\mathscr{T}_{\Gamma_C}^H\}$, $h, H \to 0+$, and we shall formulate conditions on these systems under which the constants do not depend on h.

Lemma 1.4. *The mapping Φ_{hH} is continuous in $X_+^h \times X_+^h \times \Lambda_\nu^H$ provided that (1.9)–(1.11) are satisfied.*

Proof. Let $(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}), (\varphi_1^h, \varphi_2^h, g^H) \in X_+^h \times X_+^h \times \Lambda_\nu^H$, $k \in \mathbb{N}$, be such that

$$(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}) \to (\varphi_1^h, \varphi_2^h, g^H) \text{ in } X^h \times X^h \times L^H, \quad k \to +\infty,$$

and $(\boldsymbol{u}^{h,k}, \lambda_\nu^{H,k})$ be the solutions to $(\mathscr{M}_{hH}(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}))$:

$$\left.\begin{aligned} &a(\boldsymbol{u}^{h,k}, \boldsymbol{v}^h - \boldsymbol{u}^{h,k}) + j(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}, \boldsymbol{v}_\tau^h) - j(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}, \boldsymbol{u}_\tau^{h,k}) \\ &\qquad\qquad \ge \ell(\boldsymbol{v}^h - \boldsymbol{u}^{h,k}) - (\lambda_\nu^{H,k}, v_\nu^h - u_\nu^{h,k})_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h, \\ &(\mu_\nu^H - \lambda_\nu^{H,k}, u_\nu^{h,k})_{0,\Gamma_C} \le 0, \quad \forall\, \mu_\nu^H \in \Lambda_\nu^H. \end{aligned}\right\}$$

As we know from the proof of the previous lemma, both sequences $\{\boldsymbol{u}^{h,k}\}$ and $\{\lambda_\nu^{H,k}\}$ are bounded. Thus, one can find $\{\boldsymbol{u}^{h,k_l}\} \subset \{\boldsymbol{u}^{h,k}\}$, $\{\lambda_\nu^{H,k_l}\} \subset \{\lambda_\nu^{H,k}\}$ and $\boldsymbol{u}^h \in \boldsymbol{V}^h$, $\lambda_\nu^H \in \Lambda_\nu^H$ such that

$$\boldsymbol{u}^{h,k_l} \to \boldsymbol{u}^h \text{ in } \boldsymbol{V}^h, \quad \lambda_\nu^{H,k_l} \to \lambda_\nu^H \text{ in } L^H, \quad l \to +\infty.$$

Let $\boldsymbol{v}^h \in \boldsymbol{V}^h$ and $\mu_\nu^H \in \Lambda_\nu^H$ be arbitrarily chosen. Taking into account the equivalences of all norms in the finite-dimensional spaces involved, one can easily verify that

$$\begin{aligned} &a(\boldsymbol{u}^{h,k_l}, \boldsymbol{v}^h - \boldsymbol{u}^{h,k_l}) - \ell(\boldsymbol{v}^h - \boldsymbol{u}^{h,k_l}) + (\lambda_\nu^{H,k_l}, v_\nu^h - u_\nu^{h,k_l})_{0,\Gamma_C} \\ &\qquad\qquad \stackrel{l\to+\infty}{\longrightarrow} \ a(\boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) - \ell(\boldsymbol{v}^h - \boldsymbol{u}^h) + (\lambda_\nu^H, v_\nu^h - u_\nu^h)_{0,\Gamma_C}, \\ &j(\varphi_1^{h,k_l}, \varphi_2^{h,k_l}, g^{H,k_l}, \boldsymbol{v}_\tau^h) - j(\varphi_1^{h,k_l}, \varphi_2^{h,k_l}, g^{H,k_l}, \boldsymbol{u}_\tau^{h,k_l}) \\ &\qquad\qquad \stackrel{l\to+\infty}{\longrightarrow} \ j(\varphi_1^h, \varphi_2^h, g^H, \boldsymbol{v}_\tau^h) - j(\varphi_1^h, \varphi_2^h, g^H, \boldsymbol{u}_\tau^h), \\ &(\mu_\nu^H - \lambda_\nu^{H,k_l}, u_\nu^{h,k_l})_{0,\Gamma_C} \ \stackrel{l\to+\infty}{\longrightarrow} \ (\mu_\nu^H - \lambda_\nu^H, u_\nu^h)_{0,\Gamma_C}, \end{aligned}$$

which shows that $(\boldsymbol{u}^h, \lambda_\nu^H)$ solves $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$. Since this problem admits a unique solution, the original sequences $\{\boldsymbol{u}^{h,k}\}$ and $\{\lambda_\nu^{H,k}\}$ tend to $\boldsymbol{u}^h$ and λ_ν^H.

Furthermore, from the positivity preserving assumption (1.17) and the linearity of r_h, it is readily seen that

$$|r_h(|u_{\tau,i}^{h,k}| - |u_{\tau,i}^h|)| \le r_h|u_{\tau,i}^{h,k} - u_{\tau,i}^h| \text{ on } \Gamma_C, \ i = 1, 2, \ k \in \mathbb{N}.$$

Therefore, arguing as in (1.21), one gets

$$\begin{aligned} \|(r_h|u^{h,k}_{\tau,1}|, r_h|u^{h,k}_{\tau,2}|) - (r_h|u^h_{\tau,1}|, r_h|u^h_{\tau,2}|)\|_{0,\Gamma_C} \\ \le \|(r_h|u^{h,k}_{\tau,1} - u^h_{\tau,1}|, r_h|u^{h,k}_{\tau,2} - u^h_{\tau,2}|)\|_{0,\Gamma_C} \\ \le c^{(2)}_{\mathrm{tr}}(c^{(1,0)}_{\mathrm{inv}} c_{r_h} c_\tau + 1)\|\boldsymbol{u}^{h,k} - \boldsymbol{u}^h\|_{1,\Omega}, \quad k \in \mathbb{N}, \end{aligned} \tag{1.24}$$

and the limit passage $k \to +\infty$ completes the proof. □

We have arrived at the following existence result.

Theorem 1.1. *If (1.9)–(1.11) are fulfilled, then the discretized problem given by Definition 1.1 has at least one solution.*

1.3.2 Uniqueness Result

Applying the Banach fixed-point theorem, even uniqueness of the discrete solution can be ensured. Nevertheless, to establish the Lipschitz continuity of Φ_{hH}, we shall need an additional assumption on $\boldsymbol{\mathscr{F}}$, namely

$$\begin{aligned} &\exists\, L > 0 : \\ &\quad |\mathscr{F}_i(\boldsymbol{x}, \boldsymbol{\xi}) - \mathscr{F}_i(\boldsymbol{x}, \bar{\boldsymbol{\xi}})| \le L\|\boldsymbol{\xi} - \bar{\boldsymbol{\xi}}\|, \quad \forall\, \boldsymbol{x} \in \Gamma_C,\ \forall\, \boldsymbol{\xi}, \bar{\boldsymbol{\xi}} \in \mathbb{R}^2_+,\ i = 1, 2. \end{aligned} \tag{1.25}$$

We start with a useful technical result. The matrix norm $\|.\|$ is induced by the Euclidean vector norm here and in what follows.

Lemma 1.5. *If $\boldsymbol{\mathscr{F}}$ satisfies (1.11) and (1.25), then it holds for any $\boldsymbol{u}^h, \bar{\boldsymbol{u}}^h \in \boldsymbol{V}^h$ and any $(\varphi^h_1, \varphi^h_2)$, $(\bar\varphi^h_1, \bar\varphi^h_2) \in X^h_+ \times X^h_+$ that*

$$\begin{aligned} \big|\|\boldsymbol{\mathscr{F}}(\varphi^h_1, \varphi^h_2)\bar{\boldsymbol{u}}^h_\tau\| - \|\boldsymbol{\mathscr{F}}(\varphi^h_1, \varphi^h_2)\boldsymbol{u}^h_\tau\| - (\|\boldsymbol{\mathscr{F}}(\bar\varphi^h_1, \bar\varphi^h_2)\bar{\boldsymbol{u}}^h_\tau\| - \|\boldsymbol{\mathscr{F}}(\bar\varphi^h_1, \bar\varphi^h_2)\boldsymbol{u}^h_\tau\|)\big| \\ \le L(2 + \kappa(\boldsymbol{\mathscr{F}}))\|(\varphi^h_1, \varphi^h_2) - (\bar\varphi^h_1, \bar\varphi^h_2)\|\,\|\boldsymbol{u}^h_\tau - \bar{\boldsymbol{u}}^h_\tau\| \ \textit{on}\ \Gamma_C, \end{aligned} \tag{1.26}$$

where

$$\kappa(\boldsymbol{\mathscr{F}}) := \sup_{\substack{\boldsymbol{x} \in \Gamma_C \\ \boldsymbol{\xi} \in \mathbb{R}^2_+}} \|\boldsymbol{\mathscr{F}}(\boldsymbol{x}, \boldsymbol{\xi})\|\,\|\boldsymbol{\mathscr{F}}^{-1}(\boldsymbol{x}, \boldsymbol{\xi})\| = \sup_{\substack{\boldsymbol{x} \in \Gamma_C \\ \boldsymbol{\xi} \in \mathbb{R}^2_+}} \frac{\max\{\mathscr{F}_1(\boldsymbol{x}, \boldsymbol{\xi}), \mathscr{F}_2(\boldsymbol{x}, \boldsymbol{\xi})\}}{\min\{\mathscr{F}_1(\boldsymbol{x}, \boldsymbol{\xi}), \mathscr{F}_2(\boldsymbol{x}, \boldsymbol{\xi})\}}.$$

Proof. For $\boldsymbol{x} \in \Gamma_C$, $\boldsymbol{u}^h, \bar{\boldsymbol{u}}^h \in \boldsymbol{V}^h$ and $(\varphi^h_1, \varphi^h_2), (\bar\varphi^h_1, \bar\varphi^h_2) \in X^h_+ \times X^h_+$ given, set

$$\begin{aligned} &\boldsymbol{u} := \boldsymbol{u}^h_\tau(\boldsymbol{x}), && \bar{\boldsymbol{u}} := \bar{\boldsymbol{u}}^h_\tau(\boldsymbol{x}), \\ &\boldsymbol{\varphi} = (\varphi_1, \varphi_2) := (\varphi^h_1(\boldsymbol{x}), \varphi^h_2(\boldsymbol{x})), && \bar{\boldsymbol{\varphi}} = (\bar\varphi_1, \bar\varphi_2) := (\bar\varphi^h_1(\boldsymbol{x}), \bar\varphi^h_2(\boldsymbol{x})) \end{aligned}$$

and define the function $h := G \circ \boldsymbol{F} \circ \boldsymbol{H} : \mathbb{R} \to \mathbb{R}$ with $\boldsymbol{H} : \mathbb{R} \to \mathbb{R}^2$, $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{R}$ introduced as follows:

$$\boldsymbol{H}(r) := \bar{\boldsymbol{\varphi}} + r(\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}), \quad r \in \mathbb{R},$$

$$\boldsymbol{F}(\xi_1, \xi_2) = (F_1(\xi_1, \xi_2), F_2(\xi_1, \xi_2)) := \begin{cases} (\mathscr{F}_1(\boldsymbol{x}, \xi_1, \xi_2), \mathscr{F}_2(\boldsymbol{x}, \xi_1, \xi_2)) & \text{if } 0 \le \xi_1, \xi_2, \\ (\mathscr{F}_1(\boldsymbol{x}, \xi_1, 0), \mathscr{F}_2(\boldsymbol{x}, \xi_1, 0)) & \text{if } \xi_2 < 0 \le \xi_1, \\ (\mathscr{F}_1(\boldsymbol{x}, 0, \xi_2), \mathscr{F}_2(\boldsymbol{x}, 0, \xi_2)) & \text{if } \xi_1 < 0 \le \xi_2, \\ (\mathscr{F}_1(\boldsymbol{x}, 0, 0), \mathscr{F}_2(\boldsymbol{x}, 0, 0)) & \text{if } \xi_1, \xi_2 < 0, \end{cases}$$

$$G(\xi_1, \xi_2) := \|\mathbf{Diag}(\xi_1, \xi_2)\bar{\boldsymbol{u}}\| - \|\mathbf{Diag}(\xi_1, \xi_2)\boldsymbol{u}\|, \quad (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Obviously, h is Lipschitz continuous in $\mathbb{R}$ and the left-hand side of (1.26) at the point $\boldsymbol{x}$ equals $|h(1) - h(0)|$. From the Lebourg mean-value theorem, it follows that there exists $\bar{r} \in (0, 1)$ such that

$$h(1) - h(0) \in \partial h(\bar{r}),$$

where ∂h denotes the Clarke sub-differential of h (see [11, Chapter 2]). So it suffices to estimate $|\theta|$ for any $\theta \in \partial h(r)$ and any $r \in (0, 1)$ fixed.

As both $\boldsymbol{H}$ and G are continuously differentiable, Chain Rule II for the Clarke sub-differential ∂h and the chain rule for $\partial(G \circ \boldsymbol{F})$ viewed as the generalized Jacobian imply that

$$\partial h(r) \subset (\boldsymbol{\nabla H}(r))^T \partial(G \circ \boldsymbol{F})(\boldsymbol{H}(r)),$$
$$\partial(G \circ \boldsymbol{F})(\boldsymbol{H}(r)) = (\boldsymbol{\partial F}(\boldsymbol{H}(r)))^T \nabla G(\boldsymbol{F}(\boldsymbol{H}(r)))$$

with $\boldsymbol{\partial F}$ standing for the generalized Jacobian of $\boldsymbol{F}$. Thus, any $\theta \in \partial h(r)$ is of the form

$$\theta = (\boldsymbol{\nabla H}(r))^T \boldsymbol{Z}^T \nabla G(\boldsymbol{F}(\boldsymbol{H}(r)))$$

for some $\boldsymbol{Z} = \left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}\right) \in \boldsymbol{\partial F}(\boldsymbol{H}(r))$.

Suppose first that $\boldsymbol{u}, \bar{\boldsymbol{u}} \neq \boldsymbol{0}$. If it is so then

$$(\boldsymbol{\nabla H}(r))^T \boldsymbol{Z}^T = \big((\varphi_1 - \bar{\varphi}_1)z_{11} + (\varphi_2 - \bar{\varphi}_2)z_{12}, (\varphi_1 - \bar{\varphi}_1)z_{21} + (\varphi_2 - \bar{\varphi}_2)z_{22}\big),$$

$$(\zeta_1, \zeta_2) \nabla G(\xi_1, \xi_2) = \frac{\mathbf{Diag}(\xi_1, \xi_2)\bar{\boldsymbol{u}} \cdot \mathbf{Diag}(\zeta_1, \zeta_2)\bar{\boldsymbol{u}}}{\|\mathbf{Diag}(\xi_1, \xi_2)\bar{\boldsymbol{u}}\|} - \frac{\mathbf{Diag}(\xi_1, \xi_2)\boldsymbol{u} \cdot \mathbf{Diag}(\zeta_1, \zeta_2)\boldsymbol{u}}{\|\mathbf{Diag}(\xi_1, \xi_2)\boldsymbol{u}\|}$$

and consequently,

$$\theta = \frac{\boldsymbol{A}\bar{\boldsymbol{u}} \cdot \boldsymbol{B}\bar{\boldsymbol{u}}}{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|} - \frac{\boldsymbol{A}\boldsymbol{u} \cdot \boldsymbol{B}\boldsymbol{u}}{\|\boldsymbol{A}\boldsymbol{u}\|}$$

with

$$\boldsymbol{A} := \mathbf{Diag}\big(F_1(\bar{\boldsymbol{\varphi}} + r(\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}})), F_2(\bar{\boldsymbol{\varphi}} + r(\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}))\big),$$
$$\boldsymbol{B} := \mathbf{Diag}\big((\varphi_1 - \bar{\varphi}_1)z_{11} + (\varphi_2 - \bar{\varphi}_2)z_{12}, (\varphi_1 - \bar{\varphi}_1)z_{21} + (\varphi_2 - \bar{\varphi}_2)z_{22}\big).$$

Clearly,

$$|\theta| \le \left|\frac{\boldsymbol{A}\bar{\boldsymbol{u}}\cdot\boldsymbol{B}(\bar{\boldsymbol{u}}-\boldsymbol{u})}{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|}\right| + \left|\frac{\boldsymbol{A}\bar{\boldsymbol{u}}\cdot\boldsymbol{B}\boldsymbol{u}}{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|} - \frac{\boldsymbol{A}\bar{\boldsymbol{u}}\cdot\boldsymbol{B}\boldsymbol{u}}{\|\boldsymbol{A}\boldsymbol{u}\|}\right| + \left|\frac{\boldsymbol{A}(\bar{\boldsymbol{u}}-\boldsymbol{u})\cdot\boldsymbol{B}\boldsymbol{u}}{\|\boldsymbol{A}\boldsymbol{u}\|}\right| =: s_1+s_2+s_3.$$

In virtue of the inequality $\|\boldsymbol{u}\| \le \|\boldsymbol{A}^{-1}\|\|\boldsymbol{A}\boldsymbol{u}\|$ and the fact that both $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonal matrices, one has

$$\begin{aligned}
s_1 &\le \frac{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|\,\|\boldsymbol{B}(\bar{\boldsymbol{u}}-\boldsymbol{u})\|}{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|} \le \|\boldsymbol{B}\|\,\|\bar{\boldsymbol{u}}-\boldsymbol{u}\|,\\
s_2 &= \left|\frac{(\boldsymbol{A}\bar{\boldsymbol{u}}\cdot\boldsymbol{B}\boldsymbol{u})(\|\boldsymbol{A}\boldsymbol{u}\|-\|\boldsymbol{A}\bar{\boldsymbol{u}}\|)}{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|\,\|\boldsymbol{A}\boldsymbol{u}\|}\right| \le \frac{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|\,\|\boldsymbol{B}\|\,\|\boldsymbol{u}\|\,\|\boldsymbol{A}\boldsymbol{u}-\boldsymbol{A}\bar{\boldsymbol{u}}\|\,\|\boldsymbol{A}^{-1}\|}{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|\,\|\boldsymbol{u}\|}\\
&\le \|\boldsymbol{B}\|\,\|\boldsymbol{A}\|\,\|\boldsymbol{u}-\bar{\boldsymbol{u}}\|\,\|\boldsymbol{A}^{-1}\| \le \kappa(\boldsymbol{\mathscr{F}})\|\boldsymbol{B}\|\,\|\boldsymbol{u}-\bar{\boldsymbol{u}}\|,\\
s_3 &= \left|\frac{\boldsymbol{B}(\bar{\boldsymbol{u}}-\boldsymbol{u})\cdot\boldsymbol{A}\boldsymbol{u}}{\|\boldsymbol{A}\boldsymbol{u}\|}\right| \le \|\boldsymbol{B}\|\,\|\bar{\boldsymbol{u}}-\boldsymbol{u}\|.
\end{aligned}$$

Furthermore, let $\boldsymbol{z}_i$ denote the ith row vector of $\boldsymbol{Z}$. Then $\|\boldsymbol{z}_i\| \le L$ because $\boldsymbol{z}_i \in \partial F_i(H(r))$ and the Lipschitz modulus of F_i is less than or equal to L by (1.25). Thus,

$$\|\boldsymbol{B}\| = \max_{1\le i\le 2}\{|(\varphi_1-\bar{\varphi}_1)z_{i1}+(\varphi_2-\bar{\varphi}_2)z_{i2}|\} \le \max_{1\le i\le 2}\|\boldsymbol{z}_i\|\,\|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}\| \le L\|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}\|.$$

Combining the previous estimates, we get

$$|\theta| \le L(2+\kappa(\boldsymbol{\mathscr{F}}))\|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}\|\,\|\boldsymbol{u}-\bar{\boldsymbol{u}}\|. \tag{1.27}$$

To complete the assertion, let $\boldsymbol{u}=\boldsymbol{0}\neq\bar{\boldsymbol{u}}$. In this case,

$$|\theta| = \left|\frac{\boldsymbol{A}\bar{\boldsymbol{u}}\cdot\boldsymbol{B}\bar{\boldsymbol{u}}}{\|\boldsymbol{A}\bar{\boldsymbol{u}}\|}\right| \le \|\boldsymbol{B}\|\,\|\bar{\boldsymbol{u}}-\boldsymbol{0}\| \le L\|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}\|\,\|\bar{\boldsymbol{u}}-\boldsymbol{u}\|,$$

that is, (1.27) holds as well and so it is for $\bar{\boldsymbol{u}}=\boldsymbol{0}$. □

Remark 1.6. In the case of isotropic friction with non-vanishing coefficient of friction $\mathscr{F}$, one has $\kappa(\boldsymbol{\mathscr{F}})=1$ and the previous result leads to

$$\begin{aligned}
&\big|\|\mathscr{F}(\varphi_1^h,\varphi_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\mathscr{F}(\varphi_1^h,\varphi_2^h)\boldsymbol{u}_\tau^h\| - (\|\mathscr{F}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\mathscr{F}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\boldsymbol{u}_\tau^h\|)\big|\\
&\qquad\qquad \le 3L\|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|\,\|\boldsymbol{u}_\tau^h-\bar{\boldsymbol{u}}_\tau^h\| \text{ on } \Gamma_C
\end{aligned}$$

for any $\boldsymbol{u}^h,\bar{\boldsymbol{u}}^h\in\boldsymbol{V}^h$ and any $(\varphi_1^h,\varphi_2^h)$, $(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\in X_+^h\times X_+^h$ provided that (1.25) is satisfied. Nevertheless, the estimate can be easily improved in this case. Indeed,

$$\begin{aligned}
&\big|\|\mathscr{F}(\varphi_1^h,\varphi_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\mathscr{F}(\varphi_1^h,\varphi_2^h)\boldsymbol{u}_\tau^h\| - (\|\mathscr{F}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\mathscr{F}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\boldsymbol{u}_\tau^h\|)\big|\\
&\qquad = |\mathscr{F}(\varphi_1^h,\varphi_2^h)(\|\bar{\boldsymbol{u}}_\tau^h\|-\|\boldsymbol{u}_\tau^h\|) - \mathscr{F}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)(\|\bar{\boldsymbol{u}}_\tau^h\|-\|\boldsymbol{u}_\tau^h\|)|\\
&\qquad \le L\|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|\,\|\boldsymbol{u}_\tau^h-\bar{\boldsymbol{u}}_\tau^h\| \text{ on } \Gamma_C.
\end{aligned}$$

In addition, this shows that the coefficient $\mathscr{F}$ is allowed to vanish.

Furthermore, let us mention that in the case of (1.8), (1.25) holds under the following assumption:

$$\exists\,\hat{L} > 0 : \quad |\hat{\mathscr{F}}(\boldsymbol{x},\xi) - \hat{\mathscr{F}}(\boldsymbol{x},\bar{\xi})| \le \hat{L}|\xi - \bar{\xi}|, \quad \forall\,\boldsymbol{x} \in \Gamma_C,\ \forall\,\xi,\bar{\xi} \in \mathbb{R}_+.$$

Proposition 1.1. *Let (1.9)–(1.11) and (1.25) be satisfied. For any* $R_1, R_2 > 0$, Φ_{hH} *is Lipschitz continuous in* $\mathscr{C}(R_1,R_2)$*:*

$$\exists\,c_1,c_2 > 0 : \quad \|\Phi_{hH}(\varphi_1^h,\varphi_2^h,g^H) - \Phi_{hH}(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H)\|_{X^h\times X^h\times L^H}$$
$$\le \max\Big\{\frac{\mathscr{F}_{\max}}{\sqrt{H}}c_1, \frac{L(2+\kappa(\boldsymbol{\mathscr{F}}))}{\sqrt{h_{\Gamma_C}H}}c_2R_2\Big\}\|(\varphi_1^h,\varphi_2^h,g^H) - (\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H)\|_{X^h\times X^h\times L^H},$$
$$\forall\,(\varphi_1^h,\varphi_2^h,g^H),(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H) \in \mathscr{C}(R_1,R_2). \quad (1.28)$$

Proof. For $(\varphi_1^h,\varphi_2^h,g^H),(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H) \in \mathscr{C}(R_1,R_2)$, denote by $(\boldsymbol{u}^h,\lambda_\nu^H),(\bar{\boldsymbol{u}}^h,\bar{\lambda}_\nu^H)$ the solutions to $(\mathscr{M}_{hH}(\varphi_1^h,\varphi_2^h,g^H))$ and $(\mathscr{M}_{hH}(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H))$, respectively. Inserting $\boldsymbol{v}^h := \bar{\boldsymbol{u}}^h \in \boldsymbol{K}^{hH}$ and $\boldsymbol{v}^h := \boldsymbol{u}^h \in \boldsymbol{K}^{hH}$ into $(\mathscr{P}_{hH}(\varphi_1^h,\varphi_2^h,g^H))$ and $(\mathscr{P}_{hH}(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H))$, respectively, we have

$$a(\boldsymbol{u}^h,\bar{\boldsymbol{u}}^h - \boldsymbol{u}^h) + j(\varphi_1^h,\varphi_2^h,g^H,\bar{\boldsymbol{u}}_\tau^h) - j(\varphi_1^h,\varphi_2^h,g^H,\boldsymbol{u}_\tau^h) \ge \ell(\bar{\boldsymbol{u}}^h - \boldsymbol{u}^h),$$
$$a(\bar{\boldsymbol{u}}^h,\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h) + j(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H,\boldsymbol{u}_\tau^h) - j(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H,\bar{\boldsymbol{u}}_\tau^h) \ge \ell(\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h).$$

Summing both inequalities and using (1.12), we arrive at

$$\begin{aligned}
\alpha\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{1,\Omega}^2 \\
&\le a(\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h,\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h) \\
&\le j(\varphi_1^h,\varphi_2^h,g^H,\bar{\boldsymbol{u}}_\tau^h) - j(\varphi_1^h,\varphi_2^h,g^H,\boldsymbol{u}_\tau^h) + j(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H,\boldsymbol{u}_\tau^h) - j(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H,\bar{\boldsymbol{u}}_\tau^h) \\
&= \big(g^H, \|\boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)\boldsymbol{u}_\tau^h\|\big)_{0,\Gamma_C} \\
&\quad - \big(\bar{g}^H, \|\boldsymbol{\mathscr{F}}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\boldsymbol{\mathscr{F}}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\boldsymbol{u}_\tau^h\|\big)_{0,\Gamma_C} \\
&= \big(g^H - \bar{g}^H, \|\boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)\boldsymbol{u}_\tau^h\|\big)_{0,\Gamma_C} \\
&\quad + \big(\bar{g}^H, \|\boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)\boldsymbol{u}_\tau^h\| \\
&\qquad\qquad - (\|\boldsymbol{\mathscr{F}}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\bar{\boldsymbol{u}}_\tau^h\| - \|\boldsymbol{\mathscr{F}}(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\boldsymbol{u}_\tau^h\|)\big)_{0,\Gamma_C} \\
&=: s_1 + s_2.
\end{aligned} \quad (1.29)$$

The first term can be estimated as follows:

$$\begin{aligned}
s_1 &\le \|g^H - \bar{g}^H\|_{0,\Gamma_C}\big\|\,\|\boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)\bar{\boldsymbol{u}}_\tau^h - \boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)\boldsymbol{u}_\tau^h\|\,\big\|_{0,\Gamma_C} \\
&= \|g^H - \bar{g}^H\|_{0,\Gamma_C}\|\boldsymbol{\mathscr{F}}(\varphi_1^h,\varphi_2^h)(\bar{\boldsymbol{u}}_\tau^h - \boldsymbol{u}_\tau^h)\|_{0,\Gamma_C} \le \mathscr{F}_{\max}\|g^H - \bar{g}^H\|_{0,\Gamma_C}\|\bar{\boldsymbol{u}}^h - \boldsymbol{u}^h\|_{0,\Gamma_C} \\
&\le \frac{\mathscr{F}_{\max}}{\sqrt{H}}c_{\text{inv}}^{(0,-1/2)}c_{\text{tr}}^{(2)}\|g^H - \bar{g}^H\|_{*,h}\|\bar{\boldsymbol{u}}^h - \boldsymbol{u}^h\|_{1,\Omega},
\end{aligned} \quad (1.30)$$

where $c_{\text{tr}}^{(2)}$ is the norm of the trace mapping from $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{L}^2(\partial\Omega)$ and $c_{\text{inv}}^{(0,-1/2)}$ is the constant from the equivalence of the respective norms in the finite-dimensional space L^H:

$$\|\mu^H\|_{0,\Gamma_C} \leq \frac{c_{\text{inv}}^{(0,-1/2)}}{\sqrt{H}}\|\mu^H\|_{*,h}, \quad \forall\, \mu^H \in L^H. \tag{1.31}$$

Further, from the previous lemma,

$$\begin{aligned} s_2 &\leq L(2+\kappa(\mathscr{F}))\|\bar{g}^H\|_{0,\Gamma_C}\Big\| \, \|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\| \, \|\boldsymbol{u}_\tau^h - \bar{\boldsymbol{u}}_\tau^h\| \, \Big\|_{0,\Gamma_C} \\ &\leq L(2+\kappa(\mathscr{F}))\|\bar{g}^H\|_{0,\Gamma_C}\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{0,\infty,\Gamma_C}\|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|_{0,\Gamma_C}. \end{aligned}$$

Due to the equivalence of norms in $X^h \times X^h \times X^h$, namely

$$\|\boldsymbol{\psi}^h\|_{0,\infty,\Gamma_C} \leq \frac{c_{\text{inv}}^{(\infty)}}{\sqrt{h_{\Gamma_C}}}\|\boldsymbol{\psi}^h\|_{0,4,\Gamma_C}, \quad \forall\, \boldsymbol{\psi}^h \in X^h \times X^h \times X^h \tag{1.32}$$

with an appropriate $c_{\text{inv}}^{(\infty)} > 0$, and the continuity of the trace mapping from $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{L}^4(\partial\Omega)$, whose norm is denoted by $c_{\text{tr}}^{(4)}$, one obtains

$$\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{0,\infty,\Gamma_C} \leq \frac{c_{\text{inv}}^{(\infty)} c_{\text{tr}}^{(4)}}{\sqrt{h_{\Gamma_C}}}\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{1,\Omega}.$$

Using (1.31) once again, we get

$$\|\bar{g}^H\|_{0,\Gamma_C} \leq \frac{c_{\text{inv}}^{(0,-1/2)}}{\sqrt{H}}\|\bar{g}^H\|_{*,h} \leq \frac{c_{\text{inv}}^{(0,-1/2)}}{\sqrt{H}} R_2,$$

making use of the definition of $\mathscr{C}(R_1,R_2)$. Therefore

$$s_2 \leq \frac{L(2+\kappa(\mathscr{F}))}{\sqrt{h_{\Gamma_C} H}} c_{\text{inv}}^{(0,-1/2)} c_{\text{inv}}^{(\infty)} c_{\text{tr}}^{(4)} R_2 \|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|_{0,\Gamma_C}\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{1,\Omega}. \tag{1.33}$$

The inequality (1.29) together with (1.30) and (1.33) implies that

$$\begin{aligned} &\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{1,\Omega} \\ &\quad\leq \frac{\mathscr{F}_{\max}}{\sqrt{H}}\tilde{c}_1\|g^H - \bar{g}^H\|_{*,h} + \frac{L(2+\kappa(\mathscr{F}))}{\sqrt{h_{\Gamma_C} H}}\tilde{c}_2 R_2\|(\varphi_1^h,\varphi_2^h)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h)\|_{0,\Gamma_C} \\ &\quad\leq \max\Big\{\frac{\mathscr{F}_{\max}}{\sqrt{H}}\tilde{c}_1, \frac{L(2+\kappa(\mathscr{F}))}{\sqrt{h_{\Gamma_C} H}}\tilde{c}_2 R_2\Big\}\|(\varphi_1^h,\varphi_2^h,g^H)-(\bar{\varphi}_1^h,\bar{\varphi}_2^h,\bar{g}^H)\|_{X^h\times X^h\times L^H} \end{aligned}$$

with

$$\tilde{c}_1 = \tilde{c}_1(c_{\text{inv}}^{(0,-1/2)}, c_{\text{tr}}^{(2)}, \alpha) := \frac{c_{\text{inv}}^{(0,-1/2)} c_{\text{tr}}^{(2)}}{\alpha},$$
$$\tilde{c}_2 = \tilde{c}_2(c_{\text{inv}}^{(0,-1/2)}, c_{\text{inv}}^{(\infty)}, c_{\text{tr}}^{(4)}, \alpha) := \frac{c_{\text{inv}}^{(0,-1/2)} c_{\text{inv}}^{(\infty)} c_{\text{tr}}^{(4)}}{\alpha}.$$

Following the steps in (1.24), one can see that

$$\|(r_h|u^h_{\tau,1}|, r_h|u^h_{\tau,2}|) - (r_h|\bar{u}^h_{\tau,1}|, r_h|\bar{u}^h_{\tau,2}|)\|_{0,\Gamma_C} \le c_{\text{tr}}^{(2)}(c_{\text{inv}}^{(1,0)} c_{r_h} c_\tau + 1)\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{1,\Omega}.$$

Finally, the last components of Φ_{hH} are treated similarly as in the proof of Lemma 1.3. The relations

$$a(\boldsymbol{u}^h, \boldsymbol{v}^h) = \ell(\boldsymbol{v}^h) - (\lambda_\nu^H, v_\nu^h)_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}_0^h,$$
$$a(\bar{\boldsymbol{u}}^h, \boldsymbol{v}^h) = \ell(\boldsymbol{v}^h) - (\bar{\lambda}_\nu^H, v_\nu^h)_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}_0^h$$

give

$$(\lambda_\nu^H - \bar{\lambda}_\nu^H, v_\nu^h)_{0,\Gamma_C} = a(\bar{\boldsymbol{u}}^h - \boldsymbol{u}^h, \boldsymbol{v}^h), \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}_0^h,$$
$$\|\lambda_\nu^H - \bar{\lambda}_\nu^H\|_{*,h} = \sup_{\boldsymbol{0} \ne \boldsymbol{v}^h \in \boldsymbol{V}^h} \frac{(\lambda_\nu^H - \bar{\lambda}_\nu^H, v_\nu^h)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,\Omega}} \le \sup_{\boldsymbol{0} \ne \boldsymbol{v}^h \in \boldsymbol{V}_0^h} \frac{(\lambda_\nu^H - \bar{\lambda}_\nu^H, v_\nu^h)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,\Omega}}$$
$$= \sup_{\boldsymbol{0} \ne \boldsymbol{v}^h \in \boldsymbol{V}_0^h} \frac{a(\bar{\boldsymbol{u}}^h - \boldsymbol{u}^h, \boldsymbol{v}^h)}{\|\boldsymbol{v}^h\|_{1,\Omega}} \le M\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{1,\Omega}.$$

Thus, setting

$$c_1 = c_1(c_{\text{inv}}^{(0,-1/2)}, c_{\text{inv}}^{(1,0)}, c_{r_h}, c_{\text{tr}}^{(2)}, c_\tau, M, \alpha) := \big(c_{\text{tr}}^{(2)}(c_{\text{inv}}^{(1,0)} c_{r_h} c_\tau + 1) + M\big)\tilde{c}_1,$$
$$c_2 = c_2(c_{\text{inv}}^{(0,-1/2)}, c_{\text{inv}}^{(1,0)}, c_{\text{inv}}^{(\infty)}, c_{r_h}, c_{\text{tr}}^{(2)}, c_{\text{tr}}^{(4)}, c_\tau, M, \alpha) := \big(c_{\text{tr}}^{(2)}(c_{\text{inv}}^{(1,0)} c_{r_h} c_\tau + 1) + M\big)\tilde{c}_2,$$

we have

$$\begin{aligned}
&\|\Phi_{hH}(\varphi_1^h, \varphi_2^h, g^H) - \Phi_{hH}(\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H)\|_{X^h \times X^h \times L^H} \\
&\quad = \|(r_h|u^h_{\tau,1}|, r_h|u^h_{\tau,2}|) - (r_h|\bar{u}^h_{\tau,1}|, r_h|\bar{u}^h_{\tau,2}|)\|_{0,\Gamma_C} + \|\lambda_\nu^H - \bar{\lambda}_\nu^H\|_{*,h} \\
&\quad \le \big(c_{\text{tr}}^{(2)}(c_{\text{inv}}^{(1,0)} c_{r_h} c_\tau + 1) + M\big)\|\boldsymbol{u}^h - \bar{\boldsymbol{u}}^h\|_{1,\Omega} \\
&\quad \le \max\Big\{\frac{\mathscr{F}_{\max}}{\sqrt{H}} c_1, \frac{L(2 + \kappa(\boldsymbol{\mathscr{F}}))}{\sqrt{h_{\Gamma_C} H}} c_2 R_2\Big\} \|(\varphi_1^h, \varphi_2^h, g^H) - (\bar{\varphi}_1^h, \bar{\varphi}_2^h, \bar{g}^H)\|_{X^h \times X^h \times L^H}.
\end{aligned}$$

□

Taking R_1 and R_2 from Lemma 1.3, we obtain the following uniqueness result.

Theorem 1.2. *If (1.9)–(1.11) and (1.25) are satisfied with sufficiently small* $\mathscr{F}_{\max}$ *and L, then the solution of our problem in the sense of Definition 1.1 is unique. In addition, it solves* $(\mathscr{M}_{hH}(\varphi_1^h, \varphi_2^h, g^H))$ *where the triplet* $(\varphi_1^h, \varphi_2^h, g^H) \in X_+^h \times X_+^h \times \Lambda_\nu^H$ *is the limit of the sequence generated by* the method of successive approximations*:*

$$\left.\begin{array}{l} \text{Let } (\varphi_1^{h,0}, \varphi_2^{h,0}, g^{H,0}) \in X_+^h \times X_+^h \times \Lambda_\nu^H \text{ be given;} \\ \text{for } k = 0, 1, \dots \text{ set} \\ \quad (\varphi_1^{h,k+1}, \varphi_2^{h,k+1}, g^{H,k+1}) := \Phi_{hH}(\varphi_1^{h,k}, \varphi_2^{h,k}, g^{H,k}); \end{array}\right\}$$

for any choice of $(\varphi_1^{h,0}, \varphi_2^{h,0}, g^{H,0}) \in X_+^h \times X_+^h \times \Lambda_\nu^H$*.*

Proof. Consider R_1 and R_2 given by Lemma 1.3. In view of (1.28), Φ_{hH} is contractive in $\mathscr{C}(R_1, R_2)$ for $\mathscr{F}_{\max}$ and L small enough. The assertion now follows from the Banach fixed-point theorem. □

So far, we have assumed that the partitions $\mathscr{T}_\Omega^h$ and $\mathscr{T}_{\Gamma_C}^H$ are fixed and the constants $c_{\text{inv}}^{(0,-1/2)}$, $c_{\text{inv}}^{(1,0)}$, $c_{\text{inv}}^{(\infty)}$ and c_{r_h} may eventually depend on h and H. In what follows, we present sufficient conditions under which these constants do not depend on the mesh norms. To this end we shall consider systems of partitions $\{\mathscr{T}_\Omega^h\}$ and $\{\mathscr{T}_{\Gamma_C}^H\}$ for $h, H \to 0+$. We shall suppose that

(i) $\{\mathscr{T}_\Omega^h|_{\overline{\Gamma}_C}\}$ and $\{\mathscr{T}_{\Gamma_C}^H\}$, $h, H \to 0+$, are regular systems of partitions of $\overline{\Gamma}_C$ that satisfy the so-called inverse assumption ([9, (3.2.28)]);

(ii) the Babuška-Brezzi condition is satisfied for $(\boldsymbol{V}^h, L^H)$:

$$\exists\, \beta > 0 : \quad \sup_{\mathbf{0} \neq \boldsymbol{v}^h \in \boldsymbol{V}^h} \frac{(\mu^H, v_\nu^h)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,\Omega}} \geq \beta \|\mu^H\|_{*,\Gamma_C}, \quad \forall\, \mu^H \in L^H, \forall\, h, H \to 0+,$$

where $\|.\|_{*,\Gamma_C}$ is the dual norm in X_ν' (recall that the duality pairing between X_ν and X_ν' is realized by the $L^2(\Gamma_C)$-scalar product in the discretized case):

$$\|\mu^H\|_{*,\Gamma_C} = \sup_{0 \neq \varphi \in X_\nu} \frac{(\mu^H, \varphi)_{0,\Gamma_C}}{\|\varphi\|_{X_\nu}}, \quad \mu^H \in L^H,\ H \to 0+;$$

(iii) the interpolation operator r_h is such that c_{r_h} in (1.18) does not depend on h_{Γ_C}.

From (ii), it is readily seen that

$$\beta \|\mu^H\|_{*,\Gamma_C} \leq \|\mu^H\|_{*,h} \leq \|\mu^H\|_{*,\Gamma_C}, \quad \forall\, \mu^H \in L^H, \forall\, h, H \to 0+,$$

which means that the mesh dependent norm $\|.\|_{*,h}$ can be replaced by the dual norm $\|.\|_{*,\Gamma_C}$ in all the previous estimates. In addition, taking (i) into account, the constants from the inverse inequalities (1.22), (1.31) and (1.32) are independent of h_{Γ_C}, H (see [9]). For this reason, neither R_1, R_2 from Lemma 1.3, nor c_1, c_2 from Proposition 1.1 depend on h_{Γ_C}, H.

Remark 1.7. Let (i)–(iii) hold and $\kappa(\boldsymbol{\mathscr{F}})$ be bounded. To guarantee the uniqueness of the discrete solutions for $h, H \to 0+$, we need the parameters $\mathscr{F}_{\max}$ and L to decay at least as fast as $\sqrt{H}$ and $\sqrt{h_{\Gamma_C} H}$, respectively.

Following Remark 1.6, if one considers isotropic friction, one can replace the assumption (1.11) by (1.14) to obtain an estimate analogous to (1.28) with 1 instead of the term $(2+\kappa(\boldsymbol{\mathscr{F}}))$. This ensures that the given conditions on the decay of $\mathscr{F}_{\max}$ and L remain valid under the satisfaction of (1.14), (1.25) and (i)–(iii) in this case. In particular, if $\mathscr{F}$ does not depend on $\boldsymbol{u}$, that is, $L = 0$, the classical result from [21] is recovered.

At the end of this section, let us briefly comment on the satisfaction of (ii) and (iii). It is shown in [3] that the Babuška-Brezzi condition is satisfied for (FE1) if $K_1 = K_2 = 1$. In the case of (FE2), it is satisfied provided that the ratio H/h is sufficiently large and the auxiliary linear elasticity problem:

$$\left.\begin{array}{l}\text{Find } \boldsymbol{w}_\mu \in \boldsymbol{V} \text{ such that}\\ a(\boldsymbol{w}_\mu, \boldsymbol{v}) = \langle \mu, v_\nu \rangle_\nu, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}\end{array}\right\}$$

is regular in the following sense: there exists $\varepsilon > 0$ such that for every $\mu \in X'_\nu \cap H^{-1/2+\varepsilon}(\Gamma_C)$, the solution $\boldsymbol{w}_\mu$ belongs to $\boldsymbol{H}^{1+\varepsilon}(\Omega)$ and

$$\|\boldsymbol{w}_\mu\|_{1+\varepsilon,\Omega} \le c(\varepsilon)\|\mu\|_{-1/2+\varepsilon,\Gamma_C}$$

holds with a constant $c(\varepsilon)$ depending solely on ε (see [22]).

To give an example of the interpolation operator r_h satisfying (1.17) and (1.18) with the constant c_{r_h} independent of h_{Γ_C}, we suppose that Γ_C is polygonal and $\overline{\Gamma}_C \cap \overline{\Gamma}_D$ is either empty or a union of non-degenerate segments, that is, containing no isolated points. Moreover, let $\{\mathscr{T}_\Omega^h|_{\overline{\Gamma}_C}\}$, $h \to 0+$, be a regular system of triangulations of $\overline{\Gamma}_C$ such that any two triangles from $\mathscr{T}_\Omega^h|_{\overline{\Gamma}_C}$ are either disjoint, or have a vertex or a whole side in common. If we still suppose that $\{\mathscr{T}_\Omega^h\}$ is compatible with the decomposition of $\partial\Omega$ into Γ_D, Γ_N and Γ_C then we can take the following Clément interpolation operator [12] (with $K_1 = 1$)[1]:

Let $\{\boldsymbol{x}_i\}_{1\le i\le n_c}$ be the set of all contact nodes of $\mathscr{T}_\Omega^h$, that is, the nodes of $\mathscr{T}_\Omega^h$ lying on $\overline{\Gamma}_C \setminus \overline{\Gamma}_D$, and $\{\varphi_i\}_{1\le i\le n_c}$ be the corresponding Courant basis of X^h. For each $i \in \{1, \dots, n_c\}$, denote the support of φ_i by Δ_i and define $\pi_i : L^2(\Delta_i) \to P_0(\Delta_i)$ by

$$(\pi_i\varphi)(\boldsymbol{x}) := \frac{1}{S(\Delta_i)} \int_{\Delta_i} \varphi \,\mathrm{d}S, \quad \boldsymbol{x} \in \Delta_i,\ \varphi \in L^2(\Delta_i),$$

[1] In fact, the approximation property (1.18) is shown in [12] assuming that either $\overline{\Gamma}_C \cap \overline{\Gamma}_D = \emptyset$ or the whole relative boundary of Γ_C belongs to $\overline{\Gamma}_D$. However, the same argumentation is valid also for the case considered here.

where $S(\Delta_i)$ stands for the area of Δ_i. Then r_h is defined as follows:

$$r_h\varphi = \sum_{1\le i\le n_c} (\pi_i\varphi)(\boldsymbol{x}_i)\varphi_i, \quad \varphi \in L^2(\Gamma_C).$$

Conclusion

This chapter has been devoted to the existence and uniqueness analysis of solutions to discretized contact problems with orthotropic and isotropic Coulomb friction and coefficients of friction depending on the magnitude of the tangential contact displacements. The discrete solutions have been defined as fixed points of a mapping acting on the contact part of the boundary. It has been shown that at least one solution exists for coefficients of friction represented by positive, bounded and continuous functions (which may even vanish for isotropic Coulomb friction). Uniqueness of the solution has been guaranteed provided that these functions are in addition Lipschitz continuous and upper bounds of their values together with their Lipschitz moduli are small enough. As a consequence we have obtained a justification of the method of successive approximations, which is widely used in numerical simulations of contact problems (for its application to problems with solution-dependent coefficients of friction, see [45, 24]).

Unfortunately, the bounds guaranteeing the uniqueness of the discretized solution are mesh-dependent and they have to decay in an appropriate rate depending on the mesh norms. This dependency can be understood in two ways:

- If the matrix $\boldsymbol{\mathscr{F}}$ is fixed then passing from coarser to finer meshes, one may loose unicity of the approximate solution.
- If finite-element meshes are fixed, then setting $\boldsymbol{\mathscr{F}}_\xi := \xi\boldsymbol{\mathscr{F}}$, $\xi \ge 0$, one can find $\xi_{\mathrm{crit}} > 0$ such that the discretized model has a unique solution for $\xi \le \xi_{\mathrm{crit}}$ and eventually multiple solutions if $\xi > \xi_{\mathrm{crit}}$. This behaviour has been observed in computations ([25]).

2 2D Static Problems

Unlike in the previous chapter, we focus mainly on local behaviour of solutions in the present chapter. For this purpose, we restrict ourselves to a discrete 2D Signorini problem with isotropic Coulomb friction and a coefficient of friction depending solely on the spatial variable. The forthcoming results are accepted for publication in [44].

Our exposition is organized as follows: In Sections 2.1 and 2.2, the studied discrete formulation is introduced. At the beginning of Section 2.3, it is proved, in accordance with the previous chapter, that the considered problem admits always a solution, which is unique provided that the values of $\mathscr{F}$ are below some sufficiently small bound $\mathscr{F}_{\max}$. The remaining part of Section 2.3 then deals with qualitative analysis of the solutions. First, we regard the solutions as a function of $\mathscr{F}$ and we show that this function is Lipschitz continuous with respect to all coefficients whose values are bounded by $\mathscr{F}_{\max}$. To get results of local character, we reformulate our problem as a system of generalized equations and we use a corresponding variant of the implicit-function theorem. We shall see that there exist local Lipschitz continuous branches of solutions parametrized by $\mathscr{F}$ around some reference point if there are local Lipschitz continuous branches of solutions parametrized by the load vector $\boldsymbol{f}$ around this point. For this reason, we shall consider the solutions to be a function of $\boldsymbol{f}$ for $\mathscr{F}$ fixed thereafter. Again, we show that this function is Lipschitz continuous provided that the values of $\mathscr{F}$ are lower than $\mathscr{F}_{\max}$. Further, we reformulate our problem as a system of piecewise differentiable equations and we use a version of the implicit-function theorem corresponding to this case. In this way, we arrive at a condition which ensures existence of local Lipschitz continuous branches of solutions with respect to $\boldsymbol{f}$. Finally, Section 2.4 illustrates our general approach on an elementary example whose solution structure is known analytically.

2.1 Problem Formulation

The classical formulation of the problems considered in this chapter consists of the following partial differential equation and boundary conditions:

$$\begin{aligned} -\,\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad & \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \quad & \text{on } \Gamma_D, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{h} \quad & \text{on } \Gamma_N, \\ u_\nu \leq 0, \quad \sigma_\nu(\boldsymbol{u}) \leq 0, \quad u_\nu \sigma_\nu(\boldsymbol{u}) = 0 \quad & \text{on } \Gamma_C, \end{aligned}$$

$$\left.\begin{aligned} u_\tau(\boldsymbol{x}) = 0 &\implies |\sigma_\tau(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))| \le -\mathscr{F}(\boldsymbol{x})\sigma_\nu(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})),\\ u_\tau(\boldsymbol{x}) \neq 0 &\implies \quad \sigma_\tau(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) = \mathscr{F}(\boldsymbol{x})\sigma_\nu(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))\frac{u_\tau(\boldsymbol{x})}{|u_\tau(\boldsymbol{x})|}, \end{aligned}\right\} \boldsymbol{x} \in \Gamma_C.$$

The notation is the same as in the previous chapter. The only change is that Ω is a bounded, Lipschitz domain in $\mathbb{R}^2$ and the tangential displacement u_τ and the tangential contact stress σ_τ on Γ_C are defined by $u_\tau \equiv \boldsymbol{u}\cdot\boldsymbol{\tau}$ and $\sigma_\tau(\boldsymbol{u}) \equiv (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu})\cdot\boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is a unit tangent vector along Γ_C, in this case.

