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# Finite-Size Effects for Classical Lattice Models

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I declare that I did this work on my own using only the stated  
bibliography.

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## Preface

This thesis was worked out on grounds of the results obtained during my post-graduate studies at the Faculty of Mathematics and Physics, Charles University, Prague, from September 1996 till October 2000 and during my stay at Nuclear Physics Institute, Academy Sciences of the Czech Republic, Řež near Prague, between October 1999 and October 2000.

A successful finishing of the thesis would not be possible without assistance of the people who supported me, directly or indirectly, during the whole period. Here I wish to express my sincere and deep thankfulness to all of them.

The first person whom I would like to thank is my supervisor Roman Kotecký. His guidance, views, and insights proved to be invaluable. It was a pleasure for me to be a doctoral student with him, and I hope our collaboration will continue. I am also indebted to Miloš Zahradník, my consultant, for advice and discussions about various, not only scientific topics. Jiří Hošek strongly encouraged me in the last, crucial year of my work, and made the year I spent in Řež a wonderful time.

A part of the thesis was inspired by interactions with people outside the Czech Republic. I am delighted to acknowledge the hospitality and friendly treatment of Enzo Olivieri during my two unforgettable stays in Rome. I am particularly grateful to Yvan Velenik for many fruitful and stimulating discussions we had in Prague, Marseille, and especially in Berlin. I want to thank Jean-Dominique Deuschel for making my two visits to Berlin possible. A completely new experience was the time at Microsoft Research in Seattle, where I was invited by Christian Borgs and Jennifer Chayes. Although I only spent five days in Vancouver, I shall always remember Cindy Greenwood — her hospitality, attention, and encouragement were really mighty.

The research itself would not have been sufficient for accomplishing the thesis. I must not forget the time of my life I have had with my friends and colleagues. Their list is long. Let me thus only name those with whom I shared my office(s): Karel Netočný and Tomáš Novotný; Katka Němcová, David Krejčířík, Hynek Kovařík, and Honza Kríž. At this place, I would also like to give a thought

to the AFC Řež whose soccer matches, the third halves in particular, were sometimes incredibly exhausting but so splendid.

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## Contents

Preface	v
Chapter I. Introduction	1
I.1. Preliminaries	1
I.2. Layout of the Thesis	4
I.3. Finite-Size Effects: A Non-Technical Survey	4
Bibliography	13
Chapter II. Techniques	17
II.1. Pirogov-Sinai Theory	17
II.2. Large Deviations	35
II.3. Convex Analysis	40
Bibliography	46
Chapter III. Finite-size effects for the Potts model with weak boundary conditions	49
III.1. Introduction	49
III.2. Results	51
III.3. Contour Representations	56
III.4. Proof of Theorem III.2.2	63
III.5. Proof of Theorem III.2.1	75
III.A. Auxiliary Lemmas	79
III.B. Proof of Lemma III.4.1	86
III.C. Proof of Corollary III.4.4	94
Bibliography	100
Chapter IV. Finite-Size Scaling for the 2D Ising Model with Fixed Boundary Conditions	103
IV.1. Introduction	103
IV.2. Main Result	106
IV.3. Magnetization and the Large-Deviation Rate Function	108
IV.4. Local-Limit Estimates	122
IV.A. Technical Lemmas	133
Acknowledgments	138
Bibliography	138
Chapter V. Finite-Size Effects and Large Deviations: Some Generalities	141
V.1. The Setting	141

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V.2. Main Result	146
Bibliography	150

## CHAPTER I

### Introduction

#### I.1. Preliminaries

Phase transitions are determined by non-analyticities of the free energy density. When such a non-analyticity is caused by a discontinuity of a first-order derivative, one speaks of a first-order phase transition. A non-analytic behaviour of the free energy density with its first-order derivatives kept continuous corresponds to a continuous phase transition. Here we shall only consider the former case. Accordingly, the term phase transitions will henceforth mean phase transitions of the first order.

The very introduction of the phase transitions is somewhat contradictory. On the one hand, it relies on properties of the free energy density, a quantity entirely characterizing only the *idealized*, infinitely large system (the thermodynamic limit). On the other hand, real systems are always finite. However, the finite-volume free energy is, as a rule, analytic, and no discontinuities can therefore appear in its first-order derivatives (like the magnetization or the mean energy) whatsoever. Rather, the infinite-volume jumps are smoothed out into rounded transitions (the larger the system is, the more abrupt the transition becomes), whose positions are, in general, shifted with respect to those of the infinite-volume jumps, see Fig. 1. Moreover, one may also analyze the corresponding second-order (the susceptibility, the heat capacity, etc.) and higher-order derivatives; the singularities of the  $\delta$ -function type which these quantities have in the thermodynamic limit change into sharp peaks as soon as a finite volume is considered. The points where the peaks are maximal are suitable candidates to describe the above-mentioned shifts of rounded transitions: at these points the transitions are the steepest. The phenomena connected with the asymptotic behaviour of finite systems are commonly referred to as the *finite-size effects* (at or near first-order phase transitions in our case). It is their inherent feature that they substantially depend on boundary conditions (the influence of the environment).

The study of the finite-size effects appears as a quite natural task. To this end, one constructs — as is usual in statistical mechanics — simplified microscopic models enjoying all the properties necessary for explaining their behaviour from the macroscopic point of view.

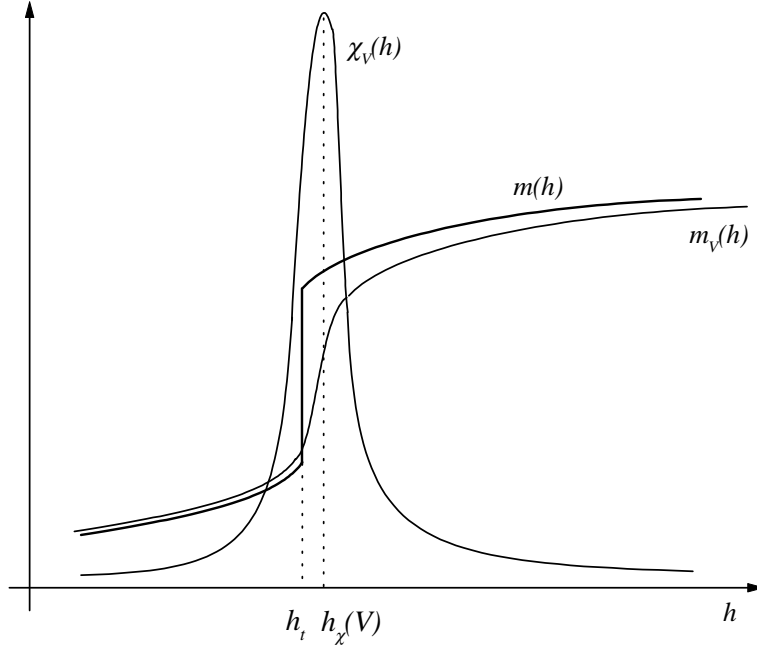


FIGURE 1. While the infinite-volume magnetization  $m(h)$  has a discontinuity at  $h_t$ , its finite-volume counterpart  $m_V(h)$  is continuous with a rounding steepest at  $h_\chi(V)$  — the maximum of the finite-volume susceptibility  $\chi_V(h)$ . In general, the point  $h_\chi(V)$  is shifted with respect to  $h_t$ .

Perhaps the most popular systems of this sort are *classical lattice models*, and in this thesis we rigorously examine the finite-size effects for the simplest of these. In particular, we are predominantly interested in the Ising and Potts ferromagnets. In order to explain the basic ideas of the finite-size analysis, let us specify a class of (classical lattice) models with which we shall work in the following.

The *lattice* is a countably infinite set of elements called *sites*. Here we always restrict ourselves to the lattice  $\mathbb{Z}^d$ , where  $d \in \mathbb{N}$  is its dimension. In order to characterise the states at a single site of the lattice, one introduces the *single-spin space*  $S$ , which we assume to be a finite set. For instance, the Ising model has  $S = \{-1, 1\}$ , and  $S = \{1, \dots, q\}$ ,  $q < \infty$ , for the Potts model. The *configuration space* of the system is then defined to be  $\Omega = S^{\mathbb{Z}^d}$ , and elements of  $\Omega$  are called *configurations*. In addition, we use  $\omega_x$  to denote the value of the spin at the site  $x \in \mathbb{Z}^d$  corresponding to the configuration  $\omega \in \Omega$ ; hence,  $\omega_x \in S$ .

When working on a subset  $\Lambda$  of  $\mathbb{Z}^d$ , we let  $\omega_\Lambda$  to be the restriction of  $\omega \in \Omega$  to  $\Lambda$ , that is  $\omega_\Lambda := \{\omega_x\}_{x \in \Lambda}$ . In case  $\Lambda$  is finite, we write  $\Lambda \Subset \mathbb{Z}^d$ . A family  $\Phi := \{\Phi_A : \Omega \rightarrow \mathbb{R} \mid A \Subset \mathbb{Z}^d\}$  is a *potential* if, for

each  $A$  under consideration, the function  $\Phi_A$  depends on the spins in  $A$  only. Throughout the thesis we only deal with a potential of *finite range*  $R < \infty$ , i.e. we set  $\Phi_A \equiv 0$  once  $\text{diam } A > R$ . Given any finite-range potential  $\Phi$ , any  $\Lambda \Subset \mathbb{Z}^d$ , and any  $\omega \in \Omega$ , the *Hamiltonian* in  $\Lambda$  associated with  $\Phi$  and with boundary conditions  $\eta \in \Omega$  is

$$H_\Lambda^{(\Phi)}(\omega|\eta) := \sum_{A \in \mathbb{Z}^d: A \not\subset \Lambda^c} \Phi_A(\omega_\Lambda \times \eta_{\Lambda^c}), \quad (\text{I.1.1})$$

where  $\omega_\Lambda \times \eta_{\Lambda^c}$  is the configuration which coincides with  $\omega$  on  $\Lambda$  and with  $\eta$  on  $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$ . The corresponding finite-volume *Gibbs measure* (or distribution or state) at an inverse temperature  $\beta > 0$  is defined as

$$\mu_\Lambda^{(\Phi)}(\eta) := \frac{e^{-\beta H_\Lambda^{(\Phi)}(\omega|\eta)}}{Z_\Lambda^{(\Phi)}(\eta)}; \quad (\text{I.1.2})$$

here  $Z_\Lambda^{(\Phi)}(\eta)$  is the normalization called the *partition function*, i.e. it is the finite sum

$$Z_\Lambda^{(\Phi)}(\eta) := \sum_{\substack{\omega \in \Omega: \\ \omega_{\Lambda^c} = \eta_{\Lambda^c}}} e^{-\beta H_\Lambda^{(\Phi)}(\omega|\eta)}. \quad (\text{I.1.3})$$

Statistical mechanics postulates that the equilibrium state of a (classical lattice) system in  $\Lambda$  interacting with its surroundings described by the configuration  $\eta$  is given by the Gibbs measure (I.1.2). In other words, expectations of all observables of the system in equilibrium are available through this measure. These expectations can usually be expressed as derivatives of the logarithm of the partition function (the finite-volume free energy).

As mentioned at the beginning, in order to describe phase transitions, one takes the thermodynamic limit. This allows to get rid of problems with boundary effects and to retain just the essential information on the original, finite but large system. Then one is interested in the corresponding infinite-volume measures, i.e. those whose conditional probabilities for finite subsystems, conditioned on the outside (boundary conditions), are of the Gibbs form (Dobrushin [Dob68] and Lanford and Ruelle [LR69]). The set  $\mathcal{G}^{(\Phi)}$  of these measures — the DLR-measures — is a simplex and its extreme points describe possible macrostates of the system; any weak limit of the finite-volume Gibbs measure (I.1.2) with arbitrary deterministic or random boundary conditions is an infinite-volume Gibbs measure. Whenever the set  $\mathcal{G}^{(\Phi)}$  is not a mere singleton,  $|\mathcal{G}^{(\Phi)}| > 1$ , the system is said to exhibit a phase transition. It means that there is a certain instability with respect to boundary conditions: a small change on the boundary leads to a dramatic change in the limiting measure on  $\Omega$ . Put another way [Isr79], this may be reformulated

by means of the lack of differentiability of the *free energy density* introduced as the limit

$$f^{(\Phi)} \equiv - \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \log Z_{\Lambda}^{(\Phi)}(\eta) := - \lim_{n \rightarrow \infty} \frac{1}{\beta|\Lambda_n|} \log Z_{\Lambda_n}^{(\Phi)}(\eta), \quad (\text{I.1.4})$$

if it exists. Here  $\{\Lambda_n; \Lambda_n \in \mathbb{Z}^d, \Lambda_n \neq \emptyset\}$  is a sequence tending to infinity in the sense of van Hove, i.e.  $\lim_{n \rightarrow \infty} |\partial_r \Lambda_n|/|\Lambda_n| = 0$  for each  $r > 0$  with  $\partial_r \Lambda_n := \{x \in \Lambda_n^c : \text{dist}(x, \Lambda_n) \leq r\}$ . It is known that for the models which we consider here the free energy density exists and does not depend on the boundary conditions  $\eta$ .

## I.2. Layout of the Thesis

In the next section we give a non-technical survey of the main ideas of the rigorous theory of the finite-size effects near first-order phase transitions worked out by Borgs and Kotecký [BK90, BK95]. After some generalities, we outline our results presented in the rest of the thesis. The next chapter is devoted to brief reviews of the techniques and methods which we shall use for the study of the finite-size effects: the Pirogov-Sinai theory and cluster expansions on the one hand, and large deviations and convex analysis on the other hand. These will be applied to the analysis of the high  $q$ -state Potts model in Chapter III and the two-dimensional Ising model in Chapter IV. Finally, in Chapter V we examine how large-deviation principles may determine, in general, the finite-size behaviour of lattice systems.

Both Chapter III and Chapter IV are actually joint papers (the former with Christian Borgs and Roman Kotecký, the latter with Roman Kotecký) being prepared for publication.

## I.3. Finite-Size Effects: A Non-Technical Survey

**I.3.1. General Results.** The theory of the finite-size effects near first-order phase transitions goes back to the work of Imry [Imr80] and, in the sequel, of Fisher and Berker [FB82], Blöte and Nightingale [BN81], Binder and co-workers [Bin81, BL84, CLB86], Privman and Fisher [PF83], and others. For a system with two coexisting phases which are related by a symmetry  $h \leftrightarrow -h$  with respect to the ordering field  $h$  (like the Ising model) in a cube of the size  $L^d$  under the *periodic boundary conditions*, these lead to the prediction that the magnetization

$$m_{\text{per}}(h, L) := \frac{1}{\beta L^d} \frac{d \log Z_{\text{per}}(h, L)}{dh} \quad (\text{I.3.1})$$

behaves as

$$m_{\text{per}}(h, L) \sim M \tanh(\beta M h L^d), \quad (\text{I.3.2})$$

where  $Z_{\text{per}}(h, L)$  is the partition function of the system,  $M$  is the (infinite-volume) spontaneous magnetization,  $\beta$  is the inverse temperature, and  $d$  is the dimension. A formula of the form (I.3.2) was also obtained for models without a symmetry relating  $h$  to  $-h$ . In the latter case, however, controversies on the shift of the susceptibility maximum with respect to the infinite-volume transition point  $h_t$  appeared. Similar (although less dramatic) controversies arose for temperature-driven first-order phase transitions as well, an example being the  $q$ -state Potts model, where  $q$  ordered low-temperature phases coexists with one disordered high-temperature phase.

The theory was later systematized in a rigorous framework by Borgs and Kotecký [BK90, BKM91]. Their results cover the finite-size effects for systems describing the coexistence of a finite number of phases with both field- and temperature-driven transitions.<sup>1</sup> As the main tool, they used the Pirogov-Sinai theory of first-order phase transitions [PS75, PS76, Zah84, BI89], and completely succeeded in resolving the above-mentioned controversies. In addition, they suggest new parameters to locate the transition point from numerical simulations. Their key point [BK90] consists in representing the partition function as (c.f. Theorem II.1.15)

$$Z_{\text{per}}(h, L) = \left( \sum_{m=1}^N e^{-\beta f_m(h) L^d} \right) [1 + O(e^{-\xi \beta L})] \quad (\text{I.3.3})$$

for some  $\xi > 0$ , where  $N$  is the number of phases. The central task concerning this formula is to introduce suitable metastable free energies  $f_1(h), \dots, f_N(h)$  so that  $f_m(h)$  coincides with the free energy  $f(h)$  of the model once the phase  $m$  is stable, while  $f_m(h) > f(h)$  if  $m$  is unstable. In addition, the metastable free energies are introduced to be differentiable to sufficient an order (although not analytic). Using (I.3.3), the finite-size behaviour of various quantities can be evaluated. For instance, in the case of the coexistence of two phases  $+$  and  $-$  ( $N = 2$ ), the magnetization (I.3.1) and the susceptibility  $\chi_{\text{per}}(h, L) := \frac{dm_{\text{per}}(h, L)}{dh}$  scale like

$$m_{\text{per}}(h, L) \sim \frac{m_+ + m_-}{2} + \frac{m_+ - m_-}{2} \tanh \left\{ \beta \frac{m_+ - m_-}{2} (h - h_{\chi_{\text{per}}}(L)) L^d \right\} \quad (\text{I.3.4})$$

and

$$\chi_{\text{per}}(h, L) \sim \left( \frac{m_+ - m_-}{2} \right)^2 L^d \cosh^{-2} \left\{ \beta \frac{m_+ - m_-}{2} (h - h_{\chi_{\text{per}}}(L)) L^d \right\} \quad (\text{I.3.5})$$

<sup>1</sup>Besides the cubic volumes, Borgs and Imbrie [BI92a, BI92b, Bor92] investigated systems in long cylinders.

when  $|h - h_t| \leq O(L^{-1})$  and  $L$  is large, the errors being of the order  $L^{-1}$  and  $L^{d-1}$ , respectively. Here  $m_+$  and  $m_-$  are the infinite-volume magnetization of the phase  $+$  and  $-$ , respectively, at the infinite-volume transition point  $h_t$  and  $h_{\chi_{\text{per}}}(L)$  is the unique point at which the susceptibility  $\chi_{\text{per}}(h, L)$  attains its maximum. For the remaining values of the field  $h$ , one may show that  $m_{\text{per}}(h, L)$  and  $\chi_{\text{per}}(h, L)$  differ from their infinite-volume counterparts by a term  $O(e^{-\xi\beta L})$ . Notice that the rounding of the infinite-volume transition takes place in a region of the width  $L^{-d}$ . Moreover, the shift of  $h_{\chi_{\text{per}}}(L)$  with respect to  $h_t$  can be explicitly found: it turns out to be of the order  $L^{-2d}$ . We formulate the statements precisely in Corollary II.1.16.

The periodic boundary conditions do not allow the description of the finite-size effects in real systems, where the influence of the surface plays a major role. In order to extend the theory to these situations, Borgs and Kotecký [BK95] studied cubic systems with the free or, more generally, with the *weak boundary conditions* — those which do not strongly favour any of the considered phases near the surface. This is necessary for applying their techniques, based again on the Pirogov-Sinai theory. This time, when the volume has a boundary, one must also take into account the metastable surface free energies  $f_m^{(d-1)}(h)$ ,  $m = 1, \dots, N$ , and the corresponding partition function can now be expressed as (c.f. Theorem II.1.18)

$$Z_{\text{weak}}(h, L) = \left( \sum_{m=1}^N e^{-\beta F_m(h, L)} \right) [1 + O(e^{-\xi\beta L})] \quad (\text{I.3.6})$$

with

$$F_m(h, L) = f_m(h)L^d + 2df_m^{(d-1)}(h)L^{d-1} + O(L^{d-2}). \quad (\text{I.3.7})$$

The metastable surface free energies are typically (for asymmetric transitions) different at  $h_t$ , which is the source of the main difference in comparison with the periodic boundary conditions. Considering the two-phase coexistence ( $N=2$ ), the formulas for the finite-volume magnetization  $m_{\text{weak}}(h, L)$  and susceptibility  $\chi_{\text{weak}}(h, L)$  when  $L$  is large and  $|h - h_t| \leq O(L^{-1})$  look formally the same as those from (I.3.4) and (I.3.5), the width of the transition being thus still proportional to  $L^{-d}$ . However, the shift of the susceptibility maximum

$$h_{\chi_{\text{weak}}}(L) = h_t + \frac{f_+^{(d-1)}(h) - f_-^{(d-1)}(h)}{m_+ - m_-} \frac{2d}{L} + O(L^{-2}). \quad (\text{I.3.8})$$

with respect to  $h_t$  is now proportional to  $L^{-1}$ . Similarly, the difference between  $m_{\text{weak}}(h, L)$  and  $\chi_{\text{weak}}(h, L)$  and their infinite-volume counterparts is of the order  $L^{-1}$  whenever  $h$  is beyond the interval

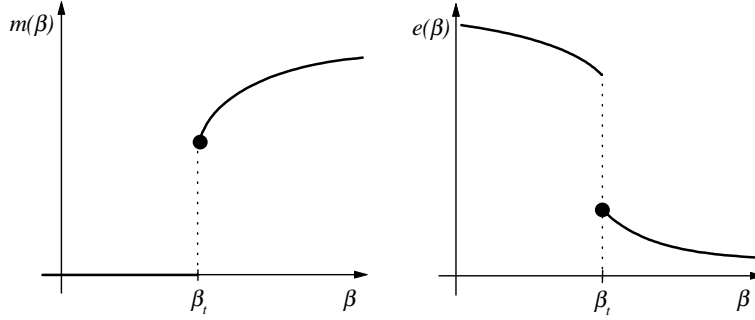


FIGURE 2. The infinite-volume magnetization and the mean energy for the high  $q$ -state Potts model. Notice that  $m(\beta) = 0$  for all  $\beta < \beta_t$ .

$|h - h_t| \leq O(L^{-1})$ . These results are precisely formulated in Corollary II.1.20.

**I.3.2. Review of the Presented Results.** Our first aim is to use the methods developed in [BK95] to study the finite-size effects for the ferromagnetic high  $q$ -state *Potts model* in a cube  $\Lambda \subset \mathbb{Z}^d$  of the size  $L^d$ ,  $d \geq 2$ , with boundary conditions interpolating between the free and the constant 1-boundary conditions. The configurations of this model are maps  $\sigma_\Lambda$  from  $\Lambda$  into  $S = \{1, \dots, q\}$ , where  $q < \infty$ . Using  $\mathbb{B} = \mathbb{B}(\Lambda)$  to denote the set of all bonds  $\langle x, y \rangle$  of nearest-neighbour sites  $x, y \in \mathbb{Z}^d$  with both end-points in  $\Lambda$  and  $\partial\mathbb{B} = \partial\mathbb{B}(\Lambda)$  to denote the set of all bonds  $\langle x, y \rangle$  such that  $x \in \Lambda$  and  $y \in \mathbb{Z}^d \setminus \Lambda$ , the corresponding Hamiltonian is given by

$$H^{(\lambda)}(\sigma_\Lambda) = -J \sum_{\langle x, y \rangle \in \mathbb{B}} \delta_{\sigma_x, \sigma_y} - \lambda \sum_{\substack{\langle x, y \rangle \in \partial\mathbb{B}: \\ x \in \Lambda}} \delta_{\sigma_x, 1}, \quad (\text{I.3.9})$$

where  $J > 0$  is the bulk coupling and  $\lambda \geq 0$  is the surface coupling. The value  $\lambda = 0$  represents the free boundary conditions, while  $\lambda = J$  represents the standard 1-boundary conditions. It is well known by now that for all  $d \geq 2$  and all  $q \geq 2$  the infinite-volume system exhibits a phase transition at some value  $\beta_t$  of the inverse temperature characterized by the appearance of a spontaneous magnetization whenever  $\beta > \beta_t$ . For  $q$  sufficiently large, this transition is known [KS82] to be first-order with a discontinuity in both the magnetization  $m(\beta)$  and the mean energy  $e(\beta)$  (see (III.2.5) and (III.2.6), respectively, for their definitions), c.f. Fig. 2.

The starting question to be answered is what boundary conditions are weak: while the free boundary conditions favour the disordered phase near the surface, the standard 1-boundary conditions favour the ordered phase. It turns out [Med96] that the weak boundary conditions correspond to  $\lambda \sim J/2$ . A direct application of the

general theory [BK95] to the considered model was done in [Med96]. However, in order to satisfy the assumptions under which the methods from [BK95] can be used, one needs to impose drastic constraints on the values of  $\lambda$  and  $\beta$ . Namely, it is necessary that  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \delta$  and  $|\frac{\beta}{\beta_t} - 1| \leq \delta$ , where  $\delta = \delta(d) < \frac{1}{144d}$ .

Our main contribution to this problem is the analysis of the asymptotic behaviour (as  $L \rightarrow \infty$ ) of the finite-volume magnetization  $M_L(\beta, \lambda)$  and the finite-volume mean energy  $E_L(\beta, \lambda)$  (for their definitions, see (III.2.7) and (III.2.8), respectively) for *any*  $\lambda \geq 0$  and  $q$  large. It turns out that the behaviour for  $\lambda \in (0, \frac{J}{2})$  and  $\beta \leq \beta_t$  is qualitatively the same as that for the free boundary conditions: the specific magnetization  $\frac{1}{|\Lambda(L)|} M_L(\beta, \lambda)$  and the specific mean energy  $\frac{1}{|\Lambda(L)|} E_L(\beta, \lambda)$  still converge to the bulk quantities in the disordered phase with corrections of the order  $L^{-1}$ . Similarly, for  $\lambda \in (\frac{J}{2}, \infty)$  and  $\beta \geq \beta_t$ , we are still in the ordered phase. Finite-size behaviour for intermediate values of  $\lambda$  — around  $\lambda = \frac{J}{2}$ , the weak boundary conditions — and any  $\beta > 0$  is governed by the competition between contributions coming from the configurations which are either in the ordered or in the disordered phase for the whole of  $\Lambda$ . Surface effects, in dependence on the particular value of  $\lambda$ , then determine the resulting finite-size rounding of the phase transition. Introducing the specific heat  $C_L(\beta, \lambda) := \beta^2 \frac{dE_L(\beta, \lambda)}{d\beta}$  and  $m^* = m(\beta_t)$ ,

$$e_0 = \left[ \lim_{\beta \rightarrow \beta_t - 0} e(\beta) + \lim_{\beta \rightarrow \beta_t + 0} e(\beta) \right] / 2, \quad (\text{I.3.10})$$

and

$$\Delta e = \left[ \lim_{\beta \rightarrow \beta_t - 0} e(\beta) - \lim_{\beta \rightarrow \beta_t + 0} e(\beta) \right] / 2, \quad (\text{I.3.11})$$

our principal results, when  $\lambda$  is around  $J/2$ , are as follows. Let  $d \geq 2$  and  $J > 0$ . Choosing  $0 \leq \mu < 1$ , one considers the interval  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$  and defines  $\nu := \frac{1}{24d} \min\{1, 3(1 - \mu)\} > 0$ . For  $q$  and  $L$  large (depending on  $d, J$ , and  $\mu$ ), the following is true:

- (1) There exists a unique point  $\beta_{\max}^{(\lambda)}(L)$  at which the specific heat  $C_L(\beta, \lambda)$  attains its maximum, and

$$\beta_{\max}^{(\lambda)}(L) = \beta_t \left[ 1 + \frac{d}{\Delta e} \left( \frac{J}{2} - \lambda + O\left(\frac{q^{-\nu}}{\log q}\right) \right) \frac{1}{L} + O(L^{-2}) \right]. \quad (\text{I.3.12})$$

(2) If  $\frac{|\beta - \beta_t|}{\beta_t} \leq \frac{8dJ}{5\Delta e} \frac{1}{L}$ , then

$$M_L(\beta, \lambda) = \frac{m^*}{2} L^d + \frac{m^*}{2} L^d \tanh(\Delta e (\beta - \beta_{\max}^{(\lambda)}(L)) L^d) + O(L^{d-1}), \quad (\text{I.3.13})$$

$$E_L(\beta, \lambda) = e_0 L^d - \Delta e L^d \tanh(\Delta e (\beta - \beta_{\max}^{(\lambda)}(L)) L^d) + O(L^{d-1}), \quad (\text{I.3.14})$$

and

$$C_L(\beta, \lambda) = \beta^2 (\Delta e)^2 L^{2d} \cosh^{-2}(\Delta e (\beta - \beta_{\max}^{(\lambda)}(L)) L^d) + O(L^{2d-1}). \quad (\text{I.3.15})$$

(3) If  $\frac{|\beta - \beta_t|}{\beta_t} > \frac{8dJ}{5\Delta e} \frac{1}{L}$ , then

$$M_L(\beta, \lambda) = m(\beta) L^d + O(L^{d-1}), \quad (\text{I.3.16})$$

$$E_L(\beta, \lambda) = e(\beta) L^d + O(L^{d-1}). \quad (\text{I.3.17})$$

In addition, there exists the derivative  $c(\beta) = -\beta^2 \frac{de(\beta)}{d\beta}$ , and

$$C_L(\beta, \lambda) = c(\beta) L^d + O(L^{d-1}). \quad (\text{I.3.18})$$

Although the proof of the above results is based on the techniques developed in [BK95], a more careful evaluation of boundary terms is crucial. As usual, the model is to this end first rewritten in terms of the Fortuin-Kasteleyn random-cluster representation [FK72], and then recast as a contour model. A part of our analysis was used in [vE00] to show that a non-robust phase transition for the high  $q$ -state Potts model occurs at  $\beta_t$ .

Our second aim here is to study the finite-volume specific magnetization  $m_L(h, \beta)$  in the two-dimensional *Ising model* in a square box  $\Lambda_L$  of the area  $L^2$  with the minus boundary conditions and a positive external magnetic field  $h$  of the order  $L^{-1}$  for all subcritical temperatures. The Hamiltonian of the model is

$$H_{L,h}(\sigma_L) = - \sum_{\substack{\langle x,y \rangle: \\ x,y \in \Lambda_L}} \sigma_x \sigma_y + \sum_{\substack{\langle x,y \rangle: \\ x \in \Lambda_L, y \in \Lambda_L^c}} \sigma_x - h |\Lambda_L| S_L(\sigma_L), \quad (\text{I.3.19})$$

where  $\sigma_L : \Lambda_L \rightarrow \{-1, 1\}$  is the spin configuration in  $\Lambda_L$ ,  $\langle x, y \rangle$  stands for a pair of nearest-neighbour sites  $x$  and  $y$  of  $\mathbb{Z}^d$ , and

$$S_L(\sigma_L) := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \sigma_x \quad (\text{I.3.20})$$

is the average spin. The magnetization  $m_L(h, \beta)$  is the expected value of  $S_L$  under the Gibbs measure at the inverse temperature  $\beta$  corresponding to the above Hamiltonian. The choice  $h \sim L^{-1}$  matches the

most interesting situation when the bulk effect of the magnetic field and the surface effect of the boundary conditions are of comparable strength. Here the methods of [BK95] are not applicable: the imposed boundary conditions are not weak and the configurations containing large contours may play a decisive role.<sup>2</sup> Instead, we employ the large-deviation results from [SS96, GS97, Vel97, IS98, BIV00] to treat this situation.

The behaviour of the Ising model at subcritical temperatures with the minus boundary conditions and  $h \sim 1/L$  has been of great interest in recent years. Considering  $h_L = B/L$ ,  $B \in \mathbb{R}$ , and  $d = 2$ , Schonmann and Shlosman [SS96] proved that there exists a unique point  $B_0 = B_0(\beta) > 0$  such that the finite-volume Gibbs measure at  $h = B/L$  converges weakly to the pure minus phase if  $B < B_0$ , while the limit is the pure plus phase if  $B > B_0$ . In both the regimes, they investigated the exponential convergence of the average spin  $S_L$  at the surface rate and established a ‘surface-order’ large-deviation principle valid at  $B = 0$ , extending the results obtained by Ioffe [Iof94, Iof95]. In the former regime ( $B < B_0$ ), the minus boundary conditions prevail, selecting the minus phase in the box  $\Lambda_L$ , and  $S_L$  converges exponentially to  $-m^* < 0$ , where  $m^*$  is the spontaneous magnetization. In the latter regime ( $B > B_0$ ), however, the magnetic field is the dominant effect, and the plus phase is outweighing in the system. The average spin now converges exponentially to a point  $m(B) \in (0, m^*)$ , see Fig. 3, and a single droplet of the plus phase within  $\Lambda_L$  immersed into the minus phase is created. The most favourable shape of such a droplet [SS96] is a squeezed version of the equilibrium crystal (or Wulff) shape — the one which minimizes the interfacial surface tension, assuming that its volume is given and that it fits the box  $\Lambda_L$ . This is caused by the fact that whenever the droplet really appears, it necessarily touches the boundary of  $\Lambda_L$ . Greenwood and Sun [GS97] pointed out (for any dimension  $d \geq 2$ ) how the large-deviation principles with  $B = 0$  and  $B \neq 0$  are related, and inspected the surface-rate exponential convergence of  $S_L$ , too.

It should be noted that the macroscopic-scale separation of pure phases along the boundary of the equilibrium crystal shape is an extremely subtle probabilistic problem. Its first rigorous study was done by Dobrushin, Kotecký, and Shlosman [DKS92, DS94] for the  $2d$  Ising model at very low temperatures. The main part of their results was extended to all subcritical temperatures by Ioffe and Schonmann [IS98]. The main results of the rigorous microscopic theory of equilibrium crystal shapes are reviewed in the recent paper by Bodineau, Ioffe, and Velenik [BIV00].

<sup>2</sup>In the previous finite-size analysis of the Potts model, for example, this is embodied in the fact that the situation when  $\lambda = 0$  and  $\lambda \geq J$  are out of scope of [BK95] if  $\beta > \beta_t$  and  $\beta < \beta_t$ , respectively.

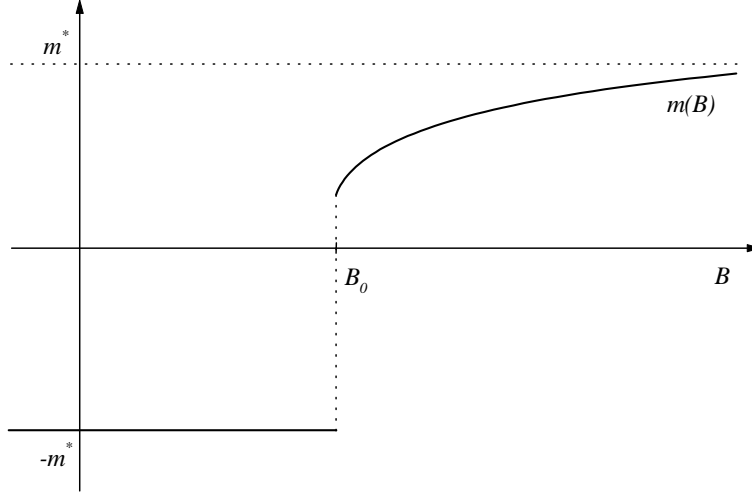


FIGURE 3. The average spin  $S_L$  converges exponentially at the surface rate to  $-m^*$  if  $B < B_0$  and to  $m(B)$  if  $B > B_0$ . The point  $B^*$  is also indicated.

Now, let us formulate our results on the behaviour of the magnetization  $m_L(h, \beta)$  as well as the susceptibility  $\chi_L(h, \beta) := \frac{1}{\beta L^2} \frac{dm_L(h, \beta)}{dh}$ . Certainly, this behaviour reflects the above-described balance between the competing influences of the magnetic field and the minus boundary conditions in our model.

We consider the interval  $|Lh - B_0| < \vartheta$ , where  $\vartheta > 0$  is arbitrary. Let  $m_+(B) = m(B)$  for  $B \geq B^*$ , whereas  $m_+(B) = m(B^*)$  for  $B \leq B^*$ , where  $B^* \in (0, B_0)$  is specified in Section IV.2 (see the text before (IV.2.7)). Introducing the shorthands

$$\bar{m}(B) := \frac{m_+(B) + (-m^*)}{2}, \quad \Delta m(B) := \frac{m_+(B) - (-m^*)}{2}, \quad (\text{I.3.21})$$

and  $\Delta := \Delta m(B_0) > 0$ , for any  $\beta > \beta_c$ ,  $0 < \delta_0 < 1/4$ , and  $L$  sufficiently large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta_0$ ) one has:

- (1) The susceptibility  $\chi_L(h, \beta)$  attains its maximal value over the interval  $|Lh - B_0| < \vartheta$  at a unique point  $h_\chi(L)$  (which is independent of  $\vartheta$ ), and  $h_\chi(L) = (B_0 + R_L^{(0)})/L$  with  $R_L^{(0)} = O(L^{-\delta_0})$ .

- (2) For any  $h$  such that  $|Lh - B_0| < \vartheta$  it follows that

$$m_L(h, \beta) = \bar{m}(Lh) + \Delta m(Lh) \tanh[\beta \Delta (h - h_\chi(L)) L^2] + R_L^{(1)}(h) \quad (\text{I.3.22})$$

and

$$\chi_L(h, \beta) = (\Delta m(Lh))^2 \cosh^{-2}[\beta \Delta (h - h_\chi(L)) L^2] + R_L^{(2)}(h), \quad (\text{I.3.23})$$

where

$$\sup_{h: |Lh - B_0| < \vartheta} R_L^{(i)}(h) = O(L^{-\delta_0}), \quad i = 1, 2. \quad (\text{I.3.24})$$

We divide the proof of these results into two parts. First, we prove their weaker version (Theorem IV.3.3) in which it is claimed that  $R_L^{(0)}$  and  $\sup_{h: |Lh - B_0| < \vartheta} R_L^{(i)}(h)$ ,  $i = 1, 2$ , tend to zero as  $L \rightarrow \infty$ ; this part is based on the large-deviation principle established in [SS96]. In order to obtain explicit uniform rates at which these errors tend to zero, we employ the local-limit estimates from [IS98, BIV00, DS94] and Theorem 7.4.3 from [Vel97]. It should be pointed out that the division of the proof into two parts is not necessary and it could be carried out solely with the help of the local-limit estimates. However, we believe that it is more transparent to examine the problem by means of the large-deviation principle at the beginning and use the more precise information of the local-limit estimates only afterwards.

The asymptotics (I.3.24) can be slightly improved if taking into account that Theorem A from [IS98] is applicable in our situation (as is remarked in [BIV00]), although the proof of this has never been written down explicitly. In any case, in the course of the proof it becomes clear that the constant  $\delta_0$  has to be smaller than  $1/2$ .

A few features of the Ising model only play an important role in the first part of the proof of its finite-size behaviour, namely, in the proof of Theorem IV.3.3. In the last chapter we therefore extract these essential features, specifying thus a group of models whose finite-size behaviour we investigate therein. This leads to a *generalization* of the results from Theorem IV.3.3 for models describing the coexistence of two phases. To be precise, we consider a sequence  $\{\Lambda_n\}$  of finite subsets of the lattice  $\mathbb{Z}^d$ ,  $d \geq 2$ , such that  $\lim_{n \rightarrow \infty} |\Lambda_n| = \infty$  and we study the models in  $\Lambda_n$  whose single spin-space is a finite set  $S$  and the Hamiltonian has the form

$$H_{n,h}(\sigma_n) = \mathcal{H}_n(\sigma_n) - h |\Lambda_n| X_n(\sigma_n) \quad (\text{I.3.25})$$

for all configurations  $\sigma_n \in S^{\Lambda_n}$ , where  $h$  is a real parameter and  $\mathcal{H}_n$  and  $X_n$  are real-valued functions on  $S^{\Lambda_n}$ . The task is to analyze the behaviour of the finite-volume ‘magnetization’ (i.e. the expected value of  $X_n$  under the corresponding Gibbs measure) in dependence on  $h$  as  $n \rightarrow \infty$ . Writing  $\text{Ran } X_n$  for the range of  $X_n$  and introducing

$$\underline{x} := \liminf_{n \rightarrow \infty} \text{Ran } X_n, \quad \bar{x} := \overline{\lim}_{n \rightarrow \infty} \text{Ran } X_n, \quad (\text{I.3.26})$$

we suppose that

$$(A) \quad \max \{|\underline{x}|, |\bar{x}|\} < \infty,$$

- (B) given a sequence  $\{h_n\}$ ,  $h_n \in \mathbb{R}$ , the distribution of  $X_n$  under the Gibbs measure with the magnetic field  $h_n$  satisfies a weak large-deviation principle with a rate  $I \not\equiv \infty$ .

Moreover, two assumptions on the form of the rate  $I$  are made; these only secure that one looks at the Ising-like situation similar to the one from Fig. 3. Then, with the help of the large-deviation theory and convex analysis, we prove results completely analogous to those concerning the two-dimensional Ising model showing that the corresponding errors  $R_n^{(0)}$  as well as  $R_n^{(i)}(h)$ ,  $i = 1, 2$ , tend to zero as  $n \rightarrow \infty$  uniformly within the corresponding interval of  $h$ .

As a matter of fact, in the assumption (B) one is interested in large-deviation principles at surface orders. Although these have been explicitly established just for the two-dimensional Ising model, it is anticipated that the Ising model in higher dimension or other models (like the Potts model) will soon be covered as well. That is why we find the problem of the connection between the finite-size effects and large deviations interesting and, hence, worth studying.

In order to carry out the finite-size analysis and prove the results presented above, we shall employ three techniques: the Pirogov-Sinai theory, basics of the theory of large deviations, and some convex analysis. We briefly review these in the next chapter.

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## CHAPTER II

### Techniques

#### II.1. Pirogov-Sinai Theory

The Pirogov-Sinai theory, whose origin goes back to the work of Pirogov and Sinai [PS75, PS76], is a general and powerful method of rigorous study of various *low-temperature*<sup>1</sup> aspects for a large class of two- and higher-dimensional statistical-mechanical models. It provides detailed control over the behaviour of all phases in all possible situations (when the phases become, as the driving parameters are changed, either stable or unstable) in infinite as well as in finite volumes. Among main achievements of the theory is its ability to treat models with or *without* symmetry, i.e. also the cases where other methods (like the reflection positivity) fail. Here we describe the basic ideas and results of the theory in the case of a Hamiltonian with a finite number of local ground states, assuming that the potential is of finite range (and mostly translation-invariant) [Zah84, BI89, Zah98]. Special emphasis is laid on the investigation of the finite-size effects [BK90, BK95].

Throughout the section we take the dimension  $d \geq 2$ .

**II.1.1. Abstract Pirogov-Sinai Model.** The starting point of the theory is to represent a lattice model in terms of an abstract contour model. Let us demonstrate how this may be carried out for a model in<sup>2</sup>  $\Lambda \Subset \mathbb{Z}^d$  whose Hamiltonian is given by (I.1.1); we shall consider constant boundary conditions.

First, let a set  $Q \subset S$  be fixed. Given a configuration  $\omega \in \Omega$ , we define  $B(\omega)$  as the union of those  $R$ -cubes in  $\mathbb{Z}^d$  in which  $\omega$  does not coincide with the constant configuration  $\omega^q := \{\omega_x = q \text{ for all } x \in \mathbb{Z}^d\}$  for any  $q \in Q$ ,

$$B(\omega) = \bigcup_{x \in \mathbb{Z}^d} \{C_R(x) : \omega_{C_R(x)} \neq (\omega^q)_{C_R(x)} \text{ for every } q \in Q\} \quad (\text{II.1.1})$$

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<sup>1</sup>This is connected with the Peierls condition (see Remark II.1.10). Strictly speaking, one does not always have to restrict oneself to the low-temperature regime. For example, it is possible to analyse the  $q$ -state Potts model, which undergoes a phase transition as the inverse temperature  $\beta$  varies, with the help of the Pirogov-Sinai theory for any  $\beta > 0$  whenever  $q$  is sufficiently large.

<sup>2</sup>Each  $\Lambda \Subset \mathbb{Z}^d$  in this section is assumed to be simply-connected.

with  $C_R(x) := \{y \in \mathbb{Z}^d : |y_i - x_i| \leq R \text{ for all } 1 \leq i \leq d\}$ . We use  $\text{cc } B(\omega)$  to denote the set of connected components of  $B(\omega)$  and, if  $B(\omega)$  is finite, we write  $\text{Ext}(\omega)$  for the unique infinite component of  $\mathbb{Z}^d \setminus B(\omega)$ . Let us introduce

$$\Psi(M, \omega) := \sum_{A \subset \mathbb{Z}^d} \frac{|M \cap A|}{|A|} \Phi_A(\omega), \quad M \in \mathbb{Z}^d, \omega \in \Omega, \quad (\text{II.1.2})$$

and

$$e_q := \sum_{A: A \ni 0} \frac{\Phi_A(\omega^q)}{|A|} = \Psi(\{0\}, \omega^q), \quad q \in Q. \quad (\text{II.1.3})$$

Moreover, let  $\Lambda^{(R)} := \cup_{x \in \Lambda} C_R(x)$  and let  $\Lambda_q^{(R)}(\omega)$ ,  $q \in Q$ , be the set of all points  $x \in \Lambda^{(R)}$  such that  $\omega_{C_R(x)} = (\omega^q)_{C_R(x)}$ . Using the shorthand  $\omega_{q, \Lambda} \equiv \omega_\Lambda \times (\omega^q)_{\Lambda^c}$ , we have

$$\begin{aligned} H_\Lambda^{(\Phi)}(\omega | \omega^q) &= \sum_{A \not\subset \Lambda^c} \sum_{x \in A} \frac{\Phi_A(\omega_{q, \Lambda})}{|A|} = \sum_{x \in \Lambda^{(R)}} \sum_{\substack{A \ni x, \\ A \not\subset \Lambda^c}} \frac{\Phi_A(\omega_{q, \Lambda})}{|A|} = \\ &= \left( \sum_{q' \in Q} \sum_{x \in \Lambda_{q'}^{(R)}(\omega_{q, \Lambda})} + \sum_{x \in B(\omega_{q, \Lambda})} \right) \left( \sum_{A \ni x} - \sum_{\substack{A \ni x, \\ A \subset \Lambda^c}} \right) \frac{\Phi_A(\omega_{q, \Lambda})}{|A|}. \end{aligned} \quad (\text{II.1.4})$$

Since

$$\begin{aligned} \sum_{q' \in Q} \sum_{x \in \Lambda_{q'}^{(R)}(\omega_{q, \Lambda})} \sum_{A \ni x} \frac{\Phi_A(\omega_{q, \Lambda})}{|A|} &= \sum_{q' \in Q} \sum_{x \in \Lambda_{q'}^{(R)}(\omega_{q, \Lambda})} \sum_{A \ni x} \frac{\Phi_A(\omega^q)}{|A|} = \\ &= \sum_{q' \in Q} e_{q'} |\Lambda_{q'}^{(R)}(\omega_{q, \Lambda})|, \end{aligned} \quad (\text{II.1.5})$$

$$\begin{aligned} \sum_{x \in B(\omega_{q, \Lambda})} \sum_{A \ni x} \frac{\Phi_A(\omega_{q, \Lambda})}{|A|} &= \sum_{\Gamma \in \text{cc } B(\omega_{q, \Lambda})} \sum_{x \in \Gamma} \sum_{A \ni x} \frac{\Phi_A(\omega_{q, \Lambda})}{|A|} = \\ &= \sum_{\Gamma \in \text{cc } B(\omega_{q, \Lambda})} \sum_{A \cap \Gamma \neq \emptyset} \sum_{x \in A \cap \Gamma} \frac{\Phi_A(\omega_{q, \Lambda})}{|A|} = \\ &= \sum_{\Gamma \in \text{cc } B(\omega_{q, \Lambda})} \Psi(\Gamma, \omega_{q, \Lambda}), \end{aligned} \quad (\text{II.1.6})$$

and

$$\sum_{x \in \Lambda^{(R)}} \sum_{\substack{A \ni x, \\ A \subset \Lambda^c}} \frac{\Phi_A(\omega_{q, \Lambda})}{|A|} = \sum_{x \in \Lambda^{(R)} \setminus \Lambda} \sum_{\substack{A \ni x, \\ A \subset \Lambda^c}} \frac{\Phi_A(\omega^q)}{|A|}, \quad (\text{II.1.7})$$

it follows that

$$H_\Lambda^{(\Phi)}(\omega|\omega^q) = \sum_{q' \in Q} e_{q'} |\Lambda_{q'}^{(R)}(\omega_{q, \Lambda})| + \sum_{\Gamma \in \text{cc } B(\omega_{q, \Lambda})} \Psi(\Gamma, \omega_{q, \Lambda}) - C(q, \Lambda, R). \quad (\text{II.1.8})$$

Here  $C(q, \Lambda, R)$  is the constant given by (II.1.7), which, for a given  $\Phi$ , depends on  $q$ ,  $\Lambda$ , and  $R$  only. The corresponding partition function (I.1.3) may now be expressed by

$$Z_\Lambda^{(\Phi)}(\omega^q) = e^{\beta C(q, \Lambda, R)} \sum_{\substack{\omega \in \Omega: \\ \omega_{\Lambda^c} = (\omega^q)_{\Lambda^c}}} \prod_{q' \in Q} e^{-\beta e_{q'} |\Lambda_{q'}^{(R)}(\omega_{q, \Lambda})|} \times \prod_{\Gamma \in \text{cc } B(\omega_{q, \Lambda})} e^{-\beta \Psi(\Gamma, \omega_{q, \Lambda})}. \quad (\text{II.1.9})$$

Let us next introduce a *contour*  $\gamma$  to be a pair  $(\text{supp } \gamma, c_\gamma(\cdot))$ , where  $\text{supp } \gamma$  is a connected subset of  $\mathbb{Z}^d$  and  $c_\gamma(\cdot)$  is an assignment of a label  $c_\gamma(\text{bd } \mathcal{K}) \in Q$  to the boundaries of each component  $\mathcal{K}$  of  $\mathbb{Z}^d \setminus \text{supp } \gamma$ . Observing that any  $\Psi(\Gamma, \omega_{q, \Lambda})$  in (II.1.9) is a function of the restriction of  $\omega_{q, \Lambda}$  to  $\Gamma$ , one may rewrite the partition function as

$$Z_\Lambda^{(\Phi)}(\omega^q) = e^{\beta C(q, \Lambda, R)} \sum_{\{\gamma_1, \dots, \gamma_n\} \sqsubset \Lambda^{(R)}} \prod_{q' \in Q} e^{-\beta e_{q'} |\Lambda_{q'}^{(R)}(\{\gamma_i\})|} \times \sum_{\omega^{(1)}} \dots \sum_{\omega^{(n)}} \prod_{i=1}^n e^{-\beta \Psi(\text{supp } \gamma_i, \omega^{(i)})}. \quad (\text{II.1.10})$$

Here the first sum is over all collections of non-overlapping contours  $\{\gamma_i\}$  such that  $\text{supp } \gamma_i \subset \Lambda^{(R)}$  for all  $i = 1, \dots, n$  and the labels of  $\gamma_1, \dots, \gamma_n$  are constant on the boundaries of each component of  $\Lambda^{(R)} \setminus \cup_{i=1}^n \text{supp } \gamma_i$ ; the set  $\Lambda_{q'}^{(R)}(\{\gamma_i\})$  is the union of all of these components which have the boundary label  $q' \in Q$ . In addition, each of the last  $n$  sums runs over all configurations  $\omega^{(i)} \in S^{\text{supp } \gamma_i}$ ,  $i = 1, \dots, n$ , satisfying the condition that  $(\omega^{(i)})_x = c_{\gamma_i}(\text{bd } \mathcal{K})$  whenever  $x \in \text{supp } \gamma_i \cap (\cup_{y \in \mathcal{K}} C_R(y))$  for any component  $\mathcal{K}$  of  $\mathbb{Z}^d \setminus \text{supp } \gamma_i$ . Realizing that

$$\prod_{i=1}^n \sum_{\omega^{(i)}} e^{-\beta \Psi(\text{supp } \gamma_i, \omega^{(i)})} = \sum_{\omega^{(1)}} \dots \sum_{\omega^{(n)}} \prod_{i=1}^n e^{-\beta \Psi(\text{supp } \gamma_i, \omega^{(i)})}, \quad (\text{II.1.11})$$

we finally obtain

$$Z_\Lambda^{(\Phi)}(\omega^q) = e^{\beta C(q, \Lambda, R)} \sum_{\{\gamma_1, \dots, \gamma_n\} \sqsubset \Lambda^{(R)}} \prod_{q' \in Q} e^{-\beta e_{q'} |\Lambda_{q'}^{(R)}(\{\gamma_i\})|} \prod_{i=1}^n \rho(\gamma_i) \quad (\text{II.1.12})$$

with  $\rho(\gamma_i) := \sum_{\omega^{(i)}} e^{-\beta\Psi(\text{supp } \gamma_i, \omega^{(i)})}$ . This is the desired contour representation of the considered lattice model. It may serve as a motivation for the general setting to be now introduced.

Let the  $Q \subset S$  be fixed (the set of *colours*). A *contour* is a pair  $\gamma = (\text{supp } \gamma, c_\gamma(\cdot))$ , where  $\text{supp } \gamma$ , the *support* of the contour  $\gamma$ , is a connected set<sup>3</sup> and  $c_\gamma(\cdot)$  is an assignment of a colour from  $Q$  to the boundaries of each component of  $\mathbb{Z}^d \setminus \text{supp } \gamma$ . Given a contour  $\gamma$  with a finite support, its *interior* is the union of all finite components of  $\mathbb{Z}^d \setminus \text{supp } \gamma$ , its  $q$ -*th interior* is the union of all the components of its interior which have the boundary colour  $q \in Q$ , and its *exterior* is the (only) infinite component of  $\mathbb{Z}^d \setminus \text{supp } \gamma$ ; we use  $\text{Int } \gamma$ ,  $\text{Int}_q \gamma$ , and  $\text{Ext } \gamma$ , respectively, to denote them. The colour on the boundaries of  $\text{Ext } \gamma$  plays a special role, and, if this colour is  $q \in Q$ , the contour  $\gamma$  is called a  $q$ -*contour*. As is common, we shall use  $|\gamma|$  to denote the size of  $\text{supp } \gamma$ .

DEFINITION II.1.1. A set of contours  $\mathfrak{G}$  is said to be *admissible* if

- (a) the contours in  $\mathfrak{G}$  are mutually non-overlapping ( $\text{supp } \gamma_1$  and  $\text{supp } \gamma_2$  are not connected for any  $\gamma_1, \gamma_2 \in \mathfrak{G}$ ),
- (b) the colours of contours in  $\mathfrak{G}$  are matching, i.e. constant on the boundaries of each component of  $\mathbb{Z}^d \setminus \cup_{\gamma \in \mathfrak{G}} \text{supp } \gamma$ .

*External* contours of an admissible family of contours are those which neighbour the infinite component of  $\mathbb{Z}^d \setminus \cup_{\gamma \in \mathfrak{G}} \text{supp } \gamma$ .

An admissible family of contours  $\mathfrak{G}$  is said to be  $q$ -*admissible* if its external contours are  $q$ -contours.

DEFINITION II.1.2. Let  $\{e_q; q \in Q\}$  and  $\{E(\gamma)\}$  be real quantities (the ‘ground-state energies’ and the ‘contour energies’, respectively). For any  $\Lambda \Subset \mathbb{Z}^d$  and any admissible family  $\mathfrak{G}$  of contours in  $\Lambda$ , let

$$H_\Lambda^{(PS)}(\mathfrak{G}) := \sum_{q \in Q} e_q |\Lambda_q(\mathfrak{G})| + \sum_{\gamma \in \mathfrak{G}} \{E(\gamma) + e |\gamma|\}, \quad (\text{II.1.13})$$

where  $\Lambda_q(\mathfrak{G})$  is the union of all the components of  $\Lambda \setminus \cup_{\gamma \in \mathfrak{G}} \text{supp } \gamma$  which have the boundary colour  $q \in Q$  and  $e := \min_{q \in Q} e_q$ . A model whose configuration space is the collection  $\{\mathfrak{G}\}$  of all admissible families of contours in  $\Lambda$  and whose Hamiltonian is given by (II.1.13) is called an *abstract Pirogov-Sinai model* in  $\Lambda$  (corresponding to  $\{\mathfrak{G}\}$ ,  $\{e_q; q \in Q\}$ , and  $\{E(\gamma)\}$ ).

<sup>3</sup>The exact meaning of connectedness can be in different situations different, taking thus into account the features and peculiarities of the inspected model. That is why we refrain from specifying this notion here. However, one may think of  $\text{supp } \gamma$  as a connected subset of  $\mathbb{Z}^d$ , for instance.

DEFINITION II.1.3. Let  $\Lambda \in \mathbb{Z}^d$ . The *partition function* of the abstract Pirogov-Sinai model in  $\Lambda$  (corresponding to  $\{\mathfrak{G}\}$ ,  $\{e_q; q \in Q\}$ , and  $\{E(\gamma)\}$ ) is given by

$$Z^{(PS)}(\Lambda) := \sum_{\mathfrak{G} \sqsubset \Lambda} \prod_{q \in Q} e^{-\beta e_q |\Lambda_q(\mathfrak{G})|} \prod_{\gamma \in \mathfrak{G}} \rho(\gamma), \quad (\text{II.1.14})$$

where the sum is over all admissible families of contours in  $\Lambda$ , the quantity  $\rho(\gamma) := e^{-\beta(E(\gamma) + e|\gamma|)}$  is the *weight* of the contour  $\gamma$ , and the  $(\mathfrak{G} = \emptyset)$ -term in (II.1.14) is set to equal  $\sum_{q \in Q} e^{-\beta e_q |\Lambda|}$ . The corresponding (specific) *free energy* is the limit

$$f^{(PS)} := - \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{\beta |\Lambda|} \log Z^{(PS)}(\Lambda), \quad (\text{II.1.15})$$

if it exists. The limit is taken in the sense of van Hove.

The  $q$ -th *partition function* in  $\Lambda$  is defined by (II.1.14) with the sum over all  $q$ -admissible families of contours in  $\Lambda$  only; we shall write  $Z_q^{(PS)}(\Lambda)$  for this quantity.<sup>4</sup>

Next, we define the central notion of the Pirogov-Sinai theory.

DEFINITION II.1.4. Let  $q \in Q$ . Given a  $q$ -contour  $\gamma$ , its *contour functional*  $\mathcal{F}_q(\gamma)$  is defined through the relation<sup>5</sup>

$$e^{-\beta \mathcal{F}_q(\gamma)} := \rho(\gamma) e^{\beta e_q |\gamma|} \prod_{q' \in Q} \frac{Z_{q'}^{(PS)}(\text{Int}_{q'} \gamma)}{Z_q^{(PS)}(\text{Int}_{q'} \gamma)}. \quad (\text{II.1.16})$$

The contour functional allows to rewrite the  $q$ -th partition function  $Z_q^{(PS)}(\Lambda)$  so that the matching condition for the contours of the  $q$ -admissible families involved in its definition is removed. One thus ends up with a system of a hard-core contour gas called the  $q$ -th *contour model*.

LEMMA II.1.5. Let  $q \in Q$  and  $\Lambda \in \mathbb{Z}^d$ . Then

$$Z_q^{(PS)}(\Lambda) = e^{-\beta e_q |\Lambda|} \sum_{\mathfrak{G}_q^* \sqsubset \Lambda} \prod_{\gamma \in \mathfrak{G}_q^*} e^{-\beta \mathcal{F}_q(\gamma)}; \quad (\text{II.1.17})$$

here the sum is over all families of non-overlapping  $q$ -contours in  $\Lambda$ . The  $(\mathfrak{G}_q^* = \emptyset)$ -term in the sum in (II.1.17) is set to equal 1.

<sup>4</sup>Clearly,  $Z^{(PS)}(\Lambda) = \sum_{q \in Q} Z_q^{(PS)}(\Lambda)$ .

<sup>5</sup>The expression (II.1.16) is meaningful since  $Z_q^{(PS)}(\Lambda) > 0$  for any  $\Lambda \in \mathbb{Z}^d$  by the very Definition II.1.3.

PROOF. We shall prove (II.1.17) by induction on the size of  $\Lambda$ . First, if  $\Lambda$  is so small that it cannot contain any contour, then (II.1.17) obviously holds.

Let (II.1.17) be true for the interiors of contours appearing in  $\Lambda$ . The summation over all those families of  $q$ -admissible contours in  $Z_q^{(PS)}(\Lambda)$  that have a fixed collection of external contours produces, for each external contour  $\gamma$ , the factor  $\prod_{q' \in Q} Z_{q'}^{(PS)}(\text{Int}_{q'} \gamma)$ . Thus,

$$Z_q^{(PS)}(\Lambda) = \sum_{\mathfrak{G}_q^{\text{ext}} \sqsubset \Lambda} e^{-\beta e_q |\text{Ext}_\Lambda(\mathfrak{G}_q^{\text{ext}})|} \prod_{\gamma \in \mathfrak{G}_q^{\text{ext}}} \left\{ \rho(\gamma) \prod_{q' \in Q} Z_{q'}^{(PS)}(\text{Int}_{q'} \gamma) \right\}, \quad (\text{II.1.18})$$

where the sum is over all families of external  $q$ -contours in  $\Lambda$  and  $\text{Ext}_\Lambda(\mathfrak{G}_q^{\text{ext}}) = \Lambda \setminus \bigcup_{\gamma \in \mathfrak{G}_q^{\text{ext}}} (\text{supp } \gamma \cup \text{Int } \gamma)$ . As

$$\rho(\gamma) \prod_{q' \in Q} Z_{q'}^{(PS)}(\text{Int}_{q'} \gamma) = e^{-\beta \mathcal{F}_q(\gamma) - \beta e_q |\gamma|} \prod_{q' \in Q} Z_{q'}^{(PS)}(\text{Int}_{q'} \gamma) \quad (\text{II.1.19})$$

for any  $q$ -contour  $\gamma$ , the inductive assumption implies

$$\begin{aligned} Z_q^{(PS)}(\Lambda) &= \sum_{\mathfrak{G}_q^{\text{ext}} \sqsubset \Lambda} e^{-\beta e_q \{ |\text{Ext}_\Lambda(\mathfrak{G}_q^{\text{ext}})| + \sum_{\gamma \in \mathfrak{G}_q^{\text{ext}}} (|\gamma| + |\text{Int } \gamma|) \}} \times \\ &\quad \times \prod_{\gamma \in \mathfrak{G}_q^{\text{ext}}} \left\{ e^{-\beta \mathcal{F}_q(\gamma)} \sum_{\mathfrak{G}_q^* \sqsubset \text{Int } \gamma} \prod_{\tilde{\gamma} \in \mathfrak{G}_q^*} e^{-\beta \mathcal{F}_q(\tilde{\gamma})} \right\}, \end{aligned}$$

which equals the right-hand side of (II.1.17). Q.E.D.

Therefore, in order to study the behaviour of contour models, a control over the contour functional is needed. Namely, if  $\mathcal{F}_q(\gamma)$  increases with the size of the support of  $\gamma$ , then one can use the *cluster-expansion theory* and express  $\log Z_q^{(PS)}(\Lambda)$  as a quickly converging series. We formulate the statement in the next lemma. It should be stressed, however, that it can be proved under very general circumstances [KP86, Dob94, BZ], in particular, the translation invariance need not be assumed.

Let us introduce the polymer partition function

$$\mathcal{Z}(\Lambda; \mathcal{F}) := \sum_{\mathfrak{G}^* \sqsubset \Lambda} \prod_{\gamma \in \mathfrak{G}^*} e^{-\beta \mathcal{F}(\gamma)}, \quad (\text{II.1.20})$$

where the sum is over all families  $\mathfrak{G}^*$  of non-overlapping contours in  $\Lambda$  with the  $(\mathfrak{G}^* = \emptyset)$ -term set to equal 1 and  $\mathcal{F}(\gamma)$  is a real-valued quantity.

