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DIPLOMOVÁ PRÁCE

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Procesy rozptylu vektorových bosonů elektroslabých interakcí

Ústav částicové a jaderné fyziky

Vedoucí diplomové práce: Prof. RNDr. Jiří Hořejší, DrSc.
Studijní program: Fyzika, Teoretická fyzika

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DIPLOMA THESIS

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Scattering of the electroweak vector bosons

Institute of Particle and Nuclear Physics

Supervisor: Prof. RNDr. Jiří Hořejší, DrSc.
Study Programme: Physics, Theoretical Physics

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Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 23. srpna 2010 Petr Morávek

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Abstrakt: V předložené práci studujeme rozptylové procesy vektorových bosonů v rámci vícedimenzionální teorie Kaluza-Kleinova typu. Tato práce poskytuje přehledný úvod do dané problematiky včetně všech technických detailů příslušných výpočtů. Základní myšlenky teorie jsou demonstrovány na sadě jednoduchých modelů. Zvláštní pozornost je pak věnována výběru konzistentních okrajových podmínek pomocí variačního principu nejmenší akce, kalibrační nezávislosti a stromové unitaritě rozptylu longitudinálních bosonů.

Klíčová slova: Narušení elektroslabé symetrie, dodatečná dimenze, teorie Kaluza-Kleinova typu, hmotné vektorové bosony, stromová unitarita

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Abstract: In the present work we study the scattering of vector gauge bosons in the framework of higher dimensional theory of Kaluza-Klein type. This work provides an instructive introduction to the given problematics with all technical details of the relevant calculations. The basic ideas of the theory are demonstrated on a set of simple models. Particular attention is paid to the selection of consistent boundary conditions using the variational principle of least action, gauge invariance and tree-level unitarity of longitudinal gauge bosons scattering.

Keywords: Electroweak symmetry breaking, extra dimension, theories of Kaluza-Klein type, massive vector bosons, tree-level unitarity

Conventions and notations

All calculations in this work are carried out in the natural system of units in which one has $\hbar = c = 1$.

Small Greek letters (e.g. μ, ν) will always denote the usual 4D Lorentz indices running through 0, 1, 2, 3, while capital Latin letters (e.g. M, N) will denote 5D indices running through 0, 1, 2, 3, 5. Thus a generic 5D index M can be decomposed as $M = (\mu, 5)$. For the sake of simplicity, when appropriate, a prime will be used to denote a derivative in the direction of the extra dimension.

We assume that there is no non-trivial gravitational background, in other words, a flat spacetime with 4D metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, in case of five dimensions $g_{MN} = \mathrm{diag}(1, -1, -1, -1, -1)$.

Letter x will be reserved for the usual $3 + 1$ space and time coordinates, for the fifth extra space coordinate we will use y, or z. We assume that the extra space dimension is a finite interval, conventionally chosen as $(0, \pi R)$, where R is an arbitrary constant corresponding to the size of the extra dimension. Thus each endpoint of this interval is actually a four dimensional hypersurface, which is in the context of the studied theory often called a *brane*.

Small Latin letters starting from a, b, etc. will denote the gauge (color) index of a field. Later on we introduce another type of index – a mode number in Kaluza-Klein (KK) expansion – for which small Latin letters will be used too, usually one of j, k, n, so they will not get mixed up with color indices. Since every 4D effective field carries both color index and KK mode number, we will often use a shorthand double-index notation $\boldsymbol{a} = (a, i)$.

Chapter 1

Introduction

1.1 Standard Model

Over the years the Standard Model has become very successful theory describing the laws governing the world at the subatomic level. Many experimental setups confirmed the predictions of the Standard Model to a high level of precision [1, 2]. Regarding the electroweak interactions, this means that the non-Abelian gauge theory with the symmetry pattern SU(2)×U(1) is a very accurate description of the nature up to 100 GeV energy scale.

One major puzzle still remains unsolved – the mechanism of the *electroweak symmetry breaking*. The hard symmetry breaking (adding the mass terms to the Lagrangian "by hand") leads to a "bad high energy behaviour" of massive vector boson scattering [3].

In the Standard Model this issue is resolved by the spontaneous symmetry breaking via the Higgs mechanism [4], which generates the masses of the weak gauge bosons $W^{\pm}, Z$ and may be also used to give mass to all the matter fields. The Higgs mechanism implies the existence of an additional massive scalar field (Higgs boson), which ensures the tree-level unitarity of scattering amplitudes [5].

The Higgs particle has not yet been observed experimentally, but there are high expectations associated with the recently launched Large Hadron Collider (LHC) built by the European Organization for Nuclear Research (CERN) near Geneva. LHC was constructed to probe a wide range of energies, including the energy scale up to 14 TeV that was so far inaccessible. Thus the measurements in the following years should either confirm that the Higgs mechanism is indeed realized in the nature, or reveal signs of new physics beyond the Standard Model [2, 6, 7].

1.2 Beyond the Standard Model

One of the motivations for developing alternative theories of electroweak symmetry breaking is associated with the search for a Grand Unification Theory, which would unify electroweak and strong force. If one does not want to employ an unnatural fine-tuning in order to get the "right" scale of Higgs mass, then some new physics beyond the Standard Model must be introduced. The problem of stabilizing the Higgs mass is commonly referred to as the *gauge-hierarchy problem* [8–10]. There are currently many attempts to come up with the theory addressing this issue [10]. Let us name just a few of the most popular ones:

- Family of the *supersymmetric models* [11, 12], which introduce additional fermion–boson symmetry; thus for each fermion of the Standard Model there is a supersymmetric boson partner and vice versa.

- *Little Higgs* models, which describe the Higgs as a pseudo-Goldstone boson arising from some global symmetry breaking at a higher energy scale [13].

- Higher dimensional theories inspired by the work of Theodor Kaluza [14] and Oskar Klein [15], thus usually referred to as *Kaluza-Klein theory.* In these models the gauge fields are described in the context of a five dimensional spacetime (the usual $3 + 1$ dimensions plus an extra compactified space dimension). This is the class of models we will study further in this work.

This class of the higher dimensional models has already been extensively studied [16–22]. This work aims to provide an instructive introduction to the given problematics with all technical details of the relevant calculations. We will demonstrate the basic ideas of the theory on a set of simple models.

The interesting feature of this class of theories is that they allow the construction of a *higgsless model* – a model of electroweak symmetry breaking without any additional physical scalar particle [17]. Scattering amplitudes are then unitarized due to the exchange of Kaluza-Klein excitations instead of the Higgs boson.

1.3 Orbifolding versus interval approach

Since in the real world around us we can see only 3 space dimensions plus time, we have to somehow justify the use of an extra space dimension and argue why we cannot see it. The easiest explanation is that the extra dimension is not infinitely large, but it has a finite and a very small size. Thus it is not directly observable.

There are two essentially equivalent approaches to the compactified extra space dimension [18].

One is a more traditional way called *orbifolding*, which starts with an infinite extra dimension and reduces the fundamental domain of the theory using a set of identifications. In our case (a single extra dimension) this is usually done in two steps: first one obtains a circle S^1 by the identification $y \to y + 2\pi R$, then the discrete symmetry Z_2 (reflection $y \to -y$) is applied. Thus one gets the orbifold S^1/Z_2, the fundamental domain of the theory is the interval $(0, \pi R)$ and the used set of identifications implies certain boundary conditions on the fields at the endpoints of the interval. In this simple example the boundary conditions are given by a choice of the parities of fields with respect to the reflection around each of the endpoints 0 and πR.

Another possibility is to start straight from the interval $(0, \pi R)$ and figure out what the consistent set of boundary conditions is by using the principle of least action. This is the procedure we follow further in this work. Although both approaches are essentially equivalent, the interval approach seems to be more natural when using a more complicated gauge structure; in the orbifolding procedure one needs to introduce fields localized at fixed points, deal with discontinuities of the fields etc. [18].

Chapter 2

Gauge theories on an interval

In this chapter we first show how to define a gauge theory in five dimensions and select a consistent set of boundary conditions at the endpoints 0 and πR of the additional space dimension. In the next step we employ Kaluza-Klein expansion of the gauge fields to obtain an effective 4D Lagrangian and identify the first four components of A^a_M with the infinite tower of 4D vector fields and the fifth component A^a_5 with the infinite tower of 4D scalars. In the unitary gauge all massive modes from the KK tower of scalars will be non-physical. Those modes are eaten by the corresponding massive modes from the tower of vector fields and supply them with the longitudinal polarization state, so they play a similar role as Goldstone bosons in the Standard Model.

2.1 Introducing an extra space dimension

We start with a straightforward generalization of the non-Abelian field strength tensor into a five dimensional spacetime. This may be written as

$$F^a_{MN}(x,y) = \partial_M A^a_N(x,y) - \partial_N A^a_M(x,y) + g_5 f^{abc} A^b_M(x,y) A^c_N(x,y)\,, \tag{2.1}$$

where g_5 is 5D gauge coupling and f^{abc} are as usual the structure constants of the gauge group (we use the adjoint representation, in which the structure constants are fully antisymmetric). The next step is to construct the gauge part of the Lagrangian:

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{4} F^a_{MN} F^{aMN} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} F^a_{\mu 5} F^{a\mu 5} \tag{2.2}$$

We have already separated the term $-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$ from the rest, this term will eventually give rise to the same type of gauge boson interactions as we know from the Standard Model.

Note that this theory can be considered only as a low-energy effective theory valid below a certain cutoff scale [18, 19]. This is due to the fact that the 5D gauge coupling g_5 has mass dimension $-1/2$, thus the theory is non-renormalizable.

2.2 Generalized R$_\xi$ and unitary gauge

The first problem we need to resolve is that our gauge Lagrangian (2.2) contains a quadratic term mixing fields[1] A^a_μ and A^a_5:

$$-\frac{1}{2}F^a_{\mu 5}F^{a\mu 5} = -\partial_5 A^{a\mu}\partial_\mu A^a_5 + \dots \tag{2.3}$$

We can eliminate this cross term by adding a suitable gauge fixing term to the Lagrangian. Since the compactification procedure generally breaks 5D Lorentz invariance, we do not need to limit ourselves to 5D invariant gauge fixing terms [20] and are free to choose

$$\mathscr{L}_{\text{g.f.}} = -\frac{1}{2\xi}\left(\partial_\mu A^{a\mu} - \xi\partial_5 A^a_5\right)^2 . \tag{2.4}$$

Such a gauge fixing term is still invariant with respect to the usual 4D Lorentz transformations and exactly cancels the cross term (we will explicitly show this later). Furthermore after KK expansion the part independent of A^a_5 agrees with the usual Lorentz gauge fixing term for each KK mode of A^a_μ, thus the propagators of vector modes have a form known from R$_\xi$ gauge of the Standard Model. The unitary gauge is given by the limit $\xi \to \infty$.

2.3 Principle of least action

2.3.1 Equations of motion

We can now derive the equations of motion using the variational principle of least action and as a by-product we obtain some consistency conditions on the fields at the endpoints 0 and πR. This is caused by the fact that we are dealing with the extra finite space dimension. In the usual four dimensional case (infinitely large space and time dimensions) we fix the initial and final state, and also require that all fields are localized in space:

$$\delta\psi(t_{\text{initial}}, \vec{x}) = \delta\psi(t_{\text{final}}, \vec{x}) = 0 \tag{2.5}$$

[1] We will always write the scalar field A^a_5 with the index pulled down ($A^{a5} = g^{55}A^a_5 = -A^a_5$), similarly with the partial derivative in the direction of the extra space dimension.

$$\forall t : \psi(t, \vec{x})\big|_{|\vec{x}|\to\infty} \longrightarrow 0 \tag{2.6}$$

We have used ψ to denote an arbitrary field in the theory. Thus the variations of fields are zero at the boundary of 4D spacetime volume, implying that an arbitrary divergence $\partial_\mu \chi^\mu$ added to the Lagrangian will not change the dynamics of the system. However, when we add the extra finite space dimension, there is no reason to assume a priori that the fields (or their variations) vanish at the endpoints of the interval. This implies that, when integrating by parts over the extra dimension, we cannot simply discard the boundary terms, because they are generally non-zero.

The variational principle requires

$$0 = \delta S = \delta \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\, \mathscr{L}_{\text{gauge}}(x,y) + \delta \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\, \mathscr{L}_{\text{g.f.}}(x,y)\,. \tag{2.7}$$

Let us start with the gauge part of action:

$$\delta S_{\text{gauge}} = \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y \left(-\frac{1}{2} F^{a\mu\nu} \delta F^a_{\mu\nu} - F^{a\mu 5} \delta F^a_{\mu 5} \right) \tag{2.8}$$

We can work out the first term as usual:

$$\begin{aligned} &-\frac{1}{2} \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\, F^{a\mu\nu} \delta F^a_{\mu\nu} = \\ &= - \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\, F^{a\mu\nu} \left(\partial_\mu \delta A^a_\nu + g_5 f^{abc} A^b_\mu \delta A^c_\nu \right) = \\ &= \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\, \left(\partial_\mu F^{a\mu\nu} - g_5 f^{abc} F^{b\mu\nu} A^c_\mu \right) \delta A^a_\nu \end{aligned} \tag{2.9}$$

In the first equality we have used the antisymmetry of $F^a_{\mu\nu}$ under the exchange of Lorentz indices, as well as the antisymmetry of structure constants f^{abc}. Then, keeping in mind the assumptions (2.5) and (2.6), we can integrate by parts the term proportional to $\partial^\mu \delta A^{a\nu}$ without picking up any boundary terms, and again make use of the total antisymmetry of structure constants. Now we turn our attention to the second term in (2.8). This can be worked out as

$$\begin{aligned} &- \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\, F^{a\mu 5} \delta F^a_{\mu 5} = \\ &= - \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\, F^{a\mu 5} \Big[\partial_\mu \delta A^a_5 - \partial_5 \delta A^a_\mu + g_5 f^{abc} \left(A^b_\mu \delta A^c_5 + \delta A^b_\mu A^c_5 \right) \Big] = \\ &= \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\, \Big[\left(\partial_\mu F^{a\mu 5} - g_5 f^{abc} F^{b\mu 5} A^c_\mu \right) \delta A^a_5 + \\ &+ \left(\partial_5 F^{a5\nu} - g_5 f^{abc} F^{b5\nu} A^c_5 \right) \delta A^a_\nu \Big] - \int \mathrm{d}^4x \Big[F^{a5\nu} \delta A^a_\nu \Big]_0^{\pi R} \end{aligned} \tag{2.10}$$

We have used the same procedure as in the previous case, but as mentioned above we have to keep the boundary terms from partial integration in the direction of the extra dimension.