To present the weak formulation of this problem, we introduce the following spaces and set:

$$\begin{aligned} \boldsymbol{V} &:= \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \,|\, \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_D\},\\ \boldsymbol{K} &:= \{\boldsymbol{v} \in \boldsymbol{V} \,|\, v_\nu \le 0 \text{ a.e. on } \Gamma_C\},\\ X_\nu &:= \{\zeta \in L^2(\Gamma_C) \,|\, \exists\, \boldsymbol{v} \in \boldsymbol{V} : \zeta = v_\nu \text{ a.e. on } \Gamma_C\},\\ X_\tau &:= \{\zeta \in L^2(\Gamma_C) \,|\, \exists\, \boldsymbol{v} \in \boldsymbol{V} : \zeta = v_\tau \text{ a.e. on } \Gamma_C\} \end{aligned}$$

and denote the (topological) duals of X_ν, X_τ by X'_ν, X'_τ and the corresponding duality pairings by $\langle .,.\rangle_\nu$, and $\langle .,.\rangle_\tau$. Moreover, we set

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &:= \int_\Omega \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}, && \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V},\\ \ell(\boldsymbol{v}) &:= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}S, && \boldsymbol{v} \in \boldsymbol{V}. \end{aligned}$$

The primal variational formulation reads as follows:

$$\left.\begin{aligned} &\text{Find } \boldsymbol{u} \in \boldsymbol{K} \text{ such that}\\ &a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \langle \mathscr{F}\sigma_\nu(\boldsymbol{u}), |v_\tau| - |u_\tau|\rangle_\nu \ge \ell(\boldsymbol{v} - \boldsymbol{u}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{K}. \end{aligned}\right\} \qquad (\mathscr{P})$$

Similarly as in the previous chapter, we consider an equivalent mixed variational formulation of the problem. In this case, it involves two Lagrange multipliers – not only the one releasing the unilateral constraint but also another one regularizing the non-smooth frictional term:

$$\left.\begin{aligned} &\text{Find } (\boldsymbol{u}, \lambda_\nu, \lambda_\tau) \in \boldsymbol{V} \times \Lambda_\nu \times \Lambda_\tau(\mathscr{F}, -\lambda_\nu) \text{ such that}\\ &a(\boldsymbol{u}, \boldsymbol{v}) = \ell(\boldsymbol{v}) + \langle \lambda_\nu, v_\nu\rangle_\nu + \langle \lambda_\tau, v_\tau\rangle_\tau, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},\\ &\langle \mu_\nu - \lambda_\nu, u_\nu\rangle_\nu + \langle \mu_\tau - \lambda_\tau, u_\tau\rangle_\tau \ge 0, \quad \forall\, (\mu_\nu, \mu_\tau) \in \Lambda_\nu \times \Lambda_\tau(\mathscr{F}, -\lambda_\nu). \end{aligned}\right\} \qquad (\mathscr{M})$$

Here, the Lagrange multiplier sets Λ_ν and $\Lambda_\tau(\mathscr{F}, -\lambda_\nu)$ are defined by

$$\begin{aligned} \Lambda_\nu &:= \{\mu_\nu \in X'_\nu \,|\, \langle \mu_\nu, v_\nu\rangle_\nu \ge 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{K}\},\\ \Lambda_\tau(\mathscr{F}, -\lambda_\nu) &:= \{\mu_\tau \in X'_\tau \,|\, \langle \mu_\tau, v_\tau\rangle_\tau - \langle -\mathscr{F}\lambda_\nu, |v_\tau|\rangle_\nu \le 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}\} \end{aligned}$$

and λ_ν and λ_τ represent $\sigma_\nu(\boldsymbol{u})$ and $\sigma_\tau(\boldsymbol{u})$ on Γ_C, respectively.

In this chapter, we shall study the following discrete problem with Coulomb friction coming from a discretization of problem $(\mathscr{M})$ (an example of an appropriate finite-element discretization is exhibited in the next section):

$$\left.\begin{array}{l} \text{Find } (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, -\boldsymbol{\lambda}_\nu) \text{ such that} \\ (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{v}) + (\boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}}, \\ (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{u}) + (\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{u}) \geq 0, \\ \qquad\qquad\qquad\qquad \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau) \in \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, -\boldsymbol{\lambda}_\nu). \end{array}\right\} \qquad (\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$$

In order to simplify notation, we use the same symbols for algebraic variables as for the corresponding continuous functions. By $(.,.)$ we denote the scalar product, by $\boldsymbol{A} \in \mathbb{M}^{n_{\boldsymbol{u}}}$, $n_{\boldsymbol{u}}$ being the number of degrees of freedom of displacements, the stiffness matrix, which is supposed to be symmetric positive definite:

$$\left.\begin{array}{l} \text{(i) } \boldsymbol{A} = \boldsymbol{A}^T, \\ \text{(ii) } \exists\, \alpha > 0: \quad (\boldsymbol{A}\boldsymbol{v}, \boldsymbol{v}) \geq \alpha \|\boldsymbol{v}\|^2, \quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}}. \end{array}\right\} \tag{2.1}$$

The matrices $\boldsymbol{B}_\nu, \boldsymbol{B}_\tau \in \mathbb{M}^{n_c, n_{\boldsymbol{u}}}$, where n_c is the number of nodes on $\overline{\Gamma}_C$ corresponding to the degrees of freedom for displacements, represent the linear mappings associating with a displacement vector its normal and tangential component on the contact zone, respectively. Hence, we assume that

$$\left.\begin{array}{rl} \text{(j)} & \text{the Euclidean norm of each row vector of } \boldsymbol{B}_\nu,\ \boldsymbol{B}_\tau \text{ is equal to one,} \\ \text{(jj)} & \text{each column of } \boldsymbol{B}_\nu,\ \boldsymbol{B}_\tau \text{ contains at most one nonzero element,} \\ \text{(jjj)} & \boldsymbol{B}_\nu^T \boldsymbol{\mu}_\nu + \boldsymbol{B}_\tau^T \boldsymbol{\mu}_\tau = \boldsymbol{0} \iff (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau) = (\boldsymbol{0}, \boldsymbol{0}) \in \mathbb{R}^{2n_c}. \end{array}\right\} \tag{2.2}$$

Note that (jjj) holds if and only if there exists $\beta > 0$ such that

$$\sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}}} \frac{(\boldsymbol{\mu}_\nu, \boldsymbol{B}_\nu \boldsymbol{v}) + (\boldsymbol{\mu}_\tau, \boldsymbol{B}_\tau \boldsymbol{v})}{\|\boldsymbol{v}\|} \geq \beta \|(\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau)\|, \quad \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau) \in \mathbb{R}^{2n_c}. \tag{2.3}$$

Further, $\boldsymbol{\mathscr{F}} = (\mathscr{F}_i) \in \mathbb{R}_+^{n_c}$ characterizes distribution of the coefficient of friction, $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ stands for the load vector and

$$\begin{aligned} \boldsymbol{\Lambda}_\nu &:= \mathbb{R}_-^{n_c}, \\ \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, \boldsymbol{g}) &:= \{\boldsymbol{\mu}_\tau = (\mu_{\tau,i}) \in \mathbb{R}^{n_c} \mid |\mu_{\tau,i}| \leq \mathscr{F}_i g_i,\ \forall\, i = 1, \ldots, n_c\}, \quad \boldsymbol{g} \in \mathbb{R}_+^{n_c}. \end{aligned}$$

In a similar way as in Chapter 1, we shall employ an equivalent fixed-point formulation of $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$ at the beginning of our theoretical analysis. For this

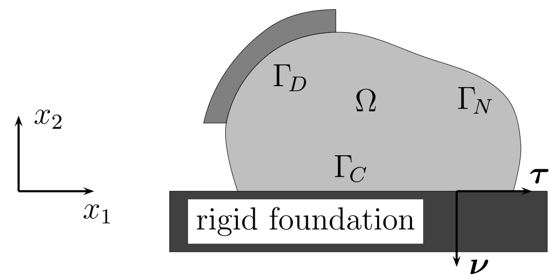


Figure 2.1: Special geometry considered in the example of discretization

reason, we associate with any $\boldsymbol{g} \in \mathbb{R}_+^{n_c}$ fixed the auxiliary problem:

$$\left.\begin{array}{l}\text{Find } (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, \boldsymbol{g}) \text{ such that} \\ (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{v}) + (\boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}}, \\ (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{u}) + (\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{u}) \geq 0, \\ \hfill \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau) \in \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, \boldsymbol{g}). \end{array}\right\} \qquad (\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$$

Again, $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$ corresponds to a contact problem with given friction and applying results from [17, Chapter VI], one can verify that it has a unique solution for any $\boldsymbol{g} \in \mathbb{R}_+^{n_c}$ provided that (2.1) and (2.2) are satisfied. Hence, one can introduce the mapping $\Phi : \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}_+^{n_c} \times \mathbb{R}_+^{n_c} \to \mathbb{R}_+^{n_c}$ by

$$\Phi(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}) := -\boldsymbol{\lambda}_\nu, \tag{2.4}$$

where $\boldsymbol{\lambda}_\nu$ is the second component of the solution to $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$.

It is readily seen that the triplet $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solves $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$ iff it is a solution of $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, -\boldsymbol{\lambda}_\nu))$, that is, $-\boldsymbol{\lambda}_\nu$ is a fixed point of $\Phi(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, .)$:

$$\Phi(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, -\boldsymbol{\lambda}_\nu) = -\boldsymbol{\lambda}_\nu.$$

2.2 An Example of Finite-Element Discretization

For better understanding, we shall describe in this section an example of discretization of $(\mathscr{M})$ leading to problem $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$ posited in the previous section. This example has been already presented in [39]. For simplicity, we assume here that $\mathscr{F} \in C(\overline{\Gamma}_C)$ and the coordinate system is chosen so that $\boldsymbol{\nu} = (0, -1)$ and $\boldsymbol{\tau} = (1, 0)$ along Γ_C (see Fig. 2.1).

Let $\mathscr{T}_h$ be a partition of $\overline{\Omega}$ that is compatible with the decomposition of $\partial\Omega$ into Γ_D, Γ_N and Γ_C and V^h be the following polynomial Lagrange finite-element space of degree $K \geq 1$:

$$V^h := \{v^h \in C(\overline{\Omega}) \,|\, v^h|_{\mathcal{T}} \in P_K(\mathcal{T}),\ \forall \mathcal{T} \in \mathscr{T}_h \ \&\ v^h = 0 \text{ on } \Gamma_D\}.$$

We set

$$
\begin{aligned}
\boldsymbol{V}^h &:= V^h \times V^h,\\
\boldsymbol{K}^h &:= \{\boldsymbol{v}^h \in \boldsymbol{V}^h \,|\, v^h_\nu(\boldsymbol{y}_i) \le 0,\ \forall\, i = 1,\dots,n_c\},\\
X^h_C &:= \{\zeta^h \in C(\overline{\Gamma}_C) \,|\, \exists\, v^h \in V^h : \zeta^h = v^h \text{ on } \overline{\Gamma}_C\},\\
\Lambda^h_\nu &:= \{\mu^h_\nu \in X^h_C \,|\, (\mu^h_\nu, v^h_\nu)_{0,\Gamma_C} \ge 0,\ \forall\, \boldsymbol{v}^h \in \boldsymbol{K}^h\},\\
\Lambda^h_\tau(\mathscr{F}, -\lambda^h_\nu) &:= \{\mu^h_\tau \in X^h_C \,|\, (\mu^h_\tau, v^h_\tau)_{0,\Gamma_C} - (-\lambda^h_\nu, r_h(\mathscr{F}|v^h_\tau|))_{0,\Gamma_C} \le 0,\ \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h\},
\end{aligned}
$$

where $\{\boldsymbol{y}_i\}_{1\le i\le n_c}$ is the set of all nodes on $\overline{\Gamma}_C$ corresponding to the degrees of freedom of V^h and r_h denotes the Lagrange interpolation operator into X^h_C.

Then the approximation of $(\mathscr{M})$ reads as follows:

$$
\left.
\begin{array}{l}
\text{Find } (\boldsymbol{u}^h, \lambda^h_\nu, \lambda^h_\tau) \in \boldsymbol{V}^h \times \Lambda^h_\nu \times \Lambda^h_\tau(\mathscr{F}, -\lambda^h_\nu) \text{ such that}\\
a(\boldsymbol{u}^h, \boldsymbol{v}^h) = \ell(\boldsymbol{v}^h) + (\lambda^h_\nu, v^h_\nu)_{0,\Gamma_C} + (\lambda^h_\tau, v^h_\tau)_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h,\\
(\mu^h_\nu - \lambda^h_\nu, u^h_\nu)_{0,\Gamma_C} + (\mu^h_\tau - \lambda^h_\tau, u^h_\tau)_{0,\Gamma_C} \ge 0,\\
\qquad\qquad\qquad\qquad\qquad\qquad \forall\, (\mu^h_\nu, \mu^h_\tau) \in \Lambda^h_\nu \times \Lambda^h_\tau(\mathscr{F}, -\lambda^h_\nu).
\end{array}
\right\} \quad (\mathscr{M}_h)
$$

For obtaining the algebraic formulation, let $\{\boldsymbol{x}_i\}_{1\le i\le n_{\boldsymbol{u}}/2}$ be the set of all nodes corresponding to the degrees of freedom of V^h, $\{\phi_i\}_{1\le i\le n_{\boldsymbol{u}}/2}$ be the set of shape functions of V^h such that

$$\phi_i(\boldsymbol{x}_j) = \delta_{ij}, \quad i, j = 1,\dots,n_{\boldsymbol{u}}/2,$$

and $\{\boldsymbol{\phi}_i\}_{1\le i\le n_{\boldsymbol{u}}}$ be the set of the shape functions of $\boldsymbol{V}^h$ of the form

$$\boldsymbol{\phi}_i = \begin{cases} (\phi_j, 0) & \text{if } i = 2j-1,\\ (0, \phi_j) & \text{if } i = 2j, \end{cases} \qquad j = 1,\dots,n_{\boldsymbol{u}}/2.$$

Further, let $\Theta : \{1,\dots,n_c\} \to \{1,\dots,n_{\boldsymbol{u}}/2\}$ be the mapping linking the local and global numeration of the nodes on $\overline{\Gamma}_C$ so that

$$\boldsymbol{y}_i = \boldsymbol{x}_{\Theta(i)}, \quad i = 1,\dots,n_c,$$

and $\{\eta_i\}_{1\le i\le n_c}$ be the set of the shape functions of X^h_C of the form

$$\eta_i = \phi_{\Theta(i)}|_{\overline{\Gamma}_C}.$$

We introduce the algebraic representatives $\boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, $\boldsymbol{\mu}_C \in \mathbb{R}^{n_c}$ of arbitrary $\boldsymbol{v}^h \in \boldsymbol{V}^h$, $\mu^h_C \in X^h_C$ as follows:

$$\boldsymbol{v} = (v_i) \text{ such that } \boldsymbol{v}^h = \sum_{1\le i\le n_{\boldsymbol{u}}} v_i \boldsymbol{\phi}_i, \tag{2.5}$$

$$\boldsymbol{\mu}_C = (\mu_{C,i}),\ \mu_{C,i} := (\mu^h_C, \eta_i)_{0,\Gamma_C}. \tag{2.6}$$

It is worth mentioning that unlike the representative $\boldsymbol{v}$ of $\boldsymbol{v}^h$, $\boldsymbol{\mu}_C$ is not the commonly used representative of μ_C^h consisting of its coordinates with respect to the corresponding finite-element basis. Though, the mapping $\mu_C^h \mapsto \boldsymbol{\mu}_C$ defined in this way is an isomorphism between X_C^h and $\mathbb{R}^{n_c}$. Indeed, if one defines

$$\tilde{\boldsymbol{B}} = (\tilde{B}_{ij}) \in \mathbb{M}^{n_c},\ \tilde{B}_{ij} := (\eta_i, \eta_j)_{0,\Gamma_C},$$
$$\tilde{\boldsymbol{\mu}}_C = (\tilde{\mu}_{C,i}) \in \mathbb{R}^{n_c} \text{ such that } \mu_C^h = \sum_{1\le i\le n_c} \tilde{\mu}_{C,i}\eta_i$$

then the matrix $\tilde{\boldsymbol{B}}$ is regular as it is the Gram matrix of the basis functions $\eta_1, \dots, \eta_{n_c}$, the mapping $\mu_C^h \mapsto \tilde{\boldsymbol{\mu}}_C$ is the commonly used isomorphism between X_C^h and $\mathbb{R}^{n_c}$ and

$$\mu_{C,i} = \Big(\eta_i, \sum_{1\le j\le n_c} \tilde{\mu}_{C,j}\eta_j\Big)_{0,\Gamma_C} = (\tilde{\boldsymbol{B}}\tilde{\boldsymbol{\mu}}_C)_i, \quad i = 1, \dots, n_c,$$

that is, $\boldsymbol{\mu}_C = \tilde{\boldsymbol{B}}\tilde{\boldsymbol{\mu}}_C$.

Moreover, we set

$$\boldsymbol{f} = (f_i) \in \mathbb{R}^{n_{\boldsymbol{u}}},\ f_i := \ell(\boldsymbol{\phi}_i), \qquad \boldsymbol{\mathscr{F}} = (\mathscr{F}_i) \in \mathbb{R}^{n_c},\ \mathscr{F}_i := \mathscr{F}(\boldsymbol{y}_i),$$
$$\boldsymbol{\Lambda}_\nu := \{\boldsymbol{\mu}_\nu \in \mathbb{R}^{n_c} \,|\, \mu_\nu^h \in \Lambda_\nu^h\}, \qquad \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, -\boldsymbol{\lambda}_\nu) := \{\boldsymbol{\mu}_\tau \in \mathbb{R}^{n_c} \,|\, \mu_\tau^h \in \Lambda_\tau^h(\mathscr{F}, -\lambda_\nu^h)\},$$

where $\mu_\nu^h, \mu_\tau^h, \lambda_\nu^h \in X_C^h$ are the functions represented by $\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau, \boldsymbol{\lambda}_\nu \in \mathbb{R}^{n_c}$ according to (2.6), respectively. Obviously, the expressions of the Lagrange multiplier sets introduced in this way can be simplified as follows:

$$\boldsymbol{\Lambda}_\nu = \{\boldsymbol{\mu}_\nu \in \mathbb{R}^{n_c} \,|\, (\mu_\nu^h, \eta_i)_{0,\Gamma_C} \le 0,\ \forall\, i = 1, \dots, n_c\} = \mathbb{R}_-^{n_c},$$

$$\boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, -\boldsymbol{\lambda}_\nu) = \Big\{\boldsymbol{\mu}_\tau \in \mathbb{R}^{n_c} \,\Big|\, (\mu_\tau^h, v_\tau^h)_{0,\Gamma_C} + \Big(\lambda_\nu^h, \sum_{1\le i\le n_c} \mathscr{F}(\boldsymbol{y}_i)|v_\tau^h(\boldsymbol{y}_i)|\eta_i\Big)_{0,\Gamma_C} \le 0,\ \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h\Big\} \tag{2.7}$$

$$= \{\boldsymbol{\mu}_\tau \in \mathbb{R}^{n_c} \,|\, |\mu_{\tau,j}| \le -\mathscr{F}_j\lambda_{\nu,j},\ \forall\, j = 1, \dots, n_c\}. \tag{2.8}$$

To see the last equality, consider $\boldsymbol{\mu}_\tau \in \mathbb{R}^{n_c}$ from the set in (2.7) and $j \in \{1, \dots, n_c\}$. Taking $\boldsymbol{v}^h \in \boldsymbol{V}^h$ with $v_\tau^h = (\mu_\tau^h, \eta_j)_{0,\Gamma_C}\eta_j$ in (2.7), one obtains

$$(\mu_\tau^h, \eta_j)_{0,\Gamma_C}^2 + \big(\lambda_\nu^h, \mathscr{F}(\boldsymbol{y}_j)|(\mu_\tau^h, \eta_j)_{0,\Gamma_C}|\eta_j\big)_{0,\Gamma_C} \le 0,$$
$$\mu_{\tau,j}^2 + \mathscr{F}_j|\mu_{\tau,j}|\lambda_{\nu,j} \le 0.$$

On the other hand, any $\mu_\tau^h \in X_C^h$ represented by $\boldsymbol{\mu}_\tau$ from (2.8) satisfies

$$(\mu_\tau^h, v_\tau^h)_{0,\Gamma_C} = \Big(\mu_\tau^h, \sum_{1\le i\le n_c} v_\tau^h(\boldsymbol{y}_i)\eta_i\Big)_{0,\Gamma_C} = \sum_{1\le i\le n_c} v_\tau^h(\boldsymbol{y}_i)\mu_{\tau,i} \le -\sum_{1\le i\le n_c} |v_\tau^h(\boldsymbol{y}_i)|\mathscr{F}_i\lambda_{\nu,i}$$
$$= -\Big(\lambda_\nu^h, \sum_{1\le i\le n_c} \mathscr{F}(\boldsymbol{y}_i)|v_\tau^h(\boldsymbol{y}_i)|\eta_i\Big)_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h.$$

Finally, we define

$$\boldsymbol{A} = (A_{ij}) \in \mathbb{M}^{n_{\boldsymbol{u}}},\ A_{ij} := a(\boldsymbol{\phi}_i, \boldsymbol{\phi}_j), \qquad \boldsymbol{B}_\nu = (B_{\nu,ij}) \in \mathbb{M}^{n_c, n_{\boldsymbol{u}}},\ B_{\nu,ij} := -\delta_{2\Theta(i),j},$$
$$\boldsymbol{B}_\tau = (B_{\tau,ij}) \in \mathbb{M}^{n_c, n_{\boldsymbol{u}}},\ B_{\tau,ij} := \delta_{2\Theta(i)-1,j}$$

so that

$$(\boldsymbol{B}_\nu \boldsymbol{v})_i = v_\nu^h(\boldsymbol{y}_i), \quad (\boldsymbol{B}_\tau \boldsymbol{v})_i = v_\tau^h(\boldsymbol{y}_i), \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h,\ i = 1, \ldots, n_c.$$

Due to our special geometry,

$$\begin{aligned}
(\mu_\nu^h, v_\nu^h)_{0,\Gamma_C} &= \Big(\mu_\nu^h, \big(\sum_{1\le i\le n_{\boldsymbol{u}}} v_i \boldsymbol{\phi}_i\big)_\nu\Big)_{0,\Gamma_C} = \sum_{1\le i\le n_{\boldsymbol{u}}} v_i \Big(\mu_\nu^h, -\sum_{1\le j\le n_c} \delta_{2\Theta(j),i}\eta_j\Big)_{0,\Gamma_C} \\
&= \sum_{1\le i\le n_{\boldsymbol{u}}} v_i \sum_{1\le j\le n_c} B_{\nu,ji}\mu_{\nu,j} = (\boldsymbol{\mu}_\nu, \boldsymbol{B}_\nu \boldsymbol{v}), \quad \forall\, \mu_\nu^h \in X_C^h,\ \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h, \\
(\mu_\tau^h, v_\tau^h)_{0,\Gamma_C} &= \Big(\mu_\tau^h, \big(\sum_{1\le i\le n_{\boldsymbol{u}}} v_i \boldsymbol{\phi}_i\big)_\tau\Big)_{0,\Gamma_C} = \sum_{1\le i\le n_{\boldsymbol{u}}} v_i \Big(\mu_\tau^h, \sum_{1\le j\le n_c} \delta_{2\Theta(j)-1,i}\eta_j\Big)_{0,\Gamma_C} \\
&= \sum_{1\le i\le n_{\boldsymbol{u}}} v_i \sum_{1\le j\le n_c} B_{\tau,ji}\mu_{\tau,j} = (\boldsymbol{\mu}_\tau, \boldsymbol{B}_\tau \boldsymbol{v}), \quad \forall\, \mu_\tau^h \in X_C^h,\ \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h.
\end{aligned}$$

All in all, the algebraic transcription of $(\mathscr{M}_h)$ is exactly $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$.

2.3 Theoretical Analysis of the Discrete Problem

As a preparation for the analysis of $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$, we shall study the discrete problem $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$ with given friction first. Let $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ and $\boldsymbol{\mathscr{F}}, \boldsymbol{g} \in \mathbb{R}_+^{n_c}$ be given and $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ be the unique solution of $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$. From the inequality in $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$ and the definitions of $\boldsymbol{\Lambda}_\nu$, $\boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, \boldsymbol{g})$, it follows that

$$\begin{gathered}
(\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{u}) = 0 \quad \& \quad \boldsymbol{u} \in \boldsymbol{K} := \{\boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}} \mid \boldsymbol{B}_\nu \boldsymbol{v} \le \boldsymbol{0}\}, \\
(\boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{u}) = \inf_{\boldsymbol{\mu}_\tau \in \boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}, \boldsymbol{g})} (\boldsymbol{\mu}_\tau, \boldsymbol{B}_\tau \boldsymbol{u}) = -\sum_{1\le i\le n_c} \mathscr{F}_i g_i |(\boldsymbol{B}_\tau \boldsymbol{u})_i|,
\end{gathered}$$

where the inequality $\boldsymbol{B}_\nu \boldsymbol{v} \le \boldsymbol{0}$ means that $(\boldsymbol{B}_\nu \boldsymbol{v})_i \le 0$ for any $i = 1, \ldots, n_c$. Thus, taking the equation in $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$ with $\boldsymbol{v} := \boldsymbol{v} - \boldsymbol{u}$, one can easily verify that $\boldsymbol{u}$ solves the following variational inequality:

$$\left.\begin{aligned}
&\text{Find } \boldsymbol{u} \in \boldsymbol{K} \text{ such that} \\
&(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \sum_{1\le i\le n_c} \mathscr{F}_i g_i \big(|(\boldsymbol{B}_\tau \boldsymbol{v})_i| - |(\boldsymbol{B}_\tau \boldsymbol{u})_i|\big) \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}), \\
&\hfill \forall\, \boldsymbol{v} \in \boldsymbol{K}.
\end{aligned}\right\} \quad (\mathcal{P}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$$

The next two lemmas summarize some other useful properties of the problem with given friction. Recall that $\|.\|$ stands for the Euclidean vector norm as well as for the corresponding matrix norm and $\|.\|_\infty$ denotes the max-norm for vectors. The mapping Φ was defined by (2.4).

Lemma 2.1. *Let (2.1) and (2.2) be satisfied. Then for any $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ and any $\boldsymbol{\mathscr{F}}, \boldsymbol{g} \in \mathbb{R}_+^{n_c}$, the following estimates hold for the solution $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ of $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$:*

$$\|\boldsymbol{u}\| \leq \frac{\|\boldsymbol{f}\|}{\alpha}, \tag{2.9}$$

$$\|(\boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)\| \leq \frac{\|\boldsymbol{f}\|}{\beta}\Big(\frac{\|\boldsymbol{A}\|}{\alpha} + 1\Big), \tag{2.10}$$

where β is the constant from (2.3).

Proof. Inserting $\boldsymbol{v} := \boldsymbol{0} \in \boldsymbol{K}$ into $(\mathcal{P}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$, one gets

$$-(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{u}) - \sum_{1 \leq i \leq n_c} \mathscr{F}_i g_i |(\boldsymbol{B}_\tau \boldsymbol{u})_i| \geq -(\boldsymbol{f}, \boldsymbol{u}).$$

Using (2.1), one has

$$\alpha\|\boldsymbol{u}\|^2 \leq (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{u}) + \sum_{1 \leq i \leq n_c} \mathscr{F}_i g_i |(\boldsymbol{B}_\tau \boldsymbol{u})_i| \leq (\boldsymbol{f}, \boldsymbol{u}) \leq \|\boldsymbol{f}\|\|\boldsymbol{u}\|,$$

which yields (2.9). To prove (2.10), we employ the equation in $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$:

$$(\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{v}) + (\boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{v}) = (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) \leq \|\boldsymbol{A}\|\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{f}\|\|\boldsymbol{v}\|, \quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}}.$$

This, (2.3) and (2.9) lead to

$$\beta\|(\boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)\| \leq \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}}} \frac{(\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{v}) + (\boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{v})}{\|\boldsymbol{v}\|} \leq \|\boldsymbol{A}\|\|\boldsymbol{u}\| + \|\boldsymbol{f}\| \leq \|\boldsymbol{A}\|\frac{\|\boldsymbol{f}\|}{\alpha} + \|\boldsymbol{f}\|.$$

□

It is worth mentioning that both bounds (2.9) and (2.10) are independent of $\boldsymbol{\mathscr{F}}$ and $\boldsymbol{g}$.

Lemma 2.2. *Assume that (2.1), (2.2) hold and $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, $\boldsymbol{\mathscr{F}}, \bar{\boldsymbol{\mathscr{F}}}, \boldsymbol{g}, \bar{\boldsymbol{g}} \in \mathbb{R}_+^{n_c}$. Let $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$, $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$ be the solutions to $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$ and $(\mathcal{M}(\boldsymbol{f}, \bar{\boldsymbol{\mathscr{F}}}, \bar{\boldsymbol{g}}))$, respectively. Then*

$$\|\boldsymbol{u} - \bar{\boldsymbol{u}}\| \leq \frac{\|\boldsymbol{\mathscr{F}}\|_\infty}{\alpha}\|\boldsymbol{g} - \bar{\boldsymbol{g}}\| + \frac{\|\bar{\boldsymbol{g}}\|}{\alpha}\|\boldsymbol{\mathscr{F}} - \bar{\boldsymbol{\mathscr{F}}}\|_\infty, \tag{2.11}$$

$$\|(\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu, \boldsymbol{\lambda}_\tau - \bar{\boldsymbol{\lambda}}_\tau)\| \leq \frac{\|\boldsymbol{A}\|\|\boldsymbol{\mathscr{F}}\|_\infty}{\alpha\beta}\|\boldsymbol{g} - \bar{\boldsymbol{g}}\| + \frac{\|\boldsymbol{A}\|\|\bar{\boldsymbol{g}}\|}{\alpha\beta}\|\boldsymbol{\mathscr{F}} - \bar{\boldsymbol{\mathscr{F}}}\|_\infty. \tag{2.12}$$

In particular, if $\mathscr{F} = \bar{\mathscr{F}}$ *then*

$$\|\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu\| \le \frac{\|\boldsymbol{A}\|\|\mathscr{F}\|_\infty}{\alpha\beta}\|\boldsymbol{g} - \bar{\boldsymbol{g}}\|, \tag{2.13}$$

that is, $\Phi(\boldsymbol{f}, \mathscr{F}, .)$ *is Lipschitz continuous in* $\mathbb{R}_+^{n_c}$.

Proof. Inserting $\boldsymbol{v} := \bar{\boldsymbol{u}} \in \boldsymbol{K}$ in $(\mathcal{P}(\boldsymbol{f}, \mathscr{F}, \boldsymbol{g}))$ and $\boldsymbol{v} := \boldsymbol{u} \in \boldsymbol{K}$ in $(\mathcal{P}(\boldsymbol{f}, \bar{\mathscr{F}}, \bar{\boldsymbol{g}}))$, we have

$$\begin{aligned}
(\boldsymbol{A}\boldsymbol{u}, \bar{\boldsymbol{u}} - \boldsymbol{u}) + \sum_{1\le i\le n_c} \mathscr{F}_i g_i\big(|(\boldsymbol{B}_\tau\bar{\boldsymbol{u}})_i| - |(\boldsymbol{B}_\tau\boldsymbol{u})_i|\big) &\ge (\boldsymbol{f}, \bar{\boldsymbol{u}} - \boldsymbol{u}),\\
(\boldsymbol{A}\bar{\boldsymbol{u}}, \boldsymbol{u} - \bar{\boldsymbol{u}}) + \sum_{1\le i\le n_c} \bar{\mathscr{F}}_i \bar{g}_i\big(|(\boldsymbol{B}_\tau\boldsymbol{u})_i| - |(\boldsymbol{B}_\tau\bar{\boldsymbol{u}})_i|\big) &\ge (\boldsymbol{f}, \boldsymbol{u} - \bar{\boldsymbol{u}}).
\end{aligned}$$

Summing both inequalities and using (2.1) and (2.2), we arrive at

$$\begin{aligned}
&\alpha\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|^2\\
&\quad\le \big(\boldsymbol{A}(\bar{\boldsymbol{u}} - \boldsymbol{u}), \bar{\boldsymbol{u}} - \boldsymbol{u}\big) \le \sum_{1\le i\le n_c} (\mathscr{F}_i g_i - \bar{\mathscr{F}}_i\bar{g}_i)\big(|(\boldsymbol{B}_\tau\bar{\boldsymbol{u}})_i| - |(\boldsymbol{B}_\tau\boldsymbol{u})_i|\big)\\
&\quad\le \sum_{1\le i\le n_c} |\mathscr{F}_i(g_i - \bar{g}_i)|\,|(\boldsymbol{B}_\tau\bar{\boldsymbol{u}} - \boldsymbol{B}_\tau\boldsymbol{u})_i| + \sum_{1\le i\le n_c} |(\mathscr{F}_i - \bar{\mathscr{F}}_i)\bar{g}_i|\,|(\boldsymbol{B}_\tau\bar{\boldsymbol{u}} - \boldsymbol{B}_\tau\boldsymbol{u})_i|\\
&\quad\le \|\mathscr{F}\|_\infty\|\boldsymbol{g} - \bar{\boldsymbol{g}}\|\|\bar{\boldsymbol{u}} - \boldsymbol{u}\| + \|\mathscr{F} - \bar{\mathscr{F}}\|_\infty\|\bar{\boldsymbol{g}}\|\|\bar{\boldsymbol{u}} - \boldsymbol{u}\|,
\end{aligned}$$

which leads to (2.11).

The difference between the equalities in $(\mathcal{M}(\boldsymbol{f}, \mathscr{F}, \boldsymbol{g}))$ and $(\mathcal{M}(\boldsymbol{f}, \bar{\mathscr{F}}, \bar{\boldsymbol{g}}))$ results in

$$(\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu, \boldsymbol{B}_\nu\boldsymbol{v}) + (\boldsymbol{\lambda}_\tau - \bar{\boldsymbol{\lambda}}_\tau, \boldsymbol{B}_\tau\boldsymbol{v}) = \big(\boldsymbol{A}(\boldsymbol{u} - \bar{\boldsymbol{u}}), \boldsymbol{v}\big) \le \|\boldsymbol{A}\|\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|\|\boldsymbol{v}\|, \quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}}.$$

From this and (2.3), it follows that

$$\beta\|(\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu, \boldsymbol{\lambda}_\tau - \bar{\boldsymbol{\lambda}}_\tau)\| \le \sup_{\boldsymbol{0}\ne\boldsymbol{v}\in\mathbb{R}^{n_{\boldsymbol{u}}}} \frac{(\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu, \boldsymbol{B}_\nu\boldsymbol{v}) + (\boldsymbol{\lambda}_\tau - \bar{\boldsymbol{\lambda}}_\tau, \boldsymbol{B}_\tau\boldsymbol{v})}{\|\boldsymbol{v}\|} \le \|\boldsymbol{A}\|\|\boldsymbol{u} - \bar{\boldsymbol{u}}\|,$$

which together with (2.11) completes the proof. □

Let

$$B_{n_c}(R) := \{\boldsymbol{\mu} \in \mathbb{R}^{n_c} \mid \|\boldsymbol{\mu}\| \le R\}, \quad R > 0.$$

The following theorem guarantees existence and under an additional assumption also uniqueness of the fixed points characterizing the solutions of the discrete contact problem with Coulomb friction $(\mathcal{M}(\boldsymbol{f}, \mathscr{F}))$.

Theorem 2.1. *Suppose that (2.1) and (2.2) are satisfied. For any* $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ *and any* $\boldsymbol{\mathscr{F}} \in \mathbb{R}^{n_c}_+$, *there exists at least one fixed point of the mapping* $\Phi(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, .)$. *All the fixed points are contained in* $B_{n_c}(R)$ *with* $R = \|\boldsymbol{f}\|/\beta \cdot (\|\boldsymbol{A}\|/\alpha + 1)$. *In addition, the fixed point is unique provided that* $\|\boldsymbol{\mathscr{F}}\|_\infty < \alpha\beta/\|\boldsymbol{A}\|$.

Proof. Making use of Lemmas 2.1 and 2.2, this follows from the Brouwer and the Banach fixed-point theorems. □

Corollary 2.1. *Let (2.1) and (2.2) be satisfied. For any* $\boldsymbol{\mathscr{F}} \in \mathbb{R}^{n_c}_+$, $\|\boldsymbol{\mathscr{F}}\|_\infty < \alpha\beta/\|\boldsymbol{A}\|$, *and any* $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$ *has a unique solution* $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$. *In addition,* $-\boldsymbol{\lambda}_\nu = \boldsymbol{g}$ *where* $\boldsymbol{g} \in \mathbb{R}^{n_c}_+$ *is the limit of the following method of successive approximations:*

$$\left.\begin{array}{l} \text{Let } \boldsymbol{g}^0 \in \mathbb{R}^{n_c}_+ \text{ be arbitrarily chosen;} \\ \text{for } k = 0, 1, \dots \text{ set} \\ \quad \boldsymbol{g}^{k+1} := \Phi(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}^k). \end{array}\right\}$$

Confining ourselves to $\boldsymbol{\mathscr{F}}$ such that $\|\boldsymbol{\mathscr{F}}\|_\infty \leq \mathscr{F}_{\max}$ for an arbitrary $\mathscr{F}_{\max} \in [0, \alpha\beta/\|\boldsymbol{A}\|)$, we shall show that the solution of the contact problem with Coulomb friction is a Lipschitz continuous function of $\boldsymbol{\mathscr{F}}$. For this purpose, we define a mapping $\mathscr{S}_{\boldsymbol{f}} : \mathbb{R}^{n_c}_+ \to \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_c}$ for a *fixed* $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ by

$$\mathscr{S}_{\boldsymbol{f}}(\boldsymbol{\mathscr{F}}) := (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau), \quad \boldsymbol{\mathscr{F}} \in \mathbb{R}^{n_c}_+, \|\boldsymbol{\mathscr{F}}\|_\infty < \frac{\alpha\beta}{\|\boldsymbol{A}\|},$$

where $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ is the unique solution to $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$.

Theorem 2.2. *Let (2.1) and (2.2) be satisfied and let* $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ *be arbitrary. Then for any* $\mathscr{F}_{\max} \in [0, \alpha\beta/\|\boldsymbol{A}\|)$ *there exists* $\gamma > 0$ *such that*

$$\|\mathscr{S}_{\boldsymbol{f}}(\boldsymbol{\mathscr{F}}) - \mathscr{S}_{\boldsymbol{f}}(\bar{\boldsymbol{\mathscr{F}}})\| \leq \gamma\|\boldsymbol{\mathscr{F}} - \bar{\boldsymbol{\mathscr{F}}}\|_\infty, \quad \forall \boldsymbol{\mathscr{F}}, \bar{\boldsymbol{\mathscr{F}}} \in \mathbb{R}^{n_c}_+, \|\boldsymbol{\mathscr{F}}\|_\infty, \|\bar{\boldsymbol{\mathscr{F}}}\|_\infty \leq \mathscr{F}_{\max}.$$

Proof. For given $\mathscr{F}_{\max} \in [0, \alpha\beta/\|\boldsymbol{A}\|)$ and $\boldsymbol{\mathscr{F}}, \bar{\boldsymbol{\mathscr{F}}} \in \mathbb{R}^{n_c}_+$ with $\|\boldsymbol{\mathscr{F}}\|_\infty, \|\bar{\boldsymbol{\mathscr{F}}}\|_\infty \leq \mathscr{F}_{\max}$, let $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) := \mathscr{S}_{\boldsymbol{f}}(\boldsymbol{\mathscr{F}})$, $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau) := \mathscr{S}_{\boldsymbol{f}}(\bar{\boldsymbol{\mathscr{F}}})$. Further, let $\{\boldsymbol{g}^k\}$ and $\{\bar{\boldsymbol{g}}^k\}$ be sequences such that

$$\boldsymbol{g}^0 = \bar{\boldsymbol{g}}^0 \in \mathbb{R}^{n_c}_+ \text{ are arbitrarily chosen so that } \|\boldsymbol{g}^0\| \leq \frac{\|\boldsymbol{f}\|}{\beta}\Big(\frac{\|\boldsymbol{A}\|}{\alpha} + 1\Big), \tag{2.14}$$

$$\boldsymbol{g}^{k+1} := \Phi(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}^k), \quad \bar{\boldsymbol{g}}^{k+1} := \Phi(\boldsymbol{f}, \bar{\boldsymbol{\mathscr{F}}}, \bar{\boldsymbol{g}}^k), \quad k = 1, 2, \dots$$

From Corollary 2.1, we know

$$\lim_{k\to\infty} \boldsymbol{g}^k = -\boldsymbol{\lambda}_\nu, \quad \lim_{k\to\infty} \bar{\boldsymbol{g}}^k = -\bar{\boldsymbol{\lambda}}_\nu.$$

First, (2.12) and (2.14) give

$$\begin{aligned}\|\boldsymbol{g}^1-\bar{\boldsymbol{g}}^1\| &= \|\Phi(\boldsymbol{f},\boldsymbol{\mathscr{F}},\boldsymbol{g}^0)-\Phi(\boldsymbol{f},\bar{\boldsymbol{\mathscr{F}}},\boldsymbol{g}^0)\| \le \frac{\|\boldsymbol{A}\|\|\boldsymbol{g}^0\|}{\alpha\beta}\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty \\ &\le \frac{\|\boldsymbol{A}\|\|\boldsymbol{f}\|}{\alpha\beta^2}\Big(\frac{\|\boldsymbol{A}\|}{\alpha}+1\Big)\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty = c\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty \qquad (2.15)\end{aligned}$$

with $c := \|\boldsymbol{A}\|\|\boldsymbol{f}\|/(\alpha\beta^2)\cdot(\|\boldsymbol{A}\|/\alpha+1)$. From (2.10), (2.12) and (2.15), it follows:

$$\begin{aligned}\|\boldsymbol{g}^2-\bar{\boldsymbol{g}}^2\| &= \|\Phi(\boldsymbol{f},\boldsymbol{\mathscr{F}},\boldsymbol{g}^1)-\Phi(\boldsymbol{f},\bar{\boldsymbol{\mathscr{F}}},\bar{\boldsymbol{g}}^1)\| \\ &\le \frac{\|\boldsymbol{A}\|\|\boldsymbol{\mathscr{F}}\|_\infty}{\alpha\beta}\|\boldsymbol{g}^1-\bar{\boldsymbol{g}}^1\| + \frac{\|\boldsymbol{A}\|\|\bar{\boldsymbol{g}}^1\|}{\alpha\beta}\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty \\ &\le q\|\boldsymbol{g}^1-\bar{\boldsymbol{g}}^1\| + c\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty \le (cq+c)\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty,\end{aligned}$$

where $q := \mathscr{F}_{\max}\|\boldsymbol{A}\|/(\alpha\beta) < 1$ by assumption. Thus by induction,

$$\begin{aligned}\|\boldsymbol{g}^{k+1}-\bar{\boldsymbol{g}}^{k+1}\| &\le c\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty + q\|\boldsymbol{g}^k-\bar{\boldsymbol{g}}^k\| \\ &\le c\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty + q(c+cq+\dots+cq^{k-1})\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty \\ &\le \frac{c}{1-q}\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty.\end{aligned}$$

Letting $k\to\infty$, we obtain

$$\|\boldsymbol{\lambda}_\nu-\bar{\boldsymbol{\lambda}}_\nu\| \le \frac{c}{1-q}\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty. \qquad (2.16)$$

Taking (2.11) with $\boldsymbol{g} := -\boldsymbol{\lambda}_\nu$ and $\bar{\boldsymbol{g}} := -\bar{\boldsymbol{\lambda}}_\nu$, using (2.16) and Theorem 2.1, we see that

$$\begin{aligned}\|\boldsymbol{u}-\bar{\boldsymbol{u}}\| &\le \frac{\|\boldsymbol{\mathscr{F}}\|_\infty}{\alpha}\|\boldsymbol{\lambda}_\nu-\bar{\boldsymbol{\lambda}}_\nu\| + \frac{\|\bar{\boldsymbol{\lambda}}_\nu\|}{\alpha}\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty \\ &\le \Big(\frac{c\mathscr{F}_{\max}}{\alpha(1-q)} + \frac{\|\boldsymbol{f}\|}{\alpha\beta}\Big(\frac{\|\boldsymbol{A}\|}{\alpha}+1\Big)\Big)\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty.\end{aligned}$$

Finally, (2.12) with $\boldsymbol{g} := -\boldsymbol{\lambda}_\nu$ and $\bar{\boldsymbol{g}} := -\bar{\boldsymbol{\lambda}}_\nu$ together with Theorem 2.1 and (2.16) ensures

$$\|\boldsymbol{\lambda}_\tau-\bar{\boldsymbol{\lambda}}_\tau\| \le q\|\boldsymbol{\lambda}_\nu-\bar{\boldsymbol{\lambda}}_\nu\| + c\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty \le \frac{c}{1-q}\|\boldsymbol{\mathscr{F}}-\bar{\boldsymbol{\mathscr{F}}}\|_\infty.$$

□

In what follows, we shall focus on local behaviour of solutions to the discrete contact problems with Coulomb friction. To this end, we restrict ourselves to $\boldsymbol{\mathscr{F}}$ with positive components solely. On the other hand, *no upper* bounds will be imposed, that is, $\boldsymbol{\mathscr{F}}$ will belong to the set $\mathscr{A}$ defined by

$$\mathscr{A} := \{\boldsymbol{\mathscr{F}} \in \mathbb{R}^{n_c} \,|\, \mathscr{F}_i > 0,\ \forall\, i = 1, \ldots, n_c\}.$$

Furthermore, we introduce alternative formulation of the discrete problems with given and Coulomb friction in which the Lagrange multiplier set $\boldsymbol{\Lambda}_\tau(.)$ does not depend on $\boldsymbol{\mathscr{F}}$.

Let $\boldsymbol{\mathscr{F}} \in \mathscr{A}$, $\boldsymbol{g} \in \mathbb{R}^{n_c}_+$ be given and set

$$\boldsymbol{\Lambda}_\tau(\boldsymbol{g}) := \{\boldsymbol{\mu}_\tau \in \mathbb{R}^{n_c} \,|\, |\mu_{t,i}| \leq g_i,\ \forall\, i = 1, \ldots, n_c\}.$$

As an alternative to $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$, a mixed formulation of the problem with given friction reads as follows:

$$\left.\begin{array}{l}
\text{Find } (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_\tau(\boldsymbol{g}) \text{ such that}\\
(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{v}) + (\boldsymbol{\mathcal{F}}\boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}},\\
(\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{u}) + (\boldsymbol{\mathcal{F}}(\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau), \boldsymbol{B}_\tau \boldsymbol{u}) \geq 0,\\
\hfill \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau) \in \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_\tau(\boldsymbol{g}),
\end{array}\right\} \quad (\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$$

where $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(\boldsymbol{\mathscr{F}}) := \mathbf{Diag}(\mathscr{F}_1, \ldots, \mathscr{F}_{n_c}) \in \mathbb{M}^{n_c}$.

Clearly, the triplet $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solves $(\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$ iff $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\mathcal{F}}\boldsymbol{\lambda}_\tau)$ is a solution of $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$. Hence, the existence and the uniqueness of the solution to $(\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$ is still guaranteed.

In the same spirit, we rewrite the discrete contact problem with Coulomb friction as follows:

$$\left.\begin{array}{l}
\text{Find } (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu) \text{ such that}\\
(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{v}) + (\boldsymbol{\mathcal{F}}\boldsymbol{\lambda}_\tau, \boldsymbol{B}_\tau \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}},\\
(\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu, \boldsymbol{B}_\nu \boldsymbol{u}) + (\boldsymbol{\mathcal{F}}(\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau), \boldsymbol{B}_\tau \boldsymbol{u}) \geq 0,\\
\hfill \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_\tau) \in \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu).
\end{array}\right\} \quad (\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$$

Due to the one-to-one correspondence between the solutions to $(\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$ and $(\mathcal{M}(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$, the existence and uniqueness results remain valid.

Next, we derive another equivalent formulation of the contact problem with Coulomb friction. Let $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ be *fixed* and let $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ be the corresponding solution of $(\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$. The inequality in $(\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$ can be replaced by

$$-\boldsymbol{B}_\nu \boldsymbol{u} \in N_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu) \quad \text{and} \quad -\boldsymbol{\mathcal{F}}\boldsymbol{B}_\tau \boldsymbol{u} \in N_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau),$$

where $N_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\mu})$, $N_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\mu})$ denote the normal cones of $\boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)$ at $\boldsymbol{\mu} \in \mathbb{R}^{n_c}$, respectively. Consequently, the solution of $(\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}))$ can be characterized as a solution to the following system of generalized equations:

$$\left.\begin{array}{l}\text{Find } \boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \text{ such that}\\ \mathbf{0} \in \boldsymbol{C_f}(\mathscr{F}, \boldsymbol{y}) + \boldsymbol{Q}(\boldsymbol{y}),\end{array}\right\} \tag{2.17}$$

where $\boldsymbol{C_f} : \mathscr{A} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \to \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ and $\boldsymbol{Q} : \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \rightrightarrows \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ are the single-valued continuously differentiable function and the set-valued mapping defined by

$$\boldsymbol{C_f}(\mathscr{F}, \boldsymbol{y}) := \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{B}_\nu^T & -\boldsymbol{B}_\tau^T \boldsymbol{\mathcal{F}} \\ \boldsymbol{B}_\nu & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\mathcal{F}} \boldsymbol{B}_\tau & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda}_\nu \\ \boldsymbol{\lambda}_\tau \end{pmatrix} - \begin{pmatrix} \boldsymbol{f} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

$$\boldsymbol{Q}(\boldsymbol{y}) := \begin{pmatrix} \mathbf{0} \\ N_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu) \\ N_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau) \end{pmatrix},$$

$$\mathscr{F} \in \mathscr{A},\ \boldsymbol{y} := (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c},$$

respectively.

Interpreting $\mathscr{F}$ as a perturbation parameter and following the technique used in [4], we shall analyse this system according to [55] (see also [15]):

Let $\mathscr{F}^0 \in \mathscr{A}$ be a reference point and let $\boldsymbol{y}^0 \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ be such that

$$\mathbf{0} \in \boldsymbol{C_f}(\mathscr{F}^0, \boldsymbol{y}^0) + \boldsymbol{Q}(\boldsymbol{y}^0).$$

Let us define the multi-valued mappings $\mathscr{S}_{\boldsymbol{f}}^* : \mathscr{A} \rightrightarrows \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ and $\Sigma_{\boldsymbol{f}} : \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \rightrightarrows \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ by

$$\mathscr{S}_{\boldsymbol{f}}^*(\mathscr{F}) := \{\boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \mid \mathbf{0} \in \boldsymbol{C_f}(\mathscr{F}, \boldsymbol{y}) + \boldsymbol{Q}(\boldsymbol{y})\}, \quad \mathscr{F} \in \mathscr{A}, \tag{2.18}$$

$$\Sigma_{\boldsymbol{f}}(\boldsymbol{\xi}) := \{\boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \mid \boldsymbol{\xi} \in \boldsymbol{C_f}(\mathscr{F}^0, \boldsymbol{y}^0) + \boldsymbol{\nabla_y C_f}(\mathscr{F}^0, \boldsymbol{y}^0)(\boldsymbol{y} - \boldsymbol{y}^0) + \boldsymbol{Q}(\boldsymbol{y})\},\quad \boldsymbol{\xi} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c},$$

where $\boldsymbol{\nabla_y C_f}(\mathscr{F}^0, \boldsymbol{y}^0)$ stands for the gradient of $\boldsymbol{C_f}$ with respect to $\boldsymbol{y}$ at $(\mathscr{F}^0, \boldsymbol{y}^0)$. In other words, $\mathscr{S}_{\boldsymbol{f}}^*(\mathscr{F})$ is the solution set of (2.17) for a given coefficient $\mathscr{F} \in \mathscr{A}$ and the load vector $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$. Furthermore, $\Sigma_{\boldsymbol{f}}(\boldsymbol{\xi})$ is the solution set to a perturbed generalized equation obtained by partial linearization of $\boldsymbol{C_f}(\mathscr{F}, \boldsymbol{y})$ in (2.17) with respect to the second variable around the reference point $(\mathscr{F}^0, \boldsymbol{y}^0)$.

