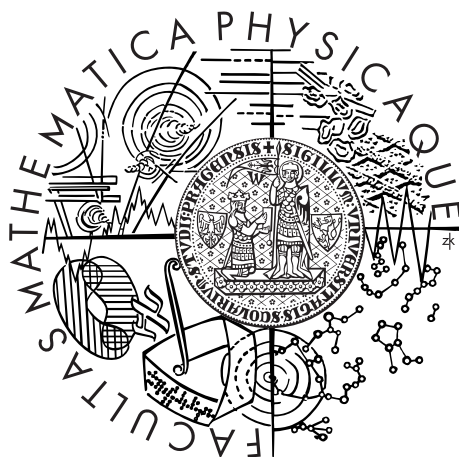


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Matematicko-fyzikální fakulta

## BAKALÁŘSKÁ PRÁCE



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## Vlastnosti metrických prostorů pomocí konvergence

Katedra matematické analýzy

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I would like to thank my parents and my girlfriend Lucie for their support, and the following companies for their products which made writing this thesis easier: Apple for the extraordinary Mac, Moleskine for the legendary notebook and Pilot for the reliable G-2 roller.

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, date 24th May 2012

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Abstrakt: V této práci se zabýváme zobecněním struktury konvergence v metrických prostorech a charakterizací některých vlastností pomocí posloupností. Na základě chování konvergentních posloupností, které společně s vybranými vlastnostmi metrických prostorů připomínáme, zavádíme dvě obecné struktury. První z nich, sekvenciální prostor, obsahuje informace o limitě posloupností, kterou uvažuje jednoznačnou. Druhá, uniformně sekvenciální prostor, zobecňuje relaci blízkosti dvou posloupností. Ukážeme, že spojitost zobrazení, topologie, kompaktnost, souvislost a separabilita se dají odvodit ze sekvenciální struktury. Dále že totální omezenost a úplnost se dají charakterizovat s využitím pojmu cauchyovské posloupnosti, kterou můžeme definovat v uniformně sekvenciálních strukturách. O omezenosti dokážeme, že ji nelze popsat ani jednou z těchto struktur.

Klíčová slova: Metrický prostor, Sekvenciální struktura, Uniformně sekvenciální struktura

Title: Properties of metric spaces by means of convergence

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Abstract: In this thesis we deal with generalisation of the structure of convergence in metric spaces and characterisation of some properties using sequences. On basis of behaviour of convergent sequences in metric spaces which we alongside with selected properties in metric spaces remind we introduce the general structures. The first of them - sequential spaces - includes the information about the limit of a sequence which we consider to be unique. The second - uniformly sequential spaces - generalises the relation of adjacency of two sequences. We show that continuity of a mapping, topology, compactness, connectedness and separability can be induced from a sequential structure. In addition, we show total boundedness and completeness can be characterised using the term of a Cauchy sequence which we can define in a uniformly sequential structure. Boundedness is shown to be independent on both of those structures.

Keywords: Metric space, Sequential structure, Uniformly sequential structure

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# Preface

In the theory of metric spaces various properties are studied. These properties usually have numerous characterisations. We shall examine whether selected ones can be described using convergent sequences.

In the first chapter we remind the definition of a metric space and some of its interesting properties without using sequences. Stated theorems show equivalent conditions or basic behaviour.

The second chapter introduces two abstract spaces: sequential and uniformly sequential spaces. Both of them provide a generalization of certain properties of sequences we examined in the first chapter. We show their relations to metric spaces and to each other.

In the third chapter properties are divided into three parts. Continuous mappings, topology, compactness, connectedness and separability are those which can be characterised using sequential spaces only. We also examine separability behaves slightly differently although equivalently in metric spaces. Total boundedness and completeness need more than this, we show that can be characterised using uniformly sequential spaces. The last part concerns properties which cannot be characterised using either of these structures, we show boundedness is one of them.

# 1. Selected facts from the theory of metric spaces

## 1.1 Metric spaces and their properties

We will briefly remind selected definitions and theorems, for proofs see [1] or [2].

**Definition 1.1.** Let  $X$  be a non-empty set. Let  $\varrho : X \times X \rightarrow \mathbb{R}$  be a mapping satisfying the following conditions:

(M1) For  $x, y \in X : \varrho(x, y) = 0 \Leftrightarrow x = y$ .

(M2) For  $x, y, z \in X : \varrho(x, z) \leq \varrho(x, y) + \varrho(z, y)$ .

We say that  $\varrho$  is a *metric on  $X$*  and  $(X, \varrho)$  is a *metric space*.

The number  $\varrho(x, y)$  is called the *distance between  $x$  and  $y$* . Condition (M1) is called *identity of indiscernibles* and (M2) is called *triangle inequality*. It is an easy exercise to show that the following holds:

(M3) For  $x, y \in X : \varrho(x, y) = \varrho(y, x)$ .

Sometimes (M2) is written with  $\varrho(y, z)$  as the last item instead of  $\varrho(z, y)$  but then (M3) must be added to the conditions.

When we set  $x = z$  in (M2) it follows that  $\varrho(x, y) \geq 0$  for arbitrary points  $x, y \in X$ .

**Definition 1.2.** Let  $(X, \varrho)$  be a metric space.

(i) We say the *distance between a point  $x \in X$  and a set  $A \subset X$*  (denoted as  $\varrho(x, A)$  or  $\varrho(A, x)$ ) is the number

$$\varrho(x, A) = \varrho(A, x) := \inf\{\varrho(x, y) : y \in A\}.$$

(ii) For  $x_0 \in X$  and  $\varepsilon > 0$  the set  $B(x_0, \varepsilon) := \{x \in X : \varrho(x_0, x) < \varepsilon\}$  is called the *open ball with centre  $x_0$  and radius  $\varepsilon$* .

(iii) Set  $A \subset X$  is called *open in  $(X, \varrho)$*  if  $\forall x \in A \exists \varepsilon > 0 : B(x, \varepsilon) \subset A$ . Set  $B \subset X$  is called *closed in  $(X, \varrho)$*  if  $X \setminus B$  is open.

(iv) A *closure of a subset  $A$  in  $(X, \varrho)$*  (denoted as  $\overline{A}$ ) is defined as

$$\overline{A} := \bigcap \{F : A \subset F \subset X, F \text{ closed in } (X, \varrho)\}.$$

(v) Let  $\sigma$  be a metric on  $X$ . We say  $\varrho$  and  $\sigma$  are *equivalent* if the family of all sets open in  $(X, \varrho)$  is the same as the family of all sets open in  $(X, \sigma)$ .

(vi) For a non-empty  $A \subset X$  the number  $\text{diam}(A) := \sup_{x, y \in A} \varrho(x, y)$  is called the *diameter of  $A$* . We define  $\text{diam}(\emptyset) := 0$ . The space  $(X, \varrho)$  is called *bounded* if  $\text{diam}(X) < \infty$ , otherwise it is called *unbounded*.

(vii) The space is called *totally bounded* if

$$\forall \varepsilon \exists N \in \mathbb{N} \exists x_1, \dots, x_N \in X : \bigcup_{n=1}^N B(x_n, \varepsilon) = X.$$

(viii) The space is called *connected* if  $X \neq \emptyset$  and whenever  $U, V \subset X$ , both  $U$  and  $V$  are open,  $X = U \cup V$  and  $U \cap V = \emptyset$  then  $U = \emptyset$  or  $V = \emptyset$ .