Deriving the variation of the gauge fixing part of action is simple – once again we integrate by parts and have to keep another boundary term. Working out this part of the action, we get

$$\begin{aligned}\delta S_{\text{g.f.}} &= -\frac{1}{\xi}\int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y\; (\partial_\mu A^{a\mu} - \xi\partial_5 A_5^a)\,(\partial_\nu \delta A^{a\nu} - \xi\partial_5 \delta A_5^a) = \\ &= \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y \left[\left(\frac{1}{\xi}\partial^\nu\partial^\mu A_\mu^a - \partial^\nu\partial_5 A_5^a \right) \delta A_\nu^a + \right. \\ &\quad \left. + \left(\xi\partial_5\partial_5 A_5^a - \partial_5\partial^\mu A_\mu^a\right)\delta A_5^a \right] + \int \mathrm{d}^4x \left[\left(\partial^\mu A_\mu^a - \xi\partial_5 A_5^a\right)\delta A_5^a \right]_0^{\pi R}\end{aligned} \tag{2.11}$$

Putting all the expressions together and requiring $\delta S = 0$, we can write the equations of motion in the form

$$\partial_M F^{aM\nu} - g_5 f^{abc} F^{bM\nu} A_M^c + \frac{1}{\xi}\partial^\nu\partial^\mu A_\mu^a - \partial^\nu\partial_5 A_5^a = 0 \tag{2.12a}$$

$$\partial_\mu F^{a\mu 5} - g_5 f^{abc} F^{b\mu 5} A_\mu^c - \partial_5\partial^\mu A_\mu^a + \xi\partial_5\partial_5 A_5^a = 0 \tag{2.12b}$$

Variational principle also requires that the two boundary terms we have obtained must vanish, i.e.

$$\left[F^{a5\nu}\delta A_\nu^a\right]_0^{\pi R} = 0 \tag{2.13a}$$

$$\left[\left(\partial^\mu A_\mu^a - \xi\partial_5 A_5^a\right)\delta A_5^a\right]_0^{\pi R} = 0 \tag{2.13b}$$

2.3.2 Selecting boundary conditions

There are many possibilities how to satisfy (2.13) – the least complicated way is to ensure that the expressions vanish for every gauge field at each endpoint separately, in other words requiring that the variation itself or its coefficient are zero. Assuming that we impose the same boundary conditions for all colors of gauge fields (but we can impose different conditions at each brane), we have three general choices of boundary conditions, namely

$$A_\nu^a = const \qquad \partial_5 A_5^a = 0 \tag{2.14a}$$

$$\partial_5 A_\nu^a = 0 \qquad A_5^a = 0 \tag{2.14b}$$

$$A_\nu^a = const \qquad A_5^a = const \tag{2.14c}$$

The last option simply means requiring that the variations vanish ($\delta A_M^a = 0$) at the boundary. For now we have no reason to discard this boundary condition, but we will show in the last section of this chapter that this choice leads to the theory with broken invariance with respect to the gauge parameter ξ. For this reason, let us concentrate on the options (2.14a) and (2.14b).

Before proceeding with our examination of possible boundary conditions, we make several assumptions to simplify the problem a little:

- The gauge group is SU(2).
- Since we want to construct two charged bosons $W_M^{\pm} = \left(A_M^1 \mp iA_M^2\right)/\sqrt{2}$ with the same masses, we will always impose the same boundary conditions on fields A_M^1 and A_M^2.
- We consider only the Dirichlet ($\psi = 0$) and Neumann ($\partial_5\psi = 0$) boundary conditions, thus we assume that all constants in (2.14) are zero.

Reading [16] one can easily get an impression that we can impose an arbitrary combination of those boundary conditions at each brane and for each color of gauge fields. This is not entirely true, because the expression (2.13a) mixes fields of different colors, so when imposing different boundary conditions on different colors of gauge fields, we need to be sure that this term still vanishes.

The above simplification leaves us with 16 different combinations of boundary conditions – 2 branes × 2 possible boundary conditions for each of 2 types of bosons, but 7 (almost a half) of them do not satisfy the condition (2.13a). Those are the cases with the condition (2.14b) for $A_M^{1,2}$ and (2.14a) for A_M^3 at the same brane. Let us explicitly write down the expression (2.13a) e.g. for the color $a = 1$:

$$\left[\partial^\nu A_5^1 - \partial_5 A^{1\nu} - g_5\left(A_5^2 A^{3\nu} - A_5^3 A^{2\nu}\right)\right]\delta A_\nu^1 = g_5 A_5^3 A^{2\nu}\delta A_\nu^1 \neq 0 \tag{2.15}$$

It is also worth noting that the boundary condition (2.14a) can be reinterpreted as coming from the Higgs field localized on the brane [18]. The interesting thing is that the limit of vacuum expectation value going to infinity does not spoil the tree-level unitarity, the Higgs field completely decouples from the theory and one is left only with the gauge fields with precisely the boundary conditions (2.14b).

Apart from these simple models there is another way to satisfy (2.13) – it is possible that the total sum of all terms is zero, although the individual terms are generally non-zero. This method is used in the more realistic models of electroweak symmetry breaking via boundary conditions [17, 18].

2.4 Kaluza-Klein expansion

To get the effective 4D fields we use Kaluza-Klein expansion – decompose all fields into the infinite series of eigenfunctions $\varphi_{a,n}(y)$ of the operator $\partial_5\partial_5$. The decomposition of the vector field $A^a_\mu(x,y)$ is then given by

$$A^a_\mu(x,y) = \sum_n A^{a,n}_\mu(x)\,\varphi_{a,n}(y)\,. \tag{2.16}$$

The $\varphi_{a,n}(y)$ is called the wave function (in the extra dimension) of the mode $A^{a,n}_\mu(x)$. Similar decomposition can be done with scalar fields, just the set of the wave functions will be different due to the different boundary conditions.

As long as the imposed boundary conditions keep the operator $\partial_5\partial_5$ hermitian with respect to the scalar product $\langle f,g\rangle = \int_0^{\pi R} \mathrm{d}y\, f^*(y)g(y)$, we are guaranteed that its eigenfunctions

$$\varphi''_{a,n}(y) = -m^2_{a,n}\varphi_{a,n}(y) \tag{2.17}$$

form a complete orthonormal basis.

We can easily check that the Dirichlet or Neumann boundary conditions indeed keep the operator hermitian. Using the integration by parts, we get:

$$\int_0^{\pi R} \mathrm{d}y\, f^*(y)g''(y) = \int_0^{\pi R} \mathrm{d}y\, f^{*\prime\prime}(y)g(y) + \left[f^*(y)g'(y) - f^{*\prime}(y)g(y)\right]_0^{\pi R} \tag{2.18}$$

The boundary term obviously vanishes, because either the value of the function (the Dirichlet condition) or its first derivative (the Neumann condition) is zero.

To understand why we have chosen this basis, let us take a look at the free field equation of motion in the unitary gauge, e.g. for the vector field:

$$\Box_5 A^a_\nu(x,y) - \partial_\nu\partial^\mu A^a_\mu(x,y) = \big(\Box_4 - \partial_5\partial_5\big)A^a_\nu(x,y) - \partial_\nu\partial^\mu A^a_\mu(x,y) = 0 \tag{2.19}$$

Employing the expansion (2.16) and the formula (2.17) we arrive at the 4D equations of motion for each mode of the infinite KK tower:

$$\big(\Box_4 + m^2_{a,n}\big)A^{a,n}_\nu(x) - \partial_\nu\partial^\mu A^{a,n}_\mu(x) = 0 \tag{2.20}$$

Thus each mode $A^{a,n}_\nu$ obeys 4D equation of motion for the vector boson with the mass $m_{a,n}$. The mass is determined by the corresponding wave function and the eigenvalue of the operator $\partial_5\partial_5$.

2.5 Effective 4D Lagrangian

Using what we have learned so far, we can rewrite the action $S = \int \mathrm{d}^5x(\mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{g.f.}})$ in a form in which one can see better what interactions of effective 4D vector and scalar fields there are.

We will do this manipulation in several separate steps. First let us take a look at the term $F^{a\mu 5}F^a_{\mu 5}$, this can be worked out as:

$$
\begin{aligned}
&-\frac{1}{2}\int \mathrm{d}^5x\, F^{a\mu 5}F^a_{\mu 5} = -\frac{1}{2}\int \mathrm{d}^5x\, F^{a\mu 5}\left(\partial_\mu A^a_5 - \partial_5 A^a_\mu + g_5 f^{abc}A^b_\mu A^c_5\right) = \\
&= -\frac{1}{2}\int \mathrm{d}^5x \left[\partial_5 F^{a\mu 5}A^a_\mu + F^{a\mu 5}\left(\partial_\mu A^a_5 + g_5 f^{abc}A^b_\mu A^c_5\right)\right] = \\
&= \frac{1}{2}\int \mathrm{d}^5x \left[(\partial_\mu A^a_5)(\partial^\mu A^a_5) - A^a_\mu \partial_5\partial_5 A^{a\mu} + A^a_\mu \partial^\mu \partial_5 A^a_5 - (\partial_5 A^a_\mu)(\partial^\mu A^a_5) + \right. \\
&\left. + 2g_5 f^{abc}A^a_5 A^b_\mu \partial_5 A^{c\mu} + 2g_5 f^{abc}A^{a\mu}A^b_5\partial_\mu A^c_5 + g_5^2 f^{abc}f^{ade}A^b_\mu A^{d\mu}A^c_5 A^e_5\right]
\end{aligned}
\tag{2.21}
$$

Apart from trivial algebraic manipulations, we have used a stronger variant of condition (2.13a) – when using only the Dirichlet and Neumann boundary conditions, the variation of the field is zero if and only if the field itself is zero at the boundary, thus we do not pick up any boundary term when integrating the term $F^{a\mu 5}\partial_5 A^a_\mu$ by parts. Similarly, we can use a stronger variant of (2.13b) when dealing with the gauge fixing part of action. In this case we get

$$
\begin{aligned}
&-\frac{1}{2\xi}\int \mathrm{d}^5x\, \left(\partial_\mu A^{a\mu} - \xi\partial_5 A^a_5\right)^2 = \\
&= -\frac{1}{2\xi}\int \mathrm{d}^5x \left[\xi A^a_5 \partial_5\left(\partial_\mu A^{a\mu} - \xi\partial_5 A^a_5\right) + \partial_\nu A^{a\nu}\left(\partial_\mu A^{a\mu} - \xi\partial_5 A^a_5\right)\right] = \\
&= \frac{1}{2}\int \mathrm{d}^5x \left[(\partial_5 A^a_\mu)(\partial^\mu A^a_5) - A^a_\mu\partial^\mu\partial_5 A^a_5 + \xi A^a_5\partial_5\partial_5 A^a_5 - (\partial_\mu A^{a\mu})^2/\xi\right]
\end{aligned}
\tag{2.22}
$$

A short look at the expressions (2.21) and (2.22) reveals that the gauge fixing term really cancels all cross terms mixing scalar and vector fields.