The following generalization of the implicit-function theorem holds ([15, Theorem 5.1]).

Theorem 2.3. *Assume that there exist: a single-valued Lipschitz continuous function $\phi_{\boldsymbol{f}}$ from a neighbourhood W of $\mathbf{0} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ into $\mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ and a neighbourhood $\tilde{Y}$ of $\boldsymbol{y}^0$ such that*

$$\phi_{\boldsymbol{f}}(\mathbf{0}) = \boldsymbol{y}^0 \qquad \textit{and} \qquad \phi_{\boldsymbol{f}}(\boldsymbol{\xi}) = \Sigma_{\boldsymbol{f}}(\boldsymbol{\xi}) \cap \tilde{Y}, \quad \forall \boldsymbol{\xi} \in W.$$

Then there exist neighbourhoods U and Y of $\mathscr{F}^0$ and $\boldsymbol{y}^0$, respectively, and a single-valued Lipschitz continuous map $\sigma_{\boldsymbol{f}} : U \to Y$ with

$$\sigma_{\boldsymbol{f}}(\mathscr{F}^0) = \boldsymbol{y}^0 \qquad \textit{and} \qquad \sigma_{\boldsymbol{f}}(\mathscr{F}) = \mathscr{S}^*_{\boldsymbol{f}}(\mathscr{F}) \cap Y, \quad \forall \mathscr{F} \in U.$$

Let us mention that if $\boldsymbol{Q} = \boldsymbol{0}$, the single-valuedness of $\Sigma_{\boldsymbol{f}}$ in a neighbourhood of $\boldsymbol{0}$ in the assumption of the previous theorem corresponds to the non-singularity of $\boldsymbol{\nabla_y C_f}(\mathscr{F}^0, \boldsymbol{y}^0)$. Hence, Theorem 2.3 is a generalization of the classical implicit-function theorem.

Next, we analyse the assumptions of the stated theorem. Obviously, $\Sigma_{\boldsymbol{f}}(\boldsymbol{\xi})$ with $\boldsymbol{\xi} := (\boldsymbol{\xi}_u, \boldsymbol{\xi}_\nu, \boldsymbol{\xi}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ is the set of all $\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ satisfying

$$\left.\begin{aligned} &\boldsymbol{0} = \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau^T \boldsymbol{\mathcal{F}}^0 \boldsymbol{\lambda}_\tau - \boldsymbol{f} - \boldsymbol{\xi}_u, \\ &\boldsymbol{0} \in \boldsymbol{B}_\nu \boldsymbol{u} - \boldsymbol{\xi}_\nu + N_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu), \\ &\boldsymbol{0} \in \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \boldsymbol{u} - \boldsymbol{\xi}_\tau + N_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau), \end{aligned}\right\} \tag{2.19}$$

where $\boldsymbol{\mathcal{F}}^0 = \boldsymbol{\mathcal{F}}^0(\mathscr{F}^0) := \mathbf{Diag}(\mathscr{F}_1^0, \ldots, \mathscr{F}_{n_c}^0)$. Substitution

$$\boldsymbol{w} := \boldsymbol{u} - \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_\tau \end{pmatrix},$$

where $\left(\begin{smallmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{smallmatrix}\right)^+$ denotes the Moore-Penrose pseudo-inverse of $\left(\begin{smallmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{smallmatrix}\right)$, leads to the following transformation of (2.19) provided that (2.2) is fulfilled:

$$\left.\begin{aligned} &\boldsymbol{0} = \boldsymbol{A}\boldsymbol{w} - \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau^T \boldsymbol{\mathcal{F}}^0 \boldsymbol{\lambda}_\tau - \boldsymbol{f} + \boldsymbol{A} \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_\tau \end{pmatrix} - \boldsymbol{\xi}_u, \\ &\boldsymbol{0} \in \boldsymbol{B}_\nu \boldsymbol{w} + N_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu), \\ &\boldsymbol{0} \in \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \boldsymbol{w} + N_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau). \end{aligned}\right\} \tag{2.20}$$

Indeed,

$$\begin{aligned} \begin{pmatrix} \boldsymbol{B}_\nu \boldsymbol{w} \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \boldsymbol{w} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{pmatrix} \boldsymbol{w} = \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{pmatrix} \boldsymbol{u} - \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_\tau \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{B}_\nu \boldsymbol{u} - \boldsymbol{\xi}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \boldsymbol{u} - \boldsymbol{\xi}_\tau \end{pmatrix}. \end{aligned}$$

Comparing (2.20) with (2.17), one can readily seen that the triplet $(\boldsymbol{w}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ satisfies (2.20) iff it is a solution to (2.17) with the coefficient $\mathscr{F}^0$ and a new load vector $\boldsymbol{\xi}_f$,

$$\boldsymbol{\xi}_f := \boldsymbol{f} - \boldsymbol{A} \begin{pmatrix} \boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0 \boldsymbol{B}_\tau \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_\tau \end{pmatrix} + \boldsymbol{\xi}_u$$

being a perturbation of $\boldsymbol{f}$. That is,

$$(\boldsymbol{w}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathscr{S}^*_{\boldsymbol{\xi}_f}(\boldsymbol{\mathscr{F}}^0),$$

where $\mathscr{S}^*_{\boldsymbol{\xi}_f}(\boldsymbol{\mathscr{F}}^0)$ is defined by (2.18) with $\boldsymbol{f} := \boldsymbol{\xi}_f$ and $\boldsymbol{\mathscr{F}} := \boldsymbol{\mathscr{F}}^0$.

To summarize the result, we now introduce for a *fixed* $\boldsymbol{\mathscr{F}} \in \mathscr{A}$ the set-valued mapping $S^*_{\boldsymbol{\mathscr{F}}} : \mathbb{R}^{n_{\boldsymbol{u}}} \rightrightarrows \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ by

$$S^*_{\boldsymbol{\mathscr{F}}}(\boldsymbol{f}) := \{\boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \,|\, \boldsymbol{0} \in \boldsymbol{C}_{\boldsymbol{f}}(\boldsymbol{\mathscr{F}}, \boldsymbol{y}) + \boldsymbol{Q}(\boldsymbol{y})\}, \quad \boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}.$$

Theorem 2.4. *Let us suppose that (2.2) is valid and $S^*_{\boldsymbol{\mathscr{F}}^0}$ has a local Lipschitz continuous branch containing $\boldsymbol{y}^0$ in a vicinity of $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, that is, there exist: a single-valued Lipschitz continuous function $\varphi_{\boldsymbol{\mathscr{F}}^0}$ from a neighbourhood O of $\boldsymbol{f}$ into $\mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ and a neighbourhood $\hat{Y}$ of $\boldsymbol{y}^0$ such that*

$$\varphi_{\boldsymbol{\mathscr{F}}^0}(\boldsymbol{f}) = \boldsymbol{y}^0 \qquad \text{and} \qquad \varphi_{\boldsymbol{\mathscr{F}}^0}(\boldsymbol{\xi}_f) = S^*_{\boldsymbol{\mathscr{F}}^0}(\boldsymbol{\xi}_f) \cap \hat{Y}, \quad \forall\, \boldsymbol{\xi}_f \in O.$$

Then there are neighbourhoods U, Y of $\boldsymbol{\mathscr{F}}^0$, $\boldsymbol{y}^0$, respectively, and a single-valued Lipschitz continuous function $\sigma_{\boldsymbol{f}} : U \to Y$ satisfying

$$\sigma_{\boldsymbol{f}}(\boldsymbol{\mathscr{F}}^0) = \boldsymbol{y}^0 \qquad \text{and} \qquad \sigma_{\boldsymbol{f}}(\boldsymbol{\mathscr{F}}) = \mathscr{S}^*_{\boldsymbol{f}}(\boldsymbol{\mathscr{F}}) \cap Y, \quad \forall\, \boldsymbol{\mathscr{F}} \in U.$$

Proof. One can easily verify the assumptions of Theorem 2.3 for

$$\phi_{\boldsymbol{f}}(\boldsymbol{\xi}) := \varphi_{\boldsymbol{\mathscr{F}}^0}\!\left(\boldsymbol{f} - \boldsymbol{A}\begin{pmatrix}\boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0\boldsymbol{B}_\tau\end{pmatrix}^{+}\begin{pmatrix}\boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_\tau\end{pmatrix} + \boldsymbol{\xi}_u\right) + \begin{pmatrix}\begin{pmatrix}\boldsymbol{B}_\nu \\ \boldsymbol{\mathcal{F}}^0\boldsymbol{B}_\tau\end{pmatrix}^{+}\begin{pmatrix}\boldsymbol{\xi}_\nu \\ \boldsymbol{\xi}_\tau\end{pmatrix} \\ \boldsymbol{0} \\ \boldsymbol{0}\end{pmatrix},$$

$$\boldsymbol{\xi} = (\boldsymbol{\xi}_u, \boldsymbol{\xi}_\nu, \boldsymbol{\xi}_\tau) \in W,$$

with some sufficiently small neighbourhood W of $\boldsymbol{0} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$. □

The previous theorem says that the analysis of local dependence of solutions to the discrete contact problem with Coulomb friction on $\boldsymbol{\mathscr{F}}$ can be converted to the analysis of local dependence of the solutions on $\boldsymbol{f}$. For this reason, we shall focus on the study of the set-valued mapping $\boldsymbol{f} \mapsto S^*_{\boldsymbol{\mathscr{F}}}(\boldsymbol{f})$, $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, for $\boldsymbol{\mathscr{F}} \in \mathscr{A}$ *fixed* hereafter.

To start with, using the same technique as in the proof of Lemma 2.2, one can get the following auxiliary result.

Lemma 2.3. *Let (2.1) and (2.2) be satisfied and let* $\boldsymbol{\mathscr{F}} = (\mathscr{F}_i) \in \mathscr{A}$, $\boldsymbol{f}, \bar{\boldsymbol{f}} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ *and* $\boldsymbol{g}, \bar{\boldsymbol{g}} \in \mathbb{R}_+^{n_c}$ *be arbitrary. If one denotes the unique solutions of* $(\mathcal{M}^*(\boldsymbol{f}, \boldsymbol{\mathscr{F}}, \boldsymbol{g}))$, $(\mathcal{M}^*(\bar{\boldsymbol{f}}, \boldsymbol{\mathscr{F}}, \bar{\boldsymbol{g}}))$ *by* $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ *and* $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\lambda}}_\nu, \bar{\boldsymbol{\lambda}}_\tau)$, *respectively, then*

$$\|\boldsymbol{u} - \bar{\boldsymbol{u}}\| \le \frac{1}{\alpha}\|\boldsymbol{f} - \bar{\boldsymbol{f}}\| + \frac{\|\boldsymbol{\mathscr{F}}\|_\infty}{\alpha}\|\boldsymbol{g} - \bar{\boldsymbol{g}}\|, \tag{2.21}$$

$$\|\boldsymbol{\lambda}_\nu - \bar{\boldsymbol{\lambda}}_\nu\| \le \frac{1}{\beta}\Big(\frac{\|\boldsymbol{A}\|}{\alpha} + 1\Big)\|\boldsymbol{f} - \bar{\boldsymbol{f}}\| + \frac{\|\boldsymbol{A}\|\|\boldsymbol{\mathscr{F}}\|_\infty}{\alpha\beta}\|\boldsymbol{g} - \bar{\boldsymbol{g}}\|, \tag{2.22}$$

$$\|\boldsymbol{\lambda}_\tau - \bar{\boldsymbol{\lambda}}_\tau\| \le \frac{1}{\beta\mathscr{F}_{\min}}\Big(\frac{\|\boldsymbol{A}\|}{\alpha} + 1\Big)\|\boldsymbol{f} - \bar{\boldsymbol{f}}\| + \frac{\|\boldsymbol{A}\|\|\boldsymbol{\mathscr{F}}\|_\infty}{\alpha\beta\mathscr{F}_{\min}}\|\boldsymbol{g} - \bar{\boldsymbol{g}}\|, \tag{2.23}$$

where $\mathscr{F}_{\min} := \min_{i=1,\dots,n_c} \mathscr{F}_i$.

Next, we shall suppose for a moment that all components of the fixed $\boldsymbol{\mathscr{F}} \in \mathscr{A}$ are strictly bounded by $\alpha\beta/\|\boldsymbol{A}\|$ from above, that is, $\boldsymbol{\mathscr{F}} \in \mathscr{B}$ with

$$\mathscr{B} := \Big\{\boldsymbol{\mathscr{F}} \in \mathbb{R}^{n_c} \,\Big|\, 0 < \mathscr{F}_i < \frac{\alpha\beta}{\|\boldsymbol{A}\|},\ \forall\, i = 1, \dots, n_c\Big\}.$$

Then $S^*_{\boldsymbol{\mathscr{F}}}$ is single-valued on $\mathbb{R}^{n_{\boldsymbol{u}}}$ for any such $\boldsymbol{\mathscr{F}}$ according to Corollary 2.1. Owing to the previous lemma, it can be proved in a similar way as in Theorem 2.2 that $S^*_{\boldsymbol{\mathscr{F}}}$ is even Lipschitz continuous on $\mathbb{R}^{n_{\boldsymbol{u}}}$.

Theorem 2.5. *Assume that (2.1) and (2.2) are satisfied and* $\boldsymbol{\mathscr{F}} \in \mathscr{B}$ *is arbitrary but fixed. Then there exists* $\hat{\gamma}_{\boldsymbol{\mathscr{F}}} > 0$ *such that*

$$\|S^*_{\boldsymbol{\mathscr{F}}}(\boldsymbol{f}) - S^*_{\boldsymbol{\mathscr{F}}}(\bar{\boldsymbol{f}})\| \le \hat{\gamma}_{\boldsymbol{\mathscr{F}}}\|\boldsymbol{f} - \bar{\boldsymbol{f}}\|, \quad \forall\, \boldsymbol{f}, \bar{\boldsymbol{f}} \in \mathbb{R}^{n_{\boldsymbol{u}}}.$$

From here, we arrive as an illustration of application of Theorem 2.4 at a result, which is analogous to Theorem 2.2.

Corollary 2.2. *Let (2.1) and (2.2) hold and let* $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ *be arbitrary but fixed. Then* $\mathscr{S}^*_{\boldsymbol{f}}$ *is locally Lipschitz continuous in* $\mathscr{B}$, *that is, for any* $\boldsymbol{\mathscr{F}}^0 \in \mathscr{B}$ *there exist a neighbourhood* $U \subset \mathscr{B}$ *of* $\boldsymbol{\mathscr{F}}^0$ *and* $\gamma_{\boldsymbol{\mathscr{F}}^0} > 0$ *such that*

$$\|\mathscr{S}^*_{\boldsymbol{f}}(\boldsymbol{\mathscr{F}}) - \mathscr{S}^*_{\boldsymbol{f}}(\bar{\boldsymbol{\mathscr{F}}})\| \le \gamma_{\boldsymbol{\mathscr{F}}^0}\|\boldsymbol{\mathscr{F}} - \bar{\boldsymbol{\mathscr{F}}}\|_\infty, \quad \forall\, \boldsymbol{\mathscr{F}}, \bar{\boldsymbol{\mathscr{F}}} \in U.$$

At the rest of this section, we shall suppose again that $\boldsymbol{\mathscr{F}} \in \mathscr{A}$, i.e. no upper bounds on $\boldsymbol{\mathscr{F}}$ are imposed. Our aim is to analyse the mapping $\boldsymbol{f} \mapsto S^*_{\boldsymbol{\mathscr{F}}}(\boldsymbol{f})$, $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, for such $\boldsymbol{\mathscr{F}}$ fixed with the aid of the implicit-function theorem for piecewise differentiable functions presented in [56] (see also Appendix).

For this purpose, we formulate the discrete contact problem with Coulomb friction as a system of non-smooth equations. Let $r > 0$ be an arbitrary parameter and

$\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$. If $\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in S^*_{\mathscr{F}}(\boldsymbol{f})$, that is, $(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solves $(\mathcal{M}^*(\boldsymbol{f}, \mathscr{F}))$, the inequality in $(\mathcal{M}^*(\boldsymbol{f}, \mathscr{F}))$ multiplied by $(-r)$ gives:

$$\left.\begin{array}{ll} (\mu_{\nu,i} - \lambda_{\nu,i})((\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i - \lambda_{\nu,i}) \le 0, & i = 1, \dots, n_c,\ \forall\, \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu, \\ (\mu_{\tau,i} - \lambda_{\tau,i})((\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i - \lambda_{\tau,i}) \le 0, & i = 1, \dots, n_c,\ \forall\, \boldsymbol{\mu}_\tau \in \boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu). \end{array}\right\} \quad (2.24)$$

Since $\boldsymbol{\lambda}_\nu \in \boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\lambda}_\tau \in \boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)$, the equivalent expression of (2.24) is

$$\boldsymbol{\lambda}_\nu = \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u}), \quad \boldsymbol{\lambda}_\tau = \boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u}).$$

Here, $\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu} : \mathbb{R}^{n_c} \to \boldsymbol{\Lambda}_\nu$ and $\boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)} : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ are vector functions with the components

$$(\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu})_i(\boldsymbol{\mu}) := P_{\mathbb{R}_-}(\mu_i), \quad i = 1, \dots, n_c,\ \boldsymbol{\mu} \in \mathbb{R}^{n_c},$$

$$(\boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)})_i(\boldsymbol{\mu}) := \begin{cases} P_{[\lambda_{\nu,i}, -\lambda_{\nu,i}]}(\mu_i) & \text{if } \lambda_{\nu,i} \le 0, \\ -P_{[-\lambda_{\nu,i}, \lambda_{\nu,i}]}(\mu_i) & \text{if } \lambda_{\nu,i} > 0, \end{cases} \quad i = 1, \dots, n_c,\ \boldsymbol{\mu} \in \mathbb{R}^{n_c},$$

where $P_{\mathbb{R}_-}$, $P_{[-\zeta,\zeta]}$ stand for the projections of $\mathbb{R}$ onto $\mathbb{R}_-$ and $[-\zeta, \zeta]$, $\zeta \ge 0$, respectively. It is readily seen that $\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}$ is the projection of $\mathbb{R}^{n_c}$ onto $\boldsymbol{\Lambda}_\nu$ and $\boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)}$ is the projection of $\mathbb{R}^{n_c}$ onto $\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)$ whenever $\boldsymbol{\lambda}_\nu \in \boldsymbol{\Lambda}_\nu$.

Let $\boldsymbol{\mathcal{H}} : \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \to \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ be defined by

$$\boldsymbol{\mathcal{H}}(\boldsymbol{f}, \boldsymbol{y}) := \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau^T \boldsymbol{\mathcal{F}} \boldsymbol{\lambda}_\tau - \boldsymbol{f} \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u}) \\ \boldsymbol{\lambda}_\tau - \boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(-\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u}) \end{pmatrix}, \quad \boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}.$$

Then for any $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, $\boldsymbol{y} \in S^*_{\mathscr{F}}(\boldsymbol{f})$ if and only if $\boldsymbol{y}$ solves the following problem:

$$\left.\begin{array}{l} \text{Find } \boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \text{ such that} \\ \boldsymbol{\mathcal{H}}(\boldsymbol{f}, \boldsymbol{y}) = \boldsymbol{0}. \end{array}\right\} \quad (2.25)$$

We shall view this problem as an equation parametrized by $\boldsymbol{f}$ and we shall verify the assumptions of the above mentioned implicit-function theorem. First, we shall demonstrate that $\boldsymbol{\mathcal{H}}$ a piecewise differentiable function. Obviously, it is continuous. Moreover, let $(\boldsymbol{f}^0, \boldsymbol{y}^0) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$, $\boldsymbol{y}^0 := (\boldsymbol{u}^0, \boldsymbol{\lambda}_\nu^0, \boldsymbol{\lambda}_\tau^0)$, be an arbitrarily chosen vector. To construct a set of selection functions for $\boldsymbol{\mathcal{H}}$ at $(\boldsymbol{f}^0, \boldsymbol{y}^0)$, we introduce in a similar way as in [5] the following index sets (see Fig. 2.2):

$$\begin{aligned} I_\nu^s(\boldsymbol{y}^0) &:= \{i \in \{1, \dots, n_c\} \,|\, (\lambda_\nu^0 - r\boldsymbol{B}_\nu \boldsymbol{u}^0)_i < 0\}, \\ I_\nu^0(\boldsymbol{y}^0) &:= \{i \in \{1, \dots, n_c\} \,|\, (\lambda_\nu^0 - r\boldsymbol{B}_\nu \boldsymbol{u}^0)_i > 0\}, \\ I_\nu^w(\boldsymbol{y}^0) &:= \{i \in \{1, \dots, n_c\} \,|\, (\lambda_\nu^0 - r\boldsymbol{B}_\nu \boldsymbol{u}^0)_i = 0\}, \\ I_\tau^+(\boldsymbol{y}^0) &:= \{i \in \{1, \dots, n_c\} \,|\, (\lambda_\tau^0 - r\boldsymbol{B}_\tau \boldsymbol{u}^0)_i < -|\lambda_{\nu,i}^0|\}, \\ I_\tau^-(\boldsymbol{y}^0) &:= \{i \in \{1, \dots, n_c\} \,|\, (\lambda_\tau^0 - r\boldsymbol{B}_\tau \boldsymbol{u}^0)_i > |\lambda_{\nu,i}^0|\}, \\ I_\tau^s(\boldsymbol{y}^0) &:= \{i \in \{1, \dots, n_c\} \,|\, |(\lambda_\tau^0 - r\boldsymbol{B}_\tau \boldsymbol{u}^0)_i| < |\lambda_{\nu,i}^0|\}, \end{aligned}$$

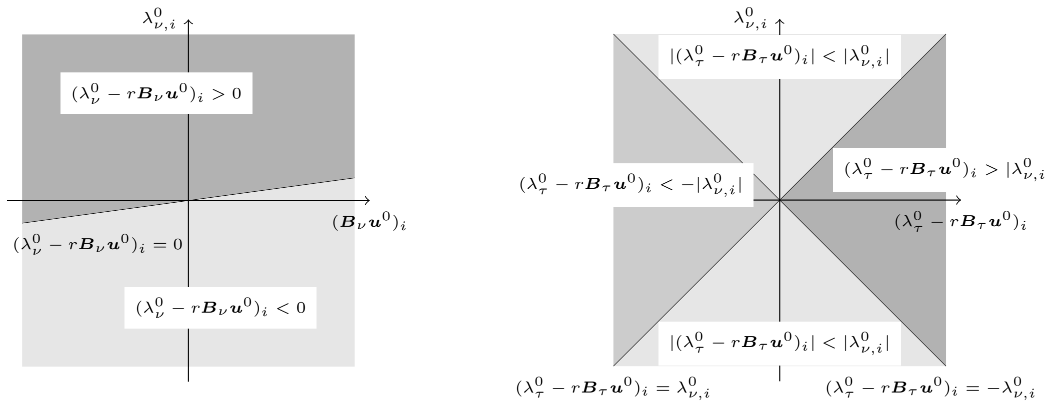


Figure 2.2: Partitions corresponding to the index sets

$$
\begin{aligned}
I_\tau^{w+}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,n_c\} \,|\, (\lambda^0_\tau - r\boldsymbol{B}_\tau \boldsymbol{u}^0)_i = \lambda^0_{\nu,i}\},\\
I_\tau^{w-}(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,n_c\} \,|\, (\lambda^0_\tau - r\boldsymbol{B}_\tau \boldsymbol{u}^0)_i = -\lambda^0_{\nu,i}\},\\
J^-(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,n_c\} \,|\, \lambda^0_{\nu,i} < 0\},\\
J^0(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,n_c\} \,|\, \lambda^0_{\nu,i} = 0\},\\
J^+(\boldsymbol{y}^0) &:= \{i \in \{1,\dots,n_c\} \,|\, \lambda^0_{\nu,i} > 0\}.
\end{aligned}
$$

Remark 2.1. To interpret the sets defined above, let us suppose for a moment that $\boldsymbol{y}^0$ solves (2.25) with the load vector $\boldsymbol{f}^0$. Then

$$
\begin{aligned}
i \in I_\nu^s(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\nu \boldsymbol{u}^0)_i = 0 \;\&\; \lambda^0_{\nu,i} < 0 \quad &&\text{(strong contact)},\\
i \in I_\nu^0(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\nu \boldsymbol{u}^0)_i < 0 \;\&\; \lambda^0_{\nu,i} = 0 \quad &&\text{(no contact)},\\
i \in I_\nu^w(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\nu \boldsymbol{u}^0)_i = \lambda^0_{\nu,i} = 0 \quad &&\text{(weak contact)}.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
&\left.\begin{aligned}
i \in I_\tau^+(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\tau \boldsymbol{u}^0)_i > 0 \;\&\; \lambda^0_{\tau,i} = \lambda^0_{\nu,i},\\
i \in I_\tau^-(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\tau \boldsymbol{u}^0)_i < 0 \;\&\; \lambda^0_{\tau,i} = -\lambda^0_{\nu,i}
\end{aligned}\right\} \quad \text{(slip)},\\
&\;\; i \in I_\tau^s(\boldsymbol{y}^0) \iff (\boldsymbol{B}_\tau \boldsymbol{u}^0)_i = 0 \;\&\; |\lambda^0_{\tau,i}| < -\lambda^0_{\nu,i} \quad \text{(strong stick)},\\
&\left.\begin{aligned}
i \in I_\tau^{w+}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\tau \boldsymbol{u}^0)_i = 0 \;\&\; \lambda^0_{\tau,i} = \lambda^0_{\nu,i},\\
i \in I_\tau^{w-}(\boldsymbol{y}^0) &\iff (\boldsymbol{B}_\tau \boldsymbol{u}^0)_i = 0 \;\&\; \lambda^0_{\tau,i} = -\lambda^0_{\nu,i}
\end{aligned}\right\} \quad \text{(weak stick)}.
\end{aligned}
$$

Let $I_\nu^{w-} \subset I_\nu^w(\boldsymbol{y}^0)$, $I_\tau^{w++} \subset I_\tau^{w+}(\boldsymbol{y}^0)$ and $I_\tau^{w--} \subset I_\tau^{w-}(\boldsymbol{y}^0)$ be arbitrary sets. For such sets, we shall denote

$$I_\nu^{w+} := I_\nu^w(\boldsymbol{y}^0) \setminus I_\nu^{w-}, \quad I_\tau^{w+-} := I_\tau^{w+}(\boldsymbol{y}^0) \setminus I_\tau^{w++}, \quad I_\tau^{w-+} := I_\tau^{w-}(\boldsymbol{y}^0) \setminus I_\tau^{w--}.$$

Furthermore, we shall associate with them the sets

$$\begin{aligned}
&\pi^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} \\
&:= \{(\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \mid \\
&\quad (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i \le 0,\ \forall\, i \in I_\nu^{w-},\ (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i \ge 0,\ \forall\, i \in I_\nu^{w+}, \\
&\quad (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i \ge \lambda_{\nu,i},\ \forall\, i \in I_\tau^{w++},\ (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i \le \lambda_{\nu,i},\ \forall\, i \in I_\tau^{w+-}, \\
&\quad (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i \le -\lambda_{\nu,i},\ \forall\, i \in I_\tau^{w--},\ (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i \ge -\lambda_{\nu,i},\ \forall\, i \in I_\tau^{w-+}\}
\end{aligned} \tag{2.26}$$

and the functions $\boldsymbol{\mathcal{H}}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} : \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \to \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ whose components are defined by

$$\mathcal{H}_i^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})}(\boldsymbol{f}, \boldsymbol{y}) := (\boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau^T \boldsymbol{\mathcal{F}} \boldsymbol{\lambda}_\tau - \boldsymbol{f})_i, \quad i = 1, \dots, n_{\boldsymbol{u}}, \tag{2.27}$$

$$\mathcal{H}_{n_{\boldsymbol{u}}+i}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})}(\boldsymbol{f}, \boldsymbol{y}) := \begin{cases} r(\boldsymbol{B}_\nu \boldsymbol{u})_i & \text{if } i \in I_\nu^s(\boldsymbol{y}^0) \cup I_\nu^{w-}, \\ \lambda_{\nu,i} & \text{if } i \in I_\nu^0(\boldsymbol{y}^0) \cup I_\nu^{w+}, \end{cases} \tag{2.28}$$

$$\begin{aligned}
&\mathcal{H}_{n_{\boldsymbol{u}}+n_c+i}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})}(\boldsymbol{f}, \boldsymbol{y}) \\
&:= \begin{cases}
r(\boldsymbol{B}_\tau \boldsymbol{u})_i & \text{if } i \in \big((I_\tau^s(\boldsymbol{y}^0) \cup I_\tau^{w++} \cup I_\tau^{w--}) \cap J^-(\boldsymbol{y}^0)\big) \\
& \qquad \cup (I_\tau^{w++} \cap I_\tau^{w--} \cap J^0(\boldsymbol{y}^0)), \\
(2\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i & \text{if } i \in \big((I_\tau^s(\boldsymbol{y}^0) \cup I_\tau^{w+-} \cup I_\tau^{w-+}) \cap J^+(\boldsymbol{y}^0)\big) \\
& \qquad \cup (I_\tau^{w+-} \cap I_\tau^{w-+} \cap J^0(\boldsymbol{y}^0)), \\
(\boldsymbol{\lambda}_\tau - \boldsymbol{\lambda}_\nu)_i & \text{if } i \in I_\tau^+(\boldsymbol{y}^0) \cup (I_\tau^{w+-} \cap J^-(\boldsymbol{y}^0)) \cup (I_\tau^{w--} \cap J^+(\boldsymbol{y}^0)) \\
& \qquad \cup (I_\tau^{w+-} \cap I_\tau^{w--} \cap J^0(\boldsymbol{y}^0)), \\
(\boldsymbol{\lambda}_\tau + \boldsymbol{\lambda}_\nu)_i & \text{if } i \in I_\tau^-(\boldsymbol{y}^0) \cup (I_\tau^{w-+} \cap J^-(\boldsymbol{y}^0)) \cup (I_\tau^{w++} \cap J^+(\boldsymbol{y}^0)) \\
& \qquad \cup (I_\tau^{w-+} \cap I_\tau^{w++} \cap J^0(\boldsymbol{y}^0)).
\end{cases}
\end{aligned} \tag{2.29}$$

Then one can easily verify that there exists a neighbourhood W of $(\boldsymbol{f}^0, \boldsymbol{y}^0)$ such that:

$$\begin{aligned}
&\boldsymbol{\mathcal{H}}(\boldsymbol{f}, \boldsymbol{y}) = \boldsymbol{\mathcal{H}}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})}(\boldsymbol{f}, \boldsymbol{y}), \\
&\qquad\qquad \forall\, (\boldsymbol{f}, \boldsymbol{y}) \in W \cap \big(\{(\boldsymbol{f}^0, \boldsymbol{y}^0)\} + \pi^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})}\big).
\end{aligned}$$

Now, consider all possible combinations of $I_\nu^{w-} \subset I_\nu^w(\boldsymbol{y}^0)$, $I_\tau^{w++} \subset I_\tau^{w+}(\boldsymbol{y}^0)$ and $I_\tau^{w--} \subset I_\tau^{w-}(\boldsymbol{y}^0)$ and their total number denote by n_s. One obtains the collections Π

and $\{\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(n_s)}\}$ of subsets of $\mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ and functions from $\mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ into $\mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$, respectively:

$$\forall\, \pi \in \Pi\ \exists\, I_\nu^{w-} \subset I_\nu^w(\boldsymbol{y}^0),\ I_\tau^{w++} \subset I_\tau^{w+}(\boldsymbol{y}^0),\ I_\tau^{w--} \subset I_\tau^{w-}(\boldsymbol{y}^0) : \quad \pi = \pi^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})}, \tag{2.30}$$

$$\forall\, j \in \{1, \ldots, n_s\}\ \exists\, I_\nu^{w-} \subset I_\nu^w(\boldsymbol{y}^0),\ I_\tau^{w++} \subset I_\tau^{w+}(\boldsymbol{y}^0),\ I_\tau^{w--} \subset I_\tau^{w-}(\boldsymbol{y}^0) : \quad \boldsymbol{\mathcal{H}}^{(j)} = \boldsymbol{\mathcal{H}}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} \quad \text{in } \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}. \tag{2.31}$$

From the construction, it immediately follows that there exists a neighbourhood W of $(\boldsymbol{f}^0, \boldsymbol{y}^0)$ such that:

$$\forall\, \pi \in \Pi\ \exists\, j_\pi \in \{1, \ldots, n_s\} : \quad \boldsymbol{\mathcal{H}}(\boldsymbol{f}, \boldsymbol{y}) = \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{f}, \boldsymbol{y}), \quad \forall\, (\boldsymbol{f}, \boldsymbol{y}) \in W \cap \big(\{(\boldsymbol{f}^0, \boldsymbol{y}^0)\} + \pi\big). \tag{2.32}$$

This implies that $\boldsymbol{\mathcal{H}}$ is a continuous selection of $\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(n_s)}$ and consequently a piecewise differentiable function in some sufficiently small neighbourhood of $(\boldsymbol{f}^0, \boldsymbol{y}^0)$.

Let us note that if $\boldsymbol{y}^0$ is such that $I_\nu^w(\boldsymbol{y}^0) = I_\tau^{w+}(\boldsymbol{y}^0) = I_\tau^{w-}(\boldsymbol{y}^0) = \emptyset$, then $n_s = 1$, $\pi = \{\mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}\}$ and $\boldsymbol{\mathcal{H}}^{(1)} = \boldsymbol{\mathcal{H}}$ in some neighbourhood of $(\boldsymbol{f}^0, \boldsymbol{y}^0)$, that is, $\boldsymbol{\mathcal{H}}$ is even differentiable therein. Otherwise, we claim that Π is a conical subdivision of $\mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$.

Indeed, let $\pi = \pi^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} \in \Pi$ be given. Let us introduce the functions

$$\begin{aligned}
\Theta_1^{(I_\nu^w(\boldsymbol{y}^0))} &: \{1, \ldots, |I_\nu^w(\boldsymbol{y}^0)|\} \ \to I_\nu^w(\boldsymbol{y}^0),\\
\Theta_2^{(I_\tau^{w+}(\boldsymbol{y}^0))} &: \{1, \ldots, |I_\tau^{w+}(\boldsymbol{y}^0)|\} \ \to I_\tau^{w+}(\boldsymbol{y}^0),\\
\Theta_3^{(I_\tau^{w-}(\boldsymbol{y}^0))} &: \{1, \ldots, |I_\tau^{w-}(\boldsymbol{y}^0)|\} \ \to I_\tau^{w-}(\boldsymbol{y}^0)
\end{aligned}$$

such that

$$\begin{aligned}
\forall\, i \in I_\nu^w(\boldsymbol{y}^0)\ \exists\, j \in \{1, \ldots, |I_\nu^w(\boldsymbol{y}^0)|\} &: \quad \Theta_1^{(I_\nu^w(\boldsymbol{y}^0))}(j) = i,\\
\forall\, i \in I_\tau^{w+}(\boldsymbol{y}^0)\ \exists\, j \in \{1, \ldots, |I_\tau^{w+}(\boldsymbol{y}^0)|\} &: \quad \Theta_2^{(I_\tau^{w+}(\boldsymbol{y}^0))}(j) = i,\\
\forall\, i \in I_\tau^{w-}(\boldsymbol{y}^0)\ \exists\, j \in \{1, \ldots, |I_\tau^{w-}(\boldsymbol{y}^0)|\} &: \quad \Theta_3^{(I_\tau^{w-}(\boldsymbol{y}^0))}(j) = i,
\end{aligned}$$

where $|I|$ stands for the cardinality of a set I. With the aid of these functions, we define the matrix $\boldsymbol{B}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} \in \mathbb{M}^{|I_\nu^w(\boldsymbol{y}^0)|+|I_\tau^{w+}(\boldsymbol{y}^0)|+|I_\tau^{w-}(\boldsymbol{y}^0)|, 2n_{\boldsymbol{u}}+2n_c}$ by

$$\boldsymbol{B}_j^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} := \begin{cases} \big(\boldsymbol{0}_{1,n_{\boldsymbol{u}}},\ (-r\boldsymbol{B}_\nu)_i,\ (\boldsymbol{I}_{n_c})_i,\ \boldsymbol{0}_{1,n_c}\big) & \text{if } i \in I_\nu^{w-},\\ \big(\boldsymbol{0}_{1,n_{\boldsymbol{u}}},\ (r\boldsymbol{B}_\nu)_i,\ (-\boldsymbol{I}_{n_c})_i,\ \boldsymbol{0}_{1,n_c}\big) & \text{if } i \in I_\nu^{w+}, \end{cases}$$
$$i = \Theta_1^{(I_\nu^w(\boldsymbol{y}^0))}(j),\ \ j = 1, \ldots, |I_\nu^w(\boldsymbol{y}^0)|, \tag{2.33}$$

$$\boldsymbol{B}_j^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})} := \begin{cases} \left(\boldsymbol{0}_{1,n_{\boldsymbol{u}}},\ (r\boldsymbol{B}_\tau)_i,\ (\boldsymbol{I}_{n_c})_i,\ (-\boldsymbol{I}_{n_c})_i\right) & \text{if } i \in I_\tau^{w++}, \\ \left(\boldsymbol{0}_{1,n_{\boldsymbol{u}}},\ (-r\boldsymbol{B}_\tau)_i,\ (-\boldsymbol{I}_{n_c})_i,\ (\boldsymbol{I}_{n_c})_i\right) & \text{if } i \in I_\tau^{w+-}, \end{cases}$$

$$i = \Theta_2^{(I_\tau^{w+}(\boldsymbol{y}^0))}(j - |I_\nu^w(\boldsymbol{y}^0)|),\ j = |I_\nu^w(\boldsymbol{y}^0)| + 1, \dots, |I_\nu^w(\boldsymbol{y}^0)| + |I_\tau^{w+}(\boldsymbol{y}^0)|, \quad (2.34)$$

$$\boldsymbol{B}_j^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})} := \begin{cases} \left(\boldsymbol{0}_{1,n_{\boldsymbol{u}}},\ (-r\boldsymbol{B}_\tau)_i,\ (\boldsymbol{I}_{n_c})_i,\ (\boldsymbol{I}_{n_c})_i\right) & \text{if } i \in I_\tau^{w--}, \\ \left(\boldsymbol{0}_{1,n_{\boldsymbol{u}}},\ (r\boldsymbol{B}_\tau)_i,\ (-\boldsymbol{I}_{n_c})_i,\ (-\boldsymbol{I}_{n_c})_i\right) & \text{if } i \in I_\tau^{w-+}, \end{cases}$$

$$i = \Theta_3^{(I_\tau^{w-}(\boldsymbol{y}^0))}(j - |I_\nu^w(\boldsymbol{y}^0)| - |I_\tau^{w+}(\boldsymbol{y}^0)|),$$
$$j = |I_\nu^w(\boldsymbol{y}^0)| + |I_\tau^{w+}(\boldsymbol{y}^0)| + 1, \dots, |I_\nu^w(\boldsymbol{y}^0)| + |I_\tau^{w+}(\boldsymbol{y}^0)| + |I_\tau^{w-}(\boldsymbol{y}^0)|. \quad (2.35)$$

Here $\boldsymbol{B}_i$ denotes the ith row vector of a matrix $\boldsymbol{B}$, $\boldsymbol{0}_{m,n}$ stands for the m-by-n zero matrix and $\boldsymbol{I}_n$ represents the identity matrix of order n.

Then we have

$$\pi^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})} = \left\{(\boldsymbol{f},\boldsymbol{y}) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \,\middle|\, \boldsymbol{B}^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{y} \end{pmatrix} \le \boldsymbol{0} \right\}, \quad (2.36)$$

where the inequality is understood componentwise. This shows that $\pi^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})}$ is a polyhedral cone with vertex at $\boldsymbol{0} \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$. By the assumption (2.2), $\boldsymbol{B}^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})}$ is a full-row-rank matrix and one can find a vector $(\bar{\boldsymbol{f}},\bar{\boldsymbol{y}}) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ with $\boldsymbol{B}^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})}\binom{\bar{\boldsymbol{f}}}{\bar{\boldsymbol{y}}} < \boldsymbol{0}$. Hence, the dimension of the linear hull of $\pi^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})}$ equals $(2n_{\boldsymbol{u}} + 2n_c)$.

The union of all cones in Π covers $\mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ as we consider all possible choices of I_ν^{w-}, I_τ^{w++} and I_τ^{w--}. Finally, the intersection of any two cones $\pi = \pi^{(I_\nu^{w-},I_\tau^{w++},I_\tau^{w--})}$, $\tilde{\pi} = \pi^{(\tilde{I}_\nu^{w-},\tilde{I}_\tau^{w++},\tilde{I}_\tau^{w--})} \in \Pi$ takes the form

$$\begin{aligned}\pi \cap \tilde{\pi} = \big\{(\boldsymbol{f},\boldsymbol{y}) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \,\big|\, \\
&(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u})_i = 0,\ \forall\, i \in (I_\nu^{w-} \cap \tilde{I}_\nu^{w+}) \cup (I_\nu^{w+} \cap \tilde{I}_\nu^{w-}), \\
&(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u})_i \le 0,\ \forall\, i \in I_\nu^{w-} \cap \tilde{I}_\nu^{w-},\ (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u})_i \ge 0,\ \forall\, i \in I_\nu^{w+} \cap \tilde{I}_\nu^{w+}, \\
&(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u})_i = \lambda_{\nu,i},\ \forall\, i \in (I_\tau^{w++} \cap \tilde{I}_\tau^{w+-}) \cup (I_\tau^{w+-} \cap \tilde{I}_\tau^{w++}), \\
&(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u})_i \ge \lambda_{\nu,i},\ \forall\, i \in I_\tau^{w++} \cap \tilde{I}_\tau^{w++}, \\
&(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u})_i \le \lambda_{\nu,i},\ \forall\, i \in I_\tau^{w+-} \cap \tilde{I}_\tau^{w+-}, \\
&(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u})_i = -\lambda_{\nu,i},\ \forall\, i \in (I_\tau^{w--} \cap \tilde{I}_\tau^{w-+}) \cup (I_\tau^{w-+} \cap \tilde{I}_\tau^{w--}), \\
&(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u})_i \le -\lambda_{\nu,i},\ \forall\, i \in I_\tau^{w--} \cap \tilde{I}_\tau^{w--}, \\
&(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u})_i \ge -\lambda_{\nu,i},\ \forall\, i \in I_\tau^{w-+} \cap \tilde{I}_\tau^{w-+}\big\}.\end{aligned}$$

Whenever π and $\tilde{\pi}$ are distinct, at least one of the sets I_ν^{w-}, I_τ^{w++} or I_τ^{w--} does not coincide with $\tilde{I}_\nu^{w-}$, $\tilde{I}_\tau^{w++}$, $\tilde{I}_\tau^{w--}$, respectively, and the set above forms a common proper face of both cones.

Next, let n_l denote the dimension of the lineality space of Π. According to the assumptions of the previously mentioned implicit-function theorem for piecewise differentiable equations, either $2n_{\boldsymbol{u}} + 2n_c - n_l \leq 1$ needs to be satisfied or there has to exist a number $k \in \{2, \ldots, 2n_{\boldsymbol{u}} + 2n_c - n_l\}$ such that the kth branching number of Π does not exceed $2k$.

The lineality space of any cone $\pi = \pi^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} \in \Pi$ is the subspace

$$\left\{ (\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \,\middle|\, \boldsymbol{B}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{y} \end{pmatrix} = \boldsymbol{0} \right\}$$

with $\boldsymbol{B}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})} \in \mathbb{M}^{|I_\nu^w(\boldsymbol{y}^0)|+|I_\tau^{w+}(\boldsymbol{y}^0)|+|I_\tau^{w-}(\boldsymbol{y}^0)|, 2n_{\boldsymbol{u}}+2n_c}$ defined by (2.33)–(2.35) (confer (2.36)). The full row rank of any $\boldsymbol{B}^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})}$ under consideration guaranteed by (2.2) yields that the dimension of the lineality space of $\pi^{(I_\nu^{w-}, I_\tau^{w++}, I_\tau^{w--})}$ (and of Π, as well) is equal to $(2n_{\boldsymbol{u}}+2n_c-(|I_\nu^w(\boldsymbol{y}^0)|+|I_\tau^{w+}(\boldsymbol{y}^0)|+|I_\tau^{w-}(\boldsymbol{y}^0)|))$. Consequently, the condition $2n_{\boldsymbol{u}}+2n_c-n_l \leq 1$ is equivalent to $|I_\nu^w(\boldsymbol{y}^0)|+|I_\tau^{w+}(\boldsymbol{y}^0)|+|I_\tau^{w-}(\boldsymbol{y}^0)| \leq 1$. If it is not satisfied, we assert that the other condition holds with $k = 2$. Indeed, the 2^{nd} branching number of Π is the maximal number of cones in Π containing a common face of dimension $(2n_{\boldsymbol{u}} + 2n_c - 2)$. Having in mind (2.2) and (2.26) with (2.30), each such face can be written as

$$\begin{aligned}
&\{(\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \mid \\
&\quad (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i = 0,\ \forall\, i \in I_\nu^{w0},\ (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i \leq 0,\ \forall\, i \in I_\nu^{w-} \setminus I_\nu^{w0}, \\
&\quad (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i \geq 0,\ \forall\, i \in I_\nu^{w+} \setminus I_\nu^{w0},\ (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i = \lambda_{\nu,i},\ \forall\, i \in I_\tau^{w+0}, \\
&\quad (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i \geq \lambda_{\nu,i},\ \forall\, i \in I_\tau^{w++} \setminus I_\tau^{w+0},\ (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i \leq \lambda_{\nu,i},\ \forall\, i \in I_\tau^{w+-} \setminus I_\tau^{w+0}, \\
&\quad (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i = -\lambda_{\nu,i},\ \forall\, i \in I_\tau^{w-0},\ (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i \leq -\lambda_{\nu,i},\ \forall\, i \in I_\tau^{w--} \setminus I_\tau^{w-0}, \\
&\quad (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i \geq -\lambda_{\nu,i},\ \forall\, i \in I_\tau^{w-+} \setminus I_\tau^{w-0}\}
\end{aligned}$$

for some $I_\nu^{w-}, I_\nu^{w0} \subset I_\nu^w(\boldsymbol{y}^0)$, $I_\tau^{w++}, I_\tau^{w+0} \subset I_\tau^{w+}(\boldsymbol{y}^0)$ and $I_\tau^{w--}, I_\tau^{w-0} \subset I_\tau^{w-}(\boldsymbol{y}^0)$ with $|I_\nu^{w0}|+|I_\tau^{w+0}|+|I_\tau^{w-0}| = 2$. From this, it easily follows that the 2^{nd} branching number of Π is equal to 4.

To conclude, the following two theorems are valid (confer Theorem 4.2.2 and Proposition 4.2.2 in [56]).

Theorem 2.6. *Let (2.2) be valid and* $(\boldsymbol{f}^0, \boldsymbol{y}^0) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ *be a vector with* $\boldsymbol{\mathcal{H}}(\boldsymbol{f}^0, \boldsymbol{y}^0) = \boldsymbol{0}$. *If all matrices* $\boldsymbol{\nabla}_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{(j)}(\boldsymbol{f}^0, \boldsymbol{y}^0)$, $j = 1, \ldots, n_s$, *where* $\boldsymbol{\mathcal{H}}^{(j)}$ *are given by (2.31) have the same non-vanishing determinant sign then*

1. *the equation* $\boldsymbol{\mathcal{H}}(\boldsymbol{f}, \boldsymbol{y}) = \boldsymbol{0}$ *determines an implicit* PC^1*-function at* $(\boldsymbol{f}^0, \boldsymbol{y}^0)$, *that is, there exist neighbourhoods* O, $\hat{Y}$ *of* $\boldsymbol{f}^0$, $\boldsymbol{y}^0$, *respectively, and a* PC^1*-function* $\varphi_{\mathscr{F}} : O \to \hat{Y}$ *such that*

$$\varphi_{\mathscr{F}}(\boldsymbol{f}^0) = \boldsymbol{y}^0 \qquad \text{and} \qquad \varphi_{\mathscr{F}}(\boldsymbol{f}) = S^*_{\mathscr{F}}(\boldsymbol{f}) \cap \hat{Y}, \quad \forall\, \boldsymbol{f} \in O;$$

2. *the implicit functions determined by the equations* $\boldsymbol{\mathcal{H}}^{(j)}(\boldsymbol{f},\boldsymbol{y}) = \mathbf{0}$ *for* $j = 1,\dots,n_s$ *form a collection of selection functions for* $\varphi_{\mathscr{F}}$ *at* $\boldsymbol{f}^0$*;*

3. *for every* $\boldsymbol{\zeta} \in \mathbb{R}^{n_{\boldsymbol{u}}}$*, the identity* $\boldsymbol{\xi} = \varphi'_{\mathscr{F}}(\boldsymbol{f}^0;\boldsymbol{\zeta})$ *holds if and only if* $\boldsymbol{\xi}$ *satisfies the piecewise linear equation* $\boldsymbol{\mathcal{H}}'\big((\boldsymbol{f}^0,\boldsymbol{y}^0);(\boldsymbol{\zeta},\boldsymbol{\xi})\big) = \mathbf{0}$.

Theorem 2.7. *Suppose that the assumptions of the previous theorem are satisfied and* $\boldsymbol{\zeta} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ *is arbitrary.*

1. *Then there exists a cone* $\pi \in \Pi$ *such that*

$$\begin{pmatrix} \boldsymbol{\zeta} \\ \mathbf{0}_{n_{\boldsymbol{u}}+2n_c,1} \end{pmatrix} \in \begin{pmatrix} \boldsymbol{I}_{n_{\boldsymbol{u}}} & \mathbf{0}_{n_{\boldsymbol{u}},n_{\boldsymbol{u}}+2n_c} \\ \boldsymbol{\nabla}_{\boldsymbol{f}}\boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{f}^0,\boldsymbol{y}^0) & \boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{f}^0,\boldsymbol{y}^0) \end{pmatrix} \pi \tag{2.37}$$

   with j_π *being given by (2.32).*

2. *The inclusion (2.37) holds if and only if*

$$\begin{pmatrix} \boldsymbol{\zeta} \\ -\big(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{f}^0,\boldsymbol{y}^0)\big)^{-1}\boldsymbol{\nabla}_{\boldsymbol{f}}\boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{f}^0,\boldsymbol{y}^0)\boldsymbol{\zeta} \end{pmatrix} \in \pi.$$

3. *If* $\boldsymbol{\zeta}$ *satisfies (2.37) then*

$$\varphi'_{\mathscr{F}}(\boldsymbol{f}^0;\boldsymbol{\zeta}) = -\big(\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{f}^0,\boldsymbol{y}^0)\big)^{-1}\boldsymbol{\nabla}_{\boldsymbol{f}}\boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{f}^0,\boldsymbol{y}^0)\boldsymbol{\zeta},$$

   where $\varphi_{\mathscr{F}}$ *is the implicit* PC^1*-function determined by the equation* $\boldsymbol{\mathcal{H}}(\boldsymbol{f},\boldsymbol{y}) = \mathbf{0}$ *at* $(\boldsymbol{f}^0,\boldsymbol{y}^0)$*.*

Applying Corollary 4.1.1 in [56], which states that every piecewise differentiable function is locally Lipschitz continuous, we get the following consequence of Theorems 2.4 and 2.6.