(ix) The space is called *separable* if there is a countable  $A \subset X$  for which  $\overline{A} = X$ .

(x) The space  $(X, \rho)$  is called *compact* if when  $X = \bigcup_{\alpha \in A} U_\alpha$ , where  $A$  is an arbitrary set and  $U_\alpha$  are open for every  $\alpha \in A$ , then there exists a finite  $B \subset A$  such that  $X = \bigcup_{\alpha \in B} U_\alpha$ .

A set is closed if and only if it is equal to its closure. A totally bounded space is evidently bounded.

**Theorem 1.3** (Properties of closure operator). *The operator assigning to  $A \subset X$  its closure  $\overline{A}$  (called Kuratowski closure operator) has the following properties:*

- (i)  $\overline{\emptyset} = \emptyset$ ,
- (ii)  $A \subset \overline{A}$ ,
- (iii)  $\forall B \subset X : \overline{A \cup B} = \overline{A} \cup \overline{B}$ ,
- (iv)  $\overline{\overline{A}} = \overline{A}$ .

**Theorem 1.4** (Generating topology by the closure operator). *Let  $X$  be a set and let  $c : A \subset X \rightarrow c(A) \subset X$  be an operator satisfying conditions (i)-(iv) from Theorem 1.3. Then  $\{X \setminus A : A = c(A)\}$  is a topology on  $X$ .*

Previous two theorems with proofs can be found in [5, Theorem 1.1.3] and [5, Proposition 1.2.7], respectively.

**Example 1.5.** Let  $X$  be a non-empty set,  $x, y \in X$  and

$$\rho(x, y) := \begin{cases} 1, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

It is evident that  $(X, \rho)$  is a metric space. It is called a *metrically discrete space*.

For our purposes uniformly discrete space will mean any metric space for which  $\exists \varepsilon > 0, \forall x \in X : B(x, \varepsilon) = \{x\}$  holds.

**Definition 1.6.** Let  $(X, \rho)$  and  $(Y, \sigma)$  be two metric spaces. We say a mapping  $f : X \rightarrow Y$  is *continuous* if

$$\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 \forall y \in X : \rho(x, y) < \delta \Rightarrow \sigma(f(x), f(y)) < \varepsilon.$$

We say  $f$  is *uniformly continuous* if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : \rho(x, y) < \delta \Rightarrow \sigma(f(x), f(y)) < \varepsilon.$$

Clearly a uniformly continuous mapping is continuous.

**Theorem 1.7.** *Let  $(X, \rho)$  and  $(Y, \sigma)$  be two metric spaces. Then  $f : X \rightarrow Y$  is continuous if and only if preimage  $f^{-1}[A]$  is open in  $(X, \rho)$  for every  $A \subset Y$  open in  $(Y, \sigma)$ .*

## 1.2 Some properties of sequences in metric spaces

**Definition 1.8.** Let  $(X, \rho)$  be a metric space,  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$ . We say that  $\{x_n\}_{n=1}^{\infty}$  converges to  $x \in X$  (or  $x$  is a limit of  $\{x_n\}_{n=1}^{\infty}$ ) if  $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$  and we will denote it as  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$ .

We shall sometimes write only  $\{x_n\}$  instead of  $\{x_n\}_{n=1}^{\infty}$ .

**Theorem 1.9** (Properties of convergent sequences). *Let  $(X, \rho)$  be a metric space,  $\{x_n\}$  be a sequence in  $X$ . Then:*

- (i) *If  $x_n \rightarrow x \in X$  and  $x_n \rightarrow y \in X$  then  $x = y$ .*
- (ii) *If  $\{x_n\}$  is convergent then it is bounded.*
- (iii) *Sequence  $\{x_n\}_{n=1}^{\infty}$  converges to  $x \in X$  if and only if every its subsequence  $\{x_{n_k}\}_{k=1}^{\infty}$  converges to  $x$ .*
- (iv) *Let  $\{x_n\} \not\rightarrow x$  then there exists a subsequence  $\{x_{n_k}\}$  such that for none of its subsequence  $\{x_{n_{k_i}}\}$  is  $\{x_{n_{k_i}}\} \rightarrow x$ .*
- (v) *Let  $\sigma$  be a metric on  $X$  equivalent with  $\rho$  and  $x \in X$  then  $\{x_n\} \rightarrow x$  in  $(X, \rho)$  if and only if  $\{x_n\} \rightarrow x$  in  $(X, \sigma)$ .*
- (vi) *Let  $A \subset X$  then  $\bar{A} = \bigcup \{x \in X : \exists \{x_n\} \subset A, x_n \rightarrow x\}$ .*

**Definition 1.10.** Let  $(X, \rho)$  be a metric space.

- (i) We say that a sequence  $\{x_n\} \subset X$  is *Cauchy* if

$$\forall \varepsilon \exists n_0 \in \mathbb{N} \forall n, m \in \mathbb{N}; n, m \geq n_0 : \rho(x_n, x_m) < \varepsilon.$$

- (ii) The space is called *complete* if every Cauchy sequence of points of  $X$  converges to a point of  $X$ .

**Theorem 1.11.** *For a metric space the following are equivalent:*

- (i) *It is compact.*
- (ii) *It is complete and totally bounded.*

**Definition 1.12.** Let  $(X, \rho)$  be a metric space. We say a sequence  $\{x_n\}$  is *adjacent* to a sequence  $\{y_n\}$  (denoted as  $\{x_n\} \sim_{\rho} \{y_n\}$ ) if  $\lim_{n \rightarrow \infty} \rho(x_n, y_n) = 0$ .

The following theorem shows that  $\sim_{\rho}$  as a relation on sequences in  $X$  is an equivalence.

**Theorem 1.13.** *Relation  $\sim_{\rho}$  is reflexive, symmetric and transitive.*

*Proof.* Let  $\{x_n\}$  be a sequence in  $(X, \rho)$ . For every  $n \in \mathbb{N}$  we have  $\rho(x_n, x_n) = 0$  so  $\{x_n\} \sim_{\rho} \{x_n\}$  and  $\sim_{\rho}$  is reflexive.

Symmetry of  $\sim_{\rho}$  follows from (M3).

Let now  $\{x_n\} \sim_{\rho} \{y_n\}$ ,  $\{x_n\} \sim_{\rho} \{z_n\}$  and  $\varepsilon > 0$ . For every  $n \in \mathbb{N}$  we have  $\rho(z_n, y_n) \leq \rho(x_n, y_n) + \rho(x_n, z_n)$ . We find  $n_0, n_1 \in \mathbb{N}$  such that for all  $n \in \mathbb{N}, n \geq n_0$  ( $n \geq n_1$ ) we have  $\rho(x_n, y_n) < \frac{\varepsilon}{2}$  (respectively  $\rho(x_n, z_n) < \frac{\varepsilon}{2}$ ). Then for all  $n \in \mathbb{N}, n \geq \max\{n_0, n_1\}$  we obtain  $\rho(x_n, z_n) < \varepsilon$ . That is  $\lim_{n \rightarrow \infty} \rho(z_n, y_n) = 0$  and  $\sim_{\rho}$  is transitive.  $\square$

# 2. Sequential and uniformly sequential spaces

## 2.1 Family of convergent sequences on a set

To study the role of convergence in metric spaces we take some properties of sequences which hold in metric spaces and create new space using these properties as conditions. It turns out that the following conditions (C1) - (C4) introduce a rich class of spaces yet relatively simple to explore.