Putting it together with the term $F^{a\mu\nu}F^a_{\mu\nu}$ we can use KK expansion (2.16) and relation (2.17), then integrate over the extra dimension to get the effective 4D Lagrangian. Due to the orthonormality of the wave functions all quadratic terms of 5D fields will be converted to the sum of quadratic terms of 4D effective fields after integration. The result can be written in the form

$$
S = \int \mathrm{d}^4x\, \mathscr{L}_{\mathrm{4D}} = \int \mathrm{d}^4x\, \left(\mathscr{L}^{\mathrm{vector}}_{\mathrm{free}} + \mathscr{L}^{\mathrm{scalar}}_{\mathrm{free}} + \mathscr{L}^{\mathrm{vector}}_{\mathrm{int}} + \mathscr{L}^{\mathrm{scalar}}_{\mathrm{int}}\right) \tag{2.23}
$$

$$
\mathscr{L}^{\mathrm{vector}}_{\mathrm{free}} = \frac{1}{2}\sum_{\boldsymbol{a}} A^{\boldsymbol{a}}_\mu \,\Box_4 A^{\boldsymbol{a}\mu} + m^2_{\boldsymbol{a}} A^{\boldsymbol{a}}_\mu A^{\boldsymbol{a}\mu} + \left(1 - \frac{1}{\xi}\right)\left(\partial_\mu A^{\boldsymbol{a}\mu}\right)^2 \tag{2.24a}
$$

$$
\mathscr{L}^{\mathrm{scalar}}_{\mathrm{free}} = \frac{1}{2}\sum_{\boldsymbol{a}} (\partial_\mu A^{\boldsymbol{a}}_5)(\partial^\mu A^{\boldsymbol{a}}_5) - \xi\widetilde{m}^2_{\boldsymbol{a}} A^{\boldsymbol{a}}_5 A^{\boldsymbol{a}}_5 \tag{2.24b}
$$

$$
\mathscr{L}^{\mathrm{vector}}_{\mathrm{int}} = \sum_{\boldsymbol{abc}} g_{\boldsymbol{abc}} f^{abc} A^{\boldsymbol{a}}_\nu A^{\boldsymbol{b}}_\mu \partial^\mu A^{\boldsymbol{c}\nu} - \frac{1}{4}\sum_{\boldsymbol{abcd}} g^2_{\boldsymbol{abcd}} f^{abe}f^{cde} A^{\boldsymbol{a}}_\mu A^{\boldsymbol{c}\mu} A^{\boldsymbol{b}}_\nu A^{\boldsymbol{d}\nu} \tag{2.24c}
$$

$$\mathscr{L}_{\text{int}}^{\text{scalar}} = \sum_{\boldsymbol{abc}} \hat{e}_{\boldsymbol{abc}} f^{abc} A_5^{\boldsymbol{a}} A_\mu^{\boldsymbol{b}} A^{\boldsymbol{c}\mu} + e_{\boldsymbol{abc}} f^{abc} A_\mu^{\boldsymbol{a}} A_5^{\boldsymbol{b}} \partial^\mu A_5^{\boldsymbol{c}} + \\ + \frac{1}{2} \sum_{\boldsymbol{abcd}} e_{\boldsymbol{abcd}}^2 f^{abe} f^{cde} A_\mu^{\boldsymbol{a}} A^{\boldsymbol{c}\mu} A_5^{\boldsymbol{b}} A_5^{\boldsymbol{d}} \tag{2.24d}$$

For the sake of clarity, we have split up the effective Lagrangian into four pieces:

- The free field Lagrangian $\mathscr{L}_{\text{free}}^{\text{vector}}$ of the infinite tower of vector fields with the Lorentz gauge fixing term for each of them.
- The free field Lagrangian $\mathscr{L}_{\text{free}}^{\text{scalar}}$ of the infinite tower of scalar fields, which are clearly all unphysical in the unitary gauge (except if there is a massless mode). The eigenvalues $\widetilde{m}_{\boldsymbol{a}}^2$ originating from the decomposition of scalars are marked with tilde to distinguish them from the ones of vector fields, we will use the same notation for the wave functions later.
- Interaction Lagrangian $\mathscr{L}_{\text{int}}^{\text{vector}}$ of vector bosons only, which has a well known Yang-Mills structure.
- Interaction Lagrangian $\mathscr{L}_{\text{int}}^{\text{scalar}}$ that involves at least one scalar particle in every vertex. In unitary gauge, where all massive scalar modes are unphysical, we are left with just a few vertices with massless scalar mode (if it even exists).

All the effective 4D couplings $g_{\boldsymbol{abc}}$, $\hat{e}_{\boldsymbol{abc}}$, $e_{\boldsymbol{abc}}$ and $g_{\boldsymbol{abcd}}^2$, $e_{\boldsymbol{abcd}}^2$ contain an integral of wave functions, explicitly:

$$g_{\boldsymbol{abc}} = g_5 \int_0^{\pi R} \mathrm{d}y \, \varphi_{\boldsymbol{a}} \, \varphi_{\boldsymbol{b}} \, \varphi_{\boldsymbol{c}} \tag{2.25a}$$

$$\hat{e}_{\boldsymbol{abc}} = \frac{g_5}{2} \int_0^{\pi R} \mathrm{d}y \, \widetilde{\varphi}_{\boldsymbol{a}} \, \left(\varphi_{\boldsymbol{b}} \, \varphi_{\boldsymbol{c}}' - \varphi_{\boldsymbol{b}}' \, \varphi_{\boldsymbol{c}}\right) \tag{2.25b}$$

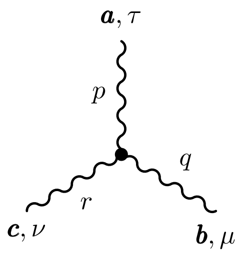
$$e_{\boldsymbol{abc}} = g_5 \int_0^{\pi R} \mathrm{d}y \, \varphi_{\boldsymbol{a}} \, \widetilde{\varphi}_{\boldsymbol{b}} \, \widetilde{\varphi}_{\boldsymbol{c}} \tag{2.25c}$$

$$g_{\boldsymbol{abcd}}^2 = g_5^2 \int_0^{\pi R} \mathrm{d}y \, \varphi_{\boldsymbol{a}} \, \varphi_{\boldsymbol{b}} \, \varphi_{\boldsymbol{c}} \, \varphi_{\boldsymbol{d}} \tag{2.25d}$$

$$e_{\boldsymbol{abcd}}^2 = g_5^2 \int_0^{\pi R} \mathrm{d}y \, \varphi_{\boldsymbol{a}} \, \widetilde{\varphi}_{\boldsymbol{b}} \, \varphi_{\boldsymbol{c}} \, \widetilde{\varphi}_{\boldsymbol{d}} \tag{2.25e}$$

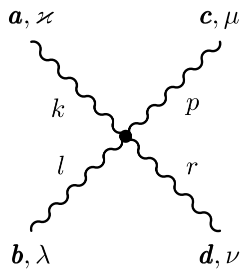
2.6 Feynman diagram rules

The propagators of both vector and scalar fields have the usual form and except for the different couplings the three and four vector boson interactions are the same as in the Standard Model. For completeness we present the rules for all the vertices involving scalar particles, although for our calculations we will need only the *vector-vector-scalar* vertex.

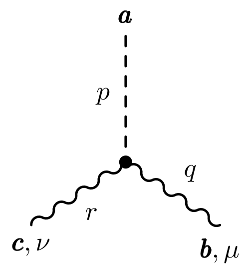


$$
\begin{aligned}
&= g_{\boldsymbol{abc}} f^{abc} V_{\tau\mu\nu}(p, q, r) = \\
&= g_{\boldsymbol{abc}} f^{abc} \Big[g_{\tau\mu}(p - q)_\nu + g_{\nu\tau}(r - p)_\mu + g_{\mu\nu}(q - r)_\tau \Big]
\end{aligned} \tag{2.26}
$$

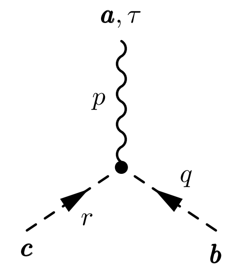
all momenta going in



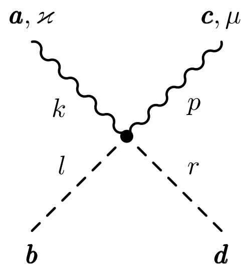
$$
\begin{aligned}
= i g^2_{\boldsymbol{abcd}} \Big[& f^{abe} f^{cde} (g_{\varkappa\nu} g_{\lambda\mu} - g_{\varkappa\mu} g_{\lambda\nu}) + \\
& + f^{ace} f^{bde} (g_{\varkappa\nu} g_{\lambda\mu} - g_{\varkappa\lambda} g_{\mu\nu}) + \\
& + f^{ade} f^{bce} (g_{\varkappa\mu} g_{\lambda\nu} - g_{\varkappa\lambda} g_{\mu\nu}) \Big]
\end{aligned} \tag{2.27}
$$



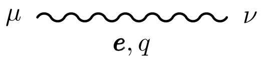
$$
= i \hat{e}_{\boldsymbol{abc}} f^{abc} 2 g_{\mu\nu} \tag{2.28}
$$



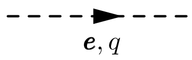
$$
= e_{\boldsymbol{abc}} f^{abc} (r - q)_\tau \tag{2.29}
$$



$$= ie^2_{\boldsymbol{abcd}} g_{\varkappa\mu} \left(f^{abe} f^{cde} - f^{ade} f^{bce} \right) \tag{2.30}$$



$$= \frac{i}{q^2 - m^2_{\boldsymbol{e}}} \left(-g_{\mu\nu} + (1-\xi) \frac{q_\mu q_\nu}{q^2 - \xi m^2_{\boldsymbol{e}}} \right) \tag{2.31a}$$



$$= \frac{i}{q^2 - \xi \widetilde{m}^2_{\boldsymbol{e}}} \tag{2.31b}$$

2.7 Gauge independence of scattering amplitudes

In this section we examine the conditions under which the scattering amplitude of the process $VV \to VV$, calculated in R_ξ gauge, does not depend on the gauge parameter ξ. The gauge parameter is present only in the part of vector propagator that is proportional to $q_\mu q_\nu$ and the mass of a scalar field (thus also in its propagator), so the only relevant (lowest order) diagrams are the s, t and u channel exchange of a scalar or vector particle. The gauge dependent terms must cancel out in each channel separately, so it is sufficient to carry out the calculations only for e.g. the s channel (see the diagrams in Fig. 2.1) and show that the terms of scalar and vector propagator containing ξ in fact cancel out.

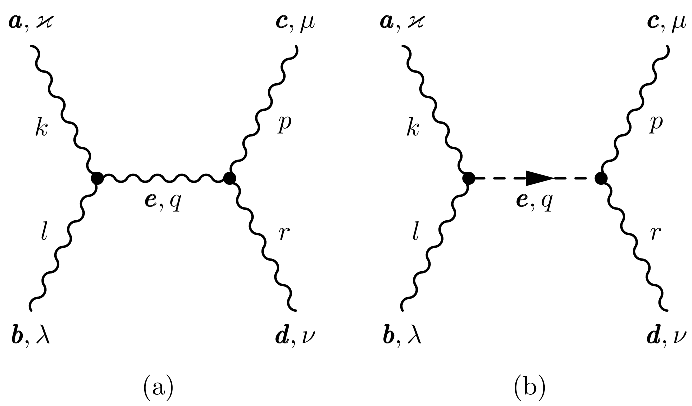


Figure 2.1: Feynman diagrams for exchange of (a) vector boson mode, (b) scalar mode, in s channel for the process $VV \to VV$ in R_ξ gauge.

First, let us write the invariant matrix elements using the rules (2.31), (2.26) and (2.28), we get

$$\begin{aligned}\mathcal{M}^{(s)}_{\text{vector}} = \sum_{\boldsymbol{e}} g_{\boldsymbol{eab}} f^{eab} V_{\alpha\varkappa\lambda}(-q,k,l)\, \frac{q^\alpha q^\beta}{q^2 - m_{\boldsymbol{e}}^2} \frac{1-\xi}{q^2 - \xi m_{\boldsymbol{e}}^2}\, g_{\boldsymbol{ecd}} f^{ecd} \times \\ \times V_{\beta\mu\nu}(q,-p,-r) \epsilon^{\varkappa}(k)\, \epsilon^{\lambda}(l)\, \epsilon^{\mu}(p)\, \epsilon^{\nu}(r) + \dots\end{aligned} \tag{2.32}$$

$$\mathcal{M}^{(s)}_{\text{scalar}} = \sum_{\boldsymbol{e}} 2i\hat{e}_{\boldsymbol{eab}} f^{eab} \frac{1}{q^2 - \xi \widetilde{m}_{\boldsymbol{e}}^2}\, 2i\hat{e}_{\boldsymbol{ecd}} f^{ecd} \left[\epsilon(k)\cdot\epsilon(l)\right] \left[\epsilon(p)\cdot\epsilon(r)\right] \tag{2.33}$$

The dots in $\mathcal{M}^{(s)}_{\text{vector}}$ correspond to the terms that do not depend on ξ. We can use the *'t Hooft identity*

$$q^\alpha\, V_{\alpha\varkappa\lambda}(q,k,l) = (l^2 g_{\varkappa\lambda} - l_\varkappa l_\lambda) - (k^2 g_{\varkappa\lambda} - k_\varkappa k_\lambda) \tag{2.34}$$

and the fact that the polarization of a particle is transverse to the corresponding momentum, and rewrite the matrix element (2.32) in a more suitable form:

$$\begin{aligned}\mathcal{M}^{(s)}_{\text{vector}} = \sum_{\boldsymbol{e}} - g_{\boldsymbol{eab}} f^{eab} \left(m_{\boldsymbol{b}}^2 - m_{\boldsymbol{a}}^2\right) \frac{1}{q^2 - m_{\boldsymbol{e}}^2} \frac{1-\xi}{q^2 - \xi m_{\boldsymbol{e}}^2}\, g_{\boldsymbol{ecd}} f^{ecd} \times \\ \times \left(m_{\boldsymbol{d}}^2 - m_{\boldsymbol{c}}^2\right) \left[\epsilon(k)\cdot\epsilon(l)\right] \left[\epsilon(p)\cdot\epsilon(r)\right] + \dots\end{aligned} \tag{2.35}$$

We have now both matrix elements in quite similar forms. In order to have any chance of cancellation between the corresponding modes of exchanged particles, we obviously need

$$m_{\boldsymbol{e}} = \widetilde{m}_{\boldsymbol{e}}\,. \tag{2.36}$$

In Tab. 2.1 there is the overview of wave functions for all four possible combinations of the Dirichlet and Neumann boundary conditions. Clearly, there are two ways to satisfy (2.36):

- Use the opposite boundary conditions for A_5^a and A_μ^a (meaning every Dirichlet condition imposed on a vector field implies the Neumann condition on the scalar field of the same color at the same brane and vice versa); this is equivalent to an arbitrary combination of (2.14a) and (2.14b).
- Impose the same boundary conditions on A_5^a as on A_μ^a, thus use the condition (2.14c) at both branes.