Corollary 2.3. *If (2.2) holds and* $\mathscr{F} \in \mathscr{A}$*,* $(\boldsymbol{f}^0,\boldsymbol{y}^0) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ *are such that the assumptions of Theorem 2.6 are fulfilled then there are neighbourhoods* U, Y *of* $\mathscr{F}$*,* $\boldsymbol{y}^0$*, respectively, and a single-valued Lipschitz continuous function* $\sigma_{\boldsymbol{f}^0} : U \to Y$ *satisfying*

$$\sigma_{\boldsymbol{f}^0}(\mathscr{F}) = \boldsymbol{y}^0 \qquad \text{and} \qquad \sigma_{\boldsymbol{f}^0}(\boldsymbol{\xi}_{\mathscr{F}}) = \mathscr{S}^*_{\boldsymbol{f}^0}(\boldsymbol{\xi}_{\mathscr{F}}) \cap Y, \quad \forall\, \boldsymbol{\xi}_{\mathscr{F}} \in U.$$

It is worth mentioning that the assertion of the previous corollary generalizes Theorem 1 in [32], which concerns discrete contact problems with Coulomb friction and a coefficient of friction represented by one real. Moreover, the latter result has been obtained from the version of the implicit-function theorem dealing with Clarke's gradient and one has to handle with generally infinite number of matrices included in the respective generalized Jacobian to verify its assumptions.

At the end of this section, we shall analyse the cases when the assumption concerning determinant signs in Theorem 2.6 is not satisfied.

1. There exists an index $j \in \{1, \dots, n_s\}$ such that

$$\boldsymbol{\mathcal{H}}^{(j)}(\boldsymbol{f}^0, \boldsymbol{y}^0) = \mathbf{0}, \tag{2.38}$$

$$\operatorname{rank}\bigl(\boldsymbol{\nabla_y \mathcal{H}}^{(j)}\bigr) = n_{\boldsymbol{u}} + 2n_c - l, \ l \geq 1. \tag{2.39}$$

Here, we denote $\boldsymbol{\nabla_y \mathcal{H}}^{(j)} := \boldsymbol{\nabla_y \mathcal{H}}^{(j)}(\boldsymbol{f}^0, \boldsymbol{y}^0)$ because $\boldsymbol{\mathcal{H}}^{(j)}$, $j = 1, \dots, n_s$, are affine functions.

From (2.27)–(2.29) and (2.31), it is readily seen that $\boldsymbol{\nabla_y \mathcal{H}}^{(j)}$ satisfies

$$\begin{aligned}
(\nabla_{\boldsymbol{y}} \mathcal{H}_i^{(j)})^T &= \bigl(\boldsymbol{A}_i, \, (-\boldsymbol{B}_\nu^T)_i, \, (-\boldsymbol{B}_\tau^T \boldsymbol{\mathcal{F}})_i\bigr), \quad i = 1, \dots, n_{\boldsymbol{u}}, \\
(\nabla_{\boldsymbol{y}} \mathcal{H}_{n_{\boldsymbol{u}}+i}^{(j)})^T &\in \bigl\{\bigl((r\boldsymbol{B}_\nu)_i, \, \mathbf{0}_{1,n_c}, \, \mathbf{0}_{1,n_c}\bigr), \, \bigl(\mathbf{0}_{1,n_{\boldsymbol{u}}}, \, (\boldsymbol{I}_{n_c})_i, \, \mathbf{0}_{1,n_c}\bigr)\bigr\}, \quad i = 1, \dots, n_c, \\
(\nabla_{\boldsymbol{y}} \mathcal{H}_{n_{\boldsymbol{u}}+n_c+i}^{(j)})^T &\in \bigl\{\bigl((r\boldsymbol{B}_\tau)_i, \, \mathbf{0}_{1,n_c}, \, \mathbf{0}_{1,n_c}\bigr), \, \bigl((-r\boldsymbol{B}_\tau)_i, \, \mathbf{0}_{1,n_c}, \, (2\boldsymbol{I}_{n_c})_i\bigr), \\
&\qquad \bigl(\mathbf{0}_{1,n_{\boldsymbol{u}}}, \, (-\boldsymbol{I}_{n_c})_i, \, (\boldsymbol{I}_{n_c})_i\bigr), \, \bigl(\mathbf{0}_{1,n_{\boldsymbol{u}}}, \, (\boldsymbol{I}_{n_c})_i, \, (\boldsymbol{I}_{n_c})_i\bigr)\bigr\}, \quad i = 1, \dots, n_c.
\end{aligned}$$

Recall that $\boldsymbol{B}_i$ stands for the ith row vector of the matrix $\boldsymbol{B}$.

Taking into account that $\boldsymbol{\mathcal{H}}^{(j)}$ is affine, (2.38) is equivalent to

$$\boldsymbol{\nabla_y \mathcal{H}}^{(j)} \boldsymbol{y}^0 = \begin{pmatrix} \boldsymbol{f}^0 \\ \mathbf{0}_{n_c,1} \\ \mathbf{0}_{n_c,1} \end{pmatrix}.$$

Making use of (2.2), one can eliminate $2n_c$ columns with the aid of the last $2n_c$ rows of the matrix $\boldsymbol{\nabla_y \mathcal{H}}^{(j)}$ and one can arrive at an equivalent system of the type

$$\boldsymbol{H}\boldsymbol{y}^0 = \begin{pmatrix} \boldsymbol{f}^0 \\ \mathbf{0}_{n_c,1} \\ \mathbf{0}_{n_c,1} \end{pmatrix}, \quad \boldsymbol{H} = \begin{pmatrix} \boldsymbol{H}_u \\ \boldsymbol{H}_\nu \\ \boldsymbol{H}_\tau \end{pmatrix}, \quad \begin{array}{l} \boldsymbol{H}_u \in \mathbb{M}^{n_{\boldsymbol{u}}, n_{\boldsymbol{u}}+2n_c}, \\ \boldsymbol{H}_\nu, \boldsymbol{H}_\tau \in \mathbb{M}^{n_c, n_{\boldsymbol{u}}+2n_c}, \end{array} \tag{2.40}$$

where the rows of the matrix $\begin{pmatrix} \boldsymbol{H}_\nu \\ \boldsymbol{H}_\tau \end{pmatrix}$ are linearly independent not only to each other but also to the rows of $\boldsymbol{H}_u$. This and (2.39) yield that $\operatorname{rank}(\boldsymbol{H}_u) = n_{\boldsymbol{u}} - l$. Moreover, the system in (2.40) is solvable if and only if $\boldsymbol{f}^0$ is contained in the range of $\boldsymbol{H}_u$. Therefore, (2.38) and (2.39) restrict $\boldsymbol{f}^0$ to some $(n_{\boldsymbol{u}} - l)$-dimensional subspace of $\mathbb{R}^{n_{\boldsymbol{u}}}$.

Since the number of all possible selection functions of $\boldsymbol{\mathcal{H}}$ is finite, the considered situation occurs generally only for $(\boldsymbol{f}^0, \boldsymbol{y}^0)$ such that $\boldsymbol{f}^0$ is from a union of some lower-dimensional subspaces of $\mathbb{R}^{n_{\boldsymbol{u}}}$.

2. Two or more selection functions with nonsingular Jacobians are active at $(\boldsymbol{f}^0, \boldsymbol{y}^0)$ satisfying $\boldsymbol{\mathcal{H}}(\boldsymbol{f}^0, \boldsymbol{y}^0) = \mathbf{0}$.

Taking one such selection function, say $\boldsymbol{\mathcal{H}}^{(j)}$, it follows that $\boldsymbol{\mathcal{H}}^{(j)}(\boldsymbol{f}^0, \boldsymbol{y}^0) = \mathbf{0}$, that is,

$$\nabla_{\boldsymbol{y}} \boldsymbol{\mathcal{H}}^{(j)} \boldsymbol{y}^0 = \begin{pmatrix} \boldsymbol{f}^0 \\ \mathbf{0}_{n_c,1} \\ \mathbf{0}_{n_c,1} \end{pmatrix}. \tag{2.41}$$

In addition to this, $|I_\nu^w(\boldsymbol{y}^0) \cup I_\tau^{w+}(\boldsymbol{y}^0) \cup I_\tau^{w-}(\boldsymbol{y}^0)| \geq 1$ (which means that at least one contact node is in weak contact or weak stick) and the following $(|I_\nu^w(\boldsymbol{y}^0)| + |I_\tau^{w+}(\boldsymbol{y}^0)| + |I_\tau^{w-}(\boldsymbol{y}^0)|)$ conditions have to be satisfied:

$$\left.\begin{aligned} (\boldsymbol{\lambda}_\nu^0 - r\boldsymbol{B}_\nu \boldsymbol{u}^0)_i &= 0, & \forall\, i \in I_\nu^w(\boldsymbol{y}^0), \\ (\boldsymbol{\lambda}_\tau^0 - r\boldsymbol{B}_\tau \boldsymbol{u}^0)_i &= \lambda_{\nu,i}^0, & \forall\, i \in I_\tau^{w+}(\boldsymbol{y}^0), \\ (\boldsymbol{\lambda}_\tau^0 - r\boldsymbol{B}_\tau \boldsymbol{u}^0)_i &= -\lambda_{\nu,i}^0, & \forall\, i \in I_\tau^{w-}(\boldsymbol{y}^0). \end{aligned}\right\} \tag{2.42}$$

Notice that if $i_1 \in I_\nu^0(\boldsymbol{y}^0) \cap I_\tau^{w+}(\boldsymbol{y}^0) \cap I_\tau^{w-}(\boldsymbol{y}^0)$ then the $(n_{\boldsymbol{u}} + i_1)$th equation in (2.41) is $\lambda_{\nu,i_1}^0 = 0$, which together with the two conditions from the second and the third line of (2.42) for i_1 yields only two linearly independent equations with respect to $\boldsymbol{y}^0$. Furthermore, if $i_1 \in I_\nu^w(\boldsymbol{y}^0) \cap I_\tau^{w+}(\boldsymbol{y}^0) \cap I_\tau^{w-}(\boldsymbol{y}^0)$ then the $(n + i_1)$th equation in (2.41) and the equation from the first line of (2.42) for i_1 are equivalent to $\lambda_{\nu,i_1}^0 = (\boldsymbol{B}_\nu \boldsymbol{u}^0)_{i_1} = 0$, which added to the two conditions from the second and the third line of (2.42) for i_1 leads only to three linearly independent equations. As a consequence, we can leave out one of the equations in the second or the third line of (2.42) for any such i_1 and (2.42) reduces in this way to a system of l equations with

$$l := |I_\nu^w(\boldsymbol{y}^0)| + |I_\tau^{w+}(\boldsymbol{y}^0)| + |I_\tau^{w-}(\boldsymbol{y}^0)| - |(I_\nu^0(\boldsymbol{y}^0) \cup I_\nu^w(\boldsymbol{y}^0)) \cap I_\tau^{w+}(\boldsymbol{y}^0) \cap I_\tau^{w-}(\boldsymbol{y}^0)| \geq 1.$$

This system extended by (2.41) can be transformed similarly as in the previous case into an equivalent system of the form

$$\boldsymbol{H}\boldsymbol{y}^0 = \begin{pmatrix} \boldsymbol{f}^0 \\ \mathbf{0}_{n_c,1} \\ \mathbf{0}_{n_c,1} \\ \mathbf{0}_{l,1} \end{pmatrix}, \quad \boldsymbol{H} = \begin{pmatrix} \boldsymbol{H}_u \\ \boldsymbol{H}_\nu \\ \boldsymbol{H}_\tau \\ \boldsymbol{H}_s \end{pmatrix}, \quad \begin{aligned} &\boldsymbol{H}_u \in \mathbb{M}^{n_{\boldsymbol{u}}, n_{\boldsymbol{u}}+2n_c},\ \boldsymbol{H}_\nu, \boldsymbol{H}_\tau \in \mathbb{M}^{n_c, n_{\boldsymbol{u}}+2n_c}, \\ &\boldsymbol{H}_s \in \mathbb{M}^{l, n_{\boldsymbol{u}}+2n_c}, \end{aligned}$$

in which the rows of the matrix $\begin{pmatrix} \boldsymbol{H}_\nu \\ \boldsymbol{H}_\tau \\ \boldsymbol{H}_s \end{pmatrix}$ are linearly independent to each other and also to the rows of $\boldsymbol{H}_u$.

Arguing in the same way as previously, one can show that (2.41) and (2.42) confine $\boldsymbol{f}^0$ to some subspace of $\mathbb{R}^{n_{\boldsymbol{u}}}$ of dimension $(n_{\boldsymbol{u}} - l)$ and that the set of all $\boldsymbol{f}^0$ corresponding to this case forms a union of some lower-dimensional subspaces of $\mathbb{R}^{n_{\boldsymbol{u}}}$ again.

We get the following remark.

Remark 2.2. Let (2.2). All vectors $(\boldsymbol{f}^0, \boldsymbol{y}^0) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ with $\boldsymbol{\mathcal{H}}(\boldsymbol{f}^0, \boldsymbol{y}^0) = \boldsymbol{0}$ which do not satisfy the assumption on determinant sign of the Jacobians in Theorem 2.6 are such that $\boldsymbol{y}^0 \in S^*_{\boldsymbol{\mathscr{F}}}(\boldsymbol{f}^0)$ with $\boldsymbol{f}^0$ being an element from a union of subspaces of dimension strictly lower than $n_{\boldsymbol{u}}$.

2.4 An Elementary Example

This section presents an elementary contact problem involving a single linear triangular finite element depicted in Fig. 2.3. This example is taken from [32] and it is nothing else than a special case of the model studied in [35].

Denoting $\boldsymbol{u} := (u_\nu, u_\tau)$ and $\boldsymbol{f} := (f_\nu, f_\tau)$, an alternative of the projection formulation (2.25) of the corresponding discrete problem reads as follows:

$$\left.\begin{array}{l} \text{Find } \boldsymbol{y} := (u_\nu, u_\tau, \lambda_\nu, \lambda_\tau) \in \mathbb{R}^4 \text{ such that} \\ \boldsymbol{\mathcal{H}}(\boldsymbol{y}) := \begin{pmatrix} au_\nu - bu_\tau - \lambda_\nu - f_\nu \\ -bu_\nu + au_\tau - \lambda_\tau - f_\tau \\ \lambda_\nu - P_{(-\infty,0]}(\lambda_\nu - ru_\nu) \\ \lambda_\tau - P_{[-\mathscr{F}|\lambda_\nu|,\mathscr{F}|\lambda_\nu|]}(\lambda_\tau - ru_\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \end{array}\right\} \tag{2.43}$$

where the constants $a := (\lambda + 3\mu)/2$ and $b := (\lambda + \mu)/2$ depend on the Lamé coefficients $\lambda, \mu > 0$ characterizing the considered homogeneous, isotropic material of the body.

We shall derive exact solutions of this problem by considering all possible situations that may occur in the last two equations of (2.43). Note that each of these situations will correspond to a particular contact mode.

(i) Let $\lambda_\nu = 0$, that is, let there be no contact forces between the body and the rigid foundation. Then the fourth equation in (2.43) implies that $\lambda_\tau = 0$.

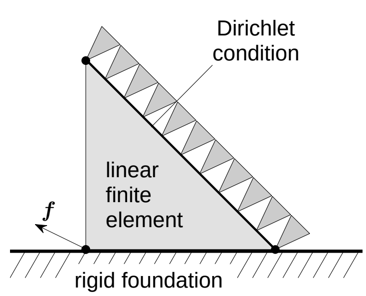


Figure 2.3: Geometry of the elementary example

Substituting the values of λ_ν and λ_τ into the first and the second equation in (2.43), one obtains a system of two linear equations with the solution

$$u_\nu = \frac{af_\nu + bf_\tau}{a^2 - b^2}, \quad u_\tau = \frac{af_\tau + bf_\nu}{a^2 - b^2}.$$

In addition, taking into account that $\lambda_\nu = 0$, it is readily seen from the third equation in (2.43) that $u_\nu \le 0$ so that

$$af_\nu + bf_\tau \le 0.$$

(ii) Suppose that $\lambda_\nu < 0$ and $u_\tau = 0$, that is, there is a strong contact with a stick between the body and the rigid foundation. Consequently, $u_\nu = 0$ by the third equation in (2.43), and the first and the second equation in (2.43) yield

$$\lambda_\nu = -f_\nu, \quad \lambda_\tau = -f_\tau.$$

Since $\lambda_\nu < 0$ and the fourth equation in (2.43) implies that $\mathscr{F}\lambda_\nu \le \lambda_\tau \le -\mathscr{F}\lambda_\nu$, one has

$$f_\nu > 0, \quad -\mathscr{F} f_\nu \le f_\tau \le \mathscr{F} f_\nu.$$

(iii) Consider $\lambda_\nu < 0$, $u_\tau > 0$, that is, a strong contact with a positive slip. Then $u_\nu = 0$, $\lambda_\tau = \mathscr{F}\lambda_\nu$ from the third and the fourth equation in (2.43), and the first and the second equation in (2.43) give

$$u_\tau = \frac{f_\tau - \mathscr{F} f_\nu}{a + b\mathscr{F}}, \quad \lambda_\nu = -\frac{af_\nu + bf_\tau}{a + b\mathscr{F}}.$$

From the conditions $\lambda_\nu < 0$ and $u_\tau > 0$, it follows that

$$af_\nu + bf_\tau > 0, \quad f_\tau - \mathscr{F} f_\nu > 0.$$

(iv) If $\lambda_\nu < 0$ and $u_\tau < 0$, which corresponds to a strong contact with a negative slip, then $u_\nu = 0$, $\lambda_\tau = -\mathscr{F}\lambda_\nu$ and the first two equations in (2.43) are equivalent to

$$\left.\begin{aligned} -bu_\tau - \lambda_\nu &= f_\nu, \\ (a - b\mathscr{F})u_\tau &= f_\tau + \mathscr{F} f_\nu. \end{aligned}\right\} \tag{2.44}$$

By assuming $\mathscr{F} \ne a/b$, this system has a unique solution

$$u_\tau = \frac{f_\tau + \mathscr{F} f_\nu}{a - b\mathscr{F}}, \quad \lambda_\nu = -\frac{af_\nu + bf_\tau}{a - b\mathscr{F}},$$

whose constraints are

$$\Big(\mathscr{F} < \frac{a}{b} \ \&\ af_\nu + bf_\tau \geq 0 \ \&\ f_\tau + \mathscr{F} f_\nu < 0 \ \&\ f_\nu \geq 0\Big) \\ \vee \quad \Big(\mathscr{F} > \frac{a}{b} \ \&\ af_\nu + bf_\tau \leq 0 \ \&\ f_\tau + \mathscr{F} f_\nu > 0 \ \&\ f_\nu \geq 0\Big).$$

If $\mathscr{F} = a/b$ then (2.44) is solvable if and only if

$$f_\tau + \mathscr{F} f_\nu = 0$$

and its solutions form the set

$$\{(u_\tau, \lambda_\nu) \in \mathbb{R}^2 \,|\, \lambda_\nu = -bu_\tau - f_\nu,\ u_\tau \in \mathbb{R}\}.$$

Due to the conditions $\lambda_\nu < 0$ and $u_\tau < 0$, u_τ has to satisfy

$$-\frac{f_\nu}{b} < u_\tau < 0.$$

From this,

$$f_\nu > 0.$$

To summarize the results, introduce the linear functions $\mathcal{S}^{(i)} : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^4$, $i = 1, 2, 3$, and the (generally) multi-valued function $\mathcal{S}^{(4)} : \mathbb{R}^2 \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^4$ by

$$\mathcal{S}^{(1)}(\boldsymbol{f}, \mathscr{F}) := \Big(\frac{af_\nu + bf_\tau}{a^2 - b^2},\ \frac{bf_\nu + af_\tau}{a^2 - b^2},\ 0,\ 0\Big), \quad \boldsymbol{f} \in \mathbb{R}^2,\ \mathscr{F} \in \mathbb{R}_+,$$

$$\mathcal{S}^{(2)}(\boldsymbol{f}, \mathscr{F}) := (0,\ 0,\ -f_\nu,\ -f_\tau), \quad \boldsymbol{f} \in \mathbb{R}^2,\ \mathscr{F} \in \mathbb{R}_+,$$

$$\mathcal{S}^{(3)}(\boldsymbol{f}, \mathscr{F}) := \Big(0,\ \frac{f_\tau - \mathscr{F} f_\nu}{a + b\mathscr{F}},\ -\frac{af_\nu + bf_\tau}{a + b\mathscr{F}},\ -\mathscr{F}\frac{af_\nu + bf_\tau}{a + b\mathscr{F}}\Big), \quad \boldsymbol{f} \in \mathbb{R}^2,\ \mathscr{F} \in \mathbb{R}_+,$$

$$\mathcal{S}^{(4)}(\boldsymbol{f}, \mathscr{F}) := \begin{cases} \Big\{\Big(0,\ \frac{f_\tau + \mathscr{F} f_\nu}{a - b\mathscr{F}},\ -\frac{af_\nu + bf_\tau}{a - b\mathscr{F}},\ \mathscr{F}\frac{af_\nu + bf_\tau}{a - b\mathscr{F}}\Big)\Big\} \\ \qquad\qquad \text{if } \boldsymbol{f} \in \mathbb{R}^2,\ \mathscr{F} \in \mathbb{R}_+ \setminus \Big\{\frac{a}{b}\Big\}, \\ \Big\{(u_\nu, u_\tau, \lambda_\nu, \lambda_\tau) \in \mathbb{R}^4 \,\Big|\, \\ \quad u_\nu = 0,\ -\frac{f_\nu}{b} \leq u_\tau \leq 0,\ \lambda_\nu = -(f_\nu + bu_\tau),\ \lambda_\tau = \mathscr{F}(f_\nu + bu_\tau)\Big\} \\ \qquad\qquad \text{if } \boldsymbol{f} \in \mathbb{R}^2,\ \mathscr{F} = \frac{a}{b}. \end{cases}$$

Moreover, for $\mathscr{F} \in \mathbb{R}_+$, define the sets

$$\rho^{(1)}(\mathscr{F}) := \{\boldsymbol{f} \in \mathbb{R}^2 \,|\, af_\nu + bf_\tau \leq 0\},$$

$$\rho^{(2)}(\mathscr{F}) := \{\boldsymbol{f} \in \mathbb{R}^2 \,|\, f_\nu \geq 0,\ -\mathscr{F} f_\nu \leq f_\tau \leq \mathscr{F} f_\nu\},$$

$$\rho^{(3)}(\mathscr{F}) := \{\boldsymbol{f} \in \mathbb{R}^2 \,|\, af_\nu + bf_\tau \geq 0,\ f_\tau - \mathscr{F} f_\nu \geq 0\},$$

$$\rho^{(4)}(\mathscr{F}) := \begin{cases} \{\boldsymbol{f} \in \mathbb{R}^2 \,|\, f_\nu \geq 0,\ af_\nu + bf_\tau \geq 0,\ f_\tau + \mathscr{F} f_\nu \leq 0\} & \text{if } \mathscr{F} \in [0, a/b], \\ \{\boldsymbol{f} \in \mathbb{R}^2 \,|\, f_\nu \geq 0,\ af_\nu + bf_\tau \leq 0,\ f_\tau + \mathscr{F} f_\nu \geq 0\} & \text{if } \mathscr{F} \in (a/b, +\infty). \end{cases}$$

Observe that only $\rho^{(1)}(\mathscr{F})$ does not depend on $\mathscr{F}$. One can easily verify that $\mathcal{S}^{(i)}(\boldsymbol{f}, \mathscr{F})$ solves (2.43) for $\boldsymbol{f} \in \rho^{(i)}(\mathscr{F})$, $\mathscr{F} \in \mathbb{R}_+$, $i = 1, 2, 3$, and $\mathcal{S}^{(4)}(\boldsymbol{f}, \mathscr{F})$ is a set of solutions to (2.43) for $\boldsymbol{f} \in \rho^{(4)}(\mathscr{F})$ and $\mathscr{F} \in \mathbb{R}_+$, which is single-point whenever $\mathscr{F} \neq a/b$.

Denote by $\mathring{\rho}^{(i)}_{\mathscr{F}}$ the interior of $\rho^{(i)}_{\mathscr{F}}$, $i = 1, \dots, 4$. It is readily seen that $\mathring{\rho}^{(3)}(\mathscr{F})$ is disjoint with $\mathring{\rho}^{(i)}(\mathscr{F})$, $i \neq 3$, for any $\mathscr{F} \in \mathbb{R}_+$. Hence, the structure of the whole solution set to (2.43) is given by the mutual position of $\rho^{(i)}(\mathscr{F})$, $i = 1, 2, 4$, which depends on the magnitude of $\mathscr{F}$. Three cases can be distinguished.

$\boxed{\mathscr{F} \in [0, a/b)}$

Suppose first that $\mathscr{F} > 0$. Then the system $\{\rho^{(1)}(\mathscr{F}), \rho^{(2)}(\mathscr{F}), \rho^{(3)}(\mathscr{F}), \rho^{(4)}(\mathscr{F})\}$ defines a partition of $\mathbb{R}^2$, that is,

$$\mathbb{R}^2 = \bigcup_{i=1}^{4} \rho^{(i)}(\mathscr{F}) \quad \text{and} \quad \mathring{\rho}^{(i)}(\mathscr{F}) \cap \mathring{\rho}^{(j)}(\mathscr{F}) = \emptyset,\ \forall\, i, j = 1, \dots, 4,\ i \neq j,$$

(see Fig. 2.4). Moreover,

$$\mathcal{S}^{(i)}(\boldsymbol{f}, \mathscr{F}) = \mathcal{S}^{(j)}(\boldsymbol{f}, \mathscr{F}), \quad \forall\, \boldsymbol{f} \in \partial\rho^{(i)}(\mathscr{F}) \cap \partial\rho^{(j)}(\mathscr{F}),\ \forall\, i, j = 1, \dots, 4.$$

Thus, (2.43) has a unique solution for any $\boldsymbol{f} \in \mathbb{R}^2$.

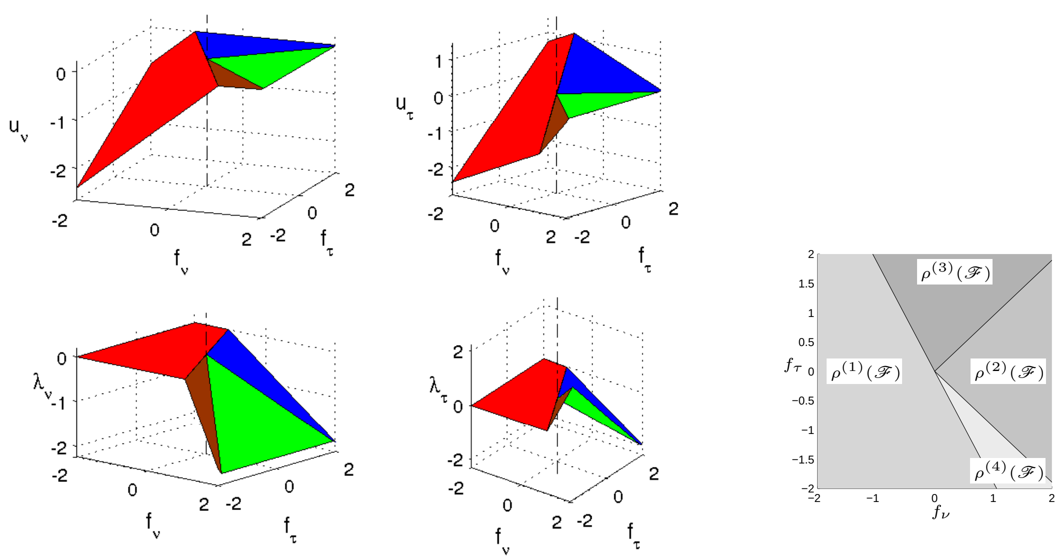


Figure 2.4: Structure of the solutions for $0 < \mathscr{F} < a/b$ with the corresponding decomposition of $\mathbb{R}^2$ into the sets $\rho^{(i)}(\mathscr{F})$

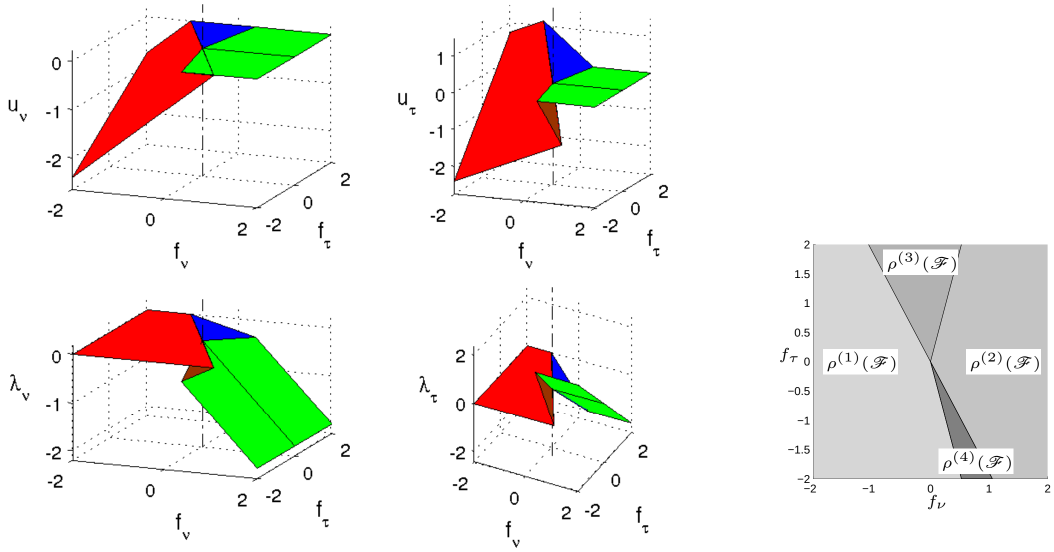


Figure 2.5: Structure of the solutions for $\mathscr{F} > a/b$

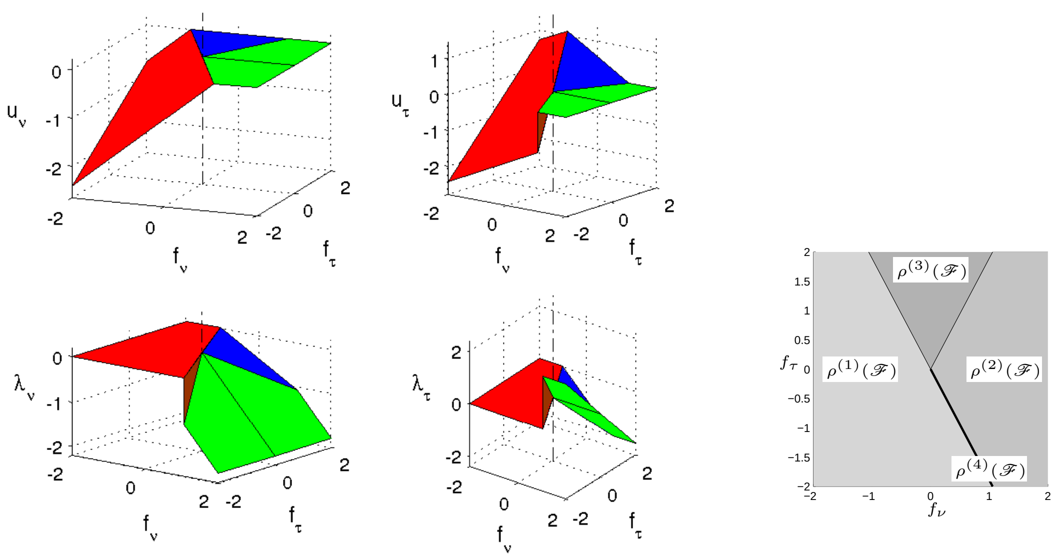


Figure 2.6: Structure of the solutions for $\mathscr{F} = a/b$

If $\mathscr{F} = 0$ then $\rho^{(2)}(\mathscr{F}) = \rho^{(3)}(\mathscr{F}) \cap \rho^{(4)}(\mathscr{F})$ and the partition of $\mathbb{R}^2$ is realized by $\{\rho^{(1)}(\mathscr{F}), \rho^{(3)}(\mathscr{F})), \rho^{(4)}(\mathscr{F}))\}$. The solution is again unique in $\mathbb{R}^2$.

Consequently, if $\mathscr{F} \in [0, a/b)$ *then (2.43) has a unique solution for any* $\boldsymbol{f} \in \mathbb{R}^2$.

$\boxed{\mathscr{F} > a/b}$

In this case, $\rho^{(4)}(\mathscr{F}) = \rho^{(1)}(\mathscr{F}) \cap \rho^{(2)}(\mathscr{F})$ and its interior is non-empty (see Fig. 2.5). It is easy to verify that *there exists a unique solution to (2.43) if* $\boldsymbol{f}$ *belongs to* $(\mathbb{R}^2 \setminus \rho^{(4)}(\mathscr{F})) \cup \{\boldsymbol{0}\}$*, there are two solutions on* $\partial\rho^{(4)}(\mathscr{F}) \setminus \{\boldsymbol{0}\}$ *and three solutions in* $\mathring{\rho}^{(4)}(\mathscr{F})$.

$\boxed{\mathscr{F} = a/b}$

This is the limit case, in which $\rho^{(4)}(\mathscr{F})) = \rho^{(1)}(\mathscr{F}) \cap \rho^{(2)}(\mathscr{F})$ is the ray emanating from the origin and separating $\rho^{(1)}(\mathscr{F})$ and $\rho^{(2)}(\mathscr{F})$ (see Fig. 2.6). *If* $\boldsymbol{f} \in (\mathbb{R}^2 \setminus \rho^{(4)}(\mathscr{F})) \cup \{\boldsymbol{0}\}$*, there exists a unique solution to (2.43). For* $\boldsymbol{f} \in \rho^{(4)}(\mathscr{F})) \setminus \{\boldsymbol{0}\}$*, the continuous branch* $\mathcal{S}^{(4)}(\boldsymbol{f}, \mathscr{F})$ *of solutions connects* $\mathcal{S}^{(1)}(\boldsymbol{f}, \mathscr{F})$ *and* $\mathcal{S}^{(2)}(\boldsymbol{f}, \mathscr{F})$.

From this analysis, we see that the solution of (2.43) is a PC^1-function of $\mathscr{F} \in [0, a/b)$ for an arbitrary $\boldsymbol{f} \in \mathbb{R}^2$ fixed. Therefore, it is Lipschitz continuous with respect to $\mathscr{F}$ in $[0, \bar{\mathscr{F}}_{\max}]$ for any $\bar{\mathscr{F}}_{\max} \in [0, a/b)$. On the other hand, we have proved the uniqueness as well as the Lipschitz continuity of the solutions with respect to $\mathscr{F}$ for $\mathscr{F}$ in $[0, \mathscr{F}_{\max}]$ with $\mathscr{F}_{\max} \in [0, \alpha\beta/\|\boldsymbol{A}\|)$ in Section 2.3. In this particular example, one has $\alpha\beta/\|\boldsymbol{A}\| = (a-b)/(a+b)$, which is strictly less than a/b. Since the situation concerning the Lipschitz continuity with respect to $\boldsymbol{f}$ is analogous, one can see that the general bounds derived before are pessimistic.

Nevertheless, this example shows that unicity of solutions depends not only on $\mathscr{F}$ but also on $\boldsymbol{f}$. Even if one takes $\mathscr{F}$ so large that there are non-unique solutions for some $\boldsymbol{f}$, for the same $\mathscr{F}$ there exist still such $\boldsymbol{f}$ that the corresponding solution is unique. Furthermore, one can verify that in this example, Theorem 2.6 guarantees local uniqueness of solutions precisely except the cases where it is actually lost. Hence, the presented local approach seems to be better suited for studying behaviour of solutions than the global one, which does not take into account the influence of $\boldsymbol{f} \in \mathbb{R}^2$.

Finally, let us mention that if one introduces selection functions $\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(n_s)}$ of the PC^1-function $\boldsymbol{\mathcal{H}}$ given by (2.43) in an analogous way as in (2.31), each mapping $(\boldsymbol{f}, \mathscr{F}) \mapsto \mathcal{S}^{(i)}(\boldsymbol{f}, \mathscr{F})$, $i = 1, \ldots, 4$, defined above is nothing else than a mapping associating $(\boldsymbol{f}, \mathscr{F})$ with the solution set (eventually single-point) of the equation $\boldsymbol{\mathcal{H}}^{(j)}(\boldsymbol{y}) = \boldsymbol{0}$ for some particular $\boldsymbol{\mathcal{H}}^{(j)}$. Since $\boldsymbol{\mathcal{H}}^{(1)}, \ldots, \boldsymbol{\mathcal{H}}^{(n_s)}$ are piecewise linear functions of the load vector $\boldsymbol{f}$, the structure of solutions to (2.43) as a function of $\boldsymbol{f}$ is quite simple. On the other hand, dependence of the solutions on the coefficient $\mathscr{F}$ is substantially more complicated, as exhibited in [32]. This confirms the benefits obtained by Theorem 2.4, which transforms the analysis of solutions with respect to

$\mathscr{F}$ into the analysis of the solutions with respect to $\boldsymbol{f}$.

Conclusion

Theoretical analysis of discrete 2D contact problems with Coulomb friction in which the coefficient of friction $\boldsymbol{\mathscr{F}}$ is assumed to be a vector has been presented in this chapter. The existence result has been obtained for any coefficient $\boldsymbol{\mathscr{F}}$ whereas to get the (global) uniqueness one, one needs the norm of $\boldsymbol{\mathscr{F}}$ to be sufficiently small. Moreover, the unique solution has been shown to be a Lipschitz continuous function of $\boldsymbol{\mathscr{F}}$ as well as of the load vector $\boldsymbol{f}$. Local analysis of potentially non-unique solutions has been based on two different but equivalent formulations of the problem – the first one has consisted of generalized equations, the second one of piecewise smooth equations. From the first formulation, we have seen that the study of local behaviour of solutions as a function of $\boldsymbol{\mathscr{F}}$ can be deduced from the study of local behaviour of the solutions as a function of $\boldsymbol{f}$. From the second one, we have concluded that the solutions are locally unique and Lipschitz continuous with respect to $\boldsymbol{f}$ if particular Jacobian matrices depending on the contact status of the solutions have the same non-vanishing determinant sign. Results determining directional derivatives to these local Lipschitz continuous branches have been also achieved. Further, it has been proved that the set of $\boldsymbol{f}$ where the existence of such branches is not guaranteed is "small". In the end, benefits of the proposed local approach have been suggested on a simple example.

3 Numerical Continuation of 2D Static Problems

In the previous chapter, we have considered solutions of discrete 2D static problems parametrized by the coefficient $\mathscr{F}$ and the load vector $\boldsymbol{f}$ and we have guaranteed that there exist local Lipschitz continuous branches of solutions with respect to these parameters. The aim of this chapter is twofold. Firstly, to develop a piecewise smooth variant of the Moore-Penrose continuation algorithm for capturing such solution branches numerically; secondly, to introduce quasi-static contact problems in finite deformations and to apply our method for computing incremental solutions that come from their discretization.

The chapter is organized as follows: In Section 3.1, the algorithm of our continuation technique is described for discrete static contact problems parametrized by one scalar parameter. More precisely, we consider these problems in the form of a system of piecewise differentiable equations and we adapt the classical Moore-Penrose numerical continuation for smooth functions to this case. In Section 3.2, we present a model of quasi-static contact problems in nonlinear elasticity. After introducing the classical formulation, we derive a weak one, taking into account the particular constitutive law considered. Full discretization of the weak formulation leads to a sequence of algebraic incremental problems. We show that these are piecewise smooth in vicinity of some appropriate points, which allows us to apply the proposed variant of numerical continuation for solving the problems. Finally, we present some numerical results.

3.1 Description of the Method

In light of the previous chapter, a formulation of discrete 2D static problems can be written as the following system of non-smooth equations (confer (2.25)):

$$\left.\begin{array}{l}\text{Find } \boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \text{ such that}\\ \boldsymbol{\mathcal{H}}(\boldsymbol{y}) = \boldsymbol{0},\end{array}\right\}$$

where $\boldsymbol{\mathcal{H}} : \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \to \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$ is defined by

$$\boldsymbol{\mathcal{H}}(\boldsymbol{y}) := \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau - \boldsymbol{f} \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u}) \\ \boldsymbol{\lambda}_\tau - \boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u}) \end{pmatrix}, \quad \boldsymbol{y} := (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}.$$

Here, $r > 0$ is a fixed parameter and the components of $\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu} : \mathbb{R}^{n_c} \to \boldsymbol{\Lambda}_\nu$ and $\boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}\boldsymbol{\lambda}_\nu)} : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ are introduced as follows:

$$(\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu})_i(\boldsymbol{\mu}) := P_{\mathbb{R}_-}(\mu_i), \quad i = 1, \ldots, n_c,\ \boldsymbol{\mu} \in \mathbb{R}^{n_c}, \tag{3.1}$$

$$(\boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}\boldsymbol{\lambda}_\nu)})_i(\boldsymbol{\mu}) := \begin{cases} P_{[\mathscr{F}_i\lambda_{\nu,i}, -\mathscr{F}_i\lambda_{\nu,i}]}(\mu_i) & \text{if } \lambda_{\nu,i} \leq 0, \\ -P_{[-\mathscr{F}_i\lambda_{\nu,i}, \mathscr{F}_i\lambda_{\nu,i}]}(\mu_i) & \text{if } \lambda_{\nu,i} > 0, \end{cases}$$
$$i = 1, \ldots, n_c,\ \boldsymbol{\mu} \in \mathbb{R}^{n_c}, \tag{3.2}$$

with $P_{\mathbb{R}_-}$, $P_{[-\zeta,\zeta]}$ being the projections of $\mathbb{R}$ onto $\mathbb{R}_-$ and $[-\zeta, \zeta]$, $\zeta \geq 0$, respectively. Recall that $\boldsymbol{\mathscr{F}} \in \mathbb{R}^{n_c}_+$ represents the coefficient of friction.

In what follows, we shall suppose that the mapping $\boldsymbol{\mathcal{H}}$ depends on an additional scalar parameter so that $\boldsymbol{\mathcal{H}} : \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I \to \mathbb{R}^{n_{\boldsymbol{u}}+2n_c}$, $I \subset \mathbb{R}$. A natural candidate for the parametrization is the load $\boldsymbol{f}$ when we are given a smooth loading path $\gamma \in I \mapsto \boldsymbol{f}(\gamma) \in \mathbb{R}^{n_{\boldsymbol{u}}}$. In this case, $\boldsymbol{\mathcal{H}}$ becomes

$$\boldsymbol{\mathcal{H}}(\boldsymbol{y}) := \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau - \boldsymbol{f}(\gamma) \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u}) \\ \boldsymbol{\lambda}_\tau - \boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u}) \end{pmatrix}, \quad \boldsymbol{y} := (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau, \gamma) \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I.$$

We may take also a smooth path $\gamma \in I \mapsto \boldsymbol{\mathscr{F}}(\gamma) \in \mathbb{R}^{n_c}_+$ and

$$\boldsymbol{\mathcal{H}}(\boldsymbol{y}) := \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} - \boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau - \boldsymbol{f} \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u}) \\ \boldsymbol{\lambda}_\tau - \boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(\boldsymbol{\mathscr{F}}(\gamma)\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u}) \end{pmatrix},$$
$$\boldsymbol{y} := (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau, \gamma) \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I. \tag{3.3}$$

Another possibility might be parametrization of a non-homogeneous Dirichlet condition (see the next section).

For definiteness, we shall consider the case (3.3), that is, the parametrization via the coefficient of friction. (As we shall see, adaptation to the other cases will be straightforward.) We are lead to the following problem:

$$\left.\begin{array}{l} \text{Find } \boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I \text{ such that} \\ \boldsymbol{\mathcal{H}}(\boldsymbol{y}) = \boldsymbol{0}. \end{array}\right\} \tag{3.4}$$

On the basis of Section 2.3, it is readily seen that $\boldsymbol{\mathcal{H}}$ is a piecewise differentiable function. Moreover, Theorem 2.2 and Corollary 2.3 establish the existence of (local) Lipschitz continuous branches of solutions to (3.4). Our present objective is to trace the solution curves numerically, using path-following (continuation) techniques.

Classical continuation techniques require $\boldsymbol{\mathcal{H}}$ in (3.4) to be *smooth*. Next, we shall show how such techniques can be adapted to our non-smooth case. In particular,

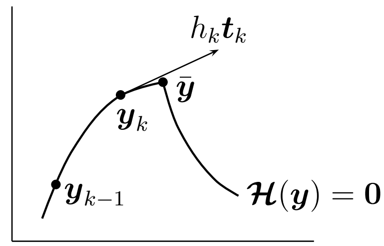


Figure 3.1: Necessity of a good prediction

we shall modify the Moore-Penrose continuation, which is presented, for instance, in [14] (see also appendix). This procedure is a *predictor-corrector* type method.

In the Newton-like correction step, it suffices to use the piecewise smooth Newton method (7.2.14 Algorithm in [18]) instead of the smooth one. In other words, the gradient $\boldsymbol{\nabla\mathcal{H}}$ is replaced by the gradient of one of its active selection functions if necessary.

On the other hand, a modification of the prediction step needs to be more sophisticated. Indeed, if one takes an initial approximation of a new point $\boldsymbol{y}_{k+1}$ of the form

$$\boldsymbol{Y}_0 := \boldsymbol{y}_k + h_k\boldsymbol{t}_k, \tag{3.5}$$

where $\boldsymbol{t}_k$ is determined from the directional derivative of $\boldsymbol{\mathcal{H}}$ as

$$\boldsymbol{\mathcal{H}}'(\boldsymbol{y}_k;\boldsymbol{t}_k) = \boldsymbol{0},$$

the continuation may fail when approaching a point of non-differentiability on the solution curve. It is caused by the fact that the Newton correction is only locally convergent and one has to take a suitable initial approximation to reach its zone of convergence (see Fig. 3.1 for illustration). In the sequel, we shall propose a special approach for passing through such points.

Recall that the non-differentiability of $\boldsymbol{\mathcal{H}}$ is caused by the functions

$$\boldsymbol{y} \mapsto \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu\boldsymbol{u}), \quad \boldsymbol{y} \mapsto \boldsymbol{\lambda}_\tau - \boldsymbol{P}_{\boldsymbol{\lambda}_\tau(\boldsymbol{\mathscr{F}}(\gamma)\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_t\boldsymbol{u}),$$

to which selection functions with the following components can be associated (confer (2.28) and (2.29)):

$$\boldsymbol{y} \mapsto r(\boldsymbol{B}_\nu\boldsymbol{u})_i, \quad \boldsymbol{y} \mapsto \lambda_{\nu,i}$$

and

$$\boldsymbol{y} \mapsto r(\boldsymbol{B}_\tau\boldsymbol{u})_i, \quad \boldsymbol{y} \mapsto (2\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau\boldsymbol{u})_i, \quad \boldsymbol{y} \mapsto \lambda_{\tau,i} - \mathscr{F}_i(\gamma)\lambda_{\nu,i}, \quad \boldsymbol{y} \mapsto \lambda_{\tau,i} + \mathscr{F}_i(\gamma)\lambda_{\nu,i},$$

$i = 1, \ldots, n_c$, respectively. We define the so-called *test functions* $\boldsymbol{\theta}_l = (\theta_{l,1}, \ldots, \theta_{l,n_c}) : \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I \to \mathbb{R}^{n_c}$, $l = 1, 2, 3$, by

$$\begin{aligned}\theta_{1,i}(\boldsymbol{y}) &:= (\boldsymbol{\lambda}_\nu - r\boldsymbol{B}_\nu \boldsymbol{u})_i,\\ \theta_{2,i}(\boldsymbol{y}) &:= (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i - \mathscr{F}_i(\gamma)\lambda_{\nu,i},\\ \theta_{3,i}(\boldsymbol{y}) &:= (\boldsymbol{\lambda}_\tau - r\boldsymbol{B}_\tau \boldsymbol{u})_i + \mathscr{F}_i(\gamma)\lambda_{\nu,i}\end{aligned}$$

for any $i = 1, \ldots, n_c$, $\boldsymbol{y} = (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau, \gamma) \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I$.

Clearly, there is a one-to-one correspondence between signs of the components of $\boldsymbol{\theta}_1(\boldsymbol{y})$, $\boldsymbol{\theta}_2(\boldsymbol{y})$ and $\boldsymbol{\theta}_3(\boldsymbol{y})$ and the selection functions for $\boldsymbol{\mathcal{H}}$ which are active at $\boldsymbol{y}$. (Possible zero components indicate that more than one selection function is active.) Suppose for a moment that $\boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I$ is a point where only one selection function is active, that is, all components of the test functions are nonzero there. Assembling the signs of the test functions into a 3-by-n_c array in such a way that the lth row corresponds to $\boldsymbol{\theta}_l(\boldsymbol{y})$, $l = 1, 2, 3$, we see that every selection function for $\boldsymbol{\mathcal{H}}$ can be represented by a 3-by-n_c array and this representation is unique.

Let $\boldsymbol{y}_k$ be a current point which is close to a point $\bar{\boldsymbol{y}}$ of non-differentiability of $\boldsymbol{\mathcal{H}}$ as illustrated in Fig. 3.1. Assume that exactly two selection functions $\boldsymbol{\mathcal{H}}^{(i_1)}$ and $\boldsymbol{\mathcal{H}}^{(i_2)}$ are active at $\bar{\boldsymbol{y}}$ and there exists a piecewise smooth curve of solutions passing through $\bar{\boldsymbol{y}}$ which consists of two smooth branches belonging to the solution sets to $\boldsymbol{\mathcal{H}}^{(i_1)}(\boldsymbol{y}) = \boldsymbol{0}$ and $\boldsymbol{\mathcal{H}}^{(i_2)}(\boldsymbol{y}) = \boldsymbol{0}$. Supposing that $\boldsymbol{y}_k$ is a root of $\boldsymbol{\mathcal{H}}^{(i_1)}$, we shall describe how to reach the unknown smooth branch of the curve corresponding to $\boldsymbol{\mathcal{H}}^{(i_2)}(\boldsymbol{y}) = \boldsymbol{0}$.