DEFINITION II.1.6. Let  $\mathfrak{C}$  be the set of all contours. For any multi-index  $\mathcal{C} : \mathfrak{C} \rightarrow \{0, 1, \dots\}$ , let  $\text{supp } \mathcal{C}$  be the union of the supports of

all contours in  $\mathfrak{C}$  with  $\mathcal{C}(\gamma) > 0$ . We say that  $\mathcal{C}$  is a *cluster* if  $\text{supp } \mathcal{C}$  is connected. The *cardinality* of a cluster  $\mathcal{C}$  is defined to be

$$|\mathcal{C}| := \sum_{\gamma \in \mathfrak{C}} \mathcal{C}(\gamma) |\gamma|. \quad (\text{II.1.21})$$

LEMMA II.1.7 (Cluster Expansion). *There exist finite positive constants  $\varepsilon_0 = \varepsilon_0(d)$  and  $K = K(d, |Q|)$  and a combinatorial coefficient  $\varphi_{\mathcal{C}}$  for each multiindex  $\mathcal{C}$  such that the following is true once*

$$\sum_{\gamma: \text{supp } \gamma = G} e^{-\beta \mathcal{F}(\gamma)} \leq \varepsilon^{|G|} \quad (\text{II.1.22})$$

for some  $0 < \varepsilon < \varepsilon_0$  and every  $G \in \mathbb{Z}^d$ .

(a) For any  $\Lambda \in \mathbb{Z}^d$ , one has

$$\log \mathcal{Z}(\Lambda; \mathcal{F}) = \sum_{\mathcal{C}: \text{supp } \mathcal{C} \subset \Lambda} \varphi_{\mathcal{C}} w_{\mathcal{C}} \quad \text{and} \quad |\varphi_{\mathcal{C}}| \leq K^{|\mathcal{C}|} \quad (\text{II.1.23})$$

with  $\varphi_{\mathcal{C}} = 0$  once  $\mathcal{C}$  is not a cluster; here  $w_{\mathcal{C}} = \prod_{\gamma} (e^{-\beta \mathcal{F}(\gamma)})^{\mathcal{C}(\gamma)}$ . The sum in (II.1.23) is quickly convergent,

$$\sum_{\mathcal{C}: \text{supp } \mathcal{C} \supset M} |\varphi_{\mathcal{C}}| w_{\mathcal{C}} \leq (2K\varepsilon)^{|M|} \quad (\text{II.1.24})$$

for any  $M \in \mathbb{Z}^d$ .

(b) Let

$$s := -\frac{1}{\beta} \sum_{\mathcal{C}: 0 \in \text{supp } \mathcal{C}} \frac{1}{|\text{supp } \mathcal{C}|} \varphi_{\mathcal{C}} w_{\mathcal{C}}. \quad (\text{II.1.25})$$

Then  $0 \geq s = O(\varepsilon)$  and

$$\log \mathcal{Z}(\Lambda; \mathcal{F}) = -\beta s |\Lambda| + O(\varepsilon) |\partial \Lambda|. \quad (\text{II.1.26})$$

REMARK II.1.8. Note that if  $\mathcal{F}(\gamma) > \tau_0 |\gamma|$  with some  $\tau_0 > 0$ , then

$$\sum_{\gamma: \text{supp } \gamma = G} e^{-\beta \mathcal{F}(\gamma)} \leq \sum_{\gamma: \text{supp } \gamma = G} e^{-\beta \tau_0 |\gamma|} \leq e^{-(\beta \tau_0 - \log |Q|) |G|}. \quad (\text{II.1.27})$$

We used that the number of all contours with the same support  $G$  can be bounded from above by  $|Q|^{|G|}$ . Hence, the condition (II.1.22) is in this case satisfied as soon as  $\beta$  is sufficiently large.

**II.1.2. Metastable (Truncated) Models.** Let us return to the abstract Pirogov-Sinai model. We introduce an additional vector parameter  $h \in \mathbb{R}^\nu$  ('external fields') with  $\nu \geq |Q| - 1$  on which the corresponding energies  $\{e_q\}$  and  $\{E(\gamma)\}$  will be dependent. Therefore, we shall now use  $Z^{(PS)}(\Lambda, h)$  and  $Z_q^{(PS)}(\Lambda, h)$  to denote the partition functions of the model (Definition II.1.3).

In order to control  $Z^{(PS)}(\Lambda, h)$ , one needs to control each partition function  $Z_q^{(PS)}(\Lambda, h)$ . Lemma II.1.5 and Lemma II.1.7 suggest that the latter may be done by a cluster-expansion analysis once the condition (II.1.22) is satisfied. To this end, the following assumption is imposed.

**ASSUMPTION II.1.9 (Peierls Condition).** Let us assume that there is a constant  $\tau > 0$  such that

$$E(\gamma) > \tau|\gamma| \quad (\text{II.1.28})$$

for any contour  $\gamma$  and all  $h \in \mathfrak{U}$ , where  $\mathfrak{U}$  is an open subset of  $\mathbb{R}^\nu$ .

**REMARK II.1.10.** In fact, one needs the inequality

$$\sum_{\gamma: \text{supp } \gamma=G} e^{-\beta E(\gamma)} \leq e^{-\beta \tilde{\tau}|G|} \quad (\text{II.1.29})$$

to be true (for some different  $\tilde{\tau} > 0$ ). In view of Remark II.1.8, this follows from (II.1.28) if  $\beta$  is sufficiently large.

Nevertheless, the Peierls condition itself does not guarantee that (II.1.22) is automatically fulfilled. On the contrary, it turns out that, in dependence on  $h$ , the inequality (II.1.22) is not always true. That is why the *metastable* (or *truncated*) *q-th contour model* is introduced [Zah84, BI89]. Namely, one constructs the metastable contour functional  $\mathcal{F}'_q(\gamma)$ , which determines the *q-th metastable partition function*

$$Z'_q{}^{(PS)}(\Lambda, h) := e^{-\beta e_q |\Lambda|} \sum_{\mathfrak{G}_q^* \sqsubset \Lambda} \prod_{\gamma \in \mathfrak{G}_q^*} e^{-\beta \mathcal{F}'_q(\gamma)} \quad (\text{II.1.30})$$

and the *q-th metastable free energy*<sup>6</sup>

$$f_q(h) := - \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{\beta |\Lambda|} \log Z'_q{}^{(PS)}(\Lambda, h), \quad (\text{II.1.31})$$

such that:

- (a)  $\mathcal{F}'_q(\gamma)$  obeys (II.1.22) for some small  $\varepsilon > 0$ ,

<sup>6</sup>The limit is taken in the sense of van Hove. Its existence will become clear later.

- (b)  $Z'_q{}^{(PS)}(\Lambda, h)$  equals the  $q$ -th partition function of the original model  $Z_q{}^{(PS)}(\Lambda, h)$  once  $f_q(h)$  coincides with  $\min_{q \in Q} f_q(h)$  (for such  $h$  the colour  $q$  is called *stable*).

Next, we illustrate the construction of the metastable models on an example of a simplified two-spin system. So, let  $Q = \{+, -\}$  and let us suppose that  $\text{Int}_\pm \gamma = \emptyset$  (i.e.  $\text{Int} \gamma = \text{Int}_\mp \gamma$ ) for every  $\pm$ -contour  $\gamma$ . We define  $\mathcal{F}'_\pm(\gamma)$  through the relation [Zah84]

$$\mathcal{F}'_\pm(\gamma) := \max\{\mathcal{F}_\pm(\gamma), \tau|\gamma|/3\}; \quad (\text{II.1.32})$$

then  $Z'_\pm{}^{(PS)}(\Lambda, h)$  and  $f_\pm(h)$  are given by (II.1.30) and (II.1.31), respectively. Besides, let

$$f(h) := \min\{f_+(h), f_-(h)\} \quad (\text{II.1.33})$$

and

$$a_\pm(h) := f_\pm(h) - f(h) \geq 0. \quad (\text{II.1.34})$$

An immediate consequence of the definition of  $\mathcal{F}'_\pm(\gamma)$  is that

$$e^{-\beta\mathcal{F}'_\pm(\gamma)} \leq e^{-\beta\tau|\gamma|/3} \quad (\text{II.1.35})$$

and  $Z'_\pm{}^{(PS)}(\Lambda, h) \leq Z_\pm{}^{(PS)}(\Lambda, h)$ . Lemma II.1.7 then yields

$$\log Z'_\pm{}^{(PS)}(\Lambda, h) = -\beta f_\pm(h)|\Lambda| + O(\delta)|\partial\Lambda| \quad (\text{II.1.36})$$

with

$$f_\pm(h) = e_\pm(h) + s_\pm(h) \quad \text{and} \quad \delta = e^{-\beta\tau/4} \quad (\text{II.1.37})$$

whenever  $\beta > 0$  is sufficiently large. Moreover,  $0 \geq s_\pm = O(\delta)$ .

Next, assuming that

$$Z_\pm{}^{(PS)}(\Lambda, h) \leq e^{-\beta f(h)|\Lambda| + O(\delta)|\partial\Lambda|}, \quad (\text{II.1.38})$$

we claim that  $\mathcal{F}'_\pm(\gamma) = \mathcal{F}_\pm(\gamma)$  as soon as  $a_\pm \text{diam} \gamma \leq \tau/2$ . Let us show this by induction on the size of the contour  $\gamma$ . When  $\text{Int} \gamma$  is too small to contain any other contours, one has

$$\begin{aligned} \mathcal{F}_\pm(\gamma) &= -\frac{1}{\beta} \log \rho(\gamma) + e_\pm|\gamma| \geq (\tau + e - e_\pm)|\gamma| \\ &= (\tau - a_\pm + O(\delta))|\gamma| \geq (\tau - \tau/2 - 1)|\gamma| \geq \tau|\gamma|/3, \end{aligned}$$

where we recalled the Peierls condition (II.1.28). Let now  $\text{Int} \gamma$  contain contours. Using the inductive assumption for the latter ones, we find

$$\begin{aligned} \mathcal{F}_\pm(\gamma) &\geq (\tau - a_\pm + O(\delta))|\gamma| - (-f|\text{Int} \gamma| + O(\delta)|\gamma|) + \\ &\quad + (-f_\pm|\text{Int} \gamma| + O(\delta)|\gamma|) \\ &\geq (\tau - 1)|\gamma| - a_\pm|V(\gamma)|, \end{aligned}$$

where  $|V(\gamma)| = |\text{supp } \gamma \cup \text{Int } \gamma|$ ; we also used (II.1.36) and the fact that  $|\partial \text{Int } \gamma| \leq O(|\gamma|)$  and  $Z'_\pm^{(PS)}(\text{Int } \gamma, h) \leq Z_\pm^{(PS)}(\text{Int } \gamma, h)$ . Observing that  $|V(\gamma)| \leq \text{diam } \gamma |\gamma|$ , we are done.

It remains to verify the bound (II.1.38). We again proceed by induction on the size of  $\Lambda$ . First, let  $\Lambda$  be so small that it contains no contours. Since  $s_\pm \leq 0$ , we get

$$Z_\pm^{(PS)}(\Lambda, h) = e^{-\beta e_\pm |\Lambda|} \leq e^{-\beta f_\pm |\Lambda|} \leq e^{-\beta f_\pm |\Lambda| + O(\delta) |\partial \Lambda|}.$$

Now, let  $\Lambda$  contain contours for whose interiors the bound (II.1.38) holds. Writing  $\text{Ext}_\Lambda(\mathfrak{G}_\pm^{\text{ext}})$  for  $\Lambda \setminus \bigcup_{\gamma \in \mathfrak{G}_\pm^{\text{ext}}} (\text{supp } \gamma \cup \text{Int } \gamma)$ , the Peierls condition (II.1.28), (II.1.18), and (II.1.36) now yield

$$\begin{aligned} Z_\pm^{(PS)}(\Lambda, h) &= \sum_{\mathfrak{G}_\pm^{\text{ext}} \sqsubset \Lambda} e^{-\beta e_\pm |\text{Ext}_\Lambda(\mathfrak{G}_\pm^{\text{ext}})|} \prod_{\gamma \in \mathfrak{G}_\pm^{\text{ext}}} \{\rho(\gamma) Z_\mp^{(PS)}(\text{Int } \gamma)\} \leq \\ &\leq \sum_{\mathfrak{G}_\pm^{\text{ext}} \sqsubset \Lambda} e^{(-\beta f_\pm + O(\delta)) |\text{Ext}_\Lambda(\mathfrak{G}_\pm^{\text{ext}})|} \times \\ &\quad \times \prod_{\gamma \in \mathfrak{G}_\pm^{\text{ext}}} \{e^{-(\beta \tau + \beta f + O(\delta)) |\gamma|} e^{-\beta f |\text{Int } \gamma| + O(\delta) |\gamma|}\} \leq \\ &\leq e^{-\beta f |\Lambda|} \sum_{\mathfrak{G}_\pm^{\text{ext}} \sqsubset \Lambda} e^{(-\beta a_\pm + O(\delta)) |\text{Ext}_\Lambda(\mathfrak{G}_\pm^{\text{ext}})|} \prod_{\gamma \in \mathfrak{G}_\pm^{\text{ext}}} e^{(-\beta \tau + 1) |\gamma|}. \end{aligned} \quad (\text{II.1.39})$$

With the help of Main Lemma from [Zah84] (see also Lemma 3.2 in [BI89]), we may bound the last sum from above by  $e^{O(\delta) |\partial \Lambda|}$  as long as  $\beta a_\pm(h) + O(\delta) \geq O(e^{-\beta \tau + 1})$ , justifying thus (II.1.38) for those  $h \in \mathfrak{U}$  for which  $\beta a_\pm(h) \geq O(\delta)$ . In order to verify the bound for any  $h \in \mathfrak{U}$ , the above procedure is to be improved. It turns out that the remedy is achieved if one also sums over all external contours in  $\mathfrak{G}_\pm^{\text{ext}}$  for which  $a_\pm \text{diam } \gamma \leq \tau/2$ . The details of this procedure may be found in [Zah84] and [BI89], for instance.

It is now not surprising that the following statement holds (see Theorem 1.7 in [Zah84] or Section 3 in [BI89]).

**THEOREM II.1.11.** *Let  $q \in Q$  and let  $\Lambda \Subset \mathbb{Z}^d$ . Let Assumption II.1.9 be satisfied. There exists a metastable contour functional  $\mathcal{F}'_q(\gamma)$  such that for all sufficiently large  $\beta > 0$  and for all  $h \in \mathfrak{U}$  we have:*

- (1) *The bound  $\mathcal{F}'_q(\gamma) \leq e^{-\tau |\gamma|/3}$  holds true.*
- (2) *If  $a_q(h) \text{diam } \gamma \leq \tau/2$ , then  $\mathcal{F}'_q(\gamma) = \mathcal{F}_q(\gamma)$ .*
- (3) *If  $a_q(h) \text{diam } \Lambda \leq \tau/2$ , then  $Z'_q^{(PS)}(\Lambda, h) = Z_q^{(PS)}(\Lambda, h)$ .*
- (4) *One has*

$$e^{-f_q(h) |\Lambda| + O(e^{-\beta \tau/4}) |\partial \Lambda|} \leq Z'_q^{(PS)}(\Lambda, h) \leq e^{-f(h) |\Lambda| + O(e^{-\beta \tau/4}) |\partial \Lambda|}. \quad (\text{II.1.40})$$

Here  $f(h) := \min_{q \in Q} f_q(h)$  and  $a_q(h) := f_q(h) - f(h)$ .

REMARK II.1.12. The part (3) is a direct consequence of the part (2) if  $\mathcal{F}'_q(\gamma)$  is defined by (II.1.32). Certainly, the definition of  $\mathcal{F}'_q(\gamma)$  is not unique, and it can be introduced slightly differently from (II.1.32), depending on the problem under consideration.

COROLLARY II.1.13. *The quantity  $f(h) := \min_{q \in Q} f_q(h)$  equals the free energy of the model  $f^{(PS)}$ .*

PROOF. Since  $Z^{(PS)}(\Lambda, h) = \sum_{q \in Q} Z_q^{(PS)}(\Lambda, h)$ , the corollary readily follows from Theorem II.1.11 (4). Q.E.D.

Few notes concerning the phase diagram of the Pirogov-Sinai models follow.

Let us consider a driving parameter  $h \in \mathfrak{U} \subset \mathbb{R}^\nu$  with  $\nu = |Q| - 1$ , where  $\mathfrak{U}$  is the open set appearing in the Peierls condition (Assumption II.1.9). Whenever the energies  $\{e_q(h)\}$  and  $\{E(\gamma)\}$  are differentiable in  $h$ , the functional  $\mathcal{F}'_q(\gamma)$  may be introduced in such a way that it is also differentiable, see Theorem II.1.15 (1). Then in a great deal of situations one can prove that for all  $h \in \mathfrak{U}$

- (†) the square matrix  $\mathbb{E}$  of the order  $|Q| - 1$  whose rows are the vectors  $\nabla(e_q(h) - e_{q_0}(h))$ ,  $q_0 \in Q$  fixed,  $q \in Q \setminus \{q_0\}$ , is regular (the *degeneracy-breaking condition*);
- (‡) its inverse  $\mathbb{E}^{-1}$  satisfies the bound

$$\max_{1 \leq i \leq \nu} \sum_{q \in Q \setminus \{q_0\}} |(\mathbb{E}^{-1})_{iq}| \leq M \quad (\text{II.1.41})$$

for some fixed constant  $M < \infty$ .

REMARK II.1.14. The condition (†) is equivalent to saying that the  $|Q|$  vectors  $\nabla e_q(h)$ ,  $q \in Q$ , are affinely independent, i.e. their convex hull is a  $(|Q| - 1)$ -dimensional simplex. Notice that for  $Q = \{+, -\}$  the two above conditions only means that  $|\frac{d}{dh}(e_+ - e_-)| \geq 1/M > 0$ .

In view of Theorem II.1.11 (1) and Lemma II.1.7, one has

$$f_q(h) = e_q(h) + O(e^{-\beta\tau/4}) \quad (\text{II.1.42})$$

for all  $h \in \mathfrak{U}$ , c.f. (II.1.37), which can be generalised to the first derivative:

$$\frac{df_q(h)}{dh} = \frac{de_q(h)}{dh} + O(e^{-\beta\tau/4}) \quad (\text{II.1.43})$$

for all  $h \in \mathfrak{U}$ . As a consequence, for all  $h \in \mathfrak{U}$

- (†') the matrix  $\mathbb{F}$  whose rows are the vectors  $\nabla(f_q(h) - f_{q_0}(h))$  is regular;

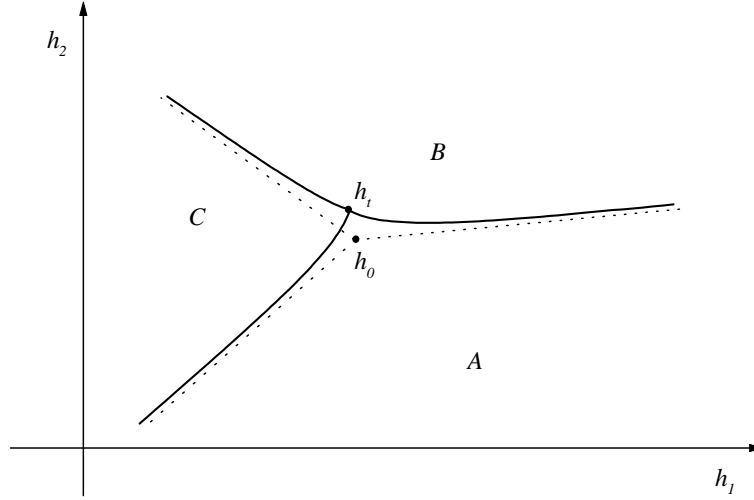


FIGURE 1. The low-temperature phase diagram (full lines) of a Pirogov-Sinai model is a deformation of the zero-temperature one (dashed lines). Here  $\mathfrak{U} = \mathbb{R}^2$ , the set  $Q = \{A, B, C\}$ , and the regions in which  $f_A, f_B$ , and  $f_C$ , respectively, is minimal, is depicted.

( $\ddagger'$ ) its inverse  $\mathbb{F}^{-1}$  satisfies a bound of the form (II.1.41), the corresponding constant on the right-hand side being slightly larger than  $M$ .

If there is a point  $h_0 \in \mathfrak{U}$  at which all the energies  $\{e_q(h)\}$  are equal, the inverse-function theorem combined with ( $\ddagger'$ ) and ( $\ddagger$ ) yields the existence of a unique point  $h_t \in \mathfrak{U}$  for which all  $\{f_q(h)\}$  are equal (all the colours are stable at  $h_t$ ), and  $h_t = h_0 + O(e^{-b\beta\tau})$ . More generally, it is possible to construct differentiable curves  $\{\ell_q(u)\}$  starting at  $h_t$  on which only  $f_q$  is not minimal (only the colour  $q$  is not stable), surfaces  $\{\ell_{q\bar{q}}(u, v)\}$  on which only  $f_q$  and  $f_{\bar{q}}$  are not minimal (only the colours  $q$  and  $\bar{q}$  are not stable), etc. Thus, the phase diagram for large inverse temperature  $\beta$  is  $|Q|$ -regular and a deformation of the order  $e^{-b\beta\tau}$  of the phase diagram at  $\beta = \infty$ , see Fig. 1.

In the end, let us point out that each stable colour  $q \in Q$  leads to a distinct pure phase (translation-invariant extrem Gibbs measure) and, moreover, these are all the pure phases of the model (completeness of the phase diagram) [Zah84]. The thermodynamics of the model can therefore be computed from its metastable free energies  $\{f_q(h)\}$ , after all.

**II.1.3. Finite-Size Effects.** When investigating the finite-size effects for the Pirogov-Sinai models, it is important to introduce the corresponding metastable models so that the free energies  $\{f_q\}$  are smooth functions of  $h$  on the open set  $\mathfrak{U} \subset \mathbb{R}^\nu$  (see Assumption

II.1.9). To this end, one needs to assume the energies  $\{e_q\}$  and  $\{E(\gamma)\}$  to be smooth and take care that the metastable functional  $\mathcal{F}'_q(\gamma)$  is introduced to be smooth as well. Such a construction, basically similar to the one outlined in the preceding subsection, is given in [BK90] and [BK95], where the models with the periodic and ‘weak’ boundary conditions, respectively, are studied. Here we formulate the main results of this finite-size analysis. We adhere to the setting used therein in which the models are equivalently defined by the energies  $\{e_q\}$  and weights  $\{\rho(\gamma)\}$ , c.f. Definition II.1.3.

In the case of the *periodic boundary conditions* [BK90], one studies translation-invariant Pirogov-Sinai models on a  $d$ -dimensional torus  $T$  with sides of the length  $L \in \mathbb{N}$  in each direction. We shall use  $Z_{\text{per}}(T, h)$  to denote the corresponding partition function. The support of a contour is a connected union of closed unit cubes in  $T$ , the connectedness being in the sense of sharing the  $(d - 1)$ -dimensional faces. The energies  $\{e_q\}$  and weights  $\{\rho(\gamma)\}$  are supposed to be  $C^4(\mathfrak{U})$  functions of  $h$  such that for all  $h \in \mathfrak{U}$  one has:

$$(a) \quad |\rho(\gamma)| \leq e^{-\beta(\tau+e)|\gamma|} \quad (\text{Assumption II.1.9}), \quad (\text{II.1.44})$$

$$(b) \quad \left| \frac{d^k \rho(\gamma)}{dh^k} \right| \leq C_{|k|} e^{-\beta(\tau+e)|\gamma|}, \quad (\text{II.1.45})$$

$$(c) \quad \left| \frac{d^k e_q}{dh^k} \right| \leq C_{|k|}. \quad (\text{II.1.46})$$

Here  $\tau > 0$  is a constant,  $k : \{1, \dots, \nu\} \rightarrow \{0, 1, \dots\}$  is a multi-index satisfying  $1 \leq |k| \equiv \sum k_i \leq 4$ , and  $C_1, \dots, C_4$  are positive constants independent of  $h$  and  $\beta$ .

The following is Theorem 4.1 from [BK90]. We note that the constant  $b$  below can be chosen arbitrarily close to 1 if  $\beta$  is taken large enough.

**THEOREM II.1.15 (Borgs, Kotecký).** *Let (II.1.44) to (II.1.46) be true. There exists a constant  $b \in (0, 1)$  depending on  $d$  and metastable free energies  $\{f_q(h); q \in Q\}$  such that for all  $\beta$  and  $L$  are sufficiently large, for every multi-index  $k$  with  $0 \leq |k| \leq 4$ , and for any  $h \in \mathfrak{U}$  the following holds.*

- (1) *The functions  $\{f_q\}$  are  $C^4(\mathfrak{U})$  in  $h$ , and*

$$\frac{d^k}{dh^k} (f_q - e_q) = O(e^{-b\beta\tau}). \quad (\text{II.1.47})$$

- (2) *The free energy  $f^{(PS)}$  equals  $\min_{q \in Q} f_q$ .*  
 (3) *We have*

$$\frac{d^k}{dh^k} \left\{ Z_{\text{per}}(T, h) - \sum_{q \in Q} e^{-\beta f_q(h) L^d} \right\} = e^{-\beta f(h) L^d} O(e^{-b\beta\tau L/2}). \quad (\text{II.1.48})$$

This theorem permits to carry out rigorous analysis of the finite-volume quantities with the periodic boundary conditions, namely, the magnetization and susceptibility,

$$m_{\text{per}}(L, h) := \frac{1}{\beta L^d} \frac{d}{dh} \log Z_{\text{per}}(T, h) \quad (\text{II.1.49})$$

and

$$\chi_{\text{per}}(L, h) := \frac{dm_{\text{per}}(L, h)}{dh}. \quad (\text{II.1.50})$$

Let us state the results for models with two ground states given in Section 3 of [BK90].

**COROLLARY II.1.16.** *Let  $Q = \{+, -\}$  and  $\nu = 1$ . Let there be a point  $h_0 \in \mathfrak{A}$  such that  $e_+(h_0) = e_-(h_0)$ . Assuming also that the signs are chosen so that  $\frac{d}{dh}(e_+ - e_-) < 0$ , for  $\beta$  and  $L$  sufficiently large we have:*

- (1) *There exists a unique point  $h_t \in \mathfrak{A}$  at which  $f_+$  and  $f_-$  coincide, and  $h_t = h_0 + O(e^{-b\beta\tau})$ .*
- (2) *There is a single point  $h_{\chi_{\text{per}}}(L) \in \mathfrak{A}$  at which the susceptibility attains its maximum over  $\mathfrak{A}$ , and*

$$h_{\chi_{\text{per}}}(L) = h_t + \frac{3\Delta\chi}{2(\Delta m)^3 L^{2d}} + O(L^{-3d}). \quad (\text{II.1.51})$$

- (3) *If  $|h - h_t| \leq O(L^{-1})$ , precisely, if  $|f_+(h) - f_-(h)| \leq \frac{\tau}{2L}$ , then*

$$m_{\text{per}}(L, h) = m_0 + \Delta m \tanh \{ \beta \Delta m (h - h_{\chi_{\text{per}}}(L)) L^d \} + O(L^{-1}) \quad (\text{II.1.52})$$

and

$$\chi_{\text{per}}(L, h) = (\Delta m)^2 L^d \cosh^{-2} \{ \beta \Delta m (h - h_{\chi_{\text{per}}}(L)) L^d \} + O(L^{d-1}). \quad (\text{II.1.53})$$

- (4) *If  $|f_+(h) - f_-(h)| \geq \frac{\tau}{2L}$ , then*

$$m_{\text{per}}(L, h) = -\frac{df(h)}{dh} + O(e^{-b\beta\tau L}) \quad (\text{II.1.54})$$

and

$$\chi_{\text{per}}(L, h) = -\frac{d^2 f(h)}{dh^2} + O(e^{-b\beta\tau L}). \quad (\text{II.1.55})$$

Here

$$m_0 := -\frac{1}{2} \left( \frac{df(h_t + 0)}{dh} + \frac{df(h_t - 0)}{dh} \right), \quad (\text{II.1.56})$$

$$\Delta m := -\frac{1}{2} \left( \frac{df(h_t + 0)}{dh} - \frac{df(h_t - 0)}{dh} \right) > 0, \quad (\text{II.1.57})$$

and

$$\Delta\chi := -\frac{1}{2} \left( \frac{d^2 f(h_t + 0)}{dh^2} - \frac{d^2 f(h_t - 0)}{dh^2} \right). \quad (\text{II.1.58})$$

REMARK II.1.17.

(i) It is exactly the problem of evaluating of the location of the maximum of the susceptibility in the part (4) why one takes the derivatives up to namely the fourth order.

(ii) The widest interval around  $h_t$  where one has a synchronal cluster-expansion control over both the  $+$ -th and the  $-$ -th partition functions in  $T$  is exactly given by the condition  $|f_+(h) - f_-(h)| \leq \frac{\tau}{2L}$ , c.f. Theorem II.1.11 (3). Thus, the system can be examined accurately here, which is crucial for deriving (II.1.52), (II.1.53), and (II.1.51).

(iii) Notice that the quantities  $m_0$ ,  $\Delta m$ , and  $\Delta\chi$  are, for a given model, universal, i.e. independent of the boundary conditions.

Let us look at the case of the *weak boundary conditions* [BK95] now. Here one considers spin models in a finite box  $\mathbf{B} := \{1, \dots, L\}^d$  with  $L \in \mathbb{N}$  and the corresponding volume  $\square := [\frac{1}{2}, L + \frac{1}{2}]^d \subset \mathbb{R}^d$  (the union of all the closed unit cubes  $c(x)$  with centres  $x \in \mathbf{B}$ ). Compared to the periodic boundary conditions, the main difference is caused by the presence of boundary effects, which makes the situation much more subtle. Introducing the set of elementary cells as the set of all closed unit cubes in  $\square$ , all closed  $(d-1)$ -dimensional faces of these cubes,  $\dots$ , and all closed edges of these cubes, two elementary cells are said to be connected if their intersection is non-empty. The support of a contour is a connected union of elementary cells.

In order to cover a wider range of realistic situations, one removes here the up-to-now used assumption of the translation invariance of the model. Namely, the energies  $\{e_q(x); q \in Q, x \in \mathbf{B}\}$  and weights  $\{\rho(\gamma)\}$  are supposed to be translation-invariant whenever  $\text{supp } \gamma$  and  $c(x)$  do not touch (have no common points with) the boundary of  $\square$ , while they are assumed translation-invariant along a  $(d-k)$ -dimensional face of  $\partial\square$  every time  $\text{supp } \gamma$  and  $c(x)$  do not touch the  $(d-k-1)$ -dimensional boundary of this face. The partition function now has the form

$$Z_{\text{weak}}(\square, h) = \sum_{\mathfrak{G} \sqsubset \square} \prod_{q \in Q} e^{-\beta \mathcal{E}_q(\square_q(\mathfrak{G}))} \prod_{\gamma \in \mathfrak{G}} \rho(\gamma), \quad (\text{II.1.59})$$

where the sum is over all admissible families of contours in  $\square$ , the set  $\square_q(\mathfrak{G})$  is the union of all the components of  $\square \setminus \cup_{\gamma \in \mathfrak{G}} \text{supp } \gamma$  with the boundary colour  $q \in Q$ ,  $\mathcal{E}_q(\square_q(\mathfrak{G})) := \sum_{x: c(x) \subset \text{cl } \square_q(\mathfrak{G})} e_q(x)$ , and the

( $\mathfrak{G} = \emptyset$ )-term is set to equal  $\sum_{q \in Q} e^{-\beta \mathcal{E}_q(\square)}$ . Moreover, one assumes that  $\{e_q(x)\}$  and  $\{\rho(\gamma)\}$  are  $C^4(\mathfrak{U})$  functions of  $h$  obeying

$$(a) \quad |\rho(\gamma)| \leq e^{-\beta(\tau|\gamma| - \mathcal{E}(\gamma))} \quad (\text{Assumption II.1.9}), \quad (\text{II.1.60})$$

$$(b) \quad \left| \frac{d^k \rho(\gamma)}{dh^k} \right| \leq |k|! (C_0 |\gamma|)^{|k|} e^{-\beta(\tau|\gamma| - \mathcal{E}(\gamma))}, \quad (\text{II.1.61})$$

$$(c) \quad \left| \frac{d^k e_q(x)}{dh^k} \right| \leq C_0^{|k|}, \text{ and} \quad (\text{II.1.62})$$

$$(d) \quad |e_q(x) - e_q| \leq \lambda \tau. \quad (\text{II.1.63})$$

for all  $h \in \mathfrak{U}$ . Here  $\tau > 0$  is a constant,  $|\gamma|$  is the number of elementary cells (not contained in higher-dimensional cells) of  $\text{supp } \gamma$ ,

$$\mathcal{E}(\gamma) := \sum_{x: c(x) \subset \text{supp } \gamma} e(x) \quad \text{with} \quad e(x) := \min_{q \in Q} e_q(x), \quad (\text{II.1.64})$$

$k$  is a multi-index satisfying  $1 \leq |k| \leq 4$ , the constant  $C_0$  is independent of  $h$  and  $\beta$ , the energy  $e_q := e_q(x)$  for any  $x$  with  $c(x)$  not touching  $\partial \square$  (the bulk energy), and  $\lambda \in (0, 1)$  is to be specified later. The condition (d) avoids a situation where the boundary conditions strongly favour some phases inside  $\square$  (hence the name weak b.c.).

Let us define the functions  $\{e_q^{(i)}; i = 0, \dots, d\}$  via

$$\sum_{x: c(x) \subset \square} e_q(x) = \sum_{i=1}^d e_q^{(i)} |\partial_i \square|, \quad (\text{II.1.65})$$

where  $|\partial_d \square| \equiv |\square|$  and  $|\partial_i \square|$ ,  $0 \leq i \leq d-1$ , is the joint  $i$ -dimensional area of all  $i$ -dimensional faces of  $\square$ , i.e.  $|\partial_i \square| = 2^{d-i} \binom{d}{i} L^i$ . Note that  $e_q^{(d)} = e_q$ . We now state Theorem 3.1 from [BK95]. Again, the constant  $b$  below can be chosen arbitrarily close to 1 if  $\beta$  is taken large enough.

**THEOREM II.1.18 (Borgs, Kotecký).** *Let (II.1.60) through (II.1.63) be satisfied. There exist constants  $b \in (0, 1)$  and  $\lambda_0 \in (0, 1/144d)$  depending on  $d$  and metastable free energies  $\{f_q(h) \equiv f_q^{(d)}(h)\}$ , surface free energies  $\{f_q^{(d-1)}(h)\}$ , ..., edge free energies  $\{f_q^{(1)}(h)\}$ , and corner free energies  $\{f_q^{(0)}(h)\}$  such that for all  $\beta$  and  $L$  sufficiently large, every multi-index  $k$  with  $0 \leq |k| \leq 4$ , any  $\lambda < \lambda_0$ , and any  $h \in \mathfrak{U}$  the following is true.*

(1) *The functions  $\{f_q^{(i)}; i = 0, \dots, d\}$  are  $C^4(\mathfrak{U})$  in  $h$ , and*

$$\frac{d^k}{dh^k} (f_q^{(i)} - e_q^{(i)}) = O(e^{-b\beta\tau}). \quad (\text{II.1.66})$$

(2) *The free energy  $f$  equals  $\min_{q \in Q} f_q$ .*

(3) We have

$$\frac{d^k}{dh^k} \left\{ Z_{\text{weak}}(\square, h) - \sum_{q \in Q} e^{-\beta F_q(\square, h)} \right\} = O(e^{-b\beta\tau L}) \max_{q \in Q} e^{-\beta F_q(\square, h)}, \quad (\text{II.1.67})$$

where

$$F_q(\square, h) := \sum_{i=1}^d f_q^{(i)}(h) |\partial_i \square|. \quad (\text{II.1.68})$$

REMARK II.1.19.

(i) The requirement that  $\lambda < \frac{1}{144d}$  follows from relations (3.17) and (4.47b) of [BK95]. It may well happen, however, that in particular cases this condition can be fairly weakened. We show this for the Potts model in Chapter III.

(ii) It is a straightforward consequence of Theorem II.1.18 (1) that for all  $h \in \mathfrak{U}$  and  $\beta$  and  $L$  sufficiently large

(†'') the matrix  $\mathbb{F}_L$  of the order  $|Q| - 1$  whose rows are the vectors  $L^{-d} \nabla(F_q(\square, h) - F_{q_0}(\square, h))$  is regular;

(‡'') its inverse  $\mathbb{F}_L^{-1}$  satisfies a bound of the form (II.1.41) with a slightly larger constant on the right-hand side.

So, if there is a point  $h_0 \in \mathfrak{U}$  at which all the energies  $\{e_q(h)\}$  are equal, the inverse-function theorem combined with (†'') and (‡'') implies that there is a unique point  $h_t(L) \in \mathfrak{U}$  for which all  $\{F_q(\square, h)\}$  are equal, and  $h_t(L) = h_t + O(L^{-1})$  (here  $h_t$  is the point introduced at the end of Subsection II.1.2). In addition, one may construct differentiable curves starting at  $h_t(L)$  on which only  $F_q(\square, h)$  is not minimal, surfaces on which only  $F_q(\square, h)$  and  $F_{\bar{q}}(\square, h)$  are not minimal, etc. The picture thus obtained is, for large  $L$ , just a deformation of the order  $L^{-1}$  of the low-temperature phase diagram at  $L = \infty$ .

Introducing the finite-volume magnetization and susceptibility

$$m_{\text{weak}}(L, h) := \frac{1}{\beta L^d} \frac{d}{dh} \log Z_{\text{weak}}(\square, h) \quad (\text{II.1.69})$$

and

$$\chi_{\text{weak}}(L, h) := \frac{dm_{\text{weak}}(L, h)}{dh}, \quad (\text{II.1.70})$$

we have this corollary (see Theorem A in [BK95]).

COROLLARY II.1.20. Let  $Q = \{+, -\}$  and  $\nu = 1$ . Let there be a point  $h_0 \in \mathfrak{U}$  such that  $e_+^{(d)}(h_0) = e_-^{(d)}(h_0)$ . Assuming also that the signs are chosen so that  $\frac{d}{dh}(e_+^{(d)} - e_-^{(d)}) < 0$ , for  $\beta$  and  $L$  sufficiently large we have:

- (1) There exists a unique point  $h_t \in \mathfrak{U}$  at which  $f_+$  and  $f_-$  coincide, and  $h_t = h_0 + O(e^{-b\beta\tau})$ .
- (2) There is a single point  $h_{\chi_{\text{weak}}}(L) \in \mathfrak{U}$  at which the susceptibility attains its maximum, and

$$h_{\chi_{\text{weak}}}(L) = h_t + \frac{\Delta F(L)}{2\Delta m L^d} (1 + O(L^{-1})) + \frac{3\Delta\chi}{2(\Delta m)^3 L^{2d}} + O(L^{-3d}). \quad (\text{II.1.71})$$

- (3) If  $|h - h_t| \leq O(L^{-1})$ , precisely, if  $|f_+(h) - f_-(h)| \leq \frac{\tau}{2L}$ , then

$$m_{\text{weak}}(L, h) = m_0 + \Delta m \tanh \{ \beta \Delta m (h - h_{\chi_{\text{weak}}}(L)) L^d \} + O(L^{-1}) \quad (\text{II.1.72})$$

and

$$\chi_{\text{weak}}(L, h) = (\Delta m)^2 L^d \cosh^{-2} \{ \beta \Delta m (h - h_{\chi_{\text{weak}}}(L)) L^d \} + O(L^{d-1}). \quad (\text{II.1.73})$$

- (4) If  $|f_+(h) - f_-(h)| \geq \frac{\tau}{2L}$ , then

$$m_{\text{weak}}(L, h) = -\frac{df(h)}{dh} + O(L^{-1}) \quad (\text{II.1.74})$$

and

$$\chi_{\text{weak}}(L, h) = -\frac{d^2 f(h)}{dh^2} + O(L^{-1}). \quad (\text{II.1.75})$$

Here  $\Delta F(L) := F_+(\square, h_t) - F_-(\square, h_t)$  and  $m_0$ ,  $\Delta m$ , and  $\Delta\chi$  are defined as in Corollary II.1.16.

REMARK II.1.21.

(i) Note that  $\Delta F(L) = O(L^{d-1})$  because of the part (1). For asymmetric models, it is typical that  $f_+^{(d-1)} \neq f_-^{(d-1)}$ , i.e.  $\Delta F(L)$  is of the order  $L^{d-1}$ , leading to the shift of  $h_{\chi_{\text{weak}}}(L)$  with respect to  $h_t$  of the order  $L^{-1}$ . On the other hand, it could also turn out that  $\Delta F(L) = 0$  in some cases. Then the shift is asymptotically the same as for the periodic boundary conditions.

(ii) The Taylor expansion reveals that  $|f_+(h) - f_-(h)| \leq \frac{\tau}{2L}$  is equivalent to the restriction of  $h$  to the interval given by the inequality  $|h - h_t| \leq \frac{\tau}{4\Delta m} \frac{1}{L} + O(L^{-2})$ . The point  $h_{\chi_{\text{weak}}}(L)$  lies within this interval. Indeed, using (II.1.63), (II.1.66), (II.1.68), and (II.1.71), we readily

get

$$\begin{aligned} |h_{\chi_{\text{weak}}}(L) - h_t| &\leq \frac{2d(2\lambda\tau + O(e^{-b\tau}))}{2\Delta m L} + O(L^{-2}) \leq \frac{3d\lambda\tau}{\Delta m} \frac{1}{L} \\ &\leq \frac{\tau}{48\Delta m} \frac{1}{L} \end{aligned} \quad (\text{II.1.76})$$

for any  $\lambda \in (0, 1/144d)$  and all  $\tau$  and  $L$  large.

(iii) Let  $\Lambda$  be either the torus  $T$  or the box  $\square$  and  $Z^{\text{b.c.}}(\Lambda, h)$  the corresponding partition function (here ‘b.c.’ stands either for ‘per’ or ‘weak’). Writing

$$Z_{\text{b.c.}}(\Lambda, h) = Z_+^{\text{b.c.}}(\Lambda, h) + Z_-^{\text{b.c.}}(\Lambda, h), \quad (\text{II.1.77})$$

an essential part of the proof of Corollary II.1.16 and Corollary II.1.20 is, arguably, to show (see Theorem 3.3 (ii) of [BK90] and Lemma 6.2 of [BK95]) that there exists a single point  $h_t^{\text{b.c.}}(L)$  such that

$$Z_+^{\text{b.c.}}(\Lambda, h_t^{\text{b.c.}}(L)) = Z_-^{\text{b.c.}}(\Lambda, h_t^{\text{b.c.}}(L)) \quad (\text{II.1.78})$$

and guarantee (by rejecting of inappropriate boundary conditions) that  $|f_+(h_t^{\text{b.c.}}(L)) - f_-(h_t^{\text{b.c.}}(L))| < \frac{\tau}{2L}$  (that is, both  $Z_+^{\text{b.c.}}(\Lambda, h)$  and  $Z_-^{\text{b.c.}}(\Lambda, h)$  can be controlled by cluster expansions around  $h_t^{\text{b.c.}}(L)$ ). As one intuitively anticipates, it is a point near which the rounded transition takes place. In fact, it turns out that

$$h_{\chi_{\text{b.c.}}}(L) - h_t^{\text{b.c.}}(L) = \frac{3\Delta\chi}{2(\Delta m)^3} \frac{1}{L^{2d}} + O(L^{-3d}). \quad (\text{II.1.79})$$

## II.2. Large Deviations

Let us consider a sequence of probability measures  $\{P_n\}$  converging weakly to the unit point measure  $\delta_{x_0}$  as  $n \rightarrow \infty$  and let  $A$  be an event for which  $x_0 \notin \text{cl } A$ . Then, according to a standard result, one has  $\lim_{n \rightarrow \infty} P_n(A) = \delta_{x_0}(\text{cl } A) = 0$ ; the point  $x_0$  is typical (the event  $A$  is deviant) with respect to the probability measures  $P_n$  as  $n$  increases. Could we say anything about the rate at which  $P_n(A)$  tends to zero? Although the answer is in general negative, in the case of extremely deviant events — for which  $P_n(A)$  decays to zero exponentially fast — the situation is more promising: a variety of techniques and methods has been developed to this end, see [DS89, DZ98, Ell85], for instance. Basically, an assertion that  $P_n(A) \rightarrow 0$  at a specific exponential rate is called a large-deviation principle for the sequence  $\{P_n\}$ .

From a physical point of view, the extremely deviant events correspond to large fluctuations of the densities of the order parameters (e.g., the mean energy or magnetization). However, it is characteristic of physical systems to exhibit phase transitions, which makes their large-deviation analysis rather complicated.

In this section we present the very basic notions and statements of the large-deviation theory. Throughout this section we restrict ourselves to the probabilities on the measurable space  $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ ,  $N \in \mathbb{N}$ , where  $\mathcal{B}(\mathbb{R}^N)$  is the Borel  $\sigma$ -field on  $\mathbb{R}^N$ , although everything can be formulated in a much more general setting [DS89, DZ98].

DEFINITION II.2.1.

- (1) A sequence  $\{(P_n)^{\varepsilon_n}\}$ , where  $P_n$  is a probability measure and  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  in  $(0, 1]$ , is called a *large-deviation sequence* (LD sequence).
- (2) A *rate* is a lower semi-continuous<sup>7</sup> function  $I : \mathbb{R}^N \rightarrow [0, \infty]$ .

DEFINITION II.2.2. We say that an LD sequence  $\{(P_n)^{\varepsilon_n}\}$  satisfies the *full large-deviation principle* with the rate  $I$  (notated as  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  fully) if

$$\sup_G e^{-I} \leq \underline{\lim}_{n \rightarrow \infty} (P_n(G))^{\varepsilon_n} \quad \text{for all } G \subset \mathbb{R}^N \text{ open and} \quad (\text{II.2.1})$$

$$\overline{\lim}_{n \rightarrow \infty} (P_n(F))^{\varepsilon_n} \leq \sup_F e^{-I} \quad \text{for all } F \subset \mathbb{R}^N \text{ closed.} \quad (\text{II.2.2})$$

A *weak large-deviation principle* for  $\{(P_n)^{\varepsilon_n}\}$  with the rate  $I$  is the statement (II.2.1) and

$$\overline{\lim}_{n \rightarrow \infty} (P_n(K))^{\varepsilon_n} \leq \sup_K e^{-I} \quad \text{for all } K \subset \mathbb{R}^N \text{ compact.} \quad (\text{II.2.3})$$

In this case the notation is  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  weakly.

REMARK II.2.3. If  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  fully or weakly, then the rate  $I$  is unique (Lemma 4.1.4 and Exercise 4.1.30 in [DZ98]).

It is now natural to ask when  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  weakly implies that  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  fully (the converse implication is clearly always true). One anticipates that this can only happen when the probability measures  $P_n$  are (on an exponential scale) concentrated on compact sets. The answer is given by the next lemma (see Lemma 2.1.5 in [DS89]).

DEFINITION II.2.4.

- (1) An LD sequence  $\{(P_n)^{\varepsilon_n}\}$  is said to be *exponentially tight* if for every  $\varepsilon > 0$  there exists a compact set  $K$  (depending on  $\varepsilon$ ) such that  $\overline{\lim}_{n \rightarrow \infty} (P_n(K^c))^{\varepsilon_n} \leq \varepsilon$ .
- (2) A rate  $I$  is *good* if each level set  $\text{lev}_r(I)$  of  $I$  is compact.

<sup>7</sup>That is, the level set  $\text{lev}_r(I) := \{x \in \mathbb{R}^N : I(x) \leq r\}$  is closed for all  $r < \infty$ .

LEMMA II.2.5. *Let  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  weakly. If  $\{(P_n)^{\varepsilon_n}\}$  is exponentially tight, then  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  fully and, moreover, the rate  $I$  is good.*

Let us introduce further useful notions and state basic properties concerning a rate  $I$ .

DEFINITION II.2.6. Let  $I$  be a rate.

- (1) Its *effective domain* is the set  $\text{dom } I := \{x \in \mathbb{R}^N : I(x) < \infty\}$ .
- (2) Its *minimum set* is defined as the set of those points where  $I$  attains its global infimum, i.e.  $\mathcal{M}(I) := \text{lev}_{\inf_{\mathbb{R}^N} I} I \subset \text{dom } I$ .
- (3) A set  $A \in \mathcal{B}(\mathbb{R}^N)$  is  *$I$ -continuous* if  $\inf_{\text{int } A} I = \inf_{\text{cl } A} I$ .

LEMMA II.2.7. *Let  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  fully.*

- (1) *If  $A \in \mathcal{B}(\mathbb{R}^N)$  is  $I$ -continuous, then*

$$\lim_{n \rightarrow \infty} (P_n(A))^{\varepsilon_n} = \sup_A e^{-I}. \quad (\text{II.2.4})$$

- (2) *The minimum set of  $I$  is closed and  $\mathcal{M}(I) = \{x \in \mathbb{R}^N : I(x) = 0\}$ , i.e.  $\inf_{\mathbb{R}^N} I = 0$ .*
- (3) *It follows that  $\text{dom } I \neq \emptyset$ . In addition, whenever  $I$  is good, then also  $\mathcal{M}(I) \neq \emptyset$ .*

PROOF. The part (1) is obvious. In order to prove the part (2), note first that each level set of  $I$  is closed, and therefore so is  $\mathcal{M}(I)$ . Moreover, as  $\mathbb{R}^N$  is always  $I$ -continuous and  $P_n(\mathbb{R}^N) = 1$  for all  $n$ , the part (1) now yields that  $\inf_{\mathbb{R}^N} I = 0$ . It remains to prove the part (3).

The fact that  $\text{dom } I \neq \emptyset$  is clear: the contrary would imply  $I \equiv \infty$ , and the global infimum of  $I$  could not be zero.

Let  $I$  be good and let  $x \in \text{dom } I$ . Then the level set  $\text{lev}_{I(x)} I$  is non-empty (it surely contains the point  $x$ ) and compact. Since a lower semi-continuous function achieves its infimum over any compact set (Theorem B.1 of [DZ98]), there exists a point  $y \in \text{lev}_{I(x)} I$  such that  $I(y) = \inf_{\text{lev}_{I(x)} I} I$ . As  $0 = \inf_{\text{dom } I} I = \inf_{\text{lev}_{I(x)} I} I$ , we see that  $I(y) = 0$ . Q.E.D.

The following lemma gives a standard expression of a rate of a weak large-deviation principle.

LEMMA II.2.8. *Let  $\mathcal{U}_\varepsilon(x)$  be the open ball of the radius  $\varepsilon > 0$  around the point  $x \in \mathbb{R}^N$ . Then  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  weakly iff*

$$\lim_{\varepsilon \rightarrow 0^+} \underline{\lim}_{n \rightarrow \infty} (P_n(\mathcal{U}_\varepsilon(x)))^{\varepsilon_n} = e^{-I(x)} = \lim_{\varepsilon \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} (P_n(\mathcal{U}_\varepsilon(x)))^{\varepsilon_n}. \quad (\text{II.2.5})$$

PROOF. (a) Let  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  weakly first, and let us consider the function  $\varphi_x(\varepsilon) : \varepsilon \in (0, \infty) \mapsto \inf_{\mathcal{U}_\varepsilon(x)} I$  with any  $x \in \mathbb{R}^N$ . Since  $\varphi_x(\varepsilon)$  is non-increasing, for every continuity point  $\varepsilon > 0$  of this function we have  $\inf_{\mathcal{U}_\varepsilon(x)} I = \inf_{\text{cl}\mathcal{U}_\varepsilon(x)} I$ . Lemma II.2.7 (1) now yields that

$$\lim_{n \rightarrow \infty} (P_n(\mathcal{U}_\varepsilon(x)))^{\varepsilon_n} = \sup_{\mathcal{U}_\varepsilon(x)} e^{-I} \quad (\text{II.2.6})$$

is true for a dense subset of  $(0, \infty)$ . Since  $I(x) = \lim_{\varepsilon \rightarrow 0^+} \inf_{\mathcal{U}_\varepsilon(x)} I$  by the lower semi-continuity of  $I$ , we arrive at (II.2.5).

(b) Let (II.2.5) holds. Given any  $G \subset \mathbb{R}^N$  open, there is  $\varepsilon > 0$  small such that  $\mathcal{U}_\varepsilon(x) \subset G$  for all  $x \in G$ . Hence,

$$\underline{\lim}_{n \rightarrow \infty} (P_n(G))^{\varepsilon_n} \geq \underline{\lim}_{n \rightarrow \infty} (P_n(\mathcal{U}_\varepsilon(x)))^{\varepsilon_n}. \quad (\text{II.2.7})$$

Taking the limit  $\varepsilon \rightarrow 0^+$  and using that  $x \in G$  is arbitrary, this yields (II.2.1). Next, let  $K \subset \mathbb{R}^N$  be compact and let  $\varepsilon > 0$ . From the open cover  $\cup_{x \in K} \mathcal{U}_\varepsilon(x)$  of  $K$  we can always extract its finite open cover. Let it be  $\cup_i \mathcal{U}_\varepsilon(x_i)$ , where  $x_i \in K$  and  $i = 1, \dots, N$  with  $N < \infty$  fixed. As

$$P_n(K) \leq P_n(\cup_i \mathcal{U}_\varepsilon(x_i)) \leq \sum_i P_n(\mathcal{U}_\varepsilon(x_i)) \leq N \max_i P_n(\mathcal{U}_\varepsilon(x_i))$$

for all  $n$ , we find

$$\overline{\lim}_{n \rightarrow \infty} (P_n(K))^{\varepsilon_n} \leq \max_i \overline{\lim}_{n \rightarrow \infty} (P_n(\mathcal{U}_\varepsilon(x_i)))^{\varepsilon_n} \quad (\text{II.2.8})$$

The limit  $\varepsilon \rightarrow 0^+$  leads to (II.2.3). Q.E.D.

Next, we state an important Varadhan's theorem of the large-deviation theory, see Section 2.1 in [DS89] or Section 4.3 in [DZ98]. We shall later use it to control the asymptotics of the partition function. First, however, let us introduce

$$P_n^{(\phi)}(A) := \int_A e^{\phi/\varepsilon_n} P_n(dx) \quad (\text{II.2.9})$$

for any LD sequence  $\{(P_n)^{\varepsilon_n}\}$ , any set  $A \in \mathcal{B}(\mathbb{R}^N)$ , and any continuous function  $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ .

**THEOREM II.2.9 (Varadhan).** *Let  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  fully with the rate  $I$  being good. Let  $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$  be a continuous function.*

(1) *As soon as  $\sup_{\mathbb{R}^N} \phi$  is finite, we have*

$$\lim_{n \rightarrow \infty} (P_n^{(\phi)}(\mathbb{R}^N))^{\varepsilon_n} = \sup_{\mathbb{R}^N} e^{\phi-I}. \quad (\text{II.2.10})$$

(2) *More generally, whenever*

$$\lim_{D \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} (P_n^{(\phi)}(\{x \in \mathbb{R}^N : \phi \geq D\}))^{\varepsilon_n} = 0, \quad (\text{II.2.11})$$

the statement (II.2.10) is true.

Given an LD sequence  $\{P_n\}$ , it is usually a non-trivial task to establish that  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$  weakly or fully for some rate  $I$ . The below-stated Gärtner-Ellis theorem (see Section 2.3 in [DZ98] for its proof) provides a solution of this task in some cases. Namely, let us consider a sequence  $\{X_n\}$  of random vectors taking values in  $\mathbb{R}^N$  and let  $\{P_n\}$  be a sequence of distributions of  $X_n$  on  $\mathbb{R}^N$ .

DEFINITION II.2.10. The *logarithmic moment-generating function* of  $X_n$  is

$$\varphi_n(\xi) := \log E_n[e^{\langle \xi, X_n \rangle}], \quad \xi \in \mathbb{R}^N, \quad (\text{II.2.12})$$

where  $E_n$  is the expectation with respect to  $P_n$  and  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product on  $\mathbb{R}^N$ .

THEOREM II.2.11 (Gärtner, Ellis). Let  $\{X_n; X_n \in \mathbb{R}^N\}$  be a sequence of random vectors. Let  $\{(P_n)^{\varepsilon_n}\}$  be an LD sequence such that  $P_n$  is a distribution of  $X_n$  on  $\mathbb{R}^N$ . For any  $\xi \in \mathbb{R}^N$ , let the limit

$$\varphi(\xi) := \lim_{n \rightarrow \infty} \varepsilon_n \varphi_n(\xi/\varepsilon_n) \quad (\text{II.2.13})$$

exists as an extended real number and let the origin belong to the interior of the set  $\text{dom } \varphi := \{\xi \in \mathbb{R}^N : \varphi(\xi) < \infty\}$ .

(1) It follows that

$$\overline{\lim}_{n \rightarrow \infty} (P_n(F))^{\varepsilon_n} \leq \sup_F e^{-\varphi^*} \quad (\text{II.2.14})$$

for all  $F \subset \mathbb{R}^N$  closed. Here  $\varphi^*$  is the Legendre-Fenchel transform of  $\varphi$ ,

$$\varphi^*(x) := \sup_{\xi \in \mathbb{R}^N} \{\langle x, \xi \rangle - \varphi(\xi)\}, \quad x \in \mathbb{R}^N. \quad (\text{II.2.15})$$

(2) If  $\varphi$  is in addition lower semi-continuous and such that

(a) it is differentiable throughout  $\text{int dom } \varphi$ ,

(b)  $\lim_{n \rightarrow \infty} |\nabla \varphi(\xi_n)| = \infty$  whenever  $\{\xi_n\}$  is a sequence of points converging to a boundary point of  $\text{int dom } \varphi$ ,

then  $(P_n)^{\varepsilon_n} \rightarrow e^{-\varphi^*}$  fully, the rate  $\varphi^*$  being good.

REMARK II.2.12.

(i) Due to Hölder's inequality, each  $\varphi_n$  is a convex function on  $\mathbb{R}^N$ ; so is  $\varphi$  as convexity is preserved under pointwise limits.

(ii) Note that the assumptions of the part (2) are in particular satisfied if  $\varphi$  is differentiable (since then the condition (b) holds vacuously).

### II.3. Convex Analysis

In this section we introduce the concept of a convex function on  $\mathbb{R}^N$ ,  $N \in \mathbb{N}$ , and several related notions needed in the following, especially the notion of the sub-differential and convex conjugate (Legendre-Fenchel transform). A standard reference to the topic is the monograph by Rockafellar [**Roc70**], and we refer it for the proofs of the forthcoming statements; another excellent source is [**HL93**].

DEFINITION II.3.1. A function  $f : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{\infty\}$  which is not identically equal to  $\infty$  is called *convex* when

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad (\text{II.3.1})$$

holds as an inequality in  $\mathbb{R} \cup \{\infty\}$  for all  $x, y \in \mathbb{R}^N$  and any  $\alpha \in (0, 1)$ . We shall write  $\text{Conv } \mathbb{R}^N$  for the class of convex functions on  $\mathbb{R}^N$ .

The (*effective*) domain of  $f \in \text{Conv } \mathbb{R}^N$  is the set

$$\text{dom } f := \{x \in \mathbb{R}^N : f(x) < \infty\} \neq \emptyset. \quad (\text{II.3.2})$$

An *affine function* is a finite function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $f$  as well as its negative  $-f$  are both convex.

We say that  $f \in \text{Conv } \mathbb{R}^N$  is *closed* if it is lower semi-continuous. We use  $\overline{\text{Conv}} \mathbb{R}^N$  to denote the set of such convex functions on  $\mathbb{R}^N$ .

REMARK II.3.2.

(i) The above definition coincides with that of the proper convex function used in [**Roc70**]. Since we exclude the value  $-\infty$  from the range of any  $f \in \text{Conv } \mathbb{R}^N$ , the distinction is superfluous.

(ii) The domain  $\text{dom } f$  of any  $f \in \text{Conv } \mathbb{R}^N$  is a *convex set* in  $\mathbb{R}^N$ , i.e. the point  $\alpha x + (1 - \alpha)y$  is in  $\text{dom } f$  for all  $x, y \in \mathbb{R}^N$  and any  $\alpha \in (0, 1)$ .

(iii) A function  $f$  is lower semi-continuous iff

$$f(x_0) \leq \liminf_{x \rightarrow x_0} f(x) = \lim_{\epsilon \rightarrow 0^+} \inf_{|x - x_0| < \epsilon} f(x) \quad (\text{II.3.3})$$

holds in  $\mathbb{R} \cup \{\infty\}$  for all  $x_0 \in \mathbb{R}^N$ .

Although the interior of a line segment or triangle embedded in  $\mathbb{R}^3$  is empty, these do have non-empty interiors in  $\mathbb{R}$  and  $\mathbb{R}^2$ , respectively (on the contrary, their closures are the same in any  $\mathbb{R}^N$ ). This fact leads to the definition of the relative interior.

DEFINITION II.3.3. The *affine hull*  $\text{aff } S$  of any  $S \subset \mathbb{R}^N$  is the affine set<sup>8</sup> given as the intersection of all affine sets containing  $S$ .

The *relative interior*  $\text{ri } C$  of a convex set  $C \subset \mathbb{R}^N$  is the interior of  $C$  for the topology relative to  $\text{aff } C$ ; in other words,  $x \in \text{ri } C$  iff  $x \in \text{aff } C$  and there is  $\delta > 0$  such that  $\text{aff } C \cap \mathcal{U}_\delta(x) \subset C$ . Its *relative boundary* is  $\text{rbd } C := \text{cl } C \setminus \text{ri } C$ .

The *dimension* of a convex set  $C$  is the dimension of its affine hull.

REMARK II.3.4. It is obvious that  $\text{ri } C \subset C \subset \text{cl } C$ . Moreover, for any  $N$ -dimensional convex set  $C$  one has  $\text{aff } C = \mathbb{R}^N$  by definition, so  $\text{ri } C = \text{int } C$ . The convex set  $C$  for which  $\text{ri } C = \text{int } C$  is *relatively open*.

EXAMPLE II.3.5. Let us consider the convex set  $C$  to be a singleton  $\{x\}$ , a line segment  $[x, y]$ ,  $x \neq y$ , and the open ball  $\mathcal{U}_\delta(x)$  in  $\mathbb{R}^N$ . Then  $\text{aff } C$ ,  $\dim C$ , and  $\text{ri } C$  are  $\{x\}$ ,  $0$ , and  $\{x\}$ ; line passing through  $x$  and  $y$ ,  $1$ , and  $(x, y)$ ; and  $\mathbb{R}^N$ ,  $N$ , and  $\mathcal{U}_\delta(x)$ , respectively.

Convex functions turn out to possess remarkable continuity and differentiability properties. We shall now list most important results concerning this point.

Let us say that a function  $f$  on  $\mathbb{R}^N$  is *continuous relative to* a set  $S \subset \mathbb{R}^N$  if the restriction of  $f$  to  $S$  is a continuous function.

THEOREM II.3.6 ([Roc70]: Theorem 10.1). *A function  $f \in \text{Conv } \mathbb{R}^N$  is continuous relative to  $\text{ri}(\text{dom } f)$ . More generally, it is continuous relative to any relatively open convex set  $C \subset \text{dom } f$ .*

COROLLARY II.3.7. *As long as  $f \in \text{Conv } \mathbb{R}^N$  is finite on the whole of  $\mathbb{R}^N$ , it is necessarily continuous.*

A convex function  $f$  does not have to be continuous up to the relative boundary of  $\text{dom } f$ . However, once  $f$  is closed, its continuity on  $\text{rbd}(\text{dom } f)$  is guaranteed. Indeed, according to the above theorem the property of being closed only involves the behaviour of  $f$  on  $\text{rbd}(\text{dom } f)$ . If  $x \in \mathbb{R}^N$  and  $y \in \text{dom } f$ , the convexity of  $f$  implies

$$\overline{\lim}_{\alpha \rightarrow 0^+} f(x + \alpha(y - x)) \leq \overline{\lim}_{\alpha \rightarrow 0^+} \{\alpha f(y) + (1 - \alpha)f(x)\} = f(x). \tag{II.3.4}$$

On the other hand, the closedness of  $f$  yields

$$\underline{\lim}_{\alpha \rightarrow 0^+} f(x + \alpha(y - x)) \geq f(x), \tag{II.3.5}$$

---

<sup>8</sup>A set  $M \subset \mathbb{R}^N$  is affine if it contains the line passing through any two points  $x, y \in M$ .

c.f. (II.3.3). As  $x + \alpha(y - x) \in \text{ri}(\text{dom } f)$  whenever  $x \in \text{rbd}(\text{dom } f)$ ,  $y \in \text{ri}(\text{dom } f)$ , and  $\alpha \in (0, 1]$  (see Theorem 6.1 in [Roc70]), we have the following statement.