After putting (2.33) and (2.35) together and factorizing out all common gauge independent parts, it remains to show that the expression

$$\mathcal{A}_{\boldsymbol{e}} = \left(g_{\boldsymbol{eab}} \left(m_{\boldsymbol{b}}^2 - m_{\boldsymbol{a}}^2\right) g_{\boldsymbol{ecd}} \left(m_{\boldsymbol{d}}^2 - m_{\boldsymbol{c}}^2\right) \frac{1-\xi}{q^2 - m_{\boldsymbol{e}}^2} + 2\hat{e}_{\boldsymbol{eab}} 2\hat{e}_{\boldsymbol{ecd}} \right) \frac{1}{q^2 - \xi m_{\boldsymbol{e}}^2} \tag{2.37}$$

BC at $y = 0$	BC at $y = \pi R$	Wave function
Dirichlet	Dirichlet	$k \geq 1$: $\sqrt{\frac{2}{\pi R}} \sin\left(\frac{2k}{R} y\right)$
Dirichlet	Neumann	$k \geq 1$: $\sqrt{\frac{2}{\pi R}} \sin\left(\frac{2k-1}{R} y\right)$
Neumann	Dirichlet	$k \geq 1$: $\sqrt{\frac{2}{\pi R}} \cos\left(\frac{2k-1}{R} y\right)$
Neumann	Neumann	$k \geq 1$: $\sqrt{\frac{2}{\pi R}} \cos\left(\frac{2k}{R} y\right)$, $k = 0$: $\frac{1}{\sqrt{\pi R}}$

Table 2.1: Wave functions for an arbitrary combination of the Dirichlet and Neumann boundary conditions (BC).

is independent of ξ for each KK mode $\boldsymbol{e} = (e, k)$.

Now we make an important observation that allows us to recast the problem only in terms of the wave functions – the following expression does not depend on the gauge parameter ξ:

$$\left(m_{\boldsymbol{e}}^2 \frac{1-\xi}{q^2 - m_{\boldsymbol{e}}^2} + 1 \right) \frac{1}{q^2 - \xi m_{\boldsymbol{e}}^2} = \frac{1}{q^2 - m_{\boldsymbol{e}}^2} \tag{2.38}$$

Thus, in order to obtain the scattering amplitudes independent of the gauge parameter, we need to show the validity of the following relation between couplings and masses:

$$g_{\boldsymbol{eab}} \left(m_{\boldsymbol{b}}^2 - m_{\boldsymbol{a}}^2 \right) g_{\boldsymbol{ecd}} \left(m_{\boldsymbol{d}}^2 - m_{\boldsymbol{c}}^2 \right) = 2\hat{e}_{\boldsymbol{eab}} 2\hat{e}_{\boldsymbol{ecd}} m_{\boldsymbol{e}}^2 \tag{2.39}$$

For this we obviously need the definitions of couplings (2.25) and the relation (2.17) for the eigenvalues of the operator $\partial_5 \partial_5$. Let us first work out the expression for the coupling and masses of a single vector boson vertex:

$$\begin{aligned} g_{\boldsymbol{eab}} \left(m_{\boldsymbol{b}}^2 - m_{\boldsymbol{a}}^2 \right) &= g_5 \int_0^{\pi R} \mathrm{d}y \, \varphi_{\boldsymbol{e}} \left(\varphi_{\boldsymbol{a}}' \varphi_{\boldsymbol{b}} - \varphi_{\boldsymbol{a}} \varphi_{\boldsymbol{b}}' \right)' = \\ &= g_5 \Big[\varphi_{\boldsymbol{e}} \left(\varphi_{\boldsymbol{a}}' \varphi_{\boldsymbol{b}} - \varphi_{\boldsymbol{a}} \varphi_{\boldsymbol{b}}' \right) \Big]_0^{\pi R} - g_5 \int_0^{\pi R} \mathrm{d}y \, \varphi_{\boldsymbol{e}}' \left(\varphi_{\boldsymbol{a}}' \varphi_{\boldsymbol{b}} - \varphi_{\boldsymbol{a}} \varphi_{\boldsymbol{b}}' \right) \end{aligned} \tag{2.40}$$

The boundary term is actually zero, because it contains the product of two wave function values and the first derivative of the wave function (each of them of a different color) and in most cases at least one of them vanishes at the boundary. As stated before we will always impose the same boundary conditions on $A_M^{1,2}$, thus the only combination of boundary conditions under which the boundary term does not vanish is the Neumann condition on $A_M^{1,2}$ and the Dirichlet condition on A_M^3, but we have already excluded this combination in the Section 3.3.2. Repeating the same procedure on the other vertex, we can rewrite the whole expression on the left-hand side of (2.39) as:

$$\begin{aligned} g_{\boldsymbol{eab}}\,(m_{\boldsymbol{b}}^2 - m_{\boldsymbol{a}}^2) g_{\boldsymbol{ecd}}\,(m_{\boldsymbol{d}}^2 - m_{\boldsymbol{c}}^2) &= \\ = g_5^2 \int_0^{\pi R} \mathrm{d}y\, \varphi_{\boldsymbol{e}}' \left(\varphi_{\boldsymbol{a}}'\, \varphi_{\boldsymbol{b}} - \varphi_{\boldsymbol{a}}\, \varphi_{\boldsymbol{b}}'\right) &\int_0^{\pi R} \mathrm{d}z\, \varphi_{\boldsymbol{e}}' \left(\varphi_{\boldsymbol{c}}'\, \varphi_{\boldsymbol{d}} - \varphi_{\boldsymbol{c}}\, \varphi_{\boldsymbol{d}}'\right) \end{aligned} \tag{2.41}$$

Using the definitions of couplings (2.25), we can also explicitly write down the product of couplings on the right-hand side of (2.39) in terms of the integrals of the wave functions as

$$2\hat{e}_{\boldsymbol{eab}} 2\hat{e}_{\boldsymbol{ecd}} = g_5^2 \int_0^{\pi R} \mathrm{d}y\, \widetilde{\varphi}_{\boldsymbol{e}} \left(\varphi_{\boldsymbol{a}}'\, \varphi_{\boldsymbol{b}} - \varphi_{\boldsymbol{a}}\, \varphi_{\boldsymbol{b}}'\right) \int_0^{\pi R} \mathrm{d}z\, \widetilde{\varphi}_{\boldsymbol{e}} \left(\varphi_{\boldsymbol{c}}'\, \varphi_{\boldsymbol{d}} - \varphi_{\boldsymbol{c}}\, \varphi_{\boldsymbol{d}}'\right) . \tag{2.42}$$

Let us examine the relation between the wave functions $\varphi_{\boldsymbol{e}}$ of the vector modes and $\widetilde{\varphi}_{\boldsymbol{e}}$ of the scalar modes. If we use the boundary condition (2.14c), then the functions are the same and obviously we cannot get gauge independent scattering amplitudes. The same conclusion was also reached in [21] using a different line of argumentation based on the requirement of consistently defined restricted class of 5D gauge transformations. On the other hand, if we use an arbitrary combination of (2.14a) and (2.14b) boundary conditions, then one of the functions is sine and the other cosine with the same arguments, thus the relation for every massive mode of color e can be conveniently written in the form

$$m_{\boldsymbol{e}}^2\, \widetilde{\varphi}_{\boldsymbol{e}}(y)\, \widetilde{\varphi}_{\boldsymbol{e}}(z) = \varphi_{\boldsymbol{e}}'(y)\, \varphi_{\boldsymbol{e}}'(z)\,. \tag{2.43}$$

Plugging this relation into (2.41) and comparing with (2.42), we conclude that the equality (2.39) is indeed satisfied for all massive modes and thus the gauge dependent terms in fact cancel out between all massive scalar and vector modes of exchanged particle.

It remains to show that the ξ-dependent terms cancel out even for massless modes. We have already established that the only consistent boundary conditions are (2.14a) and (2.14b), and the only combination of boundary conditions that allows a massless mode is the Neumann condition at both branes, thus there is only a massless scalar, or a massless vector particle, but not both of the same color. The massless mode has a simple flat wave function $\varphi_{e,0} = 1/\sqrt{\pi R}$ and obviously its mass is $m_{e,0} = 0$.

The massless scalar is not a problem, because the gauge parameter is present only in the term $\xi m_{e,0}^2 = 0$. In case of the massless vector boson, the part of invariant matrix element proportional to $q^\mu q^\nu$ in the propagator does not contribute at all. As we have seen, this part is proportional to the expression (2.41), which contains under both integrals $\varphi_{e,0}' = 0$.

The conclusion of this section is that we can impose an arbitrary combination of boundary conditions (2.14a) and (2.14b) on the gauge fields as long as it satisfies (2.13). This leaves us with nine consistent combinations (under the chosen restrictions in the Section 3.3.2) of boundary conditions resulting in theory with the scattering amplitudes independent[2] of the gauge parameter ξ. Thus, we can go to the unitary gauge, where

- all massive scalar modes are unphysical,
- massless scalar modes, if the imposed boundary conditions allow their existence, have propagator $1/q^2$,
- massive vector modes have the usual propagator $\big(-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_{\boldsymbol{e}}^2}\big)/\big(q^2-m_{\boldsymbol{e}}^2\big)$,
- massless vector modes, if the imposed boundary conditions allow their existence, have effectively[3] propagator $-g_{\mu\nu}/\big(q^2-m_{\boldsymbol{e}}^2\big)$.

[2]For completeness we should also show the gauge independence of scattering amplitudes for processes with one or more massless scalars in the initial or final state, but the line of argumentation remains the same, only the couplings are a bit different.

[3]Although we have shown this only for the vertex (2.26), similar argumentation can be used for all remaining vertices.

Chapter 3

Tree-level unitarity of $V_\mathrm{L}V_\mathrm{L} \to V_\mathrm{L}V_\mathrm{L}$ process

In this chapter we want to show that a general process of scattering $2 \to 2$ longitudinal gauge bosons is unitary at tree-level. By tree-level unitarity we mean that, after taking into account all the relevant Feynman diagrams for the process and adding together their invariant matrix elements, all terms growing as a positive power of energy are eliminated. Unless the theory is unitary at tree-level, the “bad” high energy behaviour of scattering amplitudes leads to a “rapid violation” of unitarity [5]; thus the energy scale, under which the theory may be valid, is considerably lowered.

Tree-level unitarity of course does not guarantee that there is no unitarity violation at all. The concern is that the finite value of a scattering amplitude itself could be too large and break the unitarity bound, possibly only in some of the partial waves; there are also other problems regarding the fact that this class of theories is non-renormalizable and valid only under a certain cutoff scale [18], but we will not address these issues here. Under tree-level unitarity we will always understand a simple cancellation of divergent terms in scattering amplitudes.

We calculate the energy dependence of invariant matrix elements for this generally inelastic scattering of gauge bosons using formulas derived in the previous chapter. We consider high energy limit and expand all quantities in powers of energy (more precisely in the powers of Mandelstam invariant s) keeping only the divergent parts, and show that they really cancel out automatically without introducing additional Higgs field. This is the major difference from the Standard Model, where the diagrams involving the Higgs exchange are needed for the complete cancellation of divergent terms in the scattering amplitudes of longitudinal gauge bosons.

We do not employ a hard cutoff on the spectrum of KK modes and keep the whole infinite towers of KK excitations. This is justified due to the fact that the contributions from the highest KK modes are suppressed in high energy limit, for a detailed discussion see [16].

3.1 Feynman diagrams for the process

The overview of all relevant lowest order Feynman diagrams for $V_{\mathrm{L}}V_{\mathrm{L}} \to V_{\mathrm{L}}V_{\mathrm{L}}$ process is in Fig. 3.1. The first three diagrams are present only if we have imposed the boundary conditions that give rise to the massless scalar mode. Let us stress that we make no assumptions regarding the color or KK mode number of the gauge bosons in the initial or final state.