**Definition 2.1.** Let  $X$  be a non-empty set. A *sequential structure* on  $X$  (denoted by  $\Xi_X$ ) is a set of pairs  $(\{x_n\}, x)$  consisting of a sequence  $\{x_n\} \subseteq X$  and a point  $x \in X$  satisfying the following conditions:

- (C1) If  $(\{x_n\}, x) \in \Xi_X$  and  $\{y_n\}$  is a subsequence of  $\{x_n\}$  then  $(\{y_n\}, x) \in \Xi_X$ .
- (C2) If  $(\{x_n\}, x) \in \Xi_X, (\{x_n\}, y) \in \Xi_X$  then  $x = y$ .
- (C3) If  $(\{x_n\}, x) \notin \Xi_X$  then there exists a subsequence  $\{x_{n_k}\}_{k=1}^{\infty}$  such that for none of its subsequence  $\{x_{n_{k_i}}\}$  is  $(\{x_{n_{k_i}}\}, x) \in \Xi_X$ .
- (C4) If  $x \in X$  then  $(\{x\}, x) \in \Xi_X$ .

We say  $(X, \Xi_X)$  is a *sequential space*.

In the next we will write  $\Xi$  instead of  $\Xi_X$  wherever the set  $X$  is clear from the context. Unless stated otherwise  $X$  will denote a non-empty set. When  $(\{x_n\}, x) \in \Xi$  we say that  $\{x_n\}$  is a *convergent sequence*,  $\{x_n\}$  *converges* to  $x$  or that  $x$  is a limit of  $\{x_n\}$ .

The properties in Definition 2.1 can be restated as follows:

- (C1') A subsequence of a convergent sequence is convergent and converges to the same point.
- (C2') Every sequence has at most one limit.
- (C3') A sequence that is not converging to a point  $x$  contains a subsequence such that any of its subsequences either converges to a different point (not to  $x$ ) or does not converge at all.
- (C4') A constant sequence converges to the common point.

The first two and the fourth properties are natural and follow from basic knowledge of convergence in metric spaces. To understand the condition (C3) we need to remind that a sequence  $\{x_n\}$  which does not converge to  $x$  might include a subsequence  $\{x_{n_k}\}$  which converges to  $x$ .

Our motivation to define sequential structures was to generalise convergence in metric spaces. The following shows the relation between them.

**Definition 2.2.** Let  $(X, \rho)$  be a metric space. We say  $\Xi$  is *generated by*  $X$  if

$$\Xi = \{(\{x_n\}, x) : x \in X, \forall n \in \mathbb{N}, x_n \in X, \lim_{n \rightarrow \infty} \rho(x_n, x) = 0\}.$$

A sequential structure generated by a metric space is well defined. We know that  $\lim_{n \rightarrow \infty} \varrho(x_n, x) = 0$  implies  $\lim_{n \rightarrow \infty} \varrho(x_{n_k}, x) = 0$  for every subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$ . If  $x_n \rightarrow x$  and  $x_n \rightarrow y$  for some  $x, y \in X$  we have

$$0 \leq \varrho(x, y) \leq \varrho(x_n, x) + \varrho(x_n, y) \rightarrow 0$$

as  $n \rightarrow \infty$ , and so  $\varrho(x, y) = 0$  which by (M1) means  $x = y$ . Property (C3) follows from Theorem 1.9. And last, obviously  $\{x\} \rightarrow x$ .

In metric spaces adding a finite number of members to a sequence as well as removing finite number of members from it does not affect the convergence of the sequence. The same holds for sequential structures.

**Theorem 2.3.** *Let  $\Xi$  be a sequential structure,  $(\{x_n\}, x) \in \Xi$  and  $a \in X$ . Let  $\{y_n\}$  and  $\{z_n\}$  be sequences defined as follows:*

$$(i) \quad y_1 = a, \quad y_{n+1} = x_n \quad \forall n \in \mathbb{N},$$

$$(ii) \quad z_n = x_{n+1} \quad \forall n \in \mathbb{N}.$$

Then  $(\{y_n\}, x), (\{z_n\}, x) \in \Xi$ .

*Proof.* Let for contradiction suppose  $(\{y_n\}, x) \notin \Xi$ . Then using (C3) we can find a subsequence  $\{y_{n_k}\}$  such that for every its subsequence  $(\{y_{n_{k_j}}\}, x) \notin \Xi$ . We can assume  $\{y_{n_{k_j}}\}$  does not include the first member of  $\{y_n\}$ . Then  $\{y_{n_{k_j}}\}$  is a subsequence of  $\{x_n\}$  and by (C1) we have  $(\{y_{n_{k_j}}\}, x) \in \Xi$  which is a contradiction.

Since  $\{z_n\}$  is a subsequence of  $\{x_n\}$  the conclusion follows from (C1).  $\square$

**Theorem 2.4.** *An intersection of sequential structures on a set  $X$  is a sequential structure on  $X$ .*

*Proof.* Let  $\Xi_\alpha$  be a sequential structure on  $X$  for every  $\alpha \in A$ , where  $A$  is an arbitrary non-empty set. Let  $(\{x_n\}, x) \in \bigcap_{\alpha \in A} \Xi_\alpha$  and  $\{y_n\}$  be a subsequence of  $\{x_n\}$ . Then for every  $\alpha \in A$  we have  $(\{x_n\}, x) \in \Xi_\alpha$  so by (C1) we have  $(\{y_n\}, x) \in \Xi_\alpha$ . That is,  $(\{y_n\}, x) \in \bigcap_{\alpha \in A} \Xi_\alpha$ .

Also for every  $\alpha \in A$  and every  $x \in X$  we have  $(\{x\}, x) \in \Xi_\alpha$  so  $(\{x\}, x) \in \bigcap_{\alpha \in A} \Xi_\alpha$ .

Let  $(\{x_n\}, x) \notin \bigcap_{\alpha \in A} \Xi_\alpha$  then there exists  $\alpha_0 \in A$  for which  $(\{x_n\}, x) \notin \Xi_{\alpha_0}$ . We find the subsequence  $\{x_{n_k}\}$  from (C3) for  $\Xi_{\alpha_0}$  and let  $\{x_{n_{k_i}}\}$  be its subsequence. Then  $(\{x_{n_{k_i}}\}, x) \notin \Xi_{\alpha_0}$  therefore  $(\{x_{n_{k_i}}\}, x) \notin \bigcap_{\alpha \in A} \Xi_\alpha$ .

Let  $(\{x_n\}, x) \in \bigcap_{\alpha \in A} \Xi_\alpha$  and  $y \in X$ ,  $x \neq y$ . Then for some  $\alpha \in A$  we have  $(\{x_n\}, x) \in \Xi_\alpha$  and from (C2) for  $\Xi_\alpha$  we have  $(\{x_n\}, y) \notin \Xi_\alpha$ . Therefore  $(\{x_n\}, y) \notin \bigcap_{\alpha \in A} \Xi_\alpha$ .  $\square$

**Definition 2.5.** Let  $X$  be a non-empty set. Symbol  $\Xi_0$  will denote the smallest (in the meaning of inclusion) sequential structure on  $X$ .

The definition says that  $\Xi_0$  consists of convergent sequences which are constant up to a finite number of members.

Since  $\Xi_0$  is the smallest sequential structure on the set it is wherefore interesting to see for which metric spaces  $\Xi = \Xi_0$  holds. It is easy to observe that  $\Xi$  generated by a metrically discrete space (in Example 1.5) has his property.