**THEOREM II.3.8.** *Let  $f \in \overline{\text{Conv}} \mathbb{R}^N$  and let  $x \in \text{rbd}(\text{dom } f)$ , while  $y \in \text{ri}(\text{dom } f)$ . For any  $\alpha \in (0, 1]$ , the point  $x + \alpha(y - x)$  lies in  $\text{ri}(\text{dom } f)$ , and*

$$\lim_{\alpha \rightarrow 0^+} f(x + \alpha(y - x)) = f(x). \quad (\text{II.3.6})$$

**REMARK II.3.9.** A function  $f \in \overline{\text{Conv}} \mathbb{R}$  can be easily characterized. Namely, a convex function on  $\mathbb{R}$  is closed iff it is continuous (relative to  $\text{dom } f$ ) at each end-point of  $\text{dom } f$  which lies in  $\text{dom } f$  and  $f(x)$  tends to  $\infty$  as  $x$  approaches a finite end-point not in  $\text{dom } f$ .

Let us turn to the differentiability properties of convex functions. We recall that  $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$  is differentiable at  $x \in \mathbb{R}^N$  in which  $f$  is finite if there exists a vector  $\nabla f(x) \in \mathbb{R}^N$  (necessarily unique) such that

$$f(y) = f(x) + \langle \nabla f(x), y - x \rangle + o(|y - x|) \quad (\text{II.3.7})$$

as  $y \rightarrow x$ . The vector  $\nabla f(x)$  is then called the *gradient* of  $f$  at  $x$ . Moreover, the *one-sided directional derivative*  $f'(x; y)$  of  $f$  with respect to any vector  $y$  exists, is two-sided,<sup>9</sup> and a linear function of  $y$ ,

$$f'(x; y) := \lim_{\lambda \rightarrow 0^+} \frac{f(x + \lambda y) - f(x)}{\lambda} = \langle \nabla f(x), y \rangle. \quad (\text{II.3.8})$$

Thus, the partial derivatives  $\frac{\partial f(x)}{\partial x_i}$ ,  $i = 1, \dots, N$ , exist, and it follows that

$$\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_N} \right). \quad (\text{II.3.9})$$

For convex function these implications can almost always be conversed.

**THEOREM II.3.10** ([Roc70]: Theorem 23.1, 25.2, and 25.5). *Let  $f \in \text{Conv } \mathbb{R}^N$  and let  $x \in \text{dom } f$ . For any  $y$ , the difference quotient in the definition (II.3.8) of  $f'(x; y)$  is a non-decreasing function of  $\lambda > 0$ , i.e.  $f'(x; y)$  exists, and*

$$f'(x; y) := \inf_{\lambda > 0} \frac{f(x + \lambda y) - f(x)}{\lambda} \quad (\text{II.3.10})$$

<sup>9</sup>The one-sided derivative of  $f$  is two-sided iff  $f'(x; y) = -f'(x; -y)$ , as one may easily observe from (II.3.8).

The directional derivative  $f'(x; y)$  is linear in  $y$  iff  $f$  is differentiable at  $x$ . The latter holds iff all the partial derivatives  $\frac{\partial f(x)}{\partial x_i}$ ,  $i = 1, \dots, N$ , exist at  $x$  and are finite.

Let  $\mathcal{D} \subset \text{int}(\text{dom } f)$  be the set of points  $x$  where  $f$  is differentiable. Then  $\mathcal{D}$  is a dense subset of  $\text{int}(\text{dom } f)$  and  $\text{int}(\text{dom } f) \setminus \mathcal{D}$  is a set of measure zero. Moreover, the mapping  $\nabla f : x \mapsto \nabla f(x)$  is continuous relative to  $\mathcal{D}$ .

REMARK II.3.11. In the one-dimensional case,  $N = 1$ , we have these results (Theorem 24.1 of [Roc70]). Let  $f \in \text{Conv } \mathbb{R}$  and, for convenience, let us extend the right and left derivatives  $f'_+$  and  $f'_-$  beyond the interval  $\text{dom } f$  by setting both to be  $\infty$  for the points lying to the right of  $\text{dom } f$  and  $-\infty$  for the points lying to the left of  $\text{dom } f$ . Then  $f'_+$  and  $f'_-$  exist, are non-decreasing functions on  $\mathbb{R}$ , finite on  $\text{int dom } f$  (so that  $f$  is continuous on  $\text{int dom } f$ ), and

$$f'_+(x) \leq f'_-(y) \leq f'_+(y) \leq f'_-(z) \quad (\text{II.3.11})$$

for every triple  $x < y < z$ . If  $f$  is also closed, then

$$\lim_{y \rightarrow x+0} f'_\pm(y) = f'_\pm(x) \quad \text{and} \quad \lim_{y \rightarrow x-0} f'_\pm(y) = f'_\pm(x). \quad (\text{II.3.12})$$

for any  $x \in \mathbb{R}$ .

When  $f \in \text{Conv } \mathbb{R}^N$  is differentiable at  $x \in \text{dom } f$ , then

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \quad (\text{II.3.13})$$

for all  $y \in \mathbb{R}^N$  (Theorem 25.1 in [Roc70]). The notion of sub-gradient extends the concept of differential to point at which  $f$  is not differentiable.

DEFINITION II.3.12. Let  $f \in \text{Conv } \mathbb{R}^N$  and let  $x \in \mathbb{R}^N$ . A vector  $\sigma \in \mathbb{R}^N$  is a *sub-gradient* of  $f$  at  $x$  if

$$f(y) \geq f(x) + \langle \sigma, y - x \rangle \quad (\text{II.3.14})$$

for all  $y \in \mathbb{R}^N$ . The *sub-differential* of  $f$  at  $x$  is the set

$$\partial f(x) := \{ \sigma \in \mathbb{R}^N : \sigma \text{ is a sub-differential of } f \text{ at } x \}. \quad (\text{II.3.15})$$

If  $\partial f(x) \neq \emptyset$ , we call  $f$  *sub-differentiable* at  $x$ .

REMARK II.3.13.

(i) The inequality (II.3.14) has a simple geometric interpretation if  $f$  is finite at  $x$ : the graph of the affine function  $g(y) := f(x) + \langle \sigma, y - x \rangle$  is a non-vertical supporting hyperplane in  $\mathbb{R}^{N+1}$  to the epigraph of  $f$  at the point  $(x, f(x))$ , see p. 215 in [Roc70].

(ii) For any  $f \in \overline{\text{Conv}} \mathbb{R}$  one has (p. 229 of [Roc70])

$$\partial f(x) = \{\sigma \in \mathbb{R}^N : f'_-(x) \leq \sigma \leq f'_+(x)\}, \quad (\text{II.3.16})$$

c.f. the previous remark.

(iii) The definition of the sub-gradient of a function  $f \in \text{Conv} \mathbb{R}^N$  implies that  $f$  attains its minimum at a point  $x$  iff  $0 \in \partial f(x)$ .

The following theorem relates the properties of a convex function to topological properties of its sub-differential. For  $N = 1$  the proof is a direct consequence of (II.3.16).

**THEOREM II.3.14** ([Roc70]: p. 215, Theorem 23.4 and 25.1). *Let us consider  $f \in \text{Conv} \mathbb{R}^N$ . Then:*

- (1) *For any  $x \in \mathbb{R}^N$  the sub-differential  $\partial f(x)$  is a closed convex set which may be empty.*
- (2) *If  $x \notin \text{dom } f$ , then  $\partial f(x) = \emptyset$ .*
- (3) *If  $x \in \text{ri}(\text{dom } f)$ , then  $\partial f(x) \neq \emptyset$ .*
- (4) *The set  $\partial f(x)$  is non-empty and bounded iff  $x \in \text{int}(\text{dom } f)$ .*
- (5) *The function  $f$  is differentiable at  $x$  iff  $\partial f(x)$  is a singleton. If  $\partial f(x)$  is a singleton, then  $\partial f(x) = \{\nabla f(x)\}$ .*

At the end of this section, we introduce an important notion of the convex conjugate and outline its basic properties.

**DEFINITION II.3.15.** Let  $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$  be an arbitrary function. Its *convex conjugate* (or *Legendre-Fenchel transform*)  $f^*$  is defined by the formula

$$f^*(\xi) := \sup_{x \in \mathbb{R}^N} \{\langle x, \xi \rangle - f(x)\}, \quad \xi \in \mathbb{R}^N. \quad (\text{II.3.17})$$

If  $f$  is finite at least at one point in  $\mathbb{R}^N$  and majorizes at least one closed convex function, we let its *closed convex hull*  $\overline{\text{co}}f$  be the largest closed convex function majorized by  $f$ .

**REMARK II.3.16.** Obviously, the supremum over  $\mathbb{R}^N$  can be replaced by the supremum over  $\text{dom } f := \{x \in \mathbb{R}^N : f(x) < \infty\}$ . Moreover, if  $f$  is convex, it can actually be replaced by the supremum over  $\text{ri}(\text{dom } f)$ , see Corollary 12.2.2 in [Roc70], and  $\text{dom } f^* \neq \emptyset$ . Let us prove the latter.

If  $\text{dom } f = \{x_0\}$ , then  $f(x) \geq f(x_0) + \langle \xi, x - x_0 \rangle$  for any  $x, \xi \in \mathbb{R}^N$  (notice that the inequality is surely true for  $x = x_0$ , while  $f(x) = \infty$  for  $x \neq x_0$ ). Thus, we have  $\langle x, \xi \rangle - f(x) \leq \langle x_0, \xi \rangle - f(x_0) < \infty$ , i.e.  $f^*(\xi) < \infty$ . On the other hand, if  $\text{dom } f$  contains more than a single point, then  $\text{ri}(\text{dom } f) \neq \emptyset$ . Since  $\partial f(x_0) \neq \emptyset$  for

any  $x_0 \in \text{ri}(\text{dom } f)$  according to the part (3) of the preceding theorem, it follows that  $f(x) \geq f(x_0) + \langle \xi, x - x_0 \rangle$  for all  $x \in \mathbb{R}^N$  and all  $\xi \in \partial f(x_0)$ , and  $f^*(\xi)$  is therefore finite.

**THEOREM II.3.17** ([Roc70]: p. 104). *Let  $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$  be any function finite at least at one point in  $\mathbb{R}^N$  and majorizes at least one closed convex function. Then  $f^* \in \overline{\text{Conv}} \mathbb{R}^N$ , and*

$$f^* = (\overline{\text{co}}f)^* \quad \text{and} \quad f^{**} = \overline{\text{co}}f. \quad (\text{II.3.18})$$

**REMARK II.3.18.** In view of the above theorem, the conjugacy operation  $f \rightarrow f^*$  induces a symmetric one-to-one correspondence on  $\overline{\text{Conv}} \mathbb{R}^N$ .

**THEOREM II.3.19** ([Roc70]: Theorem 23.5). *Let  $f \in \text{Conv } \mathbb{R}^N$  and  $x \in \mathbb{R}^N$ . The following three conditions on a vector  $\xi$  are equivalent:*

- (1)  $\xi \in \partial f(x)$ ;
- (2)  $\langle z, \xi \rangle - f(z)$  attains its supremum over  $z$  at  $z = x$ ;
- (3)  $f(x) + f^*(\xi) = \langle x, \xi \rangle$ .

*If  $f$  is even closed, two more conditions can be added:*

4.  $x \in \partial f^*(\xi)$ ;
5.  $\langle x, \zeta \rangle - f^*(\zeta)$  attains its supremum over  $\zeta$  at  $\zeta = \xi$ .

**REMARK II.3.20.** Since

$$f(x) + f^*(\xi) \geq \langle x, \xi \rangle \quad (\text{II.3.19})$$

holds for any  $f \in \text{Conv } \mathbb{R}^N$  and for all  $x, \xi \in \mathbb{R}^N$  (Fenchel's inequality), the condition (3) is in fact the same as  $f(x) + f^*(\xi) \leq \langle x, \xi \rangle$ .

Finally, let us prove the following useful lemma.

**LEMMA II.3.21.** *Let  $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$ . Then:*

- (1)  $x \in \partial f^*(\xi)$  iff  $\xi \in \partial f^{**}(x)$ ;
- (2)  $f$  achieves its infimum at a point  $x_0$  iff  $f^{**}$  achieves its infimum at  $x_0$  and  $f^{**}(x_0) = f(x_0)$ .

**PROOF.** (1) First, let us observe that, for any  $x \in \mathbb{R}^N$ ,

$$\begin{aligned} f^{**}(x) &= \sup_{\xi \in \mathbb{R}^N} \{ \langle x, \xi \rangle - \sup_{y \in \mathbb{R}^N} \{ \langle \xi, y \rangle - f(y) \} \} \\ &\leq \sup_{\xi \in \mathbb{R}^N} \langle \xi, x - y \rangle + f(y) \quad \forall y \in \mathbb{R}^N. \end{aligned}$$

Therefore,

$$f^{**} \leq f. \quad (\text{II.3.20})$$

Using now the definition of the sub-differential, we may write that  $x \in \partial f^*(\xi)$  iff

$$\begin{aligned} &\Leftrightarrow f^*(\zeta) \geq f^*(\xi) + \langle x, \zeta - \xi \rangle \quad \forall \zeta \in \mathbb{R}^N \\ &\Leftrightarrow \inf_{\zeta \in \mathbb{R}^N} \{f^*(\zeta) - \langle x, \zeta \rangle\} \geq f^*(\xi) - \langle x, \xi \rangle \\ &\Leftrightarrow -f^{**}(x) \geq f^*(\xi) - \langle x, \xi \rangle \geq (f^*)^{**}(\xi) - \langle x, \xi \rangle \\ &\Leftrightarrow \langle x, \xi \rangle - f^{**}(x) \geq \sup_{z \in \mathbb{R}^N} \{\langle z, \xi \rangle - f^{**}(z)\} \geq \langle z, \xi \rangle - f^{**}(z) \quad \forall z \in \mathbb{R}^N \\ &\Leftrightarrow f^{**}(x) + \langle \xi, z - x \rangle \leq f^{**}(z) \quad \forall z \in \mathbb{R}^N, \end{aligned}$$

i.e. iff  $\xi \in \partial f^{**}(x)$ .

(2) Let  $0 \in \partial f^{**}(x_0)$  (the function  $f^{**}$  achieves its infimum at  $x_0$ ) and  $f^{**}(x_0) = f(x_0)$  be true. With the help of (II.3.20) and the definition of the sub-differential, these conditions yield

$$f(z) \geq f^{**}(z) \geq f^{**}(x_0) = f(x_0) \quad (\text{II.3.21})$$

for all  $z \in \mathbb{R}^N$ . Hence, the point  $x_0$  is a minimum of  $f$ .

Let  $x_0$  be a point where  $f$  attains its minimum. Let us consider the constant function  $g(x) := f(x_0)$ ,  $x \in \mathbb{R}^N$ ; it is an affine function majorized by  $f$ . Since  $f^{**}$  is a pointwise supremum of the collection of all continuous affine functions majorized by  $f$ , we get  $f^{**} \geq g$ . Thus,  $f^{**}(x_0) \geq f(x_0)$ , but this is the same as  $f^{**}(x_0) = f(x_0)$  due to (II.3.20). Moreover,

$$f^{**}(z) \geq g(z) = f(x_0) = f^{**}(x_0) \quad (\text{II.3.22})$$

for all  $z \in \mathbb{R}^N$  so that  $0 \in \partial f^{**}(x_0)$ .

Q.E.D.

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## CHAPTER III

# Finite-size effects for the Potts model with weak boundary conditions

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ABSTRACT. Using Pirogov-Sinai theory, we study finite-size effects for the ferromagnetic  $q$ -state Potts model in a cube with boundary conditions that interpolate between free and constant boundary conditions. If the surface coupling is about half of the bulk coupling and  $q$  is sufficiently large, we show that only small perturbations of the ordered and disordered ground states are dominant contributions to the partition function in a finite but large volume. This allows us to rigorously control the finite-size effects for these “weak boundary conditions.” In particular, we give explicit formulæ for the rounding of the infinite-volume jumps of the internal energy and magnetization, as well as the position of the maximum of the finite-volume specific heat.

### III.1. Introduction

First-order phase transitions are characterized by discontinuities in the mean values of order parameters in the thermodynamic limit. However, in a finite volume the transition is rounded and, possibly, shifted with respect to the infinite volume transition point.

The details of the finite-size effects depend crucially on the choice of boundary conditions that, in turn, depend on the physical situation under consideration. The simplest and best studied case is that of classical lattice systems with periodic boundary conditions. This investigation goes back to the work of Imry [15], Fisher and Berker [12], Blötte and Nightingale [4], Binder and co-workers [1, 2], and others. Rigorous results concerning finite-size effects with periodic boundary conditions [8, 9, 11, 7, 6, 5] show a universal behaviour of the rounded transition and yield details of the asymptotics of the finite-size shift of the transition. One of the results of these papers

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is that for cubic volumes of linear size  $L$ , the intersection point  $h_t(L)$  of the mean value of the order parameter is shifted by a correction which is typically of order  $L^{-d}$ , where  $d \geq 2$  is the dimension of the lattice under consideration. For the special case of two phase coexistence, this shift is much smaller, namely of order  $O(L^{-2d})$ .

While periodic boundary conditions are studied most often (and are the easiest to implement in computer simulations), free boundary conditions, constant boundary conditions, and, more general, boundary conditions with boundary fields are more natural from the point of view of realistic systems. The case of fixed constant boundary conditions, where one has to investigate the balancing effect of boundary conditions and an opposite driving force (say, the external magnetic field) is rather difficult to control rigorously, and only results for two-dimensional Ising model are available [16]. On the other hand, when boundary conditions are sufficiently weak ("close" to the free boundary conditions), a rather general class of models was rigorously studied in [10]. The asymptotics of the rounding and the shift of the transition point were precisely evaluated. In contrast to periodic boundary conditions, the shift is of order  $L^{-1}$ , due to the contribution of the surface free energies.

Even though the results of [10] cover a rather general class of systems, the case of the temperature-driven transition for the Potts model is included only in principle. The details of the contour analysis depend on a slightly different type of contours and, in addition, the discussion of the Fortuin-Kasteleyn representation with the corresponding boundary conditions has to be included. Given also the fact that the Potts model is often used as a typical case of a weak and asymmetric first-order transition for computer simulations, we find it useful to present it separately in the present paper. Finally, it is interesting to discuss the very meaning of "weak" boundary conditions for the Potts model. While the free boundary conditions at the phase coexistence temperature actually strongly enforce the disordered phase, the role of "weak" boundary conditions turns out to be played by constant boundary conditions with a boundary coupling constant that is roughly half of the coupling constant for the bulk.

In the following section we introduce the model and present our results. It is useful to consider three separate regions for the strength  $\lambda$  of the boundary condition: the interval  $[J\frac{1-\mu}{2}, J\frac{1+\mu}{2}]$ , where  $J$  is the bulk coupling parameter and  $\mu$  an arbitrary fixed parameter  $\mu \in (0, 1)$ , and the complementary intervals  $[0, J\frac{1-\mu}{2}]$  and  $[J\frac{1+\mu}{2}, 1]$ . The behaviour for  $\lambda$  in the latter two intervals is close to the corresponding (ordered or disordered) phase, as described in Theorem III.2.1, while the former yields a transition region where the interpolation between two phases occurs. The corresponding results are presented

in Theorem III.2.2, including the claim that the maximum of the specific heat occurs at the inverse temperature

$$\beta_{\max}^{(\lambda)}(L) = \beta_t \left[ 1 + \frac{d}{\Delta e} \left( \frac{J}{2} - \lambda + O\left(\frac{q^{-\nu}}{\log q}\right) \right) \frac{1}{L} + O(L^{-2}) \right]$$

with  $\nu$  of the order  $\frac{1-\mu}{8d}$ .

Note that our results in the window  $[0, J\frac{1-\mu}{2}]$  imply that the phase transition in the Potts model is not robust. Here robustness is defined in the sense introduced by Pemantle and Steif in [18]: A phase transition is said to be robust if the different extremal states can be obtained as limit Gibbs states with arbitrarily weak boundary fields. Theorem III.2.1 immediately yields that the temperature driven first-order phase transition of the Potts model is not robust, for sufficiently small boundary fields lead to the disordered, and not the ordered Gibbs state. A proof of this statement was already sketched in [19], but the details were not provided there.

In Section 3 we use the Fortuin-Kasteleyn representation to derive a suitable contour representation of the model. Sections 4 and 5 are then devoted to proofs of Theorem III.2.2 and Theorem III.2.1, respectively. Proofs of several technical lemmas are deferred to three appendices.

### III.2. Results

In this paper, we shall consider the  $q$ -state Potts model in the  $d$ -dimensional cube

$$\Lambda = \Lambda(L) = \left\{ x \in \mathbb{Z}^d \mid -\frac{L}{2} < x_i \leq \frac{L}{2} \text{ for all } i = 1, \dots, d \right\}, \quad L \in \mathbb{N}, \quad (\text{III.2.1})$$

with boundary conditions interpolating between free and constant 1-boundary conditions.<sup>1</sup> As usual, the spin configurations of this model are maps  $\sigma_\Lambda$  from  $\Lambda$  into  $Q = \{1, \dots, q\}$ . We use  $\mathbb{B} = \mathbb{B}(\Lambda)$  to denote the set of all bonds  $\langle x, y \rangle$  of nearest-neighbour sites  $x, y \in \mathbb{Z}^d$  with both end-points in  $\Lambda$  and  $\partial\mathbb{B} = \partial\mathbb{B}(\Lambda)$  to denote the set  $\{\langle x, y \rangle \mid x \in \Lambda, y \in \mathbb{Z}^d \setminus \Lambda\}$ . The Hamiltonian with boundary conditions interpolating between the free and the 1-boundary conditions is

$$H^{(\lambda)}(\sigma_\Lambda) = -J \sum_{\langle x, y \rangle \in \mathbb{B}} \delta_{\sigma_x, \sigma_y} - \lambda \sum_{\substack{\langle x, y \rangle \in \partial\mathbb{B}: \\ x \in \Lambda}} \delta_{\sigma_x, 1}, \quad (\text{III.2.2})$$

where  $J > 0$  is the bulk coupling and  $\lambda \geq 0$  is the surface coupling. The value  $\lambda = 0$  represents *free boundary conditions*, while  $\lambda = J$  represents *standard 1-boundary conditions*. The Gibbs state corresponding

<sup>1</sup>Without loss of generality, we use 1-boundary conditions,  $\sigma_x = 1$  for all  $x \in \mathbb{Z}^d \setminus \Lambda$ , instead of general fixed boundary conditions.

to the Hamiltonian (III.2.2) is given by

$$\langle \cdot \rangle_L^{(\beta, \lambda)} = \frac{1}{Z_L(\beta, \lambda)} \sum_{\sigma_\Lambda \in Q^\Lambda} \cdot e^{-\beta H^{(\lambda)}(\sigma_\Lambda)}, \quad (\text{III.2.3})$$

where  $Z_L(\beta, \lambda)$  is the partition function,

$$Z_L(\beta, \lambda) = \sum_{\sigma_\Lambda \in Q^\Lambda} e^{-\beta H^{(\lambda)}(\sigma_\Lambda)}. \quad (\text{III.2.4})$$

It is well known by now that for all  $d \geq 2$  and all  $q \geq 2$  the infinite-volume system exhibits a phase transition at some value  $\beta_t$  characterized by the appearance of a spontaneous magnetization for  $\beta > \beta_t$ . For  $q$  sufficiently large, this transition is known to be first-order with a discontinuity in both the magnetization<sup>2</sup>

$$m(\beta) = \lim_{L \rightarrow \infty} \frac{1}{L^d} \frac{1}{q-1} \left\langle \sum_{x \in \Lambda(L)} (q \delta_{\sigma_x, 1} - 1) \right\rangle_L^{(\beta, \lambda=J)} \quad (\text{III.2.5})$$

and the mean energy

$$e(\beta) = \lim_{L \rightarrow \infty} \frac{1}{L^d} \langle H^{(J)}(\sigma_{\Lambda(L)}) \rangle_L^{(\beta, \lambda=J)}. \quad (\text{III.2.6})$$

The magnetization  $m(\beta)$  is zero for  $\beta < \beta_t$ , jumps from

$$m_{\text{dis}}(\beta_t) = 0$$

to

$$m_{\text{ord}}(\beta_t) = \lim_{\beta \downarrow \beta_t} m(\beta) = m(\beta_t) > 0$$

at  $\beta_t$ , and is strictly increasing for  $\beta > \beta_t$ , while the mean energy  $e(\beta)$  is strictly decreasing for all  $\beta$ , with a jump from

$$e_{\text{dis}}(\beta_t) = \lim_{\beta \uparrow \beta_t} e(\beta)$$

to

$$e_{\text{ord}}(\beta_t) = \lim_{\beta \downarrow \beta_t} e(\beta) = e(\beta_t)$$

at  $\beta_t$ .

Here we study the finite-volume magnetization and the mean energy defined by

$$M_L(\beta, \lambda) = \frac{1}{q-1} \left\langle \sum_{x \in \Lambda(L)} (q \delta_{\sigma_x, 1} - 1) \right\rangle_L^{(\beta, \lambda)} \quad (\text{III.2.7})$$

<sup>2</sup>The existence of the limits (III.2.5) and (III.2.6) with the constant 1-boundary conditions follows from either GKS-inequalities [14] or FK-monotonicity, see e.g. [3]. By the same methods, one can also show that the functions  $m$  and  $e$  are right continuous,  $m(\beta) = m(\beta + 0)$ ,  $e(\beta) = e(\beta + 0)$ .

and

$$E_L(\beta, \lambda) = \langle H^{(\lambda)}(\sigma_\Lambda) \rangle_L^{(\beta, \lambda)} = -\frac{\partial}{\partial \beta} \log Z_L(\beta, \lambda), \quad (\text{III.2.8})$$

respectively. The case of the free boundary conditions ( $\lambda = 0$ ) can, for  $q$  sufficiently large, be analyzed by the standard Pirogov-Sinai theory as long as  $\beta \leq \beta_t$ , while the case of the standard 1-boundary conditions ( $\lambda = J$ ) can be analyzed as long as  $\beta \geq \beta_t$ . This, in particular, gives

$$\lim_{L \rightarrow \infty} \frac{1}{L^d} M_L(\beta_t, 0) = 0, \quad \lim_{L \rightarrow \infty} \frac{1}{L^d} E_L(\beta_t, 0) = e_{\text{dis}}(\beta_t) \quad (\text{III.2.9})$$

and

$$\lim_{L \rightarrow \infty} \frac{1}{L^d} M_L(\beta_t, J) = m_{\text{ord}}(\beta_t), \quad \lim_{L \rightarrow \infty} \frac{1}{L^d} E_L(\beta_t, J) = e_{\text{ord}}(\beta_t). \quad (\text{III.2.10})$$

The main contribution of this paper is the analysis of the asymptotic behaviour (as  $L \rightarrow \infty$ ) of  $M_L(\beta, \lambda)$  and  $E_L(\beta, \lambda)$  for any  $\lambda \geq 0$  and  $q$  large. It turns out that the behaviour for  $\lambda \in (0, \frac{J}{2})$  and  $\beta \leq \beta_t$  is qualitatively the same as that for the free boundary conditions: the specific magnetization  $\frac{1}{|\Lambda(L)|} M_L(\beta, \lambda)$  and the specific mean energy  $\frac{1}{|\Lambda(L)|} E_L(\beta, \lambda)$  still converge to the bulk quantities in the disordered phase with corrections of the order  $L^{-1}$ . Similarly, for  $\lambda \in (\frac{J}{2}, \infty)$  and  $\beta \geq \beta_t$ , we are still in the ordered phase. These two cases are jointly referred to as the *strong boundary conditions*. Finite-size behaviour for intermediate values of  $\lambda$  — ‘around’  $\lambda = \frac{J}{2}$ , the *weak boundary conditions* — and any  $\beta > 0$  is governed by the competition between contributions coming from the configurations which are either in the ordered or in the disordered phase for the whole of  $\Lambda$ . Surface effects, in dependence on the particular value of  $\lambda$ , then determine the resulting finite-size rounding of the phase transition.

Our results are summarized in the following theorems. In order to state them, we first introduce the specific heat

$$\begin{aligned} C_L(\beta, \lambda) &= \beta^2 \left( \langle (H^{(\lambda)}(\sigma_\Lambda))^2 \rangle_L^{(\beta, \lambda)} - (\langle H^{(\lambda)}(\sigma_\Lambda) \rangle_L^{(\beta, \lambda)})^2 \right) = \\ &= -\beta^2 \frac{\partial E_L(\beta, \lambda)}{\partial \beta} \end{aligned} \quad (\text{III.2.11})$$

and the shorthands  $m^* = m(\beta_t)$ , the derivative  $c(\beta) = -\beta^2 \frac{de(\beta)}{d\beta}$  (known to exist as a smooth function as long as  $\beta \neq \beta_t$ ),

$$e_0 = \frac{e_{\text{dis}}(\beta_t) + e_{\text{ord}}(\beta_t)}{2}, \quad \text{and} \quad \Delta e = \frac{e_{\text{dis}}(\beta_t) - e_{\text{ord}}(\beta_t)}{2}. \quad (\text{III.2.12})$$

THEOREM III.2.1. Let  $d \geq 2$ ,  $J > 0$ , and  $\mu > 0$ . For  $q$  and  $L$  sufficiently large, we have:

(a) If  $0 \leq \lambda \leq \frac{J}{2}(1 - \mu)$  and  $\beta \leq \beta_t$ , then

$$M_L(\beta, \lambda) = O(L^{d-1}) \quad (\text{III.2.13})$$

and

$$E_L(\beta, \lambda) = e(\beta - 0)L^d + O(L^{d-1}). \quad (\text{III.2.14})$$

(b) If  $\lambda \geq \frac{J}{2}(1 + \mu)$  and  $\beta \geq \beta_t$ , then

$$M_L(\beta, \lambda) = m(\beta)L^d + O(L^{d-1}) \quad (\text{III.2.15})$$

and

$$E_L(\beta, \lambda) = e(\beta)L^d + O(L^{d-1}). \quad (\text{III.2.16})$$

Our second theorem concerns weak boundary conditions, i.e. values of  $\lambda$  in the interval  $[J\frac{1-\mu}{2}, J\frac{1+\mu}{2}]$ , where  $\mu$  is an arbitrary fixed parameter  $\mu \in [0, 1)$ . For these boundary conditions we control the finite-size behaviour of  $M_L(\beta, \lambda)$ ,  $E_L(\beta, \lambda)$ , and  $C_L(\beta, \lambda)$  for all  $\beta \in (0, \infty)$ . To state our results, it will be convenient to distinguish values of  $\beta$  inside a window of the form  $[\beta_t - \Delta\beta(L), \beta_t + \Delta\beta(L)]$ , and values of  $\beta$  outside this window. Here  $\Delta\beta(L)$  is a function which goes to zero as  $L \rightarrow \infty$  fast enough to ensure that  $\Delta\beta(L) \log L \rightarrow 0$ , and slow enough to guaranty that  $\Delta\beta(L) L$  is bounded away from zero. To be more precise, we will consider functions  $\Delta\beta(L)$  of the form

$$\Delta\beta(L) = \frac{d\nu J\beta_t}{5\Delta e} \frac{1}{\omega(L)}, \quad (\text{III.2.17})$$

where

$$\nu = \frac{1}{24d} \min\{1, 3(1 - \mu)\}, \quad (\text{III.2.18})$$

and  $\omega : \mathbb{N} \rightarrow [0, \infty)$  is chosen in such a way that

$$\limsup_{L \rightarrow \infty} \frac{\omega(L)}{L} \leq \frac{\nu}{8} \quad \text{and} \quad \liminf_{L \rightarrow \infty} \frac{\omega(L)}{\log L} = \infty. \quad (\text{III.2.19})$$

THEOREM III.2.2. Let  $d \geq 2$ ,  $J > 0$ , and  $0 \leq \mu < 1$ . If  $\omega : \mathbb{N} \rightarrow [0, \infty)$  is a function obeying (III.2.19), then for all  $q, L$  sufficiently large and  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ , we have:

(a) There exists a unique point  $\beta_{\max}^{(\lambda)}(L)$  at which the specific heat  $C_L(\beta, \lambda)$  attains its maximum. Furthermore, there exists a function  $b(J, \lambda, q)$  such that

$$\beta_{\max}^{(\lambda)}(L) = \beta_t \left[ 1 + \frac{d}{\Delta e} \left( \frac{J}{2} - \lambda + b(J, \lambda, q) \right) \frac{1}{L} + O(L^{-2}) \right] \quad (\text{III.2.20})$$

and  $b(J, \lambda, q) \leq \text{const} \frac{q^{-\nu}}{\log q}$  uniformly in  $\beta, J$ , and  $\lambda$ .

(b) If  $|\beta - \beta_t| \leq \Delta\beta(L)$ , with  $\Delta\beta(L)$  as in (III.2.17), then

$$M_L(\beta, \lambda) = \frac{m^*}{2} L^d + \frac{m^*}{2} L^d \tanh(\Delta e (\beta - \beta_{\max}^{(\lambda)}(L)) L^d) + O\left(\frac{L^d}{\omega(L)}\right), \quad (\text{III.2.21})$$

$$E_L(\beta, \lambda) = e_0 L^d - \Delta e L^d \tanh(\Delta e (\beta - \beta_{\max}^{(\lambda)}(L)) L^d) + O\left(\frac{L^d}{\omega(L)}\right), \quad (\text{III.2.22})$$

and

$$C_L(\beta, \lambda) = \beta^2 (\Delta e)^2 L^{2d} \cosh^{-2}(\Delta e (\beta - \beta_{\max}^{(\lambda)}(L)) L^d) + O\left(\frac{L^{2d}}{\omega(L)}\right). \quad (\text{III.2.23})$$

(c) If  $|\beta - \beta_t| > \Delta\beta(L)$ , then

$$M_L(\beta, \lambda) = m(\beta) L^d + O(L^{d-1}), \quad (\text{III.2.24})$$

$$E_L(\beta, \lambda) = e(\beta) L^d + O(L^{d-1}), \quad (\text{III.2.25})$$

and

$$C_L(\beta, \lambda) = c(\beta) L^d + O(L^{d-1}). \quad (\text{III.2.26})$$

REMARK III.2.3. (i) In Theorem III.2.1 (a), it is meaningful to consider  $0 < \mu \leq 1$  only. Note also that  $|\beta_{\max}^{(\lambda)}(L) - \beta_t| < \Delta\beta(L) = \frac{d\nu J \beta_t}{5\Delta e} \frac{1}{\omega(L)}$  for any  $0 < \lambda < J$  and  $q, L$  large due to (III.2.20) and the first condition in (III.2.19).

(ii) Notice that the results of Theorem III.2.1 and Theorem III.2.2 are, in the regions of overlapping parameters, in agreement. Indeed, let  $\lambda \in (0, \frac{J}{2})$  and  $\beta \leq \beta_t$  first. If  $\beta < \beta_t$ , then Theorem III.2.2 (c) yields (III.2.13) and (III.2.14), respectively, whenever one takes  $L$  such that  $\omega(L) > \frac{d\nu J}{5\Delta e} \frac{\beta_t}{\beta_t - \beta}$ . If  $\beta = \beta_t$ , the equations (III.2.13) and (III.2.14) follow from Theorem III.2.2 (a) and (b): one just uses that  $\tanh x = 1 + O(e^{-2x})$  for  $x \gg 1$  and observes that  $\beta_t - \beta_{\max}^{(\lambda)}(L)$  is negative and of the order  $L^{-1}$  by virtue of (III.2.20). The case  $\lambda \in (\frac{J}{2}, J)$  and  $\beta \geq \beta_t$  is similar.

(iii) In order to prove the above theorems, one in fact needs to exclude the values of  $\beta$  close to 0 (c.f. Lemma III.A.1), and we take  $\beta \geq 1$  where necessary. Nevertheless, the restriction of  $\beta$  to the interval  $[1, \infty)$  is not serious: if  $\beta \leq 1$ , we may use, for  $q$  large, a standard high-temperature expansion to obtain the results of Theorem III.2.1 (a) and Theorem III.2.2 (a), (c).

(iv) The techniques used in this paper do not allow to study the finite-size scaling of  $M_L(\beta, \lambda)$  and  $E_L(\beta, \lambda)$  for boundary conditions which *strongly* favour the ordered or the disordered phase near the boundary of  $\Lambda$ . In this case, the leading contributions to  $Z_L(\beta, \lambda)$  feature a dip along a large contour from one of the two phases near the boundary to the other phase within the bulk. To analyze the finite-size scaling, a control over the behaviour of this large contour would be necessary, involving, in particular, the analysis of the so-called Wulff shape of a contour filling essentially the whole volume. This is out of the scope of this paper. As will be shown in Section 4, such a detailed analysis of large contours is not necessary in the case of the weak boundary conditions.

(v) A general concept of the finite-size effects for first-order phase transitions was developed in [10] on the basis of Pirogov-Sinai theory, and one may try to apply it to our model. To this end, the model must be first rewritten in terms of contours and then the assumptions under which [10] can be used are to be satisfied; this was done in [17]. Whereas the assumptions (3.7) to (3.9) of [10] are fulfilled in our situation (if we suppose that, say,  $\beta \geq 1$ ), the assumption (3.11) of [10] imposes drastic constraints on the values of  $\lambda$  and  $\beta$ . Namely, one must assume that  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \delta$  and  $|\frac{\beta}{\beta_t} - 1| \leq \delta$ , where  $\delta = \delta(d) < \frac{1}{52d}$ , see (4.47b) in [10]. With such restrictions, the general setting of [10] enables us to establish the results of Theorem III.2.2 (with  $\omega(L) = L$ ). Here we weakened these constraints (both for  $\lambda$  and  $\beta$ ), using the methods of [10] with a more careful evaluation of boundary terms.

### III.3. Contour Representations

In order to analyze the finite-volume quantities  $M_L(\beta, \lambda)$  and  $E_L(\beta, \lambda)$ , we use the machinery of the Pirogov-Sinai theory in the form developed in [10]. To this end, we rewrite the partition function (III.2.4) in terms of contours.

Throughout this section, we assume that  $d \geq 2$ ,  $J > 0$ ,  $\lambda \geq 0$  and  $\beta > 0$ . Moreover, the cube  $\Lambda$  (and, thus, the sets  $\mathbb{B} = \mathbb{B}(\Lambda)$  and  $\partial\mathbb{B}$ ) is fixed, and we write  $\bar{\mathbb{B}}$  for  $\mathbb{B} \cup \partial\mathbb{B}$ .

First, we express  $Z_L(\beta, \lambda)$  with the help of the Fortuin-Kasteleyn random-cluster representation [13]. Modifying the approach of [11] to take into account the effect of the boundary, one obtains

$$Z_L(\beta, \lambda) = \sum_{X \subset \bar{\mathbb{B}}} e^{-G(X)} q^{-\frac{1}{2d}\|\delta X\| + C_{\text{in}}(X)}, \quad (\text{III.3.1})$$

see [17] for details. Here  $C_{\text{in}}(X)$  is the number of the connected components of  $X$  which do not include any bond of  $\partial\mathbb{B}$ ,  $\|\delta X\| =$

$|\delta_1 X| + 2|\delta_2 X|$ , where

$$\delta_i X = \{\langle x, y \rangle \in \mathbb{B} \setminus X : |\{x, y\} \cap S(X)| = i\}, \quad i = 1, 2,$$

with

$$S(X) = \{x \in \Lambda : \langle x, y \rangle \in X \text{ for some } y \in \mathbb{Z}^d\},$$

and  $G(X) = \sum_{b \in \mathbb{B}} g_X(b)$ , with

$$g_X(b) = \begin{cases} -\log(e^{J\beta} - 1) & \text{if } b \in \mathbb{B} \cap X, \\ -\log(e^{\lambda\beta} - 1) & \text{if } b \in \partial\mathbb{B} \cap X, \end{cases} \quad (\text{III.3.2})$$

$$g_X(b) = \begin{cases} -\frac{1}{d} \log q & \text{if } b \in \mathbb{B} \setminus X, \\ -\frac{1}{2d} \log q & \text{if } b \in \partial\mathbb{B} \setminus X. \end{cases} \quad (\text{III.3.3})$$

REMARK III.3.1. Note that for the free boundary conditions,  $\lambda = 0$ , the contribution of any random-cluster configuration  $X \subset \mathbb{B}$  to  $Z_L(\beta, \lambda)$  vanishes unless  $X \cap \partial\mathbb{B} = \emptyset$ .

Next, let us introduce  $V = V(L)$  as the closed  $d$ -dimensional cube in  $\mathbb{R}^d$ , of side length<sup>3</sup>  $L + 1$ , centred at the same point as  $\Lambda$ . Our aim is to rewrite every random-cluster configuration  $X \subset \mathbb{B}$  in terms of collections of contours. It turns out that it is convenient to introduce two different types of contours, depending on the boundary conditions: for weak boundary conditions it is natural to consider open contours “ending on the boundary  $\partial V$ ”, while for strong boundary conditions the natural setting is to consider closed contours only. Accordingly, the following definition distinguishes these cases. Nevertheless, the differences concern only contours “touching” the boundary  $\partial V$  — for those in the interior, all the definitions are independent of the boundary conditions.

Let  $X \subset \mathbb{B}$  be a random-cluster configuration. In order to define contours, we first identify the bonds  $\langle x, y \rangle \in X$  with the corresponding line segments in  $\mathbb{R}^d$ , and then define a closed  $k$ -dimensional unit hypercube  $c \subset \mathbb{R}^d$  with vertices in  $\mathbb{Z}^d$  to be occupied if all bonds  $b \subset c$  are bonds in  $X$ . We use  $\mathcal{P}(X)$  to denote the union of all occupied hypercubes. In a similar way, we define  $\overline{\mathcal{P}}(X) \supset \mathcal{P}(X)$  as the union of all closed unit hypercubes  $c$  with vertices in  $\mathbb{Z}^d$  for which all bonds  $b \subset c$  lie in  $X \cup \partial V$ .

Consider the sets of contours  $Y(X)$ ,  $Y_o(X)$ , and  $Y_d(X)$  defined respectively as the sets of connected components of the boundaries of the following sets (see Fig. 1):

- (a) the intersection of the  $\frac{1}{4}$ -neighbourhood of  $\overline{\mathcal{P}}(X) \setminus \partial V$  with  $V$ ,
- (b) the intersection of the  $\frac{1}{4}$ -neighbourhood of  $\overline{\mathcal{P}}(X)$  with  $V$ ,

<sup>3</sup>Note that  $V(L)$  is thus defined to be the smallest closed cube containing all bonds from  $\mathbb{B}$ .

(c) the  $\frac{1}{4}$ -neighbourhood of  $\mathcal{P}(X) \cap \{x \in V : \text{dist}(x, \partial V) \geq \frac{1}{2}\}$ .

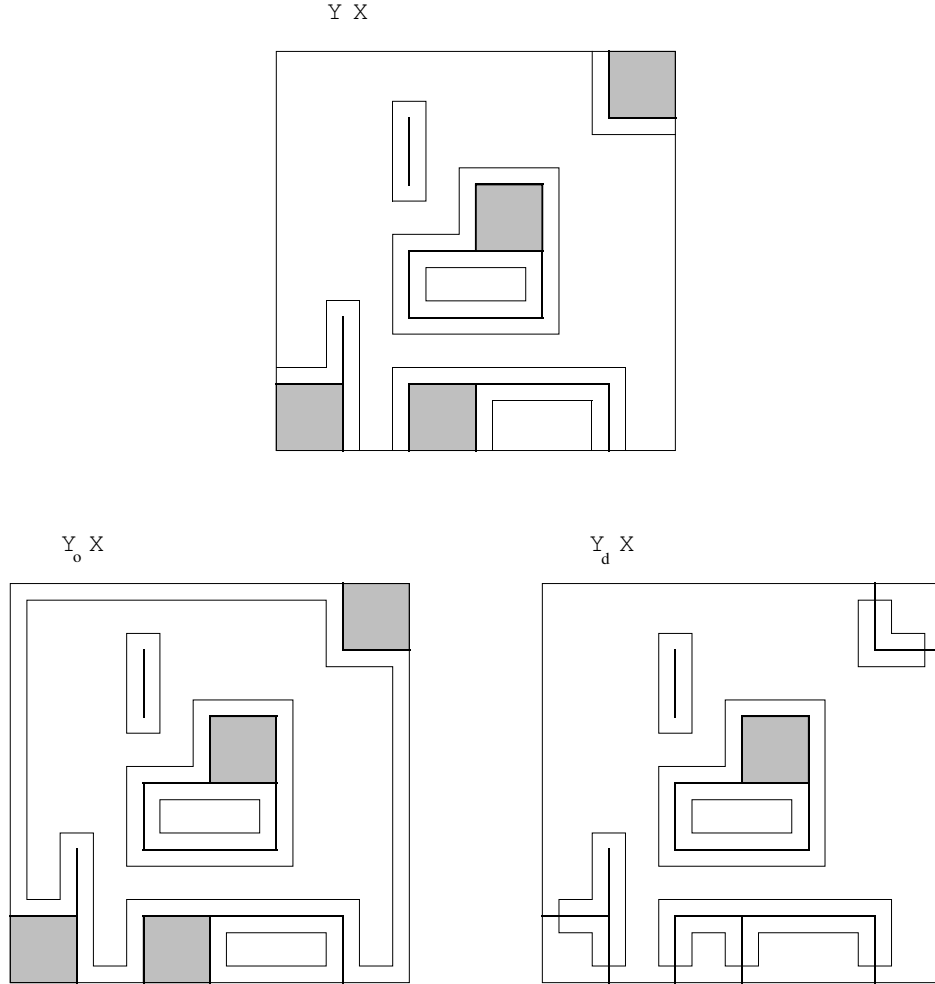


FIGURE 1. Contours under different boundary conditions.

Here the neighbourhood is defined with respect to the  $\ell_\infty$  distance,  $\text{dist}(x, y) = \max_{i=1, \dots, d} |x_i - y_i|$ . Elements of  $Y(X)$  are called *w-contours*, while elements of  $Y_o(X)$  or  $Y_d(X)$  are called *s-contours*, and we shall use  $\gamma$  to denote any of them. Furthermore, we say that a set  $\partial$  of *w-contours* (*s-contours*) is *admissible* if there exists a configuration  $X \subset \mathbb{B}$  such that  $\partial = Y(X)$  ( $\partial = Y_o(X)$  or  $\partial = Y_d(X)$ ). This configuration is necessarily unique whenever  $\partial$  is not empty, while

$$Y(\mathbb{B}) = Y(\emptyset) = Y_o(\mathbb{B}) = Y_d(\emptyset) = \emptyset.$$

If  $\partial \neq \emptyset$ , we use  $X(\partial)$  to denote the unique configuration corresponding to  $\partial$ .

Define the ‘octant’  $\mathcal{O}(k)$  of each corner  $k = [k_1, \dots, k_d]$  of the box  $V$  by

$$\mathcal{O}(k) = \{x \in \mathbb{R}^d : x_i \geq k_i \text{ if } i \in I_-, x_i \leq k_i \text{ if } i \in I_+\}, \quad (\text{III.3.4})$$

where  $i \in I_-$  as long as  $y_i \geq k_i$  for all  $y \in V$ , while  $i \in I_+$  as long as  $y_i \leq k_i$  for all  $y \in V$ . If, for a given  $\gamma$ , there is a corner  $k$  of  $V$  such that  $\gamma \cap \partial V \subset \partial \mathcal{O}(k)$ , then  $\text{int } \gamma$ , the *interior* of  $\gamma$ , is defined as the finite component of  $\mathcal{O}(k) \setminus \gamma$ .<sup>4</sup> However, if there is no such a corner, then  $\text{int } \gamma$  is defined as the smaller of the two components of  $V \setminus \gamma$ .<sup>5</sup> In addition, we set  $V(\gamma) = \gamma \cup \text{int } \gamma$  and  $\text{Ext } \gamma = V \setminus V(\gamma)$ . Any  $\gamma$  from an admissible set of  $w$ - or  $s$ -contours  $\partial$  is *external* if there is no  $\tilde{\gamma} \in \partial$  such that  $\gamma \subset \text{int } \tilde{\gamma}$ .

The set  $\{\gamma\}$  with  $\gamma$  arbitrary is admissible and non-empty, and thus there exists a unique configuration  $X_\gamma \subset \mathbb{B}$  for which  $Y(X_\gamma) = \{\gamma\}$  if  $\gamma$  is a  $w$ -contour, while  $Y_o(X_\gamma) = \{\gamma\}$  or  $Y_d(X_\gamma) = \{\gamma\}$  if  $\gamma$  is an  $s$ -contour. We call  $\gamma$  *ordered* (or *o-labelled*) if  $X_\gamma \subset \text{Ext } \gamma$  and *disordered* (or *d-labelled*) if  $X_\gamma \subset \text{int } \gamma$ . If  $X_\gamma = \emptyset$ , one necessarily has  $\{\gamma\} = Y_o(X_\gamma)$ , and we say that  $\gamma$  is *o-labelled*. Note that all the external contours of an admissible set of  $w$ - or  $s$ -contours are either ordered or disordered; for instance, the external contours of  $Y_m(X)$  with  $m = o, d$  and any  $X \subset \mathbb{B}$  are  $m$ -labelled.

Let the *length*  $\|\gamma\|$  of a contour  $\gamma$  be the number of its intersections with the bonds of  $\mathbb{B}$ . Observing that for any set of admissible contours  $\partial$ , the number of disordered contours in  $\partial$  with  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$  is  $C_{\text{in}}(X(\partial))$ , we introduce the *weight* of  $\gamma$  by

$$\rho(\gamma) = \begin{cases} q^{-\frac{1}{2d}\|\gamma\|} & \text{if } \gamma \text{ is ordered or if } \gamma \text{ is disordered with} \\ & \text{dist}(\gamma, \partial V) \leq \frac{1}{4}, \\ q^{-\frac{1}{2d}\|\gamma\|+1} & \text{if } \gamma \text{ is disordered with } \text{dist}(\gamma, \partial V) \geq \frac{3}{4}. \end{cases} \quad (\text{III.3.5})$$

If  $\partial$  is a non-empty admissible set of  $w$ -contours, then one easily sees that

$$\sum_{\gamma \in \partial} \|\gamma\| = \|\delta X(\partial)\|, \quad (\text{III.3.6})$$

<sup>4</sup>This definition clearly does not depend on the choice of  $k$  if more corners are possible.

<sup>5</sup>If both the components of  $V \setminus \gamma$  are of the same size, take the one which contains the corner  $k$  of  $V$  for which  $k_i \leq x_i, i = 1, \dots, d$ , for all  $x \in V$ .

whereas if  $\partial$  is a non-empty admissible set of  $s$ -contours, then

$$\sum_{\gamma \in \partial} \|\gamma\| = \begin{cases} \|\delta X(\partial)\| + |\partial \mathbb{B} \setminus X(\partial)| & \text{if external } s\text{-contours in } \partial \\ & \text{are ordered,} \\ \|\delta X(\partial)\| + |\partial \mathbb{B} \cap X(\partial)| & \text{if external } s\text{-contours in } \partial \\ & \text{are disordered.} \end{cases} \quad (\text{III.3.7})$$

Here  $X(\partial)$  is the unique configuration corresponding to  $\partial$ .

A set  $\{\gamma_1, \dots, \gamma_n\}$  of  $w$ -contours ( $s$ -contours) is called a *set of non-overlapping  $w$ -contours* ( $s$ -contours) if  $\text{dist}(\gamma_i, \gamma_j) \geq \frac{1}{2}$  for all  $1 \leq i < j \leq n$ . Any admissible set of  $w$ -contours ( $s$ -contours) may serve as an example.

Let  $W \subset V$  be of the form

$$V \setminus \bigcup_{\gamma \in \partial^*} V(\gamma) \quad \text{or} \quad \text{int } \gamma_0 \setminus \bigcup_{\gamma \in \tilde{\partial}^*} V(\gamma), \quad (\text{III.3.8})$$

where  $\gamma_0$  is a  $w$ -contour (or an  $s$ -contour) and  $\partial^*$  and  $\tilde{\partial}^*$  are, possibly empty, sets of non-overlapping  $w$ -contours ( $s$ -contours) with  $V(\gamma) \subset \text{int } \gamma_0$  for all  $\gamma \in \tilde{\partial}^*$ . Then  $\mathbb{B}(W)$  and  $\partial \mathbb{B}(W)$  stand for the sets of all bonds of  $\mathbb{B}$  and  $\partial \mathbb{B}$ , respectively, whose centres lie in  $W$  and  $\bar{\mathbb{B}}(W) = \mathbb{B}(W) \cup \partial \mathbb{B}(W)$ . In order to express the weight of a configuration  $X$  in terms of its contours, we also introduce the ordered and disordered “regions”,

$$\Omega_m(W, \partial) = \begin{cases} \bar{\mathbb{B}}(W) \cap X(\partial) & \text{if } m = \text{o}, \\ \bar{\mathbb{B}}(W) \setminus X(\partial) & \text{if } m = \text{d}, \end{cases} \quad (\text{III.3.9})$$

for any admissible set of  $w$ - or  $s$ -contours  $\partial$ , where we set  $X(\emptyset)$  to be equal to  $\bar{\mathbb{B}}$  if  $m = \text{o}$  and to  $\emptyset$  if  $m = \text{d}$ .

When expressing the partition function in terms of contours, we distinguish the case of weak and strong boundary conditions.

**Weak b.c.** For every  $B \subset \bar{\mathbb{B}}$  and  $m = \text{o}, \text{d}$ , we set

$$G_m(B) = \frac{g_m}{d} |\mathbb{B} \cap B| + h_m |\partial \mathbb{B} \cap B| \quad (\text{III.3.10})$$

with

$$g_o = -d \log(e^{J\beta} - 1), \quad g_d = -\log q, \quad (\text{III.3.11})$$

$$h_o = -\log(e^{\lambda\beta} - 1), \quad h_d = -\frac{1}{2d} \log q. \quad (\text{III.3.12})$$

Let us call  $\gamma$  *short* if<sup>6</sup>  $\text{diam } \gamma < \omega(L)$  and *long* otherwise; the parameter  $\omega(L)$  is supposed to be fixed so that  $1 < \omega(L) \leq L$ . Later, it will

<sup>6</sup>The *diameter* of any subset  $\mathcal{W}$  of  $\mathbb{R}^d$ ,  $\text{diam } \mathcal{W}$ , means the length of the side of the smallest square box in  $\mathbb{R}^d$  into which  $\mathcal{W}$  can fit.

be chosen to obey the condition (III.2.19) from Theorem III.2.2. For  $W$  of the form (III.3.8), we define

$$Z_{m,W}(\beta, \lambda) = \sum_{\partial \sqsubset W}^{(m)} e^{-G_o(\Omega_o(W,\partial)) - G_d(\Omega_d(W,\partial))} \prod_{\gamma \in \partial} \rho(\gamma), \quad m = o, d, \quad (\text{III.3.13})$$

where the sum is taken over all admissible sets  $\partial$  of short  $w$ -contours such that the external contours in  $\partial$  are  $m$ -labelled and  $V(\gamma) \subset W$  for all  $\gamma \in \partial$ .

As it is standard in the Pirogov-Sinai theory, one may derive another, more suitable expression for the partition function (III.3.13) (in a context similar to the present one, see e.g., Subsection 4.2 of [10]), namely,

$$Z_{m,W}(\beta, \lambda) = e^{-G_m(\mathbb{B}(W))} \sum_{\partial^* \sqsubset W}^{(m)} \prod_{\gamma \in \partial^*} K_m(\gamma), \quad m = o, d. \quad (\text{III.3.14})$$

Here the summation is over all sets  $\partial^*$  of non-overlapping short  $m$ -labelled  $w$ -contours with  $V(\gamma) \subset W$  for every  $\gamma \in \partial^*$  and

$$K_o(\gamma) = \rho(\gamma) \frac{Z_{d,\text{int } \gamma}(\beta, \lambda)}{Z_{o,\text{int } \gamma}(\beta, \lambda)}, \quad K_d(\gamma) = \rho(\gamma) \frac{Z_{o,\text{int } \gamma}(\beta, \lambda)}{Z_{d,\text{int } \gamma}(\beta, \lambda)}, \quad (\text{III.3.15})$$

where we skipped the dependence of  $K_m(\gamma)$  on  $\beta$  and  $\lambda$ .

In addition, we introduce

$$Z_{\text{big},V}(\beta, \lambda) = \sum_{\partial}^{(\text{long})} e^{-G_o(\Omega_o(V,\partial)) - G_d(\Omega_d(V,\partial))} \prod_{\gamma \in \partial} \rho(\gamma) \quad (\text{III.3.16})$$

with the sum going over all the admissible sets  $\partial$  of  $w$ -contours which contain at least one long  $\gamma \in \partial$ . Given such a set  $\partial$ , let  $\partial_l$  be the set of its long  $w$ -contours; it is obviously admissible. Then  $V \setminus \partial_l$  splits into connected components  $\mathcal{C}_1, \dots, \mathcal{C}_N$  and, for each  $i = 1, \dots, N$ , either  $\mathbb{B}(\mathcal{C}_i) \subset \Omega_o(V, \partial_l)$  or  $\mathbb{B}(\mathcal{C}_i) \subset \Omega_d(V, \partial_l)$ . We use  $W_o(\partial_l)$  to denote the union of the former components and  $W_d(\partial_l)$  to denote the union of the latter ones. Now, let us decompose  $\partial$  into the disjoint union  $\partial_l \cup \partial^o \cup \partial^d$ , where  $\partial^m$ ,  $m = o, d$ , is the set of all the short  $w$ -contours of  $\partial$  with  $V(\gamma) \subset W_m(\partial_l)$  for every  $\gamma \in \partial^m$ . Clearly, the external  $w$ -contours of  $\partial^m$  are  $m$ -labelled and

$$\Omega_m(V, \partial) = \Omega_m(W_m(\partial_l), \partial^m) \cup \Omega_m(W_{m^c}(\partial_l), \partial^{m^c}) \quad (\text{III.3.17})$$

is also a disjoint union (here  $m^c = o$  if  $m = d$  and vice versa). Re-summing all the short  $w$ -contours contributing to  $Z_{\text{big},V}(\beta, \lambda)$ , we therefore obtain

$$Z_{\text{big},V}(\beta, \lambda) = \sum_{\partial_l} Z_{o,W_o(\partial_l)}(\beta, \lambda) Z_{d,W_d(\partial_l)}(\beta, \lambda) \prod_{\gamma \in \partial_l} \rho(\gamma). \quad (\text{III.3.18})$$

Let  $\partial$  be an admissible set of  $w$ -contours. Then the number of disordered  $w$ -contours in  $\partial$  with  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$  is  $C_{\text{in}}(X(\partial))$ . Hence,

$$\prod_{\gamma \in \partial} \rho(\gamma) = q^{-\frac{1}{2d} \|\delta X(\partial)\| + C_{\text{in}}(X(\partial))} \quad (\text{III.3.19})$$

by (III.3.6) and the definition (III.3.5) of  $\rho(\gamma)$ . Combining this with (III.3.1), (III.3.13), and (III.3.16), we arrive at a contour representation for the partition function (III.2.4):

$$Z_L(\beta, \lambda) = Z_{o, V(L)}(\beta, \lambda) + Z_{d, V(L)}(\beta, \lambda) + Z_{\text{big}, V(L)}(\beta, \lambda). \quad (\text{III.3.20})$$

We shall use it to prove Theorem III.2.2.

**Strong b.c.** For these boundary conditions, our definitions are slightly more involved. This stems in part from our definition of contours, in particular from the extra terms in (III.3.7), and in part from our desire to rewrite  $Z_L(\beta, \lambda)$  as a partition function with disordered or ordered boundary conditions, depending on whether we are in the situation of Theorem III.2.1 (a) or (b).

For every  $B \subset \bar{\mathbb{B}}$ , let

$$Z_{m'}^m(B) = \frac{g_{m'}}{d} |\mathbb{B} \cap B| + h_{m'}^m |\partial \mathbb{B} \cap B|, \quad m, m' = o, d, \quad (\text{III.3.21})$$

where  $g_o$  and  $g_d$  were defined in (III.3.11) and

$$h_o^m = \begin{cases} -\log(e^{\lambda\beta} - 1) & \text{if } m = o, \\ -\log(e^{\lambda\beta} - 1) - \frac{1}{2d} \log q & \text{if } m = d, \end{cases} \quad (\text{III.3.22})$$

$$h_d^m = \begin{cases} -\frac{1}{d} \log q & \text{if } m = o, \\ -\frac{1}{2d} \log q & \text{if } m = d. \end{cases} \quad (\text{III.3.23})$$

Moreover, for any  $W$  of the form (III.3.8), we define

$$\tilde{Z}_{m, W}(\beta, \lambda) = \sum_{\partial \sqsubset W}^{(m)} e^{-G_o^m(\Omega_o(W, \partial)) - G_d^m(\Omega_d(W, \partial))} \prod_{\gamma \in \partial} \rho(\gamma), \quad m = o, d, \quad (\text{III.3.24})$$

where the sum goes over all admissible sets  $\partial$  of  $s$ -contours such that the external  $s$ -contours in  $\partial$  are  $m$ -labelled and  $V(\gamma) \subset W$  for every  $\gamma \in \partial$ . With this definition, we get

$$Z_L(\beta, \lambda) = \tilde{Z}_{o, V(L)}(\beta, \lambda) = \tilde{Z}_{d, V(L)}(\beta, \lambda). \quad (\text{III.3.25})$$

Indeed, consider  $\tilde{Z}_{o, V}(\beta, \lambda)$  defined by (III.3.24). Then every  $\partial$  contributing to it contains exactly  $C_{\text{in}}(X(\partial))$  disordered  $s$ -contours, all with  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$ . Analogously, any  $\partial$  contributing to  $\tilde{Z}_{d, V}(\beta, \lambda)$  contains  $C_{\text{in}}(X(\partial))$  disordered  $s$ -contours for which  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$ . Combining this with (III.3.1), (III.3.5), (III.3.7), and (III.3.21) — (III.3.24), we obtain (III.3.25), and hence two more contour representations for our model. They will be used to prove Theorem III.2.1.