3.2 Overview of kinematic variables

We carry out the calculation in the center of mass reference frame, where the four-vectors of momenta and polarizations have the following form:

$$k_{\varkappa} = |\vec{p}|\left(\frac{1}{\beta_{\boldsymbol{a}}}, \vec{n}\right) \qquad \epsilon_{\varkappa}(k) = \frac{|\vec{p}|}{m_{\boldsymbol{a}}}\left(1, \frac{\vec{n}}{\beta_{\boldsymbol{a}}}\right) \tag{3.1a}$$

$$l_{\lambda} = |\vec{p}|\left(\frac{1}{\beta_{\boldsymbol{b}}}, -\vec{n}\right) \qquad \epsilon_{\lambda}(l) = \frac{|\vec{p}|}{m_{\boldsymbol{b}}}\left(1, -\frac{\vec{n}}{\beta_{\boldsymbol{b}}}\right) \tag{3.1b}$$

$$p_{\mu} = |\vec{p}\,'|\left(\frac{1}{\beta_{\boldsymbol{c}}}, \vec{n}\,'\right) \qquad \epsilon_{\mu}(p) = \frac{|\vec{p}\,'|}{m_{\boldsymbol{c}}}\left(1, \frac{\vec{n}\,'}{\beta_{\boldsymbol{c}}}\right) \tag{3.1c}$$

$$r_{\nu} = |\vec{p}\,'|\left(\frac{1}{\beta_{\boldsymbol{d}}}, -\vec{n}\,'\right) \qquad \epsilon_{\nu}(r) = \frac{|\vec{p}\,'|}{m_{\boldsymbol{d}}}\left(1, -\frac{\vec{n}\,'}{\beta_{\boldsymbol{d}}}\right) \tag{3.1d}$$

The unit vectors $\vec{n}$ and $\vec{n}\,'$ of incoming and outgoing momenta directions satisfy $\vec{n} \cdot \vec{n}\,' = \cos\theta$, where θ is the scattering angle. The ratio of the energy and momentum, carried by e.g. the boson with index $\boldsymbol{a}$, is given by

$$\frac{1}{\beta_{\boldsymbol{a}}} = \frac{E_{\boldsymbol{a}}}{|\vec{p}|} = 1 + \frac{m_{\boldsymbol{a}}^2}{2|\vec{p}|^2} + \mathcal{O}(|\vec{p}|^{-4}) \tag{3.2}$$

Since in the end we want to expand all quantities in power series of Mandelstam invariant s, we will need to know the first few terms of expansion of previously introduced variables. Note that in the leading order

$$s \approx 4E^2 \approx 4|\vec{p}|^2 \approx 4|\vec{p}\,'|^2\,. \tag{3.3}$$

Combining this relation with the definition of the invariant $s = (k + l)^2$ and the chosen form of momenta (3.1), we can easily derive the expansion of momenta

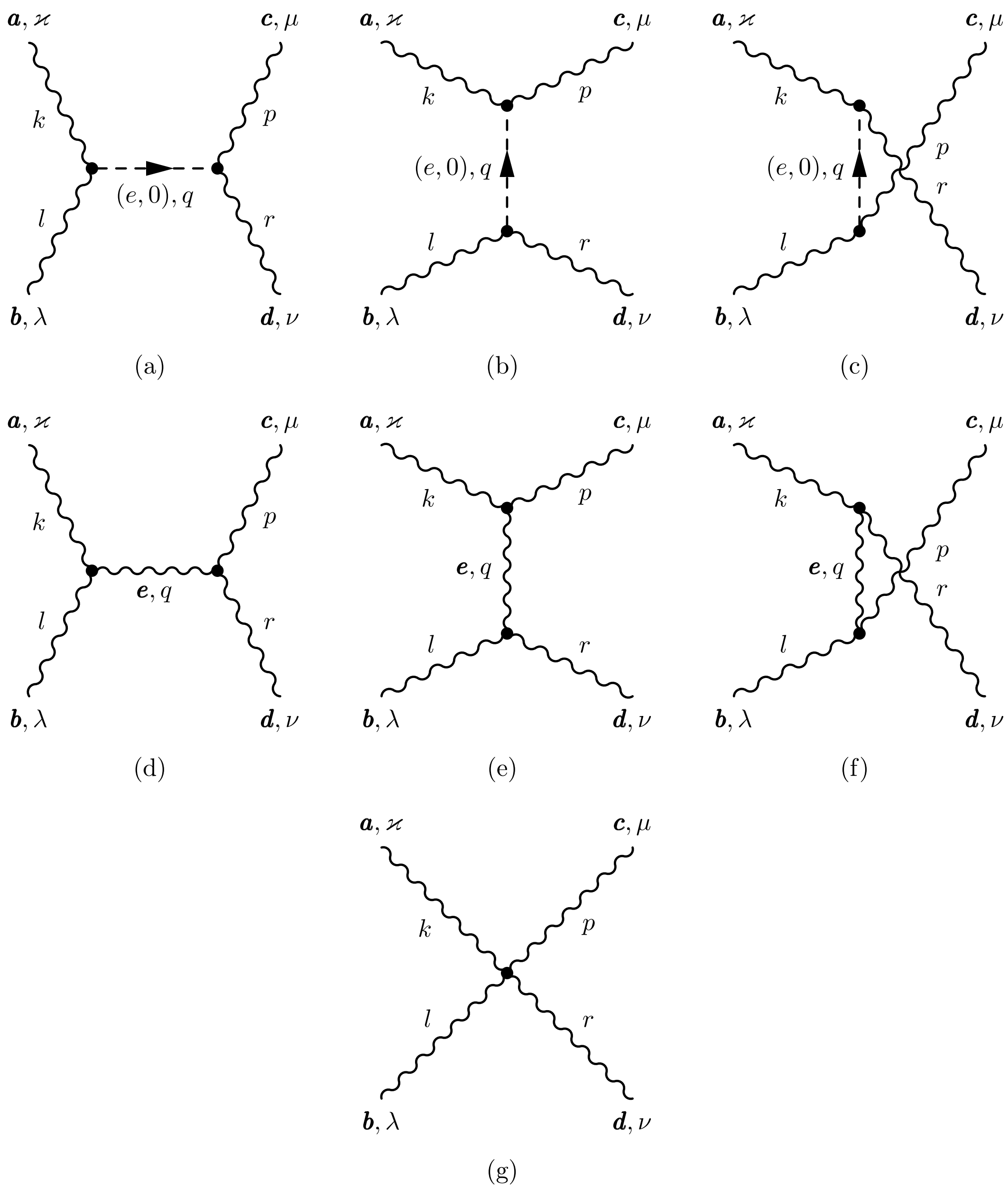


Figure 3.1: Lowest order Feynman diagrams for the process $V_L V_L \to V_L V_L$ calculated in unitary gauge – (a), (b), (c) exchange of the massless scalar mode (in some models may not exist) in s, t, u channel, (d), (e), (f) exchange of vector boson mode in s, t, u channel, and (g) contact interaction of vector bosons.

$|\vec{p}|$ and $|\vec{p}\,'|$ in terms of Mandelstam invariant s. After a trivial algebraic exercise we arrive at

$$|\vec{p}| = \frac{\sqrt{s}}{2}\left(1 - \frac{m_{\boldsymbol{a}}^2 + m_{\boldsymbol{b}}^2}{s} + \mathcal{O}(s^{-2})\right) \tag{3.4a}$$

$$|\vec{p}\,'| = \frac{\sqrt{s}}{2}\left(1 - \frac{m_{\boldsymbol{c}}^2 + m_{\boldsymbol{d}}^2}{s} + \mathcal{O}(s^{-2})\right) \tag{3.4b}$$

Now we can also recast the expansion (3.2) using the invariant s in the form

$$\frac{1}{\beta_{\boldsymbol{a}}} = 1 + \frac{2m_{\boldsymbol{a}}^2}{s} + \mathcal{O}(s^{-2})\,. \tag{3.5}$$

For certain steps of the calculation it is useful to decompose the polarization vector as

$$\epsilon_\alpha(q) = \frac{q_\alpha}{m} + \Delta_\alpha(q)\,, \tag{3.6}$$

where $\Delta_\alpha(q)$ is only of order $\mathcal{O}(s^{-1/2})$. This can be easily seen from the chosen form of the momenta and polarizations (3.1) and the expansions (3.4) and (3.5), indeed e.g. for $\Delta_\varkappa(k)$ one has

$$\Delta_\varkappa(k) = \epsilon_\varkappa(k) - \frac{k_\varkappa}{m_{\boldsymbol{a}}} = \left[\frac{1}{\beta_{\boldsymbol{a}}} - 1\right]\frac{|\vec{p}|}{m_{\boldsymbol{a}}}\left(-1, \vec{n}\right) = \frac{m_{\boldsymbol{a}}}{\sqrt{s}}\left(-1, \vec{n}\right) + \mathcal{O}(s^{-3/2}) \tag{3.7}$$

We will also need an expansion of all Mandelstam invariants:

$$s = m_{\boldsymbol{a}}^2 + m_{\boldsymbol{b}}^2 + 2k\cdot l = m_{\boldsymbol{c}}^2 + m_{\boldsymbol{d}}^2 + 2p\cdot r \tag{3.8a}$$

$$\begin{aligned} t &= m_{\boldsymbol{a}}^2 + m_{\boldsymbol{c}}^2 - 2k\cdot p = m_{\boldsymbol{b}}^2 + m_{\boldsymbol{d}}^2 - 2l\cdot r = \\ &= -s\frac{1-\cos\theta}{2}\left(1 - \frac{m_{\boldsymbol{a}}^2 + m_{\boldsymbol{b}}^2 + m_{\boldsymbol{c}}^2 + m_{\boldsymbol{d}}^2}{s} + \mathcal{O}(s^{-2})\right) \end{aligned} \tag{3.8b}$$

$$\begin{aligned} u &= m_{\boldsymbol{a}}^2 + m_{\boldsymbol{d}}^2 - 2k\cdot r = m_{\boldsymbol{b}}^2 + m_{\boldsymbol{c}}^2 - 2l\cdot p = \\ &= -s\frac{1+\cos\theta}{2}\left(1 - \frac{m_{\boldsymbol{a}}^2 + m_{\boldsymbol{b}}^2 + m_{\boldsymbol{c}}^2 + m_{\boldsymbol{d}}^2}{s} + \mathcal{O}(s^{-2})\right) \end{aligned} \tag{3.8c}$$

For the sake of clarity, let us introduce a shorthand notation for the sum of squared masses of the gauge bosons in the initial and final state:

$$4\overline{m}^2 = m_{\boldsymbol{a}}^2 + m_{\boldsymbol{b}}^2 + m_{\boldsymbol{c}}^2 + m_{\boldsymbol{d}}^2 \tag{3.9}$$

3.3 Quartic contact vertex

Let us start with the easiest part of the scattering amplitude, which is given by the diagram involving the contact four boson interaction. Using the rule (2.27), the corresponding invariant matrix element is

$$\begin{aligned}\mathcal{M}^{(4\mathrm{V})} = g^2_{\boldsymbol{abcd}} \Big\{ & f^{eab} f^{ecd} \big[\epsilon(k)\cdot\epsilon(r)\,\epsilon(l)\cdot\epsilon(p) - \epsilon(k)\cdot\epsilon(p)\,\epsilon(l)\cdot\epsilon(r)\big] \\ & + f^{eac} f^{ebd} \big[\epsilon(k)\cdot\epsilon(r)\,\epsilon(l)\cdot\epsilon(p) - \epsilon(k)\cdot\epsilon(l)\,\epsilon(p)\cdot\epsilon(r)\big] \\ & + f^{ead} f^{ebc} \big[\epsilon(k)\cdot\epsilon(p)\,\epsilon(l)\cdot\epsilon(r) - \epsilon(k)\cdot\epsilon(l)\,\epsilon(p)\cdot\epsilon(r)\big] \Big\}\end{aligned} \tag{3.10}$$

Although the expression seems quite complicated, the calculation is in fact straightforward. First, substituting for the polarization vectors defined in (3.1) and computing their scalar products, we get

$$\begin{aligned}\mathcal{M}^{(4\mathrm{V})} = & \frac{g^2_{\boldsymbol{abcd}} |\vec{p}\,|^2 |\vec{p}\,'|^2}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} \times \\ & \times \Bigg\{ f^{eab} f^{ecd} \left[\left(1 + \frac{\cos\theta}{\beta_{\boldsymbol{a}}\beta_{\boldsymbol{d}}}\right) \left(1 + \frac{\cos\theta}{\beta_{\boldsymbol{b}}\beta_{\boldsymbol{c}}}\right) - \left(1 - \frac{\cos\theta}{\beta_{\boldsymbol{a}}\beta_{\boldsymbol{c}}}\right) \left(1 - \frac{\cos\theta}{\beta_{\boldsymbol{b}}\beta_{\boldsymbol{d}}}\right) \right] + \\ & + f^{eac} f^{ebd} \left[\left(1 + \frac{\cos\theta}{\beta_{\boldsymbol{a}}\beta_{\boldsymbol{d}}}\right) \left(1 + \frac{\cos\theta}{\beta_{\boldsymbol{b}}\beta_{\boldsymbol{c}}}\right) - \left(1 + \frac{1}{\beta_{\boldsymbol{a}}\beta_{\boldsymbol{b}}}\right) \left(1 + \frac{1}{\beta_{\boldsymbol{c}}\beta_{\boldsymbol{d}}}\right) \right] + \\ & + f^{ead} f^{ebc} \left[\left(1 - \frac{\cos\theta}{\beta_{\boldsymbol{a}}\beta_{\boldsymbol{c}}}\right) \left(1 - \frac{\cos\theta}{\beta_{\boldsymbol{b}}\beta_{\boldsymbol{d}}}\right) - \left(1 + \frac{1}{\beta_{\boldsymbol{a}}\beta_{\boldsymbol{b}}}\right) \left(1 + \frac{1}{\beta_{\boldsymbol{c}}\beta_{\boldsymbol{d}}}\right) \right] \Bigg\}\end{aligned} \tag{3.11}$$

Now we can use the expansion formulas (3.4) and (3.5). Keeping only the divergent terms of the matrix element, we can write the result in the form

$$\begin{aligned}\mathcal{M}^{(4\mathrm{V})} = & \left(\frac{s}{4}\right)^2 \frac{g^2_{\boldsymbol{abcd}}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} \Big[f^{eab} f^{ecd} (4\cos\theta) + \\ & + f^{eac} f^{ebd} (-3 + 2\cos\theta + \cos^2\theta) + f^{ead} f^{ebc} (-3 - 2\cos\theta + \cos^2\theta) \Big] + \\ & + \left(\frac{s}{4}\right) \frac{4\overline{m}^2 g^2_{\boldsymbol{abcd}}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} \Big[f^{eab} f^{ecd} (-\cos\theta) + \\ & + f^{eac} f^{ebd} \frac{1 - \cos\theta}{2} + f^{ead} f^{ebc} \frac{1 + \cos\theta}{2} \Big] + \mathcal{O}(1)\end{aligned} \tag{3.12}$$

3.4 Vector boson exchange in s, t and u channels

The matrix elements of s, t, u channels are in fact infinite sums over all KK modes of exchanged gauge boson[1], which may be massive as well as massless. Thus with respect to the different form of vector propagator for massive and massless modes, it is convenient to split them in two parts – $\mathcal{M}^{(s,t,u)}_{(\mathrm{long})}$ and $\mathcal{M}^{(s,t,u)}_{(\mathrm{diag})}$ corresponding to the longitudinal part of the propagator $q^\mu q^\nu$ and the diagonal part of the propagator $g^{\mu\nu}$ respectively.