As explained before, one of the test functions, say $\boldsymbol{\theta}_l$, has a zero component at $\bar{\boldsymbol{y}}$, say the mth one, and this component changes its sign when passing through $\bar{\boldsymbol{y}}$. Continuity of $\boldsymbol{\theta}_l$ ensures that $\theta_{l,m}(\boldsymbol{y}_k)$ is close to zero. If we consider the 3-by-n_c array representing $\boldsymbol{\mathcal{H}}^{(i_1)}$, then changing the sign corresponding to $\theta_{l,m}$, we obtain the representative of the selection function $\boldsymbol{\mathcal{H}}^{(i_2)}$, hence the form of $\boldsymbol{\mathcal{H}}^{(i_2)}$ itself. This leads us to the following choice of the vector $\boldsymbol{t}_k$ for (3.5):

$$\boldsymbol{\nabla}\boldsymbol{\mathcal{H}}^{(i_2)}(\boldsymbol{y}_k)\boldsymbol{t}_k = \boldsymbol{0}, \quad \|\boldsymbol{t}_k\| = 1.$$

In the end, direction of this vector is selected so that

$$\theta_{l,m}(\boldsymbol{y}_k)\big(\nabla\theta_{l,m}(\boldsymbol{y}_k), \boldsymbol{t}_k\big) \le 0$$

as our aim is to traverse the set $\{\boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I \mid \theta_{l,m}(\boldsymbol{y}) = 0\}$ (see Fig. 3.2). Recall that $(.,.)$ stands for the scalar product.

Let us note that the expounded procedure can be also applied when the point of non-differentiability $\bar{\boldsymbol{y}}$ is met exactly, that is, $\boldsymbol{y}_k = \bar{\boldsymbol{y}}$. Nevertheless, this situation is highly improbable.

On the basis of the above considerations, we arrive at the following algorithm. By $I_{\boldsymbol{\mathcal{H}}}(\boldsymbol{Y}_j)$ we denote the active index set at $\boldsymbol{Y}_j$.

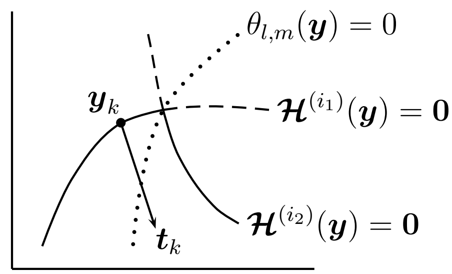


Figure 3.2: Determination of the direction of the new “tangent” vector

Algorithm 3.1. (*Piecewise smooth variant of the Moore-Penrose continuation*)

Data: $\varepsilon, \varepsilon' > 0$, $\vartheta_{\min} \leq 1$, $h_{\max} \geq h_{\text{init}} \geq h_{\min} > 0$, $h_{\text{inc}} > 1 > h_{\text{dec}} > 0$, $j_{\max} \geq j_{\text{thr}} > 0$ and $\boldsymbol{y}_0 \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c} \times I, \boldsymbol{t}_0 \in \mathbb{R}^{n_{\boldsymbol{u}}+2n_c+1}$ satisfying:

$$\|\boldsymbol{\mathcal{H}}(\boldsymbol{y}_0)\| < \varepsilon, \quad \boldsymbol{\mathcal{H}}'(\boldsymbol{y}_0; \boldsymbol{t}_0) = \boldsymbol{0}, \quad \|\boldsymbol{t}_0\| = 1.$$

Step 1: Set $h_0 := h_{\text{init}}$, $k := 0$.

Step 2: Set $n_{\text{dec}} := 0$.

Step 3 (*prediction*)**:** Set $\boldsymbol{Y}_0 := \boldsymbol{y}_k + h_k \boldsymbol{t}_k$, $\boldsymbol{T}_0 := \boldsymbol{t}_k$, $j := 0$.

Step 4 (*correction*)**:** Select an index i_j in $I_{\boldsymbol{\mathcal{H}}}(\boldsymbol{Y}_j)$ and set:

$$\boldsymbol{B} := \begin{pmatrix} \boldsymbol{\nabla}\boldsymbol{\mathcal{H}}^{(i_j)}(\boldsymbol{Y}_j) \\ (\boldsymbol{T}_j)^T \end{pmatrix}, \quad \boldsymbol{R} := \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}, \quad \boldsymbol{Q} := \begin{pmatrix} \boldsymbol{\mathcal{H}}(\boldsymbol{Y}_j) \\ 0 \end{pmatrix},$$

$$\tilde{\boldsymbol{T}} := \boldsymbol{B}^{-1}\boldsymbol{R}, \quad \boldsymbol{T}_{j+1} := \frac{\tilde{\boldsymbol{T}}}{\|\tilde{\boldsymbol{T}}\|},$$

$$\boldsymbol{Y}_{j+1} := \boldsymbol{Y}_j - \boldsymbol{B}^{-1}\boldsymbol{Q}.$$

Step 5: If $\|\boldsymbol{\mathcal{H}}(\boldsymbol{Y}_{j+1})\| < \varepsilon$ and $\|\boldsymbol{Y}_{j+1} - \boldsymbol{Y}_j\| < \varepsilon'$, go to Step 7.

Step 6: If $j < j_{\max}$, set $j := j + 1$ and go to Step 4. Otherwise, go to Step 8.

Step 7: If $(\boldsymbol{T}_{j+1}, \boldsymbol{t}_k) \geq \vartheta_{\min}$, set $\boldsymbol{y}_{k+1} := \boldsymbol{Y}_{j+1}$, $\boldsymbol{t}_{k+1} := \boldsymbol{T}_{j+1}$ and go to Step 10.

Step 8: If $h_k > h_{\min}$, set $h_k := \max\{h_{\text{dec}} h_k, h_{\min}\}$, $n_{\text{dec}} := n_{\text{dec}} + 1$ and go to Step 3.

Step 9: According to a component $\theta_{l,m}(\boldsymbol{y}_k)$, $l = 1, 2, 3$, $m = 1, \ldots, n_c$, close to 0, select a function $\boldsymbol{\mathcal{H}}^{(i)}$ which is likely to be active in a vicinity of $\boldsymbol{y}_k$ and compute the vector $\boldsymbol{t}_k$ satisfying

$$\boldsymbol{\nabla}\boldsymbol{\mathcal{H}}^{(i)}(\boldsymbol{y}_k)\boldsymbol{t}_k = \boldsymbol{0}, \quad \|\boldsymbol{t}_k\| = 1$$

and

$$\theta_{l,m}(\boldsymbol{y}_k)\big(\nabla\theta_{l,m}(\boldsymbol{y}_k), \boldsymbol{t}_k\big) \le 0.$$

Set $h_k := h_{\text{init}}$ and go to Step 2.

Step 10: Set

$$h_{k+1} := \begin{cases} \min\{h_{\text{inc}} h_k, h_{\max}\} & \text{if } j < j_{\text{thr}} \text{ and } n_{\text{dec}} = 0, \\ h_k & \text{otherwise} \end{cases}$$

and $k := k + 1$, go to Step 2.

Here ε and ε' are convergence tolerances, $h_{\min}$, $h_{\max}$ and h_{init} is the minimal, maximal and initial step length, respectively, and h_{inc}, h_{dec} are the scale factors for adjustment of the step length. Further, $j_{\max}$ stands for the maximal number of corrections allowed and n_{dec} denotes the number of the step length reductions of h_k for the current value of k. The parameter $\vartheta_{\min}$ serves for controlling changes of direction between the tangents at two consecutive points.

In Step 8, the current step length is shortened in the case of non-convergence of the corrections or too large deviation between the newly computed tangent and the previous one, which is tested in Step 7. Step 10 defines the step length for the prediction in the next iteration. The new step length h_{k+1} can be larger than h_k only if the number of corrections (Step 4) does not exceed j_{thr} given a priori and $n_{\text{dec}} = 0$. These parts of the routine together with the prediction and the corrections are taken from the classical Moore-Penrose continuation. Step 9 is added for handling the situations when the corrections do not lead to a new point even for $h = h_{\min}$. Making use of the test functions defined above, one determines a new "tangent" vector for the prediction here and then returns to the classical part of the procedure.

Remark 3.1. (i) One can use this algorithm to pass through points where more than two selection functions are active, as well. In this case, however, more components of the test functions are close to zero and one has to decide between more possibilities how to choose a new selection function when "switching" between different smooth branches.

(ii) In a similar way as one tests changes of direction between any two consecutive tangents $\boldsymbol{t}_k$ and $\boldsymbol{t}_{k+1}$, one can also monitor changes of the signs of components of $\boldsymbol{\theta}_l$, $l = 1, 2, 3$, in order to control transitions through points of non-differentiability.

3.2 Application to Quasi-Static Problems

Before we present an application of the numerical continuation described above for solving quasi-static contact problems in large deformations, we shall formulate briefly

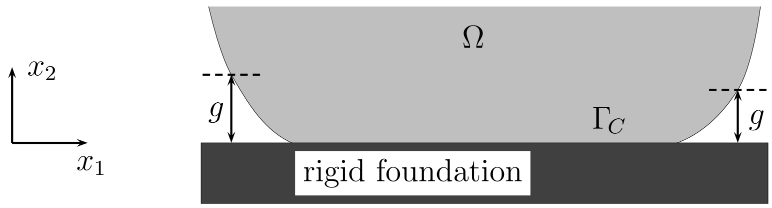


Figure 3.3: Geometry of the problem

these problems. For a thorough introduction to nonlinear elasticity, we refer the reader to [10].

3.2.1 Problem Formulation

We shall consider a contact problem between a 2D homogeneous elastic body and a rigid foundation represented by the half-plane $\{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2, \mid x_2 \leq 0\}$. For simplicity, we shall not deal with a self-contact of the body. Let us mention, however, that this would be also possible (see, for instance, [10, Section 5.6]).

The classical formulation of our problem reads as follows:

$$\left.\begin{aligned}
&\text{Find } \boldsymbol{u} : [0,T] \times \overline{\Omega} \to \mathbb{R}^2 \text{ such that } \det(\boldsymbol{I} + \boldsymbol{\nabla u}) > 0 \text{ in } [0,T] \times \overline{\Omega} \text{ and}\\
&-\mathbf{div}[(\boldsymbol{I} + \boldsymbol{\nabla u})\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{I} + \boldsymbol{\nabla u})] = \boldsymbol{f} \quad \text{in } (0,T) \times \Omega,\\
&\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } (0,T) \times \Gamma_D,\\
&(\boldsymbol{I} + \boldsymbol{\nabla u})\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{I} + \boldsymbol{\nabla u})\boldsymbol{\nu} = \boldsymbol{h} \quad \text{on } (0,T) \times \Gamma_N,\\
&u_2(t,\boldsymbol{x}) + g(\boldsymbol{x}) \geq 0, \quad \hat{T}_2(t,\boldsymbol{x},\boldsymbol{\nu}) \geq 0,\\
&\quad (u_2(t,\boldsymbol{x}) + g(\boldsymbol{x}))\hat{T}_2(t,\boldsymbol{x},\boldsymbol{\nu}) = 0 \quad \text{on } (0,T) \times \Gamma_C,\\
&\dot{u}_1(t,\boldsymbol{x}) = 0 \implies |\hat{T}_1(t,\boldsymbol{x},\boldsymbol{\nu})| \leq \mathscr{F}\hat{T}_2(t,\boldsymbol{x},\boldsymbol{\nu}) \quad \text{on } (0,T) \times \Gamma_C,\\
&\dot{u}_1(t,\boldsymbol{x}) \neq 0 \implies \hat{T}_1(t,\boldsymbol{x},\boldsymbol{\nu}) = -\mathscr{F}\hat{T}_2(t,\boldsymbol{x},\boldsymbol{\nu})\frac{\dot{u}_1(t,\boldsymbol{x})}{|\dot{u}_1(t,\boldsymbol{x})|} \quad \text{on } (0,T) \times \Gamma_C,\\
&\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) \quad \text{in } \Omega.
\end{aligned}\right\} \tag{3.6}$$

Besides the familiar notation, $T > 0$ determines the time interval of interest, $\boldsymbol{I}$ denotes the identity matrix, and $\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{I} + \boldsymbol{\nabla u})$ is the second Piola-Kirchhoff stress tensor related to the Cauchy stress tensor $\boldsymbol{\sigma}(\boldsymbol{I} + \boldsymbol{\nabla u})$ by

$$\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}) \equiv (\det \boldsymbol{F})\boldsymbol{F}^{-1}\boldsymbol{\sigma}(\boldsymbol{F})\boldsymbol{F}^{-T}, \quad \boldsymbol{F} \in \mathbb{M}^2_>, \tag{3.7}$$

where $\mathbb{M}^2_>$ stands for the set of all 2-by-2 matrices with a positive determinant. Further, $\hat{\boldsymbol{T}}(t,\boldsymbol{x},\boldsymbol{\nu}) = (\hat{T}_1(t,\boldsymbol{x},\boldsymbol{\nu}), \hat{T}_2(t,\boldsymbol{x},\boldsymbol{\nu}))$,

$$\hat{\boldsymbol{T}}(t,\boldsymbol{x},\boldsymbol{\nu}) \equiv (\boldsymbol{I} + \boldsymbol{\nabla u}(t,\boldsymbol{x}))\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{I} + \boldsymbol{\nabla u}(t,\boldsymbol{x}))\boldsymbol{\nu}, \tag{3.8}$$

represents the first Piola-Kirchhoff stress vector, $\boldsymbol{u}_D : (0,T)\times\Gamma_D \to \mathbb{R}^2$, $\boldsymbol{u}^0 : \Omega \to \mathbb{R}^2$ are known displacements and g denotes the vertical gap between the rigid foundation and the body in the reference configuration (Fig. 3.3).

Here and in what follows, we assume that $\mathscr{F} \geq 0$ is constant and the applied forces $\boldsymbol{f}$ and $\boldsymbol{h}$ are independent of the time t and of the particular deformation of the body. Moreover, $\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{I}+\boldsymbol{\nabla u}) = (\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{I}+\boldsymbol{\nabla u}))_{1\leq i,j\leq 2}$ is given by the following planar approximation of a 3D hyperelastic constitutive law with a stored energy function $W : \mathbb{M}^3_> \to \mathbb{R}$:

$$(\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}))_{1\leq i,j\leq 2} = (\hat{\hat{\boldsymbol{\sigma}}}'(\boldsymbol{F}'))_{1\leq i,j\leq 2}, \quad \boldsymbol{F}' = \begin{pmatrix} \boldsymbol{F} & \boldsymbol{0}_{2,1} \\ \boldsymbol{0}_{1,2} & 1 \end{pmatrix}, \ \boldsymbol{F} \in \mathbb{M}^2_>,$$

where

$$\hat{\hat{\boldsymbol{\sigma}}}'(\boldsymbol{F}') = (\hat{\hat{\boldsymbol{\sigma}}}'(\boldsymbol{F}'))_{1\leq i,j\leq 3} = 2\frac{\partial W}{\partial \boldsymbol{C}'}(\boldsymbol{C}'), \quad \boldsymbol{C}' = \boldsymbol{F}'^T\boldsymbol{F}' \in \mathbb{M}^3_>.$$

In particular, we consider the Ciarlet-Geymonat model:

$$W(\boldsymbol{C}') = a\,\mathrm{tr}\,\boldsymbol{C}' + b\,\mathrm{tr}\,\mathbf{Cof}\,\boldsymbol{C}' + c\det\boldsymbol{C}' - \frac{d}{2}\log\det\boldsymbol{C}' + e,$$
$$a,b,c,d>0,\ e\in\mathbb{R},\ \boldsymbol{C}'\in\mathbb{M}^3_>,$$

that is,

$$\hat{\hat{\boldsymbol{\sigma}}}'(\boldsymbol{F}') = (2a + 2b\,\mathrm{tr}\,\boldsymbol{C}')\boldsymbol{I} - 2b\boldsymbol{C}' + (2c\det\boldsymbol{C}' - d)\boldsymbol{C}'^{-1}, \quad \boldsymbol{C}' = \boldsymbol{F}'^T\boldsymbol{F}',\ \boldsymbol{F}' \in \mathbb{M}^3_>,$$

with $\mathbf{Cof}\,\boldsymbol{C}'$ being the cofactor matrix of the matrix $\boldsymbol{C}'$ ($\mathbf{Cof}\,\boldsymbol{C}' = (\det\boldsymbol{C}')\boldsymbol{C}'^{-T}$ if $\boldsymbol{C}'$ is invertible). One can easily verify that in this case,

$$\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}) = (2a+2b(\mathrm{tr}\,\boldsymbol{C}+1))\boldsymbol{I} - 2b\boldsymbol{C} + 2c\,\mathbf{Cof}\,\boldsymbol{C} - d\boldsymbol{C}^{-1}, \quad \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F},\ \boldsymbol{F}\in\mathbb{M}^2_>, \tag{3.9}$$

with

$$\mathbf{Cof}\,\boldsymbol{C} = \begin{pmatrix} C_{22} & -C_{21} \\ -C_{12} & C_{11} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in \mathbb{M}^2_>.$$

To interpret the boundary conditions on Γ_C in (3.6), we consider $t \in (0,T)$ fixed and suppose that the deformation $\boldsymbol{\varphi} \equiv \boldsymbol{id} + \boldsymbol{u}$, where $\boldsymbol{id}$ is the identity mapping, is sufficiently smooth so that $\boldsymbol{\varphi}(t,\Omega) \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz continuous boundary and $\partial\boldsymbol{\varphi}(t,\Omega) = \boldsymbol{\varphi}(t,\partial\Omega)$ (for an example of sufficient regularity, see Exercise 1.10 and Theorem 1.2-8 in [10]). Then a unit outward normal vector $\boldsymbol{\nu}^{\varphi}$ can be defined almost everywhere along $\partial\boldsymbol{\varphi}(t,\Omega)$ and one has

$$\boldsymbol{\nu}^{\varphi} = \frac{\boldsymbol{\nabla\varphi}(t,\boldsymbol{x})^{-T}\boldsymbol{\nu}}{\|\boldsymbol{\nabla\varphi}(t,\boldsymbol{x})^{-T}\boldsymbol{\nu}\|}, \quad \boldsymbol{x}\in\Gamma_C.$$

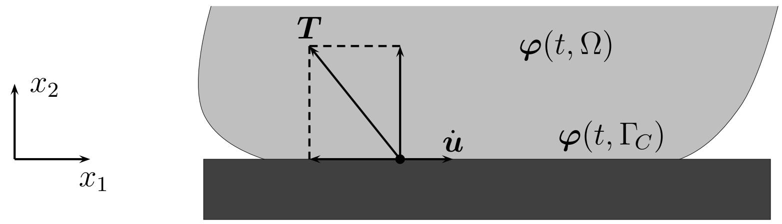


Figure 3.4: Contact with the rigid foundation

(Recall that $\boldsymbol{\nabla\varphi} = \boldsymbol{I} + \boldsymbol{\nabla u}$ is required to be regular in $[0,T] \times \overline{\Omega}$ in (3.6).)

Taking $\boldsymbol{x} \in \Gamma_C$ such that both $\boldsymbol{\nu}$ and $\boldsymbol{\nu}^{\varphi}$ are well defined and making use of (3.7) and (3.8), one obtains the following expression for the Cauchy stress vector $\boldsymbol{T}(t,\boldsymbol{x},\boldsymbol{\nu}^{\varphi})$:

$$\begin{aligned}\boldsymbol{T}(t,\boldsymbol{x},\boldsymbol{\nu}^{\varphi}) &\equiv \boldsymbol{\sigma}(\boldsymbol{\nabla\varphi}(t,\boldsymbol{x}))\boldsymbol{\nu}^{\varphi} \\ &= \frac{1}{\det \boldsymbol{\nabla\varphi}(t,\boldsymbol{x})\|\boldsymbol{\nabla\varphi}(t,\boldsymbol{x})^{-T}\boldsymbol{\nu}\|}\boldsymbol{\nabla\varphi}(t,\boldsymbol{x})\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{\nabla\varphi}(t,\boldsymbol{x}))\boldsymbol{\nu} \\ &= \frac{1}{\det \boldsymbol{\nabla\varphi}(t,\boldsymbol{x})\|\boldsymbol{\nabla\varphi}(t,\boldsymbol{x})^{-T}\boldsymbol{\nu}\|}\hat{\boldsymbol{T}}(t,\boldsymbol{x},\boldsymbol{\nu}).\end{aligned}$$

Hence, it is readily seen that the boundary conditions at $\boldsymbol{x}$ lead to the following two mutually exclusive cases:
Case (i):

$$u_2(t,\boldsymbol{x}) + g(\boldsymbol{x}) > 0, \quad \boldsymbol{T}(t,\boldsymbol{x},\boldsymbol{\nu}^{\varphi}) = \boldsymbol{0}.$$

This means that there is no contact with the rigid foundation and no surface force at the point.
Case (ii):

$$\begin{aligned}&u_2(t,\boldsymbol{x}) + g(\boldsymbol{x}) = 0, \quad T_2(t,\boldsymbol{x},\boldsymbol{\nu}^{\varphi}) \geq 0, \\ &\dot{u}_1(t,\boldsymbol{x}) = 0 \implies |T_1(t,\boldsymbol{x},\boldsymbol{\nu}^{\varphi})| \leq \mathscr{F} T_2(t,\boldsymbol{x},\boldsymbol{\nu}^{\varphi}), \\ &\dot{u}_1(t,\boldsymbol{x}) \neq 0 \implies T_1(t,\boldsymbol{x},\boldsymbol{\nu}^{\varphi}) = -\mathscr{F} T_2(t,\boldsymbol{x},\boldsymbol{\nu}^{\varphi})\frac{\dot{u}_1(t,\boldsymbol{x})}{|\dot{u}_1(t,\boldsymbol{x})|}.\end{aligned}$$

This corresponds to a contact with the rigid foundation which obeys the Coulomb law of friction in the deformed configuration (see Fig. 3.4).

Before we present a weak formulation of the problem, we shall establish some properties of the mapping $\hat{\hat{\boldsymbol{\sigma}}}$ given by (3.9). We start with analysis of the mapping $\Gamma : \mathbb{M}^2_> \to \mathbb{M}^2$ defined by

$$\Gamma(\boldsymbol{C}) := \boldsymbol{C}^{-1}, \quad \boldsymbol{C} \in \mathbb{M}^2_>.$$

Let us recall that $\mathbb{M}^2_>$ stands for the open set of all 2-by-2 matrices with a positive determinant and the matrix norm $\|.\|$ is induced by the Euclidean vector norm.

Lemma 3.1. *The mapping* Γ *is continuously differentiable on* $\mathbb{M}^2_>$ *with*

$$\Gamma'(\boldsymbol{C})\boldsymbol{E} = -\boldsymbol{C}^{-1}\boldsymbol{E}\boldsymbol{C}^{-1}, \quad \forall \boldsymbol{C} \in \mathbb{M}^2_>,\ \forall \boldsymbol{E} \in \mathbb{M}^2. \tag{3.10}$$

Moreover, for any $R > 0$*, there exist* $c_1(R), c_2(R) > 0$ *such that*

$$\|(\boldsymbol{C}+\boldsymbol{D})^{-1} - \boldsymbol{C}^{-1}\| \le c_1(R)\|\boldsymbol{D}\|,$$
$$\forall \boldsymbol{C} \in \mathbb{M}^2_>,\ \|\boldsymbol{C}^{-1}\| \le R,\ \forall \boldsymbol{D} \in \mathbb{M}^2,\ \|\boldsymbol{D}\| \le \frac{1}{2R}, \tag{3.11}$$

$$\|(\boldsymbol{C}+\boldsymbol{D})^{-1}\boldsymbol{E}(\boldsymbol{C}+\boldsymbol{D})^{-1} - \boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{C}^{-1}\| \le c_2(R)\|\boldsymbol{D}\|\|\boldsymbol{E}\|,$$
$$\forall \boldsymbol{C} \in \mathbb{M}^2_>,\ \|\boldsymbol{C}^{-1}\| \le R,\ \forall \boldsymbol{D} \in \mathbb{M}^2,\ \|\boldsymbol{D}\| \le \frac{1}{2R},\ \forall \boldsymbol{E} \in \mathbb{M}^2. \tag{3.12}$$

Proof. For any $\boldsymbol{C} \in \mathbb{M}^2_>$ and any $\boldsymbol{D} \in \mathbb{M}^2$ with $\|\boldsymbol{C}^{-1}\|\|\boldsymbol{D}\| < 1$,

$$(\boldsymbol{I} + \boldsymbol{C}^{-1}\boldsymbol{D})^{-1} = \boldsymbol{I} - \boldsymbol{C}^{-1}\boldsymbol{D} + \sum_{i\ge 2}(-\boldsymbol{C}^{-1}\boldsymbol{D})^i,$$

$$(\boldsymbol{C}+\boldsymbol{D})^{-1} = (\boldsymbol{I} + \boldsymbol{C}^{-1}\boldsymbol{D})^{-1}\boldsymbol{C}^{-1} = \boldsymbol{C}^{-1} - \boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{C}^{-1} + \sum_{i\ge 2}(-\boldsymbol{C}^{-1}\boldsymbol{D})^i\boldsymbol{C}^{-1},$$

which yields (3.10). Furthermore, (3.11) follows from

$$\|(\boldsymbol{C}+\boldsymbol{D})^{-1} - \boldsymbol{C}^{-1}\| \le \sum_{i\ge 1}\|\boldsymbol{C}^{-1}\|^i\|\boldsymbol{D}\|^i\|\boldsymbol{C}^{-1}\| \le 2\|\boldsymbol{C}^{-1}\|^2\|\boldsymbol{D}\|,$$
$$\forall \boldsymbol{C} \in \mathbb{M}^2_>,\ \forall \boldsymbol{D} \in \mathbb{M}^2,\ \|\boldsymbol{C}^{-1}\|\|\boldsymbol{D}\| \le \frac{1}{2},$$

and (3.12), ensuring the continuity of $\Gamma'(.)$, is a direct consequence of (3.11). □

Lemma 3.2. *Let the mapping* $\hat{\hat{\boldsymbol{\sigma}}} : \mathbb{M}^2_> \to \mathbb{M}^2$ *be given by (3.9) and* $R_1, R_2 > 0$ *be arbitrary. Then* $\hat{\hat{\boldsymbol{\sigma}}}$ *is continuously differentiable on* $\mathbb{M}^2_>$*, and there exist* $c_3(R_1,R_2)$*,* $c_4(R_1,R_2)$*,* $c_5(R_1,R_2)$*,* $r(R_1,R_2) > 0$ *such that for any* $\boldsymbol{F} \in \mathbb{M}^2_>$ *with* $\|\boldsymbol{F}\| \le R_1$*,* $\|\boldsymbol{F}^{-1}\| \le R_2$*,*

$$\left\|\frac{\partial(\boldsymbol{F}\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}))}{\partial \boldsymbol{F}}\boldsymbol{H}\right\| \le c_3(R_1,R_2)\|\boldsymbol{H}\| \quad \forall \boldsymbol{H} \in \mathbb{M}^2, \tag{3.13}$$

$$\left\|\frac{\partial\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}+\boldsymbol{G})}{\partial(\boldsymbol{F}+\boldsymbol{G})}\boldsymbol{H} - \frac{\partial\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F})}{\partial \boldsymbol{F}}\boldsymbol{H}\right\| \le c_4(R_1,R_2)\|\boldsymbol{G}\|\|\boldsymbol{H}\|,$$
$$\forall \boldsymbol{G} \in \mathbb{M}^2,\ \|\boldsymbol{G}\| \le r(R_1,R_2),\ \forall \boldsymbol{H} \in \mathbb{M}^2,$$

$$\left\|\frac{\partial((\boldsymbol{F}+\boldsymbol{G})\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}+\boldsymbol{G}))}{\partial(\boldsymbol{F}+\boldsymbol{G})}\boldsymbol{H} - \frac{\partial(\boldsymbol{F}\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}))}{\partial \boldsymbol{F}}\boldsymbol{H}\right\| \le c_5(R_1,R_2)\|\boldsymbol{G}\|\|\boldsymbol{H}\|,$$
$$\forall \boldsymbol{G} \in \mathbb{M}^2,\ \|\boldsymbol{G}\| \le r(R_1,R_2),\ \forall \boldsymbol{H} \in \mathbb{M}^2. \tag{3.14}$$

Proof. Let us define the mapping $\check{\hat{\boldsymbol{\sigma}}} : \mathbb{M}^2_> \to \mathbb{M}^2$ by

$$\check{\hat{\boldsymbol{\sigma}}}(\boldsymbol{C}) := (2a + 2b(\operatorname{tr}\boldsymbol{C} + 1))\boldsymbol{I} - 2b\boldsymbol{C} + 2c\,\mathbf{Cof}\,\boldsymbol{C} - d\boldsymbol{C}^{-1}, \quad \boldsymbol{C} \in \mathbb{M}^2_>,$$

so that $\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}) = \check{\hat{\boldsymbol{\sigma}}}(\boldsymbol{F}^T\boldsymbol{F})$. In view of (3.10),

$$\frac{\partial\check{\hat{\boldsymbol{\sigma}}}(\boldsymbol{C})}{\partial\boldsymbol{C}}\boldsymbol{E} = 2b(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} - 2b\boldsymbol{E} + 2c\,\mathbf{Cof}\,\boldsymbol{E} + d\boldsymbol{C}^{-1}\boldsymbol{E}\boldsymbol{C}^{-1}$$

and all the estimates result from the chain rule, (3.11) and (3.12). □

We are now at the point of introducing the weak formulation. Let $p \geq 4$, $q \geq 1$ be such that $1/p + 1/q \leq 1$. We set

$$\begin{aligned} V &:= W^{1,p}(\Omega),\\ \boldsymbol{V} &:= V \times V,\\ \boldsymbol{X}_D &:= \{\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Gamma_C) \,|\, \exists\, \boldsymbol{v} \in \boldsymbol{V} : \boldsymbol{\zeta} = \boldsymbol{v} \text{ a.e. on } \Gamma_D\},\\ X_C &:= \{\zeta \in L^2(\Gamma_C) \,|\, \exists\, v \in V : \zeta = v \text{ a.e. on } \Gamma_C\}, \end{aligned}$$

by $\boldsymbol{X}'_D$, X'_C and $\langle .,. \rangle_{\Gamma_D}$, $\langle .,. \rangle_{\Gamma_C}$ we denote the duals of $\boldsymbol{X}_D$, X_C and the corresponding duality pairings, and we define

$$\begin{aligned} \Lambda_\nu &:= \{\mu_\nu \in X'_C \,|\, \langle \mu_\nu, v\rangle_{\Gamma_C} \geq 0,\ \forall\, v \in V,\ v \leq 0 \text{ a.e. on } \Gamma_C\},\\ \Lambda_\tau(\mathscr{F}\mu_\nu) &:= \{\mu_\tau \in X'_C \,|\, \langle \mu_\tau, v\rangle_{\Gamma_C} + \langle \mathscr{F}\mu_\nu, |v|\rangle_{\Gamma_C} \leq 0,\ \forall\, v \in V\}, \quad \mu_\nu \in \Lambda_\nu,\\ A(\boldsymbol{w};\boldsymbol{v}) &:= \int_\Omega (\boldsymbol{I} + \boldsymbol{\nabla w})\hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{I} + \boldsymbol{\nabla w}) : \boldsymbol{\nabla v}\,\mathrm{d}\boldsymbol{x},\\ \ell(\boldsymbol{v}) &:= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}S, \end{aligned}$$

where $\hat{\hat{\boldsymbol{\sigma}}}$ is given by (3.9).

We shall assume that $\boldsymbol{f} \in \boldsymbol{L}^1(\Omega)$, $\boldsymbol{h} \in \boldsymbol{L}^1(\Gamma_N)$, $\boldsymbol{u}_D \in H^1(0,T;\boldsymbol{X}_D)$ and $\boldsymbol{u}^0 \in \boldsymbol{V}$ in what follows. If it is so, $\ell(\boldsymbol{v})$ is well defined for any $\boldsymbol{v} \in \boldsymbol{V}$ as Ω is bounded and the Sobolev imbedding theorem ensures that $W^{1,p}(\Omega) \subset C(\overline{\Omega})$. Moreover, Hölder's inequality implies that

$$\begin{aligned} \zeta_1\zeta_2\zeta_3\zeta_4 \in L^1(\Omega), &\quad \forall\, \zeta_1,\zeta_2,\zeta_3,\zeta_4 \in L^p(\Omega),\\ \zeta_1\zeta_2 \in L^1(\Omega), &\quad \forall\, \zeta_1 \in L^p(\Omega),\ \forall\, \zeta_2 \in L^q(\Omega), \end{aligned}$$

from which it can be easily deduced that $A(\boldsymbol{w};\boldsymbol{v})$ is well defined for any $\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}$ whenever $(\boldsymbol{I} + \boldsymbol{\nabla w})^{-1} \in \boldsymbol{L}^q(\Omega)$.

The weak formulation of (3.6) can be written as follows:

$$
\left.\begin{array}{l}
\text{Find } \boldsymbol{u} \in H^1(0,T;\boldsymbol{V}),\ \boldsymbol{\lambda}_D \in H^1(0,T;\boldsymbol{X}'_D),\ \lambda_\nu, \lambda_\tau \in H^1(0,T;X'_C) \text{ with} \\
\quad (\boldsymbol{I}+\boldsymbol{\nabla u}(t))^{-1} \in \boldsymbol{L}^q(\Omega),\ \lambda_\nu(t) \in \Lambda_\nu,\ \lambda_\tau(t) \in \Lambda_\tau(\mathscr{F}\lambda_\nu(t)) \text{ a.e. in } (0,T) \\
\quad \text{such that } \det(\boldsymbol{I}+\boldsymbol{\nabla u}(t)) > 0 \text{ a.e. in } \Omega \text{ for a.a. } t \in (0,T), \\
\quad \boldsymbol{u}(0) = \boldsymbol{u}^0 \text{ a.e. in } \Omega \text{ and} \\
A(\boldsymbol{u}(t);\boldsymbol{v}) = \ell(\boldsymbol{v}) + \langle \boldsymbol{\lambda}_D(t), \boldsymbol{v}\rangle_{\Gamma_D} + \langle \lambda_\nu(t), -v_2\rangle_{\Gamma_C} + \langle \lambda_\tau(t), v_1\rangle_{\Gamma_C}, \\
\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V} \text{ a.e. in } (0,T), \\
\langle \boldsymbol{\mu}_D, \boldsymbol{u}(t)\rangle_{\Gamma_D} = \langle \boldsymbol{\mu}_D, \boldsymbol{u}_D(t)\rangle_{\Gamma_D}, \quad \forall\, \boldsymbol{\mu}_D \in \boldsymbol{X}'_D \text{ a.e. in } (0,T), \\
\langle \mu_\nu - \lambda_\nu(t), -u_2(t) - g\rangle_{\Gamma_C} \geq 0, \quad \forall\, \mu_\nu \in \Lambda_\nu \text{ a.e. in } (0,T), \\
\langle \mu_\tau - \lambda_\tau(t), \dot{u}_1(t)\rangle_{\Gamma_C} \geq 0, \quad \forall\, \mu_\tau \in \Lambda_\tau(\mathscr{F}\lambda_\nu(t)) \text{ a.e. in } (0,T).
\end{array}\right\} \quad (\mathscr{M})
$$

Let us point out that if a solution $\boldsymbol{u}$ of (3.6) belongs to $C^1(0,T;\boldsymbol{C}^2(\overline{\Omega}))$ then it solves $(\mathscr{M})$. Indeed, let $t \in (0,T)$ be fixed. Since $(\boldsymbol{I}+\boldsymbol{\nabla u}(t)) \in \boldsymbol{C}^1(\overline{\Omega})$ and $(\boldsymbol{I}+\boldsymbol{\nabla u}(t,\boldsymbol{x})) \in \mathbb{M}^2_>$ for any $\boldsymbol{x} \in \overline{\Omega}$ according to (3.6), the continuous differentiability of the mappings Γ and $\hat{\hat{\boldsymbol{\sigma}}}$ guaranteed by Lemmas 3.1 and 3.2 implies $(\boldsymbol{I}+\boldsymbol{\nabla u}(t))^{-1}, \hat{\hat{\boldsymbol{\sigma}}}(\boldsymbol{I}+\boldsymbol{\nabla u}(t)) \in \boldsymbol{C}^1(\overline{\Omega})$. This allows us to use the Green formula and to recover $(\mathscr{M})$ in the standard manner with $\boldsymbol{\lambda}_D$ being the restriction of the first Piola-Kirchhoff stress vector $\hat{\boldsymbol{T}}$ to Γ_D and λ_ν, λ_τ being the restrictions of $-\hat{T}_2$ and $\hat{T}_1$ to Γ_C, respectively. As $(\boldsymbol{I}+\boldsymbol{\nabla u}(t))^{-1}$ is in $\boldsymbol{C}^1(\overline{\Omega})$, it is a fortiori in $\boldsymbol{L}^q(\Omega)$ for any $q \geq 1$.

Next, we present full discretization of the weak formulation. We start with semi-discretization in space. Although computations will be performed on a non-polygonal reference configuration with isoparametric finite elements in the next subsection, we restrict ourselves to the case when $\overline{\Omega}$ is a polygon for the ease of exposition here.

Let $\mathscr{T}_h$ be a triangulation of $\overline{\Omega}$ that is compatible with the decomposition of $\partial\Omega$ into Γ_D, Γ_N and Γ_C. In a similar way as in Section 2.2, we define

$$
\begin{aligned}
V^h &:= \{v^h \in C(\overline{\Omega}) \,|\, v^h|_{\mathcal{T}} \in P_2(\mathcal{T}),\ \forall\, \mathcal{T} \in \mathscr{T}_h\}, \\
\boldsymbol{V}^h &:= V^h \times V^h, \\
\boldsymbol{X}^h_D &:= \{\boldsymbol{\zeta}^h \in \boldsymbol{C}(\overline{\Gamma}_D) \,|\, \exists\, \boldsymbol{v}^h \in \boldsymbol{V}^h : \boldsymbol{\zeta}^h = \boldsymbol{v}^h \text{ on } \overline{\Gamma}_D\}, \\
X^h_C &:= \{\zeta^h \in C(\overline{\Gamma}_C) \,|\, \exists\, v^h \in V^h : \zeta^h = v^h \text{ on } \overline{\Gamma}_C\}, \\
\Lambda^h_\nu &:= \{\mu^h_\nu \in X^h_C \,|\, (\mu^h_\nu, v^h)_{0,\Gamma_C} \geq 0,\ \forall\, v^h \in V^h,\ v^h(\boldsymbol{y}_i) \leq 0,\ \forall\, i = 1,\ldots,n_c\}, \\
\Lambda^h_\tau(\mathscr{F}\mu^h_\nu) &:= \{\mu^h_\tau \in X^h_C \,|\, (\mu^h_\tau, v^h)_{0,\Gamma_C} + (\mathscr{F}\mu^h_\nu, r_h|v^h|)_{0,\Gamma_C} \leq 0,\ \forall\, v^h \in V^h\}, \quad \mu^h_\nu \in \Lambda^h_\nu,
\end{aligned}
$$

where $\{\boldsymbol{y}_i\}_{1 \leq i \leq n_c}$ is the set of nodes on $\overline{\Gamma}_C$ corresponding to the degrees of freedom of V^h and r_h denotes the Lagrange interpolation operator into X^h_C.

We suppose that we have some approximation $\boldsymbol{u}_D^h \in H^1(0,T;\boldsymbol{X}_D^h)$, $\boldsymbol{u}^{h,0} \in \boldsymbol{V}^h$ of $\boldsymbol{u}_D$ and $\boldsymbol{u}^0$, respectively, and set $g^h := r_h g$. Moreover, let us mention that we shall consider below that $\overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset$. In this case, there exists a constant $\beta > 0$ such that

$$\sup_{\mathbf{0}\neq \boldsymbol{v}^h \in \boldsymbol{V}^h} \frac{(\boldsymbol{\mu}_D^h, \boldsymbol{v}^h)_{0,\Gamma_D} + (\mu_\nu^h, -v_2^h)_{0,\Gamma_C} + (\mu_\tau^h, v_1^h)_{0,\Gamma_C}}{\|\boldsymbol{v}^h\|_{1,p,\Omega}}$$
$$\geq \beta(\|\boldsymbol{\mu}_D^h\|_{*,\Gamma_D} + \|\mu_\nu^h\|_{*,\Gamma_C} + \|\mu_\tau^h\|_{*,\Gamma_C}), \quad \forall\, \boldsymbol{\mu}_D^h \in \boldsymbol{X}_D^h,\ \forall\, \mu_\nu^h, \mu_\tau^h \in X_C^h,$$

where $\|.\|_{*,\Gamma_D}$ and $\|.\|_{*,\Gamma_C}$ stand for the dual norms in $\boldsymbol{X}_D'$ and X_C', respectively.

The spatial semi-discretization of $(\mathscr{M})$ can be introduced as follows:

$$\left.\begin{aligned}
&\text{Find } \boldsymbol{u}^h \in H^1(0,T;\boldsymbol{V}^h),\ \boldsymbol{\lambda}_D^h \in H^1(0,T;\boldsymbol{X}_D^h),\ \lambda_\nu^h, \lambda_\tau^h \in H^1(0,T;X_C^h)\\
&\quad \text{with } \lambda_\nu^h(t) \in \Lambda_\nu,\ \lambda_\tau^h(t) \in \Lambda_\tau(\mathscr{F}\lambda_\nu^h(t)) \text{ a.e. in } (0,T)\\
&\quad \text{such that } \det(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}^h(t)) > 0 \text{ in } \overline{\Omega} \text{ for a.a. } t \in (0,T),\\
&\quad \boldsymbol{u}^h(0) = \boldsymbol{u}^{h,0} \text{ in } \Omega \text{ and}\\
&A(\boldsymbol{u}^h(t);\boldsymbol{v}^h) = \ell(\boldsymbol{v}^h) + (\boldsymbol{\lambda}_D^h(t), \boldsymbol{v}^h)_{0,\Gamma_D} + (\lambda_\nu^h(t), -v_2^h)_{0,\Gamma_C}\\
&\qquad\qquad + (\lambda_\tau^h(t), v_1^h)_{0,\Gamma_C}, \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h \text{ a.e. in } (0,T),\\
&(\boldsymbol{\mu}_D^h, \boldsymbol{u}^h(t))_{0,\Gamma_D} = (\boldsymbol{\mu}_D^h, \boldsymbol{u}_D^h(t))_{0,\Gamma_D}, \quad \forall\, \boldsymbol{\mu}_D^h \in \boldsymbol{X}_D^h \text{ a.e. in } (0,T),\\
&(\mu_\nu^h - \lambda_\nu^h(t), -u_2^h(t) - g^h)_{0,\Gamma_C} \geq 0, \quad \forall\, \mu_\nu^h \in \Lambda_\nu^h \text{ a.e. in } (0,T),\\
&(\mu_\tau^h - \lambda_\tau^h(t), \dot{u}_1^h(t))_{0,\Gamma_C} \geq 0, \quad \forall\, \mu_\tau^h \in \Lambda_\tau^h(\mathscr{F}\lambda_\nu^h(t)) \text{ a.e. in } (0,T).
\end{aligned}\right\} \quad (\mathscr{M}_h)$$

The condition $\det(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}^h(t)) > 0$ in $\overline{\Omega}$ means that $\det(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}^h(t)) > 0$ in $\mathcal{T}$ for any $\mathcal{T} \in \mathscr{T}_h$. As we know, this ensures that $(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}^h(t))^{-1}|_{\mathcal{T}} \in \boldsymbol{C}(\mathcal{T})$ for any $\mathcal{T} \in \mathscr{T}_h$. Thus, $(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}^h(t))^{-1} \in \boldsymbol{L}^q(\Omega)$ for any $q \geq 1$ and the term $A(\boldsymbol{u}^h(t);\boldsymbol{v}^h)$ is well defined for any $\boldsymbol{v}^h \in \boldsymbol{V}^h$.

Let us mention, however, that we shall omit the orientation preserving condition $\det(\boldsymbol{I} + \boldsymbol{\nabla}\boldsymbol{u}^h(t)) > 0$ in $\overline{\Omega}$ hereafter for it is verified *a posteriori* in practical computations.

We shall derive algebraic formulation of $(\mathscr{M}_h)$, following Section 2.2. Still denoting the sets of the shape functions of $\boldsymbol{V}^h$ and X_C^h by $\{\boldsymbol{\phi}_i\}_{1\leq i\leq n_{\boldsymbol{u}}}$ and $\{\eta_i\}_{1\leq i\leq n_c}$, respectively, and the mapping linking the local and global numeration of the nodes on $\overline{\Gamma}_C$ by Θ, we shall denote the finite-element basis of $\boldsymbol{X}_D^h$ by $\{\boldsymbol{\xi}_i\}_{1\leq i\leq n_D}$, and in addition to the algebraic representatives of $\boldsymbol{v}^h \in \boldsymbol{V}^h$ and $\mu_C^h \in X_C^h$ defined by (2.5) and (2.6), we introduce the representative $\boldsymbol{\mu}_D \in \mathbb{R}^{n_D}$ of $\boldsymbol{\mu}_D^h \in \boldsymbol{X}_D^h$ as

$$\boldsymbol{\mu}_D = (\mu_{D,i}) \text{ such that } \boldsymbol{\mu}_D^h = \sum_{1\leq i\leq n_D} \mu_{D,i}\boldsymbol{\xi}_i.$$

Further, we set

$$
\begin{aligned}
\boldsymbol{a}(\boldsymbol{v}) &= (a_i(\boldsymbol{v})) \in \mathbb{R}^{n_{\boldsymbol{u}}},\ a_i(\boldsymbol{v}) := A\big(\textstyle\sum_{1\le j\le n_{\boldsymbol{u}}} v_j\boldsymbol{\phi}_j;\boldsymbol{\phi}_i\big),\\
\boldsymbol{B}_D &= (B_{D,ij}) \in \mathbb{M}^{n_D,n_{\boldsymbol{u}}},\ B_{D,ij} := (\boldsymbol{\xi}_i,\boldsymbol{\phi}_j)_{0,\Gamma_D},\\
\boldsymbol{B}_\nu &= (B_{\nu,ij}) \in \mathbb{M}^{n_c,n_{\boldsymbol{u}}},\ B_{\nu,ij} = -\delta_{2\Theta(i),j},\\
\boldsymbol{B}_\tau &= (B_{\tau,ij}) \in \mathbb{M}^{n_c,n_{\boldsymbol{u}}},\ B_{\tau,ij} := \delta_{2\Theta(i)-1,j},\\
\boldsymbol{f} &= (f_i) \in \mathbb{R}^{n_{\boldsymbol{u}}},\ f_i := \ell(\boldsymbol{\phi}_i),\\
\boldsymbol{u}_D(t) &= (u_{D,i}(t)) \in \mathbb{R}^{n_D},\ u_{D,i}(t) := (\boldsymbol{u}_D^h(t),\boldsymbol{\xi}_i)_{0,\Gamma_D},\quad t\in(0,T),\\
\boldsymbol{g} &= (g_i) \in \mathbb{R}^{n_c},\ g_i := g^h(\boldsymbol{y}_i),\\
\boldsymbol{\Lambda}_\nu &:= \mathbb{R}^{n_c}_-,\\
\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\mu}_\nu) &:= \{\boldsymbol{\mu}_\tau \in \mathbb{R}^{n_c} \,|\, |\mu_{\tau,i}| \le -\mathscr{F}\mu_{\nu,i},\ \forall i=1,\dots,n_c\},\quad \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu.
\end{aligned}
$$

We obtain the following problem:

$$
\left.
\begin{aligned}
&\text{Find } \boldsymbol{u}\in H^1(0,T;\mathbb{R}^{n_{\boldsymbol{u}}}),\ \boldsymbol{\lambda}_D\in H^1(0,T;\mathbb{R}^{n_D}),\ \boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_\tau\in H^1(0,T;\mathbb{R}^{n_c})\\
&\quad \text{with } \boldsymbol{\lambda}_\nu(t)\in\boldsymbol{\Lambda}_\nu,\ \boldsymbol{\lambda}_\tau(t)\in\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)) \text{ a.e. in } (0,T)\\
&\quad \text{such that } \boldsymbol{u}(0)=\boldsymbol{u}^0 \text{ in } \Omega \text{ and}\\
&\boldsymbol{a}(\boldsymbol{u}(t)) = \boldsymbol{f} + \boldsymbol{B}_D^T\boldsymbol{\lambda}_D(t) + \boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu(t) + \boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau(t) \quad \text{a.e. in } (0,T),\\
&\boldsymbol{B}_D\boldsymbol{u}(t) = \boldsymbol{u}_D(t) \quad \text{a.e. in } (0,T),\\
&(\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu(t), \boldsymbol{B}_\nu\boldsymbol{u}(t) - \boldsymbol{g}) \ge 0, \quad \forall\,\boldsymbol{\mu}_\nu\in\boldsymbol{\Lambda}_\nu \text{ a.e. in } (0,T),\\
&(\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau(t), \boldsymbol{B}_\tau\dot{\boldsymbol{u}}(t)) \ge 0, \quad \forall\,\boldsymbol{\mu}_\tau\in\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)) \text{ a.e. in } (0,T).
\end{aligned}
\right\}\qquad (\mathcal{M})
$$

Time discretization of this problem is done by dividing the interval $[0,T]$ uniformly into n_T subintervals, setting $\Delta t := T/n_T$, $t_k := k\Delta t$, $k = 0,1,\dots,n_T$, and approximating the derivative $\dot{\boldsymbol{u}}$ by the backward difference. We arrive at the sequence of the following incremental problems for $k = 0,\dots,n_T-1$:

$$
\left.
\begin{aligned}
&\text{Find } \boldsymbol{u}^{k+1}\in\mathbb{R}^{n_{\boldsymbol{u}}},\ \boldsymbol{\lambda}_D^{k+1}\in\mathbb{R}^{n_D},\ \boldsymbol{\lambda}_\nu^{k+1}\in\boldsymbol{\Lambda}_\nu,\ \boldsymbol{\lambda}_\tau^{k+1}\in\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu^{k+1}) \text{ such that}\\
&\boldsymbol{a}(\boldsymbol{u}^{k+1}) = \boldsymbol{f} + \boldsymbol{B}_D^T\boldsymbol{\lambda}_D^{k+1} + \boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu^{k+1} + \boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau^{k+1},\\
&\boldsymbol{B}_D\boldsymbol{u}^{k+1} = \boldsymbol{u}_D(t_{k+1}),\\
&(\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu^{k+1}, \boldsymbol{B}_\nu\boldsymbol{u}^{k+1} - \boldsymbol{g}) \ge 0, \quad \forall\,\boldsymbol{\mu}_\nu\in\boldsymbol{\Lambda}_\nu,\\
&\Big(\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau^{k+1}, \frac{1}{\Delta t}(\boldsymbol{B}_\tau\boldsymbol{u}^{k+1} - \boldsymbol{B}_\tau\boldsymbol{u}^k)\Big) \ge 0, \quad \forall\,\boldsymbol{\mu}_\tau\in\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu^{k+1})
\end{aligned}
\right\}
$$

or equivalently

$$
\left.
\begin{aligned}
&\text{Find } \boldsymbol{y}^{k+1} := (\boldsymbol{u}^{k+1},\boldsymbol{\lambda}_D^{k+1},\boldsymbol{\lambda}_\nu^{k+1},\boldsymbol{\lambda}_\tau^{k+1}) \in \mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c} \text{ such that}\\
&\boldsymbol{\mathcal{H}}_{k+1}(\boldsymbol{y}^{k+1}) = \boldsymbol{0},
\end{aligned}
\right\}\qquad (\mathcal{M}_{k+1})
$$

where $\boldsymbol{\mathcal{H}}_{k+1} : \mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c} \to \mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c}$ is introduced by

$$\boldsymbol{\mathcal{H}}_{k+1}(\boldsymbol{y}) := \begin{pmatrix} \boldsymbol{a}(\boldsymbol{u}) - \boldsymbol{f} - \boldsymbol{B}_D^T\boldsymbol{\lambda}_D - \boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu - \boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau \\ \boldsymbol{B}_D\boldsymbol{u} - \boldsymbol{u}_D(t_{k+1}) \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu - r\alpha(\boldsymbol{B}_\nu\boldsymbol{u} - \boldsymbol{g})) \\ \boldsymbol{\lambda}_\tau - \boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau - \frac{r}{\Delta t}(\boldsymbol{B}_\tau\boldsymbol{u} - \boldsymbol{B}_\tau\boldsymbol{u}^k)) \end{pmatrix},$$

$$\boldsymbol{y} := (\boldsymbol{u}, \boldsymbol{\lambda}_D, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau) \in \mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c}, \quad (3.15)$$

$r, \alpha > 0$ are fixed parameters and $\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}$, $\boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu)}$ are defined by (3.1) and (3.2).