Again, it will be useful to rewrite  $\tilde{Z}_{m,W}(\beta, \lambda)$  as

$$\tilde{Z}_{m,W}(\beta, \lambda) = e^{-G_m^m(\mathbb{B}(W))} \sum_{\partial^* \sqsubset W}^{(m)} \prod_{\gamma \in \partial^*} \tilde{K}_m(\gamma), \quad m = o, d, \quad (\text{III.3.26})$$

where the sum goes over all collections  $\partial^*$  of non-overlapping  $m$ -labelled  $s$ -contours such that  $V(\gamma) \subset W$  for every  $\gamma \in \partial^*$  and

$$\tilde{K}_o(\gamma) = \rho(\gamma) q^{\frac{1}{2d} |\partial \mathbb{B}(\text{int } \gamma)|} \frac{\tilde{Z}_{d, \text{int } \gamma}(\beta, \lambda)}{\tilde{Z}_{o, \text{int } \gamma}(\beta, \lambda)}, \quad (\text{III.3.27})$$

$$\tilde{K}_d(\gamma) = \rho(\gamma) q^{\frac{1}{2d} |\partial \mathbb{B}(\text{int } \gamma)|} \frac{\tilde{Z}_{o, \text{int } \gamma}(\beta, \lambda)}{\tilde{Z}_{d, \text{int } \gamma}(\beta, \lambda)}. \quad (\text{III.3.28})$$

REMARK III.3.2. Since the partition functions defined by (III.3.13) and (III.3.24) depend on  $\lambda$  only through the quantities  $G_o((\Omega_o(W, \partial))$  and  $G_o^m((\Omega_o(W, \partial))$ , respectively, they are independent of  $\lambda$  once  $\partial \mathbb{B}(W) = \emptyset$ , i.e. once  $\text{dist}(W, \partial V) \geq \frac{3}{4}$ . As a result, the activities  $K_m(\gamma)$  and  $\tilde{K}_m(\gamma)$ ,  $m = o, d$ , are independent of  $\lambda$  if  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$ .<sup>7</sup> Note also that the partition functions  $Z_{m,W}(\beta, \lambda)$  and  $\tilde{Z}_{m,W}(\beta, \lambda)$  coincide whenever  $W$  is ‘not too large and not touching the boundary’. Namely, we have  $Z_{m,W}(\beta, \lambda) = \tilde{Z}_{m,W}(\beta, \lambda)$  and  $K_m(\gamma) = \tilde{K}_m(\gamma)$  as soon as  $\text{diam } W < \omega(L)$ ,  $\text{dist}(W, \partial V) \geq \frac{3}{4}$  and  $\text{diam } \gamma < \omega(L)$ ,  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$ , respectively.

### III.4. Proof of Theorem III.2.2

The decomposition (III.3.20) of the partition function  $Z_L(\beta, \lambda)$  suggests that the finite-size scaling for the energy  $E_L(\beta, \lambda)$  and the heat capacity  $C_L(\beta, \lambda)$  may be evaluated via a cluster-expansion analysis of  $Z_{m,W}(\beta, \lambda)$ ,  $m = o, d$ . To this end, a bound  $K_m(\gamma) \leq \epsilon^{|\gamma|}$ , where  $\epsilon > 0$  is small, would be needed for every short  $w$ -contour  $\gamma$ . It turns out, though, that a bound of this form does not hold for both  $m = o, d$  and for all  $\beta > 0$ . Therefore, one first constructs [8, 10] truncated contour activities  $K_m(\gamma)$  and the corresponding partition functions<sup>8</sup>

$$\bar{Z}_{m,W}(\beta, \lambda) = e^{-G_m(\mathbb{B}(W))} \sum_{\partial^* \sqsubset W}^{(m)} \prod_{\gamma \in \partial^*} K_m(\gamma), \quad m = o, d, \quad (\text{III.4.1})$$

defined for every  $W$  of the form (III.3.8). This will be done in such a way that  $K_m(\gamma)$  are smooth functions of  $\beta$  and  $K_m(\gamma) \leq \epsilon^{|\gamma|}$  for

<sup>7</sup>Note that in this case the multiplicative factor  $q^{\frac{1}{2d} |\partial \mathbb{B}(\text{int } \gamma)|}$  in (III.3.27) vanishes.

<sup>8</sup>The sum in (III.4.1) runs over the same collections of  $w$ -contours as in (III.3.14).

some small  $\epsilon > 0$ . In addition, whenever

$$f_m(\beta) = - \lim_{L \rightarrow \infty} \frac{1}{L^d} \log \bar{Z}_{m,V(L)}(\beta, \lambda) \quad (\text{III.4.2})$$

equals

$$f(\beta) = \min\{f_o(\beta), f_d(\beta)\}, \quad (\text{III.4.3})$$

then, necessarily,  $K_m(\gamma) = K_m(\gamma)$  so that  $\bar{Z}_{m,W}(\beta, \lambda) = Z_{m,W}(\beta, \lambda)$ .

As in [10], the truncated model will be constructed inductively. However, to get weaker constraints on the surface coupling  $\lambda$ , we have to repeat the argument of [10] with a more careful evaluation of boundary terms. This will be done in Appendix III.B. Introducing

$$F_m(\bar{\mathbb{B}}(W)) = \frac{1}{d} f_m |\mathbb{B}(W)| + s_m |\partial \mathbb{B}(W)|, \quad (\text{III.4.4})$$

where

$$s_m(\beta) = - \lim_{L \rightarrow \infty} \frac{1}{2dL^{d-1}} (\bar{Z}_{m,V(L)}(\beta, \lambda) + \frac{1}{d} f_m(\beta) |\mathbb{B}|), \quad (\text{III.4.5})$$

we formulate the results in the next lemma. Its proof is also given in Appendix III.B.

LEMMA III.4.1. *Let  $d \geq 2$ ,  $J > 0$ ,  $0 \leq \mu < 1$ ,  $k_0 \in \mathbb{Z}$ ,  $k_0 \geq 0$ , and let*

$$\nu = \frac{1}{24d} \min\{1, 3(1 - \mu)\} \quad \text{and} \quad \alpha = \frac{\nu}{2} \log q - 1. \quad (\text{III.4.6})$$

*There exists a finite positive constant  $D_0$  such that, for any function  $\omega : \mathbb{N} \rightarrow [0, \infty]$  for which  $\omega(L) \leq L$ ,  $\omega(L) \rightarrow \infty$  as  $L \rightarrow \infty$ , a truncated activity  $K_m(\gamma)$  exists, for any  $\mathfrak{m}$ -labelled short  $w$ -contour  $\gamma$ ,  $\mathfrak{m} = \mathfrak{o}, \mathfrak{d}$ , satisfying the following claims (a)–(d) whenever  $q, L$  are large enough,  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ , and either  $0 \leq k \leq k_0$  and  $\beta \geq 1$  or  $k = 0$  and  $\beta > 0$ :*

(a)  $K_m(\gamma)$  is a  $C^{k_0}$  function of  $\beta$  and  $|\frac{\partial^k}{\partial \beta^k} K_m(\gamma)| \leq (D_0 q^{-2\nu})^{|\gamma|}$ .

(b) Let  $a_m(\beta) = f_m(\beta) - f(\beta)$ . If  $a_m(\beta) \text{diam } \gamma \leq \alpha$ , then one has  $K_m(\gamma) = K_m(\gamma)$ .

(c) If  $a_m(\beta) \min\{\text{diam } W, \omega(L)\} \leq \alpha$ , then  $\bar{Z}_{m,W}(\beta, \lambda) = Z_{m,W}(\beta, \lambda)$ .

(d) If  $a_m(\beta) > 0$ , then, for any  $W$  of the form (III.3.8) we have<sup>9</sup>

$$\begin{aligned} \left| \frac{\partial^k}{\partial \beta^k} Z_{m,W}(\beta, \lambda) \right| &\leq D_0 |\bar{\mathbb{B}}(W)|^k Z_{m,W}(\beta, \lambda) \leq \\ &\leq D_0 |\bar{\mathbb{B}}(W)|^k e^{-F_m(\bar{\mathbb{B}}(W)) + (\frac{\mu}{2d} \log q + O(q^{-\nu})) |\partial \mathbb{B}(W)|} \times \\ &\quad \times e^{O(q^{-\nu}) \|\partial W\| + O(q^{-\nu L})}. \end{aligned} \quad (\text{III.4.7})$$

<sup>9</sup>Given  $\mathfrak{W} \subset V$ , we write  $\|\partial \mathfrak{W}\|$  for the number of intersections of the boundary  $\partial \mathfrak{W}$  with  $\mathbb{B} \cup \partial \mathbb{B}(\mathfrak{W}^c)$ .

Here  $m^c = o$  if  $m = d$  and vice versa.

REMARK III.4.2. Lemma III.4.1 (a) and (c) allows us to analyze the functions  $\log Z_{m,W}(\beta, \lambda)$ ,  $m = o, d$ , and their derivatives up to the  $k_0$ -th order by convergent cluster expansions once we assume that  $a_m(\beta) \min\{\text{diam } W, \omega(L)\} \leq \alpha$  and  $\beta \geq 1$ . This then leads to the relations

$$\frac{\partial^k}{\partial \beta^k} \log Z_{m,W}(\beta, \lambda) = -\frac{\partial^k F_m(\bar{\mathbb{B}}(W))}{\partial \beta^k} + O(q^{-\nu}) \|\partial W\| + O(q^{-\nu L}) \quad (\text{III.4.8})$$

for all  $k = 0, \dots, k_0$  and  $q, L$  sufficiently large. Moreover,  $f_m$  and  $s_m$  are  $C^{k_0}$  functions of  $\beta$  on the interval  $[1, \infty)$ , and

$$\frac{d^k f_m}{d\beta^k} = \frac{d^k g_m}{d\beta^k} + O(q^{-\nu}), \quad \frac{d^k s_m}{d\beta^k} = \frac{d^k h_m}{d\beta^k} + O(q^{-\nu}) \quad (\text{III.4.9})$$

for any  $k = 0, \dots, k_0$  and  $q$  large enough. Combined with Lemma III.A.5 (a), one can show the existence of a unique point  $\bar{\beta}$  such that

$$f_o(\bar{\beta}) = f_d(\bar{\beta}). \quad (\text{III.4.10})$$

Moreover,

$$f(\beta) = \begin{cases} f_o(\beta) & \text{if } \beta \geq \bar{\beta}, \\ f_d(\beta) & \text{if } \beta \leq \bar{\beta}, \end{cases} \quad (\text{III.4.11})$$

and

$$\bar{\beta} = \frac{\log q}{dJ} + O(q^{-\nu}). \quad (\text{III.4.12})$$

Notice that

$$\left. \frac{df}{d\beta} \right|_{\bar{\beta}+0} - \left. \frac{df}{d\beta} \right|_{\bar{\beta}-0} = \left. \frac{d(f_o - f_d)}{d\beta} \right|_{\bar{\beta}} \leq -dJ + O(q^{-\nu}) < 0 \quad (\text{III.4.13})$$

due to (III.4.9) and (III.3.11). In Remark III.4.5 we shall show that  $\bar{\beta} = \beta_t$ .

In view of (III.4.11), the functions  $a_o(\beta)$  and  $a_d(\beta)$  vanish for  $\beta \geq \bar{\beta}$  and  $\beta \leq \bar{\beta}$ , respectively. Moreover, for  $1 \leq \beta < \bar{\beta}$ , the function  $a_o(\beta) = (f_o - f_d)(\beta) > 0$  is decreasing, whereas for  $\beta > \bar{\beta}$  the function  $a_d(\beta) = -(f_o - f_d)(\beta) > 0$  is increasing.

It is worth noting that Lemma III.A.7 applied to  $\psi_1(x) = e^{\pm x}$  and  $\psi_2(\beta) = Z_{m,W}(\beta, \lambda)$ ,  $m = o, d$ , combined with (III.4.8), (III.4.9), and Lemma III.A.1 gives

$$\left| \frac{\partial^k}{\partial \beta^k} (Z_{m,W}(\beta, \lambda))^{\pm 1} \right| \leq \tilde{D}_0 |\bar{\mathbb{B}}(W)|^k (Z_{m,W}(\beta, \lambda))^{\pm 1}, \quad 1 \leq k \leq k_0, \quad (\text{III.4.14})$$

for some finite constant  $\tilde{D}_0 > 0$  once  $a_m(\beta) \min\{\text{diam } W, \omega(L)\} \leq \alpha$  and  $\beta \geq 1$ .

For the rest of this section, we choose the function  $\omega(L)$  to satisfy the condition (III.2.19) from Theorem III.2.2. Let us provisionally introduce the energy jump at  $\bar{\beta}$ ,

$$\Delta \mathfrak{e} := -\frac{1}{2} \frac{d(f_o - f_d)}{d\beta} \Big|_{\bar{\beta}} = \frac{dJ}{2} + O(q^{-\nu}) > 0; \quad (\text{III.4.15})$$

it will turn out that it actually coincides with  $\Delta \mathfrak{e}$  from Theorem III.2.2. The next corollary immediately follows from Lemma III.4.1 and is the first step in the proof of (III.2.20).

**COROLLARY III.4.3.** *Let  $d \geq 2$ ,  $J > 0$ , and  $0 \leq \mu < 1$ . Let us define  $\nu$  and  $\alpha$  by (III.4.6) and  $\bar{\beta}$  by (III.4.10). For  $q$  and  $L$  sufficiently large and  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ , we have:*

(a) *The equation  $a_m(\beta) = \frac{\alpha}{\omega(L)}$ ,  $m = o, d$ , has a single solution  $\beta_m(L)$ , and*

$$\beta_o(L) = \bar{\beta} - \frac{\alpha}{2\Delta \mathfrak{e}} \frac{1}{\omega(L)} + O((\omega(L))^{-2}), \quad (\text{III.4.16})$$

$$\beta_d(L) = \bar{\beta} + \frac{\alpha}{2\Delta \mathfrak{e}} \frac{1}{\omega(L)} + O((\omega(L))^{-2}). \quad (\text{III.4.17})$$

*In addition,  $a_o(\beta) \leq \frac{\alpha}{\omega(L)}$  iff  $\beta \geq \beta_o(L)$ , while  $a_d(\beta) \leq \frac{\alpha}{\omega(L)}$  iff  $\beta \leq \beta_d(L)$ .*

(b) *There is a unique point  $\beta_{\equiv}^{(\lambda)}(L) \in (\beta_o(L), \beta_d(L))$  at which  $Z_{o,V}(\beta, \lambda)$  and  $Z_{d,V}(\beta, \lambda)$  coincide, and*

$$\beta_{\equiv}^{(\lambda)}(L) = \bar{\beta} \left[ 1 + \frac{d}{\Delta \mathfrak{e}} \left( \frac{J}{2} - \lambda + O\left(\frac{q^{-\nu}}{\log q}\right) \right) \frac{1}{L} + O(L^{-2}) \right]. \quad (\text{III.4.18})$$

**PROOF:** (a) Let us consider  $m = o$ , for instance. For  $0 < \beta \leq 1$ , we have

$$a_o(\beta) - \frac{\alpha}{\omega(L)} \geq \log q - d \log(e^J - 1) + O(q^{-\nu}) - \frac{\alpha}{\omega(L)} > 0$$

whenever  $q$  and  $L$  are large enough; we used (III.4.9) with  $k = 0$  and the second condition in (III.2.19). Since  $a_o(\beta)$  is continuous and decreasing on  $[1, \bar{\beta})$  once  $q$  is large, while  $a_o(\beta) = 0$  for  $\beta \geq \bar{\beta}$ , there is a single solution  $\beta_o(L) \in (1, \bar{\beta})$ . The Lagrange mean-value theorem then yields

$$\frac{\alpha}{\omega(L)} = a_o(\beta_o(L)) = (f_o - f_d)(\beta_o(L)) = (\beta_o(L) - \bar{\beta}) \frac{d(f_o - f_d)}{d\beta} \Big|_{\tilde{\beta}} \quad (\text{III.4.19})$$

for some  $\tilde{\beta}$  between  $\beta_o(L)$  and  $\bar{\beta}$ . Since the derivatives of  $f_o$  and  $f_d$  are bounded due to (III.4.9) and Lemma III.A.1, it follows that  $\beta_o(L) - \bar{\beta} = O((\omega(L))^{-1})$ . Using this, Taylor expansion around  $\bar{\beta}$

gives

$$\frac{\alpha}{\omega(L)} = (f_o - f_d)(\beta_o(L)) = (\beta_o(L) - \bar{\beta}) \left. \frac{d(f_o - f_d)}{d\beta} \right|_{\bar{\beta}} + O((\omega(L))^{-2}), \quad (\text{III.4.20})$$

which along with (III.4.15) implies the first equality of (III.4.16). One proceeds similarly for  $m = d$ .

(b) Let us introduce

$$\xi_L(\beta) = \log \frac{Z_{o,V(L)}(\beta, \lambda)}{Z_{d,V(L)}(\beta, \lambda)}.$$

If  $\beta \in [\beta_o(L), \beta_d(L)]$ , we may use the proved part (a) of this corollary, the relation (III.4.8) with  $k = 0$ , and Lemma III.A.1 to get

$$\begin{aligned} \xi_L(\beta_m(L)) &= -(f_o - f_d)(\beta_m(L)) |\mathbb{B}|/d - \\ &\quad - \left[ (s_o - s_d)(\bar{\beta}) + O\left(\frac{1}{\omega(L)}\right) \right] |\partial\mathbb{B}| + O(q^{-\nu L}) \end{aligned} \quad (\text{III.4.21})$$

for  $m = o, d$ . Since  $e^{c\bar{\beta}} - 1 = q^{\frac{c}{dJ}} (1 + O(q^{-\nu}))$  for any  $c \geq \frac{\lambda}{4}$  (so that  $\frac{c}{dJ} \geq \nu$ ), see (III.A.10), the relations (III.4.9) and (III.3.11) give

$$(s_o - s_d)(\bar{\beta}) = (h_o - h_d)(\bar{\beta}) + O(q^{-\nu}) = -\frac{1}{d} \left( \frac{\lambda}{J} - \frac{1}{2} \right) \log q + O(q^{-\nu}). \quad (\text{III.4.22})$$

Observing that

$$|\mathbb{B}| = |\mathbb{B}(\wedge(L))| = dL^{d-1}(L-1), \quad |\partial\mathbb{B}| = |\partial\mathbb{B}(\wedge(L))| = 2dL^{d-1}, \quad (\text{III.4.23})$$

and taking into account (III.4.19), (III.4.6), and (III.2.19), we eventually obtain

$$\begin{aligned} \xi_L(\beta_o(L)) &= \left( -\frac{\alpha}{\log q} \frac{L-1}{\omega(L)} + 2\left(\frac{\lambda}{J} - \frac{1}{2}\right) + O\left(\frac{q^{-\nu}}{\log q}\right) \right) L^{d-1} \log q + \\ &\quad + O\left(\frac{L^{d-1}}{\omega(L)}\right) \leq \\ &\leq \left[ -\left(\frac{\nu}{2} + \frac{1}{\log q}\right) \frac{6}{\nu} + \mu + O\left(\frac{q^{-\nu}}{\log q}\right) \right] L^{d-1} \log q + O\left(\frac{L^{d-1}}{\omega(L)}\right) \leq \\ &\leq -\frac{3}{2} L^{d-1} \log q \left[ 1 + O\left(\frac{1}{\omega(L)}\right) \right] \leq -L^{d-1} \log q < 0 \end{aligned} \quad (\text{III.4.24})$$

for  $q, L$  large. Similarly,

$$\xi_L(\beta_d(L)) \geq L^{d-1} \log q > 0 \quad (\text{III.4.25})$$

once  $q, L$  are large enough.

Next, combining the proved part (a) of the corollary, (III.4.8), (III.4.9), and Lemma III.A.1, one readily gets

$$\frac{\partial^k}{\partial \beta^k} \log Z_{m,V(L)}(\beta, \lambda) = - \frac{d^k f_m}{d \beta^k} \Big|_{\bar{\beta}} L^d + O(|\beta - \bar{\beta}| L^d) + O(L^{d-1}), \quad (\text{III.4.26})$$

$m = o, d$ , for all  $k \in \mathbb{N}$  and  $\beta \in [\beta_o(L), \beta_d(L)]$ . This and (III.4.16) yield, for any  $\beta \in [\beta_o(L), \beta_d(L)]$  and large  $q$  and  $L$ ,

$$\frac{\partial \xi_L}{\partial \beta} = 2 \Delta_e L^d \left[ 1 + O(|\beta - \bar{\beta}|) + O(L^{-1}) \right] > 0. \quad (\text{III.4.27})$$

Since  $\xi_L(\beta)$  is continuous, a result of (III.4.24), (III.4.25), and (III.4.27) is that the equation  $Z_{o,V}(\beta, \lambda) = Z_{d,V}(\beta, \lambda)$  has necessarily a unique solution  $\beta_{=}^{(\lambda)}(L)$  on the interval  $(\beta_o(L), \beta_d(L))$ . To find its position, we first use the Lagrange mean-value theorem to write

$$0 = \xi_L(\beta_{=}^{(\lambda)}(L)) = \xi_L(\bar{\beta}) + (\beta_{=}^{(\lambda)}(L) - \bar{\beta}) \frac{\partial \xi_L}{\partial \beta} \Big|_{\hat{\beta}}, \quad (\text{III.4.28})$$

where  $\hat{\beta}$  is a point between  $\beta_{=}^{(\lambda)}(L)$  and  $\bar{\beta}$ . Due to (III.4.8), (III.4.22), and (III.4.23), we have

$$\begin{aligned} \xi_L(\bar{\beta}) &= (s_d - s_o)(\bar{\beta}) |\partial \mathbb{B}| + O(q^{-\nu L}) = \\ &= 2 \left[ \left( \frac{\lambda}{J} - \frac{1}{2} \right) \log q + O(q^{-\nu}) \right] L^{d-1}. \end{aligned} \quad (\text{III.4.29})$$

In view of (III.4.27), we thus observe that  $\beta_{=}^{(\lambda)}(L) - \bar{\beta} = O(L^{-1})$ . Using this, the Taylor expansion of  $\xi_L(\beta_{=}^{(\lambda)}(L))$  around  $\bar{\beta}$  along with (III.4.26), (III.4.9), and Lemma III.A.1 imply

$$0 = \xi_L(\bar{\beta}) + (\beta_{=}^{(\lambda)}(L) - \bar{\beta}) \frac{\partial \xi_L}{\partial \beta} \Big|_{\bar{\beta}} + O(L^{d-2}). \quad (\text{III.4.30})$$

Combined with (III.4.27), (III.4.29), and (III.4.12), we get (III.4.18).  
Q.E.D.

In Appendix III.C we prove

**COROLLARY III.4.4.** *Let  $d \geq 2$ ,  $J > 0$ ,  $0 \leq \mu < 1$ , and  $k_0 = 0, 1, \dots$ . For all  $q, L$  sufficiently large,  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ , and  $0 \leq k \leq k_0$ , we have:*

(a) *If  $\beta \in [\beta_o(L), \beta_d(L)]$ , then*

$$\left| \frac{\partial^k}{\partial \beta^k} \frac{Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda) + Z_{d,V(L)}(\beta, \lambda)} \right| < q^{-\frac{1-\mu}{20d}} \omega(L). \quad (\text{III.4.31})$$

(b) *If  $\beta \geq \beta_d(L)$ , then*

$$\left| \frac{\partial^k}{\partial \beta^k} \frac{Z_{d,V(L)}(\beta, \lambda) + Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda)} \right| < q^{-\frac{1-\mu}{20d}} \omega(L). \quad (\text{III.4.32})$$

In addition, if  $1 \leq \beta \leq \beta_o(L)$ , then

$$\left| \frac{\partial^k}{\partial \beta^k} \frac{Z_{o,V(L)}(\beta, \lambda) + Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{d,V(L)}(\beta, \lambda)} \right| < q^{-\frac{1-\mu}{20d} \omega(L)}. \quad (\text{III.4.33})$$

REMARK III.4.5. Let us show here that the quantity  $f$  defined by (III.4.3) is actually the free energy of our model. Assuming first that  $\beta \geq \bar{\beta}$ , where  $\bar{\beta}$  was defined by (III.4.10), we write

$$\begin{aligned} - \lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} \log Z_L(\beta, \lambda) &= \\ &= - \lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} (\log Z_{o,V(L)}(\beta, \lambda) + X_L(\beta, \lambda)). \end{aligned} \quad (\text{III.4.34})$$

Here

$$X_L(\beta, \lambda) = \log \left( 1 + \frac{Z_{d,V(L)}(\beta, \lambda) + Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda)} \right) > 0$$

since all three terms  $Z_{o,V}(\beta, \lambda)$ ,  $Z_{d,V}(\beta, \lambda)$ , and  $Z_{\text{big},V}(\beta, \lambda)$  are positive. Hence,

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} X_L(\beta, \lambda) \geq 0.$$

On the other hand, if  $\beta \geq \beta_d(L)$ , then

$$X_L(\beta, \lambda) \leq \log(1 + q^{-\frac{1-\mu}{20d} \omega(L)})$$

due to Corollary III.4.4 (b), while if  $\bar{\beta} \leq \beta \leq \beta_d(L)$ , then

$$\begin{aligned} X_L(\beta, \lambda) &= \log \left( 1 + \frac{Z_{d,V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda)} + \right. \\ &\quad \left. + \frac{Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda) + Z_{d,V(L)}(\beta, \lambda)} \frac{Z_{o,V(L)}(\beta, \lambda) + Z_{d,V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda)} \right) \leq \\ &\leq \log(1 + e^{O(\frac{L^d}{\omega(L)}) + O(L^{d-1})} + q^{-\frac{1-\mu}{20d} \omega(L)} (1 + e^{O(\frac{L^d}{\omega(L)}) + O(L^{d-1})})) = \\ &= L^d (O((\omega(L))^{-1}) + O(L^{-1})) \end{aligned} \quad (\text{III.4.35})$$

in view of (III.4.26). Thus, using also (III.2.19),

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} X_L(\beta, \lambda) = 0 \quad \text{for all } \beta \geq \bar{\beta}.$$

As a result,

$$- \lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} \log Z_L(\beta, \lambda) = f_o(\beta) = f(\beta) \quad \text{for all } \beta \geq \bar{\beta}.$$

The case when  $\beta \leq \bar{\beta}$  is treated similarly. Finally, notice that (III.4.13) implies  $\bar{\beta} = \beta_t$ . Consequently, the quantity  $\Delta e$  introduced in Corollary III.4.3 is identical to  $\Delta e$  defined in (III.2.12).

Let us now use Lemma III.4.1 and Corollary III.4.3 and III.4.4 to study the behaviour of the finite-volume mean energy  $E_L(\beta, \lambda)$  and its derivatives (with respect to  $\beta$ ) for large values of  $q$  and  $L$ .

In the first step, we consider  $\beta \in [\beta_o(L), \beta_d(L)]$ . In this case, we use (III.3.20) and write

$$\begin{aligned} E_L(\beta, \lambda) &= -\frac{\partial}{\partial \beta} \left[ \log(Z_{o,V(L)}(\beta, \lambda) + Z_{d,V(L)}(\beta, \lambda)) + \right. \\ &\quad \left. + \log\left(1 + \frac{Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda) + Z_{d,V(L)}(\beta, \lambda)}\right) \right] = \\ &= -\frac{1}{2} \frac{\partial}{\partial \beta} \log(Z_{o,V(L)}(\beta, \lambda) Z_{d,V(L)}(\beta, \lambda)) - \\ &\quad - \frac{1}{2} \left( \frac{\partial}{\partial \beta} \log \frac{Z_{o,V(L)}(\beta, \lambda)}{Z_{d,V(L)}(\beta, \lambda)} \right) \tanh\left(\frac{1}{2} \log \frac{Z_{o,V(L)}(\beta, \lambda)}{Z_{d,V(L)}(\beta, \lambda)}\right) - \\ &\quad - \frac{\partial}{\partial \beta} \log\left(1 + \frac{Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda) + Z_{d,V(L)}(\beta, \lambda)}\right). \quad (\text{III.4.36}) \end{aligned}$$

Applying Lemma III.A.7 to the functions  $\psi_1(x) = \log x$  and  $\psi_2(\beta) = 1 + \frac{Z_{\text{big},V(\beta, \lambda)}}{Z_{o,V(\beta, \lambda)} + Z_{d,V(\beta, \lambda)}}$  and using Corollary III.4.4 (a), we get

$$\begin{aligned} \frac{\partial^k}{\partial \beta^k} \log\left(1 + \frac{Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{o,V(L)}(\beta, \lambda) + Z_{d,V(L)}(\beta, \lambda)}\right) &= \\ &= \sum_{j=1}^k (1 + O(q^{-\frac{1-\mu}{20d} \omega(L)}))^{-j} O(q^{-j \frac{1-\mu}{20d} \omega(L)}) = O(q^{-\frac{1-\mu}{20d} \omega(L)}) \end{aligned} \quad (\text{III.4.37})$$

for any  $k \in \mathbb{N}$ . Therefore, taking into account (III.4.26) and Remark III.4.5,

$$\begin{aligned} E_L(\beta, \lambda) &= e_0 L^d - \Delta e L^d \tanh\left(\frac{1}{2} \log \frac{Z_{o,V(L)}(\beta, \lambda)}{Z_{d,V(L)}(\beta, \lambda)}\right) + \\ &\quad + O(|\beta - \beta_t| L^d) + O(L^{d-1}) + O(q^{-\frac{1-\mu}{20d} \omega(L)}), \quad (\text{III.4.38}) \end{aligned}$$

where

$$e_0 = \frac{1}{2} \frac{d(f_o + f_d)}{d\beta} \Big|_{\beta_t} = -\Delta e + O(q^{-\nu}) < 0 \quad (\text{III.4.39})$$

was introduced in (III.2.12).

Next, let us find expressions for the derivatives of  $E_L(\beta, \lambda)$ . As the derivatives of  $\tanh x$  are bounded because of Lemma (III.A.13),

Lemma III.A.7 and (III.4.26) yield

$$\begin{aligned}
& \frac{\partial^k}{\partial \beta^k} \tanh \left( \frac{1}{2} \log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)} \right) = \\
& = \frac{d^k}{dx^k} \tanh x \Big|_{x=\frac{1}{2} \log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)}} \left( \frac{1}{2} \frac{\partial}{\partial \beta} \log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)} \right)^k + \\
& \quad + \sum_{j=1}^{k-1} O(L^{jd}) = \\
& = (\Delta e L^d)^k \frac{d^k}{dx^k} \tanh x \Big|_{x=\frac{1}{2} \log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)}} (1 + O(|\beta - \beta_t|) + O(L^{-1})) + \\
& \quad + O(L^{(k-1)d}) \quad (\text{III.4.40})
\end{aligned}$$

for any  $k \in \mathbb{N}$ . Along with (III.4.36) and (III.4.26), this implies

$$\begin{aligned}
& \frac{\partial^k}{\partial \beta^k} E_L(\beta, \lambda) = O(L^d) - \\
& - \frac{1}{2} \left( \frac{\partial}{\partial \beta} \log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)} \right) \frac{\partial^k}{\partial \beta^k} \tanh \left( \frac{1}{2} \log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)} \right) + \\
& \quad + \sum_{j=1}^k O(L^d) O(L^{(k-j)d}) + O(q^{-\frac{1-\mu}{20d}} \omega(L)) = \\
& = -(\Delta e L^d)^{k+1} \frac{d^k}{dx^k} \tanh x \Big|_{x=\frac{1}{2} \log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)}} [1 + O(|\beta - \beta_t|) + \\
& \quad + O(L^{-1})] + O(L^{kd}) \quad (\text{III.4.41})
\end{aligned}$$

for all  $k \in \mathbb{N}$ .

Finally, we Taylor expand  $\log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)}$  around the point  $\beta_{\equiv}^{(\lambda)}(L)$  of Corollary III.4.3 (b) and obtain

$$\begin{aligned}
\log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)} & = (\beta - \beta_{\equiv}^{(\lambda)}(L)) \left( \frac{d(f_{\text{d}} - f_{\circ})}{d\beta} \Big|_{\beta_t} L^d + O(L^{d-1}) \right) = \\
& = 2 \Delta e (\beta - \beta_{\equiv}^{(\lambda)}(L)) L^d (1 + O(L^{-1})) \quad (\text{III.4.42})
\end{aligned}$$

according to (III.4.26), (III.4.15), Lemma III.A.1, and Remark III.4.5. Using now Lemma III.A.8, we get

$$\frac{d^k}{dx^k} \tanh x \Big|_{x=\frac{1}{2} \log \frac{Z_{\circ, V(L)}(\beta, \lambda)}{Z_{\text{d}, V(L)}(\beta, \lambda)}} = \frac{d^k}{dx^k} \tanh x \Big|_{x=\Delta e (\beta - \beta_{\equiv}^{(\lambda)}(L)) L^d} + O(L^{-1}) \quad (\text{III.4.43})$$

for any  $k = 0, 1, \dots$ . Recalling that  $|\beta - \beta_i| = O((\omega(L))^{-1})$  for all  $\beta \in [\beta_o(L), \beta_d(L)]$  due to (III.4.16), the relations (III.4.38), (III.4.41), and (III.4.43) lead to

LEMMA III.4.6. *Let  $d \geq 2$ ,  $J > 0$ ,  $0 \leq \mu < 1$ , and  $k_0 = 1, 2, \dots$ . For  $q$  and  $L$  large enough,  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ ,  $\beta \in [\beta_o(L), \beta_d(L)]$ , and  $1 \leq k \leq k_0$ , we have*

$$E_L(\beta, \lambda) = e_0 L^d - \Delta e L^d \tanh(\Delta e(\beta - \beta_{\equiv}^{(\lambda)}(L))L^d) + O\left(\frac{L^d}{\omega(L)}\right), \quad (\text{III.4.44})$$

$$\begin{aligned} \frac{\partial^k}{\partial \beta^k} E_L(\beta, \lambda) &= -(\Delta e L^d)^{k+1} \frac{d^k}{dx^k} \tanh x \Big|_{x=\Delta e(\beta - \beta_{\equiv}^{(\lambda)}(L))L^d} + \\ &\quad + O\left(\frac{L^{(k+1)d}}{\omega(L)}\right). \end{aligned} \quad (\text{III.4.45})$$

Here  $\beta_{\equiv}^{(\lambda)}(L)$  is the temperature introduced in Corollary III.4.3 (b).

Let us consider now  $\beta \in [1, \beta_o(L)]$  and any  $k \in \mathbb{N}$ . This time we use (III.3.20) to write

$$\begin{aligned} \frac{\partial^{k-1} E_L(\beta, \lambda)}{\partial \beta^{k-1}} &= -\frac{\partial^k}{\partial \beta^k} \left[ \log Z_{d,V(L)}(\beta, \lambda) + \right. \\ &\quad \left. + \log \left( 1 + \frac{Z_{o,V(L)}(\beta, \lambda) + Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{d,V(L)}(\beta, \lambda)} \right) \right]. \end{aligned} \quad (\text{III.4.46})$$

The relation (III.4.8) and Lemma III.A.1 imply that

$$\frac{\partial^k}{\partial \beta^k} \log Z_{d,V(L)}(\beta, \lambda) = -\frac{d^k f_d}{d\beta^k} L^d + O(L^{d-1}). \quad (\text{III.4.47})$$

Moreover, similarly to (III.4.37),

$$\frac{\partial^k}{\partial \beta^k} \log \left( 1 + \frac{Z_{o,V(L)}(\beta, \lambda) + Z_{\text{big},V(L)}(\beta, \lambda)}{Z_{d,V(L)}(\beta, \lambda)} \right) = O(q^{-\frac{1-\mu}{20d}} \omega(L)) \quad (\text{III.4.48})$$

due to Corollary III.4.4 (b) and Lemma III.A.7. Treating the case  $\beta \in [\beta_d(L), \infty)$  analogously and using (III.4.11) and Remark III.4.5, we may state

LEMMA III.4.7. *Let  $d \geq 2$ ,  $J > 0$ ,  $0 \leq \mu < 1$ , and  $k_0 = 1, 2, \dots$ . Then, for all  $q$  and  $L$  sufficiently large,  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ ,  $\beta \in [1, \beta_o(L)] \cup [\beta_d(L), \infty)$ , and  $1 \leq k \leq k_0$ ,*

$$\frac{\partial^{k-1}}{\partial \beta^{k-1}} E_L(\beta, \lambda) = \frac{d^k f(\beta)}{d\beta^k} L^d + O(L^{d-1}). \quad (\text{III.4.49})$$

Here  $f$  is the free energy introduced by (III.4.3), and

$$\frac{d^k f(\beta)}{d\beta^k} = \begin{cases} \frac{d^k f_d(\beta)}{d\beta^k} & \text{if } \beta < \beta_t, \\ \frac{d^k f_o(\beta)}{d\beta^k} & \text{if } \beta > \beta_t. \end{cases} \quad (\text{III.4.50})$$

In the end, let us prove

LEMMA III.4.8. *Let  $d \geq 2$ ,  $J > 0$ , and  $0 \leq \mu < 1$ . For  $q$  and  $L$  sufficiently large and  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ , the specific heat  $C_L(\beta, \lambda)$  attains its maximal value at a unique temperature  $\beta_{\max}^{(\lambda)}(L)$ . Moreover,*

$$\beta_{\max}^{(\lambda)}(L) = \beta_{\leq}^{(\lambda)}(L) + O(L^{-2d}), \quad (\text{III.4.51})$$

where  $\beta_{\leq}^{(\lambda)}(L)$  was introduced in Corollary III.4.3 (b).

PROOF: Let the assumptions of the lemma be fulfilled. First, we observe that if the temperature  $\beta_{\max}^{(\lambda)}(L)$  exists, then  $\beta_{\max}^{(\lambda)}(L) \in [\beta_o(L), \beta_d(L)]$ . Indeed, the definition (III.2.11) of  $C_L(\beta, \lambda)$  along with (III.4.41) yield that  $C_L(\beta_{\leq}^{(\lambda)}(L), \lambda) = \beta^2(\Delta e L^d)^2(1 + O(L^{-1}))$ . However, as soon as  $\beta \in (0, \beta_o(L)] \cup [\beta_d(L), \infty)$ , we have  $C_L(\beta, \lambda) = O(L^d)$  in view of Lemma III.4.7, Remark III.2.3 (iii), and Lemma III.A.1. In other words,

$$C_L(\beta, \lambda) < C_L(\beta_{\leq}^{(\lambda)}(L), \lambda)$$

for all  $\beta \notin [\beta_o(L), \beta_d(L)]$  and  $q, L$  large enough.

Let us, therefore, take  $\beta \in [\beta_o(L), \beta_d(L)]$  in the following. Then Lemma III.4.6 gives

$$\frac{\partial}{\partial \beta} C_L(\beta, \lambda) = \beta^2(\Delta e L^d)^3 \frac{d^2}{dx^2} \tanh x \Big|_{x=\Delta e(\beta - \beta_{\leq}^{(\lambda)}(L))L^d} + O\left(\frac{L^{3d}}{\omega(L)}\right), \quad (\text{III.4.52})$$

$$\frac{\partial^2}{\partial \beta^2} C_L(\beta, \lambda) = \beta^2(\Delta e L^d)^4 \frac{d^3}{dx^3} \tanh x \Big|_{x=\Delta e(\beta - \beta_{\leq}^{(\lambda)}(L))L^d} + O\left(\frac{L^{4d}}{\omega(L)}\right). \quad (\text{III.4.53})$$

The function  $\frac{d^2 \tanh x}{dx^2}$  is odd and negative for  $x > 0$ , while there exists  $A > 0$  such that  $\frac{d^3 \tanh x}{dx^3} < 0$  once  $|x| < 2A$ . As a consequence, if  $q$  and  $L$  are large, the above two equations imply the existence of a unique temperature  $\beta_0(L) \in [\beta_o(L), \beta_d(L)]$  such that  $|\beta_0(L) - \beta_{\leq}^{(\lambda)}(L)| < \frac{A}{\Delta e} \frac{1}{L^d}$  and

$$C_L(\beta_0(L), \lambda) > C_L(\beta, \lambda)$$

for all  $\beta \neq \beta_0(L)$  and  $|\beta - \beta_{\equiv}^{(\lambda)}(L)| \leq \frac{A}{\Delta e} \frac{1}{L^d}$ . However, if  $|\beta - \beta_{\equiv}^{(\lambda)}(L)| > \frac{A}{\Delta e} \frac{1}{L^d}$ , then, in view of (III.2.11) and Lemma III.4.6,

$$\begin{aligned} C_L(\beta, \lambda) &= \beta^2 (\Delta e L^d)^2 \cosh^{-2}(\Delta e(\beta - \beta_{\equiv}^{(\lambda)}(L))L^d) + O\left(\frac{L^{2d}}{\omega(L)}\right) < \\ &< \beta^2 (\Delta e L^d)^2 \left[ \cosh^{-2} A + O\left(\frac{1}{\omega(L)}\right) \right] \quad (\text{III.4.54}) \end{aligned}$$

so that

$$\begin{aligned} C_L(\beta_{\equiv}^{(\lambda)}(L), \lambda) - C_L(\beta, \lambda) &\geq \\ &\geq \beta^2 (\Delta e L^d)^2 \left[ 1 - \cosh^{-2} A + O\left(\frac{1}{\omega(L)}\right) + O(L^{-1}) \right] > 0 \quad (\text{III.4.55}) \end{aligned}$$

once  $q$  and  $L$  are large. Hence,  $\beta_0(L) = \beta_{\max}^{(\lambda)}(L)$ .

It remains to prove (III.4.51). According to the Lagrange mean-value theorem, there is  $\tilde{\beta}$  between  $\beta_{\max}^{(\lambda)}(L)$  and  $\beta_{\equiv}^{(\lambda)}(L)$  such that

$$\begin{aligned} 0 = \frac{\partial C_L(\beta, \lambda)}{\partial \beta} \Big|_{\beta_{\max}^{(\lambda)}(L)} &= \frac{\partial C_L(\beta, \lambda)}{\partial \beta} \Big|_{\beta_{\equiv}^{(\lambda)}(L)} + \\ &+ (\beta_{\max}^{(\lambda)}(L) - \beta_{\equiv}^{(\lambda)}(L)) \frac{\partial^2 C_L(\beta, \lambda)}{\partial \beta^2} \Big|_{\tilde{\beta}}. \quad (\text{III.4.56}) \end{aligned}$$

Now, from (III.2.11), (III.4.41), and Corollary III.4.3, it follows that the derivative  $\frac{\partial C_L(\beta, \lambda)}{\partial \beta} \Big|_{\beta_{\equiv}^{(\lambda)}(L)} = O(L^{2d})$  as  $\frac{d^2 \tanh x}{dx^2} = 0$  at  $x = 0$ . On the other hand, the relation (III.4.53) and the fact that  $\frac{d^3 \tanh x}{dx^3} \Big|_A < 0$  implies

$$\begin{aligned} \frac{\partial^2 C_L(\beta, \lambda)}{\partial \beta^2} &\leq \beta^2 (\Delta e L^d)^4 \frac{d^3 \tanh x}{dx^3} \Big|_A + O\left(\frac{L^{4d}}{\omega(L)}\right) \leq \\ &\leq -\frac{1}{2} \beta^2 (\Delta e L^d)^4 \left| \frac{d^3 \tanh x}{dx^3} \Big|_A \quad (\text{III.4.57}) \end{aligned}$$

whenever  $|\beta - \beta_{\equiv}^{(\lambda)}(L)| \leq \frac{A}{\Delta e} \frac{1}{L^d}$  and  $q, L$  are large. Combined with (III.4.56), this yields (III.4.51). Q.E.D.

In view of the preceding lemma, Corollary III.4.3 (b), and Lemma III.A.8, we have

$$\begin{aligned} \frac{d^k}{dx^k} \tanh x \Big|_{x=\Delta e(\beta - \beta_{\equiv}^{(\lambda)}(L))L^d} &= \\ &= \frac{d^k}{dx^k} \tanh x \Big|_{x=\Delta e(\beta - \beta_{\max}^{(\lambda)}(L))L^d} + O(L^{-2d+1}) \quad (\text{III.4.58}) \end{aligned}$$

for all  $\beta \in [\beta_o(L), \beta_d(L)]$ . Moreover, with the help of (III.4.16), (III.4.6), and (III.4.12), we observe that

$$\begin{aligned} |\beta_m(L) - \beta_t| &= \frac{\alpha}{2\Delta e} \frac{1}{\omega(L)} + O((\omega(L))^{-2}) = \\ &= \frac{\nu}{2\Delta e} \left( \frac{1}{2} \log q - \frac{1}{\nu} \right) \frac{1}{\omega(L)} + O((\omega(L))^{-2}) = \\ &= \frac{d\nu J}{4\Delta e} \frac{\beta_t}{\omega(L)} \left( 1 + O\left(\frac{1}{\log q}\right) + O((\omega(L))^{-1}) \right) \geq \frac{d\nu J}{5\Delta e} \frac{\beta_t}{\omega(L)}, \end{aligned} \quad (\text{III.4.59})$$

$m = o, d$ , for  $q, L$  large. Recalling Lemma III.4.6, Lemma III.4.7, and Lemma III.4.8, we get Theorem III.2.2.

### III.5. Proof of Theorem III.2.1

We shall proceed similarly as in the proof of Theorem III.2.2. We begin with

LEMMA III.5.1. *Let  $d \geq 2, J > 0, k_0 = 0, 1, \dots$ , and  $\alpha = \frac{1}{24d} \log q - 1$ . There exist meta-stable specific free energies  $\tilde{f}_o$  and  $\tilde{f}_d$  such that the following is true for all  $q, L$  is sufficiently large,  $\beta \geq 1, 0 \leq k \leq k_0$ , and  $m = o, d$ .*

(a) *The quantity  $f_m(\beta)$  is a  $C^{k_0}$  function of  $\beta$ , and*

$$\frac{d^k \tilde{f}_m}{d\beta^k} = \frac{d^k g_m}{d\beta^k} + O(q^{-\frac{1}{24d}}), \quad (\text{III.5.1})$$

where  $g_m$  was defined in (III.3.11).

(b) *For any  $m$ -labelled  $s$ -contour  $\gamma$  with  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$ , the activity  $\tilde{K}_m(\gamma)$  does not depend on  $\lambda$ . Introducing  $\tilde{f}(\beta) = \min\{\tilde{f}_o(\beta), \tilde{f}_d(\beta)\}$  and  $\tilde{a}_m(\beta) = \tilde{f}_m(\beta) - \tilde{f}(\beta)$ , we further have: If  $\tilde{a}_m(\beta) \text{diam } \gamma \leq \alpha$ , then  $\tilde{K}_m(\gamma)$  is a  $C^{k_0}$  function of  $\beta$ , and  $|\frac{\partial^k}{\partial \beta^k} \tilde{K}_m(\gamma)| \leq (D_0 q^{-\frac{1}{12d}})^{\|\gamma\|}$ . Here  $D_0 > 0$  is the constant of Lemma III.4.1.*

(c) *For any volume  $W$  of the form (III.3.8) with  $\text{dist}(W, \partial V) \geq \frac{3}{4}$ , the partition function  $\tilde{Z}_{m,W}(\beta, \lambda)$  is independent of  $\lambda$ . In addition, whenever  $\tilde{a}_m(\beta) \text{diam } W \leq \alpha$ , then*

$$\frac{\partial^k}{\partial \beta^k} \log \tilde{Z}_{m,W}(\beta, \lambda) = -\frac{d^k \tilde{f}_m}{d\beta^k} |\mathbb{B}(W)|/d + O(q^{-\frac{1}{24d}}) \|\partial W\|. \quad (\text{III.5.2})$$

PROOF: We first observe that any  $s$ -contour  $\gamma$  with  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$  satisfies  $|\partial \mathbb{B}(\text{int } \gamma)| = 0$ . Thus, using Remark III.3.2,  $\tilde{K}_m(\gamma) = \tilde{K}_m(\gamma)$  once we choose  $\omega(L) = L$ . It then suffices to apply Lemma III.4.1 in which we may put  $\nu = \frac{1}{24}$ . Indeed, if  $|\partial \mathbb{B}(\text{int } \gamma)| = 0$  for some contour  $\gamma$ , then the bound (III.B.27) is to be skipped. Q.E.D.

REMARK III.5.2. Let  $X_o$  be the random-cluster configuration consisting of all the bonds of  $\mathbb{B}$  whose both end-points have the distance from  $\partial V$  less than or equal to 1. Moreover, let  $X_d = \mathbb{B}$  and consider the only  $s$ -contours  $\Gamma_o(L)$  and  $\Gamma_d(L)$  corresponding to  $X_o$  and  $X_d$ , respectively. Note that  $\text{dist}(\Gamma_m(L), \partial V) \geq \frac{3}{4}$  for both  $m = o, d$  and  $\text{diam } \Gamma_o(L) = L - \frac{3}{2}$ ,  $\text{diam } \Gamma_d(L) = L - \frac{1}{2}$ . Then, in view of (III.5.2),

$$-\lim_{L \rightarrow \infty} \frac{1}{|\Lambda(L)|} \log \tilde{Z}_{m, \text{int } \Gamma_m(L)}(\beta, \lambda) = \tilde{f}_m(\beta) = \tilde{f}(\beta) \quad (\text{III.5.3})$$

as soon as  $\tilde{a}_m(\beta) = 0$ . However, it is easy to observe that

$$\tilde{Z}_{o, \text{int } \Gamma_o(L)}(\beta, \lambda) = Z_{L-2}(\beta, J), \quad \tilde{Z}_{d, \text{int } \Gamma_d(L)}(\beta, \lambda) = Z_{L-2}(\beta, 0),$$

c.f. Remark III.3.1 and III.3.2. Because  $\tilde{a}_o(\beta) = 0$  or  $\tilde{a}_d(\beta) = 0$  for every  $\beta > 0$ , we may conclude that  $\tilde{f}$  introduced above is the free energy of our model and, thus, coincides with  $f$  defined by (III.4.3).

Furthermore, we may apply Lemma III.A.5 (a) to show that

$$\tilde{f}(\beta) = \begin{cases} \tilde{f}_o(\beta) & \text{if } \beta \geq \beta_t, \\ \tilde{f}_d(\beta) & \text{if } \beta \leq \beta_t \end{cases} \quad (\text{III.5.4})$$

since

$$\left. \frac{d\tilde{f}}{d\beta} \right|_{\beta_t+0} - \left. \frac{d\tilde{f}}{d\beta} \right|_{\beta_t-0} = \left. \frac{d(\tilde{f}_o - \tilde{f}_d)}{d\beta} \right|_{\beta_t} \leq -dJ + O(q^{-\frac{1}{24d}}) < 0 \quad (\text{III.5.5})$$

according to (III.5.1) and (III.3.11). It also follows that  $\tilde{f}_m = f_m$  whenever  $\tilde{a}_m = 0$ . Here  $f_m$  was defined in (III.4.2).

COROLLARY III.5.3. *Let  $d \geq 2$ ,  $J > 0$ , and  $k_0 = 0, 1, \dots$ . Let  $\alpha = \frac{1}{24d} \log q - 1$ . There exist meta-stable specific surface free energies  $\tilde{s}_o$  and  $\tilde{s}_d$  such that, for all  $q, L$  large enough,  $\lambda \geq 0$ ,  $\beta \geq 1$ ,  $0 \leq k \leq k_0$ , and  $m = o, d$ , we have:*

(a) *The quantity  $\tilde{s}_m(\beta)$  is a  $C^{k_0}$  function of  $\beta$ , and*

$$\frac{d^k \tilde{s}_m}{d\beta^k} = \frac{d^k h_m^m}{d\beta^k} + O(q^{-\nu}) = \frac{d^k h_m}{d\beta^k} + O(q^{-\nu}), \quad (\text{III.5.6})$$

where  $h_m^m$  and  $h_m$  were defined in (III.3.22) and (III.3.11), respectively.

(b) *Let  $W \subset V$  be of the form (III.3.8) with  $\text{dist}(W, \partial V) \geq \frac{1}{4}$ . Whenever  $\tilde{a}_m(\beta) \text{diam } W \leq \alpha$ , then*

$$\frac{\partial^k}{\partial \beta^k} \log \tilde{Z}_{m, W}(\beta, \lambda) = -\frac{d^k \tilde{F}_m(\mathbb{B}(W))}{d\beta^k} + O(q^{-\nu}) \|\partial W\| + O(q^{-\nu L}). \quad (\text{III.5.7})$$

Here  $\tilde{F}_m(\bar{\mathbb{B}}(W)) = \tilde{f}_m|\mathbb{B}(W)| + \tilde{s}_m|\partial\mathbb{B}(W)|$ . On the other hand, once  $\tilde{a}_m(\beta) > 0$ , then

$$\begin{aligned} \left| \frac{\partial^k}{\partial\beta^k} \tilde{Z}_{m,W}(\beta, \lambda) \right| &\leq D_0 |\bar{\mathbb{B}}(W)|^k \tilde{Z}_{m,W}(\beta, \lambda) \leq \\ &\leq D_0 |\bar{\mathbb{B}}(W)|^k e^{-\tilde{F}_{m^c}(\bar{\mathbb{B}}(W)) + \max\{0, \tilde{s}_{m^c} - \tilde{s}_m + \varepsilon \tilde{a}_m\} |\partial\mathbb{B}(W)|} \times \\ &\quad \times e^{O(q^{-\nu}) \|\partial W\| + O(q^{-\nu L})} \end{aligned} \quad (\text{III.5.8})$$

for any  $0 < \varepsilon \leq \frac{\lambda^2}{d^2}$ . Here  $m^c = \mathfrak{o}$  if  $m = \mathfrak{d}$  and vice versa.

PROOF: Let  $W$  be of the form (III.3.8) with  $\text{dist}(W, \partial V) \geq \frac{1}{4}$ . Then, for any contour  $\gamma$  contributing to  $\tilde{Z}_{m,W}(\beta, \lambda)$ , one has that  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$ . Lemma III.5.1 (b) implies that  $\log \tilde{Z}_{m,W}(\beta, \lambda)$  and its derivatives can be controlled by convergent cluster expansions if  $\tilde{a}_m(\beta) \text{diam } W \leq \alpha$ . This yields (III.5.6) and (III.5.7).

In order to prove (III.5.8), we use Lemma III.5.1 and completely follow the proof of Lemma III.B.1 (d) in Appendix III.B. Q.E.D.

REMARK III.5.4. Applying Lemma III.A.7 to  $\psi_1(x) = e^{-x}$  and the function  $\psi_2(\beta) = \log \tilde{Z}_{m,W}(\beta, \lambda)$ , where  $W$  is the volume considered in Corollary III.5.3 (b), and using (III.5.1), (III.5.6), and Lemma III.A.1, it follows that

$$\left| \frac{\partial^k}{\partial\beta^k} (\tilde{Z}_{m,W}(\beta, \lambda))^{-1} \right| \leq k! (D_0 |\bar{\mathbb{B}}(W)|)^k (\tilde{Z}_{m,W}(\beta, \lambda))^{-1} \quad (\text{III.5.9})$$

due to (III.5.7), c.f. Remark III.4.2.

Finally, let us prove

LEMMA III.5.5. Let  $d \geq 2$ ,  $J > 0$ ,  $\mu > 0$ , and  $k_0 = 0, 1, \dots$ . Let

$$\tilde{\nu} = \frac{1}{24d} \min\{1, 3\mu\}, \quad \tilde{\alpha} = \tilde{\nu} \log q - \frac{1}{2}.$$

There exists a finite constant  $\tilde{D}_0 > 0$  such that, for all  $q, L$  sufficiently large,  $\beta \geq 1$ , and  $0 \leq k \leq k_0$ , we have:

(a) If  $\tilde{a}_o(\beta) \text{diam } V \leq \tilde{\alpha}$  and  $0 \leq \lambda \leq \frac{1}{2}(1 - \mu)$ , then  $\tilde{K}_o(\gamma)$  is a  $C^{k_0}$  function of  $\beta$  for any ordered  $s$ -contour  $\gamma$ , and  $|\frac{\partial^k}{\partial\beta^k} \tilde{K}_o(\gamma)| \leq (\tilde{D}_0 q^{-\tilde{\nu}})^{\|\gamma\|}$ .

(b) If  $\tilde{a}_d(\beta) \text{diam } V \leq \tilde{\alpha}$  and  $\lambda \geq \frac{1}{2}(1 + \mu)$ , then  $\tilde{K}_d(\gamma)$  is a  $C^{k_0}$  function of  $\beta$  for any disordered  $s$ -contour  $\gamma$ , and  $|\frac{\partial^k}{\partial\beta^k} \tilde{K}_d(\gamma)| \leq (\tilde{D}_0 q^{-\tilde{\nu}})^{\|\gamma\|}$ .

PROOF: According to Lemma III.5.1 (b), the above lemma holds for any  $s$ -contour  $\gamma$  with  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$ . Hence, let us consider only those contours with  $\text{dist}(\gamma, \partial V(L)) = \frac{1}{4}$ . Given  $k_0 = 0, 1, \dots$  and  $m = o, d$ , let us assume that  $\tilde{a}_m(\beta) \text{diam } V \leq \tilde{\alpha}$  and  $\beta \geq 1$  and take any  $0 \leq k \leq k_0$ . Then (III.3.27), (III.3.5), (III.5.8), and (III.5.9) yield the bound

$$\left| \frac{\partial^k}{\partial \beta^k} \tilde{K}_m(\gamma) \right| \leq (2k_0 D_0 |\mathbb{B}(\text{int } \gamma)|)^{k_0} \tilde{K}_m(\gamma). \quad (\text{III.5.10})$$

Observing that

$$\tilde{a}_m(\beta) |\mathbb{B}(\text{int } \gamma)| \leq 2d \tilde{a}_m(\beta) \text{diam } \gamma \|\gamma\| \leq 2d \tilde{\alpha} \|\gamma\|,$$

we use (III.5.7) and (III.5.8) to get

$$\begin{aligned} \frac{\tilde{Z}_{m^c, \text{int } \gamma}(\beta, \lambda)}{\tilde{Z}_{m, \text{int } \gamma}(\beta, \lambda)} &\leq \max \left\{ \sup_{\substack{\beta: \tilde{a}_{m^c}(\beta)=0, \\ \tilde{a}_m(\beta) \text{diam } \gamma \leq \tilde{\alpha}}} e^{2\tilde{\alpha} \|\gamma\| + (\tilde{s}_m - \tilde{s}_{m^c} - \tilde{a}_m/d) |\partial \mathbb{B}(\text{int } \gamma)|}, \right. \\ &\quad \left. \sup_{\beta: \tilde{a}_{m^c}(\beta) > 0} e^{\max\{\tilde{s}_m - \tilde{s}_{m^c} + \varepsilon \tilde{a}_{m^c}, 0\} |\partial \mathbb{B}(\text{int } \gamma)|} \right\} e^{O(q^{-\tilde{\nu}}) \|\gamma\|} \quad (\text{III.5.11}) \end{aligned}$$

similarly to (III.B.23). In view of Lemma III.A.2, Lemma III.A.5 (b),(c), and (III.4.22), we obtain

$$\frac{\tilde{Z}_{m^c, \text{int } \gamma}(\beta, \lambda)}{\tilde{Z}_{m, \text{int } \gamma}(\beta, \lambda)} \leq e^{(2\tilde{\alpha} \pm \frac{1}{d}(\lambda - \frac{1}{2}) \log q + O(q^{-\tilde{\nu}})) \|\gamma\|}, \quad (\text{III.5.12})$$

where the sign '+' applies for  $m = d$ , while the sign '-' for  $m = o$ . Combined with (III.3.27), Lemma III.A.2, (III.3.5), and the definition of  $\tilde{\nu}$  and  $\tilde{\alpha}$ , it thus follows that

$$\tilde{K}_m(\gamma) \leq q^{\left(\frac{2\tilde{\alpha}+1}{\log q} - \frac{\mu}{2d}\right) \|\gamma\|} = q^{(2\tilde{\nu} - \frac{\mu}{2d}) \|\gamma\|} \leq q^{-2\tilde{\nu} \|\gamma\|}. \quad (\text{III.5.13})$$

Combined with (III.5.10) and Lemma III.A.3, the lemma is proved. Q.E.D.

We may now verify Theorem III.2.1. Let us do so for its part (a) only. By virtue of Lemma III.5.5, the function  $\log \tilde{Z}_{d,V}(\beta, \lambda)$  and its derivatives can be analyzed by convergent cluster expansions for  $q, L$  large and any  $0 \leq \lambda < \frac{1}{2}$  whenever  $\tilde{a}_d(\beta) \text{diam } V \leq \tilde{\alpha}$  and  $\beta \geq 1$ . Taking an arbitrary  $k \in \mathbb{N}$ , we then have

$$\frac{\partial^k}{\partial \beta^k} \log \tilde{Z}_{d,V}(\beta, \lambda) = -\frac{d^k \tilde{F}_d(\mathbb{B})}{d\beta^k} + O(q^{-\tilde{\nu}L}).$$

Combining this with (III.3.25) and (III.2.8), we get

$$\frac{d^{k-1}}{d\beta^{k-1}} E_L(\beta, \lambda) = \frac{d\tilde{f}_d}{d\beta} L^d + O(L^{d-1})$$

due to Lemma III.A.1, (III.5.6), and (III.4.23). Since  $\beta \leq \beta_t$  iff  $\tilde{a}_d(\beta) = 0$ , the relation (III.2.14) follows in view of (III.5.4) and the fact that  $\tilde{f}$  is the free energy of our model.

### III.A. Auxiliary Lemmas

LEMMA III.A.1. *For any  $k \in \mathbb{N}$ , there exists a finite constant  $\mathcal{D}_k > 0$  such that the  $k$ -th derivative (with respect to  $\beta$ ) of any of the functions  $g_o$ ,  $h_o$ , and  $h_o^m$ ,  $m = o, d$ , defined by (III.3.11) and (III.3.22), respectively, can be uniformly bounded by  $\mathcal{D}_k$  on the interval  $[1, \infty)$ .*

PROOF: Consider the function  $f(x) = -\log(e^{ax} - 1)$  with  $a > 0$  is arbitrary. Obviously,

$$\frac{df}{dx} = -\frac{a}{1 - e^{-ax}}$$

can be uniformly bounded on  $[1, \infty)$ . Using the identity

$$\frac{d^2f}{dx^2} = a \frac{df}{dx} + \left(\frac{df}{dx}\right)^2,$$

one shows by induction that, for any  $k = 1, 2, \dots$ , there exist positive constants  $C_{k1}, \dots, C_{kk}$  such that

$$\frac{d^k f}{dx^k} = \sum_{i=1}^k C_{ki} a^{k-i} \left(\frac{df}{dx}\right)^i.$$

Thus, any derivative of  $f$  can be uniformly bounded on  $[1, \infty)$ . Taking into account the definitions (III.3.11) and (III.3.22), the lemma is proved. Q.E.D.

LEMMA III.A.2. *Let  $\gamma$  be a contour with  $\text{diam } \gamma < \text{diam } V$ . Then  $|\partial\mathbb{B}(\text{int } \gamma)| \leq \|\gamma\|$ .*

PROOF: If  $\text{diam } \gamma < \text{diam } V$ , then there is a corner  $k$  of  $V$  for which  $\gamma \cap \partial V \subset \partial\mathcal{O}(k)$ . If  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$ , then the lemma is trivial as  $|\partial\mathbb{B}(\text{int } \gamma)| = 0$  in this case. Therefore, assume that  $\text{dist}(\gamma, \partial V) \leq \frac{1}{4}$ . Let  $b \in \partial\mathbb{B}(\text{int } \gamma)$ , let  $p(b) \subset \mathbb{R}^d$  be the line which passes through the end-points of  $b$ , and let

$$\bar{\mathbb{B}}(b) = \{\langle x, y \rangle \in \bar{\mathbb{B}} : x \in p(b), y \in p(b)\} \setminus \{b\}.$$

Necessarily, the line  $p(b)$  intersects the boundary of  $V(\gamma)$  at least twice. Once at a point which lies on the bond  $b$  (considered here as a closed unit line segment). All the other intersections are with a contour  $\gamma$  itself and cannot occur neither at points which would lie on  $\partial V$  (otherwise  $\text{diam } \gamma$  would be equal to  $\text{diam } V$  if  $\gamma$  is a  $w$ -contour, while  $V(\gamma) \cap \partial V = \emptyset$  for any  $s$ -contour), nor at the end-points of bonds of  $\bar{\mathbb{B}}(b)$  (contours are defined not to pass through any site of  $\mathbb{Z}^d$ ). In other words, to every  $b \in \partial\mathbb{B}(\text{int } \gamma)$ , there is at least

one bond of  $\bar{\mathbb{B}}(b)$  which intersects  $\gamma$ . Observing that  $\bar{\mathbb{B}}(b) \neq \bar{\mathbb{B}}(b')$  for any two different bonds  $b, b'$  of  $\partial\mathbb{B}$ , the lemma is proved. Q.E.D.