3.4.1 Terms corresponding to the longitudinal part of boson propagator

Let us write the invariant matrix elements $\mathcal{M}^{(s,t,u)}_{(\mathrm{long})}$ for all channels using the rule (2.26) and the unitary gauge propagators for massive vector KK modes with indices $\boldsymbol{e} = (e, k)$, and sum over all of them:

$$\begin{aligned}\mathcal{M}^{(s)}_{(\mathrm{long})} &= \sum_{k\geq 1} g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} f^{eab} f^{ecd} \, \frac{q^\alpha q^\beta}{m_{\boldsymbol{e}}^2 (q^2 - m_{\boldsymbol{e}}^2)} \times \\ &\qquad \times V_{\alpha\varkappa\lambda}(-q, k, l)\, \epsilon^\varkappa(k)\, \epsilon^\lambda(l)\, V_{\beta\mu\nu}(q, -p, -r)\, \epsilon^\mu(p)\, \epsilon^\nu(r)\end{aligned} \tag{3.13a}$$

$$\begin{aligned}\mathcal{M}^{(t)}_{(\mathrm{long})} &= \sum_{k\geq 1} g_{\boldsymbol{eac}} g_{\boldsymbol{ebd}} f^{eac} f^{ebd} \, \frac{q^\alpha q^\beta}{m_{\boldsymbol{e}}^2 (q^2 - m_{\boldsymbol{e}}^2)} \times \\ &\qquad \times V_{\alpha\varkappa\mu}(-q, k, -p)\, \epsilon^\varkappa(k)\, \epsilon^\mu(p)\, V_{\beta\lambda\nu}(q, l, -r)\, \epsilon^\lambda(l)\, \epsilon^\nu(r)\end{aligned} \tag{3.13b}$$

$$\begin{aligned}\mathcal{M}^{(u)}_{(\mathrm{long})} &= \sum_{k\geq 1} g_{\boldsymbol{ead}} g_{\boldsymbol{ebc}} f^{ead} f^{ebc} \, \frac{q^\alpha q^\beta}{m_{\boldsymbol{e}}^2 (q^2 - m_{\boldsymbol{e}}^2)} \times \\ &\qquad \times V_{\alpha\varkappa\nu}(-q, k, -r)\, \epsilon^\varkappa(k)\, \epsilon^\nu(r)\, V_{\beta\lambda\mu}(q, l, -p)\, \epsilon^\lambda(l)\, \epsilon^\mu(p)\end{aligned} \tag{3.13c}$$

Repeating the procedure from the previous chapter using the 't Hooft identity (2.34) and transversality of a polarization to the corresponding momentum, one can write the matrix elements[2] in the form

$$\begin{aligned}\mathcal{M}^{(s)}_{(\mathrm{long})} &= -\sum_{k\geq 1} \frac{g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} \, f^{eab} f^{ecd}}{m_{\boldsymbol{e}}^2 (q^2 - m_{\boldsymbol{e}}^2)} \times \\ &\qquad \times \left[m_{\boldsymbol{a}}^2 - m_{\boldsymbol{b}}^2\right] \left[\epsilon(k)\cdot\epsilon(l)\right] \left[m_{\boldsymbol{c}}^2 - m_{\boldsymbol{d}}^2\right] \left[\epsilon(p)\cdot\epsilon(r)\right]\end{aligned} \tag{3.14a}$$

[1]Note that due to the structure of SU(2) group the color e of exchanged gauge boson is fixed by the colors of the bosons in the initial and final state.

[2]Since the t and u channels differ only in the simultaneous exchange of indices $\boldsymbol{c}$ and $\boldsymbol{d}$, and p and r momenta (or alternatively changing the sign before $\cos\theta$), we will from now on explicitly present only the expression for t channel.

$$\begin{aligned}\mathcal{M}^{(t)}_{(\text{long})} = -\sum_{k\geq 1} \frac{g_{\boldsymbol{eac}} g_{\boldsymbol{ebd}}\, f^{eac} f^{ebd}}{m_{\boldsymbol{e}}^2 (q^2 - m_{\boldsymbol{e}}^2)} \times \\ \times \left[m_{\boldsymbol{a}}^2 - m_{\boldsymbol{c}}^2\right] \left[\epsilon(k)\cdot\epsilon(p)\right] \left[m_{\boldsymbol{b}}^2 - m_{\boldsymbol{d}}^2\right] \left[\epsilon(l)\cdot\epsilon(r)\right]\end{aligned} \tag{3.14b}$$

Using the decomposition of polarization vectors (3.6) and Mandelstam invariants (3.8), we see that the only divergent term in both matrix elements is the leading term of order $\mathcal{O}(s)$. Thus it is sufficient to substitute for all the variables only the leading terms and use the expansion $1/(q^2 - m_{\boldsymbol{e}}^2) = 1/q^2 + \mathcal{O}(s^{-2})$. All the remaining terms will be at most of order $\mathcal{O}(1)$. Thus the resulting expressions for matrix elements are

$$\mathcal{M}^{(s)}_{(\text{long})} = \sum_{\boldsymbol{e}} \frac{s}{4} \frac{g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} f^{eab} f^{ecd}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} \frac{(m_{\boldsymbol{a}}^2 - m_{\boldsymbol{b}}^2)(m_{\boldsymbol{c}}^2 - m_{\boldsymbol{d}}^2)}{m_{\boldsymbol{e}}^2} (-1) + \mathcal{O}(1) \tag{3.15a}$$

$$\mathcal{M}^{(t)}_{(\text{long})} = \sum_{\boldsymbol{e}} \frac{s}{4} \frac{g_{\boldsymbol{eac}} g_{\boldsymbol{ebd}} f^{eac} f^{ebd}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} \frac{(m_{\boldsymbol{a}}^2 - m_{\boldsymbol{c}}^2)(m_{\boldsymbol{b}}^2 - m_{\boldsymbol{d}}^2)}{m_{\boldsymbol{e}}^2} \frac{1-\cos\theta}{2} + \mathcal{O}(1) \tag{3.15b}$$

3.4.2 Terms corresponding to the diagonal part of boson propagator

Similarly as in the previous section, we write down the expression for $\mathcal{M}^{(s,t,u)}_{(\text{diag})}$ using the rule (2.26) and the diagonal part of vector propagator, but we also have to include the massless zero mode exchange (if the boundary conditions allow its existence). The matrix elements corresponding to the s and t channel are given by

$$\begin{aligned}\mathcal{M}^{(s)}_{(\text{diag})} = \sum_{k\geq 0} g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} f^{eab} f^{ecd} \frac{-g^{\alpha\beta}}{q^2 - m_{\boldsymbol{e}}^2} \times \\ \times V_{\alpha\varkappa\lambda}(-q,k,l)\, \epsilon^{\varkappa}(k)\, \epsilon^{\lambda}(l)\, V_{\beta\mu\nu}(q,-p,-r)\, \epsilon^{\mu}(p)\, \epsilon^{\nu}(r)\end{aligned} \tag{3.16a}$$

$$\begin{aligned}\mathcal{M}^{(t)}_{(\text{diag})} = \sum_{k\geq 0} g_{\boldsymbol{eac}} g_{\boldsymbol{ebd}} f^{eac} f^{ebd} \frac{-g^{\alpha\beta}}{q^2 - m_{\boldsymbol{e}}^2} \times \\ \times V_{\alpha\varkappa\mu}(-q,k,-p)\, \epsilon^{\varkappa}(k)\, \epsilon^{\mu}(p)\, V_{\beta\lambda\nu}(q,l,-r)\, \epsilon^{\lambda}(l)\, \epsilon^{\nu}(r)\end{aligned} \tag{3.16b}$$

Since all the matrix elements have a similar structure, we first derive a general expression for the product of $V_{\alpha\mu\nu}\, \epsilon^{\mu} \epsilon^{\nu}$ and substitute the momenta and polarizations for each channel later. Thus, we first get

$$\begin{aligned}L_{\alpha} = V_{\alpha\mu\nu}(q, v_p p, v_r r)\, \epsilon^{\mu}(p) \epsilon^{\nu}(r) = \\ = [\epsilon(p)\cdot\epsilon(r)]\,[v_p p_{\alpha} - v_r r_{\alpha}] + 2\,[v_r r\cdot\epsilon(p)]\,\epsilon_{\alpha}(r) - 2\,[v_p p\cdot\epsilon(r)]\,\epsilon_{\alpha}(p)\,.\end{aligned} \tag{3.17}$$

In this step we have used momentum conservation in the three boson vertex, v_p and v_r denote the direction of momenta p and r respectively (e.g. for momenta p

going into the vertex is $v_p = 1$, for outgoing $v_p = -1$). In the next step we will use the decomposition of polarization vectors (3.6) and denote m_p and m_r the masses of particles with momenta p and r respectively. In this way, we can split the whole expression with respect to the order of divergence – a leading term is of order $\mathcal{O}(s^{3/2})$, a sub-leading term of order $\mathcal{O}(s^{1/2})$. All the remaining terms are at most of order $\mathcal{O}(s^{-1/2})$ and we do not need them for our analysis. The quantity in (3.17) then becomes

$$
\begin{aligned}
L_\alpha = & \frac{1}{m_p m_r}\Big\{ [p\cdot r]\,[v_p p_\alpha - v_r r_\alpha] + 2\,[v_r r\cdot p]\, r_\alpha - 2\,[v_p p\cdot r]\, p_\alpha \Big\} + \\
& + \frac{1}{m_p m_r}\Big\{ [m_p \Delta(p)\cdot r + m_r p\cdot \Delta(r)]\,[v_p p_\alpha - v_r r_\alpha] + \\
& \qquad + 2\,[v_r r\cdot p]\, m_r \Delta_\alpha(r) - 2\,[v_p p\cdot r]\,\Delta_\alpha(p) + \\
& \qquad + 2\,[m_p v_r r\cdot\Delta(p)]\, r_\alpha - 2\,[m_r v_p p\cdot \Delta(r)]\, p_\alpha \Big\} = \\
= & \frac{1}{m_p m_r}\Big\{ [p\cdot r]\,[v_r r_\alpha - v_p p_\alpha] \Big\} + \\
& + \frac{1}{m_p m_r}\Big\{ 2\,[p\cdot r]\,[v_r m_r \Delta_\alpha(r) - m_p v_p \Delta_\alpha(p)] + \\
& \qquad + [m_p\Delta(p)\cdot r - m_r \Delta(r)\cdot p]\,[v_p p_\alpha + v_r r_\alpha] \Big\} + \mathcal{O}(s^{-1/2})
\end{aligned}
\tag{3.18}
$$

All matrix elements contain a scalar product of two quantities of this type. We are interested only in the divergent terms of the resulting matrix elements. The propagator is of order $\mathcal{O}(s^{-1})$; thus, looking at the orders of divergence of terms in (3.18), it is sufficient to keep only the leading and sub-leading term of the scalar product $L \cdot L'$. All the remaining terms are obviously at most of order $\mathcal{O}(s)$ and will not result in any divergences in the matrix elements.

The quantity (3.18) contains, besides the masses of the gauge bosons, also scalar products of a momentum with either another momentum, or the $\Delta(q)$ part of a polarization vector. The product of two arbitrary momenta can be recast in terms of the boson masses and Mandelstam invariants using (3.8).

Tab. 3.1 contains an overview of scalar products of an arbitrary combination of a momentum and the $\Delta(q)$ part of a polarization vector. Note that for the reasons mentioned above, we need explicitly only the terms of order $\mathcal{O}(s^{-1/2})$ and higher. To derive these results one needs the chosen form of momenta (3.1) and the decomposition of polarization vectors (3.6).