Let us recall that in fact, we are seeking such solutions $\boldsymbol{y}^{k+1}$ of $(\mathcal{M}_{k+1})$ for which the orientation preserving condition $\det\big(\boldsymbol{I}+\sum_{1\le j\le n_{\boldsymbol{u}}} u_j^{k+1}\boldsymbol{\nabla}\boldsymbol{\phi}_j\big) > 0$ is satisfied in $\overline{\Omega}$. The following result establishes differentiability property of the nonlinear mapping $\boldsymbol{a} : \mathbb{R}^{n_{\boldsymbol{u}}} \to \mathbb{R}^{n_{\boldsymbol{u}}}$ at the corresponding vectors $\boldsymbol{u}^{k+1}$.

Proposition 3.1. *For any $\boldsymbol{w} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ with*

$$\det\big(\boldsymbol{I} + \textstyle\sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big) > 0 \quad \text{in } \overline{\Omega}, \quad (3.16)$$

there exists an open neighbourhood $U \subset \mathbb{R}^{n_{\boldsymbol{u}}}$ of $\boldsymbol{w}$ in which $\boldsymbol{a}$ is continuously differentiable.

Proof. Let $i \in \{1, \dots, n_{\boldsymbol{u}}\}$ and $\boldsymbol{w} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ satisfying (3.16) be arbitrarily chosen. With regard to the equality

$$a_i(\boldsymbol{w}) = \int_\Omega \big(\boldsymbol{I} + \textstyle\sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big)\hat{\hat{\boldsymbol{\sigma}}}\big(\boldsymbol{I} + \sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big) : \boldsymbol{\nabla}\boldsymbol{\phi}_i \,\mathrm{d}\boldsymbol{x},$$

a natural candidate for $a_i'(\boldsymbol{w})$ is the linear mapping $L(\boldsymbol{w};.)$ defined by

$$L(\boldsymbol{w};\boldsymbol{v}) := \int_\Omega \frac{\partial\big(\big(\boldsymbol{I} + \sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big)\hat{\hat{\boldsymbol{\sigma}}}\big(\boldsymbol{I} + \sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big)\big)}{\partial\big(\boldsymbol{I} + \sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big)} \cdot \big(\textstyle\sum_{1\le j\le n_{\boldsymbol{u}}} v_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big) : \boldsymbol{\nabla}\boldsymbol{\phi}_i \,\mathrm{d}\boldsymbol{x}.$$

We shall prove that $L(\boldsymbol{w};.)$ is really a differential of a_i at $\boldsymbol{w}$ first.

Since for any $j \in \{1, \dots, n_{\boldsymbol{u}}\}$ and any $\mathcal{T} \in \mathscr{T}_h$, the basis function $\boldsymbol{\phi}_j$ restricted to $\mathcal{T}$ is in $\boldsymbol{C}^1(\mathcal{T})$, the restrictions of $\boldsymbol{\nabla}\boldsymbol{\phi}_j$, $\big(\boldsymbol{I}+\sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big)$ on $\mathcal{T}$ are in $\boldsymbol{C}(\mathcal{T})$. Moreover, in virtue of Lemma 3.1 and the assumption guaranteeing that $\det\big(\boldsymbol{I} + \sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big) > 0$ in $\mathcal{T}$, $\big(\boldsymbol{I} + \sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j\big)^{-1}\big|_{\mathcal{T}}$ belongs to $\boldsymbol{C}(\mathcal{T})$, as well. Therefore, there exist constants $c_6, R_1, R_2 > 0$ such that for any $\boldsymbol{x} \in \overline{\Omega}$,

$$\big(\textstyle\sum_{1\le j\le n_{\boldsymbol{u}}}\|\boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\|^2\big)^{1/2} \le c_6,$$
$$\big\|\big(\boldsymbol{I} + \textstyle\sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big)\big\| \le R_1, \quad \big\|\big(\boldsymbol{I} + \sum_{1\le j\le n_{\boldsymbol{u}}} w_j\boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big)^{-1}\big\| \le R_2. \quad (3.17)$$

Obviously, there is also a constant $c_7 > 0$ such that

$$|\boldsymbol{F} : \boldsymbol{G}| \leq c_7 \|\boldsymbol{F}\| \|\boldsymbol{G}\|, \quad \forall\, \boldsymbol{F}, \boldsymbol{G} \in \mathbb{M}^2.$$

From (3.13), it then follows that

$$|L(\boldsymbol{w}; \boldsymbol{v})| \leq c_3(R_1, R_2) c_6^2 c_7 \operatorname{meas}(\Omega) \|\boldsymbol{v}\|, \quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}},$$

that is, $L(\boldsymbol{w}; .)$ is continuous.

Furthermore, the Taylor-MacLaurin formula implies that for any $\boldsymbol{v} \in \mathbb{R}^{n_u}$ and any $\boldsymbol{x} \in \overline{\Omega}$ there exists $\vartheta_{\boldsymbol{v},\boldsymbol{x}} \in (0,1)$ satisfying

$$\begin{aligned}
&\big(\boldsymbol{I} + \textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + v_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big) \hat{\hat{\boldsymbol{\sigma}}}\big(\boldsymbol{I} + \textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + v_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big) : \boldsymbol{\nabla}\boldsymbol{\phi}_i(\boldsymbol{x}) \\
&\quad - \big(\boldsymbol{I} + \textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big) \hat{\hat{\boldsymbol{\sigma}}}\big(\boldsymbol{I} + \textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big) : \boldsymbol{\nabla}\boldsymbol{\phi}_i(\boldsymbol{x}) \\
&\quad = \frac{\partial\Big(\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + \vartheta_{\boldsymbol{v},\boldsymbol{x}} v_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big) \hat{\hat{\boldsymbol{\sigma}}}\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + \vartheta_{\boldsymbol{v},\boldsymbol{x}} v_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big)\Big)}{\partial\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + \vartheta_{\boldsymbol{v},\boldsymbol{x}} v_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big)} \\
&\qquad \cdot \big(\textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} v_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big) : \boldsymbol{\nabla}\boldsymbol{\phi}_i(\boldsymbol{x}).
\end{aligned}$$

In light of (3.14),

$$\begin{aligned}
&|a_i(\boldsymbol{w} + \boldsymbol{v}) - a_i(\boldsymbol{w}) - L(\boldsymbol{w}; \boldsymbol{v})| \\
&\quad = \Bigg| \int_\Omega \Bigg(\frac{\partial\Big(\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + \vartheta_{\boldsymbol{v},\boldsymbol{x}} v_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j\big) \hat{\hat{\boldsymbol{\sigma}}}\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + \vartheta_{\boldsymbol{v},\boldsymbol{x}} v_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j\big)\Big)}{\partial\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + \vartheta_{\boldsymbol{v},\boldsymbol{x}} v_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j\big)} \\
&\qquad\qquad - \frac{\partial\Big(\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j\big) \hat{\hat{\boldsymbol{\sigma}}}\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j\big)\Big)}{\partial\big(\boldsymbol{I} + \sum_{1\leq j\leq n_{\boldsymbol{u}}} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j\big)} \Bigg) \\
&\qquad\qquad \cdot \big(\textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} v_j \boldsymbol{\nabla}\boldsymbol{\phi}_j\big) : \boldsymbol{\nabla}\boldsymbol{\phi}_i \, \mathrm{d}\boldsymbol{x} \Bigg| \\
&\quad \leq c_5(R_1, R_2) c_6^3 c_7 \operatorname{meas}(\Omega) \|\boldsymbol{v}\|^2, \quad \forall\, \boldsymbol{v} \in \mathbb{R}^{n_{\boldsymbol{u}}},\ \|\boldsymbol{v}\| \leq \frac{r(R_1, R_2)}{c_6},
\end{aligned}$$

which verifies that $a_i'(\boldsymbol{w}) = L(\boldsymbol{w}; .)$.

Finally, we shall show that $a_i'(.)$ is continuous on the set

$$U := \{\boldsymbol{w}\} + \Big\{ \boldsymbol{z} \in \mathbb{R}^{n_{\boldsymbol{u}}} \,\Big|\, \|\boldsymbol{z}\| \leq \frac{1}{2R_2} \Big\}.$$

Taking any $\boldsymbol{z} \in U - \{\boldsymbol{w}\}$ and any $\boldsymbol{x} \in \overline{\Omega}$, one has

$$\begin{aligned}
&\big\| \big(\boldsymbol{I} + \textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} (w_j + z_j) \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big) \big\| \\
&\qquad \leq \big\| \big(\boldsymbol{I} + \textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} w_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x})\big) \big\| + \big\| \textstyle\sum_{1\leq j\leq n_{\boldsymbol{u}}} z_j \boldsymbol{\nabla}\boldsymbol{\phi}_j(\boldsymbol{x}) \big\| \leq R_1 + \dfrac{c_6}{2R_2} =: \tilde{R}_1
\end{aligned}$$

and from (3.11) one gets

$$\begin{aligned}
&\Big\|\big(\boldsymbol{I}+\textstyle\sum_{1\le j\le n_{\boldsymbol{u}}}(w_j+z_j)\boldsymbol{\nabla\phi}_j(\boldsymbol{x})\big)^{-1}\Big\|\\
&\qquad\le \Big\|\big(\boldsymbol{I}+\textstyle\sum_{1\le j\le n_{\boldsymbol{u}}}w_j\boldsymbol{\nabla\phi}_j(\boldsymbol{x})\big)^{-1}\Big\| + c_1(R_2)\Big\|\textstyle\sum_{1\le j\le n_{\boldsymbol{u}}}z_j\boldsymbol{\nabla\phi}_j(\boldsymbol{x})\Big\|\\
&\qquad\le R_2+\frac{c_1(R_2)c_6}{2R_2}=:\tilde{R}_2.
\end{aligned}$$

Another application of (3.14) yields

$$\begin{aligned}
&|a_i'(\boldsymbol{w}+\boldsymbol{z}+\boldsymbol{s})\boldsymbol{v}-a_i'(\boldsymbol{w}+\boldsymbol{z})\boldsymbol{v}|\\
&=\Bigg|\int_\Omega\Bigg(\frac{\partial\Big(\big(\boldsymbol{I}+\sum_{1\le j\le n_{\boldsymbol{u}}}(w_j+z_j+s_j)\boldsymbol{\nabla\phi}_j\big)\hat{\boldsymbol{\sigma}}\big(\boldsymbol{I}+\sum_{1\le j\le n_{\boldsymbol{u}}}(w_j+z_j+s_j)\boldsymbol{\nabla\phi}_j\big)\Big)}{\partial\big(\boldsymbol{I}+\sum_{1\le j\le n_{\boldsymbol{u}}}(w_j+z_j+s_j)\boldsymbol{\nabla\phi}_j\big)}\\
&\qquad\qquad-\frac{\partial\Big(\big(\boldsymbol{I}+\sum_{1\le j\le n_{\boldsymbol{u}}}(w_j+z_j)\boldsymbol{\nabla\phi}_j\big)\hat{\boldsymbol{\sigma}}\big(\boldsymbol{I}+\sum_{1\le j\le n_{\boldsymbol{u}}}(w_j+z_j)\boldsymbol{\nabla\phi}_j\big)\Big)}{\partial\big(\boldsymbol{I}+\sum_{1\le j\le n_{\boldsymbol{u}}}(w_j+z_j)\boldsymbol{\nabla\phi}_j\big)}\Bigg)\\
&\qquad\qquad\qquad\qquad\cdot\big(\textstyle\sum_{1\le j\le n_{\boldsymbol{u}}}v_j\boldsymbol{\nabla\phi}_j\big):\boldsymbol{\nabla\phi}_i\,\mathrm{d}\boldsymbol{x}\Bigg|\\
&\le c_5(\tilde{R}_1,\tilde{R}_2)c_6^3c_7\,\mathrm{meas}(\Omega)\|\boldsymbol{s}\|\|\boldsymbol{v}\|,\\
&\qquad\qquad\forall\,\boldsymbol{z}\in U-\{\boldsymbol{w}\},\ \forall\,\boldsymbol{s}\in\mathbb{R}^{n_{\boldsymbol{u}}},\ \|\boldsymbol{s}\|\le\frac{r(\tilde{R}_1,\tilde{R}_2)}{c_6},\ \forall\,\boldsymbol{v}\in\mathbb{R}^{n_{\boldsymbol{u}}},
\end{aligned}$$

and the proof is complete. □

Combining this proposition together with the analysis in Section 2.3, one can see that the mapping $\boldsymbol{\mathcal{H}}_{k+1}$ is piecewise differentiable on the open set

$$\big\{(\boldsymbol{u},\boldsymbol{\lambda}_D,\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_\tau)\in\mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c}\,\big|\,\det\big(\boldsymbol{I}+\textstyle\sum_{1\le j\le n_{\boldsymbol{u}}}u_j\boldsymbol{\nabla\phi}_j\big)>0\text{ in }\overline{\Omega}\big\}.$$

This justifies use of the piecewise smooth Newton method with a line search for solving $(\mathcal{M}_{k+1})$ (see [18, Chapter 7], [39]). Nevertheless, as we shall see in the next subsection, one can encounter situations where this method is not able to find any solution. For this reason, we propose here an adaptation of the numerical continuation described in the previous section.

To this end, we take a linear path

$$\gamma\in\mathbb{R}\mapsto\boldsymbol{u}_D^{k,k+1}(\gamma):=\boldsymbol{u}_D(t_k)+\gamma(\boldsymbol{u}_D(t_{k+1})-\boldsymbol{u}_D(t_k))$$

and define $\boldsymbol{\mathcal{H}}_{k,k+1}:\mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c+1}\to\mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c}$ by

$$\boldsymbol{\mathcal{H}}_{k,k+1}(\boldsymbol{y}):=\begin{pmatrix}\boldsymbol{a}(\boldsymbol{u})-\boldsymbol{f}-\boldsymbol{B}_D^T\boldsymbol{\lambda}_D-\boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu-\boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau\\ \boldsymbol{B}_D\boldsymbol{u}-\boldsymbol{u}_D^{k,k+1}(\gamma)\\ \boldsymbol{\lambda}_\nu-\boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu-r\alpha(\boldsymbol{B}_\nu\boldsymbol{u}-\boldsymbol{g}))\\ \boldsymbol{\lambda}_\tau-\boldsymbol{P}_{\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_\tau-\frac{r}{\Delta t}(\boldsymbol{B}_\tau\boldsymbol{u}-\boldsymbol{B}_\tau\boldsymbol{u}^k))\end{pmatrix},$$

$$\boldsymbol{y}:=(\boldsymbol{u},\boldsymbol{\lambda}_D,\boldsymbol{\lambda}_\nu,\boldsymbol{\lambda}_\tau,\gamma)\in\mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c+1}.$$

Observe that on the one hand, $(\boldsymbol{u}^{k+1}, \boldsymbol{\lambda}_D^{k+1}, \boldsymbol{\lambda}_\nu^{k+1}, \boldsymbol{\lambda}_\tau^{k+1})$ solves $(\mathcal{M}_{k+1})$ if and only if $\boldsymbol{\mathcal{H}}_{k,k+1}(\boldsymbol{u}^{k+1}, \boldsymbol{\lambda}_D^{k+1}, \boldsymbol{\lambda}_\nu^{k+1}, \boldsymbol{\lambda}_\tau^{k+1}, 1) = \boldsymbol{0}$. On the other hand, one can easily verify that $\boldsymbol{\mathcal{H}}_{k,k+1}(\boldsymbol{u}^{k}, \boldsymbol{\lambda}_D^{k}, \boldsymbol{\lambda}_\nu^{k}, \boldsymbol{\lambda}_\tau^{k}, 0) = \boldsymbol{0}$.

This leads us to the following possibility of numerical realization of $(\mathcal{M}_{k+1})$: Taking $(\boldsymbol{u}^{k}, \boldsymbol{\lambda}_D^{k}, \boldsymbol{\lambda}_\nu^{k}, \boldsymbol{\lambda}_\tau^{k}, 0)$ as a starting point, we shall apply the numerical continuation for tracing the solution set of the system:

$$\left.\begin{array}{l}\text{Find } \boldsymbol{y} \in \mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c+1} \text{ such that}\\ \boldsymbol{\mathcal{H}}_{k,k+1}(\boldsymbol{y}) = \boldsymbol{0},\end{array}\right\}$$

until we reach a point from the set $\mathbb{R}^{n_{\boldsymbol{u}}+n_D+2n_c} \times \{1\}$.

An attentive reader has surely noticed that it may be not so easy to compute a tangent $\boldsymbol{t}_0^{k+1}$ at the initial point $\boldsymbol{y}_0^{k+1} := (\boldsymbol{u}^{k}, \boldsymbol{\lambda}_D^{k}, \boldsymbol{\lambda}_\nu^{k}, \boldsymbol{\lambda}_\tau^{k}, 0)$ from the equation

$$\boldsymbol{\mathcal{H}}'_{k,k+1}(\boldsymbol{y}_0^{k+1}; \boldsymbol{t}_0^{k+1}) = \boldsymbol{0}$$

(confer the initialization of Algorithm 3.1). Indeed, more selection functions for $\boldsymbol{\mathcal{H}}_{k,k+1}$ may be active at $\boldsymbol{y}_0^{k+1}$. To see this, consider that $\boldsymbol{u}^k$ is such that the jth node is sliding. Then $|\lambda_{\tau,j}^k| = -\mathscr{F}\lambda_{\nu,j}^k$ whereas the corresponding components of the second and the third test function in the $(k+1)$th time step take the following form:

$$\begin{aligned}\theta_{2,j}^k(\boldsymbol{y}) &= \big(\boldsymbol{\lambda}_\tau - \frac{r}{\Delta t}(\boldsymbol{B}_\tau \boldsymbol{u} - \boldsymbol{B}_\tau \boldsymbol{u}^k)\big)_j - \mathscr{F}\lambda_{\nu,j},\\ \theta_{3,j}^k(\boldsymbol{y}) &= \big(\boldsymbol{\lambda}_\tau - \frac{r}{\Delta t}(\boldsymbol{B}_\tau \boldsymbol{u} - \boldsymbol{B}_\tau \boldsymbol{u}^k)\big)_j + \mathscr{F}\lambda_{\nu,j}.\end{aligned}$$

Therefore, one of them vanishes necessarily at $\boldsymbol{y}_0^{k+1}$.

To be precise, we determine the vector $\boldsymbol{t}_0^{k+1}$ in our computations simply as a solution of the system

$$\boldsymbol{\nabla}\boldsymbol{\mathcal{H}}_{k,k+1}^{(i)}(\boldsymbol{y}_0^{k+1})\boldsymbol{t}_0^{k+1} = \boldsymbol{0},$$

where the selection function $\boldsymbol{\mathcal{H}}_{k,k+1}^{(i)}$ for $\boldsymbol{\mathcal{H}}_{k,k+1}$ is determined from the 3-by-n_c array obtained from the values of $\boldsymbol{\theta}_2^{k-1}(\boldsymbol{y}_0^{k+1})$ and $\boldsymbol{\theta}_3^{k-1}(\boldsymbol{y}_0^{k+1})$ instead of $\boldsymbol{\theta}_2^{k}(\boldsymbol{y}_0^{k+1})$ and $\boldsymbol{\theta}_3^{k}(\boldsymbol{y}_0^{k+1})$, that is, from the values

$$\begin{aligned}\theta_{2,j}^{k-1}(\boldsymbol{y}_0^{k+1}) &= \big(\boldsymbol{\lambda}_\tau^k - \frac{r}{\Delta t}(\boldsymbol{B}_\tau \boldsymbol{u}^k - \boldsymbol{B}_\tau \boldsymbol{u}^{k-1})\big)_j - \mathscr{F}\lambda_{\nu,j}^k,\\ \theta_{3,j}^{k-1}(\boldsymbol{y}_0^{k+1}) &= \big(\boldsymbol{\lambda}_\tau^k - \frac{r}{\Delta t}(\boldsymbol{B}_\tau \boldsymbol{u}^k - \boldsymbol{B}_\tau \boldsymbol{u}^{k-1})\big)_j + \mathscr{F}\lambda_{\nu,j}^k\end{aligned}$$

for $j = 1, \ldots, n_c$.

Nevertheless, let us note that for the same reason, one may face difficulties also when using the piecewise smooth Newton method and taking the solution from the previous time step as the initial approximation. Since the function $\boldsymbol{\mathcal{H}}_{k+1}$ defined by (3.15) may not be smooth at this initial point, it is not clear at all due to rounding errors which gradient will be selected in the first iteration of each time step. This may cause the method not to converge in some cases although the initial approximation is not far away from the actual solution.

3.2.2 Numerical Experiments

To illustrate usefulness of the proposed continuation technique, we present here an example coming from a technical practise. First of all, we solved it with the piecewise smooth Newton method with a line search, which showed to be, however, short in some situations as we shall see later on.

The reference configuration of the elastic body is represented by a rectangle with "rounded corners" whose length and height are 20 mm and 10 mm, respectively (Fig. 3.5). The body is unilaterally supported from its lower side and loaded via the following Dirichlet condition imposed on its upper side:

$$\boldsymbol{u}_D(t,\boldsymbol{x}) = \begin{cases} \left(0, \dfrac{-0.4t}{1.1\cdot 10^{-5}}\right) & \text{if } t \le 1.1\cdot 10^{-5}, \\ \left(13800(t-1.1\cdot 10^{-5}), -0.4\right) & \text{if } t > 1.1\cdot 10^{-5}, \end{cases} \qquad \boldsymbol{x}\in\Gamma_D.$$

The body and surface forces are neglected, that is, $\boldsymbol{f}=\boldsymbol{0}$, $\boldsymbol{h}=\boldsymbol{0}$. The coefficient of friction $\mathscr{F}$ is chosen to be 1 and we set $\boldsymbol{u}^0=\boldsymbol{0}$ in Ω. The coefficients a, b, c, d in the constitutive law (3.9) are determined as follows:

$$a=\mu+\frac{\delta}{2},\ b=-\frac{\mu+\delta}{2},\ c=\frac{\lambda}{4}+\frac{\mu+\delta}{2},\ d=\frac{\lambda}{2}+\mu,$$

where

$$\lambda = 4000\,\mathrm{N/mm^2},\ \mu=120\,\mathrm{N/mm^2},\ \delta=-180\,\mathrm{N/mm^2}.$$

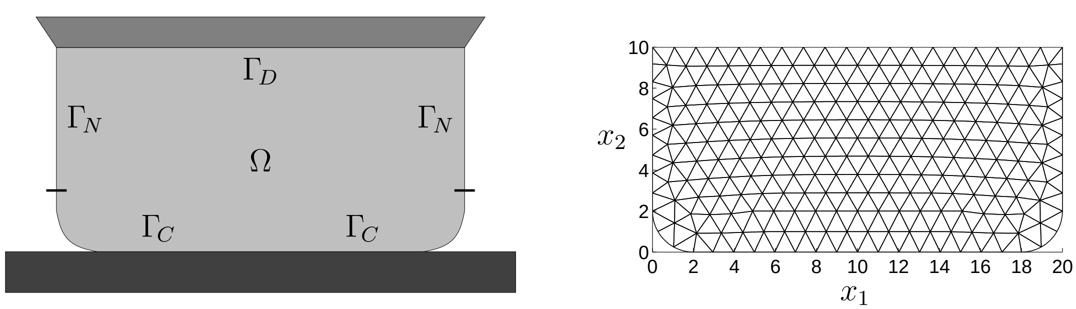


Figure 3.5: Reference configuration of the example with the unstructured finite-element mesh of the body

Isoparametric P_2 finite elements are used for the spatial semi-discretization and $\Delta t = 10^{-5}$ s is taken for the discretization in time. The programme for performing the tests employs the finite element library GetFEM++ [54].

If one solves the example on the uniform finite-element mesh depicted in Fig. 3.5, one can compute a sequence of solutions by the piecewise smooth Newton method with a line search till the time $t = 0.00051$ s (for the corresponding deformed body, see Fig. 3.6 and its zoom in the lower right-hand corner on the left of Fig. 3.7). However, this method does not converge in the next time step. Even if one diminishes the time step length Δt, there still exists some threshold where it stops converging. This is why we used the numerical continuation here. The obtained solution curve of the auxiliary problem is illustrated in Fig. 3.8(a) by the vertical displacement of a node which rebounds from the rigid foundation in course of the continuation. Notice that this curve explains the limited behaviour of the Newton method. Since it folds up, there is always a discontinuity of the solutions in time whatever small the time step is (Fig. 3.7)!

Further, we had to apply the continuation method once more for solving the problem for the time $t = 0.00061$ s (Figs. 3.8(b) and 3.9). In this case, the con-

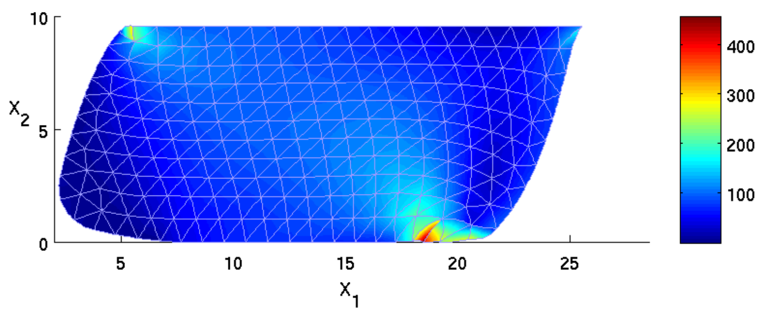


Figure 3.6: Deformed body in time $t = 0.00051$ s coloured by the values of the corresponding Von Mises stress in N/mm^2

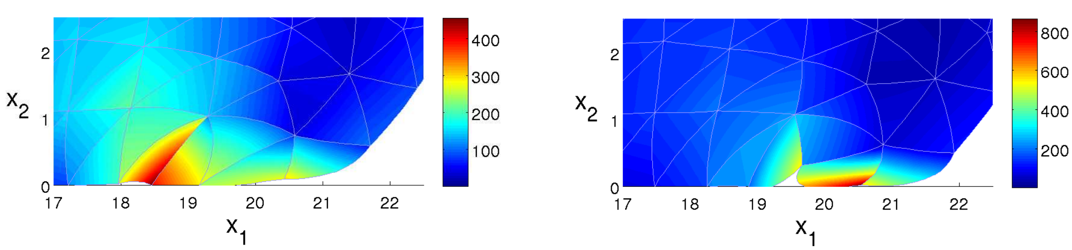
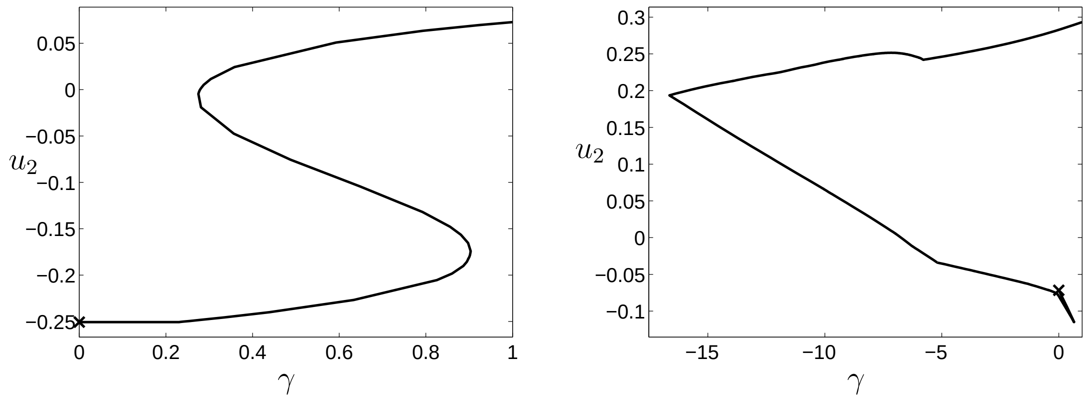


Figure 3.7: Jump of the solution between $t = 0.00051$ s (on the left) and $t = 0.00052$ s (on the right)



(a) Auxiliary problem $\boldsymbol{\mathcal{H}}_{51,52}(\boldsymbol{y}) = \mathbf{0}$ (b) Auxiliary problem $\boldsymbol{\mathcal{H}}_{60,61}(\boldsymbol{y}) = \mathbf{0}$

Figure 3.8: Vertical displacement of the node with the coordinates (18.9695 mm, 0.250673 mm) in the reference configuration in course of the continuations; the starting points are denoted by crosses

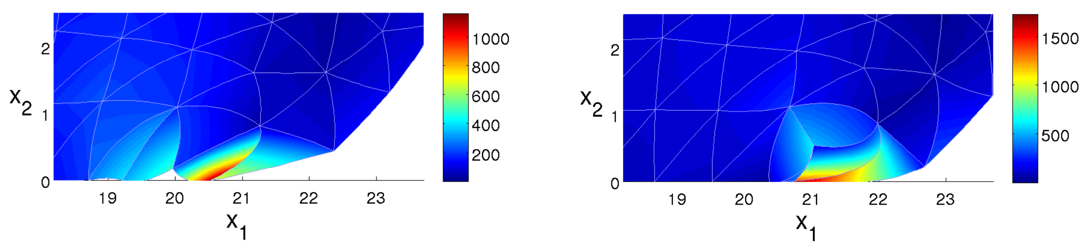


Figure 3.9: Jump of the solution between $t = 0.0006$ s (on the left) and $t = 0.00061$ s (on the right)

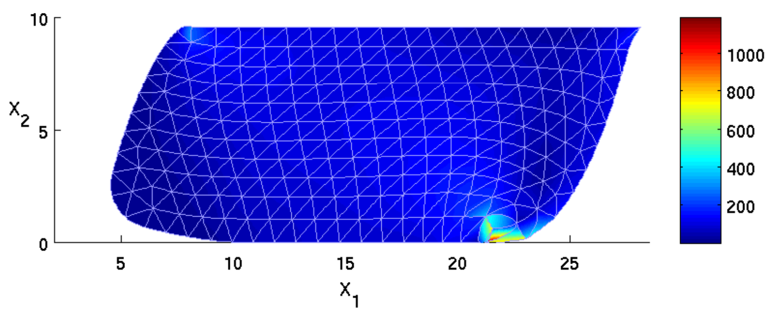


Figure 3.10: Deformed body in time $t = 0.0007$ s

tinuation parameter γ was far below zero in course of the continuation, and the resulting jump of the solution is larger than the previous one (there is even a small self-interpenetration of the body in $t = 0.00061\,\mathrm{s}$). Besides, both turning points of the solution curve are non-smooth. Despite it all, the method works well.

Let us mention that from $t = 0$ up to $t = 0.00069\,\mathrm{s}$, the body is stuck to the foundation by its lower right-hand corner. It starts sliding with its entire volume by $t = 0.0007\,\mathrm{s}$ (Fig. 3.10).

Next, we repeated the same experiment with a mesh once locally refined in its lower right-hand corner. In this case, we needed to continue two times – for $t = 0.00053\,\mathrm{s}$ and $t = 0.00056\,\mathrm{s}$. It is a bit curious that in the first case, we were not able to find any point with $\gamma > 0$ near the starting point of the continuation. Nevertheless, we arrived at a wanted point with $\gamma = 1$ in the end (Figs. 3.11 and 3.12).

In the second case, we got a circular solution curve, for a change (Fig. 3.13(a)). For this reason, we tried to continue between $t = 0.00054\,\mathrm{s}$ and $t = 0.00056\,\mathrm{s}$ in-

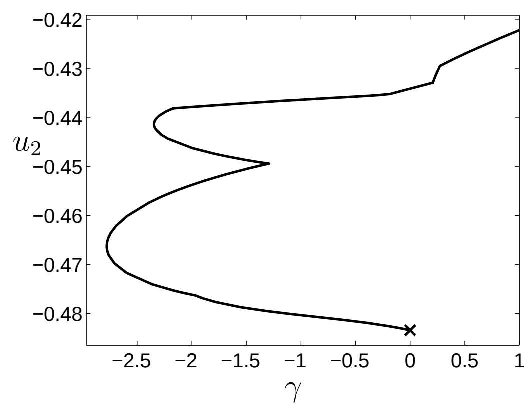


Figure 3.11: Vertical displacement of the node with the coordinates (19.4044 mm, 0.576046 mm) in the reference configuration in course of the continuation of the auxiliary problem $\boldsymbol{\mathcal{H}}_{52,53}(\boldsymbol{y}) = \mathbf{0}$

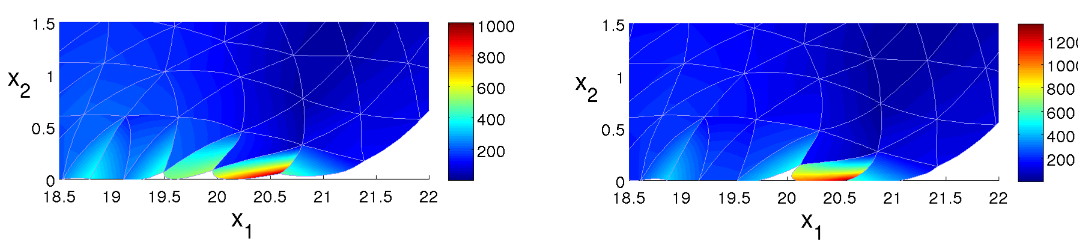
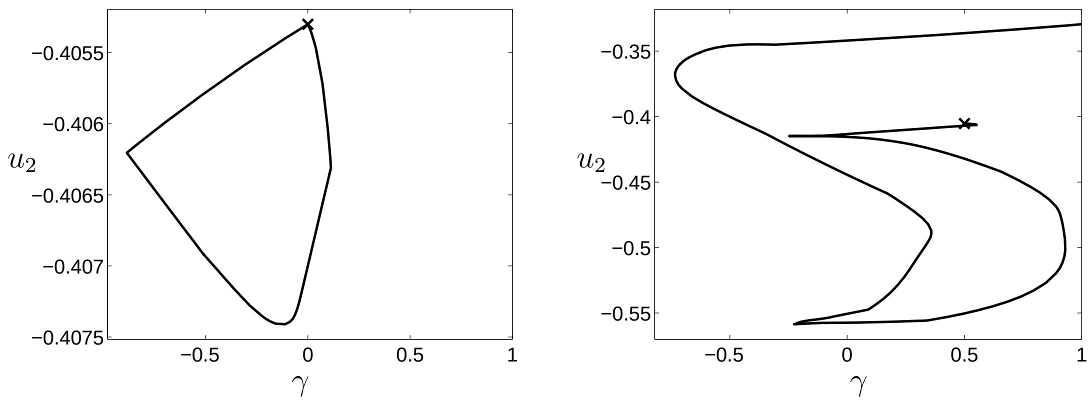


Figure 3.12: Jump of the solution between $t = 0.00052\,\mathrm{s}$ (on the left) and $t = 0.00053\,\mathrm{s}$ (on the right)



(a) Auxiliary problem $\boldsymbol{\mathcal{H}}_{55,56}(\boldsymbol{y}) = \mathbf{0}$ (b) Auxiliary problem $\boldsymbol{\mathcal{H}}_{54,56}(\boldsymbol{y}) = \mathbf{0}$

Figure 3.13: Vertical displacement of the node with the coordinates (19.4044 mm, 0.576046 mm) in the reference configuration in course of the continuations

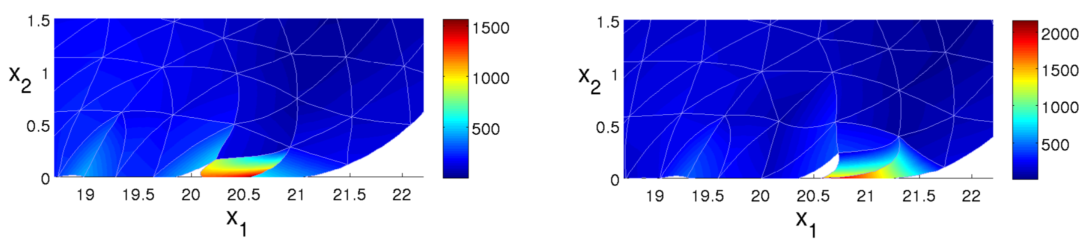


Figure 3.14: Jump of the solution between $t = 0.00055\,\mathrm{s}$ (on the left) and $t = 0.00056\,\mathrm{s}$ (on the right)

stead of $t = 0.00055\,\mathrm{s}$ and $t = 0.00056\,\mathrm{s}$ with the starting point chosen as $\boldsymbol{y}_0^{56} := (\boldsymbol{u}^{55}, \boldsymbol{\lambda}_D^{55}, \boldsymbol{\lambda}_\nu^{55}, \boldsymbol{\lambda}_\tau^{55}, 0.5)$. We arrived at a point with $\gamma = 1$ in this way and, fortunately, this point showed to give a good initial approximation for the Newton method in $t = 0.00056\,\mathrm{s}$ (Figs. 3.13(b) and 3.14).

We observed in this experiment that the Newton method itself had difficulties several times since the body started to slide with its entire volume. Moreover, its first iterations seemed to be unstable. This confirms the discussion at the end of Subsection 3.2.1.

We resolved the same experiment also with a mesh two times locally refined in its lower right-hand corner. The only remarkable change was that the structure of solutions was a little more complicated and we had to use the continuation more times. For an example, see Figs. 3.15 and 3.16.

The experiments presented so far were computed when refining the mesh while

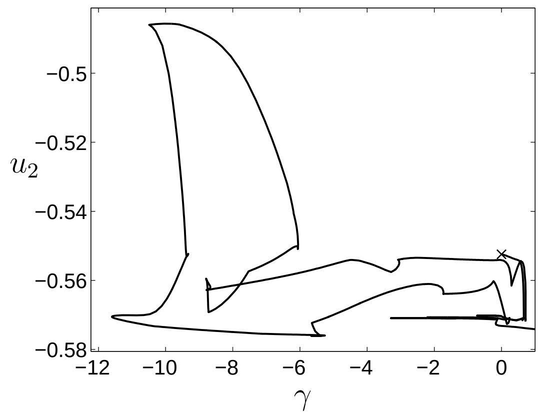


Figure 3.15: Vertical displacement of the node with the coordinates (19.4044 mm, 0.576046 mm) in the reference configuration in course of the continuation of the auxiliary problem $\boldsymbol{\mathcal{H}}_{58,59}(\boldsymbol{y}) = \boldsymbol{0}$

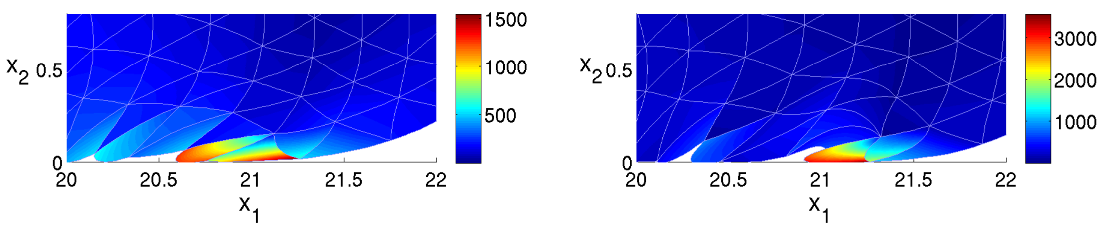


Figure 3.16: Jump of the solution between $t = 0.00058\,\text{s}$ (on the left) and $t = 0.00059\,\text{s}$ (on the right)

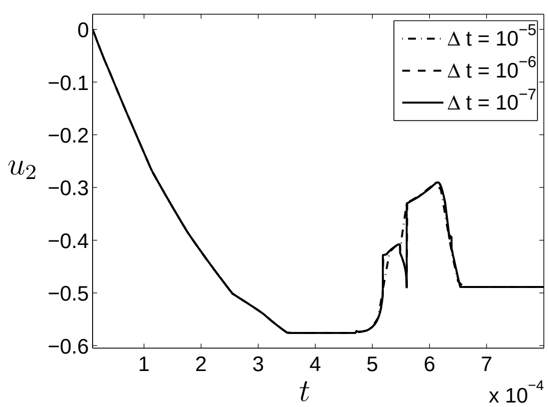


Figure 3.17: Vertical displacement of the node with the coordinates (19.4044 mm, 0.576046 mm) in the reference configuration for different time step lengths

keeping the time step length Δt fixed. It is worth mentioning that no significant changes occur when, conversely, Δt tends to zero and the mesh is fixed. In fact, the solutions will converge as illustrated in Fig. 3.17 on the mesh once locally refined (observe the two jumps in the interval $(0.0005, 0.0006)$ described above).

Conclusion

Using standard numerical methods (the Newton method, the method of successive approximations...), one is able to obtain some solution of contact problems with friction without any further information on existence of other solutions. One may face even situations when the standard solvers are not capable of finding any solution at all. This is why we have developed a piecewise smooth variant of the Moore-Penrose continuation, which allows us to follow branches of solutions parametrized by the coefficient of friction $\mathscr{F}$, the load vector $\boldsymbol{f}$, etc. (Section 3.1). In comparison with the classical Moore-Penrose continuation for smooth (differentiable) problems, we have had to do some modifications in the prediction step to provide for transitions through points of non-differentiability. In Section 3.2, we have introduced quasi-static contact problems in large deformations and their discretization leading to a sequence of incremental problems. We have explained a possible application of the proposed continuation technique for solving these incremental problems and on one example from technical practise, we have demonstrated advantages of this approach in comparison with the Newton method.

4 Spatial Semi-Discretization of Dynamic Problems

The purpose of this chapter is to present a well-posed spatial semi-discretization of dynamic contact problems with isotropic Coulomb friction, making use of the so-called mass redistribution method. This method was introduced in [38] for treating a contact condition in numerical realization of dynamic contact problems without friction. One might think that the strategy developed there is directly applicable to a friction condition as well. However, we shall see hereafter that this does not provide the well-posedness result and therefore a strategy adapted to the friction condition is needed. Let us recall in this context that the main difficulty for the unilateral contact condition is that the spatial semi-discretization by finite-element method naturally adds a mass on the nodes of the contact boundary. On the other hand, in [51] and [50], it was shown that adding a mass on the contact boundary regularizes the tangential friction problem and prevents the occurrence of multiple solutions in elastodynamics!

The method proposed here is to apply the mass redistribution method only on the unilateral contact condition not on the friction one. We show that in this case, the problem semi-discretized in space reduces to a differential inclusion with a unique Lipschitz continuous solution (not to a measure differential inclusion as in the standard semi-discretization). For the sake of simplicity, we limit ourselves to the framework of linearized elasticity. However, the same kind of difficulties exists for large deformation problems and a similar strategy can be applied. The results have been published in [46].

The outline of this chapter is the following: In Section 4.1, we present a classical finite-element spatial semi-discretization of elastodynamic contact problems with friction. In Section 4.2, we propose an application of the mass redistribution method, namely, to use it only on the normal component. The well-posedness of the obtained semi-discrete problem is proved in Section 4.3. Finally, an elementary example described in Section 4.4 shows that the well-posedness of the fully discrete problem cannot be attained when the mass redistribution method is applied both to contact and friction conditions.

4.1 A Classical Spatial Semi-Discretization

Dynamic contact problems with Coulomb friction consist in finding the displacement field $\boldsymbol{u} : [0,T] \times \overline{\Omega} \to \mathbb{R}^2$ satisfying

$$\begin{aligned}
\rho\ddot{\boldsymbol{u}} - \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in } (0,T)\times\Omega,\\
\boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{u}) && \text{in } (0,T)\times\Omega,\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } (0,T)\times\Gamma_D,\\
\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} &= \boldsymbol{h} && \text{on } (0,T)\times\Gamma_N,\\
u_\nu \le 0, \quad \sigma_\nu(\boldsymbol{u}) \le 0, \quad u_\nu\sigma_\nu(\boldsymbol{u}) &= 0 && \text{on } (0,T)\times\Gamma_C,
\end{aligned}$$

$$\left.\begin{aligned}
&\dot{u}_\tau(t,\boldsymbol{x}) = 0 \implies |\sigma_\tau(\boldsymbol{x},\boldsymbol{u}(t,\boldsymbol{x}))| \le -\mathscr{F}\sigma_\nu(\boldsymbol{x},\boldsymbol{u}(t,\boldsymbol{x})),\\
&\dot{u}_\tau(t,\boldsymbol{x}) \ne 0 \implies \sigma_\tau(\boldsymbol{x},\boldsymbol{u}(t,\boldsymbol{x})) = \mathscr{F}\sigma_\nu(\boldsymbol{x},\boldsymbol{u}(t,\boldsymbol{x}))\frac{\dot{u}_\tau(t,\boldsymbol{x})}{|\dot{u}_\tau(t,\boldsymbol{x})|}
\end{aligned}\right\}\ \text{on } (0,T)\times\Gamma_C,$$

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}), \quad \dot{\boldsymbol{u}}(0,\boldsymbol{x}) = \boldsymbol{v}^0(\boldsymbol{x}) \quad \text{in } \Omega,$$

where $T > 0$ determines the time interval of interest, ρ is the mass density and $\dot{\boldsymbol{u}}$, $\ddot{\boldsymbol{u}}$ denote the first and the second time derivative of $\boldsymbol{u}$, respectively. Further, $\boldsymbol{u}^0, \boldsymbol{v}^0 : \Omega \to \mathbb{R}^2$ are given initial displacement and velocity fields, respectively. For simplicity, we confine ourselves to a 2D case where the loads $\boldsymbol{f}$ and $\boldsymbol{h}$ do not depend on the time t and the coefficient of friction $\mathscr{F}$ is represented by a non-negative real.

Using the Green formula, this problem is formally equivalent to

$$\left.\begin{aligned}
&\text{Find } \boldsymbol{u} : [0,T] \to \boldsymbol{V} \text{ with } \dot{\boldsymbol{u}}, \ddot{\boldsymbol{u}} : [0,T] \to \boldsymbol{V},\ \lambda_\nu : [0,T] \to \Lambda_\nu,\\
&\qquad \lambda_\tau : [0,T] \to X'_\tau \text{ with } \lambda_\tau(t) \in \Lambda_\tau(\mathscr{F}\lambda_\nu(t)) \text{ a.e. in } (0,T) \text{ such that}\\
&(\rho\ddot{\boldsymbol{u}}(t), \boldsymbol{w})_{0,\Omega} + a(\boldsymbol{u}(t), \boldsymbol{w}) = \ell(\boldsymbol{w}) + \langle\lambda_\nu(t), w_\nu\rangle_\nu + \langle\lambda_\tau(t), w_\tau\rangle_\tau,\\
&\hspace{16em} \forall\,\boldsymbol{w} \in \boldsymbol{V} \ \text{ a.e. in } (0,T),\\
&\langle\mu_\nu - \lambda_\nu(t), u_\nu(t)\rangle_\nu \ge 0, \quad \forall\,\mu_\nu \in \Lambda_\nu \ \text{ a.e. in } (0,T),\\
&\langle\mu_\tau - \lambda_\tau(t), \dot{u}_\tau(t)\rangle_\tau \ge 0, \quad \forall\,\mu_\tau \in \Lambda_\tau(\mathscr{F}\lambda_\nu(t)) \ \text{ a.e. in } (0,T),\\
&\boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0,
\end{aligned}\right\} \quad (\mathscr{M})$$

where

$$\begin{aligned}
\boldsymbol{V} &:= \{\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \,|\, \boldsymbol{w} = \boldsymbol{0} \text{ a.e. on } \Gamma_D\},\\
X_\nu &:= \{\varphi \in L^2(\Gamma_C) \,|\, \exists\,\boldsymbol{w} \in \boldsymbol{V} : \varphi = w_\nu \text{ a.e. on } \Gamma_C\},\\
X_\tau &:= \{\varphi \in L^2(\Gamma_C) \,|\, \exists\,\boldsymbol{w} \in \boldsymbol{V} : \varphi = w_\tau \text{ a.e. on } \Gamma_C\},\\
\Lambda_\nu &:= \{\mu_\nu \in X'_\nu \,|\, \langle\mu_\nu, w_\nu\rangle_\nu \ge 0,\ \forall\,\boldsymbol{w} \in \boldsymbol{V},\ w_\nu \le 0 \text{ a.e. on } \Gamma_C\},\\
\Lambda_\tau(\mathscr{F}\mu_\nu) &:= \{\mu_\tau \in X'_\tau \,|\, \langle\mu_\tau, w_\tau\rangle_\tau + \langle\mathscr{F}\mu_\nu, |w_\tau|\rangle_\nu \le 0,\ \forall\,\boldsymbol{w} \in \boldsymbol{V}\}, \quad \mu_\nu \in \Lambda_\nu,
\end{aligned}$$

$$a(\boldsymbol{v},\boldsymbol{w}) := \int_\Omega \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V},$$
$$\ell(\boldsymbol{w}) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{h}\cdot\boldsymbol{w}\,\mathrm{d}S, \quad \boldsymbol{w}\in\boldsymbol{V}$$

and $\langle .,.\rangle_\nu$, $\langle .,.\rangle_\tau$ stand for the duality pairings between X_ν and X'_ν, X_τ and X'_τ, respectively.