LEMMA III.A.3. *Let  $\gamma$  be an arbitrary contour. Then  $|\bar{\mathbb{B}}(\text{int } \gamma)| \leq \|\gamma\|^d$ .*

PROOF: Let  $i_\gamma = \min\{n \in \mathbb{N} : n \leq \text{diam } \gamma\}$ . Recalling the definition of the set  $\bar{\mathbb{B}}(\text{int } \gamma)$ , we may bound its cardinality by the number of the centres of the bonds of  $\bar{\mathbb{B}}$  which are contained in the closed  $d$ -dimensional box centred at the origin and whose side length is  $i_\gamma$ . This number clearly equals to  $i_\gamma^d$ , i.e. we have

$$|\bar{\mathbb{B}}(\text{int } \gamma)| \leq i_\gamma^d.$$

Now, it suffices to observe that  $i_\gamma \leq \|\gamma\|$ . Q.E.D.

LEMMA III.A.4. *For any contour  $\gamma$ , let  $\mathcal{K}(\gamma) \geq 0$  be an arbitrary contour activity. Let us define*

$$\mathcal{Z}(W) = \sum_{\partial^* \sqsubset W} \prod_{\gamma \in \partial^*} (\mathcal{K}(\gamma) e^{\|\gamma\|})$$

for any  $W$  of the form (III.3.8), where the sum is over all families  $\partial^*$  of non-intersecting contours with  $V(\gamma) \subset W$  for every  $\gamma \in \partial^*$ . In addition, let

$$\phi = - \lim_{L \rightarrow \infty} \frac{1}{L^d} \log \mathcal{Z}(V(L)), \quad (\text{III.A.1})$$

$$\sigma = - \lim_{L \rightarrow \infty} \frac{1}{2dL^{d-1}} (\log \mathcal{Z}(V(L)) + \phi |\mathbb{B}|/d). \quad (\text{III.A.2})$$

If  $\mathcal{K}(\gamma) \leq \epsilon^{2\|\gamma\|}$  with  $\epsilon > 0$  sufficiently small, then, for any  $c_1 \geq -\phi$  and  $c_2 \geq -\sigma$ ,

$$\sum_{\partial_{\text{ext}} \sqsubset W} e^{-c_1 |\mathbb{B}(\text{Ext})|/d - c_2 |\partial\mathbb{B}(\text{Ext})|} \prod_{\gamma \in \partial_{\text{ext}}} \mathcal{K}(\gamma) \leq e^{O(\epsilon) \|\partial W\| + O(\epsilon^L)},$$

where the sum goes over sets  $\partial_{\text{ext}}$  of contours which are all external with  $V(\gamma) \subset W$  for any  $\gamma \in \partial_{\text{ext}}$  and  $\text{Ext} = W \setminus \cup_{\gamma \in \partial_{\text{ext}}} V(\gamma)$ .

PROOF: If  $\mathcal{K}(\gamma) \leq \epsilon^{2\|\gamma\|}$ , then  $\mathcal{K}(\gamma) e^{\|\gamma\|} \leq \epsilon^{\frac{3}{2}\|\gamma\|}$  for  $\epsilon$  small enough. Hence, the partition function  $\mathcal{Z}$  can be controlled by a convergent cluster expansion, yielding

$$\log \mathcal{Z}(W) = -\Phi(\bar{\mathbb{B}}(W)) + O(\epsilon) \|\partial W\| + O(\epsilon^L) \quad (\text{III.A.3})$$

with  $\Phi(\mathbb{B}(W)) = \phi|\mathbb{B}(W)|/d + \sigma|\partial\mathbb{B}(W)|$ . Assuming now that  $c_1 \geq -\phi$  and  $c_2 \geq -\sigma$  and using (III.A.3), we get

$$\begin{aligned} \sum_{\partial_{\text{ext}}\square W} e^{-c_1|\mathbb{B}(\text{Ext})|/d - c_2|\partial\mathbb{B}(\text{Ext})|} \prod_{\gamma \in \partial_{\text{ext}}} \mathcal{K}(\gamma) &\leq \\ &\leq e^{\Phi(\mathbb{B}(W))} \sum_{\partial_{\text{ext}}\square W} \prod_{\gamma \in \partial_{\text{ext}}} \left( \mathcal{K}(\gamma) e^{-\Phi(\mathbb{B}(\text{int } \gamma))} \right) = \\ &= e^{\Phi(\mathbb{B}(W))} \sum_{\partial_{\text{ext}}\square W} \prod_{\gamma \in \partial_{\text{ext}}} \left( \mathcal{K}(\gamma) \mathcal{Z}(\text{int } \gamma) e^{O(\epsilon)\|\gamma\|} \right) \leq \\ &\leq e^{\Phi(\mathbb{B}(W))} \mathcal{Z}(W) \leq e^{O(\epsilon)\|\partial W\| + O(\epsilon^L)}. \end{aligned}$$

Q.E.D.

LEMMA III.A.5. Let  $\mathfrak{m} = \mathfrak{o}, \mathfrak{d}$ , while  $\mathfrak{m}^c = \mathfrak{o}$  if  $\mathfrak{m} = \mathfrak{d}$  and vice versa. Let  $\kappa > 0$  be arbitrary and let  $g_{\mathfrak{m}}, h_{\mathfrak{m}}$  be the quantities defined in (III.3.11). Let  $\varphi_{\mathfrak{m}}(\beta)$  and  $\sigma_{\mathfrak{m}}(\beta)$  be arbitrary functions which

- (i) are continuous, and  $\varphi_{\mathfrak{m}} = g_{\mathfrak{m}} + O(q^{-\omega})$ ,  $\sigma_{\mathfrak{m}} = h_{\mathfrak{m}} + O(q^{-\kappa})$ ;
- (ii) are differentiable on  $[1, \infty)$ , and

$$\frac{d\varphi_{\mathfrak{m}}}{d\beta} = \frac{dg_{\mathfrak{m}}}{d\beta} + O(q^{-\kappa}), \quad \frac{d\sigma_{\mathfrak{m}}}{d\beta} = \frac{dh_{\mathfrak{m}}}{d\beta} + O(q^{-\kappa}). \quad (\text{III.A.4})$$

For all  $q$  sufficiently large and  $\lambda \in (0, J)$ ,  $J > 0$ , we have:

- (a) There exists a unique point  $\hat{\beta}$  such that  $\varphi_{\mathfrak{o}}(\hat{\beta}) = \varphi_{\mathfrak{d}}(\hat{\beta})$ . Moreover,

$$\varphi(\beta) = \min\{\varphi_{\mathfrak{o}}(\beta), \varphi_{\mathfrak{d}}(\beta)\} = \begin{cases} \varphi_{\mathfrak{o}}(\beta) & \text{if } \beta \geq \hat{\beta}, \\ \varphi_{\mathfrak{d}}(\beta) & \text{if } \beta \leq \hat{\beta}, \end{cases} \quad (\text{III.A.5})$$

and

$$\hat{\beta} = \frac{\log q}{dJ} + O(q^{-\kappa}). \quad (\text{III.A.6})$$

- (b) Let  $\mathcal{F}_{\mathfrak{m}} = \sigma_{\mathfrak{m}} - \sigma_{\mathfrak{m}^c} + \frac{\xi}{d}(\varphi_{\mathfrak{m}^c} - \varphi)$ . If  $\xi \leq \left(\frac{\lambda}{J}\right)^2$ , then

$$\sup_{\beta: \varphi(\beta) = \varphi_{\mathfrak{m}}(\beta)} \mathcal{F}_{\mathfrak{m}}(\beta) = \mathcal{F}_{\mathfrak{m}}(\hat{\beta}).$$

- (c) Let  $\mathcal{G}_{\mathfrak{m}} = \sigma_{\mathfrak{m}} - \sigma_{\mathfrak{m}^c} - \frac{1}{d}(\varphi_{\mathfrak{m}} - \varphi)$ . Then

$$\sup_{\beta > 0} \mathcal{G}_{\mathfrak{m}}(\beta) = \mathcal{G}_{\mathfrak{m}}(\hat{\beta}).$$

PROOF: Throughout the proof, we assume that  $q$  is large.

- (a) The function  $\varphi_{\mathfrak{o}} - \varphi_{\mathfrak{d}}$  is decreasing on  $[1, \infty)$  since

$$\frac{d}{d\beta}(\varphi_{\mathfrak{o}} - \varphi_{\mathfrak{d}}) = -\frac{dJ}{1 - e^{-J\beta}} + O(q^{-\kappa}) \leq -dJ + O(q^{-\kappa}) < 0 \quad (\text{III.A.7})$$

by (III.3.11) and (III.A.4). In addition,

$$(\varphi_o - \varphi_d)(\beta) \geq \log q - d \log(e^J - 1) + O(q^{-\kappa}) > 0$$

for all  $\beta \in (0, 1]$  and  $\lim_{\beta \rightarrow \infty} (\varphi_o - \varphi_d)(\beta) = -\infty$ . Using that  $\varphi_o - \varphi_d$  is continuous by assumption, we get the existence of a unique point  $\hat{\beta}$  for which  $\varphi_o - \varphi_d \geq 0$  if  $\beta \leq \hat{\beta}$  and  $\varphi_o - \varphi_d \leq 0$  if  $\beta \geq \hat{\beta}$ . Finally, the relation (III.A.6) is an immediate consequence of (III.A.4).

(b) Let  $\xi \leq (\frac{\lambda}{J})^2$ . We start with showing that  $\ell_1 = h_o - \frac{\xi}{d} g_o$  is a decreasing function of  $\beta$ . Indeed,

$$\frac{d\ell_1}{d\beta} = \frac{\xi J}{1 - e^{-J\beta}} - \frac{\lambda}{1 - e^{-\lambda\beta}}$$

in view of (III.3.11). Using that  $\pi_c(x) = \frac{e^{-cx}}{(1 - e^{-cx})^2}$  is a decreasing function of  $c > 0$  for all  $x > 0$ , we get

$$\frac{d^2\ell_1}{d\beta^2} = -\xi J^2 \pi_J(\beta) + \lambda^2 \pi_\lambda(\beta) > (\lambda^2 - \xi J^2) \pi_J(\beta) \geq 0.$$

Thus,

$$\frac{d\ell_1}{d\beta} \leq \frac{d\ell_1}{d\beta} \Big|_{\beta=\infty} = \xi J - \lambda \leq -\lambda \left(1 - \frac{\lambda}{J}\right) < 0$$

for all  $\lambda \in (0, J)$  as was claimed.

Now, by virtue of the assumption (i), we have

$$\begin{aligned} \mathcal{F}_d(2) - \mathcal{F}_d(\beta) &> \ell_1(\beta) - \ell_1(2) + O(q^{-\kappa}) \geq \\ &\geq \ell_1(1) - \ell_1(2) + O(q^{-\kappa}) > 0 \quad (\text{III.A.8}) \end{aligned}$$

for all  $\beta \in (0, 1]$  since  $\ell_1(1) - \ell_1(2)$  is, according to the monotonicity of  $\ell_1$ , a positive number which, in addition, does not depend on  $q$ . Next, with the help of the assumption (ii), (III.A.5), and (III.A.7), we observe that  $\mathcal{F}_m$ ,  $m = o, d$ , is differentiable in  $\beta$  on  $[1, \infty) \setminus \{\hat{\beta}\}$ . The Lagrange mean-value theorem thus yields

$$\begin{aligned} \mathcal{F}_d(\hat{\beta}) - \mathcal{F}_d(\beta) &= (\hat{\beta} - \beta) \frac{d\mathcal{F}_d}{d\beta} \Big|_{\beta_1} = \\ &= (\hat{\beta} - \beta) \left( -\frac{d\ell_1}{d\beta} \Big|_{\beta_1} + O(q^{-\kappa}) \right) \geq \frac{\lambda}{2} \left(1 - \frac{\lambda}{J}\right) (\hat{\beta} - \beta) > 0 \end{aligned}$$

for any  $\beta \in [1, \hat{\beta})$  and some  $\beta_1 \in (\beta, \hat{\beta})$ . Similarly,

$$\begin{aligned} \mathcal{F}_o(\hat{\beta}) - \mathcal{F}_o(\beta) &\geq (\hat{\beta} - \beta) \left( \frac{d\ell_1}{d\beta} \Big|_{\beta_2} + O(q^{-\kappa}) \right) \geq \\ &\geq -\frac{\lambda}{2} \left(1 - \frac{\lambda}{J}\right) (\hat{\beta} - \beta) > 0 \quad (\text{III.A.9}) \end{aligned}$$

for any  $\beta \in (\hat{\beta}, \infty)$  and some  $\beta_2 \in (\hat{\beta}, \beta)$ . The last three bounds justify the statement (b).

(c) Let us only prove the statement for  $m = o$ . First, we prove that  $\ell_2 = h_o - \frac{1}{a}g_o$  is an increasing function of  $\beta$ . To this end, one observes that  $\eta_c(x) = \frac{c e^{cx}}{e^{cx}-1}$  is an increasing function of  $c > 0$  for all  $x > 0$ , which, in view of (III.3.11), implies  $\frac{d\ell_2}{d\beta} = \eta_J(\beta) - \eta_\lambda(\beta) > 0$ . Next, using (III.A.6), we have

$$e^{a\hat{\beta}} - 1 = q^{\frac{a}{dJ}}(1 + O(\delta)), \quad \delta = \max\{q^{-\kappa}, q^{-\frac{a}{dJ}}\} \quad (\text{III.A.10})$$

for any  $0 < a \leq J$ . Combined with the assumption (i), (III.A.5), and (III.3.11), we get

$$\begin{aligned} \mathcal{G}_o(\hat{\beta}) - \mathcal{G}_o(\beta) &= \ell_2(\hat{\beta}) - \ell_2(\beta) + O(q^{-\kappa}) \geq \\ &\geq \ell_2(\hat{\beta}) - \ell_2\left(\frac{\hat{\beta}}{2}\right) + O(q^{-\kappa}) = \log\left(\frac{e^{\frac{\lambda\hat{\beta}}{2}} - 1}{e^{\lambda\hat{\beta}} - 1} \frac{e^{J\hat{\beta}} - 1}{e^{\frac{J\hat{\beta}}{2}} - 1}\right) + O(q^{-\kappa}) = \\ &= \frac{J - \lambda}{2dJ} \log q + O(\delta) > 0 \end{aligned}$$

for all  $\beta \in (0, \frac{\hat{\beta}}{2}]$ . Analogously, using also the assumption (ii),

$$\begin{aligned} \left.\frac{d\mathcal{G}_o}{d\beta}\right|_{a\hat{\beta}} &= \left.\frac{d\ell_2}{d\beta}\right|_{a\hat{\beta}} + O(q^{-\kappa}) = \eta_J(a\hat{\beta}) - \eta_\lambda(a\hat{\beta}) + O(q^{-\kappa}) = \\ &= J - \lambda + O(\delta) > 0 \end{aligned}$$

for all  $a \in [\frac{1}{2}, 1)$ , whereas

$$\frac{d\mathcal{G}_o}{d\beta} = \frac{dh_o}{d\beta} + O(q^{-\kappa}) = -\frac{\lambda}{1 - e^{-\lambda\beta}} + O(q^{-\kappa}) \leq -\lambda + O(q^{-\kappa}) < 0$$

for all  $\beta > \hat{\beta}$ . As a result,  $\mathcal{G}_o(\hat{\beta}) \geq \mathcal{G}_o(\beta)$  for any  $\beta \geq \frac{\hat{\beta}}{2}$  by the Lagrange mean-value theorem (see above). Q.E.D.

For any non-empty admissible set  $\partial$  of  $w$ -contours, let us consider the connected components  $\mathcal{C}_1, \dots, \mathcal{C}_n$  of  $V \setminus \partial$ . Observing that  $\bar{\mathbb{B}}(\mathcal{C}_i) \subset \Omega_o(V, \partial)$  or  $\bar{\mathbb{B}}(\mathcal{C}_i) \subset \Omega_d(V, \partial)$  for every  $1 \leq i \leq n$ , we define  $W_o(\partial)$  as the union of all of the former components and  $W_d(\partial)$  as the union of the latter ones.

LEMMA III.A.6. *Let  $\partial \neq \emptyset$  be an admissible set of  $w$ -contours and let  $W_o(\partial), W_d(\partial)$  be defined as above.*

(a) *For any  $m = o, d$ , we have the bound*

$$\sum_{\gamma \in \partial} \|\gamma\| \geq \left| \frac{2|\mathbb{B}(W_m(\partial))|}{L-1} - |\partial\mathbb{B}(W_m(\partial))| \right|. \quad (\text{III.A.11})$$

(b) *Let  $m = o, d$ . If  $\Theta$  is an admissible set of  $w$ -contours which are all external and  $V(\gamma) \subset W_m(\partial)$  for every  $\gamma \in \Theta$ , then*

$$\sum_{\gamma \in \partial} \|\gamma\| + \sum_{\hat{\gamma} \in \Theta} \|\hat{\gamma}\| \leq c_m |\bar{\mathbb{B}}(\text{Ext}_m)|.$$

Here  $\text{Ext}_m = W_m(\partial) \setminus \cup_{\gamma \in \Theta} V(\gamma)$  and  $c_o = 2(2d - 1)$ ,  $c_d = 2$ .

PROOF: (a) Let  $m = o$  or  $m = d$ . Given  $\mathcal{C}_i \in W_m(\partial)$ , let  $\partial\mathcal{C}_i$  be the boundary of  $\mathcal{C}_i$  and  $\partial_i = \{\gamma \in \partial : \gamma \subset \partial\mathcal{C}_i \setminus \partial V\}$ . One obviously has

$$\partial_i \cap \partial_j = \emptyset \text{ for all } 1 \leq i < j \leq n, \quad \partial = \bigcup_{i: \mathcal{C}_i \in W_m(\partial)} \partial_i.$$

In addition, let  $\tilde{\mathcal{C}}_i = \{x \in \mathcal{C}_i : \text{dist}(x, \partial V) > \frac{1}{4}\}$  and let  $\|\Gamma_i\|$  be the number of the intersections of the boundary of  $\tilde{\mathcal{C}}_i$  with the bonds of  $\mathbb{B}$ . Clearly,

$$\|\Gamma_i\| = \sum_{\gamma \in \partial_i} \|\gamma\| + |\partial\mathbb{B}(\mathcal{C}_i)|, \quad \sum_{i: \mathcal{C}_i \in W_m(\partial)} \|\Gamma_i\| = \sum_{\gamma \in \partial} \|\gamma\| + |\partial\mathbb{B}(W_m(\partial))|.$$

Finally, let  $\pi$  be the set of all the lines in  $\mathbb{R}^d$  each of which passes through the end-points of some  $b \in \partial\mathbb{B}$ . Then  $|\mathbb{B}(\mathcal{C}_i)| \leq |\pi_i \cap \mathbb{B}|$ , where  $\pi_i = \{p \in \pi : p \cap \tilde{\mathcal{C}}_i \neq \emptyset\}$ . Moreover, for any line  $p \in \pi_i$ , the set  $p \cap \mathbb{B}$  contains either at least one bond of  $\mathbb{B}$  such that the boundary of  $\tilde{\mathcal{C}}_i$  intersects it twice or at least two bonds of  $\mathbb{B}$  such that the boundary of  $\tilde{\mathcal{C}}_i$  intersects each of them once. In any case, there are at least two intersections of this boundary with  $\mathbb{B}$  corresponding to every line  $p \in \pi_i$ , i.e.  $2|\pi_i| \leq \|\Gamma_i\|$ . Since  $|p \cap \mathbb{B}| = L - 1$  for any  $p \in \pi$ , we get

$$|\mathbb{B}(\mathcal{C}_i)| \leq |\pi_i \cap \mathbb{B}| = \sum_{p \in \pi_i} |p \cap \mathbb{B}| = (L - 1)|\pi_i| \leq (L - 1) \frac{\|\Gamma_i\|}{2}.$$

Consequently,

$$\begin{aligned} \sum_{\gamma \in \partial} \|\gamma\| + |\partial\mathbb{B}(W_m(\partial))| &= \sum_{i: \mathcal{C}_i \in W_m(\partial)} \|\Gamma_i\| \geq \\ &\geq \frac{2}{L - 1} \sum_{i: \mathcal{C}_i \in W_m(\partial)} |\mathbb{B}(\mathcal{C}_i)| = \frac{2|\mathbb{B}(W_m(\partial))|}{L - 1} \end{aligned}$$

and the lemma holds as soon as  $\frac{2|\mathbb{B}(W_m(\partial))|}{L - 1} - |\partial\mathbb{B}(W_m(\partial))| \geq 0$ . However,

$$\begin{aligned} \frac{2|\mathbb{B}(W_o(\partial))|}{L - 1} - |\partial\mathbb{B}(W_o(\partial))| + \frac{2|\mathbb{B}(W_d(\partial))|}{L - 1} - |\partial\mathbb{B}(W_d(\partial))| &= \\ &= \frac{2|\mathbb{B}|}{L - 1} - |\partial\mathbb{B}| = 0 \end{aligned}$$

due to (III.4.23). Thus, the absolute value in (III.A.11) is the same for both  $m = o$  and  $m = d$  and non-negative for one of them.

(b) Let  $m = o, d$  be fixed. If a contour  $\gamma_0$  of  $\partial$  or  $\Theta$  intersects a bond  $b \in \mathbb{B}$ , then either one of the end-points of  $b$  lies in  $\Lambda \cap \text{Ext}_m$  (when  $\gamma_0$  is ordered) or the centre of  $b$  has to lie in  $\text{Ext}_m$ , i.e.  $b \in \mathbb{B}(\text{Ext}_m)$  (when  $\gamma_0$  is disordered).

Let us take, in the former case, any  $x \in \Lambda \cap \text{Ext}_m$  and consider all the  $2d$  bonds of  $\mathbb{B}$  whose one end-point is  $x$ . Necessarily, at most  $2d - 1$  of them are once intersected by some contour of  $\partial$  or  $\Theta$ , while at least one of them is in  $\mathbb{B}(\text{Ext}_m)$ . Since every bond of  $\mathbb{B}(\text{Ext}_m)$  has at most two end-points from  $\Lambda \cap \text{Ext}_m$ , we get

$$\sum_{\gamma \in \partial} \|\gamma\| + \sum_{\hat{\gamma} \in \Theta} \|\hat{\gamma}\| \leq (2d - 1)|\Lambda \cap \text{Ext}_m|, \quad |\Lambda \cap \text{Ext}_m| \leq 2|\mathbb{B}(\text{Ext}_m)|.$$

In the latter case, one just observes that any  $b \in \mathbb{B}$  can be intersected at most twice. Q.E.D.

LEMMA III.A.7. Let  $\psi_r : \mathbb{R} \rightarrow \mathbb{R}$ ,  $r = 1, 2$ , be two  $C^\infty$  functions. Then, for any  $k \in \mathbb{N}$ ,

$$\frac{d^k \psi_1(\psi_2(x))}{dx^k} = \sum_{j=1}^k \frac{d^j \psi_1(y)}{dy^j} \Big|_{y=\psi_2(x)} \sum_{\{I_1, \dots, I_j\}} \prod_{i=1}^j \frac{d^{|I_i|} \psi_2(x)}{dx^{|I_i|}},$$

where  $\{I_1, \dots, I_j\}$ ,  $j = 1, \dots, k$ , is a set of non-empty sub-sequences which partition  $\{1, \dots, k\}$  and  $|I_i|$ ,  $i = 1, \dots, j$ , is the cardinality of  $I_i$ .

PROOF: By induction on  $k \in \mathbb{N}$ . Q.E.D.

LEMMA III.A.8. Let  $x_1, x_2$  be two real numbers. For any  $k = 0, 1, \dots$ , there is a constant  $\theta_k > 0$  such that

$$\begin{aligned} \left| \left( \frac{d^k}{dx^k} \tanh x \right)_{x_1} - \left( \frac{d^k}{dx^k} \tanh x \right)_{x_2} \right| &\leq \\ &\leq \theta_k \min \left\{ \frac{\tanh x_1}{x_1}, \frac{\tanh x_2}{x_2} \right\} |x_1 - x_2|. \end{aligned} \quad (\text{III.A.12})$$

PROOF: Let  $x_1, x_2 \in \mathbb{R}$  be given. Without loss of generality, we may suppose that  $x_1 > x_2$ . Then  $\tanh x_1 > \tanh x_2$  and  $\frac{\tanh x_1}{x_1} < \frac{\tanh x_2}{x_2}$ . Thus,

$$\begin{aligned} |\tanh x_1 - \tanh x_2| \frac{x_1}{\tanh x_1} &= \left( 1 - \frac{\tanh x_2}{\tanh x_1} \right) |x_1| < \\ &< \left| 1 - \frac{x_2}{x_1} \right| |x_1| = |x_1 - x_2|, \end{aligned}$$

which verifies the lemma for  $k = 0$  with  $\theta_0 = 1$ .

Let  $k \geq 1$  be fixed now. It is easy to show by induction that there exist constants  $\Xi_{k1}, \dots, \Xi_{kk}$  such that

$$\frac{d^k}{dx^k} \tanh x = \left( \frac{d}{dx} \tanh x \right) \sum_{j=0}^k \Xi_{kj} \tanh^j x \quad (\text{III.A.13})$$

for any  $x \in \mathbb{R}$ . Using that  $|\tanh x| \leq 1$  for real  $x \in \mathbb{R}$ , we get

$$\begin{aligned} \left| \left( \frac{d^k}{dx^k} \tanh x \right)_{x_1} - \left( \frac{d^k}{dx^k} \tanh x \right)_{x_2} \right| &= \left| \int_{x_1}^{x_2} \frac{d^{k+1} \tanh x}{dx^{k+1}} dx \right| \leq \\ &\leq \left| \sum_{j=0}^{k+1} \Xi_{k+1,j} \right| \left| \int_{x_1}^{x_2} \frac{d \tanh x}{dx} dx \right| = \theta_k |\tanh x_2 - \tanh x_1|. \end{aligned}$$

Q.E.D.

### III.B. Proof of Lemma III.4.1

Let us define the *truncated activities*  $K_m(\gamma)$ ,  $m = o, d$ , for any short  $w$ -contour  $\gamma$  in the following inductive manner.

Let  $m = o, d$  and an arbitrary  $n \in \mathbb{N}$  be given. Assuming that  $K_m(\gamma)$  has already been defined for any short  $w$ -contour  $\gamma$  for which  $|\bar{\mathbb{B}}(\text{int } \gamma)| < n$ , we introduce an auxiliary contour model with the activities<sup>10</sup>

$$K_m^{(n-1)}(\gamma) = \begin{cases} K_m(\gamma) & \text{if } \gamma \text{ is short and } |\bar{\mathbb{B}}(\text{int } \gamma)| \leq n-1, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{III.B.1})$$

and define the corresponding partition function

$$Z_{m,W}^{(n-1)}(\beta, \lambda) = e^{-G_m(\bar{\mathbb{B}}(W))} \sum_{\partial^* \sqsubset W}^{(m)} \prod_{\gamma \in \partial^*} K_m^{(n-1)}(\gamma), \quad W \subset V, \quad (\text{III.B.2})$$

(here the sum is going over the same collections  $\partial^*$  of contours as in (III.3.14)) and

$$f_m^{(n-1)}(\beta) = - \lim_{L \rightarrow \infty} \frac{1}{L^d} \log Z_{m,V(L)}^{(n-1)}(\beta, \lambda), \quad (\text{III.B.3})$$

$$s_m^{(n-1)}(\beta) = - \lim_{L \rightarrow \infty} \frac{1}{2dL^{d-1}} (\log Z_{m,V(L)}^{(n-1)}(\beta, \lambda) + f_m^{(n-1)}(\beta) |\bar{\mathbb{B}}|/d). \quad (\text{III.B.4})$$

Now, let  $\tilde{\gamma}$  be a short  $w$ -contour with  $|\bar{\mathbb{B}}(\text{int } \tilde{\gamma})| = n$ . Introducing a smoothed version of the characteristic function  $\chi : \mathbb{R} \rightarrow [0, 1]$  as a  $C^\infty$  function such that

- (a)  $0 \leq \chi \leq 1$ ,
- (b)  $\chi(x) = 0$  if  $x \leq -1$ , while  $\chi(x) = 1$  if  $x \geq 1$ ,
- (c) for every  $k \in \mathbb{N}$ , there exists a positive constant  $\bar{C}_k$  such that

$$\left| \frac{d^k}{dx^k} \chi(x) \right| \leq \bar{C}_0(k), \quad (\text{III.B.5})$$

<sup>10</sup>Note that  $K_m^{(0)}(\gamma) = 0$  because there are no contours with  $|\bar{\mathbb{B}}(\text{int } \gamma)| = 0$ .

we put

$$K_m(\tilde{\gamma}) = \chi'_m(\tilde{\gamma}) \rho(\tilde{\gamma}) \frac{Z_{m^c, \text{int } \tilde{\gamma}}(\beta, \lambda)}{Z_{m, \text{int } \tilde{\gamma}}^{(n-1)}(\beta, \lambda)}, \quad (\text{III.B.6})$$

where  $m^c = \text{o}$  if  $m = \text{d}$  and vice versa and

$$\chi'_m(\tilde{\gamma}) = \chi\left((4\alpha + 2)\|\tilde{\gamma}\| - (f_m^{(n-1)} - f_{m^c}^{(n-1)})|\bar{\mathbb{B}}(\text{int } \tilde{\gamma})|/d\right). \quad (\text{III.B.7})$$

The constant  $\alpha$  was defined in (III.4.6). If

$$K_m^{(n-1)}(\gamma) \leq \epsilon^{\|\gamma\|} \quad (\text{III.B.8})$$

for some  $\epsilon > 0$  small, then  $\log Z_{m, \text{int } \tilde{\gamma}}^{(n-1)}(\beta, \lambda)$  can be controlled by a convergent cluster expansion. As a result, one is then able to establish the bound  $K_m(\tilde{\gamma}) \leq \epsilon^{\|\tilde{\gamma}\|}$ . We shall prove this as a part of the next lemma. Before stating it, let us define

$$f^{(n)}(\beta) = \min\{f_{\text{o}}^{(n)}(\beta), f_{\text{d}}^{(n)}(\beta)\}, \quad a_m^{(n)}(\beta) = f_m^{(n)}(\beta) - f^{(n)}(\beta) \quad (\text{III.B.9})$$

for every  $n = 0, 1, \dots$  and

$$v(W) = \max_{\substack{\gamma: V(\gamma) \subset W, \\ \gamma \text{ short } w\text{-contour}}} |\bar{\mathbb{B}}(\text{int } \gamma)| \quad (\text{III.B.10})$$

for every  $W \subset V$  of the form (III.3.8).

**LEMMA III.B.1.** *Let  $d \geq 2$ ,  $J > 0$ ,  $0 \leq \mu < 1$ , and  $k_0 = 0, 1, \dots$ . Let us define  $\nu$  and  $\alpha$  by (III.4.6). There exist finite constants  $D_1, D_2 \geq 1$  such that, for any function  $\omega : \mathbb{N} \rightarrow [0, \infty]$  for which  $\omega(L) \leq L$  and  $\omega(L) \rightarrow \infty$  as  $L \rightarrow \infty$ , a truncated activity  $K_m(\gamma)$  exists for any  $m$ -labelled short  $w$ -contour  $\gamma$ ,  $m = \text{o}, \text{d}$ , such that the following claims hold whenever  $q, L$  is large enough,  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ ,  $0 \leq k \leq k_0$  and  $\beta \geq 1$  or  $k = 0$  and  $\beta > 0$ , any  $n = 0, 1, \dots$ ,  $|\bar{\mathbb{B}}(\text{int } \gamma)| \leq n$  and  $v(W) \leq n$ , and  $m = \text{o}, \text{d}$ .*

(a)  $K_m(\gamma)$  is a  $C^{k_0}$ -function of  $\beta$  and  $|\frac{\partial^k}{\partial \beta^k} K_m(\gamma)| \leq (D_1 q^{-2\nu})^{\|\gamma\|}$ .

(b) If  $a_m^{(n)}(\beta) \text{diam } \gamma \leq \alpha$ , then  $\chi'_m(\gamma) = 1$  and  $K_m(\gamma) = K_m(\gamma)$ .

(c) Whenever  $a_m^{(n)}(\beta) \min\{\text{diam } W, \omega(L)\} \leq \alpha$ , then  $Z_{m, W}^{(n)}(\beta, \lambda) = Z_{m, W}(\beta, \lambda)$ .

(d) Let  $F_m^{(n)}(\bar{\mathbb{B}}(W)) = f_m^{(n)} |\bar{\mathbb{B}}(W)|/d + s_m^{(n)} |\partial \bar{\mathbb{B}}(W)|$ . If  $a_m^{(n)}(\beta) > 0$ ,

then, for any  $W$  of the form (III.3.8), we have

$$\begin{aligned} \left| \frac{\partial^k}{\partial \beta^k} Z_{m,W}(\beta, \lambda) \right| &\leq D_2 |\bar{\mathbb{B}}(W)|^k Z_{m,W}(\beta, \lambda) \leq \\ &\leq D_2 |\bar{\mathbb{B}}(W)|^k e^{-F_{m^c}^{(n)}(\bar{\mathbb{B}}(W)) + (\frac{m}{2d} \log q + O(q^{-\nu})) |\partial \bar{\mathbb{B}}(W)|} \times \\ &\quad \times e^{O(q^{-\nu}) \|\partial W\| + O(q^{-\nu L})}, \end{aligned} \quad (\text{III.B.11})$$

where  $m^c = o$  if  $m = d$  and vice versa.

**REMARK III.B.2.** As a result of the claim (a), one can analyze the logarithm  $\log Z_{m,W}^{(r)}(\beta, \lambda)$ ,  $r \leq n$ ,  $m = o, d$ , if  $\beta > 0$ , as well as its derivatives up to the  $k_0$ -th order, if  $\beta \geq 1$ , by convergent cluster expansions. This yields

$$\frac{\partial^k}{\partial \beta^k} \log Z_{m,W}^{(r)}(\beta, \lambda) = -\frac{d^k F_m^{(r)}}{d\beta^k} + O(q^{-\nu}) \|\partial W\| + O(q^{-\nu L}) \quad (\text{III.B.12})$$

and

$$\frac{d^k f_m^{(r)}}{d\beta^k} = \frac{d^k g_m}{d\beta^k} + O(q^{-\nu}), \quad \frac{d^k s_m^{(r)}}{d\beta^k} = \frac{d^k h_m}{d\beta^k} + O(q^{-\nu}) \quad (\text{III.B.13})$$

for all  $k = 0, \dots, k_0$  and  $q, L$  large. As a consequence, for any  $r \leq n$  Lemma III.A.5 (a) implies that there exists a unique point  $\bar{\beta}^{(r)}$  such that

$$f^{(r)}(\beta) = \begin{cases} f_o^{(r)}(\beta) & \text{if } \beta \geq \bar{\beta}^{(r)}, \\ f_d^{(r)}(\beta) & \text{if } \beta \leq \bar{\beta}^{(r)}. \end{cases} \quad (\text{III.B.14})$$

Note that applying Lemma III.A.7 to the functions  $\psi_1(x) = e^{\pm x}$  and  $\psi_2(\beta) = \log Z_{m,W}^{(r)}(\beta, \lambda)$  and employing Lemma III.A.1, (III.B.12), and (III.B.13), one easily finds that

$$\left| \frac{\partial^k}{\partial \beta^k} (Z_{m,W}^{(r)}(\beta, \lambda))^{\pm 1} \right| \leq D_3 |\bar{\mathbb{B}}(W)|^k (Z_{m,W}^{(r)}(\beta, \lambda))^{\pm 1}, \quad 1 \leq k \leq k_0, \quad (\text{III.B.15})$$

for some finite constant  $D_3 > 0$  whenever  $\beta \geq 1$ .

**PROOF OF LEMMA III.B.1:** We proceed by induction on  $n \in \mathbb{Z}$ , assuming that  $k_0 = 0, 1, \dots$  is given.

- *Proof of Lemma III.B.1 for  $n = 0$ .* Since there is no  $w$ -contour  $\gamma$  with  $|\bar{\mathbb{B}}(\text{int } \gamma)| = 0$ , there is nothing to prove in the parts (a) and (b) of the lemma.

Next, let  $W \subset V$  be of the form (III.3.8) with  $v(W) = 0$ .<sup>11</sup> Then (III.3.14), (III.B.2), and (III.B.3) directly yield

$$Z_{m,W}(\beta, \lambda) = e^{-G_m(\mathbb{B}(W))} = Z_{m,W}^{(0)}(\beta, \lambda), \quad f_m^{(0)} = g_m, \quad \text{and } s_m^{(0)} = h_m \quad (\text{III.B.16})$$

because  $K_m^{(0)}(\gamma) = 0$  (see the footnote on p. 86). Hence,

$$Z_{m,W}(\beta, \lambda) = e^{-f_m^{(0)}|\mathbb{B}(W)|/d - s_m^{(0)}|\partial\mathbb{B}(W)|} \leq e^{-F_{m^c}^{(0)}(\mathbb{B}(W)) + (s_{m^c}^{(0)} - s_m^{(0)})|\partial\mathbb{B}(W)|} \quad (\text{III.B.17})$$

for all  $\beta > 0$  such that  $a_m^{(0)}(\beta) > 0$ . This proves the parts (c) and (d) for  $k = 0$  because the right-hand side of (III.B.17) is smaller than or equal to the right-hand side of (III.B.11) with  $k = 0$ .

Let  $1 \leq k \leq k_0$  now. Due to Lemma III.A.1, the derivatives of  $g_m$  and  $h_m$  are bounded once  $\beta \geq 1$ . It then follows that

$$\left| \frac{\partial^k}{\partial \beta^k} G_m(\mathbb{B}(W)) \right| \leq \tilde{C} |\mathbb{B}(W)| \quad (\text{III.B.18})$$

for some finite constant  $\tilde{C} > 0$  and all  $\beta \geq 1$ . Combined with (IV.3.18) and (III.B.17), we arrive at (III.B.11) for any  $k \leq k_0$  and  $\beta \geq 1$ . This concludes the proof of the lemma for  $n = 0$ , once we choose  $D_2 > \tilde{C}$ .

Next, we shall prove the lemma for any  $n \geq 1$ , assuming that it has already been proved for all integers smaller than  $n$ .

• *Proof of Lemma III.B.1 (a) for  $n \geq 1$ .* If  $|\mathbb{B}(\text{int } \gamma)| = n$ , then  $v(\text{int } \gamma) \leq n - 1$ . By the inductive assumption (a),  $\log Z_{m,\text{int } \gamma}^{(n-1)}(\beta, \lambda)$  can be controlled by a convergent cluster expansion, and we have (III.B.12) with  $r = n - 1$  and  $k = 0$ . In order to control  $Z_{m^c,\text{int } \gamma}(\beta, \lambda)$ , we may use either the inductive assumption (c) or (d), according to the value of  $\beta$ . This gives

$$\frac{Z_{m^c,\text{int } \gamma}(\beta, \lambda)}{Z_{m,\text{int } \gamma}^{(n-1)}(\beta, \lambda)} = e^{a_m^{(n-1)}|\mathbb{B}(\text{int } \gamma)|/d + (s_m^{(n-1)} - s_{m^c}^{(n-1)})|\partial\mathbb{B}(\text{int } \gamma)| + O(q^{-\nu})\|\gamma\|} \quad (\text{III.B.19})$$

for all  $\beta > 0$  such that  $a_{m^c}^{(n-1)} \text{diam } \gamma \leq \alpha$  (the former case) and, as  $\|\partial\text{int } \gamma\| \leq \|\gamma\|$ ,

$$\frac{Z_{m^c,\text{int } \gamma}(\beta, \lambda)}{Z_{m,\text{int } \gamma}^{(n-1)}(\beta, \lambda)} \leq e^{(\frac{\mu}{2d} \log q + O(q^{-\nu}))|\partial\mathbb{B}(\text{int } \gamma)| + O(q^{-\nu})\|\gamma\|} \quad (\text{III.B.20})$$

otherwise (the latter case). Without loss of generality, we may assume that  $\chi'_m(\gamma) > 0$  (if  $K_m(\gamma) = 0$ , the statement (a) is trivial). By

<sup>11</sup>Such a volume certainly exists: take, for example,  $W = \text{int } \varphi$  with  $\varphi$  such that  $|\mathbb{B}(\text{int } \varphi)| = 1$ .

the definition of  $\chi'_m(\gamma)$ , this means that

$$(f_m^{(n-1)} - f_{m^c}^{(n-1)})|\mathbb{B}(\text{int } \gamma)|/d \leq 1 + (4\alpha + 2)\|\gamma\| \leq (4\alpha + 3)\|\gamma\|, \quad (\text{III.B.21})$$

which in turn implies

$$a_m^{(n-1)}|\mathbb{B}(\text{int } \gamma)|/d \leq (4\alpha + 3)\|\gamma\|. \quad (\text{III.B.22})$$

As a result,

$$\begin{aligned} \frac{Z_{m^c, \text{int } \gamma}(\beta, \lambda)}{Z_{m, \text{int } \gamma}^{(n-1)}(\beta, \lambda)} &\leq \\ &\leq \max \left\{ \sup_{\substack{\beta: \chi'_m(\gamma) > 0, \\ a_{m^c}^{(n-1)} \text{ diam } \gamma \leq \alpha}} e^{(4\alpha+3)\|\gamma\| + (s_m^{(n-1)} - s_{m^c}^{(n-1)} - a_m^{(n-1)}/d)|\partial\mathbb{B}(\text{int } \gamma)|}, \right. \\ &\quad \left. e^{(\frac{\mu}{2d} \log q + O(q^{-\nu}))|\partial\mathbb{B}(\text{int } \gamma)|} \right\} e^{O(q^{-\nu})\|\gamma\|}. \quad (\text{III.B.23}) \end{aligned}$$

Using Lemma III.A.5 (c), we get

$$\begin{aligned} \frac{Z_{m^c, \text{int } \gamma}(\beta, \lambda)}{Z_{m, \text{int } \gamma}^{(n-1)}(\beta, \lambda)} &\leq \max \left\{ e^{(4\alpha+3)\|\gamma\| + (s_m^{(n-1)} - s_{m^c}^{(n-1)})|\partial\mathbb{B}(\text{int } \gamma)|}, \right. \\ &\quad \left. e^{(\frac{\mu}{2d} \log q + O(q^{-\nu}))|\partial\mathbb{B}(\text{int } \gamma)|} \right\} e^{O(q^{-\nu})\|\gamma\|}. \quad (\text{III.B.24}) \end{aligned}$$

Combined with Lemma III.A.2 and (III.4.22), we finally obtain

$$\frac{Z_{m^c, \text{int } \gamma}(\beta, \lambda)}{Z_{m, \text{int } \gamma}^{(n-1)}(\beta, \lambda)} \leq e^{(4\alpha+3+O(q^{-\nu}))\|\gamma\| + \frac{\mu}{2d} \log q |\partial\mathbb{B}(\text{int } \gamma)|} \quad (\text{III.B.25})$$

since  $|\frac{\lambda}{J} - \frac{1}{2}| \leq \frac{\mu}{2}$ .

If  $|\partial\mathbb{B}(\text{int } \gamma)| = 0$ , we have  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$  and  $\rho(\gamma) \leq q^{-\frac{1}{6d}\|\gamma\|}$ .<sup>12</sup> Then (III.B.6), (III.B.25), (III.4.6), and the fact that  $\chi \leq 1$  yield

$$K_m(\gamma) \leq q^{(-\frac{1}{6d} + 2\nu)\|\gamma\|}. \quad (\text{III.B.26})$$

On the other hand, if  $|\partial\mathbb{B}(\text{int } \gamma)| \geq 1$ , then  $\text{dist}(\gamma, \partial V) = 0$  and  $\rho(\gamma) = q^{-\frac{1}{2d}\|\gamma\|}$ . Using Lemma III.A.2 again, we have

$$K_m(\gamma) \leq q^{(-\frac{1-\mu}{2d} + 2\nu)\|\gamma\|}. \quad (\text{III.B.27})$$

In view of the definition of  $\nu$ , the last two bounds justify the statement (a) of the lemma for  $k = 0$  and any  $\beta > 0$ .

Now, let us consider an arbitrary  $1 \leq k \leq k_0$  and  $\beta \geq 1$ . By virtue of the inductive assumptions (a), (c), and (d), Remark III.B.2, Lemma

<sup>12</sup>To see the latter bound, we use (III.3.5) and realize that the shortest disordered contour  $\gamma$  with  $\text{dist}(\gamma, \partial V) \geq \frac{3}{4}$  has the length  $4d - 2$ .

III.A.1, and (III.B.15), the bound (III.B.23) can be generalized to the  $k$ -th derivative. Namely,

$$\left| \frac{\partial^k}{\partial \beta^k} \frac{Z_{m^c, \text{int } \gamma}(\beta, \lambda)}{Z_{m, \text{int } \gamma}^{(n-1)}(\beta, \lambda)} \right| \leq k! (2D_2 |\bar{\mathbb{B}}(\text{int } \gamma)|)^k \frac{Z_{m^c, \text{int } \gamma}(\beta, \lambda)}{Z_{m, \text{int } \gamma}^{(n-1)}(\beta, \lambda)}. \quad (\text{III.B.28})$$

Next, Lemma III.A.1 and (III.B.13) imply the  $k$ -th derivative of  $f_m^{(n-1)}$  is bounded, which, according to the definition of  $\chi'_m(\gamma)$ , implies that

$$\left| \frac{\partial^k}{\partial \beta^k} \chi'_m(\gamma) \right| \leq \hat{C} |\bar{\mathbb{B}}(\text{int } \gamma)|^k \quad (\text{III.B.29})$$

for some  $\hat{C} > 0$  finite. In view of (III.B.6), we thus obtain

$$\left| \frac{\partial^k}{\partial \beta^k} K_m(\gamma) \right| \leq (2k_0 \hat{C} |\bar{\mathbb{B}}(\text{int } \gamma)|)^{k_0} \rho(\gamma) \frac{Z_{m^c, \text{int } \gamma}(\beta, \lambda)}{Z_{m, \text{int } \gamma}^{(n-1)}(\beta, \lambda)} \quad (\text{III.B.30})$$

with  $\hat{C} = \max\{\hat{C}, 2D_2\}$ . Lemma III.A.3 and (III.B.25) to (III.B.27) then conclude the proof of the part (a).

• *Proof of Lemma III.B.1 (b) for  $|\bar{\mathbb{B}}(\text{int } \gamma)| = r \leq n$  with  $n \geq 1$ .* Because we proved the part (a) of the lemma, we can analyze  $f_m^{(r)}$  for all  $r \leq n$  by a convergent cluster expansion (see Remark III.B.2). Any contour  $\hat{\gamma}$  contributing to the difference  $f_m^{(r)} - f_m^{(n)}$ ,  $r \leq n-1$ , obeys the lower bound

$$\|\hat{\gamma}\| \geq |\bar{\mathbb{B}}(\text{int } \hat{\gamma})|^{\frac{1}{d}} = (r+1)^{\frac{1}{d}} \geq r^{\frac{1}{d}} \quad (\text{III.B.31})$$

by Lemma III.A.3. Therefore,

$$|f_m^{(r)} - f_m^{(n)}| \leq q^{-\nu r^{1/d}}, \quad r \leq n, \quad (\text{III.B.32})$$

and we have

$$|f_m^{(r)} - f_m^{(n)}| |\bar{\mathbb{B}}(\text{int } \gamma)|/d \leq q^{-\nu r^{1/d}} \frac{r}{d} \leq \frac{1}{d} q^{-\nu} \quad (\text{III.B.33})$$

whenever  $q$  is large enough. Observing that  $a_m^{(n)} \geq f_m^{(n)} - f_{m^c}^{(n)}$  and estimating

$$|\bar{\mathbb{B}}(\text{int } \gamma)| \leq 2d \text{diam } \gamma (\|\gamma\| + |\partial \bar{\mathbb{B}}(\text{int } \gamma)|) \leq 4d \text{diam } \gamma \|\gamma\| \quad (\text{III.B.34})$$

with the help of Lemma III.A.2, we get

$$\begin{aligned} (f_m^{(r)} - f_{m^c}^{(r)}) |\bar{\mathbb{B}}(\text{int } \gamma)|/d &\leq a_m^{(n)} |\bar{\mathbb{B}}(\text{int } \gamma)|/d + \frac{2}{d} q^{-\nu} \leq \\ &\leq 4 a_m^{(n)} \text{diam } \gamma \|\gamma\| + \frac{2}{d} q^{-\nu} \end{aligned} \quad (\text{III.B.35})$$

As a result, if  $a_m^{(n)} \text{diam } \gamma \leq \alpha$ , then

$$\begin{aligned} (4\alpha + 2)\|\gamma\| - (f_m^{(r)} - f_m^{(c)}) |\mathbb{B}(\text{int } \gamma)|/d &\geq 2\|\gamma\| - \frac{2}{d} q^{-\nu} \geq \\ &\geq 2 - \frac{2}{d} q^{-\nu} > 1, \end{aligned} \quad (\text{III.B.36})$$

i.e.  $\chi'_m(\gamma) = 1$ . Moreover, it may now be easily shown that  $K_m(\gamma) = K_m(\gamma)$  once  $a_m^{(n)} \text{diam } \gamma \leq \alpha$ . A formal proof based on the induction on the sub-volumes of  $\text{int } \gamma$  is given in [8].

- *Proof of Lemma III.B.1 (c) for  $n \geq 1$ .* This part is an immediate consequence of the just proved parts (a) and (b).
- *Proof of Lemma III.B.1 (d) for  $n \geq 1$ .* Let us consider  $k = 0$  and  $\beta > 0$ . We call a contour  $\gamma$  *stable* if  $a_m^{(n)}(\beta) \text{diam } \gamma \leq \alpha$  and *unstable* if  $a_m^{(n)}(\beta) \text{diam } \gamma > \alpha$ . Splitting the external  $w$ -contours of every set  $\partial$  contributing to  $Z_{m,W}(\beta, \lambda)$  in (III.3.13) into stable and unstable and summing over non-external and stable external  $w$ -contours of  $\partial$ , we get

$$Z_{m,W}(\beta, \lambda) = \sum_{\partial_{\text{ext}} \sqsubset W}^{(m)} Z_{m,\text{Ext}}^{\text{stable}}(\beta, \lambda) \prod_{\gamma \in \partial_{\text{ext}}} [\rho(\gamma) Z_{m^c, \text{int } \gamma}(\beta, \lambda)]. \quad (\text{III.B.37})$$

Here the sum is over sets of  $m$ -labelled unstable short  $w$ -contours such that every  $\gamma \in \partial_{\text{ext}}$  is external and  $V(\gamma) \subset W$ . Moreover, we use  $\text{Ext}$  to denote  $W \setminus \cup_{\gamma \in \partial_{\text{ext}}} V(\gamma)$  and  $Z_{m,\text{Ext}}^{\text{stable}}(\beta, \lambda)$  is obtained from  $Z_{m,\text{Ext}}(\beta, \lambda)$  by dropping all the unstable external short  $w$ -contours.

Since all external  $w$ -contours contributing to  $Z_{m,\text{Ext}}^{\text{stable}}(\beta, \lambda)$  are stable, so is any other  $w$ -contour contributing it. Thus, using the inductive assumptions (a) and (c), we can control this partition function by a convergent cluster expansion, obtaining

$$Z_{m,\text{Ext}}^{\text{stable}}(\beta, \lambda) = e^{-F_m^{\text{stable}}(\mathbb{B}(\text{Ext})) + O(q^{-\nu})\|\partial\text{Ext}\|}, \quad (\text{III.B.38})$$

where  $F_m^{\text{stable}}(\mathbb{B}(\text{Ext})) = f_m^{\text{stable}} |\mathbb{B}(\text{Ext})|/d + s_m^{\text{stable}} |\partial\mathbb{B}(\text{Ext})|$  and the quantities  $f_m^{\text{stable}}$  and  $s_m^{\text{stable}}$  corresponding to the contour model with the activities

$$K_m^{\text{stable}}(\gamma) = \begin{cases} K_m(\gamma) & \text{if } \gamma \text{ is a stable short } w\text{-contour and} \\ & |\mathbb{B}(\text{int } \gamma)| \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{III.B.39})$$

Because  $2\|\gamma_0\| \geq \text{diam } \gamma_0$  for any  $w$ -contour  $\gamma_0$ , we have a lower bound on the length of every unstable  $w$ -contour, namely,  $\|\gamma\| >$

$\zeta \equiv \frac{\alpha}{2a_m^{(n)}(\beta)}$ . Hence,

$$|f_m^{(n)} - f_m^{\text{stable}}| \leq q^{-\nu\zeta} \leq \frac{2}{\alpha\nu \log q} a_m^{(n)}(\beta) \leq \frac{\varepsilon}{2} a_m^{(n)}(\beta) \quad (\text{III.B.40})$$

and, similarly,

$$|s_m^{(n)} - s_m^{\text{stable}}| \leq q^{-\nu\zeta} \leq \frac{\varepsilon}{2} a_m^{(n)}(\beta) \quad (\text{III.B.41})$$

for any  $\varepsilon > 0$  once  $q$  is sufficiently large. Consequently,

$$Z_{m,\text{Ext}}^{\text{stable}}(\beta, \lambda) \leq e^{-F_m^{(n)}(\mathbb{B}(\text{Ext})) + \frac{\varepsilon}{2} a_m^{(n)}(|\mathbb{B}(\text{Ext})|/d + |\partial\mathbb{B}(\text{Ext})|) + O(q^{-\nu})\|\partial\text{Ext}\|}. \quad (\text{III.B.42})$$

Now, since  $v(\text{int } \gamma) < v(W) \leq n$  and  $a_{m^c}^{(n)}(\beta) = 0$  by assumption, we can apply the proved parts (a) through (c) of the lemma to  $Z_{m^c, \text{int } \gamma}(\beta, \lambda)$ . In view of (III.B.12) with  $r = n$  and  $k = 0$ , this allows to establish

$$Z_{m^c, \text{int } \gamma}(\beta, \lambda) = Z_{m^c}^{(n)}(\text{int } \gamma; \beta, \lambda) = e^{-F_{m^c}^{(n)}(\mathbb{B}(\text{int } \gamma)) + O(q^{-\nu})\|\gamma\|} \quad (\text{III.B.43})$$

for all  $\beta > 0$  such that  $a_{m^c}^{(n)}(\beta) = 0$ .

Combining (III.B.37), (III.B.42), and (III.B.43) with

$$|\mathbb{B}(W)| = |\mathbb{B}(\text{Ext})| + \sum_{\gamma \in \partial_{\text{ext}}} |\mathbb{B}(\text{int } \gamma)|, \quad (\text{III.B.44})$$

$$|\partial\mathbb{B}(W)| = |\partial\mathbb{B}(\text{Ext})| + \sum_{\gamma \in \partial_{\text{ext}}} |\partial\mathbb{B}(\text{int } \gamma)| \quad (\text{III.B.45})$$

and  $\|\partial\text{Ext}\| = \|\partial W\| + \sum_{\gamma \in \partial_{\text{ext}}} \|\gamma\|$ , it follows that

$$\begin{aligned} Z_{m,W}(\beta, \lambda) &\leq e^{-F_{m^c}^{(n)}(\mathbb{B}(W)) + O(q^{-\nu})\|\partial W\|} \sum_{\partial_{\text{ext}} \sqsubset W}^{(m)} e^{-(1-\frac{\varepsilon}{2})a_m^{(n)}|\mathbb{B}(\text{Ext})|/d} \times \\ &\times e^{(s_{m^c}^{(n)} - s_m^{(n)} + \frac{\varepsilon}{2} a_m^{(n)})|\partial\mathbb{B}(\text{Ext})|} \prod_{\gamma \in \partial_{\text{ext}}} [\rho(\gamma) e^{O(q^{-\nu})\|\gamma\|}] \leq \quad (\text{III.B.46}) \end{aligned}$$

$$\begin{aligned} &\leq e^{-F_{m^c}^{(n)}(\mathbb{B}(W)) + O(q^{-\nu})\|\partial W\| + \max\{s_{m^c}^{(n)} - s_m^{(n)} + \varepsilon a_m^{(n)}, 0\}|\partial\mathbb{B}(W)|} \times \\ &\times \sum_{\partial_{\text{ext}} \sqsubset W}^{(m)} e^{-\frac{\varepsilon}{2} a_m^{(n)}(|\mathbb{B}(\text{Ext})|/d + |\partial\mathbb{B}(\text{Ext})|)} \prod_{\gamma \in \partial_{\text{ext}}} [\rho(\gamma) e^{\|\gamma\|}]. \quad (\text{III.B.47}) \end{aligned}$$

Next, let us take  $\mathcal{K}(\gamma) = \rho(\gamma) e^{\|\gamma\|}$  if  $\gamma$  is a  $w$ -contour contributing to the sum in (III.B.47), whereas  $\mathcal{K}(\gamma) = 0$  otherwise. Since  $\rho(\gamma) \leq q^{-\frac{1}{6d}\|\gamma\|}$  (c.f. the footnote on p. 90), we have  $\mathcal{K}(\gamma) \leq q^{-2\nu\|\gamma\|}$ . Because  $\|\gamma\| < \zeta$ , where  $\zeta > 0$  is the constant from (III.B.40) and (III.B.41), for  $\phi$  and  $\sigma$  introduced in Lemma III.A.4, this yields  $0 \leq -\phi \leq q^{-\nu\zeta}$  and  $0 \leq -\sigma \leq q^{-\nu\zeta}$ . As  $q^{-\nu\zeta} \leq \frac{\varepsilon}{2} a_m^{(n)}$ , Lemma III.A.4 allows to bound

the sum in (III.B.47) from above by  $e^{O(q^{-\nu})\|\partial W\|+O(q^{-\nu L})}$ . Moreover, Lemma III.A.5 (b) and (III.4.22) yield

$$\begin{aligned} \sup_{\beta: a_m^{(n)}(\beta) > 0} (s_{m^c}^{(n)} - s_m^{(n)} + \varepsilon a_m^{(n)})(\beta) &= (s_{m^c}^{(n)} - s_m^{(n)} + \varepsilon a_m^{(n)})(\bar{\beta}^{(n)}) \leq \\ &\leq \frac{\mu}{2d} \log q + O(q^{-\nu}) \quad (\text{III.B.48}) \end{aligned}$$

for all  $|\frac{\lambda}{j} - \frac{1}{2}| \leq \frac{\mu}{2}$  and  $\varepsilon \leq \frac{\lambda^2}{dj^2}$ . Since  $\varepsilon > 0$  can be chosen arbitrarily small (for  $q$  large enough) and, by assumption, we consider  $\beta$  such that  $a_m^{(n)}(\beta) > 0$ , we come to (III.B.11) with  $k = 0$ .

Let us prove the part (d) for any  $1 \leq k \leq k_0$ . Supposing that  $a_{m^c}^{(n)}(\beta) = 0$  and  $\beta \geq 1$  and recalling that all the contours contributing to  $Z_{m,\text{Ext}}^{\text{stable}}(\beta, \lambda)$  are stable, we can control  $\log Z_{m,\text{Ext}}^{\text{stable}}(\beta, \lambda)$  and  $\log Z_{m^c,\text{int}\gamma}(\beta, \lambda)$  along with their derivatives up to the  $k_0$ -th order by convergent cluster expansions due to inductive assumptions (a) and (c). Because any unstable  $w$ -contour  $\gamma$  satisfies  $\|\gamma\| > \zeta \equiv \frac{\alpha}{2a_m^{(n)}(\beta)}$ , the bounds (III.B.40) and (III.B.41) can be generalized to the  $k$ -th derivative. Hence, the functions  $f_m^{\text{stable}}$  and  $s_m^{\text{stable}}$  have uniformly bounded derivatives as  $f_m$  and  $s_m$  do (by Lemma III.A.1 and (III.B.13)). By virtue of (III.B.18), we thus get

$$\begin{aligned} \left| \frac{\partial^k}{\partial \beta^k} F_m^{\text{stable}}(\bar{\mathbb{B}}(W)) \right| &\leq \left| \frac{\partial^k}{\partial \beta^k} F_m(\bar{\mathbb{B}}(W)) \right| + O(q^{-\nu\zeta}) |\bar{\mathbb{B}}(W)| \leq \\ &\leq (2\tilde{C} + O(q^{-\nu\zeta})) |\bar{\mathbb{B}}(W)| \leq C^* |\bar{\mathbb{B}}(W)| \quad (\text{III.B.49}) \end{aligned}$$

for some  $C^* > 0$  finite since  $\zeta > 0$  by assumption. This all then yields relations for  $Z_{m,\text{Ext}}^{\text{stable}}(\beta, \lambda)$  and  $Z_{m^c,\text{int}\gamma}(\beta, \lambda)$  analogous to (III.B.15) if we choose  $D_2 = \max\{2C^*, 1\}$ , say. Using now (III.B.37), we readily get

$$\left| \frac{\partial^k}{\partial \beta^k} Z_{m,W}(\beta, \lambda) \right| \leq k! (D_2 |\bar{\mathbb{B}}(W)|)^k Z_{m,W}(\beta, \lambda) \quad (\text{III.B.50})$$

as was to be proved. Q.E.D.

**PROOF OF LEMMA III.4.1:** Since one has  $f_m = \lim_{n \rightarrow \infty} f_m^{(n)}$  and  $s_m = \lim_{n \rightarrow \infty} s_m^{(n)}$ , Lemma III.4.1 follows from Lemma III.B.1 if taking  $D_0 = \max\{D_1, D_2\}$ . Q.E.D.

### III.C. Proof of Corollary III.4.4

Let the assumptions of the corollary be satisfied and let  $k_0 = 0, 1, \dots$

(a) For any  $\beta \in [\beta_o(L), \beta_d(L)]$ , the function  $\log Z_{m,W}(\beta, \lambda)(\beta, \lambda)$ ,  $m = o, d$ , and its derivatives can be controlled by convergent cluster

expansions. Let us only consider  $\beta \in [\beta_{\neq}^{(\lambda)}(L), \beta_d(L)]$ , where  $\beta_{\neq}^{(\lambda)}(L)$  was introduced in Corollary III.4.3 (the case  $\beta \in [\beta_o(L), \beta_{\neq}^{(\lambda)}(L)]$  can be treated analogously).