At first sight one might think that  $\Xi = \Xi_0$  implies the space  $(X, \varrho)$  to be metrically discrete. The following shows a counterexample constructing a space that is not metrically discrete.

**Example 2.6.** Suppose  $X = \{\frac{1}{n}, n \in \mathbb{N}\}$ , then  $X \subset \mathbb{R}$  and with induced metric it forms a subspace of  $\mathbb{R}$ . As the sequence  $\{\frac{1}{n}\}$  has a limit in  $\mathbb{R}$  which is not included in  $X$ , every convergent sequence in  $X$  has to be constant from some index. Therefore  $\Xi = \Xi_0$ .

Another idea we had was: If  $\Xi = \Xi_0$  then either  $X$  is metrically discrete or  $X$  is not complete. The following example proves this statement to be false.

**Example 2.7.** Suppose  $X = \{(n, 0), n \in \mathbb{N}\} \cup \{(n, \frac{1}{n}), n \in \mathbb{N}\}$ , then  $X \subset \mathbb{R}^2$ . Intuitive image is the set of natural numbers as a subset of  $x$  axis united with the graph of  $\{\frac{1}{n}\}$  in the plane. This space (with induced metric) is complete and every convergent sequence is constant from some index.

**Theorem 2.8.** Let  $\Xi$  be generated by a metric space  $(X, \rho)$  then the following are equivalent:

(i)  $\Xi = \Xi_0$ ;

(ii)  $\forall x \in X, \exists \varepsilon_x > 0, \forall y \in X, x \neq y : \rho(x, y) > \varepsilon_x$ . (We say  $(X, \rho)$  is topologically discrete.)

*Proof.* Let  $X$  be a topologically discrete metric space,  $\Xi$  generated by  $X$  and for contradiction suppose there exists a sequence  $\{x_n\}$  and  $x \in X$  such that  $(\{x_n\}, x) \in \Xi \setminus \Xi_0$ . Then from (ii) we find  $\varepsilon_x$ . Because  $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$  there is  $n_0 \in \mathbb{N}$  for which  $\rho(x_{n_0}, x) \leq \frac{1}{2}\varepsilon_x$  and  $x_{n_0} \neq x$  (since  $(\{x_n\}, x) \notin \Xi_0$ ). This is a contradiction with  $X$  being topologically discrete.

Let  $\Xi = \Xi_0$  and for contradiction suppose that for some  $x \in X$  we have

$$\forall \varepsilon > 0, \exists y \in X, x \neq y : \rho(x, y) < \varepsilon.$$

For  $n \in \mathbb{N}$  we set  $\varepsilon_n := \frac{1}{n}$  and find  $y_n \neq x$  such that  $\rho(x, y_n) < \varepsilon_n$ . Then  $\{y_n\} \rightarrow x$  which is a contradiction since  $\{y_n\}$  is not constant.  $\square$

## 2.2 Uniformly sequential structure on a set

**Definition 2.9.** Let  $X$  be a non-empty set. A *uniformly sequential structure* on  $X$  is an equivalence  $\sim$  on sequences in  $X$  satisfying the following conditions:

(U1) If  $\{x_n\} \sim \{y_n\}$  and  $\{x_{n_k}\}, \{y_{n_k}\}$  are subsequences then  $\{x_{n_k}\} \sim \{y_{n_k}\}$ .

(U2) If  $x_n = x, y_n = y \forall n \in \mathbb{N}$  for some  $x, y \in X$  and  $\{x_n\} \sim \{y_n\}$  then  $x = y$ .

(U3) If  $\{x_n\} \not\sim \{y_n\}$  then there exist subsequences  $\{x_{n_k}\}, \{y_{n_k}\}$  such that for none of their subsequences  $\{x_{n_{k_i}}\}, \{y_{n_{k_i}}\}$  is  $\{x_{n_{k_i}}\} \sim \{y_{n_{k_i}}\}$ .

(U4) If  $x \in X$  then  $\{x\} \sim \{y\}$ . (Reflexivity)

We say  $(X, \sim)$  is a *uniformly sequential space*. If  $\{x_n\} \sim \{y_n\}$  we say  $\{x_n\}$  is *adjacent to*  $\{y_n\}$ .

**Theorem 2.10.** Let  $X$  have a uniformly sequential structure. Then

$$\Xi_{\sim} := \left\{ (\{x_n\}, x) : \{x_n\} \sim \{x\} \right\}$$

is a sequential structure on  $X$ .

*Proof.* Let  $\{x_n\} \sim \{x\}$  and  $\{x_{n_k}\}$  be subsequence of  $\{x_n\}$ . We know every subsequence of a constant sequence is again a constant sequence. So from (U1) we have  $\{x_{n_k}\} \sim \{x\}$ .

Let  $x, y \in X$ . Let  $\{x_n\} \sim \{x\}$  and  $\{x_n\} \sim \{y\}$ . Then from transitivity of  $\sim$  we obtain  $\{x\} \sim \{y\}$  and (U2) gives us  $x = y$ .

Let  $\{x_n\} \not\sim \{x\}$ . Then (C3) follows directly from (U3).

Because of (U4) we have  $(\{x\}, x) \in \Xi_{\sim}$ .  $\square$

**Theorem 2.11.** *Adjacent sequences in a metric space  $(X, \varrho)$  satisfy conditions (U1)-(U4).*

*Proof.* From Theorem 1.13 we know  $\sim_{\varrho}$  is an equivalence.

Let  $x_n = x, y_n = y \forall n \in \mathbb{N}$  for some  $x, y \in X$  and  $\{x_n\} \sim_{\varrho} \{y_n\}$ . If  $x \neq y$  then from (M1) we have  $\varrho(x, y) > 0$  so  $x = y$ .

Let  $\{x_n\} \sim_{\varrho} \{y_n\}$  and  $\{x_{n_k}\}, \{y_{n_k}\}$  be subsequences. Then clearly  $\{x_{n_k}\} \sim_{\varrho} \{y_{n_k}\}$ .

Let  $\{x_n\} \not\sim_{\varrho} \{y_n\}$  then there exists  $\varepsilon > 0$  such that for infinitely many  $n \in \mathbb{N}$  we have  $\varrho(x_n, y_n) \geq \varepsilon$ , these introduce a subsequence. It is easy to observe it has desired properties.  $\square$

We are not able to characterise a Cauchy sequence in terms of simple convergence. It turns out that this problem is the main motivation to develop uniformly sequential spaces in which we can define this property.

**Definition 2.12.** Let  $X$  have a uniformly sequential structure. We say a sequence  $\{x_n\}$  in  $X$  is *Cauchy* if it is adjacent to all its subsequences.

As we would expect every convergent sequence is Cauchy.

**Theorem 2.13.** *Let  $X$  have a uniformly sequential structure. If  $(\{x_n\}, x) \in \Xi_{\sim}$  then  $\{x_n\}$  is Cauchy.*

*Proof.* Let  $\{x_{n_k}\}$  be a subsequence of  $\{x_n\}$ . Then from (C1) we have  $(\{x_{n_k}\}, x) \in \Xi_{\sim}$ . That is  $\{x_n\} \sim \{x\}$  and  $\{x_{n_k}\} \sim \{x\}$  and the transitivity of  $\sim$  gives us  $\{x_{n_k}\} \sim \{x_n\}$ .  $\square$

The relation of a Cauchy sequence in a uniformly sequential space and a Cauchy sequence in a metric space is shown later (Theorem 3.10).