A spatial finite-element semi-discretization of $(\mathscr{M})$ leads to the following problem (for more details see Section 2.2):

$$\left.\begin{array}{l} \text{Find } \boldsymbol{u}:[0,T]\to\mathbb{R}^{n_{\boldsymbol{u}}},\ \boldsymbol{\lambda}_\nu:[0,T]\to\boldsymbol{\Lambda}_\nu,\ \boldsymbol{\lambda}_\tau:[0,T]\to\mathbb{R}^{n_c} \\ \qquad\qquad \text{with } \boldsymbol{\lambda}_\tau(t)\in\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)) \text{ a.e. in } (0,T) \text{ such that} \\ \boldsymbol{M}\ddot{\boldsymbol{u}}(t)+\boldsymbol{A}\boldsymbol{u}(t)=\boldsymbol{f}+\boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu(t)+\boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau(t) \quad \text{a.e. in } (0,T), \\ (\boldsymbol{\mu}_\nu-\boldsymbol{\lambda}_\nu(t),\boldsymbol{B}_\nu\boldsymbol{u}(t))\ge 0, \quad \forall\,\boldsymbol{\mu}_\nu\in\boldsymbol{\Lambda}_\nu \ \text{ a.e. in } (0,T), \\ (\boldsymbol{\mu}_\tau-\boldsymbol{\lambda}_\tau(t),\boldsymbol{B}_\tau\dot{\boldsymbol{u}}(t))\ge 0, \quad \forall\,\boldsymbol{\mu}_\tau\in\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)) \ \text{ a.e. in } (0,T), \\ \boldsymbol{u}(0)=\boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0)=\boldsymbol{v}^0. \end{array}\right\} \qquad (\mathcal{M})$$

As in the previous chapters, we use the same symbols for algebraic variables as for the corresponding continuous functions. Besides the notation introduced in Section 2.1, $\boldsymbol{M}\in\mathbb{M}^{n_{\boldsymbol{u}}}$ stands for the mass matrix, $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$ are the vectors of degrees of freedom of the discretized initial displacement and velocity fields, respectively, and

$$\boldsymbol{\Lambda}_\nu := \mathbb{R}^{n_c}_-,$$
$$\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\mu}_\nu) := \{\boldsymbol{\mu}_\tau\in\mathbb{R}^{n_c} \mid |\mu_{\tau,i}| \le -\mathscr{F}\mu_{\nu,i},\ \forall\, i=1,\dots,n_c\}, \quad \boldsymbol{\mu}_\nu\in\boldsymbol{\Lambda}_\nu.$$

We assume that both $\boldsymbol{A}$ and $\boldsymbol{M}$ are symmetric positive definite:

$$\left.\begin{array}{l} \text{(i) } \boldsymbol{A}=\boldsymbol{A}^T, \\ \text{(ii) } (\boldsymbol{A}\boldsymbol{w},\boldsymbol{w})>0, \quad \forall\,\boldsymbol{w}\in\mathbb{R}^{n_{\boldsymbol{u}}}\setminus\{\boldsymbol{0}\}, \end{array}\right\} \qquad (4.1)$$

$$\left.\begin{array}{l} \text{(j) } \boldsymbol{M}=\boldsymbol{M}^T, \\ \text{(jj) } (\boldsymbol{M}\boldsymbol{w},\boldsymbol{w})>0, \quad \forall\,\boldsymbol{w}\in\mathbb{R}^{n_{\boldsymbol{u}}}\setminus\{\boldsymbol{0}\} \end{array}\right\} \qquad (4.2)$$

and the rows $\boldsymbol{B}_{\nu,i}$, $\boldsymbol{B}_{\tau,i}$ of $\boldsymbol{B}_\nu,\boldsymbol{B}_\tau\in\mathbb{M}^{n_c,n_{\boldsymbol{u}}}$ are mutually orthonormal:

$$(\boldsymbol{B}_{\nu,i},\boldsymbol{B}_{\nu,j})=\delta_{ij}, \quad (\boldsymbol{B}_{\tau,i},\boldsymbol{B}_{\tau,j})=\delta_{ij}, \quad (\boldsymbol{B}_{\nu,i},\boldsymbol{B}_{\tau,j})=0, \quad \forall\, i,j=1,\dots,n_c. \qquad (4.3)$$

Note that from (4.3), it immediately follows that there exists $\beta>0$ such that

$$\sup_{\boldsymbol{0}\neq\boldsymbol{w}\in\mathbb{R}^{n_{\boldsymbol{u}}}} \frac{(\boldsymbol{\mu}_\nu,\boldsymbol{B}_\nu\boldsymbol{w})+(\boldsymbol{\mu}_\tau,\boldsymbol{B}_\tau\boldsymbol{w})}{\|\boldsymbol{w}\|} \ge \beta\|(\boldsymbol{\mu}_\nu,\boldsymbol{\mu}_\tau)\|, \quad \forall\,(\boldsymbol{\mu}_\nu,\boldsymbol{\mu}_\tau)\in\mathbb{R}^{2n_c}. \qquad (4.4)$$

Problem $(\mathcal{M})$ can be viewed as a measure differential inclusion (see [47, 49]). It is ill-posed unless an impact law is added on each contact node. Even in this case, the solutions have a very low regularity.

4.2 The Mass Redistribution Method

The analysis presented in [38] highlights the fact that the main cause of ill-posedness of $(\mathcal{M})$ is due to the inertia of finite-element nodes on the contact boundary. It is proposed a method that consists in the redistribution of the mass near the contact boundary. This technique ensures well-posedness of the semi-discrete problem and transforms the measure differential inclusion corresponding to $(\mathcal{M})$ into a regular Lipschitz continuous ordinary differential equation, which can be approximated by any reasonable difference scheme.

It is worth mentioning that the singular dynamic method introduced in [53] for unilateral conditions is more general than the mass redistribution method. However, we use here the latter one. The reason is that we need a differentiated treatment of unilateral and friction conditions, which would be more difficult to obtain with the singular dynamic method.

Let $\mathcal{N} := \operatorname{span}\{\boldsymbol{B}_{\nu,1}, \dots, \boldsymbol{B}_{\nu,n_c}\}$ and $\mathcal{N}^\perp$ denote the subspace of $\mathbb{R}^{n_{\boldsymbol{u}}}$ spanned by $\boldsymbol{B}_{\nu,i}$ and its orthogonal complement, respectively. We shall consider the redistributed mass matrix $\boldsymbol{M}_r \in \mathbb{M}^{n_{\boldsymbol{u}}}$ satisfying (confer (4.2)):

$$\left.\begin{aligned} &\text{(j)}\ \boldsymbol{M}_r = \boldsymbol{M}_r^T,\\ &\text{(jj)}\ \operatorname{Ker}\boldsymbol{M}_r = \mathcal{N},\\ &\text{(jjj)}\ (\boldsymbol{M}_r\boldsymbol{w}, \boldsymbol{w}) > 0, \quad \forall\, \boldsymbol{w} \in \mathcal{N}^\perp \setminus \{\boldsymbol{0}\}, \end{aligned}\right\} \tag{4.5}$$

that is, being symmetric positive semi-definite with the kernel equal to $\mathcal{N}$. In [38], a simple algorithm is proposed to build the redistributed mass matrix preserving the main characteristics of the mass matrix (total mass, center of gravity and moments of inertia).

Using the decomposition $\boldsymbol{u}(t) = \boldsymbol{u}_{\mathcal{N}^\perp}(t) + \boldsymbol{u}_{\mathcal{N}}(t)$, $\boldsymbol{u}_{\mathcal{N}^\perp}(t) \in \mathcal{N}^\perp$, $\boldsymbol{u}_{\mathcal{N}}(t) \in \mathcal{N}$, of the displacement vector for any time t and replacing $\boldsymbol{M}$ with $\boldsymbol{M}_r$, problem $(\mathcal{M})$ becomes

$$\left.\begin{aligned} &\text{Find } \boldsymbol{u}_{\mathcal{N}^\perp} : [0,T] \to \mathcal{N}^\perp,\ \boldsymbol{u}_{\mathcal{N}} : [0,T] \to \mathcal{N},\ \boldsymbol{\lambda}_\nu : [0,T] \to \boldsymbol{\Lambda}_\nu,\\ &\qquad \boldsymbol{\lambda}_\tau : [0,T] \to \mathbb{R}^{n_c} \text{ with } \boldsymbol{\lambda}_\tau(t) \in \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)) \text{ a.e. in } (0,T) \text{ such that}\\ &\boldsymbol{M}_r\ddot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t) + \boldsymbol{A}(\boldsymbol{u}_{\mathcal{N}^\perp}(t) + \boldsymbol{u}_{\mathcal{N}}(t)) = \boldsymbol{f} + \boldsymbol{B}_\nu^T\boldsymbol{\lambda}_\nu(t) + \boldsymbol{B}_\tau^T\boldsymbol{\lambda}_\tau(t)\\ &\hspace{20em}\text{a.e. in } (0,T),\\ &(\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu(t), \boldsymbol{B}_\nu\boldsymbol{u}_{\mathcal{N}}(t)) \ge 0, \quad \forall\, \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu \text{ a.e. in } (0,T),\\ &(\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau(t), \boldsymbol{B}_\tau\dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t)) \ge 0, \quad \forall\, \boldsymbol{\mu}_\tau \in \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t)) \text{ a.e. in } (0,T),\\ &\boldsymbol{u}_{\mathcal{N}^\perp}(0) = \boldsymbol{u}^0_{\mathcal{N}^\perp}, \quad \dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(0) = \boldsymbol{v}^0_{\mathcal{N}^\perp}, \end{aligned}\right\} \tag{$\mathcal{M}_r$}$$

where $\boldsymbol{u}^0_{\mathcal{N}^\perp}$, $\boldsymbol{v}^0_{\mathcal{N}^\perp}$ are the projections of the initial values of the displacement and velocity vectors into $\mathcal{N}^\perp$, respectively. Since the constraints in $\boldsymbol{\Lambda}_\nu$ as well as in

$\boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu(t))$ are separated, it is possible to express the unilateral contact and friction conditions in an equivalent way (see [39], for instance) and rewrite the problem as follows:

$$\left.\begin{aligned}
&\text{Find } \boldsymbol{u}_{\mathcal{N}^\perp} : [0,T] \to \mathcal{N}^\perp,\ \boldsymbol{u}_{\mathcal{N}} : [0,T] \to \mathcal{N},\ \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau : [0,T] \to \mathbb{R}^{n_c}\\
&\quad\text{such that}\\
&\boldsymbol{M}_r \ddot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t) + \boldsymbol{A}(\boldsymbol{u}_{\mathcal{N}^\perp}(t) + \boldsymbol{u}_{\mathcal{N}}(t))\\
&\qquad\qquad = \boldsymbol{f} + \sum_{1\le i\le n_c} \lambda_{\nu,i}(t)\boldsymbol{B}_{\nu,i} + \sum_{1\le i\le n_c} \lambda_{\tau,i}(t)\boldsymbol{B}_{\tau,i} \quad \text{a.e. in } (0,T),\\
&-\lambda_{\nu,i}(t) \in N_{\mathbb{R}_-}(\boldsymbol{B}_{\nu,i}^T \boldsymbol{u}_{\mathcal{N}}(t)), \quad \forall\, i = 1,\dots,n_c \ \text{ a.e. in } (0,T),\\
&\lambda_{\tau,i}(t) \in \mathscr{F}\lambda_{\nu,i}(t)\,\mathrm{Sgn}(\boldsymbol{B}_{\tau,i}^T \dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t)), \quad \forall\, i = 1,\dots,n_c \ \text{ a.e. in } (0,T),\\
&\boldsymbol{u}_{\mathcal{N}^\perp}(0) = \boldsymbol{u}^0_{\mathcal{N}^\perp}, \quad \dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(0) = \boldsymbol{v}^0_{\mathcal{N}^\perp},
\end{aligned}\right\} \quad (\mathcal{M}'_r)$$

where $N_{\mathbb{R}_-}$ denotes the normal cone of $\mathbb{R}_-$ and the multifunction $\mathrm{Sgn} : \mathbb{R} \rightrightarrows \mathbb{R}$ is the sub-differential of the function $r \mapsto |r|$, that is,

$$\mathrm{Sgn}(r) = \begin{cases} \frac{r}{|r|} & \text{if } r \neq 0,\\ [-1,1] & \text{if } r = 0. \end{cases}$$

4.3 Well-Posedness Result

In this section, we shall establish the well-posedness of problem $(\mathcal{M}_r)$. First, owing to (4.3) and (4.5), the first three variables of any $(\boldsymbol{u}_{\mathcal{N}^\perp}, \boldsymbol{u}_{\mathcal{N}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solving $(\mathcal{M}_r)$ have to satisfy

$$\left.\begin{aligned}
&\boldsymbol{u}_{\mathcal{N}^\perp}(t) \in \mathcal{N}^\perp,\ \boldsymbol{u}_{\mathcal{N}}(t) \in \mathcal{N},\ \boldsymbol{\lambda}_\nu(t) \in \boldsymbol{\Lambda}_\nu,\\
&\big(\boldsymbol{A}(\boldsymbol{u}_{\mathcal{N}^\perp}(t) + \boldsymbol{u}_{\mathcal{N}}(t)), \boldsymbol{w}\big) = (\boldsymbol{f}, \boldsymbol{w}) + (\boldsymbol{\lambda}_\nu(t), \boldsymbol{B}_\nu \boldsymbol{w}), \quad \forall\, \boldsymbol{w} \in \mathcal{N},\\
&(\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu(t), \boldsymbol{B}_\nu \boldsymbol{u}_{\mathcal{N}}(t)) \ge 0, \quad \forall\, \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu
\end{aligned}\right\} \qquad (4.6)$$

for almost all $t \in (0,T)$. From here, $\boldsymbol{u}_{\mathcal{N}}$ and $\boldsymbol{\lambda}_\nu$ are uniquely determined by $\boldsymbol{u}_{\mathcal{N}^\perp}$ as states the following assertion.

Lemma 4.1. *Let (4.1) and (4.3) be satisfied and $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ be arbitrary. Then there exist unique functions $\boldsymbol{g}_1 : \mathcal{N}^\perp \to \mathcal{N}$ and $\boldsymbol{g}_2 : \mathcal{N}^\perp \to \boldsymbol{\Lambda}_\nu$ such that the triplet $(\boldsymbol{u}_{\mathcal{N}^\perp}(t), \boldsymbol{u}_{\mathcal{N}}(t), \boldsymbol{\lambda}_\nu(t))$ with $\boldsymbol{u}_{\mathcal{N}}(t) := \boldsymbol{g}_1(\boldsymbol{u}_{\mathcal{N}^\perp}(t))$, $\boldsymbol{\lambda}_\nu := \boldsymbol{g}_2(\boldsymbol{u}_{\mathcal{N}^\perp}(t))$, satisfies (4.6) for any $\boldsymbol{u}_{\mathcal{N}^\perp}(t) \in \mathcal{N}^\perp$ and any $t \in [0,T]$. Moreover, the functions $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are Lipschitz continuous:*

$$\exists\, L_1, L_2 > 0 : \quad \|\boldsymbol{g}_i(\boldsymbol{w}) - \boldsymbol{g}_i(\bar{\boldsymbol{w}})\| \le L_i \|\boldsymbol{w} - \bar{\boldsymbol{w}}\|, \quad \forall\, \boldsymbol{w}, \bar{\boldsymbol{w}} \in \mathcal{N}^\perp,\ i = 1,2. \qquad (4.7)$$

Proof. In fact, it suffices to analyze the static problem:

$$\left.\begin{array}{l}\text{Find } (\tilde{\boldsymbol{u}}_{\mathcal{N}}, \tilde{\boldsymbol{\lambda}}_\nu) := (\tilde{\boldsymbol{u}}_{\mathcal{N}}(\tilde{\boldsymbol{u}}_{\mathcal{N}^\perp}), \tilde{\boldsymbol{\lambda}}_\nu(\tilde{\boldsymbol{u}}_{\mathcal{N}^\perp})) \in \mathcal{N} \times \boldsymbol{\Lambda}_\nu \text{ such that}\\ (\boldsymbol{A}\tilde{\boldsymbol{u}}_{\mathcal{N}}, \boldsymbol{w}) = (\boldsymbol{f} - \boldsymbol{A}\tilde{\boldsymbol{u}}_{\mathcal{N}^\perp}, \boldsymbol{w}) + (\tilde{\boldsymbol{\lambda}}_\nu, \boldsymbol{B}_\nu \boldsymbol{w}), \quad \forall\, \boldsymbol{w} \in \mathcal{N},\\ (\boldsymbol{\mu}_\nu - \tilde{\boldsymbol{\lambda}}_\nu, \boldsymbol{B}_\nu \tilde{\boldsymbol{u}}_{\mathcal{N}}) \geq 0, \quad \forall\, \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu\end{array}\right\} \tag{4.8}$$

for $\tilde{\boldsymbol{u}}_{\mathcal{N}^\perp} \in \mathcal{N}^\perp$ given. It is readily seen that this problem is equivalent to finding a saddle-point $(\tilde{\boldsymbol{u}}_{\mathcal{N}}, \tilde{\boldsymbol{\lambda}}_\nu)$ of the Lagrangian

$$\mathscr{L}(\boldsymbol{w}, \boldsymbol{\mu}_\nu) := \frac{1}{2}(\boldsymbol{A}\boldsymbol{w}, \boldsymbol{w}) - (\boldsymbol{f} - \boldsymbol{A}\tilde{\boldsymbol{u}}_{\mathcal{N}^\perp}, \boldsymbol{w}) - (\boldsymbol{\mu}_\nu, \boldsymbol{B}_\nu \boldsymbol{w}), \quad (\boldsymbol{w}, \boldsymbol{\mu}_\nu) \in \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_c},$$

on $\mathcal{N} \times \boldsymbol{\Lambda}_\nu$. Since $\boldsymbol{A}$ is supposed to be positive definite and

$$\beta \|\boldsymbol{\mu}_\nu\| \leq \sup_{\boldsymbol{0} \neq \boldsymbol{w} \in \mathbb{R}^{n_{\boldsymbol{u}}}} \frac{(\boldsymbol{\mu}_\nu, \boldsymbol{B}_\nu \boldsymbol{w})}{\|\boldsymbol{w}\|} = \sup_{\boldsymbol{0} \neq \boldsymbol{w} \in \mathcal{N}} \frac{(\boldsymbol{\mu}_\nu, \boldsymbol{B}_\nu \boldsymbol{w})}{\|\boldsymbol{w}\|}, \quad \forall\, \boldsymbol{\mu}_\nu \in \mathbb{R}^{n_c}$$

due to (4.3), where β is the constant from (4.4), problem (4.8) possesses a unique solution for any $\tilde{\boldsymbol{u}}_{\mathcal{N}^\perp} \in \mathcal{N}^\perp$, which depends Lipschitz continuously on the data $\tilde{\boldsymbol{u}}_{\mathcal{N}^\perp}$ (see [17] and eventually the technique of the proof of Lemma 2.2). This yields the existence, the uniqueness and the Lipschitz continuity of the functions $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$. □

From the other side, if $(\boldsymbol{u}_{\mathcal{N}^\perp}, \boldsymbol{u}_{\mathcal{N}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solves $(\mathcal{M}_r')$ then

$$\left.\begin{array}{l}(\boldsymbol{M}_r \ddot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t), \boldsymbol{w}) + \big(\boldsymbol{A}(\boldsymbol{u}_{\mathcal{N}^\perp}(t) + \boldsymbol{u}_{\mathcal{N}}(t)), \boldsymbol{w}\big)\\ \qquad\qquad = (\boldsymbol{f}, \boldsymbol{w}) + \Big(\displaystyle\sum_{1 \leq i \leq n_c} \lambda_{\tau,i}(t) \boldsymbol{B}_{\tau,i}, \boldsymbol{w}\Big), \quad \forall\, \boldsymbol{w} \in \mathcal{N}^\perp \ \text{ a.e. in } (0,T),\\ \lambda_{\tau,i}(t) \in \mathscr{F} \lambda_{\nu,i}(t)\, \mathrm{Sgn}(\boldsymbol{B}_{\tau,i}^T \dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t)), \quad \forall\, i = 1, \ldots, n_c \ \text{ a.e. in } (0,T),\\ \boldsymbol{u}_{\mathcal{N}^\perp}(0) = \boldsymbol{u}^0_{\mathcal{N}^\perp}, \quad \dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(0) = \boldsymbol{v}^0_{\mathcal{N}^\perp}.\end{array}\right\} \tag{4.9}$$

By substituting the inclusion for $\lambda_{\tau,i}(t)$ into the equality and taking $\boldsymbol{u}_{\mathcal{N}}(t) := \boldsymbol{g}_1(\boldsymbol{u}_{\mathcal{N}^\perp}(t))$, $\lambda_{\nu,i}(t) := g_{2,i}(\boldsymbol{u}_{\mathcal{N}^\perp}(t))$ according to Lemma 4.1, this becomes

$$\left.\begin{array}{l}(\boldsymbol{M}_r \ddot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t), \boldsymbol{w}) \in \big(\boldsymbol{f} - \boldsymbol{A}\boldsymbol{u}_{\mathcal{N}^\perp}(t) - \boldsymbol{A}\boldsymbol{g}_1(\boldsymbol{u}_{\mathcal{N}^\perp}(t)), \boldsymbol{w}\big)\\ \qquad\qquad + \Big(\displaystyle\sum_{1 \leq i \leq n_c} \mathscr{F} g_{2,i}(\boldsymbol{u}_{\mathcal{N}^\perp}(t))\, \mathrm{Sgn}(\boldsymbol{B}_{\tau,i}^T \dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t)) \boldsymbol{B}_{\tau,i}, \boldsymbol{w}\Big),\\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{w} \in \mathcal{N}^\perp \ \text{ a.e. in } (0,T),\\ \boldsymbol{u}_{\mathcal{N}^\perp}(0) = \boldsymbol{u}^0_{\mathcal{N}^\perp}, \quad \dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(0) = \boldsymbol{v}^0_{\mathcal{N}^\perp}.\end{array}\right\} \tag{4.10}$$

Lemma 4.2. *Let (4.1), (4.3) and (4.5) be fulfilled and $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, $\boldsymbol{u}^0_{\mathcal{N}^\perp}, \boldsymbol{v}^0_{\mathcal{N}^\perp} \in \mathcal{N}^\perp$ be arbitrary. Then there exists a unique Lipschitz continuous function $\boldsymbol{u}_{\mathcal{N}^\perp} : [0,T] \to \mathcal{N}^\perp$ with $\ddot{\boldsymbol{u}}_{\mathcal{N}^\perp} \in L^1(0,T;\mathbb{R}^{n_{\boldsymbol{u}}})$ solving (4.10).*

Proof. Introducing the matrix $\boldsymbol{P} \in \mathbb{M}^{n_{\boldsymbol{u}},\bar{n}}$, $\bar{n} := \dim \mathcal{N}^\perp$, columns of which form an orthonormal basis of $\mathcal{N}^\perp$, any vector $\boldsymbol{w} \in \mathcal{N}^\perp$ can be represented by $\bar{\boldsymbol{w}} \in \mathbb{R}^{\bar{n}}$ with

$$\bar{\boldsymbol{w}} = \boldsymbol{P}^T \boldsymbol{w}, \quad \boldsymbol{w} = \boldsymbol{P}\boldsymbol{P}^T \boldsymbol{w} = \boldsymbol{P}\bar{\boldsymbol{w}}$$

and (4.10) is equivalent to

$$\left.\begin{aligned} &(\bar{\boldsymbol{M}}_r \ddot{\bar{\boldsymbol{u}}}(t), \bar{\boldsymbol{w}}) \in \big(\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\bar{\boldsymbol{u}}(t) - \bar{\boldsymbol{g}}_1(\bar{\boldsymbol{u}}(t)), \bar{\boldsymbol{w}}\big) \\ &\qquad\qquad + \Big(\sum_{1 \le i \le n_c} \mathscr{F}\bar{g}_{2,i}(\bar{\boldsymbol{u}}(t)) \operatorname{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T \dot{\bar{\boldsymbol{u}}}(t)) \bar{\boldsymbol{B}}_{\tau,i}, \bar{\boldsymbol{w}}\Big), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \bar{\boldsymbol{w}} \in \mathbb{R}^{\bar{n}} \quad \text{a.e. in } (0,T), \\ &\bar{\boldsymbol{u}}(0) = \bar{\boldsymbol{u}}^0, \quad \dot{\bar{\boldsymbol{u}}}(0) = \boldsymbol{P}^T \boldsymbol{v}^0_{\mathcal{N}^\perp}, \end{aligned}\right\}$$

where

$$\begin{gathered} \bar{\boldsymbol{M}}_r = \boldsymbol{P}^T \boldsymbol{M}_r \boldsymbol{P}, \quad \bar{\boldsymbol{A}} = \boldsymbol{P}^T \boldsymbol{A}\boldsymbol{P}, \quad \bar{\boldsymbol{g}}_1(\bar{\boldsymbol{u}}(t)) = \boldsymbol{P}^T \boldsymbol{A}\boldsymbol{g}_1(\boldsymbol{P}\bar{\boldsymbol{u}}(t)), \\ \bar{\boldsymbol{g}}_2(\bar{\boldsymbol{u}}(t)) = (\bar{g}_{2,i}(\bar{\boldsymbol{u}}(t))) = \boldsymbol{g}_2(\boldsymbol{P}\bar{\boldsymbol{u}}(t)), \quad \bar{\boldsymbol{u}} = \boldsymbol{P}^T \boldsymbol{u}_{\mathcal{N}^\perp}, \quad \bar{\boldsymbol{u}}^0 = \boldsymbol{P}^T \boldsymbol{u}^0_{\mathcal{N}^\perp}, \quad \bar{\boldsymbol{f}} = \boldsymbol{P}^T \boldsymbol{f}, \\ \bar{\boldsymbol{B}}_{\tau,i} = \boldsymbol{P}^T \boldsymbol{B}_{\tau,i}, \ i = 1, \dots, n_c. \end{gathered}$$

With regard to (4.5), this can be written as

$$\left.\begin{aligned} &\ddot{\bar{\boldsymbol{u}}}(t) \in \bar{\boldsymbol{M}}_r^{-1} \Big[\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\bar{\boldsymbol{u}}(t) - \bar{\boldsymbol{g}}_1(\bar{\boldsymbol{u}}(t)) + \sum_{1 \le i \le n_c} \mathscr{F}\bar{g}_{2,i}(\bar{\boldsymbol{u}}(t)) \operatorname{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T \dot{\bar{\boldsymbol{u}}}(t)) \bar{\boldsymbol{B}}_{\tau,i}\Big] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } (0,T), \\ &\bar{\boldsymbol{u}}(0) = \bar{\boldsymbol{u}}^0, \quad \dot{\bar{\boldsymbol{u}}}(0) = \boldsymbol{P}^T \boldsymbol{v}^0_{\mathcal{N}^\perp}, \end{aligned}\right\}$$

and by denoting $\bar{\boldsymbol{v}} := \bar{\boldsymbol{M}}_r^{1/2} \dot{\bar{\boldsymbol{u}}}$, $\bar{\boldsymbol{v}}^0 := \bar{\boldsymbol{M}}_r^{1/2} \boldsymbol{P}^T \boldsymbol{v}^0_{\mathcal{N}^\perp}$, this leads to the following differential inclusion of the first order:

$$\left.\begin{aligned} &\begin{pmatrix} \dot{\bar{\boldsymbol{u}}}(t) \\ \dot{\bar{\boldsymbol{v}}}(t) \end{pmatrix} \in \begin{pmatrix} \bar{\boldsymbol{M}}_r^{-1/2} \bar{\boldsymbol{v}}(t) \\ \bar{\boldsymbol{M}}_r^{-1/2} \big[\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\bar{\boldsymbol{u}}(t) - \bar{\boldsymbol{g}}_1(\bar{\boldsymbol{u}}(t)) \\ \qquad + \sum_{1 \le i \le n_c} \mathscr{F}\bar{g}_{2,i}(\bar{\boldsymbol{u}}(t)) \operatorname{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T \bar{\boldsymbol{M}}_r^{-1/2} \bar{\boldsymbol{v}}(t)) \bar{\boldsymbol{B}}_{\tau,i}\big] \end{pmatrix} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } (0,T), \\ &\begin{pmatrix} \bar{\boldsymbol{u}}(0) \\ \bar{\boldsymbol{v}}(0) \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{u}}^0 \\ \bar{\boldsymbol{v}}^0 \end{pmatrix}. \end{aligned}\right\}$$

Thus, we have to solve

$$\left.\begin{aligned} &\dot{\boldsymbol{y}}(t) \in F(\boldsymbol{y}(t)) \quad \text{a.e. in } (0,T), \\ &\boldsymbol{y}(0) = \boldsymbol{y}^0 \end{aligned}\right\} \tag{4.11}$$

with the multifunction $F : \mathbb{R}^{2\bar{n}} \rightrightarrows \mathbb{R}^{2\bar{n}}$ defined by

$$F(\boldsymbol{z}) := \begin{pmatrix} \bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2 \\ \bar{\boldsymbol{M}}_r^{-1/2}\big[\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\boldsymbol{z}_1 - \bar{\boldsymbol{g}}_1(\boldsymbol{z}_1) \\ \qquad + \sum_{1\le i\le n_c} \mathscr{F}\bar{g}_{2,i}(\boldsymbol{z}_1)\,\mathrm{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2)\bar{\boldsymbol{B}}_{\tau,i}\big] \end{pmatrix}, \quad \boldsymbol{z} = (\boldsymbol{z}_1, \boldsymbol{z}_2) \in R^{2\bar{n}}, \tag{4.12}$$

and $\boldsymbol{y}^0 := (\bar{\boldsymbol{u}}^0, \bar{\boldsymbol{v}}^0)$.

Obviously, F is upper semi-continuous, that is, $F^{-1}(\mathscr{A})$ is closed whenever $\mathscr{A} \subset \mathbb{R}^{2\bar{n}}$ is closed, and $F(\boldsymbol{z})$ is a closed convex set for each $\boldsymbol{z} \in \mathbb{R}^{2\bar{n}}$. Furthermore, there exists $c > 0$ such that

$$\|F(\boldsymbol{z})\| \equiv \sup\{\|\boldsymbol{\omega}\| \,|\, \boldsymbol{\omega} \in F(\boldsymbol{z})\} \le c(1 + \|\boldsymbol{z}\|) \quad \forall\, \boldsymbol{z} \in \mathbb{R}^{2\bar{n}}. \tag{4.13}$$

Indeed,

$$\begin{aligned} \|F(\boldsymbol{z})\| &\le \|\bar{\boldsymbol{M}}_r^{-1/2}\|\Big[\|\boldsymbol{z}_2\|^2 + \big\|\bar{\boldsymbol{f}} - \bar{\boldsymbol{A}}\boldsymbol{z}_1 - \bar{\boldsymbol{g}}_1(\boldsymbol{z}_1) \\ &\qquad + \sum_{1\le i\le n_c} \mathscr{F}\bar{g}_{2,i}(\boldsymbol{z}_1)\,\mathrm{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2)\bar{\boldsymbol{B}}_{\tau,i}\big\|^2\Big]^{1/2} \\ &\le \|\bar{\boldsymbol{M}}_r^{-1/2}\|\Big[\|\boldsymbol{z}_2\|^2 + \big(\|\bar{\boldsymbol{f}}\| + \|\bar{\boldsymbol{A}}\|\|\boldsymbol{z}_1\| + \|\bar{\boldsymbol{g}}_1(\boldsymbol{z}_1)\| \\ &\qquad + \big\|\sum_{1\le i\le n_c} \mathscr{F}\bar{g}_{2,i}(\boldsymbol{z}_1)\,\mathrm{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2)\bar{\boldsymbol{B}}_{\tau,i}\big\|\big)^2\Big]^{1/2}. \end{aligned}$$

First,

$$\Big\|\sum_{1\le i\le n_c} \mathscr{F}\bar{g}_{2,i}(\boldsymbol{z}_1)\,\mathrm{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2)\bar{\boldsymbol{B}}_{\tau,i}\Big\| \le \Big(\sum_{1\le i\le n_c} (\mathscr{F}\bar{g}_{2,i}(\boldsymbol{z}_1))^2\Big)^{1/2} = \mathscr{F}\|\bar{\boldsymbol{g}}_2(\boldsymbol{z}_1)\|$$

in virtue of the orthonormality of $\bar{\boldsymbol{B}}_{\tau,i}$ and the definition of the mapping Sgn. Second, making use of (4.7) and of the form of $\boldsymbol{P}$, we have

$$\begin{gathered} \|\bar{\boldsymbol{g}}_1(\boldsymbol{z}_1)\| = \|\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{z}_1)\| \le \|\boldsymbol{A}\|\|\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{z}_1)\|, \\ \|\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{z}_1)\| - \|\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{0})\| \le L_1\|\boldsymbol{P}(\boldsymbol{z}_1 - \boldsymbol{0})\| = L_1\|\boldsymbol{z}_1\|, \end{gathered}$$

consequently

$$\|\bar{\boldsymbol{g}}_1(\boldsymbol{z}_1)\| \le \|\boldsymbol{A}\|(\|\boldsymbol{g}_1(\boldsymbol{0})\| + L_1\|\boldsymbol{z}_1\|)$$

and in a similar way one can verify that

$$\|\bar{\boldsymbol{g}}_2(\boldsymbol{z}_1)\| \le \|\boldsymbol{g}_2(\boldsymbol{0})\| + L_2\|\boldsymbol{z}_1\|.$$

Hence,

$$\|F(\boldsymbol{z})\| \le \|\bar{\boldsymbol{M}}_r^{-1/2}\| \big[\|\boldsymbol{z}_2\|^2 + \big(\|\bar{\boldsymbol{f}}\| + \|\bar{\boldsymbol{A}}\|\|\boldsymbol{z}_1\| + \|\boldsymbol{A}\|(\|\boldsymbol{g}_1(\boldsymbol{0})\| + L_1\|\boldsymbol{z}_1\|) \\ + \mathscr{F}(\|\boldsymbol{g}_2(\boldsymbol{0})\| + L_2\|\boldsymbol{z}_1\|)\big)^2\big]^{1/2},$$

from which the expression for the constant c in (4.13) follows. Therefore, Theorem 5.1 in [13] guarantees that (4.11) has an absolutely continuous solution $\boldsymbol{y}$ in $[0,T]$ for any $\boldsymbol{y}^0 \in \mathbb{R}^{2\bar{n}}$, that is, a function $\boldsymbol{y} : [0,T] \to \mathbb{R}^{2\bar{n}}$ with $\dot{\boldsymbol{y}} \in L^1(0,T;\mathbb{R}^{2\bar{n}})$ satisfying

$$\boldsymbol{y}(t) = \boldsymbol{y}^0 + \int_0^t \dot{\boldsymbol{y}}(s)\,\mathrm{d}s \quad \text{for all } t \in [0,T] \quad \text{and} \quad \dot{\boldsymbol{y}}(t) \in F(\boldsymbol{y}(t)) \quad \text{a.e. in } (0,T).$$

This gives the existence part of the assertion. To prove the uniqueness, it suffices to show that F is one-sided Lipschitz (see for instance Theorem 10.4 in [13]), that is,

$$\exists\, K \in \mathbb{R} : \quad (F(\boldsymbol{z}^1) - F(\boldsymbol{z}^2), \boldsymbol{z}^1 - \boldsymbol{z}^2) \le K\|\boldsymbol{z}^1 - \boldsymbol{z}^2\|^2, \quad \forall\, \boldsymbol{z}^1, \boldsymbol{z}^2 \in \mathbb{R}^{2\bar{n}}.$$

From the definition of F,

$$\begin{aligned}
\big(F(\boldsymbol{z}^1) - F(\boldsymbol{z}^2), \boldsymbol{z}^1 - \boldsymbol{z}^2\big) \\
&= \big(\bar{\boldsymbol{M}}_r^{-1/2}(\boldsymbol{z}_2^1 - \boldsymbol{z}_2^2), \boldsymbol{z}_1^1 - \boldsymbol{z}_1^2\big) + \big(\bar{\boldsymbol{M}}_r^{-1/2}\bar{\boldsymbol{A}}(\boldsymbol{z}_1^2 - \boldsymbol{z}_1^1), \boldsymbol{z}_2^1 - \boldsymbol{z}_2^2\big) \\
&\quad + \big(\bar{\boldsymbol{M}}_r^{-1/2}(\bar{\boldsymbol{g}}_1(\boldsymbol{z}_1^2) - \bar{\boldsymbol{g}}_1(\boldsymbol{z}_1^1)), \boldsymbol{z}_2^1 - \boldsymbol{z}_2^2\big) \\
&\quad + \Big(\bar{\boldsymbol{M}}_r^{-1/2} \sum_{1\le i\le n_c} \mathscr{F}\big(\bar{g}_{2,i}(\boldsymbol{z}_1^1)\,\mathrm{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2^1) \\
&\qquad\qquad - \bar{g}_{2,i}(\boldsymbol{z}_1^2)\,\mathrm{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2^2)\big)\bar{\boldsymbol{B}}_{\tau,i}, \boldsymbol{z}_2^1 - \boldsymbol{z}_2^2\Big) \\
&=: s_1 + s_2 + s_3 + s_4.
\end{aligned}$$

Clearly,

$$s_1 \le \|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\boldsymbol{z}^1 - \boldsymbol{z}^2\|^2, \quad s_2 \le \|\bar{\boldsymbol{M}}_r^{-1/2}\bar{\boldsymbol{A}}\|\|\boldsymbol{z}^1 - \boldsymbol{z}^2\|^2$$

and

$$\begin{aligned}
s_3 &\le \|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\bar{\boldsymbol{g}}_1(\boldsymbol{z}_1^2) - \bar{\boldsymbol{g}}_1(\boldsymbol{z}_1^1)\|\|\boldsymbol{z}^1 - \boldsymbol{z}^2\| \\
&\le \|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\boldsymbol{A}\|\|\boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{z}_1^2) - \boldsymbol{g}_1(\boldsymbol{P}\boldsymbol{z}_1^1)\|\|\boldsymbol{z}^1 - \boldsymbol{z}^2\| \le L_1\|\bar{\boldsymbol{M}}_r^{-1/2}\|\|\boldsymbol{A}\|\|\boldsymbol{z}^1 - \boldsymbol{z}^2\|^2
\end{aligned}$$

by (4.7). Furthermore,

$$\begin{aligned}
s_4 = \sum_{1\le i\le n_c} &\mathscr{F}\big(\bar{g}_{2,i}(\boldsymbol{z}_1^1)\,\mathrm{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2^1) - \bar{g}_{2,i}(\boldsymbol{z}_1^2)\,\mathrm{Sgn}(\bar{\boldsymbol{B}}_{\tau,i}^T\bar{\boldsymbol{M}}_r^{-1/2}\boldsymbol{z}_2^2)\big) \\
&\cdot \big((\bar{\boldsymbol{M}}_r^{-1/2}\bar{\boldsymbol{B}}_{\tau,i}, \boldsymbol{z}_2^1) - (\bar{\boldsymbol{M}}_r^{-1/2}\bar{\boldsymbol{B}}_{\tau,i}, \boldsymbol{z}_2^2)\big).
\end{aligned}$$

Hence, by fixing i and setting

$$p^1 := \bar{g}_{2,i}(\boldsymbol{z}_1^1), \quad p^2 := \bar{g}_{2,i}(\boldsymbol{z}_1^2), \quad q^1 := \bar{\boldsymbol{B}}_{\tau,i}^T \bar{\boldsymbol{M}}_r^{-1/2} \boldsymbol{z}_2^1, \quad q^2 := \bar{\boldsymbol{B}}_{\tau,i}^T \bar{\boldsymbol{M}}_r^{-1/2} \boldsymbol{z}_2^2,$$

the ith summand of s_4 takes the form

$$\mathscr{F}(p^1 \operatorname{Sgn}(q^1) - p^2 \operatorname{Sgn}(q^2))(q^1 - q^2).$$

The definition of $\boldsymbol{\Lambda}_\nu$ implies $p^1, p^2 \leq 0$. We claim that in this case

$$(p^1 \operatorname{Sgn}(q^1) - p^2 \operatorname{Sgn}(q^2))(q^1 - q^2) \leq |p^1 - p^2||q^1 - q^2|. \tag{4.14}$$

Indeed, for $\zeta \in \operatorname{Sgn}(q^1)$ and $\xi \in \operatorname{Sgn}(q^2)$ we get

$$(p^1\zeta - p^2\xi)(q^1 - q^2) = (p^1\zeta - p^1\xi + p^1\xi - p^2\xi)(q^1 - q^2) \leq (p^1 - p^2)\xi(q^1 - q^2)$$

due to monotonicity of the multifunction Sgn. And of course, (4.14) can be deduced from

$$(p^1 - p^2)\xi(q^1 - q^2) \leq |p^1 - p^2||q^1 - q^2|.$$

Applying this together with the Cauchy-Schwarz inequality and (4.7), we get

$$\begin{aligned} s_4 &\leq \mathscr{F} \sum_{1 \leq i \leq n_c} \left|\bar{g}_{2,i}(\boldsymbol{z}_1^1) - \bar{g}_{2,i}(\boldsymbol{z}_1^2)\right| \left|\bar{\boldsymbol{B}}_{\tau,i}^T \bar{\boldsymbol{M}}_r^{-1/2} \boldsymbol{z}_2^1 - \bar{\boldsymbol{B}}_{\tau,i}^T \bar{\boldsymbol{M}}_r^{-1/2} \boldsymbol{z}_2^2\right| \\ &\leq \mathscr{F} \|\bar{\boldsymbol{g}}_2(\boldsymbol{z}_1^1) - \bar{\boldsymbol{g}}_2(\boldsymbol{z}_1^2)\| \|\boldsymbol{B}_\tau \bar{\boldsymbol{M}}_r^{-1/2}(\boldsymbol{z}_2^1 - \boldsymbol{z}_2^2)\| \leq \mathscr{F} L_2 \|\bar{\boldsymbol{M}}_r^{-1/2}\| \|\boldsymbol{z}^1 - \boldsymbol{z}^2\|^2. \end{aligned}$$

All in all, the one-sided Lipschitz property of F is verified. □

On the basis of the previous two lemmas we arrive at the announced well-posedness result.

Theorem 4.1. *Let $\boldsymbol{f} \in \mathbb{R}^{n_{\boldsymbol{u}}}$, $\boldsymbol{u}^0_{\mathcal{N}^\perp}, \boldsymbol{v}^0_{\mathcal{N}^\perp} \in \mathcal{N}^\perp$ be arbitrary. If (4.1), (4.3) and (4.5) are satisfied then there exist a unique Lipschitz continuous function $\boldsymbol{u}_{\mathcal{N}^\perp} : [0,T] \to \mathcal{N}^\perp$ with $\ddot{\boldsymbol{u}}_{\mathcal{N}^\perp} \in L^1(0,T;\mathbb{R}^{n_{\boldsymbol{u}}})$ and unique functions $\boldsymbol{u}_{\mathcal{N}} : [0,T] \to \mathcal{N}$, $\boldsymbol{\lambda}_\nu : [0,T] \to \boldsymbol{\Lambda}_\nu$ and $\boldsymbol{\lambda}_\tau : [0,T] \to \mathbb{R}^{n_c}$ such that the quadruplet $(\boldsymbol{u}_{\mathcal{N}^\perp}, \boldsymbol{u}_{\mathcal{N}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau)$ solves $(\mathcal{M}_r)$. In addition, $\boldsymbol{u}_{\mathcal{N}}$, $\boldsymbol{\lambda}_\nu$ are Lipschitz continuous in $[0,T]$ and $\boldsymbol{\lambda}_\tau \in L^\infty(0,T;\mathbb{R}^{n_c})$.*

Proof. The existence and uniqueness as well as the Lipschitz continuity of $\boldsymbol{u}_{\mathcal{N}^\perp}$ and $\boldsymbol{u}_{\mathcal{N}}$, $\boldsymbol{\lambda}_\nu$ are ensured by Lemmas 4.2 and 4.1, respectively. Consequently, the existence of $\boldsymbol{\lambda}_\tau$ is readily seen from the relation between (4.9) and (4.10). If $(\boldsymbol{u}_{\mathcal{N}^\perp}, \boldsymbol{u}_{\mathcal{N}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau^1)$ and $(\boldsymbol{u}_{\mathcal{N}^\perp}, \boldsymbol{u}_{\mathcal{N}}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_\tau^2)$ were two solutions to $(\mathcal{M}_r)$ then

$$(\boldsymbol{\lambda}_\tau^1(t) - \boldsymbol{\lambda}_\tau^2(t), \boldsymbol{B}_\tau \boldsymbol{w}) = 0, \quad \forall\, \boldsymbol{w} \in \mathbb{R}^{n_{\boldsymbol{u}}} \text{ a.e. in } (0,T)$$

by the first equation in $(\mathcal{M}_r)$ and

$$\beta\|\boldsymbol{\lambda}_\tau^1(t) - \boldsymbol{\lambda}_\tau^2(t)\| \leq \sup_{\boldsymbol{0}\neq\boldsymbol{w}\in\mathbb{R}^{n_{\boldsymbol{u}}}} \frac{(\boldsymbol{\lambda}_\tau^1(t) - \boldsymbol{\lambda}_\tau^2(t), \boldsymbol{B}_\tau \boldsymbol{w})}{\|\boldsymbol{w}\|} = 0 \quad \text{a.e. in } (0,T)$$

in virtue of (4.4). In a similar way, one also shows that $\boldsymbol{\lambda}_\tau \in L^\infty(0,T;\mathbb{R}^{n_c})$ from the Lipschitz continuity of $\boldsymbol{\lambda}_\nu$ and the second inclusion of $(\mathcal{M}_r')$. □

Remark 4.1. It is readily seen that this theorem remains valid when the coefficient of friction is represented by an arbitrary vector from $\mathbb{R}_+^{n_c}$ as in Chapter 2. Invoking the results from [13] for non-autonomous differential inclusions, one can generalize the well-posedness result to problems with a load vector which is a Lipschitz continuous function of time. Finally, the analysis can be extended to 3D problems since its key point is the monotonicity of the multifunction Sgn. For the 3D problems, the friction condition can be expressed by means of the sub-differential of the function $r \mapsto \|r\|$, which is also monotonic. This allows to arrive at an analogous relation to (4.14).

At the end of this section, we shall take a closer look at a fully discrete problem. First, we shall examine its well-posedness. For definiteness, we shall consider time discretization by the midpoint rule, however, the analysis will be similar for other standard difference methods.

Following Chapter 6 in [37], we divide the interval $[0,T]$ uniformly into n_T subintervals and set $\Delta t := T/n_T$ and $t_k := k\Delta t$, $k = 1/2, 3/2, \ldots, n_T - 1/2$. Adapting the midpoint scheme to problem $(\mathcal{M}_r)$, we seek the approximations $\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1/2}$, $\boldsymbol{v}_{\mathcal{N}^\perp}^{k+1/2}$, $\boldsymbol{u}_{\mathcal{N}}^{k+1/2}$, $\boldsymbol{\lambda}_\nu^{k+1/2}$ and $\boldsymbol{\lambda}_\tau^{k+1/2}$ of $\boldsymbol{u}_{\mathcal{N}^\perp}(t_{k+1/2})$, $\dot{\boldsymbol{u}}_{\mathcal{N}^\perp}(t_{k+1/2})$, $\boldsymbol{u}_{\mathcal{N}}(t_{k+1/2})$, $\boldsymbol{\lambda}_\nu(t_{k+1/2})$ and $\boldsymbol{\lambda}_\tau(t_{k+1/2})$, respectively, for $k = 0, \ldots, n_T - 1$ such that

$$\left.\begin{aligned}
&\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1/2}, \boldsymbol{v}_{\mathcal{N}^\perp}^{k+1/2} \in \mathcal{N}^\perp,\ \boldsymbol{u}_{\mathcal{N}}^{k+1/2} \in \mathcal{N},\ \boldsymbol{\lambda}_\nu^{k+1/2} \in \boldsymbol{\Lambda}_\nu,\ \boldsymbol{\lambda}_\tau^{k+1/2} \in \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu^{k+1/2}),\\
&\frac{\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1} - \boldsymbol{u}_{\mathcal{N}^\perp}^{k}}{\Delta t} = \boldsymbol{v}_{\mathcal{N}^\perp}^{k+1/2},\\
&\boldsymbol{M}_r \frac{\boldsymbol{v}_{\mathcal{N}^\perp}^{k+1} - \boldsymbol{v}_{\mathcal{N}^\perp}^{k}}{\Delta t} + \boldsymbol{A}(\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1/2} + \boldsymbol{u}_{\mathcal{N}}^{k+1/2}) = \boldsymbol{f} + \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu^{k+1/2} + \boldsymbol{B}_\tau^T \boldsymbol{\lambda}_\tau^{k+1/2},\\
&(\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu^{k+1/2}, \boldsymbol{B}_\nu \boldsymbol{u}_{\mathcal{N}}^{k+1/2}) \geq 0, \quad \forall\, \boldsymbol{\mu}_\nu \in \boldsymbol{\Lambda}_\nu,\\
&(\boldsymbol{\mu}_\tau - \boldsymbol{\lambda}_\tau^{k+1/2}, \boldsymbol{B}_\tau \boldsymbol{v}_{\mathcal{N}^\perp}^{k+1/2}) \geq 0, \quad \forall\, \boldsymbol{\mu}_\tau \in \boldsymbol{\Lambda}_\tau(\mathscr{F}\boldsymbol{\lambda}_\nu^{k+1/2}),
\end{aligned}\right\}$$

where

$$\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1/2} = \frac{\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1} + \boldsymbol{u}_{\mathcal{N}^\perp}^{k}}{2}, \quad \boldsymbol{v}_{\mathcal{N}^\perp}^{k+1/2} = \frac{\boldsymbol{v}_{\mathcal{N}^\perp}^{k+1} + \boldsymbol{v}_{\mathcal{N}^\perp}^{k}}{2}.$$

Fixing k and arguing in the same way as in the study of the semi-discrete problem, one can see that $\boldsymbol{u}_{\mathcal{N}}^{k+1/2} = \boldsymbol{g}_1(\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1/2})$ and $\boldsymbol{\lambda}_\nu^{k+1/2} = \boldsymbol{g}_2(\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1/2})$, where $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are given by Lemma 4.1. Consequently, one arrives at the following discretization of (4.11):

$$\frac{\boldsymbol{y}^{k+1}-\boldsymbol{y}^k}{\Delta t} \in F(\frac{\boldsymbol{y}^{k+1}+\boldsymbol{y}^k}{2}) \tag{4.15}$$

with F defined exactly by (4.12).

Let us introduce the multi-valued map $G : \mathbb{R}^{2\bar{n}} \rightrightarrows \mathbb{R}^{2\bar{n}}$ as $G := G_1 + G_2$, where $G_1 : \mathbb{R}^{2\bar{n}} \rightrightarrows \mathbb{R}^{2\bar{n}}$ and $G_2 : \mathbb{R}^{2\bar{n}} \to \mathbb{R}^{2\bar{n}}$ are the following:

$$\begin{aligned} G_1(\boldsymbol{y}) &:= \frac{1}{2}\boldsymbol{y} - \Delta t F(\frac{\boldsymbol{y}+\boldsymbol{y}^k}{2}) - \boldsymbol{y}^k, \quad \boldsymbol{y} \in \mathbb{R}^{2\bar{n}} \\ G_2(\boldsymbol{y}) &:= \frac{1}{2}\boldsymbol{y}, \quad \boldsymbol{y} \in \mathbb{R}^{2\bar{n}}. \end{aligned}$$

Then (4.15) is nothing but

$$G(\boldsymbol{y}^{k+1}) \ni \mathbf{0}. \tag{4.16}$$

Now, take an arbitrarily fixed $\Delta t \leq 1/K$, where $K > 0$ is a constant from the one-sided Lipschitz property of F. Clearly,

$$\begin{aligned} &(G_1(\boldsymbol{y}) - G_1(\bar{\boldsymbol{y}}), \boldsymbol{y} - \bar{\boldsymbol{y}}) \\ &\quad = \frac{1}{2}(\boldsymbol{y}-\bar{\boldsymbol{y}}, \boldsymbol{y}-\bar{\boldsymbol{y}}) - 2\Delta t\Big(F(\frac{\boldsymbol{y}+\boldsymbol{y}^k}{2}) - F(\frac{\bar{\boldsymbol{y}}+\boldsymbol{y}^k}{2}), \frac{\boldsymbol{y}+\boldsymbol{y}^k}{2} - \frac{\bar{\boldsymbol{y}}+\boldsymbol{y}^k}{2}\Big) \\ &\quad \geq \Big(\frac{1}{2} - \frac{K}{2}\Delta t\Big)\|\boldsymbol{y}-\bar{\boldsymbol{y}}\|^2 \geq 0, \quad \forall\, \boldsymbol{y}, \bar{\boldsymbol{y}} \in \mathbb{R}^{2\bar{n}}, \end{aligned}$$

that is, G_1 is monotone. Moreover, it is vaguely continuous and $G_1(\boldsymbol{y})$ is closed convex for all $\boldsymbol{y} \in \mathbb{R}^{2\bar{n}}$. Hence, G_1 is maximal monotone according to [6]. Since G_2 is obviously a continuous, coercive, monotone mapping which maps bounded sets of $\mathbb{R}^{2\bar{n}}$ into bounded sets of $\mathbb{R}^{2\bar{n}}$, Theorem 1 in [7] guarantees that there exists at least one $\boldsymbol{y}^{k+1}$ solving (4.16). By the strict monotonicity of G, such $\boldsymbol{y}^{k+1}$ is unique.