We are now ready to complete the calculation of the matrix elements for each channel. We substitute the momenta specific for a given channel into (3.18), and replace the relevant scalar products according to Tab. 3.1. We also replace all scalar products of two momenta with the corresponding Mandelstam invariant

	$\Delta_\alpha(k)$	$\Delta_\alpha(l)$	$\Delta_\alpha(p)$	$\Delta_\alpha(r)$
k^α	$-m_{\boldsymbol{a}}$	$\mathcal{O}(s^{-1})$	$m_{\boldsymbol{c}}\,\frac{u}{s}+\mathcal{O}(s^{-1})$	$m_{\boldsymbol{d}}\,\frac{t}{s}+\mathcal{O}(s^{-1})$
l^α	$\mathcal{O}(s^{-1})$	$-m_{\boldsymbol{b}}$	$m_{\boldsymbol{c}}\,\frac{t}{s}+\mathcal{O}(s^{-1})$	$m_{\boldsymbol{d}}\,\frac{u}{s}+\mathcal{O}(s^{-1})$
p^α	$m_{\boldsymbol{a}}\,\frac{u}{s}+\mathcal{O}(s^{-1})$	$m_{\boldsymbol{b}}\,\frac{t}{s}+\mathcal{O}(s^{-1})$	$-m_{\boldsymbol{c}}$	$\mathcal{O}(s^{-1})$
r^α	$m_{\boldsymbol{a}}\,\frac{t}{s}+\mathcal{O}(s^{-1})$	$m_{\boldsymbol{b}}\,\frac{u}{s}+\mathcal{O}(s^{-1})$	$\mathcal{O}(s^{-1})$	$-m_{\boldsymbol{d}}$

Table 3.1: Scalar products of momenta and the $\Delta_\alpha(q)$ part of polarization vectors.

using relations (3.8). Let us start with the s channel; in this case the quantity (3.18) has the following form:

$$\begin{aligned} L^{(s1)}_\alpha &= V_{\alpha\varkappa\lambda}(-q,k,l)\,\epsilon^\varkappa(k)\epsilon^\lambda(l) = \frac{1}{m_{\boldsymbol{a}}m_{\boldsymbol{b}}}\Bigg\{\frac{s-m_{\boldsymbol{a}}^2-m_{\boldsymbol{b}}^2}{2}\,(l-k)_\alpha + \\ &\quad + s\left[m_{\boldsymbol{b}}\Delta_\alpha(l)-m_{\boldsymbol{a}}\Delta_\alpha(k)\right]\Bigg\} + \mathcal{O}(s^{-1/2}) \end{aligned} \tag{3.19}$$

$$\begin{aligned} L^{(s2)}_\beta &= V_{\beta\mu\nu}(q,-p,-r)\,\epsilon^\mu(p)\epsilon^\nu(r) = \frac{1}{m_{\boldsymbol{c}}m_{\boldsymbol{d}}}\Bigg\{\frac{s-m_{\boldsymbol{c}}^2-m_{\boldsymbol{d}}^2}{2}\,(p-r)_\beta + \\ &\quad + s\left[m_{\boldsymbol{c}}\Delta_\beta(p)-m_{\boldsymbol{d}}\Delta_\beta(r)\right]\Bigg\} + \mathcal{O}(s^{-1/2}) \end{aligned} \tag{3.20}$$

Finally, we can write down the scalar product of the above expressions keeping only the relevant divergent terms. Again, we recast scalar products of momenta and the $\Delta(q)$ parts of polarization vectors in terms of the boson masses and Mandelstam invariants and arrive at

$$L^{(s1)}\cdot L^{(s2)} = \frac{1}{m_{\boldsymbol{a}}m_{\boldsymbol{b}}m_{\boldsymbol{c}}m_{\boldsymbol{d}}}\left\{\frac{s^2}{4}(t-u)+s(t-u)\,\overline{m}^2\right\}+\mathcal{O}(s)\,. \tag{3.21}$$

The last thing we need to do before formulating the result for the s channel is a trivial expansion of the denominator of the propagator in the high energy limit ($s \approx q^2 \gg m_{\boldsymbol{e}}^2$):

$$\frac{1}{q^2-m_{\boldsymbol{e}}^2} = \frac{1}{q^2}\left(1+\frac{m_{\boldsymbol{e}}^2}{q^2}+\mathcal{O}(s^{-2})\right) \tag{3.22}$$

Using the derived formula (3.21), the expansion of propagator (3.22) and again the definitions of Mandelstam invariants (3.8), we can express the result for the s channel matrix element as

$$
\begin{aligned}
\mathcal{M}^{(s)}_{(\text{diag})} &= \sum_{k\geq 0} g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} f^{eab} f^{ecd}\, \frac{-1}{s} \left[1 + \frac{m_{\boldsymbol{e}}^2}{s}\right] L^{(s1)} \cdot L^{(s2)} + \mathcal{O}(1) = \\
&= \sum_{k\geq 0} \frac{g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} f^{eab} f^{ecd}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} (u-t) \left[\frac{s}{4} + \overline{m}^2 + \frac{m_{\boldsymbol{e}}^2}{4}\right] + \mathcal{O}(1) = \\
&= \sum_{k\geq 0} \frac{g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} f^{eab} f^{ecd}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} \left[\left(\frac{s}{4}\right)^2 (-4\cos\theta) + \frac{s}{4}(-m_{\boldsymbol{e}}^2 \cos\theta)\right] + \mathcal{O}(1)
\end{aligned}
\tag{3.23}
$$

The same procedure can be applied to the t channel matrix element – we again substitute the momenta to the expression (3.18) and then apply (3.8) and Tab. 3.1 to write down the result using the boson masses and Mandelstam invariants. Thus we first get

$$
\begin{aligned}
L^{(t1)}_{\alpha} &= V_{\alpha\varkappa\mu}(-q, k, -p)\, \epsilon^{\varkappa}(k) \epsilon^{\mu}(p) = \frac{1}{m_{\boldsymbol{a}} m_{\boldsymbol{c}}} \Bigg\{ \frac{t - m_{\boldsymbol{a}}^2 - m_{\boldsymbol{c}}^2}{2}\, (p+k)_{\alpha} + \\
&\quad + t\, [m_{\boldsymbol{c}} \Delta_{\alpha}(p) + m_{\boldsymbol{a}} \Delta_{\alpha}(k)] + (m_{\boldsymbol{a}}^2 + m_{\boldsymbol{c}}^2) \frac{u}{s}\, (k-p)_{\alpha} \Bigg\} + \mathcal{O}(s^{-1/2})
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
L^{(t2)}_{\beta} &= V_{\beta\lambda\nu}(q, l, -r)\, \epsilon^{\lambda}(l) \epsilon^{\nu}(r) = \frac{1}{m_{\boldsymbol{b}} m_{\boldsymbol{d}}} \Bigg\{ \frac{t - m_{\boldsymbol{b}}^2 - m_{\boldsymbol{d}}^2}{2}\, (r+l)_{\beta} + \\
&\quad + t\, [m_{\boldsymbol{d}} \Delta_{\beta}(r) + m_{\boldsymbol{b}} \Delta_{\beta}(l)] + (m_{\boldsymbol{b}}^2 + m_{\boldsymbol{d}}^2) \frac{u}{s}\, (l-r)_{\beta} \Bigg\} + \mathcal{O}(s^{-1/2})
\end{aligned}
\tag{3.25}
$$

The resulting expressions are in this case a bit more complicated, but their scalar product is again quite simple, explicitly

$$
L^{(t1)} \cdot L^{(t2)} = \frac{1}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} \left\{ \frac{t^2}{4}(s-u) - t(s-u)\,\overline{m}^2 + \frac{t^3}{s} 2\,\overline{m}^2 \right\} + \mathcal{O}(s)\,. \tag{3.26}
$$

After substituting the derived expression in (3.16b), expanding the propagator and replacing t and u Mandelstam invariants by the combination of s invariant and $\cos\theta$, we get the final result for the t channel matrix element:

$$
\begin{aligned}
\mathcal{M}^{(t)}_{(\text{diag})} &= \sum_{k\geq 0} \frac{g_{\boldsymbol{eac}} g_{\boldsymbol{ebd}} f^{eac} f^{ebd}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} \Bigg[\left(\frac{s}{4}\right)^2 (3 - 2\cos\theta - \cos^2\theta) + \\
&\quad + \left(\frac{s}{4}\right) \left(- m_{\boldsymbol{e}}^2\, \frac{3 + \cos\theta}{2} + 8\overline{m}^2 \cos\theta \right) \Bigg] + \mathcal{O}(1)
\end{aligned}
\tag{3.27}
$$

3.5 Massless scalar mode exchange in s, t and u channels

From the previous chapter we know that only massless scalar modes are physical in the unitary gauge and they are present only if we impose the boundary conditions (2.14a) on both branes. While studying the gauge invariance we have already written the formula (2.33) for the matrix element of s channel scattering in R_ξ gauge. In this section we are interested only in the massless mode (in unitary gauge), thus we substitute $\xi m_{\boldsymbol{e}}^2 = 0$. Clearly the leading order of divergence of the matrix element is $\mathcal{O}(s)$ and all remaining terms are at most of order $\mathcal{O}(1)$. Substituting for the scalar products of polarizations we arrive at

$$\mathcal{M}^{(s)}_{(\mathrm{scalar})} = \frac{s}{4}\,\frac{2\hat{e}_{\boldsymbol{eab}}\,2\hat{e}_{\boldsymbol{ecd}} f^{eab} f^{ecd}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}}(-1) + \mathcal{O}(1) \tag{3.28a}$$

$$\mathcal{M}^{(t)}_{(\mathrm{scalar})} = \frac{s}{4}\,\frac{2\hat{e}_{\boldsymbol{eac}}\,2\hat{e}_{\boldsymbol{ebd}} f^{eac} f^{ebd}}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}}\,\frac{1-\cos\theta}{2} + \mathcal{O}(1) \tag{3.28b}$$

In the previous chapter we have also learned that the couplings satisfy the relation (2.39) for all massive modes ($k \geq 0$) of color e, thus the combination of $\mathcal{M}^{(s,t,u)}_{(\mathrm{scalar})}$ and $\mathcal{M}^{(s,t,u)}_{(\mathrm{long})}$ gives us the sum over the complete orthonormal set of functions $\widetilde{\varphi}_{e,k}$ in each channel.

3.6 Divergent terms in scattering amplitudes

3.6.1 Quartic divergences

The terms growing as the fourth power of energy in the s channel matrix element (3.23) will cancel against the terms from the matrix element (3.12) of contact four boson interaction, if the equality

$$\mathrm{g}^2_{\boldsymbol{abcd}} = \sum_{k\geq 0} \mathrm{g}_{\boldsymbol{eab}}\mathrm{g}_{\boldsymbol{ecd}} \tag{3.29}$$

is satisfied. To show that all quartic divergences from s, t and u channels cancel out against the corresponding divergent terms of the contact four boson interaction, it is sufficient to prove only (3.29). Since the coupling $\mathrm{g}^2_{\boldsymbol{abcd}}$ is fully symmetric under an arbitrary index permutation, the cancellation of quartic divergences in other channels follows trivially from (3.29).

We can use the definitions of couplings (2.25) and the completeness of the set of functions $\varphi_{e,k}$ to demonstrate that the equality is indeed satisfied:

$$\begin{aligned}
&\sum_{k\geq 0} g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} = \\
&= g_5^2 \int_0^{\pi R} \mathrm{d}y \int_0^{\pi R} \mathrm{d}z\, \varphi_{\boldsymbol{a}}(y)\, \varphi_{\boldsymbol{b}}(y)\, \varphi_{\boldsymbol{c}}(z)\, \varphi_{\boldsymbol{d}}(z) \sum_{k\geq 0} \varphi_{\boldsymbol{e}}(y)\, \varphi_{\boldsymbol{e}}(z) = \\
&= g_5^2 \int_0^{\pi R} \mathrm{d}y \int_0^{\pi R} \mathrm{d}z\, \varphi_{\boldsymbol{a}}(y)\, \varphi_{\boldsymbol{b}}(y)\, \varphi_{\boldsymbol{c}}(z)\, \varphi_{\boldsymbol{d}}(z)\, \delta(y-z) = \\
&= g_5^2 \int_0^{\pi R} \mathrm{d}y\, \varphi_{\boldsymbol{a}}(y)\, \varphi_{\boldsymbol{b}}(y)\, \varphi_{\boldsymbol{c}}(y)\, \varphi_{\boldsymbol{d}}(y) = g_{\boldsymbol{abcd}}^2
\end{aligned} \tag{3.30}$$

3.6.2 Quadratic divergences

Besides the sum rule (3.30), we obviously need to derive two more rules that allow us to get rid completely of the summations over the KK modes in the scattering amplitudes.