# 3. Properties defined by convergence

In this chapter we will use the following convention: when we say a uniformly sequential space  $(X, \sim)$  has a property defined for a sequential space we mean that  $\Xi_{\sim}$  has this property.

## 3.1 Continuous and uniformly continuous mappings

**Definition 3.1.** Let  $\Xi_X$  and  $\Xi_Y$  be two sequential structures on sets  $X$  and  $Y$ . We say  $f : X \rightarrow Y$  is a *continuous mapping*  $\Xi_X$  to  $\Xi_Y$  if  $(\{x_n\}, x) \in \Xi_X$  implies that  $(\{f(x_n)\}, f(x)) \in \Xi_Y$ .

The following theorem is often covered in basic course of mathematical analysis. We include it for completeness.

**Theorem 3.2.** Let  $\Xi_X$  and  $\Xi_Y$  be two sequential structures generated by metric spaces  $(X, \varrho)$  and  $(Y, \sigma)$ . Let  $f : X \rightarrow Y$  be a mapping. Then the following are equivalent:

- (i)  $f : (X, \varrho) \rightarrow (Y, \sigma)$  is continuous;
- (ii)  $f : \Xi_X \rightarrow \Xi_Y$  is continuous.

*Proof.* Let  $x \in X$  be a fixed point.

Let condition (i) hold and  $x_n \rightarrow x$  (that is,  $(\{x_n\}, x) \in \Xi_X$ ). For  $\varepsilon > 0$  we find  $\delta > 0$  from Definition 1.6. Then there exists  $n_\varepsilon$  such that for  $\forall n \geq n_\varepsilon$  we have  $\varrho(x_n, x) < \delta$  and then  $\sigma(f(x_n), f(x)) < \varepsilon$ . Hence  $f(x_n) \rightarrow f(x)$  in  $Y$ , that is,  $(\{f(x_n)\}, f(x)) \in \Xi_Y$ .

Let now condition (ii) hold and for contradiction suppose that (i) does not. Then

$$\exists \varepsilon > 0 \forall \delta > 0 \forall y \in X : \varrho(x, y) < \delta \Rightarrow \sigma(f(x), f(y)) \geq \varepsilon.$$

We fix this  $\varepsilon$  and find a sequence  $\{x_n\}$  such that  $\varrho(x_1, x) < 1$  and  $\varrho(x_{n+1}, x) < \min(\frac{1}{n}, \varrho(x_n, x))$  for  $n \in \mathbb{N}$ . Then  $x_n \rightarrow x$  hence from (ii)  $f(x_n) \rightarrow f(x)$ . This is not possible since  $\sigma(f(x_n), f(x)) \geq \varepsilon$  for  $n \in \mathbb{N}$ .  $\square$

**Definition 3.3.** Let  $(X, \sim_X)$  and  $(Y, \sim_Y)$  be two uniformly sequential spaces. We say  $f : X \rightarrow Y$  is a *uniformly continuous mapping*  $\sim_X$  to  $\sim_Y$  if  $\{x_n\} \sim_X \{y_n\}$  implies that  $\{f(x_n)\} \sim_Y \{f(y_n)\}$ .

**Theorem 3.4.** Let  $\sim_X$  and  $\sim_Y$  be two uniformly sequential structures generated by metric spaces  $(X, \varrho)$  and  $(Y, \sigma)$ . Let  $f : X \rightarrow Y$  be a mapping. Then the following are equivalent:

- (i)  $f : (X, \varrho) \rightarrow (Y, \sigma)$  is uniformly continuous;
- (ii)  $f : (X, \sim_X) \rightarrow (Y, \sim_Y)$  is uniformly continuous.

*Proof.* Let (i) hold and  $\{x_n\} \sim_X \{y_n\}$ . Since  $\lim_{n \rightarrow \infty} \varrho(x_n, y_n) = 0$  we have

$$\forall \delta > 0 \exists n_0 \in \mathbb{N} \forall n \in \mathbb{N}, n \geq n_0 : \varrho(x_n, y_n) < \delta.$$

Let  $\varepsilon > 0$ . We find  $\delta$  from the definition of (i), then we find  $n_0$  from the previous formula. We obtain

$$\forall n \in \mathbb{N}, n \geq n_0 : \sigma(f(x_n), f(y_n)) < \varepsilon,$$

so  $\lim_{n \rightarrow \infty} \sigma(f(x_n), f(y_n)) = 0$ .

Let (i) does not hold. Then there exists  $\varepsilon > 0$  such that for every  $n \in \mathbb{N}$  we can find  $x_n, y_n \in X$ ,  $\varrho(x_n, y_n) < \frac{1}{n}$  and  $\sigma(f(x_n), f(y_n)) > \varepsilon$ . That is,  $\{x_n\} \sim_X \{y_n\}$  but  $\{f(x_n)\} \not\sim_Y \{f(y_n)\}$ .  $\square$

## 3.2 Open and closed sets in sequential spaces and related topological properties

If we were able to define a topology on  $\Xi$  we would obtain many interesting properties. In this section we will try to achieve this.

With knowledge of the form of closed sets obtained from Theorem 1.9(vi) it seems natural to define a closure using similar approach and then use Theorem 1.4.

**Definition 3.5.** Let  $\Xi$  be a sequential structure on  $X$ ,  $A \subset X$  and  $x \in X$  then we say  $x \in c(A)$  if  $\exists (\{x_n\}, x) \in \Xi : \{x_n\} \subset A$ . That is, a point lies in the closure of a set if there is a sequence in the set converging to the point.

However operator  $c$  defined above is not a closure operator. It is easy to show that conditions (i)-(iii) in Theorem 1.3 hold, but generally (iv) does not. In the following example we will show a space in which (iv) does not hold. Please note that when  $\Xi$  is generated by a metric space  $c$  is a closure operator.

**Example 3.6.** Let  $X = \{f, f : \mathbb{R} \rightarrow \mathbb{R}\}$  be a space of all real functions on real numbers,  $\{f_n\} \subset X$  a sequence of functions and  $f \in X$  a function. We say  $(\{f_n\}, f) \in \Xi$  if

$$\forall x \in \mathbb{R} : \lim_{n \rightarrow \infty} f_n(x) = f(x).$$

Such convergence is usually called pointwise convergence.

Clearly  $(\{f\}, f) \in \Xi$ . When  $\{f_{n_k}\} \subset X$  is a subsequence of  $\{f_n\}$  then for any  $x \in \mathbb{R}$   $\{f_{n_k}(x)\}$  is a subsequence of  $\{f_n(x)\}$ . As these are sequences of real numbers we have  $\lim_{n \rightarrow \infty} f_n(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$  and  $(\{f_{n_k}\}, f) \in \Xi$ . If  $(\{f_n\}, f) \notin \Xi$  then there is  $x_0 \in \mathbb{R}$  for which  $\lim_{n \rightarrow \infty} f_n(x_0) \neq f(x_0)$ . And again since those are real sequences we easily obtain the remaining conditions and therefore  $\Xi$  is a sequential structure on  $X$ .

Now let  $C$  be a set of all real continuous functions of real variable. Then  $C \subset X$ . Applying operator  $c$  from Definition 3.5 we have that  $c(C)$  are continuous functions and functions of Baire class 1. Accordingly  $c(c(C))$  are continuous functions, functions of Baire class 1 and functions of Baire class 2. It is known that Baire class 2 set is non-empty and therefore  $c(C) \neq c(c(C))$ .