Clearly,

$$\frac{Z_{\text{big},V}(\beta, \lambda)}{Z_{o,V}(\beta, \lambda) + Z_{d,V}(\beta, \lambda)} \leq \delta(\beta) \equiv \frac{Z_{\text{big},V}(\beta, \lambda)}{Z_{o,V}(\beta, \lambda)}. \quad (\text{III.C.1})$$

Combining (III.3.18) and (III.4.8), we get the estimate

$$\begin{aligned} \delta(\beta) \leq e^{O(q^{-\nu L})} \sum_{\partial_l} e^{(f_o - f_d)|\mathbb{B}(W_d(\partial_l))|/d + (s_o - s_d)|\partial\mathbb{B}(W_d(\partial_l))|} \times \\ \times \prod_{\gamma \in \partial_l} \rho(\gamma) e^{O(q^{-\nu})\|\gamma\|} \quad (\text{III.C.2}) \end{aligned}$$

because  $\|\partial W_m(\partial_l)\| = \sum_{\gamma \in \partial_l} \|\gamma\|$  for both  $m = o$  and  $m = d$ . We now observe that  $f_o - f_d$  and  $s_o - s_d$  are decreasing functions of  $\beta$  on  $[1, \infty)$  if  $q$  large enough:

$$\begin{aligned} \frac{d(f_o - f_d)}{d\beta} &= \frac{dg_o}{d\beta} + O(q^{-\nu}) < -dJ + O(q^{-\nu}) < 0, \\ \frac{d(s_o - s_d)}{d\beta} &< -\lambda + O(q^{-\nu}) < 0, \end{aligned}$$

where we used (III.4.9) and (III.3.11). Hence, we may bound the first exponential in (III.C.2) from above by its value at  $\beta_{\neq}^{(\lambda)}(L)$ . As  $F_o(\bar{\mathbb{B}}) = F_d(\bar{\mathbb{B}}) + O(q^{-\nu L})$  at  $\beta_{\neq}^{(\lambda)}(L)$  by virtue of Corollary III.4.3 (b), we have

$$(f_o - f_d)(\beta_{\neq}^{(\lambda)}(L))|\mathbb{B}|/d = -(s_o - s_d)(\beta_{\neq}^{(\lambda)}(L))|\partial\mathbb{B}| + O(q^{-\nu L}).$$

Along with (III.4.23) and Lemma III.A.6, this yields

$$\begin{aligned} \delta(\beta) &\leq e^{O(q^{-\nu L})} \sum_{\partial_l} e^{|(s_o - s_d)(\beta_{\neq}^{(\lambda)}(L))| \frac{2}{L-1} |\mathbb{B}(W_d(\partial_l))| - |\partial\mathbb{B}(W_d(\partial_l))|} \times \\ &\quad \times \prod_{\gamma \in \partial_l} \rho(\gamma) e^{O(q^{-\nu})\|\gamma\|} \leq \\ &\leq e^{O(q^{-\nu L})} \sum_{\partial_l} \prod_{\gamma \in \partial_l} \rho(\gamma) e^{(|(s_o - s_d)(\bar{\beta})| + O(q^{-\nu}) + O(L^{-1}))\|\gamma\|}. \end{aligned}$$

Observing that  $2\|\gamma_0\| \geq \text{diam } \gamma_0$  holds for every  $w$ -contour  $\gamma_0$ , any long  $w$ -contour  $\gamma$  satisfies  $\|\gamma\| \geq \ell_0 \equiv \frac{1}{2}\omega(L)$ . Then (III.3.5) gives

$\rho(\gamma) \leq q^{-c\|\gamma\|}$  with  $c = \frac{1}{2d} - \frac{2}{\omega(L)}$ . Combined with (III.4.22), we get

$$\begin{aligned} \delta(\beta) &\leq e^{O(q^{-\nu L})} \sum_{\partial_l} \prod_{\gamma \in \partial_l} q^{(-\frac{1-\mu}{2d} + O(\frac{q^{-\nu}}{\log q}) + O(\frac{1}{\omega(L)}) + O(L^{-1}))\|\gamma\|} \leq \\ &\leq e^{O(q^{-\nu L})} \sum_{\partial_l} \prod_{\gamma \in \partial_l} q^{-\frac{1-\mu}{4d}\|\gamma\|} \quad (\text{III.C.3}) \end{aligned}$$

for all  $q$  and  $L$  large enough. Now,

$$\begin{aligned} \sum_{\partial_l} \prod_{\gamma \in \partial_l} q^{-\frac{1-\mu}{4d}\|\gamma\|} &\leq \sum_{n=1}^{\infty} \sum_{\partial_l: |\partial_l|=n} \prod_{\gamma \in \partial_l} q^{-\frac{1-\mu}{4d}\|\gamma\|} \leq \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{\gamma} q^{-\frac{1-\mu}{4d}\|\gamma\|} \right)^n, \quad (\text{III.C.4}) \end{aligned}$$

where the last sum is over all long  $w$ -contours  $\gamma$  in  $V(L)$ . Bounding the number of  $w$ -contours in  $V(L)$  whose length is  $\ell$  by  $C^\ell L^d$ , where  $C > 0$  is a constant depending on  $d$ , it follows that

$$\begin{aligned} \sum_{\gamma} q^{-\frac{1-\mu}{4d}\|\gamma\|} &\leq \sum_{\ell=\ell_0}^{\infty} \sum_{\gamma: \|\gamma\|=\ell} q^{-\frac{1-\mu}{4d}\ell} \leq \sum_{\ell=\ell_0}^{\infty} C^\ell L^d q^{-\frac{1-\mu}{4d}\ell} \leq \\ &\leq L^d \sum_{\ell=\ell_0}^{\infty} q^{-\frac{1-\mu}{5d}\ell} \leq q^{-\frac{1-\mu}{6d}\ell_0} = q^{-\frac{1-\mu}{12d}\omega(L)} \quad (\text{III.C.5}) \end{aligned}$$

whenever  $q$  and  $L$  are taken large enough. As a result,

$$\delta(\beta) \leq e^{O(q^{-\nu L})} \left( e^{q^{-\frac{1-\mu}{12d}\omega(L)}} - 1 \right) \leq q^{-\frac{1-\mu}{16d}\omega(L)} \quad (\text{III.C.6})$$

for all  $\beta \in [\beta_{\leq}^{(\lambda)}(L), \beta_d(L)]$  as soon as  $q$  and  $L$  are sufficiently large.

Let  $0 \leq k \leq k_0$ . In view of (III.4.14), we bound

$$\begin{aligned} \left| \frac{\partial^k}{\partial \beta^k} (Z_{o,V}(\beta, \lambda) + Z_{d,V}(\beta, \lambda)) \right| &\leq \\ &\leq k! (D_0 |\mathbb{B}|)^k (Z_{o,V}(\beta, \lambda) + Z_{d,V}(\beta, \lambda)). \end{aligned}$$

Therefore, applying Lemma III.A.7 to  $\psi_1(x) = x^{-1}$  and  $\psi_2(\beta) = Z_{o,V}(\beta, \lambda) + Z_{d,V}(\beta, \lambda)$ , one gets

$$\begin{aligned} \left| \frac{\partial^k}{\partial \beta^k} (Z_{o,V}(\beta, \lambda) + Z_{d,V}(\beta, \lambda))^{-1} \right| &\leq \\ &\leq k! (D_0 |\mathbb{B}|)^k (Z_{o,V}(\beta, \lambda) + Z_{d,V}(\beta, \lambda))^{-1}. \quad (\text{III.C.7}) \end{aligned}$$

On the other hand, we may combine (III.4.14) and (III.3.18) to estimate

$$\left| \frac{\partial^k Z_{\text{big},V}(\beta, \lambda)}{\partial \beta^k} \right| \leq k! (D_0 |\mathbb{B}|)^k Z_{\text{big},V}(\beta, \lambda). \quad (\text{III.C.8})$$

Combining (III.C.7) and (III.C.8), we thus have

$$\left| \frac{\partial^k}{\partial \beta^k} \frac{Z_{\text{big},V}(\beta, \lambda)}{Z_{\text{o},V}(\beta, \lambda) + Z_{\text{d},V}(\beta, \lambda)} \right| \leq \frac{(2k_0 D_0 |\mathbb{B}|)^{k_0} Z_{\text{big},V}(\beta, \lambda)}{Z_{\text{o},V}(\beta, \lambda) + Z_{\text{d},V}(\beta, \lambda)}. \quad (\text{III.C.9})$$

Taking into account (III.C.1) and (III.C.6), this verifies the part (a) of the corollary.

(b) Let us suppose that  $\beta \geq \beta_d(L)$ . Then  $a_d(\beta) > 0$  and the logarithm  $\log Z_{\text{o},W}(\beta, \lambda)$  and its derivatives can be controlled by convergent cluster expansions. Moreover, analogously to (III.B.46), we get

$$\begin{aligned} Z_{\text{d},V}(\beta, \lambda) &\leq e^{-F_0(\mathbb{B})} \sum_{\partial_{\text{ext}}^{(\text{d})}} e^{-(1-\varepsilon)a_d|\mathbb{B}(\text{Ext})|/d+(s_0-s_d+\varepsilon a_d)|\partial\mathbb{B}(\text{Ext})|} \times \\ &\times e^{-\frac{\varepsilon}{2}a_d(|\mathbb{B}(\text{Ext})|/d+|\partial\mathbb{B}(\text{Ext})|)} \prod_{\gamma \in \partial_{\text{ext}}} [\rho(\gamma) e^{O(q^{-\nu})\|\gamma\|}]. \quad (\text{III.C.10}) \end{aligned}$$

Here the sum is running over sets  $\partial_{\text{ext}}$  of unstable disordered short  $w$ -contours,  $\text{Ext}$  stands for  $V \setminus \cup_{\gamma \in \partial_{\text{ext}}} V(\gamma)$ , and  $\varepsilon \in (0, \frac{\lambda^2}{d^2})$ . Because  $a_d$  is an increasing function of  $\beta$  on  $(\bar{\beta}, \infty)$ , see Remark III.4.2, we have  $a_d(\beta) \geq a_d(\beta_d(L)) = \frac{\alpha}{\omega(L)}$  for all  $\beta$  considered. Using also Lemma III.A.5 (b) and (III.4.22), we may write

$$\begin{aligned} Z_{\text{d},V}(\beta, \lambda) &\leq e^{-F_0(\mathbb{B})} \sum_{\partial_{\text{ext}}^{(\text{d})}} e^{-(1-\varepsilon)\frac{\alpha}{\omega(L)}|\mathbb{B}(\text{Ext})|/d} \times \\ &\times e^{\left(\frac{\mu}{2d} \log q + O(q^{-\nu})\right)|\partial\mathbb{B}(\text{Ext})| - \frac{\varepsilon}{2}a_d(|\mathbb{B}(\text{Ext})|/d+|\partial\mathbb{B}(\text{Ext})|)} \times \\ &\times \prod_{\gamma \in \partial_{\text{ext}}} [\rho(\gamma) e^{O(q^{-\nu})\|\gamma\|}]. \quad (\text{III.C.11}) \end{aligned}$$

First, let  $|\mathbb{B}(\text{Ext})| \geq \frac{3}{5}|\mathbb{B}|$ . Since  $|\mathbb{B}| = \frac{1}{2}|\partial\mathbb{B}|(L-1)$  by (III.4.23), we then have

$$\begin{aligned} Z_{\text{d},V}(\beta, \lambda) &\leq e^{-F_0(\mathbb{B})} e^{\left(-\frac{\alpha}{4d} \frac{L-1}{\omega(L)} + \frac{\mu}{2d} \log q + O(q^{-\nu})\right)|\partial\mathbb{B}|} \times \\ &\times \sum_{\partial_{\text{ext}}^{(\text{d})}} e^{-\frac{\varepsilon}{2}a_d(|\mathbb{B}(\text{Ext})|/d+|\partial\mathbb{B}(\text{Ext})|)} \prod_{\gamma \in \partial_{\text{ext}}} [\rho(\gamma) e^{O(q^{-\nu})\|\gamma\|}] \quad (\text{III.C.12}) \end{aligned}$$

for any  $0 < \varepsilon \leq \min\{\frac{1}{6}, \frac{\lambda^2}{d^2}\}$ . Observing that  $\frac{L-1}{\omega(L)} \geq \frac{6}{\nu}$  in view of the first condition in (III.2.19) and bounding the sum in (III.C.12) by  $e^{O(q^{-\nu L})}$ , c.f. (III.B.47), we get

$$\begin{aligned} Z_{\text{d},V}(\beta, \lambda) &\leq e^{-F_0(\mathbb{B})+O(q^{-\nu L})} e^{\left(-\frac{1}{2d} + \frac{\mu}{2d} + O\left(\frac{q^{-\nu}}{\log q}\right) + O(L^{-1})\right)|\partial\mathbb{B}| \log q} \leq \\ &\leq e^{-F_0(\mathbb{B})} q^{-\frac{1-\mu}{2}L^{d-1}}. \quad (\text{III.C.13}) \end{aligned}$$

for sufficiently large  $q$  and  $L$ .

On the other hand, if  $|\mathbb{B}(\text{Ext})| \leq \frac{3}{5} |\mathbb{B}|$ , then, due to (III.B.34), one has

$$\sum_{\gamma \in \partial_{\text{ext}}} \|\gamma\| \geq \frac{1}{4d} \sum_{\gamma \in \partial_{\text{ext}}} \frac{|\mathbb{B}(\text{int } \gamma)|}{\text{diam } \gamma} \geq \frac{|\mathbb{B}| - |\mathbb{B}(\text{Ext})|}{4d \omega(L)} \geq \frac{1}{20d} \frac{L-1}{\omega(L)} |\partial \mathbb{B}|$$

as every  $\gamma \in \partial_{\text{ext}}$  is short. Observing also that

$$\rho(\gamma) e^{O(q^{-\nu}) \|\gamma\|} \leq q^{-\frac{1}{6d} \|\gamma\|} \leq q^{-(\frac{1}{12d} + 2\nu) \|\gamma\|}$$

due to (III.3.5) and (III.4.6), from Lemma III.A.4 and (III.C.11) we obtain

$$\begin{aligned} Z_{d,V}(\beta, \lambda) &\leq e^{-F_0(\bar{\mathbb{B}}) + O(q^{-\nu L})} q^{-(\frac{1}{12d} \frac{1}{20d} \frac{L-1}{\omega(L)} - \frac{\mu}{2d}) + O(\frac{q^{-\nu}}{\log q})} |\partial \mathbb{B}| \leq \\ &\leq e^{-F_0(\bar{\mathbb{B}})} q^{-\frac{1-\mu}{2} L^{d-1}} \end{aligned} \quad (\text{III.C.14})$$

once  $q, L$  are large enough; we used that  $\frac{L-1}{\omega(L)} \geq \frac{6}{\nu}$  and that  $\nu \leq \frac{1}{24d}$  by (III.4.6).

Combining (III.C.13), (III.C.14), and (III.4.8), it thus follows that

$$\frac{Z_{d,V}(\beta, \lambda)}{Z_{o,V}(\beta, \lambda)} \leq q^{-\frac{1-\mu}{4} L^{d-1}} \quad (\text{III.C.15})$$

for all  $\beta \geq \beta_d(L)$  and  $q, L$  large. Consequently, taking any  $0 \leq k \leq k_0$ , we may use (III.4.14), (III.B.50), and (III.C.15) to bound

$$\left| \frac{\partial^k Z_{d,V}(\beta, \lambda)}{\partial \beta^k Z_{o,V}(\beta, \lambda)} \right| \leq (2k_0 D_0 |\bar{\mathbb{B}}|)^{k_0} \frac{Z_{d,V}(\beta, \lambda)}{Z_{o,V}(\beta, \lambda)} \leq q^{-\frac{1-\mu}{6} L^{d-1}} \quad (\text{III.C.16})$$

for all  $\beta \geq \beta_d(L)$  and  $q, L$  large.

Let us now bound the derivatives of the ratio  $\frac{Z_{\text{big},V}(\beta, \lambda)}{Z_{o,V}(\beta, \lambda)}$ . Given  $k = 0, \dots, k_0$ , by virtue of (III.4.14), (III.4.7), and (III.3.18), one finds

$$\left| \frac{\partial^k Z_{\text{big},V}(\beta, \lambda)}{\partial \beta^k Z_{o,V}(\beta, \lambda)} \right| \leq k_0! (2D_0 |\bar{\mathbb{B}}|)^{k_0} \frac{Z_{\text{big},V}(\beta, \lambda)}{Z_{o,V}(\beta, \lambda)}. \quad (\text{III.C.17})$$

Next, we rewrite the partition function (III.3.13) by summing over all those sets  $\partial$  contributing to it whose external contours are fixed. This yields

$$Z_{m,W}(\beta, \lambda) = \sum_{\partial_{\text{ext}} \sqsubset W}^{(m)} e^{-G_d(\bar{\mathbb{B}}(\text{Ext}))} Z_{m^c, \text{int}}(\beta, \lambda) \prod_{\gamma \in \partial_{\text{ext}}} \rho(\gamma), \quad (\text{III.C.18})$$

where  $\text{int} = \cup_{\gamma \in \partial_{\text{ext}}} \text{int } \gamma$  and  $\text{Ext} = W \setminus \cup_{\gamma \in \partial_{\text{ext}}} V(\gamma)$  and the summation is over sets  $\partial_{\text{ext}}$  of short  $m$ -labelled  $w$ -contours which are all external with  $V(\gamma) \subset W$ . Combined with (III.3.16), we then find

$$\frac{Z_{\text{big},V}(\beta, \lambda)}{Z_{o,V}(\beta, \lambda)} = \sum_{\partial_l} \prod_{\gamma \in \partial_l} \rho(\gamma) \sum_{\tilde{\partial}_{\text{ext}} \sqsubset W_d(\partial_l)}^{(d)} \prod_{\tilde{\gamma} \in \tilde{\partial}_{\text{ext}}} \rho(\tilde{\gamma}) e^{\xi(\beta)}, \quad (\text{III.C.19})$$

where

$$\xi(\beta) = -G_d(\mathbb{B}(\widetilde{\text{Ext}})) + \log \frac{Z_{o, W_o(\partial_l)}(\beta, \lambda) Z_{o, \widetilde{\text{int}}}(\beta, \lambda)}{Z_{o, V}(\beta, \lambda)} \quad (\text{III.C.20})$$

and  $\widetilde{\text{int}} = \cup_{\tilde{\gamma} \in \tilde{\partial}_{\text{ext}}} \text{int } \gamma$ ,  $\widetilde{\text{Ext}} = W_d(\partial_l) \setminus \cup_{\tilde{\gamma} \in \tilde{\partial}_{\text{ext}}} V(\gamma)$ . With the help of (III.4.8), one easily establishes

$$\begin{aligned} \frac{d\xi}{d\beta} &= \frac{d}{d\beta} (F_o(\mathbb{B}(\widetilde{\text{Ext}})) - G_d(\mathbb{B}(\widetilde{\text{Ext}}))) + \\ &\quad + O(q^{-\nu}) (\|\partial W_o(\partial_l)\| + \|\partial \widetilde{\text{int}}\|) + O(q^{-\nu L}). \end{aligned} \quad (\text{III.C.21})$$

However,

$$\frac{d}{d\beta} (F_o(\mathbb{B}(\widetilde{\text{Ext}})) - G_d(\mathbb{B}(\widetilde{\text{Ext}}))) < (-\lambda + O(q^{-\nu})) |\mathbb{B}(\widetilde{\text{Ext}})| \quad (\text{III.C.22})$$

due to (III.4.9) and (III.3.11). Hence,

$$\begin{aligned} \frac{d\xi}{d\beta} &< (-\lambda + O(q^{-\nu})) |\mathbb{B}(\widetilde{\text{Ext}})| + O(q^{-\nu}) (\|\partial W_o(\partial_l)\| + \|\partial \widetilde{\text{int}}\|) + \\ &\quad + O(q^{-\nu L}). \end{aligned} \quad (\text{III.C.23})$$

Since, obviously,

$$\|\partial \widetilde{\text{int}}\| = \sum_{\tilde{\gamma} \in \tilde{\partial}_{\text{ext}}} \|\tilde{\gamma}\|, \quad \|\partial W_o(\partial_l)\| = \|\partial W_d(\partial_l)\| = \sum_{\gamma \in \partial_l} \|\gamma\|,$$

Lemma III.A.6 (b) yields

$$\|\partial W_o(\partial_l)\| + \|\partial \widetilde{\text{int}}\| \leq 2 |\mathbb{B}(\widetilde{\text{Ext}})|. \quad (\text{III.C.24})$$

Observing that  $|\mathbb{B}(\widetilde{\text{Ext}})| \geq 1$ , we finally get  $\frac{d\xi}{d\beta} < 0$  for all  $q$  large.

As a consequence, the ratio  $\frac{Z_{\text{big}, V}(\beta, \lambda)}{Z_{o, V}(\beta, \lambda)}$  is a decreasing function of  $\beta$  if  $\beta \geq \beta_d(L)$ , and, in view of (III.C.17) and (III.C.1), we have

$$\begin{aligned} \left| \frac{\partial^k}{\partial \beta^k} \frac{Z_{\text{big}, V}(\beta, \lambda)}{Z_{o, V}(\beta, \lambda)} \right| &< (2k_0 D_0 |\mathbb{B}|)^{k_0} \frac{Z_{\text{big}, V}(\beta_d(L), \lambda)}{Z_{o, V}(\beta_d(L), \lambda)} = \\ &= (2k_0 D_0 |\mathbb{B}|)^{k_0} \delta(\beta_d(L)) \end{aligned} \quad (\text{III.C.25})$$

for all  $k = 0, 1, \dots, k_0$ . Together with (III.C.6) and (III.C.16), this yields (III.4.32). The estimate (III.4.33) is proved analogously.

REMARK III.C.1. In the proof of Theorem III.2.1 and III.2.2, it is crucial to have upper bound on the ratio  $\frac{Z_{m, W}(\beta, \lambda)}{Z_{m^c, W}(\beta, \lambda)}$  on condition that  $a_{m^c}(\beta) \min\{\text{diam } W, \omega(L)\} \leq \alpha$ .<sup>13</sup> To find it, we combine a

<sup>13</sup>Here again  $W$  is of the form (III.3.8) and  $m^c = o$  if  $m = d$ , while  $m^c = d$  if  $m = o$ .

cluster-expansion control over  $\log Z_{m^c, W}(\beta, \lambda)$  with (III.4.7). Nevertheless, we would like to point out that one could do without the latter bound and use convergent cluster expansions only (even when one has that  $a_m(\beta) \min\{\text{diam } W, \omega(L)\} \geq \alpha$ ). Indeed, from (III.C.18) it follows that

$$\frac{Z_{m, W}(\beta, \lambda)}{Z_{m^c, W}(\beta, \lambda)} = \sum_{\partial_{\text{ext}} \square W} \prod_{\gamma \in \partial_{\text{ext}}} \rho(\gamma) e^{\vartheta(\beta)},$$

with

$$\vartheta(\beta) = -G_m(\bar{\mathbb{B}}(\text{Ext})) + \log \frac{Z_{m^c, \text{int}}(\beta, \lambda)}{Z_{m^c, W}(\beta, \lambda)}$$

for all  $\beta > 0$ . Copying the procedure used above, Lemma III.A.6 (b) implies that  $\vartheta$  is a monotonous function of  $\beta$  such that  $\frac{Z_{m, W}(\beta, \lambda)}{Z_{m^c, W}(\beta, \lambda)} \leq \frac{Z_{m, W}(\beta_m(W), \lambda)}{Z_{m^c, W}(\beta_m(W), \lambda)}$ , where  $\beta_m(W)$  is the unique point at which one has  $a_m(\beta) \min\{\text{diam } W, \omega(L)\}$  equals  $\alpha$ . At  $\beta_m(W)$ , however, the function  $\log Z_{m, W}(\beta, \lambda)$  can be controlled by a convergent cluster expansion.

An analogous remark is true for the ratio  $\frac{Z_{m, W}(\beta, \lambda)}{Z_{m^c, W}^{(n)}(\beta, \lambda)}$ , too.

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## CHAPTER IV

# Finite-Size Scaling for the 2D Ising Model with Fixed Boundary Conditions

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ABSTRACT. We study the magnetization  $m_L(h, \beta)$  for the Ising model on a large but finite lattice square under the minus boundary conditions. Using known large-deviation results evaluating the balance between the competing effects of the minus boundary conditions and the external magnetic field  $h$ , we describe the details of its dependence on  $h$  as exemplified by the finite-size rounding of the infinite-volume magnetization discontinuity and its shift with respect to the infinite-volume transition point.

### IV.1. Introduction

The ferromagnetic nearest-neighbour Ising model on  $\mathbb{Z}^d$ ,  $d \geq 2$ , is perhaps the most familiar spin system undergoing a first-order phase transition. Its formal Hamiltonian is

$$H(\sigma) = - \sum_{\langle x, y \rangle} \sigma_x \sigma_y - h \sum_x \sigma_x, \quad (\text{IV.1.1})$$

where  $\sigma_x$  is the spin at the site  $x \in \mathbb{Z}^d$  in the configuration  $\sigma \in \{-1, 1\}^{\mathbb{Z}^d}$ ,  $h \in \mathbb{R}$  is the external magnetic field, and  $\langle x, y \rangle$  stands for a pair of nearest-neighbour sites  $x$  and  $y$  of  $\mathbb{Z}^d$ . The phase transition occurs at  $h = 0$  whenever the inverse temperature  $\beta$  is sufficiently large: there exists a point  $\beta_c < \infty$  such that, for inverse temperature  $\beta > \beta_c$ , the set of infinite-volume Gibbs states of the model at  $h = 0$  contains two distinct pure phases (called the plus and the minus phase). Physically, the phase transition is characterized by the discontinuity of the specific magnetization  $m(h, \beta) = -m(-h, \beta)$ : whereas the one-sided limit<sup>1</sup>  $m(0^+, \beta) := \lim_{h \rightarrow 0^+} m(h, \beta)$  equals zero for  $0 < \beta < \beta_c$ , it is positive once  $\beta > \beta_c$ , i.e. the *spontaneous magnetization*  $m^* := m(0^+, \beta)$  appears at sub-critical temperatures.

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<sup>1</sup>This limit exists and is non-negative for all  $\beta > 0$ , see [15], for instance.

Equivalently, since the (specific Gibbs) free energy  $f(h, \beta)$  is a concave function of  $h$  for any  $\beta \geq 0$ , it has one-sided partial derivatives  $\frac{\partial f(h, \beta)}{\partial h^\pm}$  for all  $\beta \geq 0$  and  $h \in \mathbb{R}$ , and these do not coincide if and only if  $h = 0$  and  $\beta > \beta_c$ . Clearly,  $-\frac{\partial f(h, \beta)}{\partial h^\pm} = m(h^\pm, \beta)$ .

In general, any discontinuities that arise in a system exhibiting a first-order phase transition are smoothed out once the system is of a finite size. While the limiting free energy  $f$  (as well as its one-sided derivatives) does not depend on boundary conditions, its smoothed finite-volume version is heavily depending on particular boundary conditions. In [2, 3, 5] and [4] specific cases of periodic and free boundary conditions, respectively, were considered with a rather mild and well-controlled size dependence. Here we turn to the case of fixed (minus) boundary conditions. This is the case with a rather strong influence of the boundary conditions, and (as will be clarified later) one has to take into account the competing effects of boundary conditions and “long contours”.

Let  $\Lambda_L$  be the square in  $\mathbb{Z}^2$  centred at the origin whose side-length is  $L \in \mathbb{N}$ . In the present paper we examine the ferromagnetic nearest-neighbour Ising model in  $\Lambda_L$  with the minus boundary conditions and an external field  $h \in \mathbb{R}$  at a sub-critical temperature. Writing  $\sigma_L : \Lambda_L \rightarrow \{-1, 1\}$  for a configuration in  $\Lambda_L$ , the corresponding *Hamiltonian under the fixed minus boundary conditions* is given by

$$H_{L,h}(\sigma_L) = - \sum_{\substack{\langle x,y \rangle: \\ x,y \in \Lambda_L}} \sigma_x \sigma_y + \sum_{\substack{\langle x,y \rangle: \\ x \in \Lambda_L, y \in \Lambda_L^c}} \sigma_x - h |\Lambda_L| S_L(\sigma_L). \quad (\text{IV.1.2})$$

Here

$$S_L(\sigma_L) := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \sigma_x \quad (\text{IV.1.3})$$

is the *average spin* and  $\Lambda_L^c := \mathbb{Z}^2 \setminus \Lambda_L$ . The finite-volume *Gibbs measure* at the inverse temperature  $\beta$  associated with the Hamiltonian (IV.1.2) is

$$\mu_{L,h}(\sigma_L) := \frac{e^{-\beta H_{L,h}(\sigma_L)}}{Z_{L,h}} \quad (\text{IV.1.4})$$

with the *partition function*  $Z_{L,h} := \sum_{\sigma_L \in \{-1,1\}^{\Lambda_L}} e^{-\beta H_{L,h}(\sigma_L)}$ ; in order to avoid heavy notation, we abstain here and hereafter from referring explicitly to the dependence of the various quantities on  $\beta$ , and we stress that we always take fixed minus boundary conditions. We shall use  $\langle \cdot \rangle_{L,h}$  to denote the *expected value* with respect to  $\mu_{L,h}$  and  $P_{L,h}$  to denote the *distribution* of  $S_L$  under  $\mu_{L,h}$ .

Let  $\beta > \beta_c$ . If  $h \neq 0$ , boundary effects in the bulk of  $\Lambda_L$  disappear as  $\Lambda_L$  extends to the whole lattice  $\mathbb{Z}^d$  because there is a unique Gibbs measure in the infinite volume. Nevertheless, the asymptotic behaviour of the Ising system may become rather delicate once we

consider a magnetic field  $h_L$  which depends on  $L$  and decreases to zero as  $L \rightarrow \infty$ . This time, the boundary conditions could play an important role: while they force the system to be in the minus phase, a magnetic field  $h_L \rightarrow 0^+$  draws it toward the plus phase. The situation when the influence of the magnetic field  $h_L$  (a bulk effect) is comparable to that of the minus boundary conditions (a surface effect) is of particular interest; this requires  $h_L$  to be of the order  $1/L$ . Therefore, it is natural to consider  $h_L = B/L$ ,  $B \in \mathbb{R}$ . Schonmann and Shlosman [16] proved that there exists a unique point  $B_0 = B_0(\beta) > 0$  such that  $\mu_{L,B/L}$  converges weakly to the pure minus phase if  $B < B_0$ , while the limit is the pure plus phase if  $B > B_0$ . In both regimes, they investigated the exponential convergence of the average spin  $S_L$  under  $\mu_{L,B/L}$  at the *surface* rate. To this end, they established a ‘surface-order’ large-deviation principle valid at  $B = 0$ , extending the results obtained by Ioffe [11, 12]. Greenwood and Sun [10] pointed out (for any dimension  $d \geq 2$ ) how the large-deviation principles with  $B = 0$  and  $B \neq 0$  are related, and inspected the surface-rate exponential convergence of  $S_L$  under  $\mu_{L,B/L}$ , too.

The basic picture behind these results is as follows. Let  $\Lambda_L$  be large but finite. If  $B < B_0$ , the minus boundary conditions prevail, selecting the minus phase in the box  $\Lambda_L$ , and  $S_L$  converges exponentially to  $-m^* < 0$ . If  $B > B_0$ , however, the magnetic field has the dominant effect, and the plus phase is outweighing in the system. This time, the average spin converges exponentially to a point  $0 < m(B) < m^*$ , c.f. (IV.2.6), and a single droplet of the plus phase within  $\Lambda_L$  immersed into the minus phase is created. The most favourable shape of the droplet is not the usual equilibrium crystal (or Wulff) shape,<sup>2</sup> but rather its squeezed version (see [16]): whenever the droplet really appears, it necessarily touches the boundary of  $\Lambda_L$  along four equally long segments.

As a matter of fact, the droplet fluctuates around its deterministic Wulff shape. Accordingly, the macroscopic-scale separation of pure phases along the boundary of the equilibrium crystal shape is a subtle probabilistic problem. Its first rigorous study was done by Dobrushin, Kotecký, and Shlosman [8, 9] for the  $2d$  Ising model at very low temperatures, using the cluster expansion analysis. The main part of their results was extended to all sub-critical temperatures in a non-perturbative approach of Ioffe and Schonmann [13]. In particular, they gave explicit asymptotics on the probabilities of the deviation of  $S_L$  from  $-m^*$  at  $h = 0$  under the minus boundary conditions. For a recent review of main results of the rigorous microscopic theory of equilibrium crystal shapes, see [1].

<sup>2</sup>Roughly speaking, the Wulff shape is the one which minimizes the interfacial surface tension, assuming that its volume is fixed, see [8, 16] for instance.

## IV.2. Main Result

For any  $0 < \vartheta < \infty$ , let us consider the open interval

$$J_L(\vartheta) := \{h \in \mathbb{R} : |Lh - B_0| < \vartheta\}. \quad (\text{IV.2.1})$$

Our aim here is to examine, for any  $\beta > \beta_c$ , the asymptotic behaviour of the finite-volume specific magnetization

$$m_L(h, \beta) := \langle S_L \rangle_{L,h} = \frac{1}{\beta|\Lambda_L|} \frac{\partial}{\partial h} \log Z_{L,h} \quad (\text{IV.2.2})$$

and susceptibility

$$\chi_L(h, \beta) := \left\langle S_L^2 \right\rangle_{L,h} - (\langle S_L \rangle_{L,h})^2 = \frac{1}{(\beta|\Lambda_L|)^2} \frac{\partial^2}{\partial h^2} \log Z_{L,h} \quad (\text{IV.2.3})$$

on the interval  $J_L(\vartheta)$  with  $L \rightarrow \infty$ . The resulting asymptotics presented in Theorem IV.2.2 reflects the mentioned balance between the competing influences of the magnetic field and the minus boundary conditions in our model. First, however, relying on results from [16] and [10], we explicitly show the limiting values with properly scaled external field,  $h \sim 1/L$ .

**PROPOSITION IV.2.1.** *Let  $\beta > \beta_c$ ,  $B \in \mathbb{R}$ , and let  $\{h_L\}$  be a sequence of real numbers such that  $\lim_{L \rightarrow \infty} Lh_L = B$ . Then the limit*

$$\varphi(B) := \frac{1}{\beta} \lim_{L \rightarrow \infty} \frac{1}{L} \log \frac{Z_{L,h_L}}{Z_{L,0}} \quad (\text{IV.2.4})$$

*exists and does not depend on the sequence  $h_L$ , it is a convex continuous function, and there exists a single point  $B_0 = B_0(\beta) > 0$  at which  $\varphi$  is not differentiable. Moreover,*

$$\lim_{L \rightarrow \infty} m_L(h_L, \beta) = \varphi'(B) \quad (\text{IV.2.5})$$

*for every  $B \neq B_0$  and  $\varphi'$  is explicitly given as*

$$\varphi'(B) = \begin{cases} -m^* & \text{if } B < B_0, \\ m(B) = m^* - \kappa/(2B)^2 & \text{if } B > B_0, \end{cases} \quad (\text{IV.2.6})$$

*with  $\kappa = \kappa(\beta) \in (0, 4(m^*)^2)$ .*

The first part of Proposition IV.2.1 readily follows from the Varadhan lemma [6, 7] and Theorem 1 from [16], and can be found in [10] for the special case  $h_L = B/L$  with  $B \geq 0$ ; the rest of the proposition is then easy to verify. We present the proof in the next section. It will turn out there that the point  $B_0$  from this proposition coincides with the critical point  $B_0$  of [16] mentioned before — allowing thus to use the same symbol to denote it.

Let  $B^* := B_0 \left( \frac{1}{2} + \frac{\kappa}{16m^*(B_0)^2} \right)$ . Because  $\kappa < 4m^*(B_0)^2$ , one has  $B^* \in (B_0/2, 3B_0/4)$ ; it will be shown later that  $m(B^*) > -m^*$ , see the

remark after Theorem IV.3.2. Let us extend the function  $m$  defined on the interval  $(B_0, \infty)$  by (IV.2.6). Namely, we take

$$m_+(B) := \begin{cases} m(B) & \text{for } B \geq B^*, \\ m(B^*) & \text{for } B \leq B^*. \end{cases} \quad (\text{IV.2.7})$$

It is a continuous and non-decreasing function satisfying  $m(B^*) \leq m_+ < m^*$ . Introducing the shorthands

$$\bar{m}(B) := \frac{m_+(B) + (-m^*)}{2}, \quad \Delta m(B) := \frac{m_+(B) - (-m^*)}{2}, \quad (\text{IV.2.8})$$

and  $\Delta := \Delta m(B_0) > 0$ , we now formulate our main result.

**THEOREM IV.2.2.** *Let  $\beta > \beta_c$ ,  $0 < \vartheta < \infty$ , and  $0 < \delta < 1/4$ . There exists  $L_0 = L_0(\beta, \vartheta, \delta) < \infty$  such that for all  $L > L_0$  the following is true.*

(a) *The susceptibility  $\chi_L(h)$  attains its maximal value over the interval  $J_L(\vartheta)$  at a unique point  $h_\chi(L)$  (which does not depend on  $\vartheta$ ).*

(b) *The functions  $R_L^{(0)}$ ,  $R_L^{(1)}(h)$ , and  $R_L^{(2)}(h)$  defined by the equalities*

$$h_\chi(L) = (B_0 + R_L^{(0)})/L, \quad (\text{IV.2.9})$$

$$m_L(h, \beta) = \bar{m}(Lh) + \Delta m(Lh) \tanh[\beta \Delta (h - h_\chi(L))L^2] + R_L^{(1)}(h), \quad (\text{IV.2.10})$$

and

$$\chi_L(h, \beta) = (\Delta m(Lh))^2 \cosh^{-2}[\beta \Delta (h - h_\chi(L))L^2] + R_L^{(2)}(h), \quad (\text{IV.2.11})$$

satisfy the following bounds:

$$|R_L^{(0)}| \leq 3(B_0)^3 L^{-\delta} / \kappa \quad (\text{IV.2.12})$$

and

$$\sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)| \leq CL^{-\delta}, \quad k = 1, 2, \quad (\text{IV.2.13})$$

with a fixed finite constant  $C$  (not depending on  $\beta$ ,  $\vartheta$ ,  $\delta$ , and  $L$ ).

We divide the proof of Theorem IV.2.2 into two parts. First, we prove a weaker version of the above theorem in which it is only claimed that  $R_L^{(0)}$  as well as  $\sup_{h \in J_L(\vartheta)} R_L^{(i)}(h)$ ,  $i = 1, 2$ , vanish as  $L \rightarrow \infty$ ; this part is based on the large-deviation principle established in [16] and it is the content of Section IV.3. In particular, results from [16] yield explicit values for parameters  $B_0$  and  $\kappa$  above. In order to obtain then the explicit bounds (IV.2.12) and (IV.2.13), we employ the local-limit estimates from [13, 1, 9] and Theorem 7.4.3 from [17]; this second step is presented in Section IV.4.

It should be pointed out that the division of the proof into two parts is not necessary and it could be carried out solely with the help of the local-limit estimates. However, it seems to be more transparent to examine the problem by means of the large-deviation principle at the beginning and use the more precise information of the local-limit estimates only afterwards. Moreover, once the class of models for which the surface-order large-deviation principles are established is extended (at present it only contains the two-dimensional Ising model), the first part of the proof will be readily applicable, yielding a result similar to Theorem IV.3.3 below (see [14]).

Finally, let us notice that, using more detailed analysis of the errors in the surface-order large deviations of the two-dimensional Ising model [13, 1], one could expect that the upper bounds (IV.2.12) and (IV.2.13) may be improved to be of the order  $L^{-1/4} \log^2 L$ . However, one should not expect that, for  $\sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)|$ ,  $k = 1, 2$ , an improvement of the order over  $L^{-1/2}$  is possible. This follows from the fact that surface large-deviation rate function  $\mathcal{W}_B(m)$  introduced below in (IV.3.6) behaves (for  $B \geq B_0$ ) like  $(m - m(B))^2$  around its minimum at  $m(B)$ . This results in the bound (IV.4.40) below, and inspecting it one could argue that  $L^{1/2} \sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)| \rightarrow \infty$ ,  $k = 1, 2$ .

### IV.3. Magnetization and the Large-Deviation Rate Function

The aim of this section is to analyze the asymptotic behaviour (as  $L \rightarrow \infty$ ) of the magnetization  $m_L(h, \beta)$  and susceptibility  $\chi_L(h, \beta)$  when  $h \in J_L(\vartheta)$  and  $\beta > \beta_c$ , using the large-deviation principle and the related results from [16, 10]. First, let us fix some notation.

DEFINITION IV.3.1. Let  $I : \mathbb{R} \rightarrow [0, \infty]$  be a (lower semicontinuous) function with compact level sets<sup>3</sup>,  $I \not\equiv \infty$ , and let  $\{\varepsilon_n\}$  be a sequence of positive numbers such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . We say that a sequence  $\{P_n\}$  of probability measures on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , where  $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -field on  $\mathbb{R}$ , satisfies the *large-deviation principle* with the powers  $\{\varepsilon_n\}$  and the rate  $I$ , and write  $(P_n)^{\varepsilon_n} \rightarrow e^{-I}$ , if

$$\sup_G e^{-I} \leq \underline{\lim}_{n \rightarrow \infty} (P_n(G))^{\varepsilon_n} \quad \text{for all } G \subset \mathbb{R} \text{ open and} \quad (\text{IV.3.1})$$

$$\overline{\lim}_{n \rightarrow \infty} (P_n(F))^{\varepsilon_n} \leq \sup_F e^{-I} \quad \text{for all } F \subset \mathbb{R} \text{ closed.} \quad (\text{IV.3.2})$$

In the next theorem we gather up the results of Theorem 1 from [16] and Theorem 3.3 from [10]. To this end, we introduce  $\tau = \tau(\beta)$

<sup>3</sup>That is, the level sets  $\text{lev}_r(I) := \{x \in \mathbb{R} : I(x) \leq r\}$  are compact for all  $r < \infty$ . Such a function is automatically lower semi-continuous.

and  $w = w(\beta)$  to be the zero-field surface tension in the direction  $(0, 1)$  and the Wulff functional of the minimizing Wulff shape, respectively (see e.g., [16] for precise definitions). They satisfy the relations  $0 < 4\tau/3 < w < 4\tau$  for all  $\beta > \beta_c$ .

THEOREM IV.3.2 ([16], [10]). *Let  $\beta > \beta_c$ . Setting*

$$\kappa := \frac{16\tau^2 - w^2}{2m^*} > 0 \quad (\text{IV.3.3})$$

and

$$B_0 := \frac{4\tau + w}{4m^*}, \quad (\text{IV.3.4})$$

we have:

(1) Let  $m_t := -m^*(1 - \frac{w^2}{8\tau^2}) \in (-m^*, m^*)$  and

$$\mathcal{W}_0(m) := \begin{cases} w \left(\frac{m+m^*}{2m^*}\right)^{1/2} & \text{if } -m^* \leq m \leq m_t, \\ 4\tau - [\kappa(m^* - m)]^{1/2} & \text{if } m_t \leq m \leq m^*, \\ \infty & \text{otherwise.} \end{cases} \quad (\text{IV.3.5})$$

Then  $(P_{L,0})^{1/L} \rightarrow e^{-\beta\mathcal{W}_0}$ .

(2) Let

$$\mathcal{W}_B(m) := \mathcal{W}_0(m) - Bm + \mathcal{W}_0^*(B), \quad (\text{IV.3.6})$$

where  $\mathcal{W}_0^*$  is the Legendre-Fenchel transform of  $\mathcal{W}_0$ . Let  $\{h_L\}$ ,  $h_L \in \mathbb{R}$ , be a sequence with  $\lim_{L \rightarrow \infty} Lh_L = B \in \mathbb{R}$ .<sup>4</sup> Then  $(P_{L,h_L})^{1/L} \rightarrow e^{-\beta\mathcal{W}_B}$ .

(3) The derivative of the Legendre-Fenchel transform  $\mathcal{W}_0^*$  of  $\mathcal{W}_0$  has a unique discontinuity at  $B_0$ . Moreover, for all  $B \neq B_0$ , the equation  $\mathcal{W}_B(m) = 0$  has a unique solution that equals the derivative  $\frac{d\mathcal{W}_0^*}{dB}$  of  $\mathcal{W}_0^*$ , while for  $B = B_0$  it has two solutions:  $\frac{d\mathcal{W}_0^*}{dB^+}$  and  $\frac{d\mathcal{W}_0^*}{dB^-}$ .

Proposition IV.2.1 is a quite direct consequence of Theorem IV.3.2. In particular, it turns out that one can identify the function  $\varphi$  with  $\mathcal{W}_0^*$  (this was anticipated by denoting  $B_0$  the discontinuity point of both of them). Notice also that the constant  $B^* := B_0(\frac{1}{2} + \frac{\kappa}{16m^*(B_0)^2})$  defined in the previous section actually coincides with  $\tau/m^*$ . Moreover, one has  $m(B^*) = m_t > -m^*$ .

PROOF OF PROPOSITION IV.2.1. Let  $\beta > \beta_c$  and  $B \in \mathbb{R}$ . Consider the limit

$$\psi(B) = \frac{1}{\beta} \lim_{L \rightarrow \infty} \frac{1}{L} \log \left\langle e^{\beta B L S_L} \right\rangle_{L,0}. \quad (\text{IV.3.7})$$

<sup>4</sup>In fact, Theorem 3.3 of [10] only deals with  $h_L = B/L$ , where  $B > 0$ . It is clear, however, that the arguments used there work in our slightly more general case as well.

In view of the Varadhan lemma<sup>5</sup> [6, 7] and Theorem IV.3.2 (1), the limit exists and we get

$$\psi(B) = \frac{1}{\beta} \sup \{ \beta Bx - \beta \mathcal{W}_0(x) \} = \mathcal{W}_0^*(B). \quad (\text{IV.3.8})$$

With the help of (IV.3.5), one may easily find [10] that

$$\psi(B) = \begin{cases} -m^*B & \text{if } B \leq B_0, \\ m^*B - [4\tau - \kappa/(4B)] & \text{if } B \geq B_0, \end{cases} \quad (\text{IV.3.9})$$

where  $B_0 = (4\tau + w)/(4m^*)$ . This point coincides with the critical point  $B_0$  of [16], see Theorem 2 stated therein.

We now show that the functions  $\varphi$  from (IV.2.4) and  $\psi$  defined above actually coincide. Let thus  $\{h_L\}$ ,  $h_L \in \mathbb{R}$ , be an arbitrary sequence such that  $\lim_{L \rightarrow \infty} Lh_L = B$ . Since

$$\frac{Z_{L,h_L}}{Z_{L,0}} = \sum_{\sigma_L \in \{-1,1\}^{\Lambda_L}} e^{\beta h_L |\wedge_L| S_L(\sigma_L)} \mu_{L,0}(\sigma_L) = \left\langle e^{\beta h_L |\wedge_L| S_L} \right\rangle_{L,0} \quad (\text{IV.3.10})$$

and the range of the average spin  $S_L$  is contained, by definition, in the interval  $[-1, 1]$ , we may evaluate

$$e^{-\beta |Lh_L - B|L} \left\langle e^{\beta BLS_L} \right\rangle_{L,0} \leq \left\langle e^{\beta h_L |\wedge_L| S_L} \right\rangle_{L,0} \leq e^{\beta |Lh_L - B|L} \left\langle e^{\beta BLS_L} \right\rangle_{L,0}. \quad (\text{IV.3.11})$$

As a result,

$$\frac{1}{\beta} \lim_{L \rightarrow \infty} \frac{1}{L} \log \frac{Z_{L,h_L}}{Z_{L,0}} = \psi(B). \quad (\text{IV.3.12})$$

Thus, one has  $\psi(B) = \varphi(B)$ , and in order to verify (IV.2.5), we notice that  $m_L(h_L, \beta) = \langle S_L \rangle_{L,h_L} = \frac{1}{\beta} \frac{1}{L} \frac{d \log \langle e^{\beta BLS_L} \rangle_{L,0}}{dB} \Big|_{B=Lh_L}$ , getting the limit whenever the derivative of  $\varphi$  exists. Q.E.D.

The main result of the section is this simplified version of Theorem IV.2.2.

**THEOREM IV.3.3.** *Let  $\beta > \beta_c$  and  $0 < \vartheta < \infty$ . There exists  $L_0 = L_0(\beta, \vartheta) < \infty$  such that for all  $L > L_0$  the claims a) and b) of Theorem IV.2.2 hold with (IV.2.12) and (IV.2.13) replaced by*

$$\lim_{L \rightarrow \infty} R_L^{(0)} = 0 \quad \text{and} \quad \lim_{L \rightarrow \infty} \sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)| = 0, \quad k = 1, 2, \quad (\text{IV.3.13})$$

respectively.

<sup>5</sup>Notice that  $\lim_{M \rightarrow \infty} \overline{\lim}_{L \rightarrow \infty} \frac{1}{L} \log \sum_{m \in \text{Ran } S_L: \beta Bx \geq M} e^{\beta BLm} P_{L,0}(m) = -\infty$  because the range  $\text{Ran } S_L$  of average spin  $S_L$  is bounded,  $\text{Ran } S_L \subset [-1, 1]$ . This allows us to apply an extended version of Varadhan lemma [6], Theorem 4.3.1.

**IV.3.1. Proof of Theorem IV.3.3.** Let  $\beta > \beta_c$  and  $L \in \mathbb{N}$ . Given  $h \in \mathbb{R}$  and a set  $A \in \mathcal{B}(\mathbb{R})$  (which may depend on  $h$ ) such that

$$Z_{L,h}(A) := \sum_{\substack{\sigma_L \in \Omega_L: \\ S_L(\sigma_L) \in A}} e^{-\beta H_{L,h}(\sigma_L)} > 0, \quad (\text{IV.3.14})$$

we define

$$\langle \cdot | A \rangle_{L,h} := \sum_{\substack{\sigma_L \in \Omega_L: \\ S_L(\sigma_L) \in A}} \cdot \frac{e^{-\beta H_{L,h}(\sigma_L)}}{Z_{L,h}(A)}. \quad (\text{IV.3.15})$$

In order to control the most relevant contributions to the partition function on  $J_L(\vartheta)$ ,  $\vartheta > 0$ , we split — independently of  $L$  — the interval  $J_L(\vartheta)$  into a *finite* number of disjoint sub-intervals as follows. Let  $\epsilon \in (0, \epsilon_0(\vartheta))$  with

$$\epsilon_0(\vartheta) := 2 \min\{m(B_0 + \vartheta) - m(B_0), m(B_0) - m(B^*), \Delta m(B^*)/4\} \quad (\text{IV.3.16})$$

and let us consider the sequence  $\{m_i = m_0 + i\epsilon\}$ ,  $i \in \mathbb{Z}$ , where  $m_0 = m_+(B_0) - \frac{\epsilon}{2}$ . As the function  $m_+$  is bounded, there clearly exist unique natural numbers  $N_j = N_j(\beta, \vartheta, \epsilon)$ ,  $j = 1, 2$ , for which  $m_+(B_0 - \vartheta) \in [m_{-N_1}, m_{-N_1+1})$  and  $m_+(B_0 + \vartheta) \in [m_{N_2}, m_{N_2+1})$ . Let us consider now the sequence  $B^{(i)}$  with  $B^{(-N_1)} = B_0 - \vartheta$ ,  $B^{(N_2+1)} = B_0 + \vartheta$ , and  $B^{(i)}$  for  $i = -N_1 + 1, \dots, N_1$  taken as the unique solution of the equation  $m_+(B^{(i)}) = m_i$ .<sup>6</sup> We split the interval  $J_L(\vartheta)$ ,

$$J_L(\vartheta) = \bigcup_{i=-N_1}^{N_2} \mathcal{I}_{L,i}^{(\epsilon)}, \quad (\text{IV.3.17})$$

by taking

$$\mathcal{I}_{L,i}^{(\epsilon)} := \begin{cases} (B^{(i)}/L, B^{(i+1)}/L] & \text{if } i = -N_1, \dots, -1, \\ (B^{(0)}/L, B^{(1)}/L) & \text{for } i = 0, \\ [B^{(i)}/L, B^{(i+1)}/L) & \text{if } i = 1, \dots, N_2. \end{cases} \quad (\text{IV.3.18})$$

Moreover, introducing

$$\mathcal{C}^-(\epsilon) := (-m^* - \epsilon, -m^* + \epsilon), \quad (\text{IV.3.19})$$

and

$$\mathcal{C}^+(Lh, \epsilon) := (m_i - \epsilon, m_{i+1} + \epsilon) \quad \text{for any } h \in \mathcal{I}_{L,i}^{(\epsilon)}, \quad (\text{IV.3.20})$$

we have

$$|\langle S_L | \mathcal{C}^+ \rangle_{L,h} - m_+(Lh)| \leq 2\epsilon \quad \text{and} \quad |\langle S_L | \mathcal{C}^- \rangle_{L,h} - (-m^*)| \leq \epsilon \quad (\text{IV.3.21})$$

for every  $h \in J_L(\vartheta)$ .

<sup>6</sup>Since  $\epsilon < \epsilon_0(\vartheta)$ , it follows that  $B^* < B^{(0)} < B_0 < B^{(1)} < B_0 + \vartheta$ .

Taking  $\mathcal{C}(Lh, \epsilon) := \mathcal{C}^+(Lh, \epsilon) \cup \mathcal{C}^-(\epsilon)$ , we prove Theorem IV.3.3 with the help of the following sequence of lemmas.<sup>7</sup>

LEMMA IV.3.4. *Let  $\beta > \beta_c$ ,  $\vartheta > 0$ , and  $0 < \epsilon < \epsilon_0(\vartheta)$ . For any  $L > \epsilon^{-1/2}$  and  $h \in J_L(\vartheta)$ , we have*

$$|\langle S_L \rangle_{L,h} - T(\phi^{(\epsilon)}(h); \bar{m}(Lh), \Delta m(Lh))| \leq 2P_{L,h}(\mathcal{C}^c) + 3\epsilon. \quad (\text{IV.3.22})$$

Here

$$T(x; a, b) := a + b \tanh x, \quad x, a, b \in \mathbb{R}, \quad (\text{IV.3.23})$$

and

$$\phi^{(L,\epsilon)}(h) := \frac{1}{2} \log \frac{Z_{L,h}(\mathcal{C}^+)}{Z_{L,h}(\mathcal{C}^-)} = \frac{1}{2} \log \frac{P_{L,h}(\mathcal{C}^+)}{P_{L,h}(\mathcal{C}^-)}. \quad (\text{IV.3.24})$$

PROOF. Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $\epsilon < \epsilon_0(\vartheta)$ , and  $L > \epsilon^{-1/2}$  be given. Let  $h \in J_L(\vartheta)$  be arbitrary. Evidently,

$$\langle S_L \rangle_{L,h} = \langle S_L | \mathcal{C} \rangle_{L,h} P_{L,h}(\mathcal{C}) + \langle S_L | \mathcal{C}^c \rangle_{L,h} P_{L,h}(\mathcal{C}^c) = \quad (\text{IV.3.25})$$

$$= \langle S_L | \mathcal{C} \rangle_{L,h} + (\langle S_L | \mathcal{C}^c \rangle_{L,h} - \langle S_L | \mathcal{C} \rangle_{L,h}) P_{L,h}(\mathcal{C}^c). \quad (\text{IV.3.26})$$

Thus, using that  $|S_L| \leq 1$ , one has

$$|\langle S_L \rangle_{L,h} - \langle S_L | \mathcal{C} \rangle_{L,h}| \leq 2P_{L,h}(\mathcal{C}^c). \quad (\text{IV.3.27})$$

Observing that  $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$  (since  $\epsilon < \Delta m(B^*)/2$ ) and  $Z_{L,h}(\mathcal{C}^\pm) > 0$  (since  $L > \epsilon^{-1/2}$ ), we readily get

$$\begin{aligned} \langle S_L | \mathcal{C} \rangle_{L,h} &= \frac{\langle S_L | \mathcal{C}^+ \rangle_{L,h} Z_{L,h}(\mathcal{C}^+) + \langle S_L | \mathcal{C}^- \rangle_{L,h} Z_{L,h}(\mathcal{C}^-)}{Z_{L,h}(\mathcal{C}^+) + Z_{L,h}(\mathcal{C}^-)} = \\ &= \frac{\langle S_L | \mathcal{C}^+ \rangle_{L,h} + \langle S_L | \mathcal{C}^- \rangle_{L,h}}{2} + \\ &\quad + \frac{\langle S_L | \mathcal{C}^+ \rangle_{L,h} - \langle S_L | \mathcal{C}^- \rangle_{L,h}}{2} \tanh(\phi^{(\epsilon)}(h)). \end{aligned} \quad (\text{IV.3.28})$$

Since  $|\tanh x| \leq 1$  for all  $x \in \mathbb{R}$ , in view of (IV.3.21) it follows that

$$|\langle S_L | \mathcal{C} \rangle_{L,h} - T(\phi^{(\epsilon)}(h); \bar{m}(Lh), \Delta m(Lh))| \leq 3\epsilon. \quad (\text{IV.3.29})$$

Combined with (IV.3.27), we obtain the lemma. Q.E.D.

The next lemma provides bounds on the derivatives of  $\langle S_L \rangle_{L,h}$  analogous to that from (IV.3.22). To this end, we start with the following definition.

<sup>7</sup>The fact that  $\mathcal{C}^+$  and  $\mathcal{C}^-$  are open is not important: the arguments of the proof also work if these are closed or half-open.

DEFINITION IV.3.5. Let  $k \in \mathbb{N}$ . Given a set  $\{g_1, \dots, g_k\}$  of  $k$  real-valued functions in  $\mathbb{R}$ , we introduce

$$\mathcal{F}_k(\{g_j\}) := \sum_{j=1}^k (-1)^{j-1} (j-1)! \sum_{\{I_1, \dots, I_j\}} \prod_{\ell=1}^j g_{|I_\ell|}, \quad (\text{IV.3.30})$$

where the second sum is over all partitions  $\{I_1, \dots, I_j\}$ ,  $j = 1, \dots, k$ , of the set  $\{1, \dots, k\}$  and  $|I_\ell|$ ,  $\ell = 1, \dots, j$ , is the cardinality of  $I_\ell$ .

LEMMA IV.3.6. *There exist finite constants  $C_k, K_k$ ,  $k \in \mathbb{N}$ , such that if  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $0 < \epsilon < \epsilon_0(\vartheta)$ , and  $L > \epsilon^{-1/2}$ , then*

$$\left| \frac{1}{(\beta|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathcal{F}_k(\{T(\phi^{(\epsilon)}(h); \xi_j^+, \xi_j^-)\}) \right| \leq C_k P_{L,h}(\mathcal{C}^c) + K_k \epsilon \quad (\text{IV.3.31})$$

for all  $h \in J_L(\vartheta)$ . Here

$$\xi_j^\pm \equiv \frac{(m_+(Lh))^j \pm (-m^*)^j}{2}, \quad j = 1, \dots, k. \quad (\text{IV.3.32})$$

PROOF. In order to prove (IV.3.31), it suffices — by Lemma IV.A.2 — to show that for any  $k \in \mathbb{N}$  there exists a finite positive constant  $K_k$  such that

$$\left| \mathcal{F}_k(\{\langle (S_L)^j | \mathcal{C} \rangle_{L,h}\}) - \mathcal{F}_k(\{T(\phi^{(\epsilon)}(h); \xi_j^+, \xi_j^-)\}) \right| \leq K_k \epsilon. \quad (\text{IV.3.33})$$

Indeed, taking into account (IV.3.15) and (IV.3.19) (cf. (IV.3.21)), we have

$$\begin{aligned} & \left| \langle (S_L)^n | \mathcal{C}^+ \rangle_{L,h} - (m_+(Lh))^n \right| \leq \\ & \leq \sum_{\substack{\sigma_L \in \Omega_L: \\ S_L(\sigma_L) \in \mathcal{C}^+}} |(S_L(\sigma_L))^n - (m_+(Lh))^n| \frac{e^{-\beta H_{L,h}(\sigma_L)}}{Z_{L,h}(\mathcal{C}^+)} \leq \\ & \leq \sum_{\substack{\sigma_L \in \Omega_L: \\ S_L(\sigma_L) \in \mathcal{C}^+}} |S_L(\sigma_L) - m_+(Lh)| \times \\ & \times \sum_{r=0}^{n-1} |S_L(\sigma_L)|^r |m_+(Lh)|^{n-r-1} \frac{e^{-\beta H_{L,h}(\sigma_L)}}{Z_{L,h}(\mathcal{C}^+)} \leq \tilde{K}_n \epsilon \quad (\text{IV.3.34}) \end{aligned}$$

for all  $n \in \mathbb{N}$  and  $\tilde{K}_n = 2n$ . Similarly,  $|\langle (S_L)^n | \mathcal{C}^- \rangle - (-m^*)^n| \leq \tilde{K}_n \epsilon$  for all  $n \in \mathbb{N}$  with the same constant  $\tilde{K}_n$ . Combined with the equality

$$\langle (S_L)^n | \mathcal{C} \rangle_{L,h} = \frac{\langle (S_L)^n | \mathcal{C}^+ \rangle_{L,h} Z_{L,h}(\mathcal{C}^+) + \langle (S_L)^n | \mathcal{C}^- \rangle_{L,h} Z_{L,h}(\mathcal{C}^-)}{Z_{L,h}(\mathcal{C}^+) + Z_{L,h}(\mathcal{C}^-)}, \quad (\text{IV.3.35})$$

$n \in \mathbb{N}$ , valid once  $\epsilon < \Delta m(B^*)/2$  and  $L > \epsilon^{-1/2}$ , referring to (IV.3.28) it follows that

$$\left| \langle (S_L)^n | \mathcal{C} \rangle_{L,h} - T(\phi^{(\epsilon)}(h); \xi_n^+, \xi_n^-) \right| \leq \tilde{K}_n \epsilon \quad (\text{IV.3.36})$$

for all  $n \in \mathbb{N}$ .

Let now  $k \in \mathbb{N}$ . For any  $j = 1, \dots, k$  and any partition  $\{I_1, \dots, I_j\}$  one gets

$$\begin{aligned} \prod_{\ell=1}^j \langle (S_L)^{|I_\ell|} | \mathcal{C} \rangle_{L,h} &= \prod_{\ell=1}^j T(\phi^{(\epsilon)}(h); \xi_{|I_\ell|}^+, \xi_{|I_\ell|}^-) + \\ &+ \sum_{\substack{X \subset \{1, \dots, j\}: \\ X \neq \{1, \dots, j\}}} \prod_{r \in X} T(\phi^{(\epsilon)}(h); \xi_{|I_r|}^+, \xi_{|I_r|}^-) \times \\ &\times \prod_{s \in \{1, \dots, j\} \setminus X} \left( \langle (S_L)^{|I_s|} | \mathcal{C} \rangle_{L,h} - T(\phi^{(\epsilon)}(h); \xi_{|I_s|}^+, \xi_{|I_s|}^-) \right). \end{aligned} \quad (\text{IV.3.37})$$

By virtue of (IV.3.36), the obvious bound  $|\xi_n^\pm| \leq 1$  for any  $n \in \mathbb{N}$ , and the fact that  $\epsilon \leq \Delta m(B^*)/2 \leq 1$ , we arrive at (IV.3.33). Q.E.D.

Next, let us examine the behaviour of the  $\phi^{(L,\epsilon)}(h)$  defined by (IV.3.24).

LEMMA IV.3.7. *Let  $\beta > \beta_c$ ,  $\vartheta > 0$ , and  $0 < \epsilon < \epsilon_0(\vartheta)$ .*

- (a) *Let  $L > \epsilon^{-1/2}$ . Then the function  $\phi^{(L,\epsilon)}(h)$  is finite on  $J_L(\vartheta)$ . In addition, it is analytic and increasing on  $\mathcal{I}_{L,i}^{(\epsilon)}$  for each  $i = -N_1, \dots, N_2$ .*
- (b) *There exists a constant  $L_1 = L_1(\beta, \vartheta, \epsilon) < \infty$ ,  $L_1 \geq \epsilon^{-1/2}$ , such that for  $L > L_1$  the function  $\phi^{(L,\epsilon)}(h)$  vanishes inside  $J_L(\vartheta)$  at a unique point  $h_0(L, \epsilon)$ . Moreover, the limit  $\lim_{L \rightarrow \infty} Lh_0(L, \epsilon)$  exists, it is independent of  $\epsilon$ , and<sup>8</sup>*

$$\lim_{L \rightarrow \infty} Lh_0(L, \epsilon) = B_0. \quad (\text{IV.3.38})$$

- (c) *Let<sup>9</sup>*

$$\omega(h) := \beta (h - h_0(L, \epsilon)) |\Lambda_L| \Delta \quad (\text{IV.3.39})$$

*There exist finite constants  $L_2 = L_2(\beta, \vartheta, \epsilon)$ ,  $M_k$ ,  $k \in \mathbb{N}$ , such that if  $L > L_2$ , then*

$$\begin{aligned} \left| \mathcal{F}_k(\{T(\phi^{(L,\epsilon)}(h); \xi_j^+, \xi_j^-)\}) - \mathcal{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-)\}) \right| &\leq \\ &\leq M_k \left( |\Delta m(Lh_0(L, \epsilon) + L^{-1/2}) - \Delta| + \epsilon \right) / \Delta \end{aligned} \quad (\text{IV.3.40})$$

<sup>8</sup>Notice that (IV.3.38) implies  $(P_{L,h_0(L,\epsilon)})^{1/L} \rightarrow e^{-\beta \mathcal{W}_{B_0}}$  by Theorem IV.3.2 (2), where  $\mathcal{W}_{B_0}$  is given by (IV.3.6).

<sup>9</sup>Recall that  $\Delta = \Delta m(B_0)$ .

for all  $h \in J_L(\vartheta)$  and  $k \in \mathbb{N}$ .

PROOF. Let  $\beta > \beta_c$ ,  $\vartheta > 0$ , and  $0 < \epsilon < \epsilon_0(\vartheta)$ .

(a) Let  $L > \epsilon^{-1/2}$ . First, if  $h \in J_L(\vartheta)$ , then  $Z_{L,h}(\mathcal{C}^\pm) > 0$  (since  $L > \epsilon^{-1/2}$ ), and  $\phi^{(L,\epsilon)}(h)$  is finite by its very definition (IV.3.24). Second, let  $i = -N_1, \dots, N_2$  and let us consider the interval  $\mathcal{I}_{L,i}^{(\epsilon)}$ . By its definition, the set  $\mathcal{C}^+$  is independent of  $h$  on  $\mathcal{I}_{L,i}^{(\epsilon)}$ , and, hence, the function  $Z_{L,h}(\mathcal{C}^+)$  is analytic in  $h$  on  $\mathcal{I}_{L,i}^{(\epsilon)}$ . The same is thus true for the function  $\phi^{(L,\epsilon)}$ . Moreover, for any  $h$  in the interior of  $\mathcal{I}_{L,i}^{(\epsilon)}$  one has

$$\begin{aligned} \frac{1}{\beta|\Lambda_L|} \frac{\partial \phi^{(L,\epsilon)}(h)}{\partial h} &= \frac{\langle S_L | \mathcal{C}^+ \rangle_{L,h} - \langle S_L | \mathcal{C}^- \rangle_{L,h}}{2} \geq \\ &\geq \frac{m_+(Lh) + m^*}{2} - \frac{3}{2} \epsilon > \frac{\epsilon_0(\vartheta)}{2} \end{aligned} \quad (\text{IV.3.41})$$

with the help of (IV.3.21). Thus, the function  $\phi^{(L,\epsilon)}$  is increasing on  $\mathcal{I}_{L,i}^{(\epsilon)}$ .

(b) Let  $i = -N_1, \dots, -1$ . Then  $(P_{L,B^{(i+1)}}/L)^{1/L} \rightarrow e^{-\beta \mathcal{W}_{B^{(i+1)}}$  in view of Theorem IV.3.2 (2). Observing further that, due to Theorem IV.3.2 (3), one has  $\inf_{m \in \mathcal{C}^-} \mathcal{W}_B(m) = 0$  whenever  $B < B_0$ , we get

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{L} \phi^{(L,\epsilon)}(B^{(i+1)}/L) &= -\frac{\beta}{2} \inf_{x \in [m_i - \epsilon, m_{i+1} + \epsilon]} \mathcal{W}_{B^{(i+1)}}(x) = \\ &= -\frac{\beta}{2} \mathcal{W}_{B^{(i+1)}}(x_i) \end{aligned} \quad (\text{IV.3.42})$$

for some  $x_i \in [m_i - \epsilon, m_{i+1} + \epsilon]$ . Taking into account the bounds

$$m_i - \epsilon > m_+(B_0 - \vartheta) - 2\epsilon \geq -m^* + 4\epsilon_0(\vartheta) - 2\epsilon > -m^* + 2\epsilon_0(\vartheta) \quad (\text{IV.3.43})$$

as well as the fact that  $B^{(i+1)} < B_0$ , we can use Theorem IV.3.2 (3) once more to get  $\mathcal{W}_{B^{(i+1)}}(x) > 0$  and thus

$$\phi^{(L,\epsilon)}(B^{(i+1)}/L) < -\beta L \mathcal{W}_{B^{(i+1)}}(x_i)/4 < 0 \quad (\text{IV.3.44})$$

once  $L$  is sufficiently large (depending on  $\beta$ ,  $\vartheta$ ,  $\epsilon$ , and  $i$ ). In a similar way,

$$\phi^{(L,\epsilon)}(B^{(i)}/L) > \beta L \mathcal{W}_{B^{(i)}}(x_i)/4 > 0 \quad (\text{IV.3.45})$$

for any  $i = 1, \dots, N_2$  and some  $x_i \in [m^* - \epsilon, m^* + \epsilon]$  once  $L$  is sufficiently large (depending on  $\beta$ ,  $\vartheta$ ,  $\epsilon$ , and  $i$ ). Referring to the fact that  $\phi^{(L,\epsilon)}$  is increasing on  $\mathcal{I}_{L,i}^{(\epsilon)}$  for every  $i = -N_1, \dots, N_2$  by the claim (a) of this lemma and that  $N_1 + N_2$  is finite, one concludes that  $\phi^{(L,\epsilon)}(h) \neq 0$  for all  $h \in J_L(\vartheta) \setminus \mathcal{I}_{L,0}^{(\epsilon)}$  as soon as  $L$  is large enough (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ).

Let  $h \in \mathcal{I}_{L,0}^{(\epsilon)}$  now. According to the mean-value theorem,

$$\phi^{(L,\epsilon)}(h) = \phi^{(L,\epsilon)}(B_0/L) + (h - B_0/L) \frac{\partial \phi^{(L,\epsilon)}(\bar{h})}{\partial h} \quad (\text{IV.3.46})$$

for some  $\bar{h}$  between  $h$  and  $B_0/L$ . Since  $\mathcal{W}_{B_0}(m_+(B_0)) = 0$  and thus  $\inf_{m \in \mathcal{C}^+} \mathcal{W}_{B_0}(m) = 0$ , we get

$$\lim_{L \rightarrow \infty} \frac{\phi^{(L,\epsilon)}(B_0/L)}{L} = \frac{1}{2} \lim_{L \rightarrow \infty} \frac{1}{L} \log \frac{P_{L,B_0/L}(\mathcal{C}^+)}{P_{L,B_0/L}(\mathcal{C}^-)} = 0 \quad (\text{IV.3.47})$$

according to Theorem IV.3.2 (3). With the help of (IV.3.46) we thus get

$$\begin{aligned} \phi^{(L,\epsilon)}(h) &\leq \phi^{(L,\epsilon)}(B_0/L) - \frac{B_0 - B^{(0)}}{2L} \frac{\beta |\wedge_L| \epsilon_0(\vartheta)}{2} < \\ &< L \left( \frac{\phi^{(L,\epsilon)}(B_0/L)}{L} - \frac{\beta}{4} (B_0 - B^{(0)}) \epsilon_0(\vartheta) \right) < 0 \end{aligned} \quad (\text{IV.3.48})$$

for any  $h \in \mathcal{I}_{L,0}^{(\epsilon)}$  such that  $h \leq (B_0 + B^{(0)})/(2L)$  once  $L$  is large enough (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ). Analogously, one proves that  $\phi^{(L,\epsilon)}(h) > 0$  for any  $h \in \mathcal{I}_{L,0}^{(\epsilon)}$  such that  $h \geq (B_0 + B^{(1)})/(2L)$  if  $L$  is sufficiently large (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ). Since  $\phi^{(L,\epsilon)}$  is continuous (it is analytic) and increasing on  $\mathcal{I}_{L,0}^{(\epsilon)}$  for  $L > \epsilon^{-1/2}$ , this means that, for  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ), a unique point  $h_0(L, \epsilon)$  at which  $\phi^{(L,\epsilon)}(h_0(L, \epsilon)) = 0$  exists, and

$$(B_0 + B^{(0)})/2 < Lh_0(L, \epsilon) < (B_0 + B^{(1)})/2. \quad (\text{IV.3.49})$$

Moreover, the relation (IV.3.46) with  $h = h_0(L, \epsilon)$  along with (IV.3.41) and (IV.3.47) readily implies (IV.3.38).

(c) Let  $i = -N_1, \dots, N_2$  be such that  $i \neq 0$  and let  $h_i \in \mathcal{I}_{L,i}^{(\epsilon)}$ . Using that  $\phi^{(L,\epsilon)}$  is increasing on  $\mathcal{I}_{L,i}^{(\epsilon)}$  for  $L > L_1$  and recalling the bounds (IV.3.44) and (IV.3.45) valid for  $L > L_1$ , we get  $|\phi^{(L,\epsilon)}(h_i)| \geq \beta L \alpha_i(\beta, \vartheta, \epsilon)/4$  for  $L > L_1$ , where  $\alpha_i > 0$  stands for  $\mathcal{W}_{B^{(i+1)}}(x_i)$  if  $i \leq -1$  or  $\mathcal{W}_{B^{(i)}}(x_i)$  if  $i \geq 1$ . Moreover,

$$\begin{aligned} |\omega(h_i)| &\geq \beta \min\{Lh_0(L, \epsilon) - B^{(0)}, B^{(1)} - Lh_0(L, \epsilon)\} L\Delta > \\ &> \beta \min\{B_0 - B^{(0)}, B^{(1)} - B_0\} \frac{L}{2} \Delta \end{aligned} \quad (\text{IV.3.50})$$

for  $L > L_1$  by (IV.3.49). Hence, since  $1 - 2e^{-2|x|} \leq \tanh|x| \leq 1$ , we have

$$\begin{aligned} |\tanh(\phi^{(L,\epsilon)}(h_i)) - \tanh(\omega(h_i))| &\leq 2e^{-2 \min\{|\phi^{(L,\epsilon)}(h_i)|, |\omega(h_i)|\}} \leq \\ &\leq 2e^{-\beta L \alpha(\beta, \vartheta, \epsilon)/2} \end{aligned} \quad (\text{IV.3.51})$$

for  $L$  larger than some  $\tilde{L}_2 = \tilde{L}_2(\beta, \vartheta, \epsilon)$ ,  $\tilde{L}_2 \geq L_1$ , with

$$\alpha(\beta, \vartheta, \epsilon) := \min \left\{ \min_{\substack{-N_1 \leq i \leq N_2 \\ i \neq 0}} \alpha_i(\beta, \vartheta, \epsilon), \right. \quad (\text{IV.3.52})$$

$$\left. 2 \min\{B_0 - B^{(0)}, B^{(1)} - B_0\} \Delta \right\}, \quad (\text{IV.3.53})$$

which is positive. Now, let us consider  $h \in \mathcal{I}_{L,0}^{(\epsilon)}$  and take  $L$  so large that  $L \geq \tilde{L}_2$  and

$$\mathfrak{J}_L := \{h' \in \mathbb{R} : |h' - h_0(L, \epsilon)| L \leq L^{-1/2}\} \subset \mathcal{I}_{L,0}^{(\epsilon)}. \quad (\text{IV.3.54})$$

If  $h \in \mathcal{I}_{L,0}^{(\epsilon)} \setminus \mathfrak{J}_L$ , then  $|\omega(h)| \geq \beta \Delta L^{1/2}$  and

$$|\phi^{(L,\epsilon)}(h)| = |h - h_0(L, \epsilon)| \frac{\partial \phi^{(L,\epsilon)}(\hat{h})}{\partial h} \geq \beta \epsilon_0 L^{1/2} / 2 \quad (\text{IV.3.55})$$

according to the mean-value theorem and (IV.3.41) (here  $\hat{h}$  is some point between  $h$  and  $h_0(L)$ ). Consequently,

$$\begin{aligned} |\tanh(\phi^{(L,\epsilon)}(h)) - \tanh(\omega(h))| &\leq 2e^{-2 \min\{|\phi^{(L,\epsilon)}(h)|, |\omega(h)|\}} \leq \\ &\leq 2e^{-\beta \epsilon_0 L^{1/2}}. \end{aligned} \quad (\text{IV.3.56})$$

On the other hand, if  $h \in \mathfrak{J}_L$ , then Lemma IV.A.3, an upper bound similar to (IV.3.41) (c.f. (IV.3.21)), and the mean-value theorem yield

$$\begin{aligned} |\tanh(\phi^{(L,\epsilon)}(h)) - \tanh(\omega(h))| &\leq \\ &\leq \frac{|\tanh(\omega(h))|}{|\omega(h)|} |\phi^{(L,\epsilon)}(h) - \omega(h)| \leq \\ &\leq \frac{1}{|\omega(h)|} \beta |h - h_0(L, \epsilon)| |\Lambda_L| \left| \frac{1}{\beta |\Lambda_L|} \frac{\partial \phi^{(L,\epsilon)}(\hat{h})}{\partial h} - \Delta \right| \leq \\ &\leq \frac{1}{\Delta} (|\Delta m(Lh) - \Delta| + \frac{3}{2} \epsilon) \leq \\ &\leq \frac{1}{\Delta} (|\Delta m(Lh_0(L, \epsilon)) + L^{-1/2} - \Delta| + \frac{3}{2} \epsilon). \end{aligned} \quad (\text{IV.3.57})$$

Combined with (IV.3.1), (IV.3.56) (the right-hand side is bounded by  $\frac{\epsilon}{4\Delta}$  for  $L$  sufficiently large), and the obvious bound

$$\begin{aligned} |T(\phi^{(L,\epsilon)}(h); \xi_j^+, \xi_j^-) - T(\omega(h); \xi_j^+, \xi_j^-)| &\leq \\ &\leq |\xi_j^-| |\tanh(\phi^{(L,\epsilon)}(h)) - \tanh(\omega(h))|, \end{aligned} \quad (\text{IV.3.58})$$

where  $j \in \mathbb{N}$ , we arrive at (IV.3.40) with  $k = 1$  and  $M_1 = 2$  when we recall that  $|\xi_j^\pm| \leq 1$  and realize that  $\mathcal{F}_1(\{g_1\}) = g_1$ . In addition,

similarly to (IV.3.37), one has

$$\begin{aligned} \prod_{\ell=1}^j T(\phi^{(L,\epsilon)}(h); \xi_{|I_\ell|}^+, \xi_{|I_\ell|}^-) &= \prod_{\ell=1}^j T(\omega(h); \xi_{|I_\ell|}^+, \xi_{|I_\ell|}^-) + \\ &+ \sum_{\substack{X \subset \{1, \dots, j\}: \\ X \neq \{1, \dots, j\}}} \prod_{r \in X} T(\omega(h); \xi_{|I_r|}^+, \xi_{|I_r|}^-) \times \\ &\times \prod_{s \in \{1, \dots, j\} \setminus X} \xi_j^- (\tanh(\phi^{(L,\epsilon)}(h)) - \tanh(\omega(h))) \end{aligned} \quad (\text{IV.3.59})$$

for any  $j = 1, \dots, k$  with  $k = 2, 3, \dots$ , and any partition  $\{I_1, \dots, I_j\}$ . In view of (IV.3.58), we get (IV.3.40) with a suitable  $M_k$  for  $k \geq 2$ .  
Q.E.D.

Let us now show that the probability  $P_{L,h}(\mathcal{C}^c)$  appearing on the right-hand side of (IV.3.22) and (IV.3.31) converges — within the interval  $J_L(\vartheta)$  — exponentially fast to zero and that the convergence is uniform in  $h$ .

LEMMA IV.3.8. *Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $0 < \epsilon < \epsilon_0(\vartheta)$ . There exist finite constants  $\lambda = \lambda(\beta, \vartheta, \epsilon) > 0$  and  $L_3 = L_3(\beta, \vartheta, \epsilon)$  such that*

$$P_{L,h}(\mathcal{C}^c) \leq e^{-\beta\lambda L} \quad (\text{IV.3.60})$$

whenever  $L > L_3$  and  $h \in J_L(\vartheta)$ .

PROOF. Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $0 < \epsilon < \epsilon_0(\vartheta)$ ,  $L \in \mathbb{N}$ , and  $h \in J_L(\vartheta)$ . Moreover, let

$$\tilde{\mathcal{C}}(Lh, \epsilon) := (m_+(Lh) - \epsilon, m_+(Lh) + \epsilon) \cup (-m^* - \epsilon, -m^* + \epsilon); \quad (\text{IV.3.61})$$

it clearly follows that  $\tilde{\mathcal{C}} \subset \mathcal{C}$ , i.e.  $P_{L,h}(\mathcal{C}^c) \leq P_{L,h}(\tilde{\mathcal{C}}^c)$ .

With the help of Lemma IV.A.5 one has

$$\begin{aligned} &\inf_{m \in (\tilde{\mathcal{C}}(B, \epsilon))^c} \mathcal{W}_B(m) \geq \\ &\geq \min\{\mathcal{W}_B(-m^* + \epsilon), \mathcal{W}_B(m(B) - \epsilon), \mathcal{W}_B(m(B) + \epsilon)\} \end{aligned} \quad (\text{IV.3.62})$$

for any  $B \in \mathbb{R}$ . Since  $\mathcal{W}_B(m)$  as well as  $m(B)$  is continuous in  $B$ , the infimum over  $\overline{J_L(\vartheta)} = [B_0 - \vartheta, B_0 + \vartheta]$  is attained,

$$\inf_{B \in [B_0 - \vartheta, B_0 + \vartheta]} \mathcal{W}_B(\xi(B)) = \mathcal{W}_{B_\xi}(\xi(B_\xi)) \quad (\text{IV.3.63})$$

for some point  $B_\xi \in [B_0 - \vartheta, B_0 + \vartheta]$ , where  $\xi(B)$  stands for  $-m^* + \epsilon$ ,  $m(B) - \epsilon$ , or  $m(B) + \epsilon$  (the value  $B_\xi$  may differ for each of these three

functions). Thus, there exist values  $B_i \in [B_0 - \vartheta, B_0 + \vartheta]$ ,  $i = 1, 2, 3$ , depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$  such that

$$\begin{aligned} & \inf_{m \in (\tilde{\mathcal{C}}(Lh, \epsilon))^c} \mathcal{W}_{Lh}(m) \geq \\ & \geq \inf_{h \in J_L(\vartheta)} \inf_{m \in (\tilde{\mathcal{C}}(Lh, \epsilon))^c} \mathcal{W}_{Lh}(m) = \inf_{B \in [B_0 - \vartheta, B_0 + \vartheta]} \inf_{m \in (\tilde{\mathcal{C}}(B, \epsilon))^c} \mathcal{W}_B(m) \geq \\ & \geq \min\{\mathcal{W}_{B_1}(-m^* + \epsilon), \mathcal{W}_{B_2}(m(B_2) - \epsilon), \mathcal{W}_{B_3}(m(B_3) + \epsilon)\} > 0, \end{aligned} \quad (\text{IV.3.64})$$

where we used Theorem IV.3.2 (3) in the last step. Denoting the last minimum by  $2\lambda$ , Lemma IV.A.4 implies

$$P_{L,h}(\tilde{\mathcal{C}}^c) \leq e^{-\beta\lambda L} \quad (\text{IV.3.65})$$

for all  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ).

Q.E.D.

Using the preceding four lemmas, we shall now prove Theorem IV.3.3.

PROOF OF THEOREM IV.3.3. Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $h \in J_L(\vartheta)$ , and  $0 < \epsilon < \epsilon_0(\vartheta)$ . We may collect the relations (IV.3.22), (IV.3.31), (IV.3.40), and (IV.3.60) to conclude that there exists  $L_4 = L_4(\beta, \vartheta, \epsilon)$  such that for  $L > L_4$  the following holds: there exist finite positive constants  $\lambda(\beta, \vartheta, \epsilon)$  and  $D_k$ ,  $k \in \mathbb{N}$ , such that

$$\left| \frac{1}{(\beta|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathcal{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-)\}) \right| \leq D_k E_L(\epsilon) \quad (\text{IV.3.66})$$

for all  $k \in \mathbb{N}$ , where

$$E_L(\epsilon) := e^{-\beta\lambda L} + |\Delta m(Lh_0(L, \epsilon) + L^{-1/2}) - \Delta| + (1 + 1/\Delta)\epsilon > 0. \quad (\text{IV.3.67})$$

Next, we shall use (IV.3.66) to show that the susceptibility  $\chi_L(h, \beta)$  attains its maximum over  $J_L(\vartheta)$  at a unique point  $h_\chi(L)$  for any  $L$  sufficiently large (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ), and thus evaluate its position.

First, notice that

$$\mathcal{F}_k(\{T(x; \xi_j^+, \xi_j^-)\}) = (\Delta m(Lh))^k \frac{d^{k-1} \tanh x}{dx^{k-1}} \quad (\text{IV.3.68})$$

for  $k = 2, 3, 4$ . Now, let us show that if the point  $h_\chi(L)$  exists, then, necessarily, one has  $|h_\chi(L) - h_0(L, \epsilon)| < \alpha/|\Lambda_L|$ , where  $\alpha > 0$  will be specified later.<sup>10</sup> Namely, let us show that, for  $L$  sufficiently large

<sup>10</sup>We assume that  $L$  is large enough (depending on  $\beta$ ,  $\vartheta$ , and  $\alpha$ ) to ensure that the interval  $|h - h_0(L)| \geq \alpha/|\Lambda_L|$  fits into  $J_L(\vartheta)$ .

(depending on  $\beta, \vartheta, \epsilon$ , and  $\alpha$ ),

$$\chi_L(h_0(L, \epsilon), \beta) > \chi_L(h, \beta) \quad (\text{IV.3.69})$$

once  $h \in J_L(\vartheta)$  such that  $|h - h_0(L, \epsilon)| \geq \alpha/|\Lambda_L|$ . This is clear if  $|h - h_0(L, \epsilon)| \geq L^{-3/2}$ : then  $|\omega(h)| \geq \beta\Delta L^{1/2}$ , and (IV.3.66) with  $k = 2$  yields

$$\begin{aligned} \chi_L(h_0(L, \epsilon), \beta) - \chi_L(h, \beta) &\geq \\ &\geq (\Delta m(Lh_0(L, \epsilon)))^2 - (\Delta m(Lh))^2 \cosh^{-2}(\omega(h)) - 2D_2E_L(\epsilon) \geq \\ &\geq (2\epsilon_0(\vartheta))^2 - \cosh^{-2}(\beta\Delta L^{1/2}) - 2D_2E_L(\epsilon) > 0 \quad (\text{IV.3.70}) \end{aligned}$$

once  $\epsilon > 0$  is small enough (depending on  $\beta$ ) and  $L$  is large (depending on  $\beta, \vartheta$ , and  $\epsilon$ ). On the other hand, if  $\alpha/|\Lambda_L| \leq |h - h_0(L, \epsilon)| \leq L^{-3/2}$ , then we have  $|\omega(h)| \geq \beta\alpha\Delta$ , and

$$\begin{aligned} \chi_L(h_0(L, \epsilon), \beta) - \chi_L(h, \beta) &\geq \\ &\geq (\Delta m(Lh_0(L, \epsilon)))^2 - (\Delta m(L(h_0(L, \epsilon) + L^{-3/2})))^2 \cosh^{-2}(\omega(h)) - \\ &\quad - 2D_2E_L(\epsilon) \geq \\ &\geq (\Delta m(Lh_0(L, \epsilon)))^2 \left[ 1 - \left( \frac{\Delta m(Lh_0(L, \epsilon) + L^{-1/2})}{\Delta m(Lh_0(L, \epsilon))} \right)^2 \cosh^{-2}(\beta\alpha\Delta) \right] \\ &\quad - 2D_2E_L(\epsilon) \geq \\ &\geq (2\epsilon_0(\vartheta))^2 \left[ 1 - \left( \frac{\Delta m(Lh_0(L, \epsilon) + L^{-1/2})}{\Delta m(Lh_0(L, \epsilon))} \right)^2 \cosh^{-2}(\beta\alpha\Delta) \right] - \\ &\quad - 2D_2E_L(\epsilon). \quad (\text{IV.3.71}) \end{aligned}$$

Taking  $L$  so large that

$$\left( \frac{\Delta m(Lh_0(L, \epsilon) + L^{-1/2})}{\Delta m(Lh_0(L, \epsilon))} \right)^2 < 1 + (\cosh^2(\beta\alpha\Delta) - 1)/2 \quad (\text{IV.3.72})$$

(note that the left-hand side above must always be larger than 1), we obtain

$$\begin{aligned} \chi_L(h_0(L, \epsilon), \beta) - \chi_L(h, \beta) &\geq \\ &\geq (2\epsilon_0)^2(1 - \cosh^{-2}(\beta\alpha\Delta))/2 - 2D_2E_L(\epsilon) > 0 \quad (\text{IV.3.73}) \end{aligned}$$

whenever  $\epsilon > 0$  is small enough (depending on  $\beta$  and  $\alpha$ ) and  $L$  is large (depending on  $\beta, \vartheta, \epsilon$ , and  $\alpha$ ). This and (IV.3.70) verify (IV.3.69).

Next, we shall show that the susceptibility  $\chi_L(h, \beta)$  is concave on the interval  $[h_0(L, \epsilon) - \frac{\alpha}{|\Lambda_L|}, h_0(L, \epsilon) + \frac{\alpha}{|\Lambda_L|}]$  and that its derivative is positive at  $h_0(L, \epsilon) - \frac{\alpha}{|\Lambda_L|}$  and negative at  $h_0(L, \epsilon) + \frac{\alpha}{|\Lambda_L|}$ . Indeed, let us consider  $h \in J_L(\vartheta)$  such that  $|h - h_0(L, \epsilon)| \leq \alpha/|\Lambda_L|$ . In view of

(IV.3.66) with  $k = 3$ , we have

$$\begin{aligned} \frac{1}{\beta|\Lambda_L|} \frac{\partial \chi_L(h, \beta)}{\partial h} \Big|_{h=h_0(L, \epsilon) + \alpha/|\Lambda_L|} &\leq (\Delta m(Lh))^3 \frac{d^2 \tanh x}{dx^2} \Big|_{x=\beta\alpha\Delta} + \\ + D_3 E_L(\epsilon) &\leq -(2\epsilon_0)^3 \left| \frac{d^2 \tanh x}{dx^2} \right|_{x=\beta\alpha\Delta} + D_3 E_L(\epsilon) < 0 \end{aligned} \quad (\text{IV.3.74})$$

and

$$\begin{aligned} \frac{1}{\beta|\Lambda_L|} \frac{\partial \chi_L(h, \beta)}{\partial h} \Big|_{h=h_0(L, \epsilon) - \alpha/|\Lambda_L|} &\geq \\ &\geq (2\epsilon_0)^3 \left| \frac{d^2 \tanh x}{dx^2} \right|_{x=\beta\alpha\Delta} - D_3 E_L(\epsilon) > 0 \end{aligned} \quad (\text{IV.3.75})$$

for  $\epsilon$  small (depending on  $\beta$  and  $\alpha$ ) and  $L$  large (depending on  $\beta$ ,  $\vartheta$ ,  $\epsilon$ , and  $\alpha$ ). Here we used that  $\frac{d^2 \tanh x}{dx^2}$  is odd and negative for  $x > 0$ . Observing that  $\frac{d^3 \tanh x}{dx^3} < 0$  once  $|x| < 2A$  for some  $A > 0$ , we choose  $\alpha = \frac{A}{\beta\Delta}$ : then  $|\omega(h)| \leq A$ , and, using (IV.3.66) with  $k = 4$ , we get

$$\frac{1}{(\beta|\Lambda_L|)^2} \frac{\partial^2}{\partial h^2} \chi_L(h) \leq -(2\epsilon_0)^4 \left| \frac{d^3 \tanh x}{dx^3} \right|_{x=A} - D_4 E_L(\epsilon) < 0 \quad (\text{IV.3.76})$$

for all  $|h - h_0(L, \epsilon)| \leq \alpha/|\Lambda_L| = A/(\beta\Delta|\Lambda_L|)$  once  $\epsilon$  is sufficiently small (depending on  $\beta$ ) and  $L$  is sufficiently large (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ). Combined with the fact that the susceptibility  $\chi_L(h, \beta)$  is analytic in  $h$ , we thus see that the point  $h_\chi(L)$  exists, it is unique, and  $|h_\chi(L) - h_0(L, \epsilon)| < A/(\beta\Delta|\Lambda_L|)$ . Thus,

$$\lim_{L \rightarrow \infty} L h_\chi(L) = B_0 \quad (\text{IV.3.77})$$

due to (IV.3.38), which verifies the first part of (IV.3.13).

Further, let us prove that

$$|h_0(L, \epsilon) - h_\chi(L)| \leq \frac{2D_3}{\beta\theta(\epsilon_0/2)^4} \frac{E_L(\epsilon)}{|\Lambda_L|} \quad (\text{IV.3.78})$$

for  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ) with  $\theta := -\frac{d^3 \tanh x}{dx^3} \Big|_{x=A} > 0$ . This is trivial if  $h_\chi(L)$  happens to coincide with  $h_0(L, \epsilon)$ . So, let us assume that  $h_\chi(L) \neq h_0(L, \epsilon)$ . Then the Lagrange mean-value theorem yields

$$\frac{\partial}{\partial h} \chi_L(h, \beta) \Big|_{h=h_0(L, \epsilon)} = (h_0(L, \epsilon) - h_\chi(L)) \frac{\partial^2}{\partial h^2} \chi_L(h, \beta) \Big|_{h=\bar{h}} \quad (\text{IV.3.79})$$

for some  $\bar{h}$  between  $h_0(L, \epsilon)$  and  $h_\chi(L)$  and  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ ). By virtue of (IV.3.66) and the bound  $|\omega(\bar{h})| \leq |\omega(h_\chi(L))| <$

$A$ , for  $L$  large we have

$$\begin{aligned} \frac{1}{(\beta|\Lambda_L|)^2} \frac{\partial^2}{\partial h^2} \chi_L(h, \beta) \Big|_{h=\bar{h}} &\leq (2\epsilon_0)^4 \frac{d^3 \tanh x}{dx^3} \Big|_{x=A} + D_4 E_L(\epsilon) \leq \\ &\leq -(2\epsilon_0)^4 \theta/2, \end{aligned} \quad (\text{IV.3.80})$$

yielding a lower bound on its absolute value. On the other hand, the absolute value of  $\frac{\partial}{\partial h} \chi_L(h, \beta)$  at  $h_0(L, \epsilon)$  can be bounded from above by  $D_3 \beta |\Lambda_L| E_L(\epsilon)$ . Both last bounds are valid for  $L$  large enough — depending on  $\beta$ ,  $\vartheta$ , and  $\epsilon$ . As a result, the bound (IV.3.78) follows with the help of (IV.3.79).

Using (IV.3.78), it readily follows that

$$|\tanh(\omega(h)) - \tanh(\tilde{\omega}(h))| \leq |\omega(h) - \tilde{\omega}(h)| < \frac{2D_3\Delta}{\theta(\epsilon_0/2)^4} E_L(\epsilon) \quad (\text{IV.3.81})$$

with

$$\tilde{\omega}(h) := \beta\Delta(h - h_{\chi}(L))|\Lambda_L|, \quad (\text{IV.3.82})$$

for all  $L$  large (depending on  $\beta$ ,  $\vartheta$ ,  $\epsilon$ ). This in turn implies that for any  $k \in \mathbb{N}$  there is a finite positive constant  $\tilde{D}_k$  such that

$$|\mathcal{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-\})\}) - \mathcal{F}_k(\{T(\tilde{\omega}(h); \xi_j^+, \xi_j^-\})\})| \leq \tilde{D}_k E_L(\epsilon) \quad (\text{IV.3.83})$$

for all  $L$  large (depending on  $\beta$ ,  $\vartheta$ ,  $\epsilon$ ), c.f. (IV.3.59). Since the absolute value of the error term  $R_L^{(k)}(h)$ ,  $k = 1, 2$ , can be bounded by the sum of the left-hand side of (IV.3.66) and the left-hand side of (IV.3.83), it follows from (IV.3.67) that

$$\begin{aligned} 0 \leq \overline{\lim}_{L \rightarrow \infty} \sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)| &\leq (D_k + \tilde{D}_k) \lim_{L \rightarrow \infty} E_L(\epsilon) \leq \\ &\leq (D_k + \tilde{D}_k)(1 + 1/\Delta)\epsilon \end{aligned} \quad (\text{IV.3.84})$$

for any  $\epsilon > 0$ ,  $0 < \epsilon < \epsilon_0(\vartheta)$ . As a result,

$$\overline{\lim}_{L \rightarrow \infty} \sup_{h \in J_L(\vartheta)} |R_L^{(k)}(h)| = 0,$$

$k = 1, 2$ , and the second claim of (IV.3.13) is verified.

Q.E.D.

#### IV.4. Local-Limit Estimates

In this section, we use the local-limit estimates established in [1, 9, 13, 17] to prove Theorem IV.2.2. Namely, we shall employ the following statements (we formulate them in the form in which we shall need them here).

THEOREM IV.4.1 ([1], [17]). *Let  $\beta > \beta_c$  and let  $\delta \in (0, 1/4)$  be given. There exists  $L_5 = L_5(\beta, \delta) < \infty$  such that for any  $L > L_5$  and for any sequence  $\{m_L\}$  such that  $m_L \in \text{Ran } S_L$ ,  $m_L \geq -m^* + L^{-\delta}$ , and  $\lim_{L \rightarrow \infty} m_L \in [-m^*, m^*)$  exists, one has*

$$\log P_{L,0}(m_L) \leq -\beta(\mathcal{W}_0(m_L) + s_L(m_L)) L, \quad (\text{IV.4.1})$$

where

$$s_L(m_L) = \begin{cases} O(L^{-(1/4+\delta)/4}) & \text{if } |m_L + m^* - CL^{-\bar{\delta}}| \leq L^{-(1/4+\delta)/2} \\ & \text{for some } C > 0 \text{ and } \bar{\delta} \in (0, \delta], \\ O(L^{-1/2} \log L) & \text{otherwise.} \end{cases} \quad (\text{IV.4.2})$$

The first case in (IV.4.2) is a consequence of Theorem 7.4.3 from [17], while the second case is a special case of Theorem 4.3.1 from [1].

The next theorem is a consequence of Theorem 1.5.1 from [9] (c.f. (1.1.2) in [13]) and the bound (1.1.1) of [13].

THEOREM IV.4.2 ([9], [13]). *Let  $\beta \neq \beta_c$ . There exist constants  $L_6 = L_6(\beta) < \infty$  and  $c_1 = c_1(\beta) > 0$  such that for  $L > L_6$  and any  $m_L \leq \langle S_L \rangle_{L,0}$  one has  $\langle S_L \rangle_{L,0} = -m^* + O(L^{-1})$  and*

$$\log P_{L,0}(m_L) \leq -c_1 [(\langle S_L \rangle_{L,0} - m_L)L]^2. \quad (\text{IV.4.3})$$

In addition, we also need this consequence of Theorem C from [13].

THEOREM IV.4.3 ([13]). *Let  $\beta > \beta_c$  be given. There exist constants  $L_7 = L_7(\beta) < \infty$  and  $c_2 = c_2(\beta) > 0$  such that*

$$P_{L,0}(m_L) = \frac{c_2}{L} (1 + o_L(1)) \quad (\text{IV.4.4})$$

for all  $L > L_7$  and an arbitrary  $m_L = \langle S_L \rangle_{L,0} + o(L^{-1}) \in \text{Ran } S_L$  such that  $m_L > \langle S_L \rangle_{L,0}$ .

Finally, we shall make use of the following lemma.

LEMMA IV.4.4. *Let  $\beta > \beta_c$  and  $\vartheta > 0$ . Introducing*

$$\eta(\beta, \vartheta) := [m^* - m_+(B_0 + \vartheta)]/2 = \kappa(B_0 + \vartheta)^{-2}/8, \quad (\text{IV.4.5})$$

there exist constants  $L_8 = L_8(\beta, \vartheta) < \infty$  and  $c_3 = c_3(\beta, \vartheta) > 0$  such that

$$\log P_{L,h}([m^* - \eta, 1]) \leq -c_3 \beta L \quad (\text{IV.4.6})$$

for any  $h \in J_L(\vartheta)$  and all  $L > L_8$ .

PROOF. With the help of (IV.3.6) and (IV.3.5), one may easily observe (c.f. the proof of Lemma IV.A.5) that the first derivative of  $\mathcal{W}_0$  is strictly decreasing on  $(-m^*, m_t)$ , whereas it is strictly increasing on  $(m_t, m^*)$ . This implies that  $\mathcal{W}_{Lh}$  is strictly increasing on  $(-m^*, m^*)$  once  $Lh \leq \tau/m^*$ :

$$\frac{d\mathcal{W}_{Lh}(m)}{dm} \geq \frac{d\mathcal{W}_0(m)}{dm} - \beta\tau/m^* > \frac{d\mathcal{W}_0(m_t)}{dm} - \beta\tau/m^* = 0 \quad (\text{IV.4.7})$$

for all  $m \in (-m^*, m^*)$ . On the other hand, if  $Lh > \tau/m^*$ , then  $\mathcal{W}_{Lh}$  is strictly increasing on  $(m(Lh), m^*)$ . Indeed, for  $Lh > \tau/m^*$ , we have  $m(Lh) > m_t$ . Hence

$$\frac{d\mathcal{W}_{Lh}(m)}{dm} = \frac{d\mathcal{W}_0(m)}{dm} - \beta Lh > \frac{d\mathcal{W}_0(m(Lh))}{dm} - \beta Lh = 0 \quad (\text{IV.4.8})$$

for all  $m \in (m(Lh), m^*)$ . Since  $m^* - \eta \in (-m^*, m^*)$  and  $m^* - \eta > m(Lh)$  as soon as  $Lh \leq B_0 + \vartheta$ , it follows that

$$\begin{aligned} \inf_{m \in [m^* - \eta, 1]} \mathcal{W}_{Lh}(m) &= \mathcal{W}_{Lh}(m^* - \eta) \geq \inf_{h \in J_L(\vartheta)} \mathcal{W}_{Lh}(m^* - \eta) = \\ &= \inf_{B \in [B_0 - \vartheta, B_0 + \vartheta]} \mathcal{W}_B(m^* - \eta) \end{aligned} \quad (\text{IV.4.9})$$

for any  $L \in \mathbb{N}$  and  $h \in J_L(\vartheta)$ . As  $\mathcal{W}_B$  is continuous in  $B$  by (IV.3.6) and Proposition IV.2.1, the infimum on the right-hand side of (IV.4.9) is attained, i.e. it equals  $\mathcal{W}_{\tilde{B}}(m^* - \eta)$  for some  $\tilde{B} \in [B_0 - \vartheta, B_0 + \vartheta]$ . Using  $c_3(\beta, \vartheta)$  to denote  $\mathcal{W}_{\tilde{B}}(m^* - \eta)/2$ , Lemma IV.A.4 implies the proposition. Q.E.D.

We are now ready to prove Theorem IV.2.2.

**IV.4.1. Proof of Theorem IV.2.2.** The proof goes along the same lines as that of Theorem IV.3.3. Nevertheless, instead of the large-deviation principle for the sequence  $\{P_{L,0}\}$ , here we take into account a more accurate information on the asymptotic behaviour of the distribution  $P_{L,0}$  given above. This will enable to get explicit rates at which the error terms  $R_L^{(0)}$ ,  $R_L^{(1)}(h)$ , and  $R_L^{(2)}(h)$  tend to zero as  $L \rightarrow \infty$ . For this reason, the parameter  $\epsilon > 0$  appearing in the definition of the sets  $\mathcal{C}^+$  and  $\mathcal{C}^-$ , will now be chosen dependent on  $L$ , and we use  $\epsilon_L$  to denote it. Only later shall we specify this dependence precisely — the choice will minimize the above error terms.

Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $\delta \in (0, 1/4)$ , and let  $\eta(\beta, \vartheta) > 0$  be defined by (IV.4.5). We consider a fixed sequence  $\{\epsilon_L\}$ ,  $\epsilon_L > 0$ , which may depend on  $\delta$  and  $\beta$  but not on  $\vartheta$  such that  $\lim_{L \rightarrow \infty} \epsilon_L = 0$  and  $\epsilon_L \geq L^{-\delta}$  for all  $L \in \mathbb{N}$ . It will actually turn out that an optimal choice for our purposes is  $\epsilon_L = L^{-\delta}$ . Using  $\epsilon_L$  in the place of  $\epsilon$ , for a given  $L$  we divide the interval  $J_L(\vartheta)$  into a *finite* number of sub-intervals as at the beginning of Subsection IV.3.1. Consequently, the points  $m_i$ ,

$i \in \mathbb{Z}$ , will now depend on  $\beta$  and  $\epsilon_L$ , while the finite numbers  $N_1$  and  $N_2$  as well as the points  $B^{(i)}$ ,  $i = -N_1, \dots, N_2$ , will depend on  $\beta$ ,  $\vartheta$ , and  $\epsilon_L$ . We again have

$$J_L(\vartheta) = \bigcup_{i=-N_1}^{N_2} \mathcal{I}_{L,i}^{(\epsilon_L)}. \quad (\text{IV.4.10})$$

Furthermore, for any  $h \in J_L(\vartheta)$  we set  $\mathcal{C}^+(Lh, \epsilon_L) := (m_i - \epsilon_L, m_{i+1} + \epsilon_L)$  if  $h \in \mathcal{I}_{L,i}^{(\epsilon_L)}$ , while  $\mathcal{C}^-(\epsilon_L) := (-m^* - \epsilon_L/4, -m^* + \epsilon_L/4)$ . As before, we shall write  $\mathcal{C}(Lh, \epsilon_L)$  for the union  $\mathcal{C}^+(Lh, \epsilon_L) \cup \mathcal{C}^-(\epsilon_L)$ .

In order to verify Theorem IV.2.2, we first prove three auxiliary lemmas. The first is just an expression of the distribution  $P_{L,h}$ ,  $h \in \mathbb{R}$ , in terms of  $P_{L,0}$ .

LEMMA IV.4.5. *Let  $\beta > 0$ ,  $h \in \mathbb{R}$ ,  $L \in \mathbb{N}$ , and let us take any  $A \in \mathcal{B}(\mathbb{R})$  (the set  $A$  may depend on  $\beta$ ,  $h$ , and  $L$ ). Then*

$$P_{L,h}(A) = \frac{\sum_{m \in A \cap \text{Ran } S_L} e^{\beta h |\wedge_L| m} P_{L,0}(m)}{\sum_{m' \in \text{Ran } S_L} e^{\beta h |\wedge_L| m'} P_{L,0}(m')}. \quad (\text{IV.4.11})$$

PROOF. Let  $\beta > 0$ ,  $h \in \mathbb{R}$ ,  $L \in \mathbb{N}$ , and  $A \in \mathcal{B}(\mathbb{R})$ . It suffices to realize that  $P_{L,h}(A) = \sum_{m \in A \cap \text{Ran } S_L} P_{L,h}(m)$  and combine the obvious equality

$$P_{L,h}(m) = \frac{e^{\beta h |\wedge_L| m}}{Z_{L,h}} \sum_{\substack{\sigma_L \in \{-1,1\}^{\wedge_L}: \\ S_L(\sigma_L) = m}} e^{-\beta H_{L,0}(\sigma_L)} = e^{\beta h |\wedge_L| m} P_{L,0}(m) \frac{Z_{L,0}}{Z_{L,h}} \quad (\text{IV.4.12})$$

with the fact that  $\sum_{m' \in \text{Ran } S_L} P_{L,h}(m') = 1$ . Q.E.D.

The behaviour of the function  $\phi^{(L, \epsilon_L)}(h)$  is inspected in the next lemma.

LEMMA IV.4.6. *Let  $\beta > \beta_c$ ,  $\vartheta > 0$ , and  $\delta \in (0, 1/4)$ . There exists a finite constant  $L_9 = L_9(\beta, \vartheta, \delta)$  such that for  $L > L_9$  the function  $\phi^{(L, \epsilon_L)}(h)$  equals zero within  $J_L(\vartheta)$  at a unique point<sup>11</sup>  $h_0(L)$ . Moreover,*

$$|Lh_0(L) - B_0| \leq 2(B_0)^3 \epsilon_L / \kappa. \quad (\text{IV.4.13})$$

<sup>11</sup>We are not denoting the dependence of  $h_0(L)$  on  $\epsilon_L$ . Since we assume that the sequence  $\{\epsilon_L\}$  is fixed, this dependence is actually a dependence on  $L$ .

PROOF. Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $\delta \in (0, 1/4)$ , and  $i = -N_1, \dots, -1$ . With the help of Lemma IV.4.5 and Theorem IV.4.1 we may bound

$$\begin{aligned}
& \frac{P_{L,B^{(i+1)}/L}(\mathcal{C}^+(B^{(i+1)}, \epsilon_L))}{P_{L,B^{(i+1)}/L}(\mathcal{C}^-(\epsilon_L))} = \\
& = \frac{\sum_{m \in \mathcal{C}^+(B^{(i+1)}, \epsilon_L) \cap \text{Ran } S_L} e^{\beta B^{(i+1)} L m} P_{L,0}(m)}{\sum_{\hat{m} \in \mathcal{C}^-(\epsilon_L) \cap \text{Ran } S_L} e^{\beta B^{(i+1)} L \hat{m}} P_{L,0}(\hat{m})} \leq \\
& \leq \frac{8L^2 \epsilon_L \max_{m \in \mathcal{C}^+(B^{(i+1)}, \epsilon_L) \cap \text{Ran } S_L} \{e^{\beta B^{(i+1)} L m} P_{L,0}(m)\}}{e^{\beta B^{(i+1)} L \hat{m}_L} P_{L,0}(\hat{m}_L)} \leq \\
& \leq \frac{8L^2 \epsilon_L}{P_{L,0}(\hat{m}_L)} \max_{m \in \mathcal{C}^+(B^{(i+1)}, \epsilon_L)} \{e^{\beta B^{(i+1)} L m - \beta [\mathcal{W}_0(m) + O(L^{-1/2} \log L)]} L\} \times \\
& \quad \times e^{-\beta B^{(i+1)} L \hat{m}_L} \quad (\text{IV.4.14})
\end{aligned}$$

for all  $L > L_5$  and arbitrary  $\hat{m}_L \in \mathcal{C}^-(\epsilon_L) \cap \text{Ran } S_L$ . Choosing  $\hat{m}_L > \langle S_L \rangle_{L,0}$  such that  $\hat{m}_L = \langle S_L \rangle_{L,0} + o(L^{-1}) = -m^* + O(L^{-1})$ , Theorem IV.4.3 and (IV.3.6) yield

$$\begin{aligned}
& \frac{P_{L,B^{(i+1)}/L}(\mathcal{C}^+(B^{(i+1)}, \epsilon_L))}{P_{L,B^{(i+1)}/L}(\mathcal{C}^-(\epsilon_L))} \leq \\
& \leq \frac{16L^3 \epsilon_L}{c_2} \max_{m \in [m_i - \epsilon_L, m_{i+1} + \epsilon_L]} e^{-\beta L [\mathcal{W}_0(m) - B^{(i+1)} m - B^{(i+1)} (-m^*) + O(L^{-1/2} \log L)]} \\
& \leq e^{O(\log L)} \max_{m \in [m_i - \epsilon_L, m_{i+1} + \epsilon_L]} e^{-\beta L [\mathcal{W}_{B^{(i+1)}}(m) - \mathcal{W}_{B^{(i+1)}}(-m^*) + O(L^{-1/2} \log L)]} \\
& \quad (\text{IV.4.15})
\end{aligned}$$

once  $L$  is large enough (depending on  $\beta$  and  $\delta$ ); we also used the fact that  $\mathcal{W}_B(-m^*) = 0$  if  $B \leq B_0$  by Theorem IV.3.2 (3). Since  $B^{(i+1)} < B_0$ , we have

$$\begin{aligned}
& \phi^{(L, \epsilon_L)}(B^{(i+1)}/L) \leq \\
& \leq -\beta L \left[ \min_{m \in [m_i - \epsilon_L, m_{i+1} + \epsilon_L]} \mathcal{W}_{B^{(i+1)}}(m) + O(L^{-1/2} \log L) \right] / 2 = \\
& = -\beta L [\mathcal{W}_{B^{(i+1)}}(\tilde{m}_i) + O(L^{-1/2} \log L)] / 2 \quad (\text{IV.4.16})
\end{aligned}$$

for all  $L$  large (depending on  $\beta$  and  $\delta$ ) and some  $\tilde{m}_i \in [m_i - \epsilon_L, m_{i+1} + \epsilon_L]$ . When  $B^{(i+1)} \leq \tau/m^*$ , where  $\tau/m^* < B_0$ , then there is a constant  $\zeta(\beta, \vartheta, i) > 0$  such that  $\mathcal{W}_{B^{(i+1)}}(x_i) \geq \zeta$ ; this follows from Theorem IV.3.2 (3) and the bound (IV.3.43). Thus,  $\phi^{(L, \epsilon_L)}(B^{(i+1)}/L) \leq -\beta \zeta L / 4 < 0$  for any  $L$  large (depending on  $\beta$ ,  $\vartheta$ ,  $\delta$ , and  $i$ ). On the other hand, if  $B^{(i+1)} > \tau/m^*$ , then  $\tilde{m}_i = m(B^{(i+1)})$ , for the function

$\mathcal{W}_B(x)$  has a local minimum at  $m(B)$  once  $B > \tau/m^*$ . Taking into account (IV.3.5) and (IV.3.9), one finds

$$\begin{aligned} \phi^{(L,\epsilon_L)}(B^{(i+1)}/L) &\leq -\frac{\beta L}{2} [\mathcal{W}_{B^{(i+1)}}(m(B^{(i+1)})) + O(L^{-1/2} \log L)] = \\ &= -\frac{\beta L}{2} [4\tau - \frac{\kappa}{4B^{(i+1)}} - 2m^*B^{(i+1)} + O(L^{-1/2} \log L)] \quad (\text{IV.4.17}) \end{aligned}$$

for all  $L$  large enough. Introducing  $g(B) := 4\tau - \kappa/(4B) - 2m^*B$  and recalling that  $w > 4\tau/3$ , one has

$$\frac{dg(B)}{dB} = -(m^* + m(B)) \leq -m^*w^2/(8\tau^2) < -2m^*/9 < 0 \quad (\text{IV.4.18})$$

whenever  $B \geq \tau/m^*$ . Observing that  $g(B_0) = 0$ , we thus get

$$\begin{aligned} \phi^{(L,\epsilon_L)}(B^{(i+1)}/L) &\leq -\beta L [g(B^{(0)}) + O(L^{-1/2} \log L)]/2 = \\ &= \beta L [(B^{(0)} - B_0)(m^* + m(\tilde{B})) + O(L^{-1/2} \log L)]/2 \quad (\text{IV.4.19}) \end{aligned}$$

for  $L$  large and some  $\tilde{B} \in (B^{(0)}, B_0)$ . From the construction of the interval  $\mathcal{I}_{L,-1}^{(\epsilon_L)}$  it is clear that  $B_0 - B^{(0)} \geq a\epsilon_L$  for some  $a(\beta) > 0$  and  $L$  large (depending on  $\beta$  and  $\delta$ ). More accurately, the Taylor expansion of  $m(B^{(0)})$  around  $B_0$  and the fact that  $m(B^{(0)}) = m(B_0) - \epsilon_L/2$  yield  $B_0 - B^{(0)} = (B_0)^3\epsilon_L/\kappa + O((\epsilon_L)^2)$ . Therefore, we finally obtain

$$\begin{aligned} \phi^{(L,\epsilon_L)}(B^{(i+1)}/L) &\leq \\ &\leq -\beta L [(B_0)^3(m^* + m(\tilde{B}))L^{-\delta}/(2\kappa) + O(L^{-1/2} \log L)]/2 < 0 \quad (\text{IV.4.20}) \end{aligned}$$

for all  $L$  large enough (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). As a result, we see that  $\phi^{(L,\epsilon_L)}(h) < 0$  for all  $h \in \mathcal{I}_{L,i}^{(\epsilon_L)}$ ,  $i = -N_1, \dots, -1$ , once  $L$  is sufficiently large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ) because  $\phi^{(L,\epsilon_L)}$  is increasing on each  $\mathcal{I}_{L,i}^{(\epsilon_L)}$  according to Lemma IV.3.7 (a). Notice that one may use the above arguments to show that  $\phi^{(L,\epsilon_L)}(h) < 0$  for all  $h \in \mathcal{I}_{L,0}^{(\epsilon_L)}$  such that  $h \leq (B_0 + B^{(0)})/(2L)$  whenever  $L$  is large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ).

Next, let us consider  $i = 1, \dots, N_2$ . Taking  $\hat{m}_L \in \mathcal{C}^+(B^{(i)}, \epsilon_L) \cap \text{Ran } S_L$  such that  $\hat{m}_L = m(B^{(i)}) + O(L^{-2})$ , by virtue of Theorem

IV.4.1, Lemma IV.4.5, and the relations (IV.3.6) and (IV.3.9), we bound

$$\begin{aligned}
\frac{P_{L,B^{(i)}/L}(\mathcal{C}^-(\epsilon_L))}{P_{L,B^{(i)}/L}(\mathcal{C}^+(B^{(i)}, \epsilon_L))} &\leq \frac{e^{\beta B^{(i)}L(-m^* + \epsilon_L/4)} P_{L,0}(\mathcal{C}^-(\epsilon_L))}{e^{\beta B^{(i)}L\hat{m}_L} P_{L,0}(\hat{m}_L)} \leq \\
&\leq e^{\beta L[\mathcal{W}_{B^{(i)}}(m(B^{(i)})) - \mathcal{W}_{B^{(i)}}(-m^*) + B^{(i)}\epsilon_L/4 + O(L^{-1/2}\log L)]} = \\
&= e^{-\beta L[\mathcal{W}_{B^{(i)}}(-m^*) - B^{(i)}\epsilon_L/4 + O(L^{-1/2}\log L)]} = \\
&= e^{-\beta L[-g(B^{(i)}) - B^{(i)}\epsilon_L/4 + O(L^{-1/2}\log L)]} \quad (\text{IV.4.21})
\end{aligned}$$

for all  $L > L_5$ . Since

$$\begin{aligned}
\frac{d}{dB}(-g(B) - B\epsilon_L/4) &= m^* + m(B) - \epsilon_L/4 \geq \\
&\geq 2m^*/9 - \epsilon_L/4 \geq m^*/9 > 0 \quad (\text{IV.4.22})
\end{aligned}$$

for all  $L$  large (depending on  $\beta$  and  $\delta$ ), it follows that

$$\frac{P_{L,B^{(i)}/L}(\mathcal{C}^-(\epsilon_L))}{P_{L,B^{(i)}/L}(\mathcal{C}^+(B^{(i)}, \epsilon_L))} \leq e^{-\beta L[-g(B^{(1)}) - B^{(1)}\epsilon_L/4 + O(L^{-1/2}\log L)]} \quad (\text{IV.4.23})$$

once  $L$  is large enough (depending on  $\beta$  and  $\delta$ ). Because  $m(B^{(1)}) = m(B_0) + \epsilon_L/2$ , we have  $B^{(1)} - B_0 = O(\epsilon_L)$ . Namely, the Taylor expansion of  $m(B^{(1)})$  around  $B_0$  yields  $B^{(1)} - B_0 = (B_0)^3\epsilon_L/\kappa + O((\epsilon_L)^2)$ . Hence,

$$\begin{aligned}
-g(B^{(1)}) - B^{(1)}\epsilon_L/4 &= \\
&= (B^{(1)} - B_0)(m^* + m(B_0)) - B_0\epsilon_L/4 + O((\epsilon_L)^2) = \\
&= (B_0)^3(2m^* - \kappa/(2B_0)^2)\epsilon_L/\kappa - B_0\epsilon_L/4 + O((\epsilon_L)^2) \quad (\text{IV.4.24})
\end{aligned}$$

by the Taylor expansion of  $g(B^{(1)})$  around  $B_0$ . Recalling that  $\kappa < 4m^*(B_0)^2$ , the inequality  $(2m^* - \kappa/(2B_0)^2)\kappa > 1/(2B_0)^2$  is true. So,

$$\begin{aligned}
-g(B^{(1)}) - B^{(1)}\epsilon_L/4 &= \\
&= B_0[(B_0)^2(2m^* - \kappa/(2B_0)^2)/\kappa - 1/4]\epsilon_L + O((\epsilon_L)^2) = \\
&> B_0[(B_0)^2(2m^* - \kappa/(2B_0)^2)/\kappa - 1/4]\epsilon_L/2 \quad (\text{IV.4.25})
\end{aligned}$$

for all  $L$  large (depending on  $\beta$  and  $\delta$ ). Combined with (IV.4.23), we obtain that  $\phi^{(L,\epsilon_L)}(h) > 0$  for all  $h \in \mathcal{I}_{L,i}^{(\epsilon_L)}$ ,  $i = 1, \dots, N_2$ , once  $L$  is sufficiently large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ).

In addition, the above may also be used to show that  $\phi^{(L,\epsilon_L)}(h) > 0$  for any  $h \in \mathcal{I}_{L,0}^{(\epsilon_L)}$  such that  $h \geq (B^{(1)} - (\epsilon_L)^2)/L$ , say, and any  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). Recalling that, for  $L$  large, we have  $\phi^{(L,\epsilon_L)}(h) < 0$  for all  $h \in \mathcal{I}_{L,0}^{(\epsilon_L)}$  such that  $h \leq (B_0 + B^{(0)})/(2L)$ , the

fact that  $\phi^{(L, \epsilon_L)}$  is analytic and increasing on  $\mathcal{I}_{L,0}^{(\epsilon_L)}$ , see Lemma IV.3.7

(a), implies that the point  $h_0(L) \in \mathcal{I}_{L,0}^{(\epsilon_L)}$  exists, it is unique, and

$$B_0/L - h_0(L) < (B_0 - B^{(0)})/(2L) \leq (B_0)^3 \epsilon_L / (\kappa L) \quad (\text{IV.4.26})$$

and

$$h_0(L) - B_0/L < (B^{(1)} - B_0)/L \leq 2(B_0)^3 \epsilon_L / (\kappa L) \quad (\text{IV.4.27})$$

once  $L$  is large enough (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). Q.E.D.

Finally, in the following lemma, we establish a uniform bound on the probability  $P_{L,h}(\mathcal{C}^c)$  for all  $h \in J_L(\vartheta)$  analogous to (IV.3.60).

LEMMA IV.4.7. *Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $\delta \in (0, 1/4)$ , and  $h \in J_L(\vartheta)$ . There exists  $L_{10} = L_{10}(\beta, \vartheta, \delta) < \infty$  such that*

$$P_{L,h}(\mathcal{C}^c) \leq 3e^{-\beta(\tau/m^*)^3 L (\epsilon_L)^2 / \kappa} \quad (\text{IV.4.28})$$

as long as  $L > L_{12}$  and  $h \in J_L(\vartheta)$ .

PROOF. Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $\delta \in (0, 1/4)$ , and  $h \in J_L(\vartheta)$ . Clearly, we have  $m^* - \eta > \sup \mathcal{C}^+(Lh, \epsilon_L)$  once  $L$  is large (depending on  $\beta$  and  $\vartheta$ ), where  $\eta(\beta, \vartheta)$  is given by (IV.4.5). Then, obviously,

$$P_{L,h}(\mathcal{C}^c) \leq P_{L,h}([-1, -m^* - \epsilon_L/4]) + P_{L,h}(A_{Lh}(\epsilon_L)) + P_{L,h}([a_2, 1]) \quad (\text{IV.4.29})$$

with

$$A_{Lh}(\epsilon_L) := [a_1, m_+(Lh) - \epsilon_L] \cup [m_+(Lh) + \epsilon_L, a_2], \quad (\text{IV.4.30})$$

where we used the shorthands  $a_1 := -m^* + \epsilon_L/4$  and  $a_2 := m^* - \eta$ . Next, we shall uniformly bound each of the three above probabilities separately.

Let  $\tilde{m}_L \in \text{Ran } S_L$  be such that  $\tilde{m}_L > \langle S_L \rangle_{L,0}$  and  $\tilde{m}_L = \langle S_L \rangle_{L,0} + O(L^{-2})$ . Restricting the sum in the denominator of (IV.4.11) to a single term with  $m = \tilde{m}_L$  and using Theorem IV.4.2, Theorem IV.4.3, and Lemma IV.4.5, we get

$$\begin{aligned} P_{L,h}([-1, -m^* - \epsilon_L/4]) &\leq \\ &\leq \frac{L^2 \max_{m \in [-1, -m^* - \epsilon_L/4] \cap \text{Ran } S_L} \{e^{\beta h |\Lambda_L| m - c_1 [(\langle S_L \rangle_{L,0} - m)L]^2}\}}{e^{\beta h |\Lambda_L| \tilde{m}_L} P_{L,0}(\tilde{m}_L)} \leq \\ &\leq \frac{2L^3}{c_2} \max_{m \in [-1, -m^* - \epsilon_L/4]} \{e^{\beta h |\Lambda_L| m - c_1 [(\langle S_L \rangle_{L,0} - m)L]^2}\} \times \\ &\quad \times e^{-\beta h |\Lambda_L| (\langle S_L \rangle_{L,0} + O(L^{-2}))} \quad (\text{IV.4.31}) \end{aligned}$$

for all  $L > \max\{L_6, L_7\}$ . The auxiliary function  $q(m) := \beta h |\Lambda_L| m - c_1 [(\langle S_L \rangle_{L,0} - m)L]^2$  is increasing on  $(-\infty, -m^* - \epsilon_L/4]$  if  $L$  is large enough (depending on  $\beta$  and  $\vartheta$ ): then

$$\begin{aligned} \frac{dq(m)}{dm} &= \beta h |\Lambda_L| + 2c_1 (\langle S_L \rangle_{L,0} - m)L^2 \geq \\ &\geq \beta(B_0 - \vartheta)L + 2c_1 (\epsilon_L/4 + O(L^{-1}))L^2 = \\ &= [\beta(B_0 - \vartheta) + c_1 \epsilon_L L/2 + O(1)]L > 0 \quad (\text{IV.4.32}) \end{aligned}$$

because  $\langle S_L \rangle_{L,0} = -m^* + O(L^{-1})$  and  $\epsilon_L L > L^{3/4}$ . As a consequence,

$$\begin{aligned} P_{L,h}([-1, -m^* - \epsilon_L/4]) &\leq \frac{2L^3}{c_2} e^{q(-m^* - \epsilon_L/4) - \beta h |\Lambda_L| (\langle S_L \rangle_{L,0} + O(L^{-2}))} = \\ &= e^{O(\log L)} e^{\beta h |\Lambda_L| (\epsilon_L/4 + O(L^{-1})) - c_1 (\epsilon_L/4 + O(L^{-1}))^2 L^2} \leq \\ &\leq e^{O(\log L) + \beta(B_0 + \vartheta)L\epsilon_L/2 - c_1(L\epsilon_L/6)^2} \leq e^{-c_1(L\epsilon_L/8)^2} \leq e^{-L^{3/2}}. \quad (\text{IV.4.33}) \end{aligned}$$

for all  $L$  large enough (depending on  $\beta$  and  $\vartheta$ ).