From this and Lemma 4.1, the existence and uniqueness of $\boldsymbol{u}_{\mathcal{N}^\perp}^{k+1/2}$, $\boldsymbol{v}_{\mathcal{N}^\perp}^{k+1/2}$, $\boldsymbol{u}_{\mathcal{N}}^{k+1/2}$ and $\boldsymbol{\lambda}_\nu^{k+1/2}$ follows. Finally, $\boldsymbol{\lambda}_\tau^{k+1/2}$ can be treated in an analogous way as in the proof of Theorem 4.1.

Moreover, convergence of a quite general class of difference methods can be established (for a fixed mesh) in view of the results in [42], for instance. Indeed, if one constructs a sequence of piecewise linear continuous interpolants of the grid functions $(\boldsymbol{y}^0, \ldots, \boldsymbol{y}^{n_T})$ on the basis of an appropriate discretization of (4.11), all the interpolants are Lipschitz continuous with the same Lipschitz constant and the sequence is guaranteed to converge uniformly to the unique solution $\boldsymbol{y}$ of (4.11) for $n_T \to +\infty$. From here, uniform convergence of the corresponding approximations of the components $\boldsymbol{u}_{\mathcal{N}^\perp}$, $\boldsymbol{u}_{\mathcal{N}}$ and $\boldsymbol{\lambda}_\nu$ of the solution of $(\mathcal{M}_r)$ easily follows.

Remark 4.2. For most of the classical difference schemes, the fully discrete problem is also ensured to be well-posed provided that the time step is sufficiently small. Moreover, the sequences of piecewise linear continuous interpolants of the grid functions $(\boldsymbol{u}^0_{\mathcal{N}^\perp}, \dots, \boldsymbol{u}^{n_T}_{\mathcal{N}^\perp})$, $(\boldsymbol{u}^0_{\mathcal{N}}, \dots, \boldsymbol{u}^{n_T}_{\mathcal{N}})$ and $(\boldsymbol{\lambda}^0_\nu, \dots, \boldsymbol{\lambda}^{n_T}_\nu)$ converge uniformly to $\boldsymbol{u}_{\mathcal{N}^\perp}$, $\boldsymbol{u}_{\mathcal{N}}$ and $\boldsymbol{\lambda}_\nu$.

4.4 An Elementary Example

This section concerns the mass redistribution method for the dynamic case of the elementary contact problem studied in Section 2.4. The aim is to show that an undifferentiated treatment of the contact and friction conditions may lead to an ill-posedness of the fully discrete problem whatever the length of the time step is.

Denoting the lengths of the sides of the considered triangle by l, l and $\sqrt{2}l$ (confer Fig. 2.3), we obtain the following formulation of the dynamic elementary problem in inclusions:

$$\left.\begin{array}{l}\text{Find } \boldsymbol{u} : [0,T] \to \mathbb{R}^2, \lambda_\nu, \lambda_\tau : [0,T] \to \mathbb{R} \text{ such that}\\ \boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{A}\boldsymbol{u}(t) = \boldsymbol{f}(t) + \boldsymbol{B}_\nu^T \lambda_\nu(t) + \boldsymbol{B}_\tau^T \lambda_\tau(t) \quad \text{a.e. in } (0,T),\\ -\lambda_\nu(t) \in N_{\mathbb{R}_-}(\boldsymbol{B}_\nu \boldsymbol{u}(t)) \quad \text{a.e. in } (0,T),\\ \lambda_\tau(t) \in \mathscr{F}\lambda_\nu(t)\,\mathrm{Sgn}(\boldsymbol{B}_\tau \dot{\boldsymbol{u}}(t)) \quad \text{a.e. in } (0,T),\\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}^0,\end{array}\right\}$$

where

$$\boldsymbol{M} = \begin{pmatrix} \frac{\rho l^2}{12} & 0 \\ 0 & \frac{\rho l^2}{12}\end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} \frac{\lambda+3\mu}{2} & -\frac{\lambda+\mu}{2} \\ -\frac{\lambda+\mu}{2} & \frac{\lambda+3\mu}{2}\end{pmatrix}, \quad \boldsymbol{B}_\nu = \begin{pmatrix}1 & 0\end{pmatrix}, \quad \boldsymbol{B}_\tau = \begin{pmatrix}0 & 1\end{pmatrix}.$$

Here $\rho > 0$ is constant, $\lambda, \mu > 0$ are the Lamé coefficients and $\boldsymbol{f}$ is assumed to depend on t.

Obviously, the mass redistribution method consists in replacing the matrix $\boldsymbol{M}$ by $\boldsymbol{M}_r := \left(\begin{smallmatrix} m_\nu & 0 \\ 0 & m_\tau \end{smallmatrix}\right)$ with $m_\nu, m_\tau \geq 0$. The time discretisation will be done by the midpoint scheme considered already at the end of the previous section. In the case of general mass redistribution, we seek $\boldsymbol{u}^{k+1/2}$, $\boldsymbol{v}^{k+1/2} \in \mathbb{R}^2$ and $\boldsymbol{\lambda}_\nu$, $\boldsymbol{\lambda}_\tau \in \mathbb{R}$ for $k = 0, \dots, n_T - 1$ such that

$$\left.\begin{array}{l}\dfrac{\boldsymbol{u}^{k+1} - \boldsymbol{u}^k}{\Delta t} = \boldsymbol{v}^{k+1/2},\\ \boldsymbol{M}_r \dfrac{\boldsymbol{v}^{k+1} - \boldsymbol{v}^k}{\Delta t} + \boldsymbol{A}\boldsymbol{u}^{k+1/2} = \boldsymbol{f}(t_{k+1/2}) + \boldsymbol{B}_\nu^T \lambda_\nu^{k+1/2} + \boldsymbol{B}_\tau^T \lambda_\tau^{k+1/2},\\ -\lambda_\nu^{k+1/2} \in N_{\mathbb{R}_-}(\boldsymbol{B}_\nu \boldsymbol{u}^{k+1/2}),\\ \lambda_\tau^{k+1/2} \in \mathscr{F}\lambda_\nu^{k+1/2}\,\mathrm{Sgn}(\boldsymbol{B}_\tau \boldsymbol{v}^{k+1/2}),\end{array}\right\} \tag{4.17}$$

where $\Delta t := T/n_T$, $t_k := k\Delta t$, $k = 1/2, 3/2, \ldots, n_T - 1/2$, and

$$\boldsymbol{u}^{k+1/2} = \frac{\boldsymbol{u}^{k+1} + \boldsymbol{u}^k}{2}, \quad \boldsymbol{v}^{k+1/2} = \frac{\boldsymbol{v}^{k+1} + \boldsymbol{v}^k}{2}. \tag{4.18}$$

From the first equation in (4.17) and (4.18), one can express $\boldsymbol{v}^{k+1/2}$ and $\boldsymbol{v}^{k+1}$ as

$$\boldsymbol{v}^{k+1/2} = \frac{2}{\Delta t}\boldsymbol{u}^{k+1/2} - \frac{2}{\Delta t}\boldsymbol{u}^k, \quad \boldsymbol{v}^{k+1} = \frac{4}{\Delta t}\boldsymbol{u}^{k+1/2} - \frac{4}{\Delta t}\boldsymbol{u}^k - \boldsymbol{v}^k, \tag{4.19}$$

which inserted into (4.17) leads to

$$\left.\begin{array}{l}
\Big(\frac{4}{\Delta t^2}\boldsymbol{M}_r + \boldsymbol{A}\Big)\boldsymbol{u}^{k+1/2} = \hat{\boldsymbol{f}}^{k+1/2} + \boldsymbol{B}_\nu^T \lambda_\nu^{k+1/2} + \boldsymbol{B}_\tau^T \lambda_\tau^{k+1/2}, \\
-\lambda_\nu^{k+1/2} \in N_{\mathbb{R}_-}(\boldsymbol{B}_\nu \boldsymbol{u}^{k+1/2}), \\
\lambda_\tau^{k+1/2} \in \mathscr{F}\lambda_\nu^{k+1/2}\,\mathrm{Sgn}\big(\frac{2}{\Delta t}\boldsymbol{B}_\tau(\boldsymbol{u}^{k+1/2} - \boldsymbol{u}^k)\big)
\end{array}\right\}$$

with

$$\hat{\boldsymbol{f}}^{k+1/2} := \boldsymbol{f}(t_{k+1/2}) + \frac{4}{\Delta t^2}\boldsymbol{M}_r\boldsymbol{u}^k + \frac{2}{\Delta t}\boldsymbol{M}_r\boldsymbol{v}^k.$$

Finally, we consider the decomposition

$$\boldsymbol{u}^i = (u_\nu^i, u_\tau^i), \quad \hat{\boldsymbol{f}}^i = (\hat{f}_\nu^i, \hat{f}_\tau^i)$$

and denote

$$a := \Big(\frac{4}{\Delta t^2}m_\nu + \frac{\lambda + 3\mu}{2}\Big), \quad b := \frac{\lambda + \mu}{2}, \quad c := \Big(\frac{4}{\Delta t^2}m_\tau + \frac{\lambda + 3\mu}{2}\Big).$$

In each time step we obtain the following problem:

$$\left.\begin{array}{l}
\text{Find } (u_\nu^{k+1/2}, u_\tau^{k+1/2}, \lambda_\nu^{k+1/2}, \lambda_\tau^{k+1/2}) \in \mathbb{R}^4 \text{ such that} \\
au_\nu^{k+1/2} - bu_\tau^{k+1/2} = \hat{f}_\nu^{k+1/2} + \lambda_\nu^{k+1/2}, \\
-bu_\nu^{k+1/2} + cu_\tau^{k+1/2} = \hat{f}_\tau^{k+1/2} + \lambda_\tau^{k+1/2}, \\
-\lambda_\nu^{k+1/2} \in N_{\mathbb{R}_-}(u_\nu^{k+1/2}), \\
\lambda_\tau^{k+1/2} \in \mathscr{F}\lambda_\nu^{k+1/2}\,\mathrm{Sgn}(u_\tau^{k+1/2} - u_\tau^k),
\end{array}\right\} \tag{4.20}$$

after resolution of which the values of $\boldsymbol{u}^{k+1}$, and $\boldsymbol{v}^{k+1}$ are determined by (4.18) and (4.19).

Exact solutions of problem (4.20) for an arbitrary $k \in \{0, \ldots, n_T - 1\}$ can be derived in the same way as those of problem (2.43). In a similar way as in Section 2.4, we introduce the linear functions $\mathcal{S}^{(i)}_{k+1/2} : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^4$, $i = 1, 2, 3$, and the set-valued mapping $\mathcal{S}^{(4)}_{k+1/2} : \mathbb{R}^2 \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^4$ by

$$\mathcal{S}^{(1)}_{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) := \Big(\frac{c\hat{f}_\nu + b\hat{f}_\tau}{ac - b^2},\ \frac{a\hat{f}_\tau + b\hat{f}_\nu}{ac - b^2},\ 0,\ 0\Big), \quad \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} \in \mathbb{R}_+,$$
$$\mathcal{S}^{(2)}_{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) := \big(0,\ u^k_\tau,\ -(\hat{f}_\nu + bu^k_\tau),\ cu^k_\tau - \hat{f}_\tau\big), \quad \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} \in \mathbb{R}_+,$$
$$\mathcal{S}^{(3)}_{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) := \Big(0,\ \frac{\hat{f}_\tau - \mathscr{F}\hat{f}_\nu}{c + b\mathscr{F}},\ -\frac{c\hat{f}_\nu + b\hat{f}_\tau}{c + b\mathscr{F}},\ -\mathscr{F}\frac{c\hat{f}_\nu + b\hat{f}_\tau}{c + b\mathscr{F}}\Big), \quad \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} \in \mathbb{R}_+,$$
$$\mathcal{S}^{(4)}_{k+1/2}(\hat{\boldsymbol{f}}, \mathscr{F}) := \begin{cases} \Big\{\Big(0,\ \frac{\hat{f}_\tau + \mathscr{F}\hat{f}_\nu}{c - b\mathscr{F}},\ -\frac{c\hat{f}_\nu + b\hat{f}_\tau}{c - b\mathscr{F}},\ \mathscr{F}\frac{c\hat{f}_\nu + b\hat{f}_\tau}{c - b\mathscr{F}}\Big)\Big\} \\ \qquad\qquad \text{if } \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} \in \mathbb{R}_+ \setminus \Big\{\frac{c}{b}\Big\}, \\ \Big\{(u_\nu, u_\tau, \lambda_\nu, \lambda_\tau) \in \mathbb{R}^4 \,\Big|\, \\ \quad u_\nu = 0,\ -\frac{\hat{f}_\nu}{b} \le u_\tau \le u^k_\tau,\ \lambda_\nu = -(\hat{f}_\nu + bu_\tau),\ \lambda_\tau = \mathscr{F}(\hat{f}_\nu + bu_\tau)\Big\} \\ \qquad\qquad \text{if } \hat{\boldsymbol{f}} \in \mathbb{R}^2,\ \mathscr{F} = \frac{c}{b} \end{cases}$$

and for $\mathscr{F} \in \mathbb{R}_+$ define the sets

$$\rho^{(1)}_{k+1/2}(\mathscr{F}) := \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, c\hat{f}_\nu + b\hat{f}_\tau \le 0\},$$
$$\rho^{(2)}_{k+1/2}(\mathscr{F}) := \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, \hat{f}_\nu \ge -bu^k_\tau,\ (c - b\mathscr{F})u^k_\tau - \mathscr{F}\hat{f}_\nu \le \hat{f}_\tau \le (c + b\mathscr{F})u^k_\tau + \mathscr{F}\hat{f}_\nu\},$$
$$\rho^{(3)}_{k+1/2}(\mathscr{F}) := \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, c\hat{f}_\nu + b\hat{f}_\tau \ge 0,\ \hat{f}_\tau \ge (c + b\mathscr{F})u^k_\tau + \mathscr{F}\hat{f}_\nu\},$$
$$\rho^{(4)}_{k+1/2}(\mathscr{F}) := \begin{cases} \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, \hat{f}_\nu \ge -bu^k_\tau,\ c\hat{f}_\nu + b\hat{f}_\tau \ge 0,\ \hat{f}_\tau \le (c - b\mathscr{F})u^k_\tau - \mathscr{F}\hat{f}_\nu\} \\ \qquad\qquad \text{if } \mathscr{F} \in [0, c/b], \\ \{\hat{\boldsymbol{f}} \in \mathbb{R}^2 \,|\, \hat{f}_\nu \ge -bu^k_\tau,\ c\hat{f}_\nu + b\hat{f}_\tau \le 0,\ \hat{f}_\tau \ge (c - b\mathscr{F})u^k_\tau - \mathscr{F}\hat{f}_\nu\} \\ \qquad\qquad \text{if } \mathscr{F} \in (c/b, +\infty). \end{cases}$$

Again, $\mathcal{S}^{(i)}_{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ solves (4.20) for $\hat{\boldsymbol{f}}^{k+1/2} \in \rho^{(i)}_{k+1/2}(\mathscr{F})$, $\mathscr{F} \in \mathbb{R}_+$, $i = 1, 2, 3$, and $\mathcal{S}^{(4)}_{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ is the set of solutions of (4.20) for $\hat{\boldsymbol{f}}^{k+1/2} \in \rho^{(4)}_{k+1/2}(\mathscr{F})$, $\mathscr{F} \in \mathbb{R}_+$. For this reason, the structure of the solution set to (4.20) depends on the mutual position of $\rho^{(i)}_{k+1/2}(\mathscr{F})$, which depend on the magnitude of $\mathscr{F}$.

If $\mathscr{F} \in [0, c/b)$ then the interiors $\mathring{\rho}^{(i)}_{k+1/2}(\mathscr{F})$ are mutually disjoint for all $1 \le i \le 4$ and (4.20) has a unique solution for any $\hat{\boldsymbol{f}}^{k+1/2} \in \mathbb{R}^2$ (see Fig. 4.1; we visualize here only the component $\lambda^{k+1/2}_\nu$, which determines uniquely the other components of the solution). If $\mathscr{F} > c/b$ then $\rho^{(4)}_{k+1/2}(\mathscr{F}) = \rho^{(1)}_{k+1/2}(\mathscr{F}) \cap \rho^{(2)}_{k+1/2}(\mathscr{F})$ and its

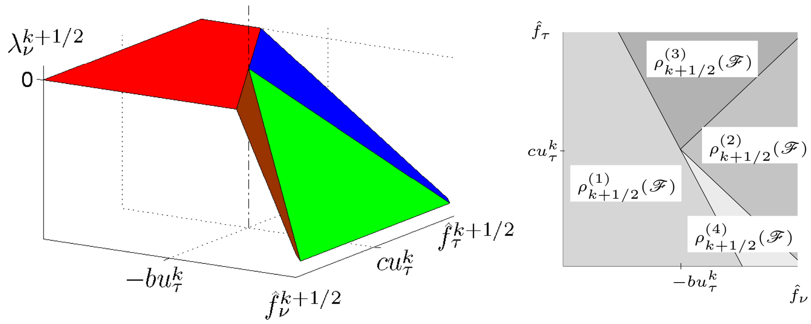


Figure 4.1: Structure of the solutions for $\mathscr{F} \in (0, c/b)$

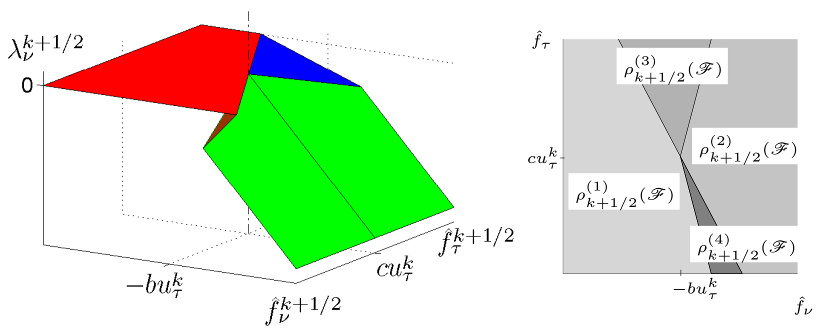


Figure 4.2: Structure of the solutions for $\mathscr{F} \in (c/b, +\infty)$

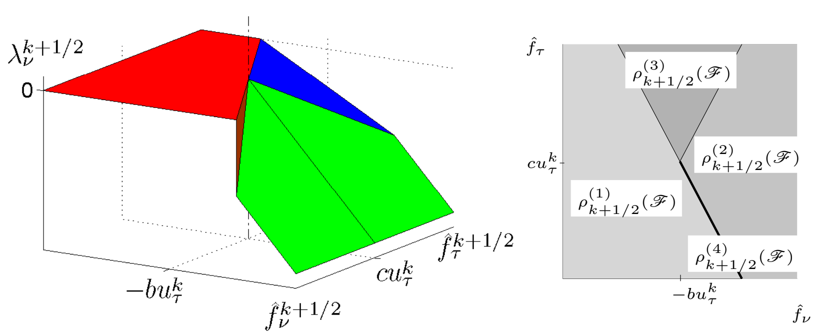


Figure 4.3: Structure of the solutions for $\mathscr{F} = c/b$

interior is non-empty. In this case, there exists a unique solution of the problem if $\hat{\boldsymbol{f}}^{k+1/2} \in (\mathbb{R}^2 \setminus \rho^{(4)}_{k+1/2}(\mathscr{F})) \cup \{(-bu^k_\tau, cu^k_\tau)\}$, there are two solutions on $\partial\rho^{(4)}_{k+1/2}(\mathscr{F}) \setminus \{(-bu^k_\tau, cu^k_\tau)\}$ and three solutions in $\mathring{\rho}^{(4)}_{k+1/2}(\mathscr{F})$ (Fig. 4.2). Finally, if $\mathscr{F} = c/b$, $\rho^{(4)}_{k+1/2}(\mathscr{F}) = \rho^{(1)}_{k+1/2}(\mathscr{F}) \cap \rho^{(2)}_{k+1/2}(\mathscr{F})$ is a half-line and there exists a unique solution of (4.20) for $\hat{\boldsymbol{f}}^{k+1/2} \in (\mathbb{R}^2 \setminus \rho^{(4)}_{k+1/2}(\mathscr{F})) \cup \{(-bu^k_\tau, cu^k_\tau)\}$ whereas the continuous branch $\mathcal{S}^{(4)}_{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ of solutions connects $\mathcal{S}^{(1)}_{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ and $\mathcal{S}^{(2)}_{k+1/2}(\hat{\boldsymbol{f}}^{k+1/2}, \mathscr{F})$ for $\hat{\boldsymbol{f}}^{k+1/2} \in \rho^{(4)}_{k+1/2}(\mathscr{F}) \setminus \{(-bu^k_\tau, cu^k_\tau)\}$ (Fig. 4.3).

Now take the redistributed mass matrix $\boldsymbol{M}_r$ such that $m_\nu = 0$ and $m_\tau > 0$, that is, (4.5) is fulfilled. Then, for any $\mathscr{F} \geq 0$ given, one can find $\Delta t_0 > 0$ satisfying

$$\frac{c}{b} = \frac{\frac{4}{\Delta t^2} m_\tau + \frac{\lambda+3\mu}{2}}{\frac{\lambda+\mu}{2}} > \mathscr{F}, \quad \forall\, \Delta t \in (0, \Delta t_0)$$

and the analysis above ensures the unique solvability of (4.20) for any $\hat{\boldsymbol{f}}^{k+1/2} \in \mathbb{R}^2$ and any $\Delta t \in (0, \Delta t_0)$. (Of course, this follows directly from the well-posedness result established in the previous section.)

On the contrary, consider $\boldsymbol{M}_r$ with $m_\nu = m_\tau = 0$, which corresponds to the total elimination of the mass from the contact zone. If the coefficient $\mathscr{F}$ is larger than $(\lambda + 3\mu)/(\lambda + \mu) = c/b$, one can always find $\hat{\boldsymbol{f}}^{k+1/2}$ such that (4.20) possesses multiple solutions whatever small Δt is. Hence, the well-posedness is not reached in this case.

Conclusion

We have adapted the mass redistribution method for elastodynamic contact problems with friction in this chapter. The proposed strategy, which is to apply the mass redistribution only on the normal component corresponding to the contact condition, allows to transform the semi-discrete problem into a regular one-sided Lipschitz differential inclusion. The advantage is that any reasonable time discretization scheme is then convergent, at least for a fixed mesh. Moreover, the fully discrete problem is also well-posed for a sufficiently small time step. The simple example described in Section 4.4 has shown that this is not the case when the mass redistribution is applied on both the contact and friction conditions. To add, let us note that in [46], a numerical test has been performed for demonstrating that the proposed strategy leads to stable time discretization schemes.

Conclusions

The aim of this thesis was to analyze discretizations of contact problems with Coulomb friction theoretically and to propose algorithms for their numerical realization, making use of the obtained theoretical results.

First, we have studied discretized 3D elastostatic contact problems with orthotropic and isotropic Coulomb friction and solution-dependent coefficients of friction (Chapter 1). We have guaranteed existence of at least one solution for a large class of coefficients. In addition, we have ensured that the solution is unique provided that the coefficients are Lipschitz continuous and their upper bounds as well as Lipschitz moduli are lower than some critical values. Unfortunately, these critical values have been shown to vanish when norms of the corresponding finite-element meshes tend to zero. As a consequence, the uniqueness result does not provide any information for larger coefficients.

To understand better the structure of discrete solutions, we have analyzed conditions guaranteeing the existence of local Lipschitz continuous branches of solutions as functions of the coefficient of friction and the load vector in the case of 2D static contact problems with isotropic Coulomb friction and a coefficient represented by a vector independent of the solution. This has been done in Chapter 2 by using variants of the implicit-function theorem for generalized equations and piecewise differentiable equations. Moreover, we have described in details a structure of solutions of an example with very small number of degrees of freedom, which can be solved analitically "by hand".

To trace the solution branches and eventually to capture multiple solutions of problems studied in Chapter 2 numerically, we have considered these problems written as a system of non-smooth equations parametrized by one scalar parameter and we have proposed a variant of a path-following algorithm adapted to the piecewise differentiable character of this system (Chapter 3). We have then successfully tested the algorithm in large deformation problems.

In the last chapter, we have focused on approximation of elastodynamic contact problems with isotropic Coulomb friction and a coefficient independent of the solution. Making use of the mass redistribution method, we have introduced a well-posed semi-discretization of these problems, which shows to be essential for obtaining stable numerical schemes. We have restricted ourselves to 2D problems, nevertheless, the extension to the 3D case is straightforward.

A Piecewise Differentiable Functions

For the sake of completeness, we give here a brief introduction to the theory of piecewise differentiable functions. The exposition is extracted from [56].

We start with some basic notions. Let $\pi := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{0}\}$, where $\boldsymbol{B} \in \mathbb{M}^{m,n}$ and the inequality has to be understood componentwise, be a polyhedral cone with vertex at $\boldsymbol{0} \in \mathbb{R}^n$. Recall that *the dimension* of π is defined as the dimension of its linear hull and nonempty faces of π can be represented as the sets

$$\{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{B}_i\boldsymbol{x} = 0,\ \forall\, i \in I,\ \boldsymbol{B}_j\boldsymbol{x} \leq 0,\ \forall\, j \in \{1, \dots, m\} \setminus I\}$$

for some index set $I \in \mathcal{I}(\boldsymbol{B}, \boldsymbol{0})$, where

$$\begin{aligned}&\mathcal{I}(\boldsymbol{B}, \boldsymbol{0})\\ &:= \{I \subset \{1, \dots, m\} \,|\, \exists\, \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{B}_i\boldsymbol{x} = 0,\ \forall\, i \in I,\ \boldsymbol{B}_j\boldsymbol{x} < 0,\ \forall\, j \in \{1, \dots, m\} \setminus I\}\end{aligned}$$

([56, Proposition 2.1.3]). Here $\boldsymbol{B}_i$ is the ith row vector of the matrix $\boldsymbol{B}$. A nonempty face of π which does not coincide with π is called *a proper face*. Further, the lineality space of π is the linear subspace $\{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}\}$.

A finite collection Π of convex polyhedral cones in $\mathbb{R}^n$ is called *a conical subdivision* of a polyhedral cone $\rho \subset \mathbb{R}^n$ if

1. all polyhedral cones in Π are subsets of ρ;

2. the dimension of the cones in Π coincides with the dimension of ρ;

3. the union of all cones in Π covers ρ;

4. the intersection of any two distinct cones in Π is either empty or a common proper face of both cones.

It holds that if Π is a conical subdivision of a polyhedral cone then all polyhedral cones $\pi \in \Pi$ have the same lineality space ([56, Proposition 2.2.4]). Hence *the lineality space* of Π is introduced as the common lineality space of the polyhedral cones in Π.

The kth branching number of a conical subdivision Π of a polyhedral cone ρ is defined as the maximal number of cones in Π containing a common face of dimension $(\dim \rho - k)$, where $k \in \{1, \dots, \dim \rho - n_l\}$ and n_l is the dimension of the lineality space of Π.

Finally, let U be a subset of $\mathbb{R}^n$ and let $\boldsymbol{\mathcal{H}}^{(j)} : U \to \mathbb{R}^m$, $j = 1, \dots, n_s$, be a collection of continuous functions. A function $\boldsymbol{\mathcal{H}} : U \to \mathbb{R}^m$ is said to be *a continuous selection* of the functions $\boldsymbol{\mathcal{H}}^{(1)}, \dots, \boldsymbol{\mathcal{H}}^{(n_s)}$ on the set $O \subset U$ if it is continuous on O

and $\boldsymbol{\mathcal{H}}(\boldsymbol{x}) \in \{\boldsymbol{\mathcal{H}}^{(1)}(\boldsymbol{x}), \dots, \boldsymbol{\mathcal{H}}^{(n_s)}(\boldsymbol{x})\}$ for every $\boldsymbol{x} \in O$. A function $\boldsymbol{\mathcal{H}} : U \to \mathbb{R}^m$ defined on an open set $U \subset \mathbb{R}^n$ is called *a PC^r-function* for some $r \in \{1, 2, \dots\} \cup \infty$ if for every $\boldsymbol{x}^0 \in U$, there exist an open neighbourhood $O \subset U$ of $\boldsymbol{x}^0$ and C^r-functions $\boldsymbol{\mathcal{H}}^{(1)}, \dots, \boldsymbol{\mathcal{H}}^{(n_s)} : O \to \mathbb{R}^m$ for some n_s such that $\boldsymbol{\mathcal{H}}$ is a continuous selection of $\boldsymbol{\mathcal{H}}^{(1)}, \dots, \boldsymbol{\mathcal{H}}^{(n_s)}$ on O. The functions $\boldsymbol{\mathcal{H}}^{(j)} : O \to \mathbb{R}^m$, $j = 1, \dots, n_s$, are termed *selection functions* for $\boldsymbol{\mathcal{H}}$ at $\boldsymbol{x}^0$ in this case. The set

$$I_{\boldsymbol{\mathcal{H}}}(\boldsymbol{x}^0) := \{j \in \{1, \dots, n_s\} \,|\, \boldsymbol{\mathcal{H}}^{(j)}(\boldsymbol{x}^0) = \boldsymbol{\mathcal{H}}(\boldsymbol{x}^0)\}$$

is known as *the active index set* and the selection functions $\boldsymbol{\mathcal{H}}^{(j)}$, $j \in I_{\boldsymbol{\mathcal{H}}}(\boldsymbol{x}^0)$, are said to be *active selection functions* at $\boldsymbol{x}^0$. PC^1-functions are also called *piecewise differentiable* functions. The directional derivative of $\boldsymbol{\mathcal{H}}$ at the point $\boldsymbol{x}$ in the direction $\boldsymbol{\xi}$ is denoted by $\boldsymbol{\mathcal{H}}'(\boldsymbol{x}; \boldsymbol{\xi})$.

Theorem A.1 ([56, Theorem 4.2.2]). *Let $U \subset \mathbb{R}^n \times \mathbb{R}^m$ be open, $\boldsymbol{\mathcal{H}} : U \to \mathbb{R}^m$ be a PC^r-function and let $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in U$ be a point with $\boldsymbol{\mathcal{H}}(\boldsymbol{x}^0, \boldsymbol{y}^0) = \mathbf{0}$. Further, let $\boldsymbol{\mathcal{H}}^{(1)}, \dots, \boldsymbol{\mathcal{H}}^{(n_s)} : O \to \mathbb{R}^m$ be a collection of selection functions for $\boldsymbol{\mathcal{H}}$ at $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in O \subset U$ and Π be a conical subdivision of $\mathbb{R}^n \times \mathbb{R}^m$ with a lineality space of dimension n_l. If*

1. *for every $\pi \in \Pi$, there exists an index $j_\pi \in \{1, \dots, n_s\}$ such that $\boldsymbol{\mathcal{H}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}, \boldsymbol{y})$ for every $(\boldsymbol{x}, \boldsymbol{y}) \in O \cap (\{(\boldsymbol{x}^0, \boldsymbol{y}^0)\} + \pi)$;*
2. *either $n + m - n_l \le 1$ or there exists a number $k \in \{2, \dots, n + m - n_l\}$ such that the kth branching number of Π does not exceed $2k$;*
3. *all matrices $\boldsymbol{\nabla_y} \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}^0, \boldsymbol{y}^0)$, $\pi \in \Pi$, have the same non-vanishing determinant sign*

then

1. *the equation $\boldsymbol{\mathcal{H}}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0}$ determines an implicit PC^r-function $\boldsymbol{y}(\boldsymbol{x})$ at the point $(\boldsymbol{x}^0, \boldsymbol{y}^0)$;*
2. *the implicit functions $\boldsymbol{y}^{(j_\pi)}(\boldsymbol{x})$ determined by the equations $\boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0}$, $\pi \in \Pi$, form a collection of selection functions for the PC^r-function $\boldsymbol{y}(\boldsymbol{x})$ at $\boldsymbol{x}^0$;*
3. *for every $\boldsymbol{\zeta} \in \mathbb{R}^n$, the identity $\boldsymbol{\xi} = \boldsymbol{y}'(\boldsymbol{x}^0; \boldsymbol{\zeta})$ holds if and only if $\boldsymbol{\xi}$ satisfies the piecewise linear equation $\boldsymbol{\mathcal{H}}'((\boldsymbol{x}^0, \boldsymbol{y}^0); (\boldsymbol{\zeta}, \boldsymbol{\xi})) = \mathbf{0}$.*

Theorem A.2 ([56, Proposition 4.2.2]). *Suppose that the assumptions of the previous theorem are satisfied and $\boldsymbol{\zeta} \in \mathbb{R}^n$ is arbitrary.*

1. *Then there exists a cone* $\pi \in \Pi$ *such that*

$$\begin{pmatrix} \boldsymbol{\zeta} \\ \mathbf{0}_{m,1} \end{pmatrix} \in \begin{pmatrix} \boldsymbol{I}_n & \mathbf{0}_{n,m} \\ \boldsymbol{\nabla_x} \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}^0, \boldsymbol{y}^0) & \boldsymbol{\nabla_y} \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}^0, \boldsymbol{y}^0) \end{pmatrix} \pi. \tag{A.1}$$

2. *The inclusion (A.1) holds if and only if*

$$\begin{pmatrix} \boldsymbol{\zeta} \\ -\big(\boldsymbol{\nabla_y} \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}^0, \boldsymbol{y}^0)\big)^{-1} \boldsymbol{\nabla_x} \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}^0, \boldsymbol{y}^0) \boldsymbol{\zeta} \end{pmatrix} \in \pi.$$

3. *If* $\boldsymbol{\zeta}$ *satisfies (A.1), then*

$$\boldsymbol{y}'(\boldsymbol{x}^0; \boldsymbol{\zeta}) = -\big(\boldsymbol{\nabla_y} \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}^0, \boldsymbol{y}^0)\big)^{-1} \boldsymbol{\nabla_x} \boldsymbol{\mathcal{H}}^{(j_\pi)}(\boldsymbol{x}^0, \boldsymbol{y}^0) \boldsymbol{\zeta}.$$

B The Moore-Penrose Continuation

Referring to [14], we present here briefly the classical Moore-Penrose continuation method.

Let $\boldsymbol{\mathcal{H}} : \mathbb{R}^{n+1} \to \mathbb{R}^n$, n a positive integer, be a smooth function. The aim of *numerical continuation* is to approximate the solution set of the equation $\boldsymbol{\mathcal{H}}(\boldsymbol{y}) = \mathbf{0}$. More precisely, following a chosen branch of solutions, one computes a sequence of consecutive points $\boldsymbol{y}_k$, $k = 1, 2, \dots$, satisfying $\|\boldsymbol{\mathcal{H}}(\boldsymbol{y}_k)\| < \varepsilon$ for a given $\varepsilon > 0$.

To describe the Moore-Penrose continuation, we suppose that we have found a point $\boldsymbol{y}_k$ satisfying the chosen tolerance criterion. We also suppose that we have a unit tangent vector $\boldsymbol{t}_k$ at $\boldsymbol{y}_k$:

$$\boldsymbol{\nabla\mathcal{H}}(\boldsymbol{y}_k)\boldsymbol{t}_k = \mathbf{0}, \quad \|\boldsymbol{t}_k\| = 1.$$

The next point is calculated in two steps – prediction and correction.

In *the prediction*, an initial approximation $\boldsymbol{Y}_0$ of the new point is given by

$$\boldsymbol{Y}_0 := \boldsymbol{y}_k + h_k \boldsymbol{t}_k,$$

where $h_k > 0$ is a step size. Its choice will be discussed later on.

The correction consists of a Newton-like procedure, which leads not only to the point $\boldsymbol{y}_{k+1}$ but also to the corresponding tangent vector $\boldsymbol{t}_{k+1}$. The algorithm is the following.

Algorithm B.1. (*Moore-Penrose continuation*)

Step 1: Set $\boldsymbol{T}^0 := \boldsymbol{t}_k$, $j := 0$.

Step 2: Set:

$$\boldsymbol{B} := \begin{pmatrix} \boldsymbol{\nabla\mathcal{H}}(\boldsymbol{Y}_j) \\ (\boldsymbol{T}_j)^T \end{pmatrix}, \quad \boldsymbol{R} := \begin{pmatrix} \boldsymbol{\nabla\mathcal{H}}(\boldsymbol{Y}_j)\boldsymbol{T}_j \\ 0 \end{pmatrix}, \quad \boldsymbol{Q} := \begin{pmatrix} \boldsymbol{\mathcal{H}}(\boldsymbol{Y}_j) \\ 0 \end{pmatrix},$$
$$\tilde{\boldsymbol{T}} := \boldsymbol{T}_j - \boldsymbol{B}^{-1}\boldsymbol{R}, \quad \boldsymbol{T}_{j+1} := \frac{\tilde{\boldsymbol{T}}}{\|\tilde{\boldsymbol{T}}\|},$$
$$\boldsymbol{Y}_{j+1} := \boldsymbol{Y}_j - \boldsymbol{B}^{-1}\boldsymbol{Q}.$$

Step 3: If $\|\boldsymbol{\mathcal{H}}(\boldsymbol{Y}_{j+1})\| < \varepsilon$ and $\|\boldsymbol{Y}_{j+1} - \boldsymbol{Y}_j\| < \varepsilon'$, set $\boldsymbol{y}_{k+1} := \boldsymbol{Y}_{j+1}$, $\boldsymbol{t}_{k+1} := \boldsymbol{T}_{j+1}$, else if $j < j_{\max}$, set $j := j + 1$ and go to Step 2.

Here $\varepsilon' > 0$ is a convergence tolerance and $j_{\max} > 0$ is the maximal number of corrections allowed.

Finally, the step size h_{k+1} in the next prediction depends on convergence of this Newton correction. Denoting the number of iterations needed by j, it is selected as

$$h_{k+1} = \begin{cases} h_{\mathrm{dec}} h_k & \text{if not converged,} \\ h_{\mathrm{inc}} h_k & \text{if converged and } j < j_{\mathrm{thr}}, \\ h_k & \text{otherwise,} \end{cases}$$

where $0 < h_{\mathrm{dec}} < 1 < h_{\mathrm{inc}}$ as well as $0 < j_{\mathrm{thr}} \leq j_{\max}$ are experimentally determined constants. At the beginning, one sets $h_1 = h_{\mathrm{init}}$ for some $h_{\mathrm{init}} > 0$.

Remark B.1. More precisely, finding the couple $(\boldsymbol{Y}_{j+1}, \tilde{\boldsymbol{T}})$ in the jth step of Algorithm B.1 corresponds to computing one iteration of the Newton method applied to the equation $\boldsymbol{H}_j(\boldsymbol{Y}, \boldsymbol{T}) = \boldsymbol{0}$, where $\boldsymbol{H}_j : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is defined by

$$\boldsymbol{H}_j(\boldsymbol{Y}, \boldsymbol{T}) = \begin{pmatrix} \boldsymbol{\mathcal{H}}(\boldsymbol{Y}) \\ (\boldsymbol{T}_j)^T (\boldsymbol{Y} - \boldsymbol{Y}_j) \\ \boldsymbol{\nabla} \boldsymbol{\mathcal{H}}(\boldsymbol{Y}_j) \boldsymbol{T} \\ (\boldsymbol{T}_j)^T \boldsymbol{T} - (\boldsymbol{T}_j)^T \boldsymbol{T}_j \end{pmatrix}, \quad (\boldsymbol{Y}, \boldsymbol{T}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.$$

Furthermore, one can easily verify that the auxiliary vector $\tilde{\boldsymbol{T}}$ can be equivalently calculated as

$$\boldsymbol{R} := \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}, \quad \tilde{\boldsymbol{T}} := \boldsymbol{B}^{-1} \boldsymbol{R}.$$

Bibliography

[1] F. Armero and E. Petőcz: *Formulation and analysis of conserving algorithms for frictionless dynamic contact/impact problems*, Comput. Methods Appl. Mech. Eng., **158** (1998), pp. 269–300.

[2] J.-P. Aubin: *Applied Functional Analysis*, Pure and Applied Mathematics. John Wiley & Sons, New York, 1979.

[3] F. Ben Belgacem and Y. Renard: *Hybrid finite element methods for the Signorini problem*, Math. Comput. **72** (2003), 1117–1145.

[4] P. Beremlijski, J. Haslinger, M. Kočvara, R. Kučera and J. V. Outrata: *Shape optimization in three-dimensional contact problems with Coulomb friction*, SIAM J. Optim. **20** (2009), 416–444.

[5] P. Beremlijski, J. Haslinger, M. Kočvara and J. V. Outrata: *Shape optimization in contact problems with Coulomb friction*, SIAM J. Optim. **13** (2002), 561–587.

[6] F. E. Browder: *Multivalued Monotone Nonlinear Mappings and Duality Mappings in Banach Spaces*, Trans. Am. Math. Soc. **118** (1965), 338–351.

[7] F. E. Browder: *Nonlinear Maximal Monotone Operators in Banach Space*, Math. Ann. **175** (1968), 89–113.

[8] N. J. Carpenter, R. L. Taylor and M. G. Kantona: *Lagrange constraints for transient finite element surface contact*, Int. J. Numer. Methods Eng. **32** (1991), 103–128.

[9] P. G. Ciarlet: *The finite element method for elliptic problems*, Studies in Mathematics and its Applications, Vol. 4. North-Holland, Amsterdam - New York - Oxford, 1978.

[10] P. G. Ciarlet: *Mathematical Elasticity*, Vol. I: Three-Dimensional Elasticity, Studies in Mathematics and it Applications, Vol. 20. North-Holland, Amsterdam, 1988.

[11] F. H. Clarke: *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, New York, 1983.

[12] P. Clément: *Approximation by finite element functions using local regularization*, Rev. Franc. Automat. Inform. Rech. Operat. **9**, R-2 (1975) 77–84.

[13] K. Deimling: *Multivalued differential equations*, De Gruyter Studies in Nonlinear Analysis and Applications 1. Walter de Gruyter, Berlin, 1992.

[14] A. Dhooge, W. Govaerts and Yu. A. Kuznetsov: *MATCONT: A MATLAB Package for Numerical Bifurcation Analysis of ODEs*, ACM Trans. Math. Softw. **29** (2003), 141–164.

[15] A. L. Dontchev and W. W. Hager: *Implicit functions, Lipschitz maps, and stability in optimization*, Math. Oper. Res. **19** (1994), 753–768.

[16] C. Eck, J. Jarušek and M. Krbec: *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics 270. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[17] I. Ekeland and R. Temam: *Analyse convexe et problèmes variationnels*, Études mathématiques. Dunod, Gauthier-Villars, Paris - Bruxelles - Montréal, 1974.

[18] F. Facchinei and J.-S. Pang: *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vol. II, Springer Series in Operations Research. Springer, New York, 2003.

[19] C. Hager, S. Hüeber and B. I. Wohlmuth: *A stable energy-conserving approach for frictional contact problems based on quadrature formulas*, Int. J. Numer. Methods. Eng. **73** (2008), 205–225.

[20] J. O. Hallquist, K. Schweizerhof and D. Stillman: *Efficiency refinements of contact strategies and algorithms in explicit finite element programming*, in: Computational Plasticity: Fundamentals and Applications. Proceedings of COMPLAS III (D. R. J. Owen, E. Onate and E. Hinton eds.), Pineridge Press, Swansea, 1992, pp. 457–482.

[21] J. Haslinger: *Approximation of the Signorini problem with friction, obeying the Coulomb law*, Math. Methods Appl. Sci. **5** (1983), 422–437.

[22] J. Haslinger, I. Hlaváček and J. Nečas: *Numerical Methods for Unilateral Problems in Solid Mechanics*, Handbook of Numerical Analysis, Vol. IV (P. G. Ciarlet et al., eds.). North-Holland, Amsterdam, 1996, pp. 313–485.

[23] J. Haslinger, V. Janovský and T. Ligurský: Qualitative analysis of solutions to discrete static contact problems with Coulomb friction, accepted to Math. Comput. Methods Appl. Mech. Eng. (2010).

[24] J. Haslinger, R. Kučera and T. Ligurský: *Qualitative analysis of 3D elastostatic contact problems with orthotropic Coulomb friction and solution-dependent coefficients of friction*, J. Comput. Appl. Math. **235** (2011), 3464–3480.

[25] J. Haslinger, R. Kučera and O. Vlach: *Bifurcations in Contact Problems with Local Coulomb Friction*, in: Numerical Mathematics and Advanced Applications. Proceedings of ENUMATH 2007 (K. Kunisch et al., eds.). Springer, Berlin, 2008, pp. 811–818.

[26] P. Hauret: *Méthodes numériques pour la dynamique des structures non-linéaires incompressibles à deux échelles*, PhD. thesis, Ecole Polytechnique, 2004.

[27] P. Hauret: *Mixed interpretation and extensions of the equivalent mass matrix approach for elastodynamics with contact*, Comput. Methods Appl. Mech. Eng. **199** (2010) 2941–2957.

[28] P. Hauret and P. Le Tallec: *Energy-controlling time integration methods for non-linear elastodynamics and low-velocity impact*, Comput. Methods Appl. Mech. Eng. **195** (2006), 4890–4916.

[29] P. Hild: *On finite element uniqueness studies for Coulomb's frictional contact model*, Int. J. Appl. Math. Comput. Sci. **12** (2002), 41–50.

[30] P. Hild: *An example of nonuniqueness for the continuous static unilateral contact model with Coulomb friction*, C. R., Math., Acad. Sci. Paris **337** (2003), 685-–688.

[31] P. Hild: *Non-unique slipping in the Coulomb friction model in two-dimensional linear elasticity*, Q. J. Mech. Appl. Math. **57** (2004), 225–235.

[32] P. Hild and Y. Renard: *Local uniqueness and continuation of solutions for the discrete Coulomb friction problem in elastostatics*, Quart. Appl. Math. **63** (2005), 553–573.

[33] I. Hlaváček: *Finite element analysis of a static contact problem with Coulomb friction*, Appl. Math. **45** (2000) 357-–379.

[34] T. J. R. Hughes, R. L. Taylor, J. L. Sackman, A. Curnier and W. Kanoknukulchai: *A finite element method for a class of contact-impact problems*, Computer Methods Appl. Mech. Engin. **8** (1976), 249–276.

[35] V. Janovský: *Catastrophic features of Coulomb friction model*, in: The Mathematics of Finite Elements and Applications IV. Proceedings of MAFELAP IV (J. R. Whiteman eds.). Academic Press, New York, 1982, pp. 259–264.

[36] V. Janovský and T. Ligurský: *Computing non unique solutions of the Coulomb friction problem*, accepted to Math. Comput. Simul. (2010).

[37] H. B. Khenous: *Problèmes de contact unilatéral avec frottement de Coulomb en élastostatique et élastodynamique. Etude mathématique et résolution numérique*, PhD. thesis, INSA de Toulouse, 2005.

[38] H. B. Khenous, P. Laborde and Y. Renard: *Mass redistribution method for finite element contact problems in elastodynamics*, Eur. J. Mech., A, Solids **27** (2008), 918–932.

[39] H. B. Khenous, J. Pommier and Y. Renard: *Hybrid discretization of the Signorini problem with Coulomb friction. Theoretical aspects and comparison of some numerical solvers*, Appl. Numer. Math. **56** (2006), 163–192.

[40] T. A. Laursen and V. Chawla: *Design of energy conserving algorithms for frictionless dynamic contact problems.* Int. J. Numer. Methods Eng. **40** (1997), 863–886.

[41] T. A. Laursen and G. R. Love: *Improved implicit integrators for transient impact problems – geometric admissibility within the conserving framework*, Int. J. Numer. Methods Eng. **53** (2002), 245–274.

[42] F. Lempio: *Difference Methods for Differential Inclusions*, in: Modern Methods of Optimization. Proceedings of the Summer School "Modern Methods of Optimization", Bayreuth, 1990 (W. Krabs and J. Zowe eds.). Lect. Notes Econ. Math. Syst. **378**, Springer-Verlag, Berlin-Heidelberg-New York, 1992, pp. 236–273.

[43] T. Ligurský: *Approximation and numerical realization of 3D contact problems with given friction and a coefficient of friction depending on the solution*, Master's thesis, Charles University in Prague, Faculty of Mathematics and Physics, 2007 (`http://artax.karlin.mff.cuni.cz/~ligut2am/tl21.pdf`).

[44] T. Ligurský: *Theoretical analysis of discrete contact problems with Coulomb friction*, accepted to Appl. Math. (2010).

[45] T. Ligurský, J. Haslinger and R. Kučera: *Approximation and numerical realization of 3D contact problems with Coulomb friction and a solution-dependent coefficient of friction*, Int. J. Numer. Methods Eng. **82** (2010), 1180–1206.

[46] T. Ligurský and Y. Renard: *A Well-Posed Semi-Discretization of Elastodynamic Contact Problems With Friction*, Q. J. Mech. Appl. Math. **64** (2011), 215–238.

[47] J. J. Moreau: *Numerical aspects of the sweeping process*, Comput. Methods Appl. Mech. Eng. **177** (1999), 329–349.

[48] J. Nečas, J. Jarušek and J. Haslinger: *On the solution of the variational inequality to the Signorini problem with small friction*, Boll. Unione Mat. Ital., V. Ser., B **17** (1980), 796–811.

[49] L. Paoli: *Time discretization of vibro-impact*, Philos. Trans. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **359** (2001), 2405–2428.

[50] J.-C. Paumier, Y. Renard: *Surface perturbation of an elastodynamic contact problem with friction*, Eur. J. Appl. Math. **14** (2003), 465–483.

[51] Y. Renard: *Modélisation des instabilités liées au frottement sec des solides élastiques, aspects théoriques et numériques*, PhD. thesis, LMC-IMAG Grenoble, 1998.

[52] Y. Renard: *A uniqueness criterion for the Signorini problem with Coulomb friction*, SIAM J. Math. Anal. **38** (2006), 452—467.

[53] Y. Renard: *The singular dynamic method for constrained second order hyperbolic equations: application to dynamic contact problems*, J. Comput. Appl. Math. **234** (2010), 906–923.

[54] Y. Renard and J. Pommier: *GetFEM++*, An open source generic C++ library for finite element methods (`http://home.gna.org/getfem`).

[55] S. M. Robinson: *Strongly regular generalized equations*, Math. Oper. Res. **5** (1980), 43–62.

[56] S. Scholtes: *Introduction to piecewise differentiable equations*, Preprint No. 53/1994, Institut für Statistik und Mathematische Wirtschaftstheorie, Universität Karlsruhe, 1994.