Operator that satisfies all conditions but (iv) is sometimes called a *Čech closure operator*.

**Definition 3.7.** Let  $\Xi$  be a sequential structure on  $X$ ,  $A \subset X$  and  $x \in X$  then we say  $x \in \text{Int}A$  if  $\forall (\{x_n\}, x) \in \Xi \exists \{x_{n_k}\}$  subsequence of  $\{x_n\} : \{x_{n_k}\} \subset A$ . Set  $A$  is *open in*  $(X, \Xi)$  if  $A = \text{Int}A$ . That is, a set is open when all sequences converging to a point in the set eventually lie in the set.

A set  $A \subset X$  is *closed in*  $(X, \Xi)$  if  $X \setminus A$  is open in  $(X, \Xi)$ .

To show that our first guess in Definition 3.5 can lead to a topology we introduce the next theorem, which we state without proof.

**Theorem 3.8.** *Operator  $c_{\omega_1}$  defined by a transfinite induction*

$$c_\alpha(A) := c \left( \bigcup_{\beta < \alpha} c_\beta(A) \right),$$

where  $\alpha$  is an ordinal number,  $A \subset X$  and  $c$  is operator from Definition 3.5, is a Kuratowski closure operator. That is, it meets the condition (iv) in Theorem 1.3:  $c(c_{\omega_1}(A)) = c_{\omega_1}(A)$ . Moreover  $c_{\omega_1}$  generates the same closed sets (and therefore the same topology) as in Definition 3.8.

### 3.3 Complete and totally bounded spaces

**Definition 3.9.** Let  $X$  have a uniformly sequential structure. We say  $X$  is *complete* if for every Cauchy sequence  $\{x_n\}$  in  $X$  we have  $(\{x_n\}, x) \in \Xi_\sim$  for some  $x \in X$ .

Completeness is very closely related to Cauchyness of sequences. That is why we need to define it using uniformly sequential spaces. The definition follows the definition of complete metric space. The next theorem therefore seems natural.

**Theorem 3.10.** *Let  $(X, \sim_\varrho)$  be a uniformly sequential space generated by metric space  $(X, \varrho)$ . Then  $(X, \varrho)$  is complete if and only if  $(X, \sim_\varrho)$  is complete.*

*Proof.* It is sufficient to show that a sequence  $\{x_n\}$  is Cauchy in  $(X, \varrho)$  if and only if it is Cauchy in  $(X, \sim)$ .

Let  $\varepsilon > 0$ , let  $\{x_n\}$  be a Cauchy sequence in  $(X, \varrho)$  and  $\{x_{n_k}\}$  be its subsequence. Because  $\{x_n\}$  is Cauchy we find  $m_0$  from Definition 1.10. That is, for every  $n, m \in \mathbb{N}$ ,  $n, m \geq m_0$  we have  $\varrho(x_n, x_m) < \varepsilon$ . Specially  $\lim_{k \rightarrow \infty} \varrho(x_k, x_{n_k}) = 0$ .

Let us suppose that there exists  $\{x_{n_k}\}$  such that  $\{x_n\} \not\sim_\varrho \{x_{n_k}\}$ . That is,

$$\exists \varepsilon > 0 \forall k_0 \in \mathbb{N} \exists k \in \mathbb{N}, k \geq k_0 : \varrho(x_k, x_{n_k}) \geq \varepsilon.$$

Then if we choose  $n = n_k, m = k$  in the definition of Cauchy sequence in metric space we see that  $\{x_n\}$  is not Cauchy.  $\square$

**Definition 3.11.** Let  $X$  have a uniformly sequential structure. We say  $X$  is *totally bounded* if for every sequence  $\{x_n\}$  in  $X$  there exists a subsequence  $\{x_{n_k}\}$  which is Cauchy.

It is uncommon to characterise totally bounded metric spaces using this or similar condition. It is however equivalent as the next shows us.

**Theorem 3.12.** *Let  $(X, \sim_\varrho)$  be a uniformly sequential space generated by a metric space  $(X, \varrho)$ . Then  $(X, \varrho)$  is totally bounded if and only if  $(X, \sim_\varrho)$  is totally bounded.*

*Proof.* Let  $(X, \varrho)$  be totally bounded and  $\{x_{0,n}\}$  be a sequence in  $X$ . Then for every  $m \in \mathbb{N}$  there exists a finite family of open balls  $B_1, \dots, B_M$  with radius  $\frac{1}{m}$  such that  $X = \bigcup_{i=1}^M B_i$ . Since a sequence is infinite there exists a ball which contains an infinite number of members of sequence  $\{x_{m-1,n}\}$ . We choose one such ball and denote  $\{x_{m,n}\}$  a subsequence of  $\{x_{m-1,n}\}$  which consists of points in the ball only. Then  $\{x_{n,n}\}$  is Cauchy because for every  $m \in \mathbb{N}$ ,  $n > m$  we have  $\varrho(x_{n,n}, x_{m,m}) < \frac{2}{m}$ .

To prove the sufficiency let us suppose that  $(X, \varrho)$  is not totally bounded. That is, there exists  $\varepsilon > 0$  for which no finite family of open balls with diameter  $\varepsilon$  covers whole  $X$ . We select an arbitrary point  $x_1 \in X$ . Then for  $n \in \mathbb{N}$  we find  $x_{n+1} \in X \setminus \bigcup_{i=1}^n B(x_i, \varepsilon) \neq \emptyset$ . Then any subsequence of  $\{x_n\}$  is not Cauchy because for every  $n, m \in \mathbb{N}$ ,  $n \neq m$  we have  $\varrho(x_n, x_m) \geq \varepsilon$ , so  $(X, \sim)$  is not totally bounded.  $\square$

### 3.4 Compact spaces

Characterisation of compactness in metric space in term of convergence is widely known. We use this principle to define compact sequential space.

**Definition 3.13.** We say a sequential space  $(X, \Xi)$  is compact if for any arbitrary sequence  $\{x_n\} \subset X$  there exists a subsequence  $\{x_{n_k}\}$  and a point  $x \in X$  such that  $(\{x_{n_k}\}, x) \in \Xi$ .

Direct proof of equivalence is covered in basic courses. We present a proof which follows from previously showed relation between total boundedness and completeness in metric and uniformly sequential spaces. The following theorem together with Theorems 1.11, 3.10 and 3.12 give us the equivalence indirectly.

**Theorem 3.14.** *A uniformly sequential space  $(X, \sim)$  is compact if and only if it is totally bounded and complete.*

*Proof.* Let  $(X, \sim)$  be totally bounded and complete. Let  $\{x_n\}$  be an arbitrary sequence in  $X$ . Because  $(X, \sim)$  is totally bounded we can find a subsequence  $\{x_{n_k}\}$  which is Cauchy. Because  $(X, \sim)$  is complete we can find a point  $x \in X$  such that  $(\{x_{n_k}\}, x) \in \Xi_\sim$ .