The combination of matrix elements $\mathcal{M}^{(s,t,u)}_{\text{(scalar)}}$ and $\mathcal{M}^{(s,t,u)}_{\text{(long)}}$ gives rise to the sum of terms containing two couplings of the type $\hat{e}_{\boldsymbol{eab}}$. The index of KK mode is present only through those couplings; thus remembering that the wave functions $\widetilde{\varphi}_{e,k}$ form a complete orthonormal set, we can write down the first sum rule:

$$\begin{aligned}
&\sum_{k\geq 0} 2\hat{e}_{\boldsymbol{eab}}\, 2\hat{e}_{\boldsymbol{ecd}} = \\
&= g_5^2 \sum_{k\geq 0} \int_0^{\pi R} \mathrm{d}y\, \widetilde{\varphi}_{\boldsymbol{e}}\, (\varphi_{\boldsymbol{a}}\, \varphi_{\boldsymbol{b}}' - \varphi_{\boldsymbol{a}}'\, \varphi_{\boldsymbol{b}}) \int_0^{\pi R} \mathrm{d}z\, \widetilde{\varphi}_{\boldsymbol{e}}\, (\varphi_{\boldsymbol{c}}\, \varphi_{\boldsymbol{d}}' - \varphi_{\boldsymbol{c}}'\, \varphi_{\boldsymbol{d}}) = \\
&= g_5^2 \int_0^{\pi R} \mathrm{d}y\, (\varphi_{\boldsymbol{a}}\, \varphi_{\boldsymbol{b}}' - \varphi_{\boldsymbol{a}}'\, \varphi_{\boldsymbol{b}})\, (\varphi_{\boldsymbol{c}}\, \varphi_{\boldsymbol{d}}' - \varphi_{\boldsymbol{c}}'\, \varphi_{\boldsymbol{d}})
\end{aligned} \tag{3.31}$$

The second type of sum contains the KK index not only in the couplings, but also in the mass of an exchanged vector mode, explicitly

$$\begin{aligned}
&\sum_{k\geq 0} m_{\boldsymbol{e}}^2\, g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} = \sum_{k\geq 0} g_5^2 m_{\boldsymbol{e}}^2 \int_0^{\pi R} \mathrm{d}y\, \varphi_{\boldsymbol{a}}\, \varphi_{\boldsymbol{b}}\, \varphi_{\boldsymbol{e}} \int_0^{\pi R} \mathrm{d}z\, \varphi_{\boldsymbol{c}}\, \varphi_{\boldsymbol{d}}\, \varphi_{\boldsymbol{e}} = \\
&= \sum_{k\geq 0} (-g_5^2) \int_0^{\pi R} \mathrm{d}y\, \varphi_{\boldsymbol{a}}\, \varphi_{\boldsymbol{b}}\, \varphi_{\boldsymbol{e}}'' \int_0^{\pi R} \mathrm{d}z\, \varphi_{\boldsymbol{c}}\, \varphi_{\boldsymbol{d}}\, \varphi_{\boldsymbol{e}} = \\
&= \sum_{k\geq 0} (-g_5^2) \int_0^{\pi R} \mathrm{d}y\, (\varphi_{\boldsymbol{a}}\, \varphi_{\boldsymbol{b}})'' \varphi_{\boldsymbol{e}} \int_0^{\pi R} \mathrm{d}z\, \varphi_{\boldsymbol{c}}\, \varphi_{\boldsymbol{d}}\, \varphi_{\boldsymbol{e}} + [\ldots]_0^{\pi R}
\end{aligned} \tag{3.32}$$

In the first equality we have used the couplings definitions (2.25), then in the next step we have applied the relation for eigenvalues (2.17) in order to get rid of the mass $m_{\boldsymbol{e}}$. And finally we have used the partial integration and denoted the boundary terms simply by dots; they are once again of the type $\varphi_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\,\varphi'_{\boldsymbol{e}}$ and thus vanish for all consistent sets of boundary conditions. Now we have the expression in a form suitable for using the completeness relation for the set of wave functions $\varphi_{\boldsymbol{e}}$, which leaves us with

$$\begin{aligned}\sum_{k\geq 0} m_{\boldsymbol{e}}^2\, g_{\boldsymbol{eab}} g_{\boldsymbol{ecd}} &= -g_5^2 \int_0^{\pi R} \mathrm{d}y\, \left(\varphi_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\right)'' \varphi_{\boldsymbol{c}}\,\varphi_{\boldsymbol{d}} = \\ &= g_5^2 \int_0^{\pi R} \mathrm{d}y\, \left(\varphi_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\right)' \left(\varphi_{\boldsymbol{c}}\,\varphi_{\boldsymbol{d}}\right)' + \left[\left(\varphi_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\right)' \varphi_{\boldsymbol{c}}\,\varphi_{\boldsymbol{d}}\right]_0^{\pi R}.\end{aligned} \tag{3.33}$$

In order to obtain a similar structure under the integral as in the previous sum rule, we have integrated by parts one more time. Note that the whole sum in the scattering amplitude is proportional to $f^{abe} f^{cde}$, thus the structure of SU(2) group requires either $a = c$ and $b = d$, or $a = d$ and $b = c$. This implies that the boundary term contains the product of the first derivative and the value of wave functions with the same boundary conditions; thus the whole boundary term is zero.

Similarly, one can use the relation for eigenvalues (2.17) and integrate by parts without picking up any non-zero boundary terms to show

$$4\overline{m}^2 g_{\boldsymbol{abcd}}^2 = 2g_5^2 \int_0^{\pi R} \mathrm{d}y\, \left(\varphi_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\right)' \left(\varphi_{\boldsymbol{c}}\,\varphi_{\boldsymbol{d}}\right)' + \varphi_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\,\varphi'_{\boldsymbol{c}}\,\varphi'_{\boldsymbol{d}} + \varphi'_{\boldsymbol{a}}\,\varphi'_{\boldsymbol{b}}\,\varphi_{\boldsymbol{c}}\,\varphi_{\boldsymbol{d}}\,. \tag{3.34}$$

If we gather all the remaining divergent terms from the contact four boson interaction (3.12) and s, t, u channel exchange of vector and scalar modes (3.15), (3.23), (3.27) and (3.28), apply the derived sum rules and the relation (3.34), we can express the overall invariant matrix element of the process in quite a simple form given by

$$\begin{aligned}\mathcal{M} = \frac{s}{4} \frac{g_5^2}{m_{\boldsymbol{a}} m_{\boldsymbol{b}} m_{\boldsymbol{c}} m_{\boldsymbol{d}}} &\left(f^{abe} f^{cde} - f^{ace} f^{bde} + f^{ade} f^{bce}\right) \times \\ &\times \Bigg\{ (1 - 3\cos\theta) \int_0^{\pi R} \mathrm{d}y\, \left(\varphi'_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\,\varphi_{\boldsymbol{c}}\,\varphi'_{\boldsymbol{d}} + \varphi_{\boldsymbol{a}}\,\varphi'_{\boldsymbol{b}}\,\varphi'_{\boldsymbol{c}}\,\varphi_{\boldsymbol{d}}\right) \\ &\quad - (1 + 3\cos\theta) \int_0^{\pi R} \mathrm{d}y\, \left(\varphi'_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\,\varphi'_{\boldsymbol{c}}\,\varphi_{\boldsymbol{d}} + \varphi_{\boldsymbol{a}}\,\varphi'_{\boldsymbol{b}}\,\varphi_{\boldsymbol{c}}\,\varphi'_{\boldsymbol{d}}\right) \\ &\quad - 2\cos\theta \int_0^{\pi R} \mathrm{d}y\, \left(\varphi'_{\boldsymbol{a}}\,\varphi'_{\boldsymbol{b}}\,\varphi_{\boldsymbol{c}}\,\varphi_{\boldsymbol{d}} + \varphi_{\boldsymbol{a}}\,\varphi_{\boldsymbol{b}}\,\varphi'_{\boldsymbol{c}}\,\varphi'_{\boldsymbol{d}}\right) \Bigg\} + \mathcal{O}(1)\end{aligned} \tag{3.35}$$

The whole divergent part of the scattering amplitude is proportional to the expression $f^{abe}f^{cde} - f^{ace}f^{bde} + f^{ade}f^{bce}$. However, this is zero due to the *Jacobi identity*.

Thus we conclude that $2 \to 2$ scattering amplitude of longitudinal gauge bosons contains no terms growing indefinitely with the energy. We have shown this fact without any assumptions regarding the colors or KK mode numbers of the gauge bosons in the initial and final state.

3.7 Elastic scattering of vector bosons

We would like to review briefly the case of elastic scattering of two identical longitudinal vector modes studied in [16]. The important assumption here is that all fields in the initial and final state satisfy the same boundary conditions and have the same KK mode number. Thus all four bosons have the same wave function and mass. In such a case the sum rules and general expressions for the invariant matrix elements derived in the previous sections may be significantly simplified.

From the couplings definitions (2.25) we can see that the coupling in the *vector-vector-scalar* vertex is zero, if the vector modes have the same wave functions. Thus in this case there will be no contribution from the exchange of a massless scalar. There will be no contribution from $\mathcal{M}^{(s,t,u)}_{\text{(long)}}$ either, because of the equality of masses $m_{\boldsymbol{a}} = m_{\boldsymbol{b}} = m_{\boldsymbol{c}} = m_{\boldsymbol{d}}$.

To reproduce the results from [16] let us denote n the mode number of the four bosons in the initial and final state, m_n and φ_n their mass and wave function. Obviously they all carry the same energy $E = \sqrt{s/4}$. Although the exchanged vector mode may come from the KK tower with a different set of boundary conditions, we keep its color e only in the structure constants and denote it only by the KK mode number k.

The contact four boson coupling g^2_{nnnn} is a simple integral of the fourth power of φ_n. And using the Jacobi identity, we can rewrite the matrix element (3.12) of the contact interaction as

$$\begin{aligned}
\mathcal{M}^{(4\text{V})}_{\text{(elastic)}} = \left(\frac{E}{m_n}\right)^4 g^2_{nnnn} \Big[& f^{eab}f^{ecd}(3 + 6\cos\theta - \cos^2\theta) + \\
& + f^{eac}f^{ebd}(-6 + 2\cos^2\theta) \Big] + \\
+ \left(\frac{E}{m_n}\right)^2 g^2_{nnnn} & \left[f^{eab}f^{ecd}(-2 - 6\cos\theta) + 4\, f^{eac}f^{ebd} \right] + \mathcal{O}(1)
\end{aligned} \tag{3.36}$$

The coupling g_{nnk} in a three boson vertex is the same for all channels; this is again the consequence of the fact that all bosons in the initial and final state have the same wave function regardless of their color. The sum of matrix elements of all three channels, again using the Jacobi identity, is given by

$$\begin{aligned}\mathcal{M}^{(3\mathrm{V})}_{(\mathrm{elastic})} = \left(\frac{E}{m_n}\right)^4 \sum_k (g_{nnk})^2 \Big[& f^{eab} f^{ecd} (-3 - 6\cos\theta + \cos^2\theta) + \\ & + f^{eac} f^{ebd} (6 - 2\cos^2\theta) \Big] + \\ + \left(\frac{E}{m_n}\right)^2 \sum_k (g_{nnk})^2 \Big[& f^{eab} f^{ecd} \Big(\frac{3}{2}\frac{m_k^2}{m_n^2}(1-\cos\theta) + 8\cos\theta\Big) + \\ & + f^{eac} f^{ebd} \Big(-3\frac{m_k^2}{m_n^2}\Big)\Big] + \mathcal{O}(1)\end{aligned} \tag{3.37}$$

These results agree with the expression presented in [16]. The sum rule (3.29) can be in this case written as

$$\sum_k (g_{nnk})^2 = g^2_{nnnn} \,. \tag{3.38}$$

To cancel all the divergent terms in the scattering amplitude of elastic process, we obviously do not need the sum rule (3.31) as there are no summations of this type. On the other hand we need the sum rule (3.33) and we can recast its right-hand side using the mass m_n and the coupling of four boson interaction. With the formula (3.34) one can work out the sum rule as

$$\sum_k m_k^2 (g_{nnk})^2 = 4 g_5^2 \int_0^{\pi R} \mathrm{d}y \; (\varphi_n \, \varphi_n')^2 = \frac{4}{3} m_n^2 \, g^2_{nnnn} \,. \tag{3.39}$$

Thus we have also reproduced both of the sum rules presented in [16] from our more general formulas.

Chapter 4

Conclusions

We have demonstrated how the electroweak symmetry breaking may be achieved in the higher dimensional theory of Kaluza-Klein type. A class of simple models was used to show the main ideas of the theory in the vector boson sector. The consistent set of boundary conditions was selected by the interval approach to the extra space dimension in combination with the principle of least action and the requirement of gauge invariant scattering amplitudes. The important lesson is the possibility of constructing a higgsless model that is unitary at tree-level. This has been demonstrated by the explicit calculation of the scattering amplitude of longitudinal vector bosons. In contrast with other available literature, where results of this calculation are usually presented only for somehow simplified cases[1], we have calculated the scattering amplitude of a generally inelastic process without any assumptions regarding the colors or KK mode numbers of gauge bosons in the initial and final state. We have presented the intermediate steps and technical details of all the relevant calculations, thus this work may serve as an instructive introduction to the given problematics for a reader without any prior knowledge of higher dimensional theories.

Although the studied class of models provides a good insight into the theory, it is far from a realistic model that could replace the Standard Model. In order to get a complete theory one obviously needs to incorporate the quark and lepton sector [22]. Another pitfall is the requirement to fit the model's predictions to the electroweak precision tests while keeping the new physics hidden from the detection on the accelerators prior to LHC. In consequence, when constructing a realistic higher dimensional model, one first has to abandon the idea of a flat extra dimension and use the warped five dimensional spacetime, explicitly the anti de Sitter background. Another required change regards the need for the custodial

[1]Most commonly is assumed that all gauge fields satisfy the same boundary conditions, or the case of an elastic boson scattering.

symmetry to get the right ratio of the W and Z boson masses, thus a more complex gauge structure must be set in place. One possibility was presented in [18], namely the symmetry pattern $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$. This symmetry suggests that each of the effective 4D gauge fields in fact lives in several fundamental higher dimensional fields.

Besides the possibility of a viable higgsless model there is another approach to the hierarchy problem arising from the higher dimensions. The *Gauge-Higgs Unification* models use the extra components of the higher dimensional gauge fields to give rise to an effective 4D Higgs field [23–25].

The higher dimensional models have become increasingly popular in the past decade, because they offer an elegant solution to the hierarchy problem and allow to incorporate the mechanisms of the Standard Model without introducing an excessive amount of new parameters to the theory. Currently there is no higher dimensional candidate as a complete replacement of the Standard Model, there still remain many unresolved problems and questions. Nevertheless we should keep our eyes opened and look for the signs of higher dimensions at LHC [6, 26].

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