Next, Theorem IV.4.1 and Lemma IV.4.5 yield

$$\begin{aligned} P_{L,h}(A_{Lh}(\epsilon_L)) &\leq 2L^2 \frac{\max_{m \in A_{Lh}(\epsilon_L) \cap \text{Ran } S_L} e^{\beta h |\Lambda_L| m} P_{L,0}(m)}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} \leq \\ &\leq 2L^2 \frac{\max_{m \in A_{Lh}(\epsilon_L) \cap \text{Ran } S_L} e^{-\beta L [\mathcal{W}_0(m) - Lhm + s_L(m)]}}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} \quad (\text{IV.4.34}) \end{aligned}$$

for any  $m_L \in \text{Ran } S_L$  and  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). Introducing  $f_{Lh}(m) := \mathcal{W}_0(m) - Lhm$  and denoting  $a^* := -m^* + \epsilon_0(\vartheta)/2$ , we may now write

$$\begin{aligned} P_{L,h}(A_{Lh}(\epsilon_L)) &\leq \frac{2L^2}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} \times \\ &\times \max \left\{ \max_{m \in [a_1, a^*]} e^{-\beta L [f_{Lh}(m) + O(L^{-(1/4+\delta)/4})]}, \right. \\ &\left. \max_{m \in [a^*, a_2] \cap A_{Lh}(\epsilon_L)} e^{-\beta L [f_{Lh}(m) + O(L^{-1/2} \log L)]} \right\} \quad (\text{IV.4.35}) \end{aligned}$$

if  $L$  is large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). Using Lemma IV.A.5, for  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ) we obtain

$$P_{L,h}(A_{Lh}(\epsilon_L)) \leq \frac{2L^2}{e^{\beta h |\Lambda_L| m_L} P_{L,0}(m_L)} e^{-\beta L [f_{Lh}(a_1) + O(L^{-(1/4+\delta)/4})]} \quad (\text{IV.4.36})$$

if  $Lh \leq \tau/m^*$ , whereas

$$\begin{aligned}
P_{L,h}(A_{Lh}(\epsilon_L)) &\leq \frac{2L^2}{e^{\beta h|\Lambda_L| m_L} P_{L,0}(m_L)} \times \\
&\quad \times \max \left\{ \max_{m \in [a_1, a^*]} e^{-\beta L[f_{Lh}(m) + O(L^{-(1/4+\delta)/4})]}, \right. \\
&\quad \left. \max_{m \in [a^*, a_2] \cap A_{Lh}(\epsilon_L)} e^{-\beta L[f_{Lh}(m) + O(L^{-1/2} \log L)]} \right\} = \\
&= \frac{2L^2}{e^{\beta h|\Lambda_L| m_L} P_{L,0}(m_L)} \max \left\{ e^{-\beta L[f_{Lh}(a_1) + O(L^{-(1/4+\delta)/4})]}, \right. \\
&\quad \left. e^{-\beta L[f_{Lh}(m(Lh) \pm \epsilon_L) + O(L^{-1/2} \log L)]} \right\} \quad (\text{IV.4.37})
\end{aligned}$$

whenever  $Lh \geq \tau/m^*$ . In the former case, let us take  $m_L > \langle S_L \rangle_{L,0}$  such that  $m_L = \langle S_L \rangle_{L,0} + O(L^{-2}) = -m^* + O(L^{-1})$ . Then, in view of Theorem IV.4.1 and (IV.3.5),

$$\begin{aligned}
P_{L,h}(A_{Lh}(\epsilon_L)) &\leq \\
&\leq \frac{2L^2 e^{-\beta L[w\sqrt{\epsilon_L/(8m^*)} - Lh(-m^* + \epsilon_L/4) + O(L^{-(1/4+\delta)/4})]}}{c_2 e^{\beta h|\Lambda_L|(-m^* + O(L^{-1}))} (1 + o_L(1))/L} = \\
&= e^{-\beta L[w\sqrt{\epsilon_L/(8m^*)} - Lh(\epsilon_L/4 + O(L^{-1})) + O(L^{-(1/4+\delta)/4}) + O(L^{-1} \log L)]} \leq \\
&\leq e^{-\beta wL\sqrt{\epsilon_L/(8m^*)} [1 + O(\sqrt{\epsilon_L}) + O(L^{-(1/4-\delta)/4})]} \leq e^{-\beta wL\sqrt{\epsilon_L/(12m^*)}} \quad (\text{IV.4.38})
\end{aligned}$$

for all  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). In the latter case, if the maximum in (IV.4.37) coincides with the first term, we use the same procedure as above to get (IV.4.38). However, if the maximum in (IV.4.37) coincides with the second term, we then take  $m_L = m(Lh) + O(L^{-2})$  and, by virtue of Theorem IV.4.1 and (IV.3.5) we find

$$\begin{aligned}
P_{L,h}(A_{Lh}(\epsilon_L)) &\leq \\
&\leq 2L^2 e^{-\beta L\{[\mathcal{W}_0(m(Lh) \pm \epsilon_L) - \mathcal{W}_0(m(Lh) + O(L^{-2}))] \pm Lh\epsilon_L + O(L^{-1/2} \log L)\}} = \\
&= e^{-\beta L\{-\frac{\kappa}{2Lh}[\sqrt{1+(2Lh)^2\epsilon_L/\kappa} - \sqrt{1+O(L^{-2})}] \pm Lh\epsilon_L + O(L^{-1/2} \log L)\}} \leq \\
&\leq e^{-\beta L[2(Lh)^3(\epsilon_L)^2/\kappa + O((\epsilon_L)^3) + O(L^{-1/2} \log L)]} \leq \\
&\leq e^{-2\beta(Lh)^3 L(\epsilon_L)^2/\kappa [1 + O(\epsilon_L) + O(L^{-1/4} \log L)]} \leq e^{-\beta(\tau/m^*)^3 L(\epsilon_L)^2/\kappa} \quad (\text{IV.4.39})
\end{aligned}$$

once  $L$  is large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). Taking into account Lemma IV.4.4, we may conclude that

$$\begin{aligned} P_{L,h}(\mathcal{C}^c) &\leq 3 \max\{e^{-\beta\omega L\sqrt{\epsilon_L/(12m^*)}}, e^{-\beta(\tau/m^*)^3 L(\epsilon_L)^2/\kappa}\} = \\ &= 3e^{-\beta(\tau/m^*)^3 L(\epsilon_L)^2/\kappa} \end{aligned} \quad (\text{IV.4.40})$$

for all sufficiently large  $L$  (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). Q.E.D.

Now, we are ready to prove Theorem IV.2.2.

PROOF OF THEOREM IV.2.2. Due to Lemma IV.3.4 and Lemma IV.3.6, there exist finite constants  $C_k, K_k, k \in \mathbb{N}$ , and  $L_{11} = L_{11}(\beta, \vartheta, \delta)$  such that for all  $\beta > \beta_c, \vartheta > 0$ , and  $\delta \in (0, 1/4)$  we have

$$\begin{aligned} \left| \frac{1}{(\beta|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathcal{F}_k(\{T(\phi^{(L,\epsilon_L)}(h); \xi_j^+, \xi_j^-)\}) \right| \leq \\ \leq C_k P_{L,h}(\mathcal{C}^c) + K_k \epsilon_L \end{aligned} \quad (\text{IV.4.41})$$

whenever  $L > L_{11}$  and  $h \in J_L(\vartheta)$ . Here  $T(x; a, b)$  and  $\xi_j^\pm$  are given by (IV.3.23) and (IV.3.32), respectively. Moreover, Lemma IV.4.6 yields

$$\begin{aligned} |\Delta m(Lh_0(L) + L^{-1/2}) - \Delta| = \\ = |m(Lh_0(L)) - m(B_0)|/2 + O(L^{-1/2}) = \epsilon_L/4 + O(L^{-1/2}) \end{aligned} \quad (\text{IV.4.42})$$

for all  $L > L_9$  by the Taylor expansion and the fact that  $h_0(L) \in \mathcal{I}_{L,0}^{(\epsilon_L)}$ . Combined with Lemma IV.3.7 (c), we may conclude that there exists finite constants  $M_k, k \in \mathbb{N}$ , and  $L_{12} = L_{12}(\beta, \vartheta, \delta)$  such that for all  $\beta > \beta_c, \vartheta > 0$ , and  $\delta \in (0, 1/4)$  one has

$$\begin{aligned} \left| \mathcal{F}_k(\{T(\phi^{(L,\epsilon_L)}(h); \xi_j^+, \xi_j^-)\}) - \mathcal{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-)\}) \right| \leq \\ \leq 2M_k \epsilon_L / \Delta \end{aligned} \quad (\text{IV.4.43})$$

once  $L > L_{12}$  and  $h \in J_L(\vartheta)$ , where  $\omega(h) := \beta\Delta(h - h_0(L))|\Lambda_L|$ . Taking into account Lemma IV.4.7, we may therefore conclude that there exist finite constants  $D_k, k \in \mathbb{N}$ , and  $L_{13} = L_{13}(\beta, \vartheta, \delta)$  such that

$$\left| \frac{1}{(\beta|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathcal{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-)\}) \right| \leq D_k E_L(\epsilon_L) \quad (\text{IV.4.44})$$

for all  $\beta > \beta_c, \vartheta > 0$ , and  $\delta \in (0, 1/4)$  as long as  $L > L_{13}$  and  $h \in J_L(\vartheta)$ , where

$$E_L(\epsilon_L) := e^{-\beta(\tau/m^*)^3 L(\epsilon_L)^2/\kappa} + (1 + 1/\Delta)\epsilon_L > 0. \quad (\text{IV.4.45})$$

Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $\delta \in (0, 1/4)$ , and  $h \in J_L(\vartheta)$ . At this point, we shall choose  $\epsilon_L$  to minimize  $E_L(\epsilon_L)$ . Namely, let us take

$$\epsilon_L = L^{-\delta}. \quad (\text{IV.4.46})$$

As a consequence,

$$E_L(\epsilon_L) \leq 2(1 + 1/\Delta)L^{-\delta} \quad (\text{IV.4.47})$$

for all  $L$  sufficiently large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ).

According to Theorem IV.3.3, there is a unique point  $h_\chi(L) \in J_L(\vartheta)$  such that if  $L$  is large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ), then the susceptibility  $\chi_L(h, \beta)$  attains maximum over  $J_L(\vartheta)$  at  $h_\chi(L)$ . In addition, with the help of (IV.4.44) and (IV.4.47) and using the same arguments that led to (IV.3.78), one may bound

$$|h_0(L) - h_\chi(L)| \leq \frac{4D_3(1 + 1/\Delta)}{\beta\theta(\epsilon_0/2)^4} \frac{L^{-\delta}}{|\Lambda_L|}, \quad (\text{IV.4.48})$$

for  $L$  large enough (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ). Hence,

$$|\tanh(\omega(h)) - \tanh(\tilde{\omega}(h))| \leq |\omega(h) - \tilde{\omega}(h)| < \frac{4D_3(1 + \Delta)}{\theta(\epsilon_0/2)^4} L^{-\delta}, \quad (\text{IV.4.49})$$

where

$$\tilde{\omega}(h) := \beta\Delta(h - h_\chi(L))|\Lambda_L|, \quad (\text{IV.4.50})$$

for all  $L$  large (depending on  $\beta$ ,  $\vartheta$ ,  $\delta$ ). This in turn implies that for any  $k \in \mathbb{N}$  there is a finite constant  $\tilde{D}_k$  such that

$$|\mathcal{F}_k(\{T(\omega(h); \xi_j^+, \xi_j^-\})\}) - \mathcal{F}_k(\{T(\tilde{\omega}(h); \xi_j^+, \xi_j^-\})\})| \leq \tilde{D}_k L^{-\delta} \quad (\text{IV.4.51})$$

for all  $L$  large (depending on  $\beta$ ,  $\vartheta$ , and  $\delta$ ), c.f. (IV.3.59). In view of (IV.4.44), (IV.4.47) and (IV.3.68), Theorem IV.2.2 follows. Q.E.D.

### IV.A. Technical Lemmas

Here we collect various technical lemmas, some of them rather standard.

LEMMA IV.A.1. *Let  $\psi_r : \mathbb{R} \rightarrow \mathbb{R}$ ,  $r = 1, 2$ , be two  $C^\infty$  functions. Then, for any  $k \in \mathbb{N}$ ,*

$$\frac{d^k \psi_2(\psi_1(x))}{dx^k} = \sum_{i=1}^k \frac{d^i \psi_2(y)}{dy^i} \Big|_{y=\psi_1(x)} \sum_{\{I_1, \dots, I_i\}} \prod_{j=1}^i \frac{d^{|I_j|} \psi_1(x)}{dx^{|I_j|}},$$

where the second sum runs over all partitions  $\{I_1, \dots, I_i\}$ ,  $i = 1, \dots, k$ , of the set  $\{1, \dots, k\}$  and  $|I_j|$ ,  $j = 1, \dots, i$ , is the cardinality of  $I_j$ .

PROOF. By induction on  $k \in \mathbb{N}$ .

Q.E.D.

LEMMA IV.A.2. Let  $h \in \mathbb{R}$ ,  $L \in \mathbb{N}$ ,  $\beta > 0$ . Given any  $k = 2, 3, \dots$ , there exists a finite positive constant  $C_k$  such that

$$\left| \frac{1}{(\beta|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} - \mathcal{F}_k(\{ \langle (S_L)^i | A \rangle_{L,h} \}) \right| \leq C_k P_{L,h}(A^c) \quad (\text{IV.A.1})$$

for any set  $A \in \mathcal{B}(\mathbb{R})$  for which  $Z_{L,h}(A) > 0$  (the set  $A$  may depend on  $h$ ).

PROOF. Let  $h \in \mathbb{R}$ ,  $L \in \mathbb{N}$ ,  $\beta > 0$ , and  $k \in \mathbb{N}$ . Let  $A \in \mathcal{B}(\mathbb{R})$  be given (it may depend on  $h$ ) such that  $Z_{L,h}(A) > 0$ . Since

$$\frac{\partial^n}{\partial h^n} Z_{L,h} = (\beta|\Lambda_L|)^n \langle (S_L)^n \rangle_{L,h} Z_{L,h} \quad (\text{IV.A.2})$$

for all  $n \in \mathbb{N}$ , Lemma IV.A.1 applied to  $\psi_1(h) = Z_{L,h}$  and  $\psi_2(x) = \log x$  readily yields

$$\begin{aligned} \frac{1}{(\beta|\Lambda_L|)^{k-1}} \frac{\partial^{k-1}}{\partial h^{k-1}} \langle S_L \rangle_{L,h} &= \frac{1}{(\beta|\Lambda_L|)^k} \frac{\partial^k}{\partial h^k} \log Z_{L,h} = \\ &= \mathcal{F}_k(\{ \langle (S_L)^i \rangle_{L,h} \}). \end{aligned} \quad (\text{IV.A.3})$$

Observing that

$$\begin{aligned} \langle (S_L)^n \rangle_{L,h} &= \langle (S_L)^n | A \rangle_{L,h} P_{L,h}(A) + \langle (S_L)^n | A^c \rangle_{L,h} P_{L,h}(A^c) = \\ &= \langle (S_L)^n | A \rangle_{L,h} + (\langle (S_L)^n | A^c \rangle_{L,h} - \langle (S_L)^n | A \rangle_{L,h}) P_{L,h}(A^c) \end{aligned} \quad (\text{IV.A.4})$$

for any  $n \in \mathbb{N}$ , it follows that

$$\begin{aligned} \prod_{j=1}^i \langle (S_L)^{|I_j|} \rangle_{L,h} &= \prod_{j=1}^i \langle (S_L)^{|I_j|} | A \rangle_{L,h} + \\ &\quad + \sum_{\substack{X \subset \{1, \dots, i\}: \\ X \neq \{1, \dots, i\}}} \prod_{r \in X} \langle (S_L)^{|I_r|} | A \rangle_{L,h} \times \\ &\quad \times \prod_{s \in \{1, \dots, i\} \setminus X} (\langle (S_L)^{|I_s|} | A^c \rangle_{L,h} - \langle (S_L)^{|I_s|} | A \rangle_{L,h}) P_{L,h}(A^c) \end{aligned} \quad (\text{IV.A.5})$$

for all partitions  $\{I_1, \dots, I_i\}$  of  $\{1, \dots, k\}$ . Since  $|\langle S_L | A \rangle_{L,h}| \leq 1$  as well as  $|\langle S_L | A^c \rangle_{L,h}| \leq 1$  and that  $P_{L,h}(A^c) \leq 1$ , the relations (IV.A.3) and (IV.A.5) imply the lemma. Q.E.D.

LEMMA IV.A.3 (Lemma 6.1 from [4]). Let  $x_1, x_2 \in \mathbb{R}$  be either both positive or both negative. Then

$$|\tanh x_1 - \tanh x_2| \leq \min \left\{ \frac{\tanh x_1}{x_1}, \frac{\tanh x_2}{x_2} \right\} |x_1 - x_2|. \quad (\text{IV.A.6})$$

PROOF. Let  $x_1, x_2 \in \mathbb{R}$  be given. Without loss of generality, we may suppose that  $x_1 > x_2 > 0$ . Then  $\tanh x_1 > \tanh x_2$  and  $\frac{\tanh x_1}{x_1} < \frac{\tanh x_2}{x_2}$ . Thus,

$$\begin{aligned} |\tanh x_1 - \tanh x_2| \frac{x_1}{\tanh x_1} &= \left(1 - \frac{\tanh x_2}{\tanh x_1}\right) |x_1| < \\ &< \left|1 - \frac{x_2}{x_1}\right| |x_1| = |x_1 - x_2|. \end{aligned} \quad (\text{IV.A.7})$$

Q.E.D.

LEMMA IV.A.4. Let  $\beta > \beta_c$ ,  $\vartheta > 0$ ,  $L \in \mathbb{N}$ , and  $F \subset \mathbb{R}$  be a closed set (it may depend on  $h$  and  $L$ ). Given  $h \in J_L(\vartheta)$ , let us assume that there exists a constant  $c > 0$  independent of  $L$  for which

$$\inf_{m \in F} \mathcal{W}_{Lh}(m) \geq c \quad \text{for all } L \in \mathbb{N}. \quad (\text{IV.A.8})$$

Then there exists  $L_0(\beta, \vartheta, c)$  such that

$$P_{L,h}(F) \leq e^{-\beta c L/2} \quad \text{once } L > L_0. \quad (\text{IV.A.9})$$

PROOF. Let  $\delta > 0$ ,  $B \in \mathbb{R}$ ,  $h \in J_L(\vartheta)$ , and a closed  $F \subset \mathbb{R}$  be given.

First, let us show that there exists a finite positive integer  $1 \leq N(\delta) \leq 3/\delta$  such that

$$P_{L,h}(F) \leq N(\delta) e^{2\beta\delta\vartheta L} \frac{\sup_{m \in F} \{e^{\beta|\Lambda_L|(h-B/L)m} P_{L,B/L}(\mathcal{U}_\delta(m))\}}{\sup_{m \in \mathbb{R}} \{e^{\beta|\Lambda_L|(h-B/L)m} P_{L,B/L}(\mathcal{U}_\delta(m))\}} \quad (\text{IV.A.10})$$

with  $\mathcal{U}_\delta(m) := (m - \delta, m + \delta)$ . In view of (IV.3.14),

$$\begin{aligned} Z_{L,h} &\geq \sum_{\sigma_L \in \Omega_L: S_L(\sigma_L) \in \mathcal{U}_\delta(m)} e^{-\beta H_{L,B/L}(\sigma_L) + \beta|\Lambda_L|(h-B/L)S_L(\sigma_L)} \geq \\ &\geq e^{\beta|\Lambda_L|((h-B/L)m - \delta\vartheta/L)} Z_{L,B/L}(\mathcal{U}_\delta(m)) \end{aligned}$$

for any  $m \in \mathbb{R}$ . Hence,

$$Z_{L,h} \geq e^{-\beta\delta\vartheta L} \sup_{m \in \mathbb{R}} \{e^{\beta|\Lambda_L|(h-B/L)m} Z_{L,B/L}(\mathcal{U}_\delta(m))\}. \quad (\text{IV.A.11})$$

On the other hand, one may find  $1 \leq N(\delta) \leq 3/\delta$  and the values  $m_1, \dots, m_{N(\delta)} \in [-1, 1]$  such that  $[-1, 1] \subset \cup_{i=1}^{N(\delta)} \mathcal{U}_\delta(m_i)$ . As a consequence,

$$\begin{aligned} Z_{L,h}(F) &= Z_{L,h}(F \cap [-1, 1]) \leq \\ &\leq \sum_{i=1}^{N(\delta)} \sum_{\substack{\sigma_L \in \Omega_L: \\ S_L(\sigma_L) \in F \cap \mathcal{U}_\delta(m_i)}} e^{-\beta H_{L,B/L}(\sigma_L) + \beta |\Lambda_L|(h-B/L) S_L(\sigma_L)} \leq \\ &\leq N(\delta) \max_{1 \leq i \leq N(\delta)} e^{\beta |\Lambda_L|(m_i(h-B/L) + \delta \vartheta/L)} Z_{L,B/L}(F \cap \mathcal{U}_\delta(m_i)). \end{aligned} \quad (\text{IV.A.12})$$

Thus,

$$Z_{L,h}(F) \leq N(\delta) e^{\beta \delta \vartheta L} \sup_{m \in F} \{e^{\beta |\Lambda_L|(m(h-B/L))} Z_{L,B/L}(\mathcal{U}_\delta(m))\}. \quad (\text{IV.A.13})$$

Combining (IV.A.11) with (IV.A.13) and (IV.3.15), we get (IV.A.10).

Further, let  $m \in \mathbb{R}$  be arbitrary. The function  $g : \delta \mapsto \inf_{\mathcal{U}_\delta(m)} \mathcal{W}_B$  is obviously non-increasing. So, considering the closure  $\overline{\mathcal{U}_\delta(m)} = [m - \delta, m + \delta]$ , the equality  $\inf_{\mathcal{U}_\delta(m)} \mathcal{W}_B = \inf_{\overline{\mathcal{U}_\delta(m)}} \mathcal{W}_B$  holds for all the continuity points of  $g$ . Recalling that  $(P_{L,B/L})^{1/L} \rightarrow e^{-\beta \mathcal{W}_B}$  with  $\mathcal{W}_B$  defined by (IV.3.6), we get that the limit

$$\beta \inf_{\mathcal{U}_\delta(m)} \mathcal{W}_B = - \lim_{L \rightarrow \infty} \frac{1}{L} \log P_{L,B/L}(\mathcal{U}_\delta(m)) \quad (\text{IV.A.14})$$

exists for almost all  $\delta > 0$ . Because  $\mathcal{W}_B(m) - \delta \leq \inf_{\mathcal{U}_\delta(m)} \mathcal{W}_B \leq \mathcal{W}_B(m)$  (the former bound is due to the lower semi-continuity of  $\mathcal{W}_B$ ), for almost all  $\delta > 0$  it follows that, given any  $\varepsilon > 0$ , there is  $L_0(\varepsilon)$  finite such that

$$-\mathcal{W}_B(m) - \varepsilon < \frac{1}{\beta L} \log P_{L,B/L}(\mathcal{U}_\delta(m)) < \mathcal{W}_B(m) + \varepsilon + \delta \quad (\text{IV.A.15})$$

whenever  $L > L_0(\varepsilon)$ . In view of (IV.A.10) and (IV.3.6), we may conclude that, for almost all  $\delta > 0$  and any  $\varepsilon > 0$ , the bound

$$\begin{aligned} \frac{1}{\beta L} \log P_{L,h}(F) &\leq \\ &\leq \sup_{m \in F} \{ \beta L(h - B/L)m + \mathcal{W}_B(m) - \mathcal{W}_B^*(\beta L(h - B/L)) \} \\ &\quad + (2\beta\vartheta + 1)\delta + 2\varepsilon + \frac{\log(3/\delta)}{L} = \\ &= - \inf_F \mathcal{W}_{Lh} + (2\beta\vartheta + 1)\delta + 2\varepsilon + \frac{\log(3/\delta)}{L} \end{aligned} \quad (\text{IV.A.16})$$

holds for all  $h \in J_L(\vartheta)$  on condition that  $L > L_0(\varepsilon)$ . Taking, for instance,  $\varepsilon = c/12$ ,  $(2\beta\vartheta + 1)\delta \in [c/7, c/6]$ , and  $L$  so large that one has  $(\log(3/\delta))/L \leq c/6$  and  $L > L_0(c/12)$ , the lemma follows due to (IV.A.8). Q.E.D.

LEMMA IV.A.5. Let  $\beta > \beta_c$  and let  $\{a_L\}$  and  $\{b_L\}$  be two sequences of positive numbers smaller than  $\Delta m(B^*)/2$ . Defining the function  $f_B(m) := \mathcal{W}_0(m) - Bm$ ,  $B \in \mathbb{R}$ , and the set  $A_L(B) := [-m^* + a_L, m_+(B) - b_L] \cup [m_+(B) + b_L, m^*]$ , we have

$$\min_{m \in A_L(B)} f_B(m) = f_B(-m^* + a_L) \quad (\text{IV.A.17})$$

for  $B \leq \tau/m^*$ , whereas

$$\min_{m \in A_L(B)} f_B(m) \geq \min\{f_B(-m^* + a_L), f_B(m(B) - b_L), f_B(m(B) + b_L)\} \quad (\text{IV.A.18})$$

for  $B \geq \tau/m^*$ .

PROOF. Let  $\beta > \beta_c$  and let sequences  $\{a_L\}$  and  $\{b_L\}$  be given. Realizing that  $m_+ \geq -m^* + \varepsilon_0$ , it follows that  $[-m^* + a_L, m_+(B) - b_L] \neq \emptyset$  since

$$m_+(B) - b_L - (-m^* + a_L) = m_+(B) - (-m^*) - (a_L + b_L) > 2\varepsilon_0 > 0. \quad (\text{IV.A.19})$$

Notice, however, that, taking into account (IV.2.6), one has  $m_+(B) + b_L > m^*$  when  $B > (\sqrt{\kappa/b_L})/2$ .

We shall use the following properties of the the function  $f_B$ :

- (a) it is strictly concave on  $[-m^*, m_t]$ ,
- (b) it is strictly convex on  $[m_t, m^*]$ ,
- (c) it is increasing on  $[-m^*, m^*]$  for any  $B \leq \tau/m^*$ , and
- (d) it has a local minimum at  $m(B)$  for all  $B > \tau/m^*$ ;

they all directly follow from (IV.3.5). Recall that  $m_t = -m^*(1 - \frac{w^2}{8\tau^2}) \in (-m^*, m^*)$ .

Since a continuous and strictly concave function attains its minimum over an interval at the end-point(s) of the interval, the property (a) implies

$$\min_{m \in [-m^* + a_L, m_t]} f_B(m) = \min\{f_B(-m^* + a_L), f_B(m_t)\} \quad \text{for all } B \in \mathbb{R}. \quad (\text{IV.A.20})$$

Moreover, in view of (c), we have

$$\min_{m \in A_L(B)} f_B(m) = f_B(-m^* + a_L) \quad \text{for all } B \leq \tau/m^*. \quad (\text{IV.A.21})$$

With the help of (b) and (d), obviously

$$\min_{m \in [m_t, m^*]} f_B(m) = \min_{m \in [m(B) - b_L, m(B) + b_L] \cap [m_t, m^*]} f_B(m) \text{ for all } B \geq \tau/m^*. \quad (\text{IV.A.22})$$

Using, further, the fact that  $B^* = \tau/m^*$  (see the remark after Theorem IV.3.2), in view of (IV.2.7) we have  $m_+(B) = m(B)$  whenever  $B \geq \tau/m^*$  and thus

$$\min_{m \in A_L(B) \cap [m_t, m^*]} f_B(m) = \min_{m \in \{m(B) - b_L, m(B) + b_L\} \cap [m_t, m^*]} f_B(m) \quad (\text{IV.A.23})$$

for  $B \geq \tau/m^*$ . Finally, observing that

$$\begin{aligned} \min_{m \in A_L(B)} f_B(m) &= \\ &= \min \left\{ \min_{m \in A_L(B) \cap [-m^* + a_L, m_t]} f_B(m), \min_{m \in A_L(B) \cap [m_t, m^*]} f_B(m) \right\} \geq \\ &\geq \min \left\{ \min_{m \in [-m^* + a_L, m_t]} f_B(m), \min_{m \in A_L(B) \cap [m_t, m^*]} f_B(m) \right\}, \end{aligned} \quad (\text{IV.A.24})$$

the lemma follows from (IV.A.20), (IV.A.23), and (IV.A.21). Q.E.D.

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## CHAPTER V

### Finite-Size Effects and Large Deviations: Some Generalities

The goal of this chapter is to investigate the finite-size behaviour of lattice models determined by the surface-order large-deviation principles from a general point of view. In particular, we shall be interested in the models describing the coexistence of two phases. To this end, we extract the features of the two-dimensional Ising model which are essential for its finite-size analysis carried out in the previous chapter, namely, those necessary to prove Theorem IV.3.3. This will specify the group of models that we shall consider in the following.

At present there only exists a very small repertoire of models (limited practically to the two-dimensional Ising model) for which large-deviation principles at surface orders have been established. Nevertheless, it may be expected that it will soon be extended to cover, for instance, the Ising model in higher dimensions as well as other simple lattice models (like the Potts model). Then our analysis could also be readily applied to these models, yielding explicit asymptotic formulas for the corresponding finite-volume quantities such as the magnetization or the mean energy.

#### V.1. The Setting

Let  $\{\Lambda_n\}$  be a sequence of finite subsets of the lattice  $\mathbb{Z}^d$ ,  $d \geq 2$ , such that  $\lim_{n \rightarrow \infty} |\Lambda_n| = \infty$ . Let us consider a spin model in  $\Lambda_n$  whose single spin-space is a *finite* set  $S$  and the *Hamiltonian* has the form

$$H_{n,h}(\sigma_n) = \mathcal{H}_n(\sigma_n) - h |\Lambda_n| X_n(\sigma_n) \quad (\text{V.1.1})$$

for all configurations  $\sigma_n \in S^{\Lambda_n}$ . Here  $h$  is a real parameter and  $\mathcal{H}_n, X_n$  are real-valued functions on  $S^{\Lambda_n}$ . The corresponding finite-volume *Gibbs measure* is

$$\mu_{n,h} := \frac{e^{-H_{n,h}(\sigma_n)}}{Z_{n,h}}, \quad (\text{V.1.2})$$

where  $Z_{n,h} := \sum_{\sigma_n \in S^{\wedge n}} e^{-H_{n,h}(\sigma_n)}$  is the *partition function*. We shall use  $\langle \cdot \rangle_{n,h}$  to denote the *expected value* with respect to  $\mu_{n,h}$  and  $P_{n,h}$  to denote the *distribution* of  $X_n$  under  $\mu_{n,h}$ . In addition, we also introduce

$$Z_{n,h}(A) := \sum_{\substack{\sigma_n \in S^{\wedge n}: \\ X_n(\sigma_n) \in A}} e^{-H_{n,h}(\sigma_n)} \quad \text{and} \quad \langle \cdot | A \rangle_{n,h} := \sum_{\substack{\sigma_n \in S^{\wedge n}: \\ X_n(\sigma_n) \in A}} \cdot \frac{e^{-H_{n,h}(\sigma_n)}}{Z_{n,h}} \tag{V.1.3}$$

for any set  $A \in \mathcal{B}(\mathbb{R})$ , and the points

$$\underline{x} := \varliminf_{n \rightarrow \infty} \text{Ran } X_n, \quad \bar{x} := \overline{\varlimsup_{n \rightarrow \infty} \text{Ran } X_n}, \tag{V.1.4}$$

where  $\text{Ran } X_n$  stands for the range of  $X_n$ . We shall assume that

- (A)  $\max \{|\underline{x}|, |\bar{x}|\} < \infty$ ,
- (B) there is a sequence  $\{h_n^0\}, h_n^0 \in \mathbb{R}$ , such that an LD sequence  $\{(P_{n,h_n^0})^{\varepsilon_n}\}$  satisfies a weak large-deviation with a rate  $I \not\equiv \infty$ .

Moreover, let us suppose that there exists a point  $\xi_0 \in \mathbb{R}$  such that

- (C)  $(I^*)'_-(\xi) < (I^*)'_+(\xi)$  iff  $\xi = \xi_0$ ,
- (D)  $I > \overline{\text{co}}I$  on the interval  $((I^*)'_-(\xi_0), (I^*)'_+(\xi_0))$ .

Here  $(I^*)_{\pm}(\xi)$  are the one-sided derivatives of the Legendre-Fenchel transform  $I^*$  of  $I$  at  $\xi \in \mathbb{R}$  and  $\overline{\text{co}}I$  is the closed convex hull of  $I$ .<sup>1</sup>

Let us now demonstrate that the above setting means that we consider a situation analogous to the two-dimensional Ising model, see Fig. I.3 and Remark V.1.4 (iii). We start with the following lemma.

LEMMA V.1.1. *Let (A) and (B) be true. We have:*

- (1)  $(P_{n,h_n^0})^{\varepsilon_n} \rightarrow e^{-I}$  fully and the rate  $I$  is good.
- (2)  $\text{dom } I \subset [\underline{x}, \bar{x}]$  and  $\text{dom } I \neq \emptyset$ , whereas  $\text{dom } I^* = \mathbb{R}$ .
- (3) For any  $\xi$  and  $x$  in  $\mathbb{R}$ , let

$$I_{\xi}(x) := I(x) - \xi x + I^*(\xi). \tag{V.1.5}$$

Then  $I_{\xi}$  is a good rate, and  $\text{dom } I_{\xi} = \text{dom } I$ . Moreover, let  $\{h_n\}$  be a sequence of real numbers such that the limit

$$\varliminf_{n \rightarrow \infty} (h_n - h_n^0) \varepsilon_n |\Lambda_n| \tag{V.1.6}$$

exists and equals  $\xi$ . Then  $(P_{n,h_n})^{\varepsilon_n} \rightarrow e^{-I_{\xi}}$  fully.

REMARK V.1.2. Notice that  $I^*$  is a finite function: since  $I^*$  is convex, one has  $I^* > -\infty$ , while  $I^* < \infty$  as  $\text{dom } I^* = \mathbb{R}$  by the part (2).

<sup>1</sup>Since  $I \not\equiv \infty$  and  $I \geq 0$ , the closed convex hull of  $I$  is well-defined.

PROOF OF LEMMA V.1.1.

(1) In view of (B) and Lemma II.2.5, it suffices to show that the LD sequence  $\{(P_{n,h_n^0})^{\varepsilon_n}\}$  is exponentially tight. This, however, follows from (A).

(2) Let  $x \in \mathbb{R}$  and  $\varepsilon > 0$ . Let  $\mathcal{U}_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ . Due to the definition of  $\underline{x}$  and  $\bar{x}$ , for every  $\delta > 0$  there exists  $n_1 > 0$  such that  $\text{Ran } X_n \subset (\underline{x} - \delta, \bar{x} + \delta)$  for all  $n > n_1$ . Hence, given any  $x \notin [\underline{x}, \bar{x}]$ , there always exist  $\varepsilon_0 > 0$  and  $\delta_0 > 0$  such that  $\mathcal{U}_\varepsilon(x) \cap (\underline{x} - \delta, \bar{x} + \delta) = \emptyset$  if  $\varepsilon < \varepsilon_0$  and  $\delta < \delta_0$ . Then  $P_{n,h_n^0}(\mathcal{U}_\varepsilon(x)) = 0$ , and Lemma II.2.8 implies that  $I(x) = \infty$ . Hence,  $\text{dom } I \subset [\underline{x}, \bar{x}]$ . In addition, one has  $\text{dom } I \neq \emptyset$  by Lemma II.2.7. Finally,  $\text{dom } I^* = \mathbb{R}$  as  $I^*(\xi) = \sup_{x \in \text{dom } I} \{x\xi - I(x)\}$ , and, clearly,  $I^* \leq \sup_{x \in \text{dom } I} x\xi < \infty$  for any  $\xi \in \mathbb{R}$  because  $I \geq 0$ .

(3) The function  $I_\xi$  is a rate with  $\text{dom } I_\xi = \text{dom } I$  due to its very definition (V.1.5) and the fact that  $\text{dom } I^* = \mathbb{R}$ . Let  $x \in \mathbb{R}$  and  $\varepsilon > 0$  be arbitrary. Observing that

$$\begin{aligned} Z_{n,h_n}(\mathcal{U}_\varepsilon(x)) &= \sum_{\substack{\sigma_n \in S^{\wedge n}: \\ X_n(\sigma_n) \in \mathcal{U}_\varepsilon(x)}} e^{-H_{n,h_n^0}(\sigma_n) + |\wedge_n|(h_n - h_n^0)X_n(\sigma_n)} \\ &= e^{|\wedge_n|[(h_n - h_n^0)x + |h_n - h_n^0|O(\varepsilon)]} Z_{n,h_n^0}(\mathcal{U}_\varepsilon(x)), \end{aligned} \quad (\text{V.1.7})$$

we have

$$\begin{aligned} P_{n,h_n}(\mathcal{U}_\varepsilon(x)) &= \frac{Z_{n,h_n}(\mathcal{U}_\varepsilon(x))}{Z_{n,h_n}} = \\ &= P_{n,h_n^0}(\mathcal{U}_\varepsilon(x)) e^{|\wedge_n|[(h_n - h_n^0)x + |h_n - h_n^0|O(\varepsilon)]} \frac{Z_{n,h_n^0}}{Z_{n,h_n}}. \end{aligned} \quad (\text{V.1.8})$$

Finding that

$$\frac{Z_{n,h_n}}{Z_{n,h_n^0}} = \sum_{\sigma_n \in S^{\wedge n}} e^{(h_n - h_n^0)|\wedge_n|X_n(\sigma_n)} \frac{e^{-H_{n,h_n^0}(\sigma_n)}}{Z_{n,h_n^0}} = \langle e^{(h_n - h_n^0)|\wedge_n|X_n(\sigma_n)} \rangle_{n,h_n^0} \quad (\text{V.1.9})$$

and, with the help of (A),

$$\langle e^{(h_n - h_n^0)|\wedge_n|X_n(\sigma_n)} \rangle_{n,h_n^0} = \langle e^{\xi X_n(\sigma_n)/\varepsilon_n} \rangle_{n,h_n^0} e^{O(|(h_n - h_n^0)\varepsilon_n|\wedge_n| - \xi|/\varepsilon_n)} \quad (\text{V.1.10})$$

whenever  $n$  is large, we may use Theorem II.2.9 to get

$$\lim_{n \rightarrow \infty} \left( \frac{Z_{n,h_n}}{Z_{n,h_n^0}} \right)^{\varepsilon_n} = \sup_{x \in \mathbb{R}} e^{\xi x - I(x)} = e^{I^*(\xi)}. \quad (\text{V.1.11})$$

Recalling that  $(P_{n,h_n^0})^{\varepsilon_n} \rightarrow e^{-I}$  weakly, it thus follows that

$$\lim_{\varepsilon \rightarrow 0^+} \underline{\lim}_{n \rightarrow \infty} (P_{n,h_n}(\mathcal{U}_\varepsilon(x)))^{\varepsilon_n} = \lim_{\varepsilon \rightarrow 0^+} \overline{\lim}_{n \rightarrow \infty} (P_{n,h_n}(\mathcal{U}_\varepsilon(x)))^{\varepsilon_n} = e^{-I_\xi(x)}, \quad (\text{V.1.12})$$

i.e.  $(P_{n,h_n})^{\varepsilon_n} \rightarrow e^{-I_\xi}$  weakly according to Lemma II.2.8. As in the part (2), one may show that  $P_{n,h_n}$  is exponentially tight. Lemma II.2.5 then says that  $(P_{n,h_n})^{\varepsilon_n} \rightarrow e^{-I_\xi}$  fully and the rate  $I_\xi$  is good. Q.E.D.

The next lemma describes the minimum set  $\mathcal{M}_\xi$  of the rate  $I_\xi$  given by (V.1.5) for any  $\xi \in \mathbb{R}$ . To this end, we introduce the countable set  $\mathcal{D}(I^*) = \{\xi \in \mathbb{R} : (I^*)'_-(\xi) < (I^*)'_+(\xi)\}$  of points at which the derivative of  $I^*$  does not exist.

LEMMA V.1.3. *Let (A) and (B) be true. We have:*

- (1) *The minimum set  $\mathcal{M}_\xi$  is a non-empty compact subset of  $\text{dom } I_\xi$ , and contains all the points at which  $I_\xi$  equals zero. In addition,*

$$\mathcal{M}_\xi = \{x \in \partial I^*(\xi) : I(x) = \overline{\text{co}}I(x)\}. \quad (\text{V.1.13})$$

- (2) *The set  $\mathcal{M}_\xi$  is a singleton iff  $\xi \in \mathbb{R} \setminus \mathcal{D}(I^*)$ . In this case one has  $\mathcal{M}_\xi = \{(I^*)'(\xi)\}$ .*  
 (3) *If  $\xi \in \mathcal{D}(I^*)$ , then  $\min \mathcal{M}_\xi = (I^*)'_-(\xi)$ ,  $\max \mathcal{M}_\xi = (I^*)'_+(\xi)$ , and  $\mathcal{M}_\xi$  contains also all other points of the interval  $\partial I^*(\xi)$  at which  $I$  and  $\overline{\text{co}}I$  coincide.*  
 (4)  $\omega_- : \mathbb{R} \ni \xi \mapsto \min \mathcal{M}_\xi$  and  $\omega_+ : \mathbb{R} \ni \xi \mapsto \max \mathcal{M}_\xi$  are non-decreasing and lower and upper semi-continuous, respectively, with the only discontinuities at the points of  $\mathcal{D}(I^*)$ . Moreover,

$$\lim_{\xi' \rightarrow \xi - 0} \omega_\pm(\xi') = \omega_\pm(\xi), \quad \lim_{\xi' \rightarrow \xi + 0} \omega_\pm(\xi') = \omega_\pm(\xi), \quad (\text{V.1.14})$$

for all  $\xi \in \mathbb{R}$  and

$$\lim_{\xi \rightarrow -\infty} \omega_\pm(\xi) = \inf \text{dom } I, \quad \lim_{\xi \rightarrow \infty} \omega_\pm(\xi) = \sup \text{dom } I. \quad (\text{V.1.15})$$

REMARK V.1.4.

- (i) Notice that  $\overline{\text{co}}I$  is affine on the interval  $\partial I^*(\xi)$  once  $\xi \in \mathcal{D}(I^*)$ .  
 (ii) If the rate  $I$  is convex, then  $\mathcal{M}_\xi = \partial I^*(\xi)$ , and  $\{[\xi, x] \in \mathbb{R}^2 : x \in \mathcal{M}_\xi\}$  is thus a complete non-decreasing curve in  $\mathbb{R} \times \text{dom } I$  [1]. In particular, if  $I$  is strictly convex, it is a graph of a continuous non-decreasing function.  
 (iii) Taking into account that the rate  $I$  satisfies the conditions (C) and (D), the above lemma yields

$$\mathcal{M}_\xi = \begin{cases} \{(I^*)'(\xi)\} & \text{for } \xi \neq \xi_0, \\ \{(I^*)'_-(\xi), (I^*)'_+(\xi)\} & \text{for } \xi = \xi_0. \end{cases} \quad (\text{V.1.16})$$

This is precisely the situation of the two-dimensional Ising model studied in the previous chapter (c.f. Fig. I.3). Obviously, the conditions (C) and (D) characterize the very class of all the rates with the same form (V.1.16) of the corresponding minimum set  $\mathcal{M}_\xi$ .

PROOF OF LEMMA V.1.3.

(1) The fact that  $\mathcal{M}_\xi = \{x \in \mathbb{R} : I_\xi(x) = 0\} \neq \emptyset$  follows from the preceding lemma and Lemma II.2.7. The set  $\mathcal{M}_\xi$  is compact, for it coincides with  $\text{lev}_0(I_\xi)$ .

Let us verify (V.1.13). According to Lemma II.3.21 (2) and Theorem II.3.17, a point  $x$  is a minimum of  $I_\xi$  iff  $x$  is a minimum of  $\overline{\text{co}}I_\xi$  and  $I_\xi(x) = \overline{\text{co}}I_\xi(x)$ . Since

$$I_\xi^*(\xi') = \sup_{x \in \mathbb{R}} \{(\xi' + \xi)x - I(x)\} - I^*(\xi) = I^*(\xi' + \xi) - I^*(\xi) \quad (\text{V.1.17})$$

by (V.1.5), we have

$$\begin{aligned} \overline{\text{co}}I_\xi(x) &= (I_\xi)^{**}(x) = \sup_{\xi' \in \mathbb{R}} \{\xi'x - I^*(\xi' + \xi)\} + I^*(\xi) = \\ &= \sup_{\xi'' \in \mathbb{R}} \{(\xi'' - \xi)x - I^*(\xi'')\} + I^*(\xi) = I^{**}(x) - \xi x + I^*(\xi). \end{aligned} \quad (\text{V.1.18})$$

Thus,  $I_\xi(x) = \overline{\text{co}}I_\xi(x)$  iff  $I(x) = I^{**}(x) = \overline{\text{co}}I(x)$ . Moreover, the point  $x$  is a minimum of  $\overline{\text{co}}I_\xi$  iff  $x \in \partial(I_\xi)^*(0)$  by Remark II.3.13 (iii). Observing that  $\partial(I_\xi)^*(0) = \partial I^*(\xi)$  due to (V.1.5), we obtain (V.1.13).

(2), (3) The statements readily follows from (V.1.13) once one realizes that  $(\overline{\text{co}}I)((I^*)'_\pm(\xi)) = I((I^*)'_\pm(\xi))$ .

(4) Using (2) and (3), one gets  $\omega_\pm(\xi) = (I^*)'_\pm(\xi)$ . Thus, in view of Remark II.3.11, the monotonicity of  $\omega_\pm$  is implied by the monotonicity of one-sided derivatives of convex functions on  $\mathbb{R}$ . In addition, since  $I^*$  is convex and lower semi-continuous, we have

$$\begin{aligned} \lim_{\xi' \rightarrow \xi - 0} (I^*)'_\pm(\xi') &= (I^*)'_-(\xi), \\ \lim_{\xi' \rightarrow \xi + 0} (I^*)'_\pm(\xi') &= (I^*)'_+(\xi) \end{aligned}$$

for all  $\xi \in \mathbb{R}$ , which proves (V.1.14). The latter also obviously implies that  $\omega_-$  and  $\omega_+$  is lower and upper semi-continuous, respectively. It remains to prove the two relations of (V.1.15). Let us only show the second one, say. Taking into account that  $\text{Ran } \omega_\pm \subset \text{Ran } \partial I^*$  by (V.1.13) and  $\text{Ran } \partial I^* = \text{dom } \overline{\text{co}}I$  (by Theorem II.3.19 (1) and (4)), we see that  $\omega_\pm \leq \sup \text{dom } \overline{\text{co}}I$ . However, the latter equals  $\sup \text{dom } I$  as one may easily observe. Hence,  $\omega_\pm \leq \sup \text{dom } I$ . It thus suffices to show that  $\lim_{\xi \rightarrow \infty} (I^*)'_-(\xi) \geq \sup \text{dom } I$  because  $\omega_\pm \geq (I^*)'_-$ . So, let  $\epsilon > 0$  be arbitrary. Then there surely exists a point  $x_0 > \sup \text{dom } I - \epsilon$  which is in  $\text{dom } \overline{\text{co}}I = \text{Ran } \partial I^*$ , i.e.  $x_0 \in \partial I^*(\xi_0)$  for

some  $\xi_0 \in \mathbb{R}$  (dependent on  $\epsilon$ ). Using again the monotonicity of the one-sided derivatives of convex functions on  $\mathbb{R}$ , we get  $\sup \text{dom } I - \epsilon < x_0 \leq (I^*)'_+(\xi_0) \leq (I^*)'_-(\xi)$  for all  $\xi > \xi_0$ , and we are done. Q.E.D.

### V.2. Main Result

The main result of the chapter is the following theorem. Before stating it, let us first define

$$x_+(\xi) := \begin{cases} (I^*)'_+(\xi) & \text{for } \xi \geq \xi_0, \\ x_+(\xi) & \text{for } \xi \leq \xi_0, \end{cases} \quad x_-(\xi) := \begin{cases} (I^*)'_-(\xi) & \text{for } \xi \leq \xi_0, \\ x_-(\xi) & \text{for } \xi \geq \xi_0, \end{cases} \tag{V.2.1}$$

where  $\xi_0$  is the point appearing in the conditions (C) and (D) and  $x_{\pm} : \mathbb{R} \rightarrow \mathbb{R}$  are two continuous non-decreasing functions such that  $x_{\pm}(\xi_0) = (I^*)'_{\pm}(\xi_0)$  and  $x_+ - x_- \geq \epsilon_0 > 0$ . Notice that  $x_+$  as well as  $x_-$  is a non-decreasing function with the range in  $[\underline{x}, \bar{x}]$ . Moreover, let

$$\bar{x} := (x_+ + x_-)/2, \quad \Delta x := (x_+ - x_-)/2, \tag{V.2.2}$$

and  $\Delta := \Delta x(\xi_0) = [(I^*)'_+(\xi_0) - (I^*)'_-(\xi_0)]/2$ .

**THEOREM V.2.1.** *Let  $0 < \vartheta < \infty$ . There exists  $n_0 = n_0(\vartheta) \in \mathbb{N}$  such that for all  $n > n_0$  the following holds.*

- (1) *The quantity  $\frac{1}{|\Lambda_n|} \frac{\partial}{\partial h} \langle X_n \rangle_{n,h}$  attains its maximum over the open interval  $\mathcal{J}_n(\vartheta) := \{h \in \mathbb{R} : |h - h_n^*| \epsilon_n |\Lambda_n| < \vartheta\}$  at a unique point  $h_{\max}(n)$ , and*

$$\lim_{n \rightarrow \infty} (h_{\max}(n) - h_n^*) \epsilon_n |\Lambda_n| = 0. \tag{V.2.3}$$

Here  $h_n^* := h_n^0 + \frac{\xi_0}{\epsilon_n |\Lambda_n|}$ .

- (2) *Introducing the functions  $R_n^{(1)}(h)$  and  $R_n^{(2)}(h)$  through the relations*

$$\langle X_n \rangle_{n,h} = \bar{x}(\xi_{n,h}) + \Delta x(\xi_{n,h}) \tanh[\Delta(h - h_{\max}(n))|\Lambda_n|] + R_n^{(1)}(h) \tag{V.2.4}$$

and

$$\begin{aligned} \frac{\partial}{\partial h} \langle X_n \rangle_{n,h} &= \\ &= |\Lambda_n| \left\{ (\Delta x(\xi_{n,h}))^2 \cosh^{-2}[\Delta(h - h_{\max}(n))|\Lambda_n|] + R_n^{(2)}(h) \right\}, \end{aligned} \tag{V.2.5}$$

where  $\xi_{n,h} := (h - h_n^0) \epsilon_n |\Lambda_n|$ , it follows that

$$\lim_{n \rightarrow \infty} \sup_{h \in \mathcal{J}_n(\vartheta)} R_n^{(i)}(h) = 0, \quad i = 1, 2. \tag{V.2.6}$$

PROOF. The theorem can be readily proved with the help of the same arguments as those used to prove Theorem 3.3 in the previous chapter once we take into account Lemma V.2.2 and introduce the corresponding sets  $\mathcal{C}^+$  and  $\mathcal{C}^-$  as follows. First, there obviously exist unique finite numbers  $N_1, N_2 \in \mathbb{N}$  (depending on  $\vartheta$  and  $\epsilon$ ) and unique points  $\xi^{(-N_1)}, \dots, \xi^{(N_2)}$  in  $\mathcal{J}_n(\vartheta)$  satisfying

- (a)  $\xi^{(0)} < \xi_0 < \xi^{(1)}$ ,
- (b)  $x_{\pm}(\xi^{(i+1)}) - x_{\pm}(\xi^{(i)}) \leq \epsilon$  for all  $i = -N_1, \dots, N_2 - 1$ , and
- (c)  $\xi^{(i+1)} - \xi^{(i)} > 0, i = -N_1, \dots, N_2 - 1$ , is 'maximal possible', i.e. either  $x_+(b) - x_+(a) > \epsilon$  or  $x_-(b) - x_-(a) > \epsilon$  for any  $a$  and  $b$  such that  $(a, b) \supset [\xi^{(i)}, \xi^{(i+1)}]$ .

Notice that  $x_{\pm}(\tilde{\xi}) - x_{\pm}(\tilde{\xi}) \leq \epsilon$  for all  $\tilde{\xi}, \tilde{\xi} \in [\xi^{(i)}, \xi^{(i+1)}], \tilde{\xi} < \tilde{\xi}$ , and any  $i = -N_1, \dots, N_2 - 1$  due to the non-decreasness of both  $x_+$  and  $x_-$ . We now set

$$\mathcal{C}^{\pm}(\xi_{n,h}, \epsilon) := (x_{\pm}(\xi^{(i)}) - \epsilon, x_{\pm}(\xi^{(i+1)}) + \epsilon) \quad (\text{V.2.7})$$

for any  $h \in \mathcal{J}_n(\vartheta)$  for which  $\xi_{n,h}$  falls within the interval

$$\mathcal{I}_i^{(\epsilon)} := \begin{cases} (\xi^{(i)}, \xi^{(i+1)}] & \text{if } i = -N_1, \dots, -1, \\ (\xi^{(0)}, \xi^{(1)}) & \text{if } i = 0, \\ [\xi^{(i)}, \xi^{(i+1)}) & \text{if } i = 1, \dots, N_2 - 1. \end{cases} \quad (\text{V.2.8})$$

Q.E.D.

LEMMA V.2.2. *Let  $\vartheta > 0, \epsilon \in (0, \epsilon_0/4)$ , and  $h \in \mathcal{J}_n(\vartheta)$ . There exists a finite positive constant  $\lambda = \lambda(\vartheta, \epsilon)$  and  $n_2 = n_2(\vartheta, \epsilon) \in \mathbb{N}$  such that*

$$P_{n,h}((\mathcal{C}^+ \cup \mathcal{C}^-)^c) \leq e^{-\lambda n} \quad (\text{V.2.9})$$

as soon as  $n > n_2$ .

PROOF. Let

$$\mathcal{J}_{\xi}^{(1)} := [x_-, x_-(\xi) - \epsilon], \mathcal{J}_{\xi}^{(2)} := [x_-(\xi) + \epsilon, x_+(\xi) - \epsilon], \quad (\text{V.2.10})$$

$$\mathcal{J}_{\xi}^{(3)} := [x_+(\xi) + \epsilon, \bar{x}], \quad (\text{V.2.11})$$

and let

$$\tilde{\mathcal{C}}(\xi_{n,h}, \epsilon) := (x_+(\xi_{n,h}) - \epsilon, x_+(\xi_{n,h}) + \epsilon) \cup \quad (\text{V.2.12})$$

$$\cup (x_-(\xi_{n,h}) - \epsilon, x_-(\xi_{n,h}) + \epsilon). \quad (\text{V.2.13})$$

Since  $I_\xi(x) = \infty$  if  $x \notin [\underline{x}, \bar{x}]$  and since lower semi-continuous functions attain their infima over compact sets, Lemma V.2.3 yield

$$\begin{aligned} \inf_{(\tilde{\mathcal{C}}(\xi_{n,h}, \epsilon))^c} I_{\xi_{n,h}} &\geq \inf_{h \in \mathcal{J}_n(\vartheta)} \inf_{(\tilde{\mathcal{C}}(\xi_{n,h}, \epsilon))^c} I_{\xi_{n,h}} = \inf_{\xi \in [\xi_0 - \vartheta, \xi_0 + \vartheta]} \inf_{(\tilde{\mathcal{C}}(\xi, \epsilon))^c} I_\xi \\ &= \min_{i=1,2,3} \inf_{\xi \in [\xi_0 - \vartheta, \xi_0 + \vartheta]} \inf_{\mathcal{J}_\xi^{(i)}} I_\xi = \min_{i=1,2,3} \inf_{\mathcal{J}_{\xi_i}^{(i)}} I_{\xi_i} = \min_{i=1,2,3} I_{\xi_i}(x_i) > 0 \end{aligned} \quad (\text{V.2.14})$$

for all  $n \in \mathbb{N}$ , some points  $\xi_i = \xi_i(\vartheta)$  from  $[\xi_0 - \vartheta, \xi_0 + \vartheta]$ , and some points  $x_i = x_i(\vartheta, \epsilon)$  from  $\mathcal{J}_{\xi_i}^{(i)}$ , where  $i = 1, 2, 3$ . Defining  $\lambda(\vartheta, \epsilon)$  as  $\min_{i=1,2,3} I_{\xi_i}(x_i)/2$ , one may use the arguments of Lemma  $\circ$  to bound  $P_{n,h}(\tilde{\mathcal{C}}^c) \leq e^{-\lambda n}$  whenever  $n$  is large enough (depending on  $\vartheta$  and  $\epsilon$ ). It now suffices to realize that  $\tilde{\mathcal{C}} \subset \mathcal{C}^+ \cup \mathcal{C}^-$ . Q.E.D.

LEMMA V.2.3. *Let  $0 < \epsilon < \epsilon_0/2$  and  $\xi \in \mathbb{R}$ . Let  $\mathcal{J}_\xi^{(i)}$ ,  $i = 1, 2, 3$ , be the intervals given by (V.2.10). The functions  $g^{(i)}(\xi) := \inf_{\mathcal{J}_\xi^{(i)}} I_\xi$ ,  $i = 1, 2, 3$ , are lower semi-continuous in  $\xi$ .*

PROOF. Let  $\xi \in \mathbb{R}$  and  $i = 1, 2, 3$ . In order to show that  $g^{(i)}$  is lower semi-continuous, by Remark II.3.2 (iii) it suffices to show that

$$g^{(i)}(\xi) \leq \lim_{\delta \rightarrow 0^+} \inf_{\mathcal{U}_\delta(\xi)} g^{(i)}, \quad (\text{V.2.15})$$

where  $\mathcal{U}_\delta(\xi) := (\xi - \delta, \xi + \delta)$ . Thus, let us prove the latter. To this end, we introduce the set

$$\mathcal{M}_\xi^{(i)} := \{x \in \mathcal{J}_\xi^{(i)} : I_\xi(x) = \inf_{\mathcal{J}_\xi^{(i)}} I_\xi\}.$$

Since lower semi-continuous functions attain their infima over compact sets, the set  $\mathcal{M}_\xi^{(i)}$  is non-empty. Moreover, it is closed because it equals an intersection of two closed sets:  $\mathcal{M}_\xi^{(i)} = \text{lev}_{\inf_{\mathcal{J}_\xi^{(i)}} I_\xi} \cap \mathcal{J}_\xi^{(i)}$ .

In the first place, let us show that

$$\max_{\xi_1} \mathcal{M}_{\xi_1}^{(i)} \leq \min_{\xi_2} \mathcal{M}_{\xi_2}^{(i)} \quad \text{every time } \xi_1 < \xi_2. \quad (\text{V.2.16})$$

Given any  $\xi_1 < \xi_2$  and any  $x_\ell^{(i)} \in \mathcal{M}_{\xi_\ell}^{(i)}$ ,  $\ell = 1, 2$ , such that  $x_1^{(i)} > x_2^{(i)}$ , the definition of  $I_\xi$  implies

$$\begin{aligned} \inf_{\mathcal{J}_{\xi_2}^{(i)}} I_{\xi_2} - I_{\xi_2}(x_1^{(i)}) &= I_{\xi_2}(x_2^{(i)}) - I_{\xi_2}(x_1^{(i)}) = I_{\xi_1}(x_2^{(i)}) - I_{\xi_1}(x_1^{(i)}) - \\ &\quad - (\xi_2 - \xi_1)(x_2^{(i)} - x_1^{(i)}) > I_{\xi_1}(x_2^{(i)}) - \inf_{\mathcal{J}_{\xi_1}^{(i)}} I_{\xi_1}. \end{aligned} \quad (\text{V.2.17})$$

As  $\min \mathcal{J}_{\xi_1}^{(i)} \leq \min \mathcal{J}_{\xi_2}^{(i)} \leq x_2^{(i)} < x_1^{(i)} \leq \max \mathcal{J}_{\xi_1}^{(i)} \leq \max \mathcal{J}_{\xi_2}^{(i)}$  because of the monotonicity of  $\min \mathcal{J}_{\xi}^{(i)}$  and  $\max \mathcal{J}_{\xi}^{(i)}$  in  $\xi$ , we have  $x_1^{(i)} \in \mathcal{J}_{\xi_2}^{(i)}$  as well as  $x_2^{(i)} \in \mathcal{J}_{\xi_1}^{(i)}$ . Therefore, the assumption  $x_1^{(i)} > x_2^{(i)}$  leads to a contradiction, for the inequality (V.2.17) cannot be true. The bound (V.2.16) is thus justified.

Let us introduce the function

$$M^{(i)}(\xi) := \min \mathcal{M}_{\xi}^{(i)}. \quad (\text{V.2.18})$$

In view of (V.2.16), we have  $M^{(i)}(\xi_1) \leq \max \mathcal{M}_{\xi_1}^{(i)} \leq M^{(i)}(\xi_2)$  for any  $\xi_1 < \xi_2$ , i.e. the function  $M^{(i)}$  is non-decreasing. Hence, the one-sided limits

$$M^{(i)}(\xi \pm 0) := \lim_{\xi' \rightarrow \xi \pm 0} M^{(i)}(\xi') \quad (\text{V.2.19})$$

exist and  $M^{(i)}(\xi - 0) \leq M^{(i)}(\xi) \leq M^{(i)}(\xi + 0)$ . Moreover, the equality  $M^{(i)}(\xi' - 0) = M^{(i)}(\xi' + 0)$  fails to hold for at most countable many points  $\xi' \in \mathbb{R}$ . We shall now use the function  $M^{(i)}$  to prove (V.2.15).

The definition of  $I_{\xi}$  yields

$$\begin{aligned} \inf_{\xi' \in \mathcal{U}_{\delta}(\xi)} g^{(i)}(\xi') &= \inf_{\xi' \in \mathcal{U}_{\delta}(\xi)} I_{\xi'}(M^{(i)}(\xi')) \geq \inf_{\xi' \in \mathcal{U}_{\delta}(\xi)} I_{\xi}(M^{(i)}(\xi')) + \\ &+ \inf_{\xi' \in \mathcal{U}_{\delta}(\xi)} (-(\xi' - \xi)M^{(i)}(\xi')) + \inf_{\xi' \in \mathcal{U}_{\delta}(\xi)} I_{\xi}^*(\xi' - \xi). \end{aligned} \quad (\text{V.2.20})$$

As  $I_{\xi}^*(\xi' - \xi) = I^*(\xi') - I^*(\xi)$  and  $I^*$  is continuous (by Remark II.3.11 and the fact that  $\text{dom } I^* = \mathbb{R}$ ), we get

$$\lim_{\delta \rightarrow 0^+} \inf_{\xi' \in \mathcal{U}_{\delta}(\xi)} I_{\xi}^*(\xi' - \xi) = \lim_{\delta \rightarrow 0^+} \inf_{\xi' \in \mathcal{U}_{\delta}(\xi)} I^*(\xi') - I^*(\xi) = 0. \quad (\text{V.2.21})$$

Next, using that  $|M^{(i)}| \leq \max\{|\underline{x}|, |\bar{x}|\}$ , one finds

$$-\max\{|\underline{x}|, |\bar{x}|\} |\xi' - \xi| \leq (\xi' - \xi)M^{(i)}(\xi') \leq \max\{|\underline{x}|, |\bar{x}|\} |\xi' - \xi|. \quad (\text{V.2.22})$$

Consequently,  $\lim_{\delta \rightarrow 0^+} \inf_{\xi' \in \mathcal{U}_\delta(\xi)} (-(\xi' - \xi)M^{(i)}(\xi')) = 0$ . Finally, observing that

$$\begin{aligned} & \inf_{\xi' \in \mathcal{U}_\delta(\xi)} I_\xi(M^{(i)}(\xi')) = \\ & = \min \left\{ \inf_{\xi' \in (\xi - \delta, \xi)} I_\xi(M^{(i)}(\xi')), \inf_{\xi' \in (\xi, \xi + \delta)} I_\xi(M^{(i)}(\xi')), I_\xi(M^{(i)}(\xi)) \right\} \\ & = \min \left\{ \inf_{(M^{(i)}(\xi - \delta), M^{(i)}(\xi - 0))} I_{\xi'}, \inf_{(M^{(i)}(\xi + 0), M^{(i)}(\xi + \delta))} I_{\xi'}, I_\xi(M^{(i)}(\xi)) \right\} \end{aligned} \tag{V.2.23}$$

for  $\delta > 0$  small, we obtain

$$\begin{aligned} & \lim_{\delta \rightarrow 0^+} \inf_{\xi' \in \mathcal{U}_\delta(\xi)} I_\xi(M^{(i)}(\xi')) = \\ & = \min \{ I_\xi(M^{(i)}(\xi - 0)), I_\xi(M^{(i)}(\xi + 0)), I_\xi(M^{(i)}(\xi)) \}. \end{aligned} \tag{V.2.24}$$

Realizing that  $M^{(i)}(\xi \pm 0) \in \mathcal{J}^{(i)}(\xi)$ , it follows that

$$g^{(i)}(\xi) = \inf_{\mathcal{J}_\xi^{(i)}} I_\xi = I_\xi(M^{(i)}(\xi)) \leq I_\xi(M^{(i)}(\xi \pm 0)).$$

Combined with the above, we arrive at (V.2.15).

Q.E.D.

### Bibliography

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