Let  $(X, \sim)$  be compact. From Theorem 2.13 we have that it is totally bounded. Let  $\{x_n\}$  be a Cauchy sequence, we can find a subsequence  $\{x_{n_k}\}$  and a point  $x \in X$  such that  $(\{x_{n_k}\}, x) \in \Xi_\sim$ . That is,  $\{x_{n_k}\} \sim \{x\}$  and since  $\{x_n\}$  is Cauchy we have  $\{x_n\} \sim \{x_{n_k}\}$ , from transitivity of  $\sim$  we obtain  $\{x_n\} \sim \{x\}$ . So  $(\{x_n\}, x) \in \Xi_\sim$  and  $(X, \sim)$  is complete.  $\square$

**Corollary 3.15.** *Let  $(X, \varrho)$  be a metric space and  $\Xi$  is generated by  $(X, \varrho)$ . Then  $(X, \varrho)$  is compact if and only if  $(X, \Xi)$  is compact.*

**Theorem 3.16.** *Let  $(X, \sim_X), (Y, \sim_Y)$  be two uniformly sequential spaces,  $f : X \rightarrow Y$  be a continuous mapping and  $(X, \sim_X)$  be compact. Then  $f$  is uniformly continuous.*

*Proof.* Let  $\{x_n\} \sim_X \{y_n\}$  and let us for contradiction suppose that  $\{f(x_n)\} \not\sim_Y \{f(y_n)\}$ . Then from (U3) we find subsequences  $\{f(x_{n_k})\}$  and  $\{f(y_{n_k})\}$  such that for none their subsequences we have  $\{f(x_{n_{k_i}})\} \sim_Y \{f(y_{n_{k_i}})\}$ . Because  $X$  is compact we can find subsequences  $\{x_{n_{k_i}}\}$  of  $\{x_{n_k}\}$ ,  $\{y_{n_{k_i}}\}$  of  $\{y_{n_k}\}$  and points  $x, y \in X$  such that  $(\{x_{n_{k_i}}\}, x) \in \Xi_{\sim_X}$  and  $(\{y_{n_{k_i}}\}, y) \in \Xi_{\sim_X}$ . Since  $\{x_n\} \sim_X \{y_n\}$  we have  $\{x_{n_k}\} \sim_X \{y_{n_k}\}$  and therefore  $x = y$ . Mapping  $f$  is continuous so  $(\{f(x_{n_{k_i}})\}, f(x)) \in \Xi_{\sim_Y}$  and  $(\{f(y_{n_{k_i}})\}, f(x)) \in \Xi_{\sim_Y}$ . That is,  $\{f(x_{n_{k_i}})\} \sim_Y \{f(x)\}$  and  $\{f(y_{n_{k_i}})\} \sim_Y \{f(x)\}$  which gives us  $\{f(x_{n_{k_i}})\} \sim_Y \{f(y_{n_{k_i}})\}$  which is the contradiction.  $\square$

The following example proves that the implication in the previous theorem can not be reversed.

**Example 3.17.** We say two infinite sets are *almost disjoint* if their intersection is finite. Let us assume an infinite maximal family of almost disjoint subsets of  $\mathbb{N}$  and denote it as  $\text{MAD}(\mathbb{N})$ .

For a reader's image of some infinite family of almost disjoint subsets of  $\mathbb{N}$  we will show a construction of one. Let  $f : \mathbb{N} \rightarrow \mathbb{Q}$  be a bijection of natural numbers onto rational numbers. For every irrational  $x \in \mathbb{R} \setminus \mathbb{Q}$  we will choose one rational sequence  $\{r_n\}$  such that  $r_n \rightarrow x$ . Then  $\mathcal{S}_x := f^{-1}[\{r_n\}] \subset \mathbb{N}$  is a preimage of that sequence. Then  $\{\mathcal{S}_x : x \in \mathbb{R} \setminus \mathbb{Q}\}$  is an infinite family of almost disjoint subsets of  $\mathbb{N}$  because if two sets had an infinite intersection the corresponding rational sequences would have an infinite intersection too and since they are both convergent it follows they would converge to the same point.

Please note that using this construction we do not obtain a  $\text{MAD}(\mathbb{N})$  as for example the preimage of the set of all even numbers is almost disjoint with all constructed sets.

We denote  $\text{MAD}(\mathbb{N}) = \{\mathcal{S}_\alpha : \alpha \in A\}$  for some set  $A$ . Let  $X := \mathbb{N} \cup \text{MAD}(\mathbb{N})$  with the smallest sequential structure such that

$$\mathcal{S}_\alpha \rightarrow \{\mathcal{S}_\alpha\}, \quad \{\mathcal{S}_\alpha\} \in \text{MAD}(\mathbb{N}).$$

This condition can be rephrased: members of  $\mathcal{S}_\alpha$  as points in  $\mathbb{N}$  converge to  $\mathcal{S}_\alpha$  as point in  $\text{MAD}(\mathbb{N})$ . Then every continuous mapping to  $\mathbb{R}$  is bounded but the space is evidently not compact.

To prove this let  $f : X \rightarrow \mathbb{R}$  be a continuous unbounded mapping. That is, there exists a sequence  $\{x_n\}$  for which  $f(x_n) \rightarrow \infty$ . Now we examine two cases.

First, infinitely many members of  $\{x_n\}$  lie in  $\mathbb{N}$ , so we can find a subsequence  $\{x_{n_k}\} \subset \mathbb{N}$ . Then there exists  $\mathcal{S}_\alpha \in \text{MAD}(\mathbb{N})$  such that  $\{x_{n_k}\} \rightarrow \mathcal{S}_\alpha$ . Then we have  $\{f(x_{n_k})\} \rightarrow f(\mathcal{S}_\alpha)$  and  $\{f(x_{n_k})\} \rightarrow \infty$  which is a contradiction.

Second, only finite number of members of  $\{x_n\}$  lie in  $\mathbb{N}$ . Without loss of generality we suppose that  $\{x_n\} = \{\mathcal{S}_{\alpha_n}\} \subset \text{MAD}(\mathbb{N})$ . For every  $n \in \mathbb{N}$  we can find  $y_n \in \mathbb{N}$  such that  $|f(y_n) - f(\mathcal{S}_{\alpha_n})| \leq 1$ . The first case yields  $\{f(y_n)\}$  is bounded and so  $\{f(\mathcal{S}_{\alpha_n})\}$  is bounded.

### 3.5 Bounded spaces

Diameter of a set extremely depends on used metric. At first, it might seem natural that when a set has a finite (infinite) diameter it will have it finite (infinite) in every equivalent metric. The following theorem uses quite known definition of equivalent metrics to prove otherwise.

**Theorem 3.18.** *When  $(X, \varrho)$  is an unbounded metric space then there exists a metric  $\sigma$  on  $X$  that is equivalent with  $\varrho$  and  $(X, \sigma)$  is bounded.*

*Proof.* For  $x, y \in X$  we define

$$\sigma_1(x, y) := \frac{\varrho(x, y)}{1 + \varrho(x, y)}, \quad \sigma_2(x, y) := \min\{\varrho(x, y), 1\}.$$

See [2, p. 22] for  $\sigma_1$  and [5, p. 250] for  $\sigma_2$  for proofs that they satisfy the conditions for a metric. Then evidently both  $\sigma_1, \sigma_2$  are equivalent with  $\varrho$  and are bounded.  $\square$

The previous theorem not only shows a diameter of a space in an equivalent metric can be finite, it also shows it can be any positive real number  $a > 0$  we want. We just take  $\sigma(x, y) := a \cdot \sigma_i(x, y)$  where  $\sigma_i$  is from Theorem 3.18 and  $i = 1$  or  $i = 2$ .

This result is enough to answer the problem. We also show the not-so-known theorem that under some quite unrestrictive conditions a space with finite diameter in some metric can have the infinite diameter in an other equivalent metric. The proof is constructive, that is, we give exact form of such metric.

**Theorem 3.19.** *When  $(X, \varrho)$  is a non-compact bounded metric space then there exists a metric  $\sigma$  on  $X$  that is equivalent with  $\varrho$  and  $(X, \sigma)$  is unbounded.*

*Proof.* Had we constructed a function  $f : X \rightarrow \mathbb{R}$  continuous and unbounded we would define

$$\sigma(x, y) := \varrho(x, y) + |f(x) - f(y)|.$$

Evidently  $\sigma(x, x) = 0$ . We have  $\sigma(x, z) \leq \sigma(x, y) + \sigma(z, y)$  because when rewritten it is

$$\varrho(x, z) + |f(x) - f(z)| \leq \varrho(x, y) + \varrho(z, y) + |f(x) - f(y)| + |f(z) - f(y)|.$$

This holds because  $|f(y) - f(z)| = |f(z) - f(y)|$ , the triangle inequality holds for  $\varrho$  and

$$|f(x) - f(z)| = |f(x) - f(y) + f(y) - f(z)| \leq |f(x) - f(y)| + |f(y) - f(z)|.$$

Let  $\{x_n\} \subset X$  and  $x_n \rightarrow x \in X$  in metric  $\varrho$ . Since  $f$  is continuous we know

$$\lim_{n \rightarrow \infty} |f(x_n) - f(x)| = 0$$

and therefore  $x_n \rightarrow x$  also in metric  $\sigma$ .

Let now  $\{y_n\} \subset X$  and  $y_n \rightarrow y \in X$  in metric  $\sigma$ . Then from

$$0 \leq \lim_{n \rightarrow \infty} \varrho(y_n, y) \leq \lim_{n \rightarrow \infty} (\varrho(y_n, y) + |f(y_n) - f(y)|) = \lim_{n \rightarrow \infty} \sigma(y_n, y) = 0$$

follows  $y_n \rightarrow y$  also in metric  $\rho$ . That is,  $\sigma$  and  $\rho$  are equivalent and  $(X, \sigma)$  is unbounded.

When a metric space is non-compact it means there exists a sequence for which no subsequence converges, let  $\{x_n\}$  be such sequence. Then  $\{x_n\}$  is closed in  $X$ . Without loss of generality we suppose that  $x_n \neq x_m$  if  $n \neq m$ .

Then for every  $n \in \mathbb{N}$  there exists  $a_n > 0$  such that  $\forall m \in \mathbb{N}, n \neq m : \rho(x_n, x_m) \geq 3a_n$ . We define a function  $f : X \rightarrow \mathbb{R}$ :

$$f(x) = \begin{cases} n \cdot \rho(x, X \setminus B(x_n, a_n)), & \text{if } x \in B(x_n, a_n) \text{ for some } n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

We can easily see that  $f$  is continuous and unbounded. □

The previous theorems show that we (generally) can not distinguish between  $\Xi$  generated by a bounded and an unbounded metric on the same set. Because of this we do not define any kind of boundedness of sequential spaces.

As a side product we obtain another interesting characterisation of compactness in metric spaces.

**Corollary 3.20.** *A metric space  $(X, \rho)$  is bounded in every equivalent metric if and only if it is compact.*

## 3.6 Connected spaces

Since we introduced topology on sequential spaces we could delegate the problem of characterisation of connectedness to it. It is however useful to define connected spaces with continuous mapping.

In the following we take  $D = \{0, 1\}$  with  $(\Xi_D)_0$ , that is, a two-point space with discrete (in meaning of Theorem 2.8) sequential structure.

**Definition 3.21.** A sequential space  $(X, \Xi)$  is *connected* if every continuous mapping  $f : X \rightarrow D$  is constant, that is,  $f[X] \subset \{0\}$  or  $f[X] \subset \{1\}$ .

**Theorem 3.22.** *Let  $(X, \rho)$  be a metric space and  $\Xi$  be generated by  $(X, \rho)$ . Then  $(X, \rho)$  is connected if and only if  $(X, \Xi)$  is connected.*

*Proof.* Let  $(X, \Xi)$  be connected. For contradiction let us suppose  $U, V$  are subsets of  $X$ , both open non-empty,  $X = U \cup V$  and  $U \cap V = \emptyset$ . We define  $f : X \rightarrow D$  as follows

$$f(x) := \begin{cases} 1, & \text{if } x \in U; \\ 0, & \text{if } x \in V. \end{cases}$$

Then  $f$  is continuous and  $f[X] = D$ . That is a contradiction with  $(X, \Xi)$  being connected.

Let now  $(X, \rho)$  be connected. For contradiction let us suppose  $f : X \rightarrow D$  is a continuous mapping such that  $f[X] = D$ . Then preimages of one-point (open) sets  $U := f^{-1}[\{0\}]$ ,  $V := f^{-1}[\{1\}]$  are open in  $(X, \rho)$ . Moreover  $X = U \cup V$ ,  $U \cap V = \emptyset$  and  $U, V$  are non-empty because  $\{0, 1\} \subset f[X]$ . Which is a contradiction with  $(X, \rho)$  being connected. □

## 3.7 Separable spaces

Again we have a topology so we could use the closure operator and define separability analogically with Definition 1.2. In metric spaces we obtain an equivalent characterisation as it is showed afterwards.

**Definition 3.23.** Let  $\Xi$  be a sequential structure on set  $X$ . We say  $\Xi$  is *separable* if there exists a countable subset  $A \subset X$  such that

$$\forall x \in X \exists \{x_n\} \subset A : (\{x_n\}, x) \in \Xi.$$

**Corollary 3.24.** Let  $(X, \rho)$  be a metric space and  $\Xi$  be generated by  $(X, \rho)$ . Then  $(X, \rho)$  is separable if and only if  $\Xi$  is separable.

*Proof.* We use Theorem 1.9(vi). □

It is known that a compact metric space is always separable. The following example shows that this implication does not hold for sequential spaces.

**Example 3.25.** Let  $X := \bigcup_{\beta < \omega_1} X_\beta$  where  $X_\beta$  are defined by a transfinite induction:

$$X_1 := \mathbb{N}, \quad X_\alpha := \text{MAD}(X_{\alpha-1}), \quad X_\gamma := \text{MAD}(\mathcal{K}_\gamma),$$

where  $\alpha$  is a successor ordinal,  $\gamma$  is a limit ordinal and  $\mathcal{K}_\gamma$  denotes the family of sequences that have a finite intersection with  $X_\beta$  for every  $\beta < \gamma$ . We say  $X_\beta$  is a *level* for every  $\beta$ . On  $X$  we take a sequential structure defined analogically to Example 3.17.

Then  $X$  is compact. Really, let  $\{x_n\}$  be a sequence in  $X$ . It follows that either  $\{x_n\}$  contains an infinite number of members in some level or it has a finite intersection with every level. In the first case we can easily find a convergent subsequence. In the second case, since  $\{x_n\}$  has countably many members there exists an  $\alpha_0 < \omega_1$  such that  $\{x_n\} \subset \bigcup_{\beta < \alpha_0} X_\beta$  so it also has a convergent subsequence.

Let  $A \subset X$  be a countable set. Then again there exists an  $\alpha_0 < \omega_1$  such that  $A \subset \bigcup_{\beta < \alpha_0} X_\beta$ . For every  $x \in X_{\alpha_0+1}$  no subsequence  $\{x_n\} \subset A$  converges to  $x$ . That is,  $X$  is not separable